Existence and nonexistence of global solutions of quasilinear parabolic equations

Dedicated to Professor Kunihiko Kajitani on his sixtieth birthday

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Abstract. We consider nonnegative solutions to the Cauchy problem for the quasilinear parabolic equations $u_t = \Delta u^m + K(x)u^p$ where $x \in \mathbf{R}^N$, $1 \le m < p$ and $K(x) \ge 0$ has the following properties: $K(x) \sim |x|^\sigma$ ($-\infty \le \sigma < \infty$) as $|x| \to \infty$ in some cone D and $K(x) = 0$ in the complement of D, where for $\sigma = -\infty$ we define that $K(x)$ has a compact support. We find a critical exponent $p^*_{m,\sigma} = p^*_{m,\sigma}(N)$ such that if $p \le p^*_{m,\sigma}$, then every nontrivial nonnegative solution is not global in time; whereas if $p > p^*_{m,\sigma}$ then there exits a global solution. We also find a second critical exponent, which is another critical exponent on the growth order α of the initial data $u_0(x)$ such that $u_0(x) \sim |x|^{-\alpha}$ as $|x| \to \infty$ in some cone D' and $u_0(x) = 0$ in the complement of D'.

1. Introduction.

In this paper we shall consider the Cauchy problem

$$(1.1) \qquad u_t = \Delta u^m + K(x)u^p \quad (x, t) \in \mathbf{R}^N \times (0, T),$$

$$(1.2) \qquad u(x, 0) = u_0(x) \quad x \in \mathbf{R}^N,$$

where $u_t = \partial u / \partial t$, $m \ge 1$, $p > 1$, $K(x) \ge 0$, $\in L^\infty_{loc}$ and $u_0(x) \ge 0$, $\in C(\mathbf{R}^N)$. We shall only consider nonnegative solutions u. We are interested in the existence and nonexistence of global solutions.

When $K(x) \equiv 1$, the next results are well known to hold: When $u_0(x) \in L^\infty(\mathbf{R}^N)$ a unique nonnegative weak solution of (1.1), (1.2) exists locally in time and can be extended as the time increases as far as $u(\cdot, t) \in L^\infty(\mathbf{R}^N)$. Further,

(I) Let $1 < p \le m + 2/N$. Then all nontrivial solutions $u(x, t)$ of (1.1), (1.2) do not exist globally in time. Namely $\lim_{t \uparrow T} \|u(t)\|_\infty = \infty$ for some $T > 0$.

(II)   Let $p > m + 2/N$.   Then there exists a constant $A > 0$ such that if

$$(1.3) \qquad \liminf_{|x| \to \infty} |x|^{2/(p-m)} u_0(x) \geq A,$$

then the solution of (1.1), (1.2) does not exist globally in time.

(III)   Let $p > m + 2/N$.   Then, for any $\alpha > 2/(p-m)$ there exists $h > 0$ such that if

$$(1.4) \qquad u_0(x) \leq h \langle x \rangle^{-\alpha} \quad \text{in } \boldsymbol{R}^N$$

then the problem (1.1), (1.2) has a global solution, where

$$(1.5) \qquad \langle x \rangle = \sqrt{1 + |x|^2}.$$

In case $p \neq m + 2/N$, (I) is due to Fujita [10] for $m = 1$ and Galaktionov et al. [12] for $m > 1$.   In case $p = m + 2/N$, (I) is due to Hayakawa [15] for $m = 1$, $N = 1, 2$, Kobayashi et al. [20] for $m = 1$, $N \geq 3$ and Galaktionov [12] (see also [19] and [27]) for $m > 1$.   (II) is due to Lee and Ni [23] for $m = 1$ and Mukai, Mochizuki and Huang [29] (see also [34]) for $m > 1$.   (III) is due to Lee and Ni [23] (including the case when $\alpha = 2/(p-m)$) for $m = 1$ and Kawanago [19] (see also [34] and [29]) for $m > 1$.

Case (I) is called the blow-up case, case (III) is called the global existence case.   The cut off number

$$(1.6) \qquad p_m^* = m + \frac{2}{N}$$

is called the critical exponent.   When the critical exponent is in the blow-up case we say the blow-up is the critical blow-up.   Also we see, from (II) and (III), that under the condition $p > m + 2/N$, the number

$$(1.7) \qquad \alpha^* = \frac{2}{p-m}$$

is another critical exponent on the growth order of the initial data $u_0(x)$.   It is called the second critical exponent ([23], [25]).   Namely, when we assume

$$(1.8) \qquad \lim_{|x| \to \infty} |x|^\alpha u_0(x) = A$$

for some $\alpha \in \boldsymbol{R}$ and $A > 0$, the following results hold: When $\alpha < 2/(p-m)$, the solution of (1.1), (1.2) does not exist globally in time.   On the other hand, when $\alpha > 2/(p-m)$, there exists a global solution of (1.1), (1.2) with the initial data εu_0 where $\varepsilon > 0$ is small enough.

So, we shall study about these critical exponents to more general $K(x)$.

When

(1.9)
$$K(x) = |x|^\sigma \quad (|x| \geq R)$$

for some $\sigma \in [-\infty, \infty)$ and $R > 0$ where we define that $K(x) = 1$ in $|x| \leq R$ and $K(x) = 0$ in $|x| > R$ in case $\sigma = -\infty$, Andreuichi and DiBenedetto [2] (see Wang [35] for $m = 1$) showed that if $\langle x \rangle^{a_\sigma^*} u_0(x) \in L^\infty$ then a solution exists locally in time, where

(1.10)
$$a_\sigma^* = \max\left\{\frac{\sigma}{p-1}, \frac{-2}{m-1}\right\}.$$

Moreover, when $m = 1$, the problems concerning the existence and nonexistence of global solutions have been studied by many authors ([4], [35], [13], [14]). Further, Pinsky [32] recently showed the very interesting results about them. We combine these results as follows: Put

(1.11)
$$p_{1,\sigma}^* = 1 + \frac{2 + \max\{\sigma, -N\}}{N}.$$

$(I_{1,\sigma})$ Let $1 < p \leq p_{1,\sigma}^*$. Then all nontrivial solutions $u(x, t)$ of (1.1), (1.2) do not exist globally in time.

$(II_{1,\sigma})$ Let $p > \max\{p_{1,\sigma}^*, 1\}$. Then there exists a constant $A > 0$ such that when

(1.12)
$$\liminf_{|x| \to \infty} |x|^{[2+\sigma]_+/(p-1)} u_0(x) > A,$$

any solution of (1.1), (1.2) does not exist globally in time. Especially, when $N = 2$, $\sigma \leq -2$ or $N \geq 3$, $\sigma = -2$ we can take $A = 0$ in (1.12).

$(III_{1,\sigma})$ Let $p > \max\{p_{1,\sigma}^*, 1\}$. If $u_0(x) \leq \delta e^{-k|x|^2}$ for small $\delta > 0$ and $k > 0$ then a global solution of (1.1), (1.2) exists. Further let $\sigma \geq 0$ or $\sigma < -2$, $N \geq 3$. Then, for any $\alpha > (2 + \sigma)/(p - 1)$ there exists a constant $h > 0$ such that if

(1.13)
$$u_0(x) \leq h \langle x \rangle^{-[\alpha]_+} \quad \text{in } \mathbf{R}^N$$

then a global solution of (1.1), (1.2) exists.

Namely, when $p_{1,\sigma}^* > 1$, $p_{1,\sigma}^*$ is the critical exponent. When $\sigma \geq 0$ or $\sigma < -2$, $N \geq 3$, $[2+\sigma]_+/(p-1)$ is the second critical exponent. $(II_{1,\sigma})$ with $\sigma > -2$ is due to Wang [35], $(III_{1,\sigma})$ with $\sigma \geq 0$ is due to Hamada [14] and the rest is due to Pinsky [32] (see also Zhang [36] for $(III_{1,\sigma})$ with $N \geq 3$, $\sigma < -2$, and [13] and [4] for $(I_{1,\sigma})$). But we do not see yet what is the second critical exponent when $N \geq 3$, $-2 \leq \sigma < 0$ or $N = 1, 2$, $\sigma < 0$. Since Pinsky's analyses are essentially based on the expression of solution by the heat kernel entering in the semilinear equation, his methods of the proof can not be applied to case $m > 1$.

When $m > 1$ there are a few works studying these problems. In the case whrere $K(x) = |x|^\sigma$ ($\sigma \geq 0$), Mukai [28] (see also [25]) showed that when $0 \leq \sigma < N(p-1)$, $m + (2+\sigma)/N$ is the critical exponent and belongs to the blow-up case (He also obtained the results about the global existence case when $\sigma \geq N(p-1)$), and $(2+\sigma)/(p-m)$ is the second critical exponent. But it is not established what is the critical exponent in case $\sigma \geq N(p-1)$ and what is the second critical exponent in case $\sigma < 0$. His methods of the proof in the blow-up case are based on the Jensen's inequality for an integration in $\mathbf{R}^N$ and can not be applied to general cases, for example, (1.9) with $R > 0$ or $K(x) = K_D(x)$ bellow.

Thus, we have the following three questions when $K(x)$ satisfies (1.9):

QUESTION 1. In the case $m = 1$, what is the second critical exponent when $N \geq 3$, $-2 \leq \sigma < 0$ or $N = 1, 2$, $\sigma < 0$?

QUESTION 2. In the case $m > 1$, what is the critical exponent when $\sigma \geq N(p-1)$?

QUESTION 3. In the case $m > 1$, what is the second critical exponent when $\sigma < 0$?

Our purpose is to solve these problems Question $1 \sim 3$ in the case where $p > m$ and $K(x)$ satisfies (1.9), and to extend the above results to more general $K(x)$, for example, $K(x) = K_D(x)$ which vanishes in some region of $\mathbf{R}^N$ as follows: For $\sigma \in (-\infty, \infty)$

$$(1.14) \qquad K_D(x) = \begin{cases} |x|^\sigma & \text{if } x \in D \cap \{|x| > 1\}, \\ 0 & \text{otherwise}, \end{cases}$$

and for $\sigma = -\infty$

$$(1.15) \qquad K_D(x) = \begin{cases} 0 & \text{if } |x| > 1, \\ 1 & \text{if } |x| \leq 1, \end{cases}$$

where $D = \mathbf{R}^N$ or a cone with vertex at the origin, that is, $D = \{x \in \mathbf{R}^N \backslash \{0\}; x/|x| \in \Omega\}$ and Ω ($\neq \varnothing$) $\subset S^{N-1}$ is an open connected subset with smooth boundary. In this paper, we obtain the following results: Put

$$(1.16) \qquad p_{m,\sigma}^* = m + \frac{2 + \max\{\sigma, -N\}}{N},$$

$$(1.17) \qquad \alpha_\sigma^* = \frac{2 + \max\{\sigma, -N\}}{p - m} = \frac{(p_{m,\sigma}^* - m)N}{p - m}.$$

THEOREM 1. (i) *Let* $m < p \leq p_{m,\sigma}^*$. *Then all nontrivial solutions* $u(x,t)$ *of* (1.1), (1.2) *do not exist globally in time.*

(ii)   *Let $p > \max\{p^*_{m,\sigma}, m\}$.   Then, if there exist an open subset $V \subset S^{N-1}$ with $|V| \neq 0$ and a constant $A > 0$ such that*

$$(1.18) \qquad \liminf_{r \to \infty} r^{[\alpha^*_\sigma]_+} u_0(r\xi) > A \quad \text{for } \xi \in V,$$

any solution of (1.1), (1.2) does not exist globally in time, where $|V|$ is the Lebesgue measure of V. Especially, in case $\alpha^*_\sigma = 0$, (adding the assumption $\Omega = S^{N-1}$ when $N \geq 3$) we can replace assumption (1.18) by the following condition:

$$(1.19) \qquad \lim_{r \to \infty} \inf_{|x| \geq r} u_0(x) > 0.$$

(iii)   *Let $p > \max\{p^*_{m,\sigma}, m\}$.   Then, for any $\alpha > \alpha^*_\sigma$ and $A > 0$ there exists a constant $\varepsilon > 0$ such that if*

$$(1.20) \qquad u_0(x) \leq \min\{\varepsilon, A|x|^{-[\alpha]_+}\}$$

then a global solution of (1.1), (1.2) exists.

Namely, $p^*_{m,\sigma}$ is the critical exponent when $p^*_{m,\sigma} > m$, and $[\alpha^*_\sigma]_+$ is the second critical exponent.   We note that in Theorem 1 (ii) we do not require the assumption $V \subset \Omega$.

REMARK 1.1.   As in Pinsky [**32**], in the assumptions of (i), (ii) of Theorem 1 (blow-up case), no growth restrictions as $|x| \to \infty$ are made on the solution.

REMARK 1.2.   In this paper, we do not consider the case where $1 < p \leq m$, since it is difficult to apply our methods to this case.

Our proof of the blow-up case (i) and (ii) of Theorem 1 is simpler and more united than that of other papers.   The methods of this proof are based on the Jensen's inequality for the integration in a bounded domain, the scaling argument for the equation and the correct asymptotic behavior of a solution of (1.1) with $K(x) = 0$.   In the proof of (iii) of Theorem 1 we must divide it into three cases.   In the case $\sigma \geq 0$, the methods of the proof are similar to those of Mukai, Mochizuki and Huang [**29**].   Namely, we use a supersolution constructed by the solution of equation (1.1) with $K(x) \equiv 0$.   In the case $\sigma < 0$, $\alpha^*_\sigma \geq 0$ we use the $L^\infty - L^\ell$ estimates for solutions due to Kawanago [**19**].   In the case $\sigma < 0$, $\alpha^*_\sigma < 0$, we construct a supersolution by stationary solutions for the proof.

The rest of the paper is organized as follows.   In the next Section 2, we define a weak solution of (1.1) and state main results (Theorem 2.4 and 2.5).   Further we prepare the fundamental propositions and several preliminary lemmas.   Theorem 2.4 (a) is the generalization of Theorem 1 (i) and proven in Section 3 (in the case $N \geq 2$ and the case $N = 1$, $\sigma \geq -1$) and Section 4 (in the

case $N = 1$, $\sigma \leq -1$). Theorem 2.4 (b) is the generalization of Theorem 1 (ii) and proven in Section 5 (in the general case) and Section 6 (in the special case $\alpha_\sigma^* = 0$). Theorem 2.5 is the generalization of Theorem 1 (iii) and proven in Section 7 (in the case $\sigma \geq 0$ and the case $\sigma < 0$, $\alpha_\sigma^* < 0$) and Section 8 (in the case $\sigma < 0$, $\alpha_\sigma^* \geq 0$).

2. Definitions and main results.

Let $u_0(x) \in C(\mathbf{R}^N)$, $u_0 \geq 0$ in $\mathbf{R}^N$ and $K(x) \in L_{loc}^\infty(\mathbf{R}^N)$. In this section we state the definition of a weak solution of (1.1) and the main results.

We begin with the definition of a weak solution of (1.1).

DEFINITION 2.1. Let G be a domain in $\mathbf{R}^N$. By a weak solution of equation (1.1) in $G \times [0, T)$, we mean a function $u(x, t)$ in $G \times [0, T)$ such that
   (i) $u(x, t) \geq 0$ in $\bar{G} \times [0, T)$ and $\in C(\bar{G} \times [0, \tau])$ for each $0 < \tau < T$.
   (ii)   For any bounded domain $\Omega \subset G$, $0 < \tau < T$ and nonnegative $\varphi(x, t) \in C^{2,1}(\bar{\Omega} \times [0, T))$ which vanishes on the boundary $\partial\Omega$,

$$(2.1) \qquad \int_\Omega u(x, \tau)\varphi(x, \tau)\, dx - \int_\Omega u(x, 0)\varphi(x, 0)\, dx$$

$$= \int_0^\tau \int_\Omega \{u\partial_t\varphi + u^m \Delta\varphi + K(x)u^p\varphi\}\, dxdt - \int_0^\tau \int_{\partial\Omega} u^m \partial_n\varphi\, dSdt$$

where n denotes the outer unit normal to the boundary.

A supersolution [or subsolution] is similarly defined with equality of (2.1) replaced by $\geq$ [or $\leq$].

Here, we note that for each $t \geq 0$, any restriction on the growth order of a weak solution $u(x, t)$ in $\mathbf{R}^N \times [0, T)$ as $|x| \to \infty$ is not required in the above definition. Hence, we do not know whether or not the uniqueness of weak solutions of (1.1), (1.2) holds.

The following comparison theorem is due to Bertsch, Kersner and Peletier [6] (see Appendix of [6]).

PROPOSITION 2.2 (comparison theorem). *Let G be a bounded domain with smooth boundary in $\mathbf{R}^N$ or let $G = \mathbf{R}^N$ and $K(x) \in L^\infty(\mathbf{R}^N)$. Let u (or v) be a supersolution (or a subsolution) of (1.1) in $G \times [0, T)$. If $u \geq v$ on the parabolic boundary of $G \times (0, T)$ and $u, v \in L^\infty(G \times (0, T))$, then we have $u \geq v$ in the whole $\bar{G} \times [0, T)$.*

In order to state our results we shall use the following spaces of functions. For $\alpha \in (-\infty, \infty)$ let $L_\alpha^\infty = \{f \in L_{loc}^\infty; \|f\|_{\infty,\alpha} \equiv \sup_{x \in \mathbf{R}^N} \langle x \rangle^\alpha |f| < \infty\}$, which is a Banach space with norm $\|\cdot\|_{\infty,\alpha}$. We set for $\alpha \in (-\infty, \infty)$,

(2.2) $$I^\alpha = L^\infty_\alpha \cap \{f \geq 0\}$$

and for $\alpha = \infty$,

(2.3) $\quad I^\alpha = \{f \in L^\infty; f \not\equiv 0, \text{supp} f \text{ (the support of } f) \text{ is compact in } \boldsymbol{R}^N\}$.

We further set for $\alpha \in (-\infty, \infty)$,

(2.4) $$I_{\alpha,\Omega} = \left\{ f \in L^\infty_{loc}(\boldsymbol{R}^N); f \geq 0, \liminf_{r \to \infty} \inf_{\xi \in \Omega} r^\alpha f(r\xi) > 0 \right\},$$

and for $\alpha = \infty$,

(2.5) $$I_{\alpha,\Omega} = \{f \in L^\infty_{loc}(\boldsymbol{R}^N); f \geq 0, f(x) > 0$$

$$\text{in } D \text{ for some nonempty open set } D \text{ in } \boldsymbol{R}^N\},$$

where $\Omega \subset \boldsymbol{S}^{N-1}$ is a nonempty open connected subset in $\boldsymbol{S}^{N-1}$ with smooth boundary.

REMARK 2.3. When $K(x) \in L^\infty(\boldsymbol{R}^N)$ satisfies $K(x) = |x|^\sigma$ ($|x| \geq R$), $K \in I^{-\sigma} \cap I_{-\sigma, \boldsymbol{S}^{N-1}}$. When $K(x) = K_D(x)$ where $K_D(x)$ is defined by (1.14), $K(x) \in I^{-\sigma} \cap I_{-\sigma, \Omega}$.

We note that if $K(x) \in I^{-\sigma}$ and $u_0(x) \in I^{a^*_\sigma}$ for some $\sigma \in [-\infty, \infty)$ then a solution of (1.1), (1.2) exists locally in time (see Theorem 3.1 and 3.2 in [**2**]), where a^*_σ is as in (1.10).

We now state our main results: Let D be $\boldsymbol{R}^N$ or a cone with vertex at the origin, that is, $D = \{x \in \boldsymbol{R}^N \backslash \{0\}; x/|x| \in \Omega\}$, where Ω ($\neq \emptyset$) $\subset \boldsymbol{S}^{N-1}$ is an open connected subset with smooth boundary.

THEOREM 2.4. *Let* $p > m$ *and* $K(x) \in I_{-\sigma,\Omega}$ *for some* $\sigma \in [-\infty, \infty)$. *Then, the following results hold*:

(a) *Let* $m < p \leq p^*_{m,\sigma}$ *where* $p^*_{m,\sigma}$ *is as in* (1.15). *Then all nontrivial solutions* $u(x,t)$ *of* (1.1), (1.2) *do not exist globally in time*.

(b) *Let* $p > \max\{p^*_{m,\sigma}, m\}$. *Then, if there exists an open subset* $V \subset \boldsymbol{S}^{N-1}$ *with* $|V| \neq 0$ ($|V|$ *is the Lebesgue measure of* V) *and a constant* $A > 0$ *such that*

(2.6) $$\liminf_{r \to \infty} r^{[\alpha^*_\sigma]_+} u_0(r\xi) > A \quad \text{for } \xi \in V,$$

any solution of (1.1), (1.2) *does not exist globally in time, where* α^*_σ *is as in* (1.17). *Especially, when* $\alpha^*_\sigma = 0$ *(namely,* $\sigma \leq -2$ *when* $N = 2$ *and* $\sigma = -2$ *when* $N \geq 3$), *adding the condition* $K(x) \in I_{-\sigma, \boldsymbol{S}^{N-1}}$ ($= I_{2, \boldsymbol{S}^{N-1}}$) *in case* $N \geq 3$, *we can replace the assumption* (2.6) *by*

(2.7) $$\liminf_{r \to \infty} \inf_{|x| \geq r} u_0(x) > 0.$$

THEOREM 2.5.   *Let $p > \max\{m, p_{m,\sigma}^*\}$ and $K(x) \in I^{-\sigma}$.   Then, for any $\alpha > \alpha_\sigma^*$ and $A > 0$, there exits a constant $\varepsilon > 0$ such that if*

$$(2.8) \qquad\qquad u_0(x) \le \min\{\varepsilon, A|x|^{-[\alpha]_+}\}$$

then a global solution u of (1.1), (1.2) in $\mathbf{R}^N \times (0, \infty)$ exists.

REMARK 2.6.   When $N = 1$,

$$(2.9) \qquad\qquad p_{m,\sigma}^* = \begin{cases} m + 2 + \sigma & \text{for } \sigma \in (-1, \infty) \\ m + 1 & \text{for } \sigma \in [-\infty, -1], \end{cases}$$

when $N = 2$,

$$(2.10) \qquad\qquad p_{m,\sigma}^* = \begin{cases} m + (2 + \sigma)/2 & \text{for } \sigma \in (-2, \infty) \\ m & \text{for } \sigma \in [-\infty, -2], \end{cases}$$

and when $N \ge 3$

$$(2.11) \qquad p_{m,\sigma}^* = \begin{cases} m + (2 + \sigma)/N \ (>m) & \text{for } \sigma \in (-2, \infty) \\ m & \text{for } \sigma = -2 \\ m + (2 + \max\{\sigma, -N\})/N \ (<m) & \text{for } \sigma \in [-\infty, -2). \end{cases}$$

When $N = 1$

$$(2.12) \qquad\qquad \alpha_\sigma^* = \begin{cases} (2 + \sigma)/(p - m) & \text{for } \sigma \in (-1, \infty) \\ 1/(p - m) & \text{for } \sigma \in [-\infty, -1], \end{cases}$$

when $N = 2$

$$(2.13) \qquad\qquad \alpha_\sigma^* = \begin{cases} (2 + \sigma)/(p - m) & \text{for } \sigma \in (-2, \infty) \\ 0 & \text{for } \sigma \in [-\infty, -2], \end{cases}$$

and when $N \ge 3$

$$(2.14) \qquad \alpha_\sigma^* = \begin{cases} (2 + \sigma)/(p - m) \ (>0) & \text{for } \sigma \in (-2, \infty) \\ 0 & \text{for } \sigma = -2 \\ (2 + \max\{\sigma, -N\})/(p - m) \ (<0) & \text{for } \sigma \in [-\infty, -2). \end{cases}$$

REMARK 2.7.   Let $p > \max\{m, p_{m,\sigma}^*\}$ and $K(x) \in I^{-\sigma}$.   Assume $\alpha_\sigma^* \ge 0$ and let $u(x, t)$ be a global solution of (1.1), (1.2) constructed in Theorem 2.5, where the initial value $u_0(x)$ satisfies (2.8) with $\alpha \in (\alpha_\sigma^*, N)$, $A > 0$, $\varepsilon > 0$ and $\varepsilon > 0$ is small enough.   Then, we can see, from the proof of Theorem 2.5, that in case $\sigma \ge 0$,

$$(2.15) \qquad\qquad \|u(t)\|_\infty \le C t^{-\alpha/(\alpha(m-1)+2)} \quad \text{for } t > 1$$

for some $C > 0$. Also, in case $\sigma < 0$ we can see that for $\alpha_\sigma^* < \alpha' < \min\{2/(p-m), \alpha\}$ there exits a small $\varepsilon > 0$ such that

$$(2.16) \qquad \|u(t)\|_\infty \le Ct^{-\alpha'/(\alpha'(m-1)+2)} \quad \text{for } t > 0.$$

In the rest of this section we state the fundamental tools and lemmas which are used later.

By the next proposition, in Theorem 2.4 we shall not need the restriction on the growth order of the initial data, except for condition (2.6).

PROPOSITION 2.8 (construction of solutions). *Let $v(x,t)$ be a supersolution of (1.1) in $\mathbf{R}^N \times (0,T)$. If $u_0(x) \le v(x,0)$ then there exists a weak solution of (1.1), (1.2) in $\mathbf{R}^N \times (0,T)$ such that*

$$(2.17) \qquad u(x,t) \le v(x,t) \quad \text{in } \mathbf{R}^N \times (0,T).$$

PROOF. Put $B_n = \{|x| < n\}$. Let $u_{0,n} \in C_0^\infty(B_n)$ satisfy that $0 \le u_{0,n} \le u_0$ in B_n and $u_{0,n} \uparrow u_0$ locally uniformly in $\mathbf{R}^N$ as $n \to \infty$ and let $u_n(x,t)$ be a unique solution of the initial boundary value problem

$$(2.18) \qquad \begin{cases} u_t = \Delta u^m + K(x)u^p & \text{in } B_n \times (0,T), \\ u(x.t) = 0 & \text{on } |x| = n, \\ u(x,0) = u_{0,n}(x) & \text{in } B_n. \end{cases}$$

Then we see from the comparison theorem (Proposition 2.2),

$$(2.19) \qquad u_n \le u_{n+1} \le v \quad \text{in } B_n \times (0,T),$$

and hence there exits $u \in L^\infty_{loc}(\mathbf{R}^N)$ such that $u_n(x,t) \uparrow u(x,t)$ as $n \to \infty$ for each $(x,t) \in \mathbf{R}^N \times (0,T)$. It follows from DiBenedetto [8] that $u_n(x,t)$ is equicontinuous in each compact set of $\mathbf{R}^N \times [0,T)$. Noting that $u_n(x,t)$ satisfies the integral equality like (2.1) we see that $u(x,t) \in C(\mathbf{R}^N \times [0,T))$ and $u_n(x,t) \uparrow u(x,t)$ locally uniformly in $\mathbf{R}^N \times [0,T)$ as $n \to \infty$, and so u is a weak solution of (1.1), (1.2) in $\mathbf{R}^N \times [0,T)$. $\qquad\square$

Next, we shall construct a supersolution of (1.1) by the methods of Mukai, Mochizuki and Huang [29]. Let $w(x,t) \in L^\infty(\mathbf{R}^N \times [0,T))$ be the classical solution of the problem

$$(2.20) \qquad w_t - \Delta w^m = 0 \quad (x,t) \in \mathbf{R}^N \times (0,\infty),$$

$$(2.21) \qquad w(x,0) = \varphi(x) \quad x \in \mathbf{R}^N,$$

where $\varphi(x) > 0$ in $x \in \mathbf{R}^N$. Let $k(t) > 0$ be a continuous decreasing function satisfying

$$(2.22) \qquad \|K(\cdot)w(\cdot,t)^{p-1}\|_\infty \le k(t) \quad \text{for all } t \ge 0.$$

Let $\alpha(t)$ be the solution of the ordinary equation $\alpha'(t) = k(t)\alpha(t)^{p-m+1}$ $(t > 0)$ with the initial data $\alpha(0) = 1$, that is

(2.23)
$$\alpha(t) = \left\{ 1 - (p-m) \int_0^t k(t)\, dt \right\}^{-1/(p-m)}.$$

Let $b(t)$ be the solution of the ordinary equation

(2.24)
$$\begin{cases} b'(t) = \{\alpha(b(t))\}^{m-1}, \\ b(0) = 0, \end{cases}$$

and put

(2.25)
$$\tilde{w}(x, t) = \alpha(b(t))w(x, b(t)).$$

PROPOSITION 2.9 (construction of supersolution).   *If*

(2.26)
$$(p-m) \int_0^\infty k(t)\, dt \leq \frac{1}{2} \quad \text{for } t \in (0, \infty),$$

then the problem (2.24) has a unique solution $b(t)$ in $[0, \infty)$ and $\tilde{w}$ is a supersolution of (1.1) in $\mathbf{R}^N \times (0, \infty)$.

PROOF.   The methods of the proof are the same as those of [**29**] (see the proof of Lemma 5 in [**29**]). By (2.26) we see that $1 < \alpha(t) \leq 2^{1/(p-m)}$ for $t \in [0, \infty)$. Hence, since $\alpha(\xi)$ is a C^1-function in $\xi \in [0, \infty)$, the problem (2.24) has a unique solution $b(t)$ in $[0, \infty)$, which is increasing in $[0, \infty)$. Further, we see

$$\tilde{w}_t - \Delta \tilde{w}^m = k(b(t))\alpha(b(t))^p w(x, b(t)) \geq K(x)w(x, b(t))^p \alpha(b(t))^p = K(x)\tilde{w}(x, t)$$

and so $\tilde{w}$ is a supersolution of (1.1) in $\mathbf{R}^N \times (0, \infty)$. $\square$

Next, we give several concrete solutions of (2.20). Let $E_m(x, t; L)$ $(L > 0)$ be the weak solution of (2.20) with $E_m(x, 0; L) = L\delta(x)$ (δ is Dirac's δ-function). Then, it is well known that

(2.27)
$$E_m(x, t; L) = L(L^{m-1}t)^{-N/((m-1)N+2)}g(\eta)$$

where $\eta = x/(L^{m-1}t)^{1/[N(m-1)+2]}$, and when $m > 1$

(2.28)
$$g(\eta) = [A - B|\eta|^2]_+^{1/(m-1)}$$

with $[y]_+ = \max\{0, y\}$, $B = (m-1)/2m\{(m-1)N + 2\}$ and A chosen to satisfy

(2.29)
$$\int_{\mathbf{R}^N} [A - B|x|]_+^{1/(m-1)}\, dx = 1,$$

and when $m = 1$

(2.30) $$g(\eta) = (4\pi)^{-N/2} e^{-|\eta|^2/4}.$$

That is, $E_m(x, t; L)$ is the Barenblatt solution in case $m > 1$ (see [**31**]) and the usual heat kernel in case $m = 1$.

PROPOSITION 2.10. *Let $\varphi \in L^\infty(\mathbf{R}^N) \cap L^1(\mathbf{R}^N)$ and let $w(x, t)$ be a weak solution of (2.20), (2.21). If we put for $k \geq 1$*

(2.31) $$w_k(x, t) = k^N w(kx, k^{2+N(m-1)} t)$$

then

(2.32) $$w_k(x, t) \rightarrow E_m(x, t; L)$$

locally uniformly in $\mathbf{R}^N \times (0, \infty)$ as $k \rightarrow \infty$ where

(2.33) $$L = \int_{\mathbf{R}^N} \varphi(x) \, dx.$$

PROOF. See Theorem 1.1 in [**9**]. □

The following lemma follows immediately from a simple calculation.

LEMMA 2.11. *Let $N \geq 2$, $\sigma > -2$ or $N = 1$, $\sigma > -1$. Then if $p = p_{m,\sigma}^* = m + (2 + \sigma)/N$ and $L > 0$,*

(2.34) $$\int_0^\tau \int_{\mathbf{R}^N} \{E_m(x, t; L)\}^p |x|^\sigma \, dx dt = \infty \quad \text{for } \tau > 0.$$

The next solutions of (2.20) have the initial data decaying more slowly. Let $A > 0$, $0 \leq a < N$ and $V \subset S^{N-1}$ be an open subset with $|V| \neq 0$ and let $W(x, t; A, a, V)$ be the weak solution of (2.20) with $W_m(x, 0; A, a, V) = A_V(x/|x|)|x|^{-a}$, where

(2.35) $$A_V(\xi) = \begin{cases} A & \xi \in V \\ 0 & \xi \notin V. \end{cases}$$

Then, it is well known (see [**5**], [**7**], [**17**] and [**18**]) that

(2.36) $\quad W_m(x, t; A, a, V) = t^{-a/(a(m-1)+2)} h(\eta; A, a, V) \quad \text{with } \eta = x/t^{1/(a(m-1)+2)},$

where $h(\eta) = h(\eta; A, a, V) \in C(\mathbf{R}^N)$ is a weak solution of the problem

(2.37) $$\begin{cases} \Delta h^m + \dfrac{1}{(m-1)a+2} \eta \cdot \nabla h + \dfrac{a}{(m-1)a+2} h = 0 & \text{in } \eta \in \mathbf{R}^N, \\ \lim_{r \to \infty} r^a h(r\xi) = A_V(\xi) & \text{for } \xi \in S^{N-1}, \\ h(0) > 0, \; \sup_{\eta \in \mathbf{R}^N} \langle \eta \rangle^a h(\eta) < \infty. \end{cases}$$

Here, we note that when $V = S^{N-1}$, $h(\eta)$ is radially symmetric and $h(\eta) > 0$ in R^N. Further, the uniqueness theorem ([5], [7]) implies

$$(2.38) \qquad W_m(x, t; AB, a, V) = A W_m(x, A^{m-1}t; B, a, V) \quad \text{for } A, B > 0,$$

$$(2.39) \qquad W_m(x, t; A, a, V) = k^a W_m(kx, k^{2+a(m-1)}t; A, a, V).$$

Hence

$$(2.40) \qquad W_m(x, t; A, a, V) = A W_m(x, A^{m-1}t; 1, a, V)$$

$$= A(A^{m-1}t)^{-a/(a(m-1)+2)} h(\eta; 1, a, V)$$

where $\eta = x/(A^{m-1}t)^{1/(a(m-1)+2)}$. Then, we can see the asymptotic behavior of solutions of (2.20) with the initial data decaying slowly. Let $\varphi(x) \in L^\infty(R^N) \cap C(R^N)$ satisfy that for some $0 \le a < N$,

$$(2.41) \qquad \sup_{x \in R^N} |x|^a \varphi(x) < \infty$$

and

$$(2.42) \qquad \lim_{r \to \infty} r^a \varphi(r\xi) = A_V(\xi) \quad \text{for } \xi \in S^{N-1}.$$

PROPOSITION 2.12. *Assume (2.41) and (2.42). Let $w(x, t)$ be the weak solution of (2.20), (2.21). If we put for each $k \ge 0$,*

$$(2.43) \qquad w_k(x, t) = k^a w(kx, k^{2+a(m-1)}t),$$

then

$$(2.44) \qquad w_k(x, t) \to W_m(x, t; A, a, V)$$

locally uniformly in $R^N \times (0, \infty)$ as $k \to \infty$.

PROOF. See Theorem 2 in [18] and Theorem B in [1]. $\qquad\qquad\square$

Next, we give the $L^\infty - L^\ell$ estimate for solutions of (1.1) due to Kawanago [19], which is used in the proof of Theorem 2.5 in the case where $\sigma \le 0$ and $\alpha_\sigma^* \ge 0$. Let Ω be a domain in R^N and let $u(x, t)$ be a weak solution of the initial boundary value problem

$$(2.45) \qquad \begin{cases} u_t - \Delta u^m = K(x)u^p & \text{in } \Omega \times (0, T), \\ u(x, 0) = u_0(x) & \text{in } \Omega, \\ u(x, t) = 0 & \text{on } \partial\Omega \times (0, T). \end{cases}$$

PROPOSITION 2.13. *Assume $u(x, t) \in L^\infty(\Omega \times (0, T))$. Let $1 \le q, r, s, \le \infty$. Then for any $\delta > 0$ and $t > 0$,*

$$(2.46) \qquad \|u(t)\|_\infty \leq 2\delta + B(\delta^{m-1}t)^{-N/2q}\|u_0\|_{L^q(\Omega)}$$

$$+ B(\delta^{m-1}t)^{-N/2r} \int_0^{t/2} \|K(x)u^p(\tau)\|_{L^r(\Omega)} \, d\tau$$

$$+ B\delta^{-N(m-1)/2s} \int_0^{t/2} \tau^{-N/2s}\|K(x)u^p(t-\tau)\|_{L^s(\Omega)} \, d\tau,$$

where $B = B(m, N, q, r, s)$ is a constant independent of Ω.

PROOF. See Proposition 2.4 in [19]. □

Finally, we give the well known blow-up theorem. Let $G \subset \mathbf{R}^N$ be a bounded nonempty domain with the smooth boundary. Let $\lambda = \lambda_G$ be the first eigenvalue of $-\Delta$ in G with Dirichlet boundary condition and $s(x) = s_G(x)$ the corresponding eigenfunction ($s(x)$ is normalized: $\int_G s(x)\,dx = 1$). Further, Let nonnegative $\tilde{K}(x) \in L_{loc}^\infty(\mathbf{R}^N)$ satisfy that for some $c_0 > 0$,

$$(2.47) \qquad\qquad \tilde{K}(x) \geq c_0 \quad \text{for } x \in G.$$

PROPOSITION 2.14. *Let $p > m$ and let $u(x,t)$ be a weak solution of (1.1), (1.2) with $K(x) = \tilde{K}(x)$. Then, if*

$$(2.48) \qquad\qquad \int_G s(x)u_0(x)\,dx > \left(\frac{\lambda}{c_0}\right)^{1/(p-m)},$$

$u(x,t)$ is not global in $\mathbf{R}^N \times (0, \infty)$.

PROOF. See Theorem 1.1 and Example 1.2 in [16] and references of [24]. □

3. Proof of Theorem 2.4(a) in the case $N \geq 2$ and the case $N = 1$, $\sigma \geq -1$.

In this section we shall show Theorem 2.4(a) in the case $N \geq 2$ and the case $N = 1$, $\sigma \geq -1$. In these cases, if $m < p \leq p_{m,\sigma}^*$ then $\sigma > -2$ when $N \geq 2$ and

$$(3.1) \qquad\qquad p_{m,\sigma}^* = m + \frac{2+\sigma}{N}.$$

The next proposition is a key proposition.

PROPOSITION 3.1. *Assume $N \geq 2$ or $N = 1$, $\sigma \geq -1$. Let $K(x) \in I_{-\sigma, \Omega}$ ($-\infty \leq \sigma < \infty$) and $u(x,t)$ be a global weak solution of (1.1), (1.2).*
    (a) If $m < p < p_{m,\sigma}^*$ then

$$(3.2) \qquad\qquad u(x,t) = 0 \quad \text{for } (x,t) \in \mathbf{R}^N \times [0, \infty).$$

(b) *If $p = p_{m,\sigma}^*$ ($>m$) then there exists a constant $M > 0$ depending only on σ, Ω such that*

$$(3.3) \qquad \int_{R^N} u(x,t)\,dx \leq M \quad \text{for } t \geq 0.$$

In order to prove this proposition we need the next lemma.

LEMMA 3.2. *Let $p > m$. Let $K(x) \in I_{-\sigma, \Omega}$ ($\sigma > -2$) and $u(x,t)$ be a global weak solution of* (1.1), (1.2). *Let G be a bounded nonempty domain in R^N with the smooth boundary satisfying*

$$(3.4) \qquad G \subset \left\{ x \in R^N; \frac{x}{|x|} \in \Omega, 1 \leq |x| \leq 2 \right\}.$$

Then, for large k

$$(3.5) \qquad \int_G s(x) k^{(2+\sigma)/(p-m)} u(kx,t)\,dx \leq \left(\frac{\lambda}{c_0} \right)^{1/(p-m)} \quad \text{for } t \geq 0,$$

where $\lambda = \lambda_G$, $s(x) = s_G(x)$ (λ_G and s_G are as in §2) and c_0 is a positive constant depending only on $K(x)$.

PROOF. Since $K(x) \in I_{-\sigma, \Omega}$ there exist $R_1 \geq 1$ and $k_0 > 0$ such that

$$(3.6) \qquad K(x) \geq k_0 |x|^\sigma \quad \text{for } |x| \geq R_1, \frac{x}{|x|} \in \Omega.$$

Put

$$(3.7) \quad u_k(x,t) = k^{(2+\sigma)/(p-m)} u(kx, k^\ell t) \quad (k \geq 1) \text{ with } \ell = \frac{2(p-1) + \sigma(m-1)}{p-m}.$$

Then u_k is a global solution of equation

$$(3.8) \qquad u_t - \Delta u^m = \tilde{K}(x) u^p \quad \text{in } R^N \times (0, \infty),$$

where $\tilde{K}(x) = k^{-\sigma} K(kx)$. Note by (3.6) that for large k

$$(3.9) \qquad \tilde{K}(x) = k^\sigma K(kx) \geq k_0 |x|^\sigma \geq c_0 \quad \text{for } x \in G,$$

where $c_0 = k_0 \min_{1 \leq |x| \leq 2} |x|^\sigma$. Hence, applying Proposition 2.14 to u_k we have

$$(3.10) \qquad \int_G s(x) k^{(2+\sigma)/(p-m)} u(kx, k^\ell t)\,dx \leq \left(\frac{\lambda}{c_0} \right)^{1/(p-m)} \quad \text{for } t \geq 0,$$

which implies (3.5).   □

PROOF OF PROPOSITION 3.1. Let $u(x, t)$ be a global weak solution of (1.1), (1.2). Assume $u_0(x) \not\equiv 0$. Let $v_R(x, t)$ be the weak solution of the problem

$$(3.11) \qquad \begin{cases} v_t - \Delta v^m = 0 & \text{in } \mathbf{R}^N \times (0, \infty), \\ v(x, 0) = u_{0,R}(x) & \text{in } \mathbf{R}^N, \end{cases}$$

where

$$(3.12) \qquad u_{0,R} = \begin{cases} u_0(x) & \text{for } |x| \leq R \\ 0 & \text{for } |x| > R. \end{cases}$$

We take R large enough to satisfy $u_{0,R}(x) \not\equiv 0$. By the proof of Proposition 2.8 and the uniqueness of solutions of (3.11) (see Theorem 2 in [7]) we have

$$(3.13) \qquad v_R(x, t) \leq u(x, t) \quad \text{for } (x, t) \in \mathbf{R}^N \times (0, \infty).$$

Putting $v_{R,k}(x, t) = k^N v_R(kx, k^{(m-1)N+2}t)$ we obtain by Lemma 3.2,

$$(3.14) \quad \int_G s(x) v_{R,k}(x, t)\, dx \leq k^{N-(2+\sigma)/(p-m)} \int_G s(x) k^{(2+\sigma)/(p-m)} u(kx, k^{(m-1)N+2}t)\, dx$$

$$\leq k^{N-(2+\sigma)/(p-m)} \left(\frac{\lambda}{c_0} \right)^{1/(p-m)} \quad \text{for large } k \geq 1,$$

where G is as in Lemma 3.2. Therefore, it follows from Proposition 2.10 that if $k \to \infty$ then in case $m < p < p^*_{m,\sigma}$ (namely, $N - (2+\sigma)/(p-m) < 0$),

$$(3.15) \qquad \int_G s(x) E_m(x, t; L_R)\, dx \leq 0 \quad \text{for } t \geq 0$$

and in case $p = p^*_{m,\sigma}$ (namely, $N - (2+\sigma)/(p-m) = 0$),

$$(3.16) \qquad \int_G s(x) E_m(x, t; L_R)\, dx \leq \left(\frac{\lambda}{c_0} \right)^{1/(p-m)} \quad \text{for } t \geq 0,$$

where $E_m(x, t; L)$ is defined by (2.27) and

$$(3.17) \qquad L_R = \int_{|x| \leq R} u_0(x)\, dx.$$

We note that for some $r_1 > 0$ and $g_0 > 0$,

$$(3.18) \qquad g(\eta) \geq g_0 > 0 \quad \text{for } |\eta| \leq r_1$$

and if we choose t_R to satisfy

$$(3.19) \qquad (L_R^{m-1} t_R)^{1/((m-1)N+2)} = \frac{2}{r_1},$$

that is,

$$(3.20) \qquad t_R = \left(\frac{2}{r_1}\right)^{(m-1)N+2} L_R^{1-m},$$

then we see

$$(3.21) \qquad E_m(x, t_R; L_R) = L_R\left(\frac{2}{r_1}\right)^{-N} g(r_1 x/2) \geq L_R\left(\frac{2}{r_1}\right)^{-N} g_0 \quad \text{for } |x| \leq 2.$$

Put $t = t_R$ in (3.15) and (3.16) respectively. Then, when $p < p^*_{m,\sigma}$, we get $L_R = 0$, namely $u_{0,R} \equiv 0$ in $\mathbf{R}^N$. This is a contradiction to the assumption and so $u_0(x) \equiv 0$. When $p = p^*_{m,\sigma}$ we obtain by letting $R \to \infty$,

$$(3.22) \qquad \int_{\mathbf{R}^N} u_0(x)\,dx \leq \left(\frac{\lambda}{c_0}\right)^{1/(p-m)} \left(\frac{2}{r_1}\right)^N g_0^{-1}.$$

Thus, considering $u(x, t)$ as the initial data for each $t > 0$ we have (3.3). The proof is complete. □

Thus, when $N \geq 2$ or $N = 1$, $\sigma \geq -1$, Theorem 2.4 (a) is shown by Proposition 3.1(a) in case $m < p < p^*_{m,\sigma}$. In case $p = p^*_{m,\sigma}$ $(>m)$ we further need the next proposition to prove the theorem. As is seen in the proof of Lemma 3.2, when $K(x) \in I_{-\sigma,\Omega}$ there exist $R_1 > 1$ and $k_0 > 0$ satisfying (3.6). Let $a \in \Omega$ and put

$$E(a; 2r) = \{x \in \mathbf{S}^{N-1}; |x - a| < 2r\}.$$

We choose $r > 0$ small to satisfy

$$(3.23) \qquad\qquad E(a; 2r) \subset \Omega.$$

Further put

$$(3.24) \qquad D_{E(a;r)} = \left\{x \in \mathbf{R}^N; x \neq 0, \frac{x}{|x|} \in E(a; r)\right\},$$

let $v_0(x) \in C_0(D_{E(a;r)})$ satisfy

$$(3.25) \qquad\qquad 0 \leq v_0(x) \leq u_0(x) \quad \text{in } \mathbf{R}^N$$

and let $v(x, t)$ be a weak solution of the problem

$$(3.26) \qquad \begin{cases} u_t - \Delta u^m = K_0(x)u^p & \text{in } \mathbf{R}^N \times (0, T), \\ u(x, 0) = v_0(x) & \text{in } \mathbf{R}^N, \end{cases}$$

where

(3.27) $\qquad K_0(x) = \begin{cases} k_0|x|^\sigma & \text{for } |x| \geq R_1, x/|x| \in E(a; 2r), \\ 0 & \text{otherwise.} \end{cases}$

Here we note

(3.28) $\qquad K_0(x) \leq K(x) \quad \text{in } \boldsymbol{R}^N.$

PROPOSITION 3.3. *Let $u(x, t)$ be a global weak solution of (1.1), (1.2) as in Proposition 3.1. Let $K(x)$ satisfy (3.6) for some $R_1 \geq 1$ and $k_0 > 0$. Then there exists a global weak solution $v(x, t)$ ($\not\equiv 0$) of (3.26) such that*

(3.29) $\qquad v(x, t) \leq u(x, t) \quad \text{in } \boldsymbol{R}^N \times (0, \infty),$

(3.30) $\qquad \displaystyle\int_{D_{E(a; 2r)}} \varphi(|x|)v^p(x, t)\, dx \geq c_1 \int_{\boldsymbol{R}^N} \varphi(|x|)v^p(x, t)\, dx \quad \text{for } t \geq 0,$

where $c_1 = c_1(r)$ is a constant and $\varphi(\xi) \geq 0$ in $\xi \in \boldsymbol{R}$.

PROOF OF PROPOSITION 3.3. The methods of the proof are similar to those of Proposition 2.8. Let $v_n(x, t)$ be the solution of the problem

(3.31) $\qquad \begin{cases} u_t - \Delta u^m = K_0(x)u^p & \text{in } B_n \times (0, T), \\ u(x, t) = 0 & \text{on } \partial B_n \times (0, T), \\ u(x, 0) = v_0(x) & \text{in } B_n, \end{cases}$

with $B_n = \{|x| < n\}$ ($n > 1$). Noting (3.28) we have by Proposition 2.2, $T = \infty$ and

(3.32) $\qquad v_n(x, t) \leq u(x, t) \quad \text{in } B_n \times (0, T).$

Now, we need several definitions and Lemmas concerning the 'reflection'. For $v \in \boldsymbol{S}^{N-1}$ (i.e. $|v| = 1$), we put

(3.33) $\qquad A = A(v) = \{x \in \boldsymbol{R}^N \mid v \cdot x = 0\},$

where "$\cdot$" means the inner product in $\boldsymbol{R}^N$. A forms a hyperplane in $\boldsymbol{R}^N$. The upper [or lower] half space of B_n with respect to A is defined as

(3.34) $\qquad B_{n,A}^+ = \{x \in B_n \mid v \cdot x > 0\} \quad [\text{or } B_{n,A}^- = \{x \in B_n \mid v \cdot x < 0\}].$

For any $x \notin A$, the reflection of x in A is denoted by $\sigma_A x$. Thus, we have for each $\zeta \in A$,

(3.35) $\qquad \zeta \cdot (\sigma_A x - x) = \dfrac{1}{2}(\sigma_A x + x) \cdot (\sigma_A x - x).$

For any set $K \subset \boldsymbol{R}^N$, we define the reflection of K in A as

(3.36) $\qquad \sigma_A K = \{\sigma_A x \mid x \in K\},$

and for any function v in B_n, we define the reflection of v in A as

$$(3.37) \qquad\qquad \sigma_A v(x) = v(\sigma_A x) \quad x \in B_n.$$

We note $\sigma_A B_n = B_n$.

Since $a \in \Omega$, we can choose $r > 0$ small enough to satisfy $\{x \in \mathbf{R}^N; |x - a| < 2r\} \subset \mathbf{R}^N \setminus \{0\}$. Then, we give two lemmas. $\square$

LEMMA 3.4. *For any $b \in \mathbf{S}^{N-1} \setminus E(a; 2r)$ there exists a hyperplain $A = A(v_b)$ such that $E(a; r) \subset B_{n,A}^+$ and $\sigma_A E(a; r) = E(b; r)$ $(\subset B_{n,A}^-)$. Further, we obtain*

$$(3.38) \qquad\qquad \sigma_A v_0(x) \geq v_0(x) \quad \text{in } B_{n,A}^-,$$

$$(3.39) \qquad\qquad \sigma_A K_0(x) \geq K_0 \quad \text{in } B_{n,A}^-.$$

PROOF. This lemma follows immediately from the definitions of $v_0(x)$ and $K_0(x)$. $\square$

LEMMA 3.5. *Put $\tilde{v}_n(x, t) = \sigma_A v_n(x, t)$ $(= v_n(\sigma_A x, t))$ in B_n, where $A = A(v_b)$. Then*

$$(3.40) \qquad\qquad v_n(x, t) \leq \tilde{v}_n(x, t) \quad \text{in } B_{n,A}^- \times (0, \infty).$$

Hence

$$(3.41) \qquad\qquad v_n(x, t) \leq \tilde{v}_n(x, t) \quad \text{in } D_{E(b;r)} \cap B_n \times (0, \infty).$$

PROOF. We can easily see that $\tilde{v}_n(x, t)$ is a weak solution of the equation

$$(3.42) \qquad\qquad u_t - \Delta u^m = \sigma_A K_0(x) u^p.$$

Since $v_n(x, 0) = v_0(x) \leq \sigma_A v_0(x) = \tilde{v}_n(x, 0)$ in $B_{n,A}^-$, Lemma 3.4 and the comparison theorem (Proposition 2.2) imply (3.40). $\square$

PROOF OF PROPOSITION 3.3 (CONTINUE). Similarly, as in the proof of Proposition 2.8, we see that $v(x, t) = \lim_{n \to \infty} v_n(x, t)$ is a weak solution of (3.26), and (3.32) is reduced to (3.29). Let $b \in \mathbf{S}^{N-1} \setminus E(a; 2r)$. Let $A_b = A(v_b)$ be as in Lemma 3.4 and put $\tilde{v}_b(x, t) = \sigma_{A_b} v(x, t)$. Then, we have by Lemma 3.5,

$$(3.43) \qquad\qquad v(x, t) \leq \tilde{v}_b(x, t) \quad \text{in } D_{E(b;r)} \times (0, \infty).$$

Since

$$\mathbf{S}^{N-1} \setminus E(a; 2r) \subset \bigcup_{b \in \mathbf{S}^{N-1} \setminus E(a; 2r)} E(b; r)$$

and $\mathbf{S}^{N-1} \setminus E(a; 2r)$ is a compact set in $\mathbf{S}^{N-1}$, there exist $b_1, b_2, \ldots, b_\ell \in \mathbf{S}^{N-1} \setminus E(a; 2r)$ such that

$$S^{N-1} \setminus E(a; 2r) \subset \bigcup_{i=1}^{\ell} E(b_i; r).$$

Hence, by putting $\tilde{v}_i = \tilde{v}_{b_i}$,

$$(3.44) \quad \int_{\mathbf{R}^N} \varphi(|x|)v^p(x,t)\,dx \le \int_{D_{E(a;2r)}} \varphi(|x|)v^p\,dx + \sum_{i=1}^{\ell} \int_{D_{E(b_i;r)}} \varphi(|x|)\tilde{v}_i^p\,dx$$

$$\le (\ell+1)\int_{D_{E(a;2r)}} \varphi(|x|)v^p\,dx,$$

where $\varphi(\xi) \ge 0$ in $\mathbf{R}$. Here we used $\varphi(|\sigma_A x|) = \varphi(|x|)$ and $\sigma_{A_b} D_{E(a;r)} = D_{E(b;r)}$. Thus, putting $c_1 = 1/(\ell+1)$ we obtain (3.30). The proof is complete. $\qquad \square$

PROPOSITION 3.6. *Assume* $p = p_{m,\sigma}^*$. *Let* v *be as in Proposition* 3.3 *and put* $v_k(x,t) = k^N v(kx, k^{(m-1)N+2}t)$. *Then there exists* $C > 0$ *such that*

$$(3.45) \quad \int_0^\tau \int_{\{|x| \ge R_1/k\}} |x|^\sigma v_k^p(x,t)\,dxdt \le C \quad \text{for all } k \ge 1.$$

PROOF. When $p = p_{m,\sigma}^* = m + (2+\sigma)/N$, for large k v_k is a global weak solution of the problem

$$(3.46) \quad \begin{cases} u_t - \Delta u^m = K_{0,k}(x)u^p & \text{in } \mathbf{R}^N \times (0, \infty), \\ u(x,0) = k^N v_0(kx) & \text{in } \mathbf{R}^N, \end{cases}$$

where

$$(3.47) \quad K_{0,k}(x) = \begin{cases} k_0|x|^\sigma & \text{for } |x| \ge R_1/k, \, x/|x| \in E(a; 2r), \\ 0 & \text{otherwise.} \end{cases}$$

Since $K_{0,k}(x) \in I_{-\sigma, E(a;2r)}$, Proposition 3.1 implies that for some $M > 0$

$$(3.48) \quad \int_{\mathbf{R}^N} v_k(x,t) \le M \quad \text{for } k \ge 1, t \ge 0.$$

Let consider $\varphi(x,t) = \rho_\varepsilon(x) = \xi_1(\varepsilon|x|)$ as a test function in the integral equality satisfied by v_k (see (2.1)), where $\xi_1(r) \in C^2([0,2])$ is a nonnegative non-increasing function satisfying

$$(3.49) \quad \xi_1(r) = \begin{cases} 0 & \text{on } r = 2 \\ 1 & \text{in } 0 \le r \le 1 \end{cases}$$

and for some $\lambda_1 > 0$

$$(3.50) \quad \xi_{1,rr} + \frac{N-1}{r}\xi_{1,r} \begin{cases} \ge -\lambda_1\xi_1 & \text{for } 1 \le r \le 2 \\ = 0 & \text{for } 0 \le r \le 1. \end{cases}$$

We note that

$$(3.51) \qquad \Delta\rho_\varepsilon(x) \begin{cases} \geq -\lambda_1\varepsilon^2\rho_\varepsilon(x) & \text{for } 1/\varepsilon \leq |x| \leq 2/\varepsilon \\ = 0 & \text{for } 0 \leq |x| \leq 1/\varepsilon \end{cases}$$

and by (3.30)

$$(3.52) \qquad \int_{|x|\leq 2/\varepsilon} K_{0,k}(x)v_k^p\rho_\varepsilon\,dx \geq c_1\int_{2/\varepsilon\geq|x|\geq R_1/k} k_0|x|^\sigma v_k^p\rho_\varepsilon\,dx \quad \text{for } k\geq 1, t\geq 0.$$

Hence

$$(3.53) \qquad M \geq \int_{|x|\leq 2/\varepsilon} v_k(x,\tau)\rho_\varepsilon(x)\,dx \geq \int_0^\tau\int_{|x|\leq 2/\varepsilon}\{v_k^m\Delta\rho_\varepsilon + K_{0,k}v_k^p\rho_\varepsilon\}\,dxdt$$

$$\geq \int_0^\tau\int_{1/\varepsilon\leq|x|\leq 2/\varepsilon}\{-\lambda_1\varepsilon^2 v_k^m + c_1k_0|x|^\sigma v_k^p\}\rho_\varepsilon\,dxdt$$

$$+ c_1k_0\int_0^\tau\int_{R_1/k\leq|x|\leq 1/\varepsilon}|x|^\sigma v_k^p\,dxdt.$$

By using the inequality $a^m \leq ca + a^p$ ($c = c(m,p) > 0$) and $\sigma > -2$, the first term of the right side of the above inequality is estimated as follows:

$$(3.54) \qquad \int_{1/\varepsilon\leq|x|\leq 2/\varepsilon}\{-\lambda_1\varepsilon^2 v_k^m + c_1k_0|x|^\sigma v_k^p\}\rho_\varepsilon\,dx$$

$$\geq \int_{1/\varepsilon\leq|x|\leq 2/\varepsilon}\{-\lambda_1\varepsilon^2(cv_k + v_k^p) + c_1k_0|x|^\sigma v_k^p\}\rho_\varepsilon\,dx$$

$$\geq -\lambda_1 c\varepsilon^2\int_{1/\varepsilon\leq|x|\leq 2/\varepsilon} v_k\,dx$$

$$+ \int_{1/\varepsilon\leq|x|\leq 2/\varepsilon}\{-\lambda_1\times 4|x|^{-2}v_k^p + c_1k_0|x|^\sigma v_k^p\}\rho_\varepsilon\,dx$$

$$\geq -\lambda_1 c\varepsilon^2 M + \int_{1/\varepsilon\leq|x|\leq 2/\varepsilon}|x|^\sigma v_k^p\rho_\varepsilon\{c_1k_0 - 4\lambda_1|x|^{-(2+\sigma)}\}\,dx \quad \text{(by (3.48))}$$

$$\geq -\lambda_1 c\varepsilon^2 M \quad \text{for small } \varepsilon > 0.$$

Thus, combining this and (3.53) we have

$$(3.55) \qquad M \geq -\lambda_1 c\varepsilon^2 M\tau + c_1k_0\int_0^\tau\int_{R_1/k\leq|x|\leq 1/\varepsilon}|x|^\sigma v_k^p\,dxdt,$$

and therefore by letting $\varepsilon \downarrow 0$ we obtain (3.45). The proof is complete. $\qquad\square$

Proof of Theorem 2.4 (a) in the case $N \geq 2$ and the case $N = 1$, $\sigma \geq -1$. We shall prove this theorem only in case $p = p_{m,\sigma}^*$. Let $K(x) \in I_{-\sigma,\Omega}$ satisfy (3.6) with $R_1 \geq 1$ and $k_0 > 0$. Let $u(x,t)$ be a global solution of (1.1), (1.2). Assume contrary $u_0(x) \not\equiv 0$ in $\mathbf{R}^N$. Then, without loss of generality we can assume $u_0 \not\equiv 0$ in $D_{E(a;r)}$ where $D_{E(a;r)}$ is as in Proposition 3.3. In fact, let $v_R(x,t)$ be as in the proof of Proposition 3.1. Then, since $v_R(x,0) = u_{0,R}(x) \not\equiv 0$ and $v_R(x,t)$ is a weak solution of (3.11), $\operatorname{supp} v_R(\cdot,t)$ (the support of $v_R(x,t)$ in $\mathbf{R}^N$) spreads out to whole $\mathbf{R}^N$ as $t \to \infty$. Hence, by (3.13) we see that for some $t_1 > 0$ $u(x,t_1) \not\equiv 0$ in $D_{E(a;r)}$, and we can consider this $u(x,t_1)$ as the initial data.

So, let $v_0(x) \in C_0(D_{E(a;r)})$ satisfy that $v_0 \not\equiv 0$ in $D_{E(a;r)}$ and $0 \leq v_0 \leq u_0$ in $\mathbf{R}^N$. Then, when $p = p_{m,\sigma}$, by Proposition 3.6 we see that there exists a global weak solution $v(x,t)$ of (3.26) satisfying

$$(3.56) \qquad \int_0^\tau \int_{\{|x| \geq R_1/k\}} |x|^\sigma v_k^p(x,t)\,dxdt \leq C \quad \text{for } k \geq 1, \tau > 0,$$

where $v_k(x,t) = k^N v(kx, k^{(m-1)N+2}t)$. Further, similarly, as in the proof of Proposition 3.1, there is a weak solution $w(x,t)$ of (3.11) with $w(x,0) = v_0(x)$ satisfying $w(x,t) \leq v(x,t)$ in $\mathbf{R}^N \times (0,\infty)$. Hence, if we put $w_k(x,t) = k^N w(kx, k^{(m-1)N+2}t)$ then $w_k \leq v_k$ and

$$(3.57) \qquad \int_0^\tau \int_{\{|x| \geq R_1/k\}} |x|^\sigma w_k^p(x,t)\,dxdt \leq C \quad \text{for } k \geq 1, \tau > 0.$$

Letting $k \to \infty$ we have by Proposition 2.10,

$$(3.58) \qquad \int_0^\tau \int_{\mathbf{R}^N} |x|^\sigma E_m^p(x,t;L)\,dxdt \leq C \quad \text{for } \tau > 0,$$

where $L = \int_{\mathbf{R}^N} v_0(x)\,dx > 0$, however, this contradicts (2.34). Therefore, we obtain $u_0 \equiv 0$ in $\mathbf{R}^N$ and, hence if we consider $u(x,t)$ as the initial data for each $t > 0$ then we see $u(x,t) \equiv 0$ in $\mathbf{R}^N \times (0,\infty)$. The proof is complete. $\qquad \square$

4. Proof of Theorem 2.4(a) in the case $N = 1$, $\sigma \leq -1$.

In this section we shall show Theorem 2.4(a) in the case $N = 1$, $\sigma \leq -1$. In this case

$$(4.1) \qquad p_{m,\sigma}^* = m + 1.$$

Let $K(x) \in I_{-\sigma,\Omega}$. Then, without loss of generality we can assume

$$(4.2) \qquad K(x) \geq K_1(x) \quad \text{in } \mathbf{R},$$

where

(4.3)
$$K_1(x) = \begin{cases} \delta & \text{in } |x| \le \delta \\ 0 & \text{in } |x| > \delta \end{cases}$$

for small $\delta > 0$.   Let $v_0(x) \in C_0(\mathbf{R})$ satisfy that $v_0(x) = v_0(|x|)$ is a radially symmetric function in $\mathbf{R}$ and a nonincreasing function in $r = |x|$, and

(4.4)
$$0 \le v_0(x) \le u_0(x) \quad \text{in } \mathbf{R}.$$

Let $v(x,t)$ be a weak solution of the problem

(4.5)
$$\begin{cases} u_t - \Delta u^m = K_1(x) u^p & \text{in } \mathbf{R} \times (0, \infty), \\ u(x,0) = v_0(x) & \text{in } \mathbf{R}. \end{cases}$$

As in §3, in order to prove the theorem we need several propositions.

PROPOSITION 4.1.   *Let $u(x,t)$ be a global weak solution of (1.1), (1.2) with $K(x)$ satisfying (4.2).   Then there exists a global weak solution $v(x,t)$ of (4.5) such that for each $t > 0$ $v(x,t) = v(|x|,t)$ is a radially symmetric function in $x \in \mathbf{R}$ and a nonincreasing function in $r = |x|$, and*

(4.6)
$$v(x,t) \le u(x,t) \quad \text{in } \mathbf{R} \times (0, \infty).$$

PROOF.   The methods of the proof are the same as those of Proposition 3.3 and so we omit the proof.   □

PROPOSITION 4.2.   *Let $N = 1$, $\sigma \le -1$ and let $v(x,t)$ be as in Proposition 4.1.*
   (a)   *If $m < p < p^*_{m,\sigma} = m + 1$ then*

(4.7)
$$v(x,t) = 0 \quad \text{for } (x,t) \in \mathbf{R} \times [0, \infty).$$

   (b)   *If $p = p^*_{m,\sigma} = m + 1$ then there exists a constant $M > 0$ depending only on σ and δ such that*

(4.8)
$$\int_{\mathbf{R}} v(x,t) \, dx \le M \quad \text{for } t \ge 0.$$

Hence

(4.9)
$$\int_0^\infty \int_{B(\delta)} v^{m+1}(x,t) \le \frac{M}{\delta} \quad \text{for } t \ge 0,$$

where $B(\delta) = \{|x| < \delta\}$.

We need the next lemma to prove this proposition.

LEMMA 4.3.   *Let $N = 1$ and $\sigma \le -1$.   Let v be as in Proposition 4.1.   Then, when $p > m$,*

$$(4.10) \qquad \int_{B(\delta)} s(x) k^{1/(p-m)} v(kx, t)\, dx \le \left(\frac{\lambda}{\delta} \right)^{1/(p-m)} \quad \text{for } t \ge 0, k \ge 1,$$

where $\lambda = \lambda_{B(\delta)}$ and $s(x) = s_{B(\delta)}(x)$ are defined in §2.

PROOF. If we put $v_k(x, t) = k^{1/(p-m)} v(kx, k^{(2p-m-1)/(p-m)} t)$ then v_k is a global weak solution of the equation

$$(4.11) \qquad u_t - \Delta u^m = k K_1(kx) u^p.$$

Hence, considering $s(x) = s_{B(\delta)}(x)$ as a test function $\varphi(x, t)$ in the integral equation satisfied by v_k (see (2.1)) we have

$$(4.12) \qquad \int_{B(\delta)} v_k s(x)\, dx \Big|_0^t \ge \int_0^t \int_{B(\delta)} -\lambda v_k^m s(x)\, dx dt + k \int_0^t \int_{|x| \le \delta/k} \delta v_k^p s(x)\, dx dt.$$

Here, we note that

$$(4.13) \qquad k \int_0^t \int_{|x| \le \delta/k} \delta v_k^p s(x)\, dx dt \ge \int_0^t \int_{|x| \le \delta} \delta v_k^p s(x)\, dx dt,$$

since for each $t > 0$ $s(x) v_k(x, t)$ is a nonincreasing function in $x > 0$. Therefore, we have

$$(4.14) \qquad \int_{B(\delta)} v_k s(x)\, dx \Big|_0^t \ge \int_0^t \int_{|x| \le \delta} \{ -\lambda v_k^m + \delta v_k^p \} s(x)\, dx dt \quad \text{for each } t \in [0, \infty)$$

to obtain by the similar methods to those of the proof of Proposition 2.14,

$$(4.15) \qquad \int_{B(\delta)} v_k(x, 0) s(x)\, dx \le \left(\frac{\lambda}{\delta} \right)^{1/(p-m)} \quad \text{for } k \ge 1.$$

Thus, considering $v_k(x, t)$ as the initial data for each $t > 0$ we get

$$(4.16) \qquad \int_{B(\delta)} v_k(x, t) s(x)\, dx \le \left(\frac{\lambda}{\delta} \right)^{1/(p-m)} \quad \text{for } k \ge 1 \text{ and } t > 0.$$

The proof is complete. $\qquad \square$

PROOF OF PROPOSITION 4.2. The methods of the proof are the same as those of the proof of Proposition 3.1. Let $v(x, t)$ be as in Proposition 4.1. Put for each $R > 0$

$$(4.17) \qquad v_{0, R}(x) = \begin{cases} v_0(x) & \text{for } |x| \le R, \\ 0 & \text{for } |x| > R, \end{cases}$$

and let $w_R(x,t)$ be a weak solution of (2.20) with $w(x,0) = v_{0,R}(x)$. Then we have $w_R(x,t) \leq v(x,t)$ for $(x,t) \in \mathbf{R} \times (0,\infty)$, and hence putting $w_{R,k}(x,t) = kw_R(kx, k^{m+1}t)$ and using Lemma 4.3 we obtain

$$(4.18) \quad \int_{B(\delta)} s(x)w_{R,k}(x,t)\,dx \leq k \int_{B(\delta)} s(x)v(kx, k^{m+1}t)\,dx \leq k^{1-1/(p-m)}\left(\frac{\lambda}{\delta}\right)^{1/(p-m)}.$$

If $k \to \infty$, then it follows from Proposition 2.10 that when $p < m+1$

$$(4.19) \qquad\qquad \int_{B(\delta)} s(x)E_m(x,t;L_R)\,dx = 0 \quad \text{for } t \geq 0$$

and when $p = m+1$

$$(4.20) \qquad\qquad \int_{B(\delta)} s(x)E_m(x,t;L_R)\,dx \leq \left(\frac{\lambda}{\delta}\right)^{1/(p-m)} \quad \text{for } t \geq 0,$$

where $L_R = \int_{|x|<R} v_0(x)\,dx$. Therefore, similarly, as in the proof of Proposition 3.1, we conclude (4.7) and (4.8).

Finally, in case $p = m+1$ we shall show (4.9). Since $v(x,t) \in L^\infty(\mathbf{R} \times (0,\tau)) \cap L^1(\mathbf{R} \times (0,\tau))$ for $\tau > 0$ by (4.8) and the monotonicity of $v(\cdot,t)$, we can consider $\varphi(x,t) \equiv 1$ as a test function φ in the integral equation satisfied by v (see (2.1)). Hence, we have

$$(4.21) \qquad\qquad M \geq \int_{\mathbf{R}} v(x,t)\,dx \geq \int_0^t \int_{B(\delta)} \delta v^{m+1}\,dxdt \quad \text{for } t \geq 0. \qquad \square$$

PROOF OF THEOREM 2.4 (a) IN THE CASE $N = 1$, $\sigma \leq -1$. Let $K(x)$ satisfy (4.2). Let $u(x,t)$ be a global weak solution of (1.1), (1.2). Assume contrary $u_0(x) \not\equiv 0$ in $\mathbf{R}$. Then, without loss of generality we can assume

$$(4.22) \qquad\qquad u_0(x) > 0 \quad \text{in } B(2\delta).$$

Let $v_0(x) \in C_0(\mathbf{R})$ satisfy that $v_0(x) = v_0(|x|)$ is a radially symmetric function in $x \in \mathbf{R}$ and a nonincreasing function in $x > 0$, and

$$(4.23) \qquad 0 \leq v_0(x) \leq u_0(x) \quad \text{in } \mathbf{R} \quad \text{and} \quad 0 < v_0(x) \quad \text{in } B(\delta).$$

Then, there exists a global solution $v(x,t)$ of (4.5) satisfying (4.7) (when $p < p_{m,\sigma}^*$) and (4.9) (when $p = p_{m,\sigma}^*$) because of Proposition 4.2. So this is a contradiction to $u_0 \not\equiv 0$ when $m < p < p_{m,\sigma}^* = m+1$.

When $p = p_{m,\sigma}^* = m+1$, we can also drive a contradiction. In fact, as is seen in the proof of Lemma 3.4 of Mochizuki and Suzuki [27], the following results hold: When $m = 1$, for any $t_0 > 0$ there exists a constant $c(t_0) > 0$ such that $v(x,t) \geq c(t_0)E_1(x,t/2;1)$ in $\mathbf{R} \times [t_0,\infty)$, and when $m > 1$ there exist con-

stants $t_1 > 0$ and $L_1 > 0$ such that $v(x, t) \geq E_m(x, t + t_1; L_1)$ in $\mathbf{R} \times [t_0, \infty)$. Hence, when $m = 1$

$$(4.24) \quad \int_0^\infty \int_{B(\delta)} v^2(x, t)\, dxdt \geq c(t_0)^2 \int_{t_0}^\infty \int_{B(\delta)} \left(\frac{t}{2}\right)^{-1} g^2(\eta)\, dxdt \quad [\eta = x/(t/2)^{1/2}]$$

$$\geq c(t_0)^2 \int_{t_0}^\infty \left(\frac{t}{2}\right)^{-1} dt \int_{B(\delta)} g^2(x/(t_0/2)^{1/2})\, dx = \infty,$$

and when $m > 1$

$$(4.25) \quad \int_0^\infty \int_{B(\delta)} v^{m+1}(x, t)\, dxdt$$

$$\geq \int_0^\infty \int_{B(\delta)} L_1^{m+1}(L_1^{m-1}(t + t_1))^{-(m+1)/(m+1)} g^{m+1}(\eta)\, dxdt$$

$$[\eta = x/(L_1^{m-1}(t + t_1))^{1/(m+1)}]$$

$$\geq L_1^2 \int_0^\infty (t + t_1)^{-1}\, dt \int_{B(\delta)} g^{m+1}(x/(L_1^{m-1} t_1)^{1/(m+1)})\, dx = \infty,$$

where $g(\eta)$ is defined by (2.28) and (2.30). In any case, these results contradict (4.9).

Thus we obtain $u_0(x) \equiv 0$, and hence $u(x, t) \equiv 0$ in $\mathbf{R} \times (0, \infty)$ if we consider $u(x, t)$ as the initial data for each $t > 0$. The proof is complete. □

5. Proof of Theorem 2.4(b) in general case.

In this section, we assume (2.6) and we shall show the first part of Theorem 2.4(b). The methods of the proof are similar to those of Theorem 2.4 (a).

First, we consider the case $N \geq 2$, $\sigma > -2$ and the case $N = 1$, $\sigma > -1$. Then

$$(5.1) \qquad \alpha_\sigma^* = \frac{2 + \sigma}{p - m} \quad (>0).$$

PROOF OF THEOREM 2.4 (b) IN THE CASE $N \geq 2$, $\sigma > -2$ AND THE CASE $N = 1$, $\sigma > -1$. Let $K \in I_{-\sigma, \Omega}$ and let $V \subset S^{N-1}$ with $|V| \neq 0$. Let $u(x, t)$ be a global weak solution of (1.1), (1.2). Assume (2.6) for some $A > 0$. Then, there exists a function $\varphi \in C(\mathbf{R}^N)$ such that

$$(5.2) \qquad \begin{cases} 0 \leq \varphi(x) \leq u_0(x) & \text{in } \mathbf{R}^N, \\ \sup_{x \in \mathbf{R}^N} |x|^{\alpha_\sigma^*} \varphi(x) < \infty, \\ \lim_{r \to \infty} r^{\alpha_\sigma^*} \varphi(r\omega) = A_V(\omega) & \text{for each } \omega \in S^{N-1}, \end{cases}$$

where $A_V(\omega)$ is defined by (2.35). Let $w(x,t)$ be a unique weak solution (2.20), (2.21). Then, it follows from the proof of Proposition 2.8 and the uniqueness of the solution of (2.20) that

$$(5.3) \qquad\qquad w(x,t) \le u(x,t) \quad \text{in } \boldsymbol{R}^N \times (0, \infty).$$

Putting $u_k(x,t) = k^{\alpha_\sigma^*} u(kx, k^{2+\alpha_\sigma^*(m-1)}t)$ and $w_k(x,t) = k^{\alpha_\sigma^*} w(kx, k^{2+\alpha_\sigma^*(m-1)}t)$, and noting $\alpha_\sigma^* = (2+\sigma)/(p-m)$, we have by Lemma 3.2,

$$(5.4) \qquad \int_G s(x) w_k(x,t)\, dx \le \int_G s(x) u_k(x,t)\, dt \le \left(\frac{\lambda}{c_0}\right)^{1/(p-m)} \quad \text{for large } k,$$

where G, $s(x)$, λ and c_0 are as in Lemma 3.2. Therefore, since $0 < \alpha_\sigma^* = (2+\sigma)/(p-m) < N$ in case $p > p_{m,\sigma}^*$, letting $k \to \infty$ and using Proposition 2.12 we obtain

$$(5.5) \qquad\qquad \int_G s(x) W_m(x,t; A, \alpha_\sigma^*, V)\, dx \le \left(\frac{\lambda}{c_0}\right)^{1/(p-m)}.$$

Hence, because of (2.40),

$$(5.6) \qquad A \int_G (A^{m-1}t)^{-\alpha_\sigma^*/(\alpha_\sigma^*(m-1)+2)} h_1(\eta) s(x)\, dx \le \left(\frac{\lambda}{c_0}\right)^{1/(p-m)} \quad \text{for } t \ge 0,$$

where $h_1(\eta) = h(\eta; 1, \alpha_\sigma^*, V)$ and $\eta = x/(A^{m-1}t)^{1/[\alpha_\sigma^*(m-1)+2]}$. We note by $h(0) > 0$ that for some $r_1 > 0$ and $\varepsilon_0 > 0$,

$$(5.7) \qquad\qquad h_1(\eta) \ge \varepsilon_0 \quad \text{for } |\eta| \le r_1.$$

Further, we choose t_1 to satisfy $(A^{m-1}t_1)^{1/[\alpha_\sigma^*(m-1)+2]} = 2/r_1$, namely, $t_1 = A^{1-m}(2/r_1)^{\alpha_\sigma^*(m-1)+2}$ and put $t = t_1$ in (5.6). Since $|\eta| = (r_1/2)|x| \le r_1$ in G, we get

$$(5.8) \qquad \left(\frac{\lambda}{c_0}\right)^{1/(p-m)} \ge A \int_G \left(\frac{2}{r_1}\right)^{-\alpha_\sigma^*} \times \varepsilon_0 s(x)\, dx = A\left(\frac{2}{r_1}\right)^{-\alpha_\sigma^*} \varepsilon_0.$$

That is,

$$(5.9) \qquad\qquad A \le \left(\frac{\lambda}{c_0}\right)^{1/(p-m)} \left(\frac{2}{r_1}\right)^{\alpha_\sigma^*} \varepsilon_0^{-1}.$$

So, if $A > (\lambda/c_0)^{1/(p-m)}(2/r_1)^{\alpha_\sigma^*}\varepsilon_0^{-1}$ then there is no global solutions. The proof is complete. $\qquad\square$

Next, we consider the case where $N = 1$ and $\sigma \le -1$. In this case,

(5.10) $$\alpha_\sigma^* = \frac{1}{p-m}.$$

Similarly, as in §4, we can assume (4.2) with $K_1(x)$ defined by (4.3). Further, since $V\ (\neq \varnothing) \subset S^0$, we can also assume (2.6) with $V = \{1\}$. Then, there exists a function $\varphi \in C(\mathbf{R}^N)$ such that

(5.11) $$\begin{cases} 0 \le \varphi(x) \le u_0(x) & \text{in } \mathbf{R}, \\ \sup_{x\in\mathbf{R}}|x|^{\alpha_\sigma^*}\varphi(x) < \infty, \\ \lim_{r\to\infty} r^{\alpha_\sigma^*}\varphi(r) = A, \\ \varphi(x) = 0 & \text{in } x \le 0, \end{cases}$$

and we have the following

PROPOSITION 5.1. *Let $u(x,t)$ be a global weak solution (1.1), (1.2). Then, there exists a global weak solution v of (4.5) with $v(x,0) = \varphi(x)$ such that for each $t > 0$ $v(x,t)$ is a nondecreasing function in $x < 0$ and*

(5.12) $$v(x,t) \le u(x,t) \quad \text{in } x \in \mathbf{R}^N \times (0,\infty).$$

Further, if we put $v_k(x,t) = k^{1/(p-m)}v(kx,k^{(2p-m-1)/(p-m)}t)$ then we have

(5.13) $$\int_{-\delta}^0 s(x)v_k\,dx \le \frac{1}{2}\left(\frac{\lambda}{\delta}\right)^{1/(p-m)} \quad \text{for } t \ge 0,$$

where $\lambda = \lambda_{(-\delta,\delta)}$ and $s(x) = s_{(-\delta,\delta)}(x)$ are defined in §2.

PROOF. The methods of the proof are the same as those of Proposition 4.1 and Lemma 4.3. So, we only prove (5.13). As is seen in the proof of Lemma 4.3, $v_k(x,t) = k^{1/(p-m)}v(kx,k^{(2p-m-1)/(p-m)}t)$ is a global weak solution of (4.11). Hence, we have

(5.14) $$\int_{-\delta}^0 v_k s(x)\,dx\Big|_0^t \ge -\int_0^t\int_{-\delta}^0 \lambda v_k^m s(x)\,dxdt + k\int_0^t\int_{-\delta/k}^0 \delta v_k^p s(x)\,dxdt,$$

where $\lambda = \lambda_{(-\delta,\delta)}$ and $s(x) = s_{(-\delta,\delta)}(x)$. In fact, since $v_k(x,t)$ is a nondecreasing function in $x < 0$ for each $t \ge 0$, we see that the above inequality holds for the classical approximate solutions of (4.11) if we multiply (4.11) by $s(x)$ and integrate by parts in $(-\delta,0)$. Hence, we also see that (5.14) holds for a weak solution v_k. Thus, as in the proof of Lemma 4.3, we obtain (5.13). □

PROOF OF THEOREM 2.4 (b) IN THE CASE $N = 1$, $\sigma \le -1$. We shall drive the proof when $V = \{1\}$ and $K(x)$ satisfy (4.2) with (4.3). Let $v(x,t)$ be a global solution as in Proposition 5.1. Let $w(x,t)$ be a unique solution of (2.20) with $w(x,0) = \varphi(x)$ and put $w_k(x,t) = k^{\alpha_\sigma^*}w(kx,k^{2+\alpha_\sigma^*(m-1)})$ with $\alpha_\sigma^* = 1/(p-m)$. Then, it is not difficult to show that

(5.15) $w_k(x,t) \le v_k(x,t)$ in $\boldsymbol{R} \times (0, \infty)$.

Further, it follows from Proposition 5.1

(5.16) $\displaystyle\int_{-\delta}^{0} s(x)w_k(x,t)\,dx \le \int_{-\delta}^{0} s(x)v_k(x,t)\,dx \le \frac{1}{2}\left(\frac{\lambda}{\delta}\right)^{1/(p-m)}$ for $t \ge 0$,

where $s(x)$ and λ are as in Proposition 5.1.   We note that $\alpha_\sigma^* = 1/(p-m) < 1$ by $p > p_{m,\sigma}^* = m+1$.   Therefore, if $k \to \infty$, Proposition 2.12 implies that

(5.17) $\displaystyle\int_{-\delta}^{0} s(x)W_m(x,t; A, \alpha_\sigma^*, V)\,dx \le \frac{1}{2}\left(\frac{\lambda}{\delta}\right)^{1/(p-m)}.$

Hence, because of (2.36),

(5.18) $A\displaystyle\int_{-\delta}^{0} (A^{m-1}t)^{-\alpha_\sigma^*/(\alpha_\sigma^*(m-1)+2)} h_1(\eta)s(x)\,dx \le \frac{1}{2}\left(\frac{\lambda}{\delta}\right)^{1/(p-m)}$ for $t \ge 0$,

where $h_1(\eta) = h(\eta; 1, \alpha_\sigma^*, V)$ and $\eta = x/(A^{m-1}t)^{1/[\alpha_\sigma^*(m-1)+2]}$.   Let $0 < \delta < 2$. Similarly, as in the proof when $N \ge 2$, $\sigma > -2$ or $N = 1$, $\sigma > -1$, we have

(5.19) $A \le \dfrac{1}{2}\left(\dfrac{\lambda}{\delta}\right)^{1/(p-m)}\left(\dfrac{2}{r_1}\right)^{\alpha_\sigma^*}\varepsilon_0^{-1},$

where r_1 and ε_0 are as in (5.7).   The proof is complete. $\square$

Finally, we show the theorem in the case $N \ge 2$, $\sigma \le -2$.   Then

(5.20) $\alpha_\sigma^* \le 0.$

  PROOF OF THEOREM 2.4 (b) IN THE CASE $N \ge 2$, $\sigma \le -2$.   The proof is also the same as that in the case $N \ge 2$, $\sigma > -2$.   Let $K(x) \in I_{-\sigma,\Omega}$ and let $u(x,t)$ be a global weak solution of (1.1), (1.2).   Assume (2.6) with $[\alpha_\sigma^*]_+ = 0$ for some $A > 0$.   Choose $\varphi(x) \in C(\boldsymbol{R}^N)$ to satisfy (5.2) with $\alpha_\sigma^* = 0$.   Let $w(x,t)$ be a unique weak solution of (2.20), (2.21).   Then, we can see $w(x,t) \le u(x,t)$ in $\boldsymbol{R}^N \times (0, \infty)$.   Put $w_k(x,t) = w(kx, k^2t)$.   It follows from Proposition 2.12 that

(5.21) $w_k(x,t) \to W_m(x,t; A, 0, V)$

locally uniformly in $\boldsymbol{R}^N \times (0, \infty)$ as $k \to \infty$.   Further, in virtue of the same methods as those of Friedman and Kamin [9],

(5.22) $|w(x,t) - W_m(x,t; A, 0, V)| \to 0$

locally uniformly in $\boldsymbol{R}^N$ as $t \to \infty$.   Therefore, if we note that

(5.23) $W_m(x,t; A, 0, V) \to Ah_0$

locally uniformly in $\boldsymbol{R}^N$ as $t \to \infty$ where $h_0 = h(0; 1, 0, V) > 0$, then we obtain

$$(5.24) \qquad\qquad w(x, t) \to Ah_0$$

locally uniformly in $\boldsymbol{R}^N$ as $t \to \infty$.

On the other hand, by $K(x) \in I_{-\sigma, \Omega}$ there exist a domain $G \ (\neq \varnothing) \subset \boldsymbol{R}^N$ and a constant $m_0 > 0$ such that

$$(5.25) \qquad\qquad K(x) \geq m_0 \quad \text{in } G.$$

It follows form Proposition 2.14 that

$$(5.26) \qquad \int_G s(x)w(x, t)\, dx \leq \int_G s(x)u(x, t)\, dx \leq \left(\frac{\lambda}{m_0}\right)^{1/(p-m)}$$

where $s(x) = s_G(x)$ and $\lambda = \lambda_G$. So, since $\int_G s(x)\, dx = 1$, if $t \to \infty$ then $Ah_0 \leq (\lambda/m_0)^{1/(p-m)}$, that is,

$$(5.27) \qquad\qquad A \leq \left(\frac{\lambda}{m_0}\right)^{1/(p-m)} h_0^{-1}.$$

The proof is complete. □

6. Proof of Theorem 2.4(b) in the case $\alpha_\sigma^* = 0$.

In this section, in the case $\alpha_\sigma^* = 0$ we shall show the last part of Theorem 2.4(b) under assumption (2.7), where we add assumption $K(x) \in I_{-\sigma, \boldsymbol{S}^{N-1}} = I_{2, \boldsymbol{S}^{N-1}}$ when $N \geq 3$. Then, we note that $\sigma = -2$ when $N \geq 3$. We need several propositions and lemmas for the proof.

LEMMA 6.1. *Let $u(x, t)$ be a weak global solution of (1.1), (1.2) with the initial data u_0 satisfying (2.7). Then, there exist constants $h > 0$ and $t_0 > 0$ such that*

$$(6.1) \qquad\qquad u(x, t_0) \geq h \quad \text{for } x \in \boldsymbol{R}^N.$$

PROOF. Assume (2.7). Let $u(x, t)$ be a global weak solution of (1.1), (1.2). Then, for some $R_1 > 0$ and $a > 0$,

$$(6.2) \qquad\qquad u_0(x) \geq a \quad \text{in } |x| \geq R_1.$$

First, we consider the case where $m = 1$. Let $\varphi(x)$ satisfy

$$(6.3) \qquad\qquad \varphi(x) = \begin{cases} a & \text{in } |x| > R_1, \\ 0 & \text{in } |x| \leq R_1, \end{cases}$$

and let $w(x, t)$ be a weak solution of (2.20), (2.21) with $m = 1$. Then, as in the proof of Proposition 2.8, we have

(6.4) $w(x, t) \leq u(x, t)$ in $\mathbf{R}^N \times (0, \infty)$.

Hence,

(6.5) $u(x, t) \geq w(x, t) = a(4\pi t)^{-N/2} \int_{|y| \geq R_1} e^{-|x-y|^2/4t} \, dy.$

Therefore, when $|x| \geq R_1 + 1$,

(6.6) $u(x, t) \geq a(4\pi t)^{-N/2} \int_{|x-y| \leq 1} e^{-|x-y|^2/4t} \, dy = a(4\pi t)^{-N/2} \int_{|z| \leq 1} e^{-|z|^2/4t} \, dz$

and when $|x| < R_1 + 1$,

(6.7) $u(x, t) \geq a(4\pi t)^{-N/2} e^{-|x|^2/2t} \int_{|y| \geq R_1} e^{-|y|^2/2t} \, dy$

$$\geq a(4\pi t)^{-N/2} e^{-(R_1+1)^2/2t} \int_{|y| \geq R_1} e^{-|y|^2/4t} \, dy.$$

Thus, we obtain (6.1) in case $m = 1$.

   Next, we consider the case where $m > 1$. Then, it is not difficult to see that for some $t_1 > 0$ and $L_1 > 0$,

(6.8) $E_m(x, t_1; L_1) < a$ in $\mathbf{R}^N$ and $\operatorname{supp} E_m(x, t_1; L_1) \subset B(1) = \{|x| < 1\}$,

where $E_m(x, t_1; L_1)$ is defined by (2.27). For each $x_0 \in \mathbf{R}^N$, put $w_{x_0}(x, t) = E_m(x - x_0, t + t_1; L_1)$. It follows from (6.2) and the comparison theorem that if $|x_0| \geq R_1 + 1$ then

(6.9) $w_{x_0}(x, t) \leq u(x, t)$ in $\mathbf{R}^N \times (0, \infty)$.

Hence,

(6.10) $0 < E_m(0, t + t_1; L_1) \leq u(x_0, t)$ for $t > 0$.

On the other hand, since $\operatorname{supp} w_{x_0}(\cdot, t)$ spreads out to whole $\mathbf{R}^N$ as $t \to \infty$, there exist $t_0 > 0$ and $\varepsilon_0 > 0$ such that

(6.11) $u(x, t_0) \geq \varepsilon_0$ for $|x| < R_1 + 1$.

Thus, putting $h = \min\{\varepsilon_0, E_m(0, t_1 + t_0; L_1)\}$ we obtain (6.1).   □

   Hence, if we assume (2.7), then we can assume for some $h > 0$

(6.12) $u_0(x) \geq h$ in $\mathbf{R}^N$.

And, if we assume $\alpha_\sigma^* = 0$, and when $N \geq 3$ we further assume $K(x) \in I_{-\sigma, \mathbf{S}^{N-1}} = I_{2, \mathbf{S}^{N-1}}$, then, without loss of generality we can assume

(6.13) $K(x) \geq K_3(x)$ in $\mathbf{R}^N$,

where $K_3(x) = K_3(|x|) \in C^\infty$ is a radially symmetric function satisfying the following properties: When $N = 2$, $K_3(r) \in C_0^\infty[0, \infty)$, $K_3(r)$ is nonincreasing in $r > 0$, $0 \le K_3(x) \le \delta$ in $\mathbf{R}^N$ for some $\delta > 0$ and

$$(6.14) \qquad K_3(x) = \begin{cases} \delta & |x| \le \delta \\ 0 & |x| > 2\delta; \end{cases}$$

When $N \ge 3$, $K_3(x) \ge 0$ in $\mathbf{R}^N$ and for some $r_0 > 1$ and $k_0 > 0$,

$$(6.15) \qquad K_3(x) = k_0|x|^{-2} \quad \text{in } |x| \ge r_0.$$

Let $v(x, t)$ be a weak solution of the problem

$$(6.16) \qquad \begin{cases} u_t - \Delta u^m = K_3(x)u^p & \text{in } \mathbf{R}^N \times (0, \infty), \\ u(x, 0) = h & \text{in } \mathbf{R}^N. \end{cases}$$

PROPOSITION 6.2. *Assume* (6.12) *and* (6.13). *Let* $u(x, t)$ *be a global weak solution of* (1.1), (1.2). *Then, there exists a classical global weak solution* $v(x, t)$ *of* (6.16) *such that for each* $t > 0$ $v(x, t) = v(|x|, t)$ *is a radially symmetric function in* $x \in \mathbf{R}^N$ *(and is nonincreasing in* $r = |x|$ *when* $N = 2$) *and*

$$(6.17) \qquad (0<) \, h \le v(x, t) \le u(x, t) \quad \text{in } \mathbf{R}^N \times (0, \infty),$$

$$(6.18) \qquad v_t \ge 0 \quad \text{in } \mathbf{R}^N \times (0, \infty).$$

PROOF. Noting that h is a solution of (2.20), we have by the comparison theorem,

$$(6.19) \qquad h \le u(x, t) \quad \text{in } \mathbf{R}^N \times (0, \infty).$$

Let $v_n(x, t)$ be a classical solution of the initial boundary value problem

$$(6.20) \qquad \begin{cases} u_t - \Delta u^m = K_3(x)u^p & \text{in } B(n) \times (0, T), \\ u = h & \text{on } \partial B(n) \times (0, T), \\ u(x, 0) = h & \text{in } B(n), \end{cases}$$

where $B(n) = \{|x| < n\}$. Then, as in the proof of Proposition 2.8, we see that $v(x, t) = \lim_{n \to \infty} v_n$ is a classical solution of (6.20) satisfying (6.17), and for each $t > 0$ $v(x, t) = v(|x|, t)$ is radially symmetric in $\mathbf{R}^N$ (and is nonincreasing in $r = |x|$ when $N = 2$). Noting that $v_t(x, 0) = \Delta v^m(x, 0) + K_3(x)v^p(x, 0) = K_3(x)h^p \ge 0$ in $\mathbf{R}^N$, we obtain (6.18) by virtue of the comparison theorem for the equation satisfied by v_t. The proof is complete. $\qquad \square$

PROPOSITION 6.3. *Let* $p > m$. *Put* $G = \mathbf{R}^N \backslash \{0\}$ *if* $N = 2$ *and* $G = \mathbf{R}^N \backslash \bar{B}(r_0 + 1)$ *if* $N \ge 3$. *Let* $v(x, t)$ *be as in Proposition 6.2 and put* $\tilde{v}(x) = \lim_{t \to \infty} v(x, t)$. *Then,* $\tilde{v} \in L_{loc}^\infty(G) \cap C^2(G)$, $\tilde{v}(x) = \tilde{v}(|x|)$ *is a radially symmetric function in* $x \in \mathbf{R}^N$ *(and is nonincreasing in* $r = |x|$ *when* $N = 2$) *and*

$$(6.21) \qquad\qquad (0<)\ h \le \tilde{v}(x) \quad \text{in } \boldsymbol{R}^N,$$

$$(6.22) \qquad\qquad -\varDelta\tilde{v} = K_3(x)\tilde{v}^p \quad \text{in } G.$$

In order to prove this proposition we need the following

LEMMA 6.4. *Let $p > m$ and let $v(x,t)$ be as in Proposition 6.2. Then, the next results hold*:
  (i) *When $N = 2$, if $r \ge \delta$ then*

$$(6.23) \qquad\qquad v(x,t) \le \left(\frac{\lambda_\delta}{\delta}\right)^{1/(p-m)} \quad \text{on } |x| = r, t > 0$$

and if $0 < r < \delta$ then

$$(6.24) \qquad\qquad v(x,t) \le \left(\frac{\lambda_r}{\delta}\right)^{1/(p-m)} \quad \text{on } |x| = r, t > 0,$$

where $\lambda_r = \lambda_{B(r)}$ is defined in §2.
  (ii) *When $N \ge 3$*

$$(6.25) \qquad\qquad v(x,t) \le 2\left(\frac{(r_0+1)^2\tilde{\lambda}_{r_0}}{k_0}\right)^{1/(p-m)} \quad \text{for } |x| \ge r_0 + 1,$$

where $\tilde{\lambda}_{r_0} = \lambda_D$ and $D = B(r_0+1)\backslash\bar{B}(r_0)$.

PROOF. (i) Put $s_r(x) = s_{B(r)}(x)$. Since $v(x,t)$ is a global solution of (6.16), Proposition 2.14 implies that for $0 < r \le \delta$,

$$(6.26) \qquad\qquad \int_{B(r)} s_r(x)v(x,t)\,dx \le \left(\frac{\lambda_r}{\delta}\right)^{1/(p-m)} \quad \text{for } t > 0.$$

Hence, noting that $v(x,t) \ge v(r,t)$ in $B(r)$ we have

$$(6.27) \qquad\qquad v(r,t) \le \left(\frac{\lambda_r}{\delta}\right)^{1/(p-m)}$$

and so (6.24). (6.23) is clear, because $v(r,t)$ is nonincreasing in $r \ge 0$.
  (ii) Let $r_1 \ge r_0 + 1$. Then, there exists $t_0 > 0$ such that

$$(6.28) \qquad\qquad v(x,t_0) \ge \frac{1}{2}v(r_1,0) \quad \text{in } |x| \le r_1.$$

In fact, let $w(x,t)$ be a weak solution of the problem

$$(6.29) \quad \begin{cases} w_t - \Delta w^m = 0 & \text{in } B(r_1) \times (0, \infty), \\ w(x, 0) = 0 & \text{in } B(r_1), \\ w(x, t) = v(r_1, 0) & \text{on } \partial B(r_1) \times (0, \infty). \end{cases}$$

Then, since $v(x, t) \geq v(x, 0) = v(r_1, 0)$ on $\partial B(r_1) \times (0, \infty)$, it follows from the comparison theorem that $v(x, t) \geq w(x, t)$ in $B(r_1) \times (0, \infty)$. Further, noting that $w(x, t) \to v(r_1, 0)$ uniformly in $B(r_1)$ as $t \to \infty$, we have (6.28).

Put $\tilde{s}_{r_0}(x) = S_D(x)$ with $D = B(r_0 + 1) \backslash \bar{B}(r_0)$. By virtue of Proposition 2.14 and the condition on $K_3(x)$ we get

$$(6.30) \qquad \int_{B(r_0+1) \backslash \bar{B}(r_0)} \tilde{s}_{r_0} v(x, t) \, dx \leq \left(\frac{\tilde{\lambda}_{r_0}}{k_0 (r_0 + 1)^{-2}} \right)^{1/(p-m)} \quad \text{for } t \geq 0$$

to obtain

$$(6.31) \qquad \frac{1}{2} v(r_1, 0) \leq \left(\frac{\tilde{\lambda}_{r_0} (r_0 + 1)^2}{k_0} \right)^{1/(p-m)}. \qquad \square$$

PROOF OF PROPOSITION 6.3. It is clear that $\tilde{v}(x) = \lim_{t \to \infty} v(x, t) < \infty$ in $x \in G$, $\tilde{v} \in L^\infty_{loc}(G)$, $\tilde{v}(x) = \tilde{v}(|x|)$ is a radially symmetric function in $x \in \mathbf{R}^N$ (and is nonincreasing in $r = |x|$ when $N = 2$) and (6.21) holds. Further, $\tilde{v} \in C^2(G)$ and (6.22) immediately follow from the methods of Kröner and Rodrigues [21]. In fact, let $\psi(t) \in C_0^\infty(0, 1)$ satisfying $\int_0^\infty \psi(t) \, dt = 1$ and let $\xi(x) \in C_0^\infty(G)$. Put $\varphi(x, t) = \psi(t)\xi(x)$. Then, if we consider $\varphi(x, t)$ as a test function in the integral equation satisfied by $v(x, t)$, for each $\tau > 0$

$$(6.32)$$

$$\int_0^1 \int_G \{ v(x, t + \tau)\psi'(t)\xi(x) + v^m(x, t + \tau)\psi \Delta \xi + K_3(x) v^p(x, t + \tau)\psi \xi \} \, dx dt = 0.$$

Letting $\tau \to \infty$ we have

$$(6.33) \qquad \int_0^1 \psi' \, dt \int_G \tilde{v}(x)\xi(x) \, dx + \int_0^1 \psi \, dt \int_G \{ \tilde{v}^m(x)\Delta \xi + K_3(x)\tilde{v}^p(x)\xi \} \, dx = 0,$$

that is,

$$(6.34) \qquad \int_G \{ \tilde{v}^m \Delta \xi + K_3 \tilde{v}^p \xi \} \, dx = 0 \quad \text{for } \xi \in C_0^\infty(G).$$

Hence,

$$(6.35) \qquad \Delta \tilde{v}^m + K_3(x)\tilde{v}^p = 0 \quad \text{in } D'(G).$$

Noting $\tilde{v} \in L^\infty_{loc}(G)$ and (6.21) we obtain $\tilde{v} \in C^2(G)$ by the regularity theorem and so we get (6.22). The proof is complete. □

PROOF OF THEOREM 2.4 (b) IN THE CASE $\alpha^*_\sigma = 0$. As above-mentioned, it is enough to show this theorem when $u_0(x)$ and $K(x)$ satisfy (6.12) and (6.13) respectively.

Assume contrary that there exists a global weak solution $u(x,t)$ of (1.1), (1.2). Then, there exists $\tilde{v}$ such as in Proposition 6.3. If we put $\tilde{v}(r) = \tilde{v}(x)$ $(r = |x|)$ then (6.22) is reduced to

$$(6.36) \qquad \tilde{v}_{rr} + \frac{N-1}{r}\tilde{v}_r = -K_3(r)\tilde{v}^p,$$

where $K_3(r) = K_3(x)$ $(r = |x|)$.

Hence,

$$(6.37) \qquad (r^{N-1}\tilde{v}_r)_r = -r^{N-1}K_3(r)\tilde{v}^p.$$

Therefore, when $N = 2$, for each $r > \delta$ and $r_1 \in (0, \delta/2)$

$$(6.38) \qquad r\tilde{v}_r = r_1\tilde{v}_r(r_1) - \int_{r_1}^r rK_3(r)\tilde{v}^p\, dr \le -h^p \int_{r_1}^r rK_3(r)\, dr,$$

from which,

$$(6.39) \qquad r\tilde{v}_r \le -h^p \int_{\delta/2}^\delta rK_3(r)\, dr = -h^p\delta \int_{\delta/2}^\delta r\, dr \equiv -c \ (<0),$$

and so

$$(6.40) \qquad \tilde{v}_r \le -\frac{c}{r} \quad \text{for } r > \delta.$$

Hence, if $r > \delta$ then

$$(6.41) \qquad \tilde{v}(r) \le \tilde{v}(\delta) - c\int_\delta^r \frac{1}{r}\, dr = \tilde{v}(\delta) - c\{\log r - \log(\delta)\} \to -\infty \quad \text{as } r \to \infty.$$

When $N \ge 3$, if $r \ge r_1 \ge r_0 + 1$ then we have by (6.37) and (6.15),

$$(6.42) \qquad r^{N-1}\tilde{v}_r \le -k_0 h^p \int_{r_1}^r r^{N-3}\, dr + r_1^{N-1}\tilde{v}_r(r_1)$$

$$= -\frac{k_0 h^p}{N-2}(r^{N-2} - r_1^{N-2}) + r_1^{N-1}\tilde{v}_r(r_1).$$

Hence,

$$(6.43) \qquad \tilde{v}_r \leq -\frac{k_0 h^p}{N-2} r^{-1} + \frac{C}{r^{N-1}} \quad \text{for } r \geq r_1,$$

which implies

$$(6.44) \qquad \tilde{v}(r) \leq \tilde{v}(r_1) - \frac{k_0 h^p}{N-2}\{\log r - \log r_1\}$$

$$+ \frac{C}{-N+2}\{r^{-N+2} - r_1^{-N+2}\} \rightarrow -\infty \quad \text{as } r \rightarrow \infty.$$

Thus, (6.41) and (6.44) are contradictions to (6.21), and so global solutions of (1.1), (1.2) never exist. The proof is complete. $\qquad \square$

7. Proof of Theorem 2.5 in the case $\sigma \geq 0$ and the case $\sigma < 0$, $\alpha_\sigma^* < 0$.

In this section we shall show Theorem 2.5 in the case $\sigma \geq 0$ and the case $\sigma < 0$, $\alpha_\sigma^* < 0$. We first show the theorem in the case $\sigma \geq 0$. The methods of the proof are the same as those of Mukai, Mochizuki and Huang [29] (see the proof of Lemma 5 in [29]). We need the next lemma. Let $W_m(x,t; L, \alpha, V)$ and $h(\eta; L, \alpha, V)$ be as in §2.

LEMMA 7.1. *Let* $\sigma \geq 0$, $(2+\sigma)/(p-m) < \alpha < N$ *and* $K(x) \in I^{-\sigma}$. *Put* $W_m(x,t) = W_m(x,t; L, \alpha, S^{N-1})$. *Then, for some* $c_1 > 0$

$$(7.1) \qquad K(x) W_m^{p-1}(x,t) \leq c_1 t^{-((p-1)\alpha-\sigma)/(2+\alpha(m-1))} \quad \text{for } t \geq 1.$$

Further, for some $t_1 \geq 1$

$$(7.2) \qquad c_1(p-m)\int_{t_1}^{\infty} t^{-((p-1)\alpha-\sigma)/(2+\alpha(m-1))}\, dt \leq \frac{1}{2}.$$

PROOF. Since $K(x) \in I^{-\sigma}$, there is a constant $C > 0$ such that

$$(7.3) \qquad K(x)^{1/(p-1)} \leq C(1 + |x|^{\sigma/(p-1)}) \quad \text{in } \mathbf{R}^N.$$

Putting $h(\eta) = h(\eta; L, \alpha, S^{N-1})$ we have for $t \geq 1$,

$$(7.4) \quad K(x)^{1/(p-1)} W_m(x,t)$$

$$\leq C t^{-\{(p-1)\alpha-\sigma\}/(p-1)\{\alpha(m-1)+2\}}(1 + |\eta|^{\sigma/(p-1)})h(\eta) \quad (\eta = x/t^{1/(\alpha(m-1)+2)}).$$

Further, noting that by $\sigma/(p-1) < (2+\sigma)/(p-m) < \alpha$

$$(7.5) \qquad 1 + |\eta|^{\sigma/(p-1)} \leq C'\langle\eta\rangle^\alpha \quad \text{and} \quad \langle\eta\rangle^\alpha h(\eta) \leq C'' \quad \text{for } \eta \in \mathbf{R}^N,$$

we obtain

$$(7.6) \quad K(x)^{1/(p-1)} W_m(x,t) \le CC'C'' t^{-\{(p-1)\alpha-\sigma\}/(p-1)\{\alpha(m-1)+2\}} \quad \text{for } t \ge 1,$$

which is reduced to (7.1). (7.2) is obvious, since $-((p-1)\alpha-\sigma)/(\alpha(m-1)+2) < -1$ by the assumptions on α. $\qquad\square$

PROOF OF THEOREM 2.5 IN THE CASE $\sigma \ge 0$. Let $\sigma \ge 0$ and $p > \max\{m, p_{m,\sigma}^*\}$. Then, we note $p_{m,\sigma}^* = m + (2+\sigma)/N$, $\alpha_\sigma^* = (2+\sigma)/(p-m)$ and $\alpha_\sigma^* < N$. Further, let $\alpha_\sigma^* < \alpha < N$ and assume

$$(7.7) \qquad u_0(x) \le \min\{\varepsilon, A|x|^{-\alpha}\} \quad \text{for } x \in \mathbf{R}^N,$$

where $\varepsilon > 0$ will be chosen later. Put $w(x,t) = W_m(x, t + t_1; L, \alpha, \mathbf{S}^{N-1})$ $(L > A)$ and $k(t) = c_1(t + t_1)^{-((p-1)\alpha-\sigma)/(\alpha(m-1)+2)}$ where $t_1 \ge 1$ and $c_1 > 0$ are as in (7.1) and (7.2). Further, we put $\tilde{w}(x,t) = \alpha(t)w(x, b(t))$ where $\alpha(t)$ and $b(t)$ are defined by (2.23) and (2.24) respectively. Noting that $k(t)$ satisfies (2.22) and (2.26), we see by Proposition 2.9, that $\tilde{w}$ is a supersolution of (1.1) in $\mathbf{R}^N \times (0, \infty)$.

On the other hand, we have

$$(7.8) \quad \tilde{w}(x,0) = w(x,0) = W_m(x, t_1; L, \alpha, \mathbf{S}^{N-1}) = t_1^{-\alpha/(2+\alpha(m-1))} h(\eta) = |x|^{-\alpha}|\eta|^\alpha h(\eta)$$

with $h(\eta) = h(\eta; L, \alpha, \mathbf{S}^{N-1})$ and $\eta = x/t_1^{1/[\alpha(m-1)+2]}$, and there exists $R > 0$ such that

$$(7.9) \qquad |\eta|^\alpha h(\eta) \ge A \quad \text{for } |x| \ge R,$$

since $|\eta|^\alpha h(\eta) \to L$ as $|\eta| \to \infty$. Hence,

$$(7.10) \qquad \tilde{w}(x,0) \ge A|x|^{-\alpha} \ge u_0(x) \quad \text{for } |x| \ge R,$$

and if ε is small enough then

$$(7.11) \qquad \tilde{w}(x,0) = t_1^{-\alpha/(2+\alpha(m-1))} h(\eta) \ge \varepsilon \ge u_0(x) \quad \text{for } |x| < R.$$

Namely,

$$(7.12) \qquad \tilde{w}(x,0) \ge u_0(x) \quad \text{in } \mathbf{R}^N.$$

Therefore, it follows form Proposition 2.8 that there is a global weak solution of (1.1), (1.2) satisfying

$$(7.13) \qquad u(x,t) \le \tilde{w}(x,t) \le Ct^{-\alpha/(2+\alpha(m-1))} \quad \text{for } t \ge 0.$$

The proof is complete. $\qquad\square$

Next, we consider the case where $\sigma < 0$ and $\alpha_\sigma^* < 0$. Then, we see $N \ge 3$, $\sigma < -2$ and $p_{m,\sigma}^* < m$. If $K(x) \in I^{-\sigma}$ then there exists $k_0 > 0$ such that

$$(7.14) \qquad K(x) \le k_0 \langle x \rangle^\sigma \quad \text{in } \mathbf{R}^N, \quad \text{where } \langle x \rangle = \sqrt{1 + |x|^2}.$$

We consider the following problem for each $\varepsilon > 0$:

(7.15)
$$\begin{cases} u_t - \Delta u^m = k_0 \langle x \rangle^\sigma u^p & \text{in } \mathbf{R}^N \times (0, T), \\ u(x, 0) = \varepsilon & \text{in } \mathbf{R}^N. \end{cases}$$

Then, by virtue of the usual existence and uniqueness theorem we see that a unique weak solution of (7.15) exists locally in time and can be extended uniquely as the time increases as far as $u(\cdot, t) \in L^\infty(\mathbf{R}^N)$.

LEMMA 7.2. *Let* $N \geq 3$ *and* $\sigma < -2$. *Then, if* $\varepsilon > 0$ *is small enough, there exists a global weak solution* $u(x, t)$ *of* (7.15) *such that*

(7.16)
$$\sup_{\mathbf{R}^N \times (0, \infty)} u(x, t) < \infty.$$

PROOF. First, it follows from the uniqueness of solutions that for each $t > 0$ the solution $u(x, t)$ of (7.15) is a radially symmetric function in $x \in \mathbf{R}^N$ and $u(x, t) = u(|x|, t)$ is nonincreasing in $r = |x|$ as far as $u(\cdot, t) \in L^\infty(\mathbf{R}^N)$.

Let $a \neq 0$ ($a \in \mathbf{R}^N$). Then, there exists $A > 0$ such that

(7.17)
$$k_0 \langle x \rangle^\sigma \leq A |x - a|^\sigma.$$

Further, let $\varepsilon > 0$ and

(7.18)
$$\max\{2 - N, \sigma + 2\} < q < 0,$$

and put

(7.19)
$$v_a(x) = \varepsilon(|x - a|^q + 1)^{1/m}.$$

Then, we see that for small $\varepsilon > 0$,

(7.20)
$$\Delta v_a^m + k_0 \langle x \rangle^\sigma v_a^p \leq 0 \quad \text{in } |x - a| \geq \frac{|a|}{2}.$$

In fact, from the inequality $(a + b)^{q'} \leq C(q')(a^{q'} + b^{q'})$ for $q' > 1$,

(7.21)
$$\Delta v_a^m + k_0 \langle x \rangle^\sigma v_a^p \leq \left(\frac{\partial^2}{\partial r^2} + \frac{N - 1}{r} \frac{\partial}{\partial r} \right) v_a^m + A |x - a|^\sigma v_a^p \quad (r = |x - a|)$$

$$\leq \varepsilon^m q(N + q - 2) r^{q-2} + A r^\sigma \varepsilon^p C(r^{pq/m} + 1)$$

$$= \varepsilon^m r^{q-2} \{ q(N + q - 2) + A C \varepsilon^{p-m} (r^{q(p/m-1)+\sigma+2} + r^{\sigma-q+2}) \}.$$

Here, we note from (7.18), that $q(N + q - 2) < 0$, $q(p/m - 1) + \sigma + 2 < 0$ and $\sigma - q + 2 < 0$. Hence, there exists a small $\varepsilon > 0$ such that for any $r = |x - a| \geq |a|/2$,

$$(7.22) \qquad \Delta v_a^m + k_0 \langle x \rangle^\sigma v_a^p$$

$$\leq \varepsilon^m r^{q-1} [q(N + q - 2)$$

$$+ AC\varepsilon^{p-m}\{(|a|/2)^{q(p/m-1)+\sigma+2} + (|a|/2)^{\sigma-q+2}\}] \leq 0.$$

So, (7.20) is obtained.

Now, we shall show

$$(7.23) \qquad u(0, t) \leq v_a(0) \quad \text{for } t \geq 0.$$

Put

$$(7.24) \qquad \tilde{T} = \sup\{T \mid u(0, t) \leq v_a(0) \text{ for } t \in [0, T]\}.$$

We note $\tilde{T} > 0$. Assume contrary $\tilde{T} < \infty$. Then,

$$(7.25) \qquad u(x, t) \leq u(0, t) \leq v_a(0) < v_a(a/2) \quad \text{for } t \in [0, \tilde{T}], x \in \mathbf{R}^N.$$

Further, as above-mentioned, for some $T_1 > 0$ $u(x, t)$ is extended uniquely in $\mathbf{R}^N \times [0, T_1 + \tilde{T})$, and for each $t \in [0, T_1 + \tilde{T})$ $u(x, t) = u(|x|, t)$ is radially symmetric in $x \in \mathbf{R}^N$ and nonincreasing in $r = |x|$. Hence, there exists $T_2 \in (0, T_1)$ such that $u(0, t) < v_a(a/2)$ in $t \in [0, \tilde{T} + T_2]$ since $u(x, t)$ is continuous in $\mathbf{R}^N \times [0, T_1 + \tilde{T})$, and so

$$(7.26) \qquad u(x, t) \leq u(0, t) < v_a(a/2) = v_a(x) \quad \text{for } |x - a| = \frac{|a|}{2}, t \in [0, \tilde{T} + T_2].$$

We see also that $u(x, 0) = \varepsilon \leq v_a(x)$ in $\mathbf{R}^N$ and $v_a(x)$ is supersolution of (7.15) in $|x - a| > |a|/2$, $t \in (0, \tilde{T} + T_2]$ by (7.20). Therefore, applying the usual comparison theorem to u and v_a we have

$$(7.27) \qquad u(x, t) \leq v_a(x) \quad \text{in } |x - a| \geq \frac{|a|}{2}, t \in [0, \tilde{T} + T_2],$$

namely,

$$(7.28) \qquad u(0, t) \leq v_a(0) \quad \text{for } t \in [0, \tilde{T} + T_2].$$

This is a contradiction to the definition of $\tilde{T}$ and so we obtain $\tilde{T} = \infty$, that is, (7.23).

Thus

$$(7.29) \qquad u(x, t) \leq u(0, t) \leq v_a(0) < \infty \quad \text{in } \mathbf{R}^N \times [0, \infty).$$

The proof is complete. □

PROOF OF THEOREM 2.5 IN THE CASE WHERE $\sigma < 0$ AND $\alpha_\sigma^* < 0$. In this case, the theorem follows from Lemma 7.2 and Theorem 2.8. □

8. Proof of Theorem 2.5 in the case $\sigma < 0$, $\alpha_\sigma^* \geq 0$.

In this section we shall show Theorem 2.5 in the case where $\sigma < 0$ and $\alpha_\sigma^* \geq 0$. Then, $N = 1, 2$, $\sigma < 0$ or $N \geq 3$, $-2 \leq \sigma < 0$. We first show the next key proposition. Let $L^q(\mathbf{R}^N)$ $(1 \leq q \leq \infty)$ be the usual space of L^q-functions in $\mathbf{R}^N$ with norm $\|f\|_q \equiv \|f\|_{L^q(\mathbf{R}^N)}$. When $K(x) \in I^{-\sigma}$, there exists $k_0 > 0$ such that

$$(8.1) \qquad K(x) \leq k_0 \langle x \rangle^\sigma \quad \text{in } \mathbf{R}^N.$$

PROPOSITION 8.1. *Let* $\sigma < 0$, $\alpha_\sigma^* \geq 0$, $p > p_{m,\sigma}^*$ *and* $K(x) \in I^{-\sigma}$. *Further, assume* $u_0 \in C(\mathbf{R}^N) \cap L^\infty(\mathbf{R}^N) \cap L^q(\mathbf{R}^N)$ *for some* $q \in (p_0, N/\alpha_\sigma^*)$ *with* $p_0 = \max\{1, N(p-m)/2\}$. *Then, there exists* $\delta_0 = \delta_0(m, p, N, q, \sigma, k_0)$ *such that if* $\|u_0\|_q < \delta_0$ *then there exists a global weak solution* $u(x,t)$ *satisfying*

$$(8.2) \qquad \|u(t)\|_\infty \leq C_1 t^{-N/(N(m-1)+2q)} \quad \text{for } t > 0,$$

where $C_1 = C_1(m, p, N, q, \sigma, k_0)$.

The methods of the proof of this proposition are similar to those of Kawanago [19]. Namely, we use several energy estimates for solutions and use Proposition 2.13. But, in our case, it is not easy to obtain such energy estimates. So, we need the next lemma.

LEMMA 8.2. *Let* $p > m \geq 1$. *For any* $u \in C_0^\infty(\mathbf{R}^N) \cap \{u \geq 0\}$, *the following two inequalities hold*:
(i)

$$(8.3) \qquad \|u(t)\|_\ell \leq C_2 \|u(t)\|_\beta^{\beta[N(m-1)+2\ell]/\ell[\beta(2-N)+N(m+\ell-1)]}$$

$$\times \|\nabla u^{(m+\ell-1)/2}\|_2^{2N(\ell-\beta)/\ell[\beta(2-N)+N(m+\ell-1)]}$$

where $C_2 = C_2(m, N, \beta, \ell) > 0$ *is a constant and* $0 < \beta \leq \ell$.
(ii) *Let* $K(x) \in I^{-\sigma}$ $(\sigma < 0)$, $\alpha_\sigma^* = (2 + \max\{\sigma, -N\})/(p-m) \geq 0$ *and* $q \in (N(p-m)/2, N/\alpha_\sigma^*)$. *Then,*

$$(8.4) \qquad \int_{\mathbf{R}^N} K(x) u^{p+\ell-1} \, dx \leq A_\ell \|u\|_q^{p-m} \|\nabla u^{(m+\ell-1)/2}\|_2^2,$$

where $A_\ell = A_\ell(m, N, p, q, \sigma, k_0) > 0$ *is a constant,* $\ell > \max\{0, 1 - m + q[N-2]_+/N\}$ *and* k_0 *is as in* (8.1).

PROOF. (8.4) is some version of the Gagliardo-Nirenberg inequality (see [19] and Lemma 2.8 of [33]). We shall show (8.4).
Let $0 < \sigma_1 < \min\{N, 2, -\sigma\}$. Then, by the Hölder inequality

$$(8.5) \qquad \int_{\mathbf{R}^N} K(x) u^{p+\ell-1} \, dx \le k_0 \int_{\mathbf{R}^N} \langle x \rangle^\sigma u^{p+\ell-1} \, dx$$

$$\le k_0 \left\{ \int_{\mathbf{R}^N} \langle x \rangle^{N\sigma/\sigma_1} \, dx \right\}^{\sigma_1/N} \left\{ \int_{\mathbf{R}^N} u^{(p+\ell-1)N/(N-\sigma_1)} \, dx \right\}^{(N-\sigma_1)/N}$$

$$\le C \left\{ \int_{\mathbf{R}^N} u^{(p+\ell-1)N/(N-\sigma_1)} \, dx \right\}^{(N-\sigma_1)/N}.$$

Here, we shall use another version of the Gagliardo-Nirenberg inequality: Let $1 < s < N/[N-2]_+$.

$$(8.6) \qquad \left[\int_{\mathbf{R}^N} u^{s(p+\ell-1)} \, dx \right]^{1/s} \le C \left[\int_{\mathbf{R}^N} u^{sN(p-m)/[N-s(N-2)]} \, dx \right]^{[N-s(N-2)]/sN}$$

$$\times \| \nabla u^{[m+\ell-1]/2} \|_2^2,$$

where $\ell > 0$ must satisfy

$$(8.7) \qquad s < \frac{N}{[N-2]_+} \times \frac{\ell+m-1}{p+\ell-1}.$$

The above inequality is obtained if we put $f = u^{(m+\ell-1)/2}$, $r = [2(p-m)/(m+\ell-1)] \times [sN/(N-s(N-2))]$ and $\tilde{r} = 2s(p+\ell-1)/(m+\ell-1)$ in the Gagliardo-Nirenberg inequality

$$(8.8) \qquad \| f \|_{\tilde{r}} \le C\tilde{r} \| f \|_r^{1-\theta} \| \nabla f \|_2^\theta \quad \text{where} \quad \theta = \frac{r^{-1} - \tilde{r}^{-1}}{N^{-1} - 2^{-1} + r^{-1}}$$

where $0 < r \le \max\{1, r\} < \tilde{r} < 2N/[N-2]_+$.

Put

$$(8.9) \qquad s = \frac{N}{N-\sigma_1}.$$

Then, $1 < s < N/[N-2]_+$ by the assumption of σ_1. Further, putting $q = sN(p-m)/[N-s(N-2)]$, by (8.5) and (8.6) we have

$$(8.10) \qquad \int_{\mathbf{R}^N} K(x) u^{p+\ell-1} \, dx \le C \left[\int_{\mathbf{R}^N} u^{sN(p-m)/[N-s(N-2)]} \, dx \right]^{[N-s(N-2)]/sN}$$

$$\times \| \nabla u^{[m+\ell-1]/2} \|_2^2$$

$$= C \| u \|_q^{p-m} \| \nabla u^{(m+\ell-1)/2} \|_2^2,$$

where $\ell > 0$ must satisfy (8.7).

Note $q = N(p - m)/(2 - \sigma_1)$ by a simple calculation. Then we can easily see by $\alpha_\sigma^* \geq 0$, that $0 < \sigma_1 < \min\{N, 2, -\sigma\}$ if and only if $N(p - m)/2 < q < N/\alpha_\sigma^*$. Also, it is not difficult to see that inequality $\ell > \max\{0, 1 - m + q[N - 2]_+/N\}$ implies (8.7). The proof is complete. $\qquad\square$

PROOF OF PROPOSITION 8.1. Let $p > p_{m,\sigma}^*$, $\sigma < 0$, $\alpha_\sigma^* \geq 0$ and $K(x) \in I^{-\sigma}$. Further, let k_0 be as in (8.1). Then, we note $N/\alpha_\sigma^* > 1$ since $p > p_{m,\sigma}^* = m + \alpha_\sigma^*(p - m)/N$. Hence, we see $p_0 < N/\alpha_\sigma^*$ with $p_0 = \max\{1, N(p - m)/2\}$.

First, we construct a approximate solution $u_n(x, t)$ as follows: Let $\{u_{0,n}\} \subset C_0(B(n))$ with $B(n) = \{|x| < n\}$ satisfy that $0 \leq u_{0,n} \leq u_{0,n+1} \leq u_0$ in $\mathbf{R}^N$ and $u_{0,n}(x) = u_0(x)$ in $B(n - 1)$. Let $u_n(x, t)$ be the weak solution of the initial boundary problem

$$
(8.11) \qquad
\begin{cases}
u_t - \Delta u^m = K(x)u^p & \text{in } B(n) \times (0, T_n), \\
u(x, 0) = u_{0,n}(x) & \text{in } B(n), \\
u(x, t) = 0 & \text{on } \partial B(n) \times (0, T_n).
\end{cases}
$$

Then, similarly, as in the proof of Lemma 4.1 of [19], we obtain the next estimate for u_n: Put

$$
(8.12) \qquad B_\ell = \frac{1}{2} \left[\frac{4m(\ell - 1)}{A_\ell(m + \ell - 1)^2} \right]^{1/(p-m)},
$$

where A_ℓ is as in (8.4). We define $u_n(x, t) = 0$ in $x \in \mathbf{R}^N \backslash B(n)$. $\qquad\square$

LEMMA 8.3. *Let* $q \in (p_0, N/\alpha_\sigma^*)$ *with* $p_0 = \max\{1, N(p - m)/2\}$ *and* $\ell \geq q$. *Then, if* $\|u_{0,n}\|_q < \min\{B_q, B_\ell\}/2$,

$$
(8.13) \qquad \|u_n(t)\|_\ell \leq C_\ell t^{-N(1 - q/\ell)/\{N(m-1)+2q\}} \quad \text{in } 0 < t < T_n,
$$

where $C_\ell = C_\ell(m, p, N, q, \sigma, k_0)$.

PROOF. Put $u = u_n$. By the similar methods to those of Suzuki [33] (see (5.9) of [33]), we have for $\ell > 1$ and $0 \leq \tau \leq s \leq T_n$,

$$
(8.14) \qquad \int_{\mathbf{R}^N} u^\ell \, dx \Big|_\tau^s + \frac{4m\ell(\ell - 1)}{(m + \ell - 1)^2} \int_\tau^s \|\nabla u^{(m+\ell-1)/2}\|_2^2 \, dt
$$

$$
\leq \ell \int_\tau^s \int_{\mathbf{R}^N} K(x)u^{p+\ell-1} \, dxds.
$$

Hence, by (8.4)

$$
(8.15) \qquad \int_{\mathbf{R}^N} u^\ell \, dx \Big|_\tau^s + \int_\tau^s \left(\frac{4m\ell(\ell - 1)}{(m + \ell - 1)^2} - \ell A_\ell \|u\|_q^{p-m} \right) \|\nabla u^{(m+\ell-1)/2}\|_2^2 \, dt \leq 0
$$

for $\ell > \max\{1, 1 - m + q[N-2]_+/N\}$. Put $\ell = q$ in the above inequality. Since $\|u(t)\|_q$ is continuous in $[0, T_n)$, if $\|u_{0,n}\|_q < B_q$ then $\|u(t)\|_q$ is nonincreasing in $t \geq 0$ and so $\|u(t)\|_q < B_q$ in $t \geq 0$. Therefore, if $\|u_{0,n}\|_q < \min\{B_q, B_\ell\}$ and $\ell > \max\{1, 1 - m + q[N-2]_+/N\}$, then

$$(8.16) \qquad \int_{R^N} u^\ell \, dx \Big|_\tau^s + C \int_\tau^s \|\nabla u^{(m+\ell-1)/2}\|_2^2 \, dt \leq 0$$

for some $C > 0$, and we see that $\|u(t)\|_\ell$ is nonincreasing in $t \geq 0$. Hence, using (8.3) with $\beta = q$ we have for $\ell \geq q$ and $0 \leq \tau \leq s \leq T_n$,

$$(8.17) \qquad \|u(s)\|_\ell^\ell + C \int_\tau^s \{\|u(t)\|_\ell^\ell\}^{1+(N(m-1)+2q)/N(\ell-q)} \, dt \leq \|u(\tau)\|_\ell^\ell.$$

Thus, by Lemma 5.2 of [**33**] we get

$$(8.18) \quad \|u(s)\|_\ell^\ell \leq \left\{ C \times \frac{N(m-1)+2q}{N(\ell-q)} s \right\}^{-N(\ell-q)/\{N(m-1)+2q\}} \qquad \text{for } 0 < s < T_n$$

to obtain (8.13). The proof is complete.   □

The next lemma is useful.

LEMMA 8.4. *Let* $\sigma < 0$ *and* $K(x) \in I^{-\sigma}$. *Further, let* $r > 1$ *and* $0 < \eta < \min\{N, -r\sigma\}$. *Then, for any* $u \in C_0^\infty \cap \{u \geq 0\}$ *the following inequality holds: for some* $C_3 > 0$

$$(8.19) \qquad \|K(x)u^p\|_r \leq C_3 \|u\|_{Nrp/(N-\eta)}^p.$$

PROOF. By the Hölder inequality we have

$$(8.20) \quad \int_{R^N} \{K(x)u^p\}^r \, dx \leq k_0^r \int_{R^N} \langle x \rangle^{r\sigma} u^{rp} \, dx$$

$$\leq k_0^r \left\{ \int_{R^N} \langle x \rangle^{Nr\sigma/\eta} \, dx \right\}^{\eta/N} \left\{ \int_{R^N} u^{Nrp/(N-\eta)} \, dx \right\}^{(N-\eta)/N},$$

where k_0 is as in (8.1). Hence, we obtain (8.19) since $r\sigma + \eta < 0$.   □

PROOF OF PROPOSITION 8.1 (CONTINUE). Let $q \in (p_0, N/\alpha_\sigma^*)$. We shall show the next results: There exists a constant $\delta_0 = \delta_0(m, p, N, q, \sigma, k_0)$ such that if $\|u_{0,n}\|_q < \delta_0$ then

$$(8.21) \qquad \|u_n(t)\|_\infty \leq C_4 t^{-N/[N(m-1)+2q]} \quad \text{for } 0 < t < T_n,$$

where $C_4 = C_4(m, p, N, q, \sigma, k_0)$.

For this aim we apply Proposition 2.13 to $u = u_n$. Choose real numbers r and s to satisfy

(8.22)

$$\max\left\{1, \frac{qN}{Nm+2q}\right\} < r < \min\left\{\frac{N}{\sigma_1}, q\right\} \quad \text{and} \quad \max\left\{\frac{N}{2}, \frac{qN}{Nm+2q}\right\} < s < \frac{N}{\sigma_1},$$

where σ_1 satisfies equation

(8.23)
$$q = N(p-m)/(2-\sigma_1).$$

Here, we note that $q \in (p_0, N/\alpha_\sigma^*)$ implies $0 < \sigma_1 < \min\{N, 2, -\sigma\}$ as in the proof of Lemma 8.2, and hence $\max\{1, qN/(Nm+2q)\} < \min\{N/\sigma_1, q\}$ and $\max\{N/2, qN/(Nm+2q)\} < N/\sigma_1$. Furthermore, put $\delta = t^{-N/[N(m-1)+2q]}$ in (2.46), set

(8.24)
$$\delta_0 = \frac{1}{2} \times \min\{B_q, B_{Nrp/(N-r\sigma_1)}, B_{Nsp/(N-s\sigma_1)}\}$$

and let $\|u_{0,n}\|_q < \delta_0$. Here, we note $\max\{r\sigma_1, s\sigma_1\} < N$ and $q \leq \min\{Nrp/(N-r\sigma_1), Nsp/(N-s\sigma_1)\}$ by the relation

(8.25)
$$q\sigma_1 = 2q - N(p-m).$$

Therefore, by means of Lemma 8.3 and Lemma 8.4 we have for some $C > 0$ and $C' > 0$,

(8.26) $\quad \|u_n(t)\|_\infty \leq 2t^{-N/(N(m-1)+2q)} + B(t^{-N(m-1)/(N(m-1)+2q)+1})^{-N/2q}\|u_{0,n}\|_q$

$$+ B(t^{-N(m-1)/(N(m-1)+2q)+1})^{-N/2r} \int_0^{t/2} C_3 \|u_n(\tau)\|_{Nrp/(N-r\sigma_1)}^p \, d\tau$$

$$+ B(t^{-N/(N(m-1)+2q)})^{-N(m-1)/2s} \times$$

$$\times \int_0^{t/2} C_3 \|u_n(t-\tau)\|_{Nsp/(N-s\sigma_1)}^p \tau^{-N/2s} \, d\tau$$

$$\leq (2 + B\delta_0) t^{-N/(N(m-1)+2q)}$$

$$+ BCC_3 t^{(-Nq/r)/(N(m-1)+2q)} \int_0^{t/2} \tau^{(-Nm-2q+qN/r)/(N(m-1)+2q)} \, d\tau$$

$$+ BCC_3 t^{N^2(m-1)/2s(N(m-1)+2q)} \times$$

$$\times \int_0^{t/2} (t-\tau)^{(-Nm-2q+qN/s)/(N(m-1)+2q)} \tau^{-N/2s} \, d\tau$$

$$\leq C' t^{-N/[N(m-1)+2q]} \quad \text{for } t > 0.$$

Here, we used (8.25) and inequalities $(-Nm - 2q + qN/r)/(N(m-1) + 2q) > -1$, $-N/2s > -1$ and $-Nm - 2q + qN/s < 0$. Thus, we have proven (8.21).

So, let $\|u_0\|_q < \delta_0$. Then, we get (8.21) because of $\|u_{0,n}\|_q \leq \|u_0\|_q < \delta_0$. Hence, it follows from the uniqueness and existence theorem for solutions of (8.11) that $T_n = \infty$ and (8.21) holds with $T_n = \infty$. Thus, by the same methods as those of [19] (see also the proof of Theorem 2 in [33]) we see that $u(x,t) = \lim_{n\to\infty} u_n(x,t)$ is a global weak solution of (1.1), (1.2) satisfying (8.2). $\qquad\square$

PROOF OF THEOREM 2.5 IN THE CASE $\sigma < 0$, $\alpha_\sigma^* \geq 0$. Let $\alpha > \alpha_\sigma^*$ and $A > 0$. For $\varepsilon > 0$ we assume

$$(8.27) \qquad u_0(x) \leq \min\{\varepsilon, A|x|^{-\alpha}\} \quad \text{in } \mathbf{R}^N.$$

Then, if ε is small enough,

$$(8.28) \qquad u_0(x) \leq \min\{\varepsilon, (A+1)\langle x\rangle^{-\alpha}\} \equiv h_\varepsilon(x) \quad \text{in } \mathbf{R}^N.$$

We choose q to satisfy $\max\{p_0, N/\alpha\} < q < N/\alpha_\sigma^*$, where p_0 is as in Proposition 8.1. Then, by inequality $-\alpha q + N - 1 < -1$ we have

$$(8.29) \qquad \int_{\mathbf{R}^N} (A+1)^q \langle x\rangle^{-\alpha q}\, dx < \infty.$$

Hence, the Lebesgue dominated theorem implies that $\|h_\varepsilon\|_q \to 0$ as $\varepsilon \to 0$. Therefore, if ε is small enough further, then $\|u_0\|_q < \delta_0$ where δ_0 is as in Proposition 8.1. So, applying Proposition 8.1 we get the existence of a global weak solution $u(x,t)$ of (1.1), (1.2) satisfying (8.2). The proof is complete.

$\qquad\square$

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