

On a method of estimating derivatives in complex differential equations

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Abstract. By a recent method to estimate the derivatives $|w^{(k)}(z_i)|$, $k > 1$, at certain a -points of a meromorphic function $w(z)$ in terms of the Ahlfors-Shimizu characteristic and of $|w'(z_i)|$, we improve some classical results on the growth of meromorphic solutions of certain algebraic differential equations. Moreover, we offer similar results for equations involving inverse derivatives and derivatives of a power w^t of a meromorphic function w .

Introduction.

Recently, a method has been established to estimate $|w^{(k)}(z_i)|$ at certain a -points z_i of a meromorphic function $w(z)$ in terms of the Ahlfors-Shimizu characteristic $A(r, w)$ for $k = 1$ and in terms of $|w'(z_i)|$ for $k > 1$, see [3]. A natural idea is to apply these estimates in the field of complex differential equations. In fact, if w is a solution of $P(z, w, w', \dots, w^{(k)}) = 0$, meromorphic in \mathbf{C} , then by considering this equation in the sets $z_i(a, w)$ of “good” a -points of w , restrictions for $A(r, w)$ appear, making possible conclusions on the growth of meromorphic solutions for some classes of algebraic differential equations.

In Section 1 we apply the above method to improve a recent result due to W. Bergweiler [9] and G. Frank and Y. Wang [10]. Section 2 is devoted to considering similar equations, where the usual derivatives have been complemented by the new notion of inverse derivatives, see [8]. We obtain similar upper bounds for the growth of meromorphic solutions as in Section 1. Section 3 extends upper bound considerations to more complicated equations including derivatives of the power w^t of w in addition to derivatives and inverse derivatives.

We assume that the reader is familiar with the Nevanlinna theory including its geometric version due to Ahlfors, see e.g. [13], as well as with the results and

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notations of the preceding articles [3], [7] and [8]. However, for the convenience of the reader, we shall repeat some of the key results needed below.

Concerning the notation, if $s(r), t(r), \varphi(r) > 0$ are real-valued functions defined in the real axis, the notation $s(r) \stackrel{\varphi(r)}{\iff} t(r)$ will be applied for the double inequality $(1/(\varphi(r)))s(r) \leq t(r) \leq \varphi(r)s(r)$.

1. Algebraic differential equations.

1.1. Revisiting a result due to Gol'dberg.

An algebraic differential equation is of the form

$$P(z, w, w', \dots, w^{(k)}) = 0, \quad (1.1)$$

where P is a polynomial in each of its variables. The equation is of order k , if $w^{(k)}$ is the highest derivative appearing in P . An important part of the theory of algebraic differential equations is to investigate the order $\rho(w)$ of solutions w meromorphic in \mathbf{C} , preferably in terms of P only. For $k = 1$, A. A. Gol'dberg proved [11] that $w(z)$ must be of finite order.

The method of estimating derivatives arose in [3] where it was applied to give a new proof for the above result of A. A. Gol'dberg. We shortly recall the idea, see [3], Theorem 1. For certain a -points $z_j(a, w)$ of w lying in a disk $D(r) := \{z : |z| < r\}$ and for an arbitrary increasing real function $\varphi(r) \rightarrow \infty$ as $r \rightarrow \infty$, we have

$$|w'(z_j(a, w))| \geq \frac{A^{1/2}(r, w)}{\varphi(r)r}, \quad r \notin E, \quad (1.2)$$

where E is a set of finite logarithmic measure. By considering now the equation $P(z, w, w') = 0$ on the sets of such a -points we obtain $P(z_j(a, w), a, w'(z_j(a, w))) = 0$ which immediately yields

$$|w'(z_j(a, w))| \leq \text{const. } |z_j(a, w)|^p \leq \text{const. } r^p \quad (1.3)$$

with a rational exponent p . Now, combining (1.2) and (1.3) results in

$$A(r, w) \leq O(r^{2p+1}), \quad r \notin E,$$

hence $\rho_w \leq 2p + 1$.

1.2. Some higher order differential equations.

Concerning higher order algebraic differential equations, $k \geq 2$, the situation is more complicated, and most of the existing results are restricted to special types of equations or to meromorphic solutions under special assumptions, see e.g. [1], [2], [12], [15]. In [4], order estimates for meromorphic solutions of some classes

of second order algebraic differential equations have been given. This article seems to be the first one to apply the Ahlfors' theory of covering surfaces to studying complex differential equations. A bit later, G. Barsegian extended the results in [4] to some classes of algebraic differential equations of any order k , simultaneously improving the estimates for $\rho(w)$ in [4]. These results were first published in a short communication [5], while complete proofs appeared much later in [6]. To this end, let us consider

$$P_0(z, w)(w')^m + \sum_{j=0}^{m-1} P_j(z, w, w', \dots, w^{(k)})(w')^{m-j} = 0, \tag{1.4}$$

where P_0, \dots, P_m are polynomials in each of their variables of the form

$$P_n = \sum_{j(n)} a_{j(n)} z^{c(z, j(n))} w^{c(w, j(n))} (w'')^{c(w'', j(n))} \dots (w^{(k)})^{c(w^{(k)}, j(n))}$$

for $n = 1, \dots, m$ with constant coefficients $a_{j(n)}$. Defining now

$$p_n := \max_{j(n)} \{2c(w'', j(n)) + \dots + kc(w^{(k)}, j(n))\}, \quad n = 1, \dots, m,$$

we have the following

THEOREM A ([5], [6]). *All meromorphic solutions w of (1.4) are of finite order of growth, provided $p_n < n$ for $n = 1, \dots, m$.*

In fact, the proof in [6] implies that

$$\rho(w) \leq 2 \frac{\gamma}{1 - \delta} + 2,$$

where

$$\gamma := \frac{1}{n} \max_{1 \leq n \leq m} \{c(z, j(n))\},$$

$$\delta := \frac{1}{n} \max_{1 \leq n \leq m} p_n.$$

Recently, W. Bergweiler [9] and simultaneously G. Frank and Y. Wang [10] applied the Zalcman lemma [16] from the theory of normal families to obtain results similar to Theorem A. More precisely, they considered slightly restricted algebraic differential equations of the form

$$P_0(z, w)(w')^m - P(z, w, w', \dots, w^{(k)}) = 0, \tag{1.5}$$

where

$$P = \sum_{n=1}^N P_n(z, w) D_n[w] = \sum_{n=1}^N P_n(z, w) (w')^{c(1,n)} \dots (w^{(k)})^{c(k,n)},$$

$P_n(z, w)$ are polynomials in z and w with constant coefficients and $P_0(z, w) \neq 0$. The Zalcman lemma may be applied to obtain estimates for $\rho(w)$, as shown in [9] and [10]. In fact, define

$$p(D_n) := c(1, n) + 2c(2, n) + \dots + kc(k, n), \quad n = 1, \dots, N,$$

$$p(P) := \max_{1 \leq n \leq N} p(D_n),$$

$$c(z, n) := \deg_z P_n(z, w), \quad n = 0, \dots, N,$$

$$\alpha_n := \max \left(0, \frac{c(z, n) - c(z, 0)}{m - p(D_n)} \right),$$

$$\alpha := \max_{1 \leq n \leq N} \alpha_n,$$

$$\beta := \max_{1 \leq n \leq N} (c(z, n) - c(z, 0)).$$

With these notations we get

THEOREM B ([5], [6], [9], [10]). *Let w be a meromorphic solution of (1.5). If $m > p(P)$, then $\rho(w) \leq 2\alpha + 2$, while if $m = p(P)$ and $\beta < 0$, then $\rho(w) \leq 2$.*

Below we now apply our method to get an improvement to Theorem B. To this end, we arrange the terms in

$$P = \sum_{n=1}^H P_n(z, w) D_n[w] + \sum_{n=H+1}^N P_n(z, w) D_n[w]$$

so that $p(D_1) = \dots = p(D_H) = p(P)$ while $p(D_j) < p(P)$ for $j = H + 1, \dots, N$. Define now

$$\alpha^* := \max_{1 \leq n \leq N} \alpha_n, \quad \text{if } m > p(P)$$

and

$$\alpha^* := \max_{H+1 \leq n \leq N} \alpha_n, \quad \text{if } m = p(P) \quad \text{and} \quad \beta^* := \max_{1 \leq n \leq H} (c(z, n) - c(z, 0)) < 0.$$

THEOREM 1. *Let f be a meromorphic solution to (1.5). If $m > p(P)$ or if $m = p(P)$ and $\beta^* < 0$, then $\rho(f) \leq 2\alpha^* + 2$.*

REMARK. Obviously, if $\beta < 0$, then $\beta^* < 0$ and $\alpha^* = 0$. However, we may have $\beta^* < 0$ with $\alpha^* > 0$. Therefore, Theorem 1 is a slight improvement of Theorem B.

Before proving Theorem 1, we repeat here a key result from [8] as well as a few related notions from the earlier reference [7]. To this end, let f be meromorphic and let a_1, \dots, a_q be distinct complex numbers. Given $r > 0$, let $\Omega(r, f)$ denote a subset of the a_ν -points, $\nu = 1, \dots, q$, of f in $|z| \leq r$, and use the notation $n_0(\Omega(r, f), a_\mu)$ for the number of simple a_μ -points, in $\Omega(r, f)$. We now call the family of sets $\Omega(r, f)$ the Ahlfors set of a -points, resp. the Ahlfors set of simple a -points, if for any $r \notin E$, where E is an exceptional set of finite logarithmic measure,

$$\sum_{\nu=1}^q n_0(\Omega(r, f), a_\nu) \geq (q - 2)A(r, f) - o(A(r, f))$$

as $r \rightarrow \infty$, where

$$A(r, f) = \frac{1}{\pi} \int_0^r \rho \int_0^{2\pi} \frac{|f'(\rho e^{i\varphi})|^2}{(1 + |f(\rho e^{i\varphi})|^2)^2} d\varphi d\rho.$$

Using these notations, the essential parts of [8], Theorem 2, now read as follows:

THEOREM C. *Given a meromorphic function f , a monotone increasing function $\varphi(r) \rightarrow \infty$ as $r \rightarrow \infty$, and distinct complex numbers a_1, \dots, a_q , there exists an Ahlfors set of simple a -points. Moreover, there exist pairwise disjoint, simply connected domains $E_j(r, f)$, $j = 1, \dots, \Phi(r, f)$, in $|z| < r$ for $r \notin E$, E being an exceptional set of finite logarithmic measure, such that the following properties hold:*

- (1) *All simple a_1, \dots, a_q -points in $\bigcup_{j=1}^{\Phi(r, f)} E_j(r, f)$ form an Ahlfors set in $|z| < r$, $r \notin E$.*
- (2) *The number $\Phi(r, f)$ of the domains $E_j(r, f)$ satisfies $\Phi(r, f)/A(r, f) \rightarrow 1$ as $r \rightarrow \infty$ for $r \notin E$.*
- (3) *For any $b \in \mathbf{C}$ and a b -point $z_j(b, f) \in E_j(r, f)$,*

$$|f'(z_j(b, f))| \stackrel{\varphi(r)}{\iff} \frac{1}{d(E_j(r, f))} \geq \frac{A^{1/2}(r, f)}{r\varphi(r)},$$

where $d(E_j(r, f))$ stands for the diameter of $E_j(r, f)$.

- (4) *Given $k > 1$ and $b \in \mathbf{C}$, $z_j(b, f) \in E_j(r, f)$ as in (3) above,*

$$|f^{(k)}(z_j(b, f))| \leq (\varphi(r))^k |f'(z_j(b, f))|^k$$

for all r sufficiently large.

PROOF OF THEOREM 1. Let a_1, \dots, a_q be pairwise distinct values in \mathbf{C} , and let $E_i(r, f)$, $i = 1, \dots, \Phi(r, f)$ be pairwise disjoint simply connected domains as defined in Theorem C. We restrict our consideration to those w which belong to

$f(E_i(r, f))$ for all $i = 1, 2, \dots, \Phi(r, f)$. We now select ε and R by making use of the geometry of $E_i(r, f)$. By [7], Proposition 1, there exists a curve

$$\gamma \subset \bigcap_{i=1}^{\Phi(r, f)} f(E_i(r, f))$$

such that

$$\sup_{z', z'' \in \gamma} |z' - z''| = c_0 > 0.$$

In fact, we may take for γ the boundary of $B_0(n)$ in [7], Proposition 1. Clearly, the constant c_0 is determined by the geometry of the domains $f(E_i(r, f))$, depending on a_1, \dots, a_q only. On the other hand, by [7], Proposition 2, any of the domains $f(E_i(r, f))$ is contained in $\{w \mid |w| < c_1\}$, the constant c_1 depending on a_1, \dots, a_q only. We may now choose $\varepsilon < c_0(N + 1)^{-1}$ and $R := c_1$. By the Cartan lemma, we find some discs

$$C_j^{(n)} := \{w \mid |w - w_j^{(n)}| < r_j^{(n)}\} \subset \{w \mid |w| < R\}$$

with $\sum_j r_j^{(n)} < \varepsilon$ such that for a constant $c(\varepsilon, R, P_n)$

$$|P_n(z, w)| \stackrel{c(\varepsilon, R, P_n)}{\iff} |z|^{c(z, n)}$$

as soon as $|w| < R$, $w \notin \bigcup_j C_j^{(n)}$ and $|z| > r(\varepsilon, R, P_n)$.

Fixing now $c^* := \max_{0 \leq n \leq N} \{1, c(\varepsilon, R, P_n)\}$ we arrive at the conclusion that there exists a point

$$b \in \gamma \setminus \bigcup_{j, n} C_j^{(n)} \subset \bigcap_{i=1}^{\Phi(r, f)} f(E_i(r, f))$$

and a constant c^* , depending on P_0, \dots, P_N and a_1, \dots, a_q only such that for $z_i(b) := z_i(b, f) \in E_i(r, f)$ and for all $n = 0, \dots, N$,

$$|P_n(z_i(b), b)| \stackrel{c^*}{\iff} |z_i(b)|^{c(z, n)} \tag{1.6}$$

holds as soon as $|z_i(b)| > \max_{0 \leq n \leq N} \{1, r(\varepsilon, R, P_n)\}$. Since $z_i(b) \in E_i(r, f)$, we have $|z_i(b)| \rightarrow \infty$ as $i \rightarrow \infty$ and so (1.6) holds for all $i > i_0$ and all $n = 0, \dots, N$.

Given $z \in \mathbf{C}$, consider two terms, say $P_{n_1}(z, f(z))$ and $P_{n_2}(z, f(z))$ in (1.5) with greatest moduli. Clearly,

$$\frac{1}{N + 1} |P_{n_2}| \leq |P_{n_1}| \leq (N + 1) |P_{n_2}|, \tag{1.7}$$

since otherwise a contradiction to (1.5) would follow immediately. We may

assume $|P_{n_1}| \leq |P_{n_2}|$. If one of P_{n_1} or P_{n_2} coincides with $P_0(f')^m$, say $P_{\tilde{n}}$, then $|P_0| |f'|^m \leq (N + 1)P_{\tilde{n}}$. Otherwise, (1.7) implies that $|P_0| |f'|^m \leq (N + 1)|P_{n_2}|$. Therefore, we always find $\tilde{n} \in (1, \dots, N)$ such that

$$|P_0(z, f(z))| |f'(z)|^m \leq (N + 1)|P_{\tilde{n}}(z, f(z))|. \tag{1.8}$$

Assume now $m > p(P)$. By (1.6) and Theorem C(4) above, which gives upper bounds for higher derivatives of f , we get

$$\frac{1}{c^*} |z_i(b)|^{c(z,0)} |f'(z_i(b))|^m \leq c^*(N + 1) |z_i(b)|^{c(z,\tilde{n})} |f'(z_i(b))|^{p(D_{\tilde{n}})} \varphi^{p(D_{\tilde{n}})}(r) \tag{1.9}$$

for all $i > i_0$. Therefore,

$$\begin{aligned} |f'(z_i(b))| &\leq [(c^*)^2(N + 1)]^{1/(m-p(D_{\tilde{n}}))} |z_i(b)|^{\alpha_{\tilde{n}}} (\varphi(r))^{p(D_{\tilde{n}})/(m-p(D_{\tilde{n}}))} \\ &\leq [(c^*)^2(N + 1)]^{1/(m-p(D_{\tilde{n}}))} r^{\alpha_{\tilde{n}}} (\varphi(r))^{p(D_{\tilde{n}})/(m-p(D_{\tilde{n}}))} \end{aligned} \tag{1.10}$$

and so

$$\begin{aligned} |f'(z_i(b))| &\leq [(c^*)^2(N + 1)]^{1/(m-p(D_{\tilde{n}}))} r^{\alpha^*} (\varphi(r))^{p(D_{\tilde{n}})/(m-p(D_{\tilde{n}}))} \\ &\leq [(c^*)^2(N + 1)]^{1/(m-p(P))} r^{\alpha^*} (\varphi(r))^{p(P)/(m-p(P))}. \end{aligned}$$

By Theorem C(3), we now obtain

$$A(r, f) \leq [(c^*)^2(N + 1)]^{1/(m-p(P))} r^{2\alpha^*+2} (\varphi(r))^{2+p(P)/(m-p(P))}$$

as $r \rightarrow \infty$, $r \notin E$, where E is an exceptional set of finite logarithmic measure as in Theorem C. Choosing now $\varphi(r) = r^\varepsilon$ we get the assertion.

Finally, we consider the case $m = p(P)$ with $\beta^* < 0$. If the dominant term in (1.8) now would be such that $m = p(D_{\tilde{n}})$, (1.9) would take the form

$$1 \leq (c^*)^2(N + 1) |z_i(b)|^{\beta^*} (\varphi(r))^m.$$

Assuming that $|z_i(b)| \geq r/2$ and taking $\varphi(r) = r^\varepsilon$, this is a contradiction, provided ε is small enough, as $r \rightarrow \infty$. Therefore, we have $m > p(D_{\tilde{n}})$ in (1.8), and so the preceding part may be applied with minor modifications only, i.e. by considering the terms $P_j(z, f)$ with $j = H + 1, \dots, N$ only. □

REMARK. Provided the solution $w(z)$ of (1.4) is the derivative of a meromorphic function $W(z)$, a variant of Theorem 1 follows. Of course, this is the case if $w(z)$ is entire. In fact, (1.5) may be rewritten as

$$P_0(z, W')(W'')^m - P(z, W', W'', \dots, W^{(k+1)}) = 0,$$

where

$$P(z, W', W'', \dots, W^{(k+1)}) = \sum_{n=1}^N \hat{P}_n(z) \hat{D}_n[W],$$

$$\hat{D}_n[W] = (W')^{c(0,n)} (W'')^{c(1,n)} \dots (W^{(k+1)})^{c(k,n)}.$$

This results in a modified weight

$$\hat{p}(\hat{D}_n) := c(0, n) + 2c(1, n) + \dots + (k + 1)c^{(k, n)} \quad (>p(D_n))$$

to be applied for defining modified quantities $\hat{p}(P), \hat{c}(z, n) = \deg \hat{P}_n(z), \hat{\alpha}_n, \hat{\alpha}$ and $\hat{\beta}$. These modified weights have been applied by W. Hayman in [14]. In fact, [14] is the most comprehensive description for the growth of entire solutions of algebraic differential equations. As usual for entire solutions, the Wiman-Valiron method was applied in [14].

Comparing Theorem A, resp. Theorem 1, above to the results offered by Hayman in [14], we observe that the term of highest weight in (1.5), resp. (1.4), is the first term, while the highest term in [14], Theorem C, is of no specific form. Since [14], Theorem C, is restricted to entire solutions only, it is natural to ask whether meromorphic solutions of an algebraic differential equation of the general form (1.1) permit a counterpart to Theorem 1. This question remains open.

2. Complex differential equations with inverse meromorphic derivatives.

In [8], a new type of meromorphic functions $F^{(U)}(w(z))$ associated with $w(z)$ has been introduced. $F^{(U)}(w(z))$ is the composition of $F_i^{(U)}(w)$ and $w(z)$, where for any point z_i with $w(z_i) = a$ we define $F_i(a)$ to be that of the branches $F_i(w)$ of the inverse function $F(w)$ to $w(z)$ for which $F_i(w(z_i)) = z_i$ and we denote by $F_i^{(U)}(w)$ the U^{th} derivative of $F_i(w)$ with respect to w . These functions $F_i^{(U)}(w(z))$ were called “inverse meromorphic derivatives”; they are meromorphic functions. Indeed, if z_0 is an ordinary point then clearly in a small neighbourhood of z_0 , the function $F_i^{(U)}(w(z))$ is single-valued; if z is a multiple point with multiplicity k then $F_i(w)$ has a representation of the form $z_0 + a_1(w - w(z_0))^{1/k} + a_2(w - w(z_0))^{2/k} + \dots$ and $w(z) - w(z_0)$ a representation of the form $b_k(z - z_0)^k + b_{k+1}(z - z_0)^{k+1} + \dots$ so that $F_i^{(U)}(w(z))$ is single-valued in a small neighbourhood of z_0 . Moreover, z_0 is a pole of multiplicity $k - 1$ for $F_i^{(U)}(w(z))$. Thus the composition $F_i^{(U)}(w(z))$ of $F_i^{(U)}(w)$ and $w(z)$ is a meromorphic function. In particular, $F_i^{(1)}(w(z))$ equals to $1/w'(z)$.

In this section we consider algebraic differential equations involving inverse meromorphic derivatives together with usual derivatives. To this end, consider an equation of the form

$$P_0(z, w)(w')^m - P(z, w, w', \dots, w^{(k)}, F', \dots, F^{(s)}) = 0, \tag{2.1}$$

where

$$P = \sum_{n=1}^N P_n(z, w) G_n[w, F],$$

and $G_n[w, F]$, $n = 1, 2, \dots, N$, are some differential monomials of the form

$$(w')^{c(1,n)} \dots (w^{(k)})^{c(k,n)} (F')^{d(1,n)} \dots (F^{(s)})^{d(s,n)}.$$

Here $m, N, k, s \in \mathbb{N}$, $P_n(z, w)$ are polynomials in z and w with constant coefficients and $P_0(z, w) \neq 0$.

The weight $p(G_n)$ of the monomial $G_n[w, F]$ is now defined as

$$p(G_n) := c(1, n) + 2c(2, n) + \dots + kc(k, n) - [d(1, n) + d(2, n) + \dots + d(s, n)];$$

and the weight of P as

$$\tilde{p}(P) := \max_{1 \leq n \leq N} p(G_n).$$

Now, similarly as to Section 1 we may arrange the terms of P to those monomials $G_n^*[w, F]$ with the highest weight $\tilde{p}(P)$ and to $G_n[w, F]$ with the weights $p(G_n) < \tilde{p}(P)$ so that (2.1) takes the form

$$P_0(z, w)(w')^m - \sum_{n=1}^H P_n^{(*)}(z, w) G_n^{(*)}[w] + \sum_{n=H+1}^N P_n(z, w) G_n[w] = 0. \quad (2.2)$$

Defining $c(z, 0) := \deg_z P_0(z, w)$, $c(z, n) := \deg_z P_n^{(*)}(z, w)$ for $n = 1, 2, \dots, H$ and $c(z, n) := \deg_z P_n(z, w)$ for $n = H + 1, \dots, N$, we obtain

$$\tilde{\alpha}_n := \max \left[\frac{c(z, n) - c(z, 0)}{m - p(G_n)}, 0 \right].$$

Finally, denote

$$\tilde{T} := \max_{1 \leq n \leq N} \tilde{\alpha}_n, \quad \text{if } m > p$$

and

$$\tilde{T} := \max_{H+1 \leq n \leq N} \tilde{\alpha}_n, \quad \text{if } m = p(P) \quad \text{and} \quad \tilde{\beta}^* := \max_{1 \leq n \leq H} (c(z, n) - c(z, 0)) < 0.$$

With these notations, the following generalized version of Theorem 1 follows:

THEOREM 2. *Let $w(z)$ be a solution of (2.2) satisfying either a) $m > \tilde{p}(P)$, or b) $m = \tilde{p}(P)$ and $\tilde{\beta}^* < 0$. Then $\rho_w \leq 2T + 2 \leq \infty$.*

PROOF. While deriving the inequality (1.9) we have used Theorem C(4), to obtain upper bounds for the higher order derivatives $w^{(u)}(z_i(b, w))$ in terms of $|w'(z_i(b, w))|$. In the present situation, we now apply [8], Theorem 2(7), which gives upper bounds for $|F_i^{(U)}(w(z_i(b, w)))|$ in terms of $|w'(z_i(b))|$. In fact, cor-

responding to Theorem C(4), the same inequality now applies with $f^{(k)}$, f' replaced by the corresponding inverse derivatives $F^{(k)}$, F' , see [8], Theorem 2(7), for details and some further inequalities. Then, instead of the monomials $D_{\tilde{n}}$, we deal with $G_{\tilde{n}}$, and instead of (1.9) we obtain

$$\frac{1}{c^*} |z_i(b)|^{c(z,0)} |w'(z_i(b))|^m \leq c^*(N+1) |z_i(b)|^{c(z,\tilde{n})} |w'(z(b))|^{p(G_{\tilde{n}})} \varphi^{p(G_{\tilde{n}})}(r).$$

Similarly as to the proof of Theorem 1, we now complete the proof of Theorem 2.

3. Complex differential equations having composite terms.

Recently, composite entire and meromorphic functions have been under a considerable interest. Moreover, several papers have been devoted to the (pseudo)primeness of meromorphic solutions of certain classes of differential equations. On the other hand, it is well-known that for certain differential equations, their meromorphic solutions turn out to be composite functions.

In this section, we consider differential equations involving as variables derivatives of the special composite function w^t , $t \in N$. More precisely, let us consider

$$P^*(z, w, w', \dots, w^{(k)}, F', \dots, F^{(s)}, [w^t]', \dots, [w^t]^{(c)}) = 0, \tag{3.1}$$

where P^* is a polynomial in all of its variables, $k, s, t, c \in N$ and $t \leq c$. Suppose further that for $t < c$, substituting $w, [w^t]', \dots, [w^t]^{(c)}$ by zero we obtain the equation

$$P^*(z, 0, w', \dots, w^{(k)}, F', \dots, F^{(s)}, 0, \dots, 0) = 0 \tag{3.2}$$

which is of the form (2.1) and for $t = c$, substituting $w, [w^t]', \dots, [w^t]^{(c-1)}$ by zero and $[w^t]^{(c)}$ by $c!|w'|^c$ we obtain the equation

$$P^*(z, 0, w', \dots, w^{(k)}, F', \dots, F^{(s)}, 0, \dots, 0, c!|w'|^c) = 0, \tag{3.3}$$

also of the form (2.1). As in Section 2, (3.2) and (3.3) may be written in the form

$$P_0(z, 0)(w')^m - \sum_{n=1}^H P_n^{(*)}(z, 0)G_n^{(*)}[w] + \sum_{n=H+1}^N P_n(z, 0)G_n[w] = 0 \tag{3.4}$$

corresponding to (2.2). Here $G_n^{(*)}[w]$ are differential monomials with coefficients $P_n^{(*)}(z, 0)$ and with the highest weight

$$\tilde{p}_0(P) := \max \left[\max_{1 \leq n \leq H} p(G_n^{*}), \max_{H+1 \leq n \leq N} p(G_n) \right],$$

and $G_n[w]$ are differential monomials with coefficients $P_n(z, 0)$ and the weight of the $p(G_n) < \tilde{p}_0(P)$. Set $c_0(z, 0) = \deg_z P_n(z, 0)$ for $n = 0$, $c_0(z, n) = \deg_z P_n^{(*)}(z, 0)$ for $n = 1, 2, \dots, H$, $c_0(z, n) = \deg_z P_n(z, 0)$ for $n = H + 1, \dots, N$, and

$$\tilde{\alpha}_n(0) := \max \left[\frac{c_0(z, n) - c_0(z, 0)}{m - p(G_n)}, 0 \right].$$

Denote now

$$T(0) := \max_{q \leq n \leq N} \tilde{\alpha}_n(0), \quad \text{if } m > \tilde{p}_0(P)$$

and

$$T(0) := \max_{H+1 \leq n \leq N} \tilde{\alpha}_n(0), \quad \text{if } m = \tilde{p}_0(P) \quad \text{and}$$

$$\tilde{\beta}^*(0) := \max_{1 \leq n \leq H} (c_0(z, n) - c_0(z, 0)) < 0.$$

It turns out that if for such a reduced form of (3.2) and (3.3), the conditions of Theorem 2 are satisfied, then any meromorphic solution w of (3.4) has to be of finite order ρ_w , provided that the value 0 is “good” in the sense of the theory of covering surfaces. To this notion, let Y_0 be a domain containing the origin and let $n_0(r, Y_0, w)$ be the number of simple islands of the covering surface $F_r := \{w(z) \mid |z| < r\}$ over the domain Y_0 . Then we say that $0 \in \mathbb{C}$ is good if

$$\delta_0(Y_0) := \liminf_{r \rightarrow \infty} \frac{n_0(r, Y_0, w)}{A(r, w)} > 0. \tag{3.5}$$

If $\delta_0(Y_0) = 0$, this qualitatively means that the ramification of F_r in a neighbourhood of the value 0 is maximal, so that the value 0 is exceptional indeed in the Ahlfors theory sense.

With these notations, the following result is valid.

THEOREM 3. *Consider the equation (3.1), reduced over (3.2) or (3.3) to the form (3.4) with a) $m > \tilde{p}_0(P)$ or b) $m = \tilde{p}_0(P)$ and $\tilde{\beta}^*(0) < 0$. If $w(z)$ is a meromorphic solution of (3.1) such that 0 is a good value, then $\rho_w \leq 2T(0) + 2 < \infty$.*

COROLLARY. *Suppose that in the differential equation*

$$P_0(z, w)(w^m)^{(m)} - P(z, w, w', \dots, w^{(k)}, F', \dots, F^{(s)}) = 0 \tag{3.6}$$

we have either a) $m > \tilde{p}_0(P)$ or b) $m = \tilde{p}_0(P)$ and $\tilde{\beta}^(0) < 0$. If $w(z)$ is a meromorphic solution of the equation (3.6) such that 0 is a good value, then $\rho_w \leq 2T(0) + 2 \leq \infty$.*

THEOREM 4. *Suppose that a differential equation*

$$P(z, w, w', \dots, w^{(k)}, F', \dots, F^{(s)}) = 0$$

can be rewritten in the form

$$P^*(z, w, w', \dots, w^{(k)}, F', \dots, F^{(s)}, [\phi(w)]', \dots, [\phi(w)]^{(c)}) = 0, \tag{3.7}$$

where P is a polynomial in all its variables, $k, s, c \in \mathbf{N}$ and ϕ is a meromorphic function satisfying $\phi(0) = \phi'(0) = \dots = \phi^{(c)}(0) = 0$. Suppose also that by substituting w by zero in this equation we get an equation of the form (2.1) with a) $m > \tilde{p}_0(P)$ or b) $m = \tilde{p}_0(P)$ and $\tilde{\beta}^*(0) < 0$. If $w(z)$ is a meromorphic solution of (3.7) such that 0 is a good value, then $\rho_w \leq 2T(0) + 2 < \infty$.

PROOF OF THEOREM 3. We now make use of [8], Theorem 3. By this theorem, given $\varepsilon > 0$, if w is a meromorphic function satisfying (3.5) for a domain Y_0 , then by taking $a_1 = 0$ we may choose a_1, \dots, a_q such that Theorem C is true. By Theorem C(1) and the definition of the Ahlfors set, we get

$$n_0(r, 0, w) \geq (\delta_0(Y_0) - \varepsilon)A(r, w) \rightarrow 1, \quad r \notin E, \quad r > r_0(\varepsilon, w, \varphi),$$

where $n_0(r, 0, w)$ is the number of simple 0-points $z_i(0, w)$ of w belonging to $\bigcup_{k=1}^{\Phi(r, w)} A_i(r, w)$.

Consequently, assuming $0 < \varepsilon < \delta_0(Y_0)$, we have some simple zeros $z_i(0, w) \in E_i(r, w)$ of w in $\{z : |z| < r\}$, $r \notin E$. Consider the equation (3.1) on the set of these simple 0-points $z_i = z_i(0, w)$. Thus for $z_i = z_i(0, w)$, we have $w(z_i) = 0$ and for $[w^t(z_i)]^{(c)}$ simple calculations yield $[w^t(z_i)]^{(c)} = 0$ when $t < c$ and $[w^c(z_i)]^{(c)} = c![w'(z_i)]^c$ when $t = c$. Therefore, in the case when $t < c$, the equation (3.1) being considered at the points $z_i = z_i(0, w)$ takes the form

$$P^*(z_i, 0, w'(z_i), \dots, w^{(k)}(z_i), F'(w(z_i)), \dots, F^{(s)}(w(z_i)), 0, \dots, 0) = 0.$$

Due to the conditions of our theorem, the equation can be rewritten in the form (3.4). If $t = c$, the equation (3.1) being considered at the points $a_i = z_i(0, w)$ takes the form

$$P^*(z_i, 0, w'(z_i), \dots, w^{(k)}(z_i), F'(w(z_i)), \dots, F^{(s)}(w(z_i)), 0, \dots, 0, c![w'(z_i)]^c) = 0.$$

Again, the equation can be rewritten in the same form (3.4). Now we are in the same situation as for Theorem 2 with the only difference that instead of the constants $c(z, n)$ we deal with the $c_0(z, n)$. Therefore, similarly as to the proofs of Theorem 1 and Theorem 2, we obtain Theorem 3.

To prove the Corollary, it is enough to note that $[w^m(z_i)]^m = c![w'(z_i)]^m$ so that the equation (3.6) being considered at the points $z_i = z_i(0, w)$ can be rewritten in the form (3.4) and we may proceed as to above in the proof of Theorem 3.

For Theorem 4, it is enough to note that the equation (3.7) can be rewritten in the same form (3.4).

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