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On Julia sets of postcritically finite branched coverings Part I—coding of Julia sets

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Abstract. We define Julia sets for (topological) expanding postcritically finite branched coverings on S^2, and show the existence and the uniqueness of Julia sets. Our main aim is the investigation of codings of Julia sets (i.e. semiconjugacies between symbolic dynamics and Julia sets). In particular, it is proved that if two expanding branched coverings are combinatorially equivalent, then their Julia sets are topologically conjugate.

1. Introduction.

We are concerned with the topological properties of 'Julia sets' for topological branched coverings on S^2. Though Julia sets are usually defined for holomorphic dynamics, we do not assume the analyticity of the branched coverings for the following reasons. First, for subhyperbolic maps, one often proceeds the study of the topological properties of Julia sets without the analyticity of maps, assuming only the expandingness. For example, recall the fact that the Julia set of $f(z) = z^2 + c$ for c outside the Mandelbrot set is totally disconnected. That is deduced from the combinatorial data and the expandingness. Another example is the proof of the connectedness of the Julia sets for postcritically finite rational maps. Second, expanding branched coverings are a generalization of subhyperbolic rational maps. We will see an example of expanding branched coverings not conjugate to any rational maps.

Thus we may expect that even if we forget the holomorphic structure, we can find mathematical richness in the dynamics of branched coverings. In the present paper and the succeeding [**6**], we investigate various aspects of Julia sets from the topological standpoint.

We first define expandingness of topological branched covering on S^2 and the Julia set for an expanding branched covering in Section 2. In Section 3 we construct a semiconjugacy map from the full shift to the Julia set, which is called a coding map. We also show the uniqueness of Julia sets. As well as in other hyperbolic dynamics, the concept of symbolic dynamics plays a crucial role in our systems. We state the condition that the codes of distinct points in the Julia set coincide in Section 4. As a consequence, it is proved that two equivalent expanding branched coverings are topologically conjugate on neighborhoods of their Julia sets. In Section 5 we introduce groups and homomorphisms which describe the behavior of the dynamics of branched cov-

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erings, and restate the result on codings of Section 4. We construct an example of expanding branched coverings not equivalent to any rational map in Section 6. Some conditions on which expanding branched coverings are always equivalent to rational maps are also given.

Remark 1.1. The construction of coding maps in Sections 3 and the conditions for the code coincidence in Section 4 are also discussed by Kimura [**8**] in the hyperbolic rational map case. By chance, Kimura and the author independently presented their results in the same conference.

2. Definition of Julia sets.

In this section, we give the notion of the expandingness of branched coverings, and then the definition of Julia sets. We consider only postcritically finite branched coverings.

Definition 2.1. Suppose $f : S^2 \to S^2$ is a topological branched covering. We say

$$C_f = \{c \,|\, c \text{ is a critical point}\}$$

is the *critical set* of f and

$$P_f = \overline{\{f^n(c) \,|\, c \in C_f, n > 0\}}$$

is the *postcritical set* of f. We call f *postcritically finite* if P_f is a finite set.

Throughout this paper, we assume that f is a postcritically finite branched covering of degree $d \geq 2$.

Definition 2.2. Let f be a postcritically finite branched covering. We say a periodic cycle $\{x_1, x_2, \ldots, x_k\}$ is a *critical cycle* of period k if it contains a critical point. A point of a periodic cycle of period k is called a critical periodic point of period k. We divide P_f into P_f^a and P_f^r.

$$P_f^a = \{x \in P_f \,|\, \exists k > 0, f^k(x) \text{ is contained in a critical cycle}\}, \quad P_f^r = P_f - P_f^a.$$

Definition 2.3. A smooth branched covering f is said to be *expanding* if there exists a Riemannian metric $\|\cdot\|$ on $S^2 - P_f$ which satisfies:

(1) Any compact piecewise smooth curve inside $S^2 - P_f^a$ has finite length.
(2) The distance $d(\cdot,\cdot)$ on $S^2 - P_f^a$ determined by the curve length is complete.
(3) For some constants $C > 0$ and $0 < \lambda < 1$,

$$\|v\| < C\lambda^k \|df^k(v)\|$$

for any $k > 0$ and any tangent vector $v \in T_p(S^2)$ if $f^k(p) \in S^2 - P_f$.

Then $|l| < C\lambda^k |f^k(l)|$ for any piecewise smooth curve l with $f^k(l) \subset S^2 - P_f^a$, where $|\cdot|$ means the length of a curve.

Definition 2.4. A non-empty compact set $J \subset S^2 - P_f^a$ is called a *Julia set* of an expanding postcritically finite branched coverings f if $f^{-1}(J) = J = f(J)$.

Proposition 2.5. *An expanding postcritically finite branched covering f has a Julia set.*

Proof. Let $c = \{p_1, p_2, \dots, p_n\}$ be a critical cycle of period n. Since the derivative of f^n at p_1 vanishes, there is a neighborhood U of c such that for $x \in U$, $f^k(x)$ is attracted to c as $k \to \infty$. Therefore $A(c) = \{x \mid f^k(x) \text{ is attracted to } c \text{ as } k \to \infty\}$ is open. Since f has infinitely many periodic points (for example, see [7] §8.6), $J = S^2 - \bigcup_{c:\ \text{critical cycle}} A(c)$ is not empty. Thus J is a Julia set. □

Since a postcritically finite rational map R is subhyperbolic (see [**10**] §19), we can easily see that R is topologically conjugate to an expanding branched covering. Therefore the Julia set of R in the usual sense is a Julia set in our sense.

Example 2.6. Consider $f(z) = z^2 - 2$. This map has the critical set $C_f = \{0, \infty\}$ and the postcritical set $P_f = \{-2, 2, \infty\}$. Let $T(x) = 2\cos x$ and $h(z) = |dT^{-1}(z)/dz| = |4 - z^2|^{-1/2}$. Then the positive function h is smooth in $\boldsymbol{C} - \{-2, 2\}$, and the metric $h(z)|dz|$ satisfies (1) and (2) of Definition 2.3. In view of the relation $T(2x) = f(T(x))$, we can see that f is expanding for the metric $h(z)|dz|$.

3. Coding maps for Julia sets.

In this section, we suppose that f is an expanding postcritically finite branched covering of degree d. In general, expansiveness often leads correspondences between subshifts and invariant sets of dynamical systems. In our case, we construct semi-conjugacies, which is called coding maps, between full shifts and the Julia sets of expanding branched coverings. This depends on the choice of inverse branches of f, which is described by a graph in S^2 called a 'radial.' Using coding maps, we show the uniqueness of Julia sets in 3.1. Various topological properties are showed in 3.2. In 3.3 we see that the inverse branches of f are lifted to contraction maps on the universal covering, and hence the lift of the Julia set are considered as a self-similar set.

3.1. Construction of coding map.

Notation 3.1. We denotes by (Σ_d, σ) the one-sided symbolic dynamics with d symbols, that is,

$$\Sigma_d = \{1, 2, \dots, d\}^{\boldsymbol{N}} = \{a_1 a_2 \cdots \mid a_i \in \{1, 2, \dots, d\}\}, \quad \sigma : \Sigma_d \ni a_1 a_2 \cdots \mapsto a_2 \cdots \in \Sigma_d.$$

The set of sequences of the form $i \cdots \in \Sigma_d$ is denoted by $S(i)$ for $i \in \{1, 2, \dots, d\}$. Then the restriction of σ on $S(i)$ is invertible. We write $\sigma_i = (\sigma \mid S(i))^{-1}$:

$$\sigma_i : \Sigma \ni a_1 a_2 \cdots \mapsto i a_1 a_2 \cdots \in S(i).$$

Consider the set of *words* $W_k = \{1, 2, \dots, d\}^k$ for $k = 1, 2, \dots$ and $W_0 = \{\varnothing\}$. For $w = a_1 a_2 \cdots a_k \in W_k$, we write $\sigma_w = \sigma_{a_1} \circ \sigma_{a_2} \circ \cdots \circ \sigma_{a_k}$ and $S(w) = \sigma_w(\Sigma_d)$. On W_k, the shift map σ and the inverse maps are defined by

$$\sigma : W_k \ni a_1 a_2 \cdots a_k \mapsto a_2 \cdots a_k \in W_{k-1}$$

and

$$\sigma_i : W_k \ni a_1 a_2 \cdots a_k \mapsto i a_2 \cdots a_k \in W_{k+1}.$$

An overlined word means its infinite repeat, which is a member of Σ_d. For example, $\overline{1} = 11 \cdots$ and $\overline{123} = 123123 \cdots$.

Definition 3.2. Let Q_d denote the graph in the plane

$$\{te^{k\cdot 2\pi i/d} \in \boldsymbol{C} \mid 0 \le t \le 1, k = 1, 2, \dots, d\}.$$

A *radial* is a piecewise smooth map $r : Q_d \to S^2 - P_f$ such that

$$f^{-1}(r(0)) = \{r(e^{k\cdot 2\pi i/d}) \mid k = 1, 2, \dots, d\}.$$

We say $r(0)$ is the *base point* of r and a point of $r(e^{k\cdot 2\pi i/d})$ is a *radial point* of r. The arc $l_k : [0,1] \ni t \mapsto r(te^{k\cdot 2\pi i/d}) \in S^2 - P_f$ is called the k-th *spoke* of r.

For two arcs $\alpha, \beta : [0,1] \to X$ with $\alpha(1) = \beta(0)$, we define the arc $\alpha + \beta : [0,1] \to X$ such that $(\alpha + \beta)(t) = \alpha(2t)$ if $0 \le t < 1/2$ and $(\alpha + \beta)(t) = \beta(2t - 1)$ if $1/2 \le t \le 1$. If $\alpha(1) = \beta(1)$, we set $\alpha - \beta = \alpha + (-\beta)$, where $(-\beta)(t) = \beta(1 - t)$.

Definition 3.3. Suppose r is a radial with base point x, radial points x_k and spokes l_k. Let $\gamma : [0,1] \to S^2$ be an arc joining the base point x and $y \in S^2 - P_f^a$ such that $\gamma(0) = x$, $\gamma(t) \in S^2 - P_f$ $(0 < t < 1)$, $\gamma(1) = y$. We define an arc $\omega_k(\gamma)$ by the lift of γ by f (i.e. $f \circ \omega_k(\gamma) = \gamma$) which is an arc joining x_k and y', where y' is a point of $f^{-1}(y)$.

Let $\tilde{U}(f, x)$ be the set

$$\tilde{U}(f,x) = \left\{ \gamma : [0,1] \to S^2 \;\middle|\; \gamma \text{ is continuous, } \begin{array}{c} \gamma(0) = x, \\ \gamma(t) \in S^2 - P_f \;\; (0 < t < 1), \\ \gamma(1) \in S^2 - P_f^a \end{array} \right\}.$$

Then $L_k : \gamma \mapsto l_k + \omega_k(\gamma)$ is a map of $\tilde{U}(f, x)$ to itself. We say L_k is the *lift* of f^{-1} with respect to r. Let $\rho : \tilde{U}(f, x) \to S^2 - P_f^a$ be the natural projection $\rho(\gamma) = \gamma(1)$.

Theorem 3.4. *Let J be a Julia set of f. Then there exists a continuous surjective map $\pi : \Sigma_d \to J$ such that*

$$\begin{array}{ccc} \Sigma_d & \xrightarrow{\;\sigma\;} & \Sigma_d \\ {\scriptstyle \pi}\big\downarrow & & \big\downarrow{\scriptstyle \pi} \\ J & \xrightarrow[\;f\;]{} & J \end{array}$$

commutes.

Proof. Take a base point x and a radial r with spokes l_i. For $a_1 a_2 \cdots a_k \in W_k$, we define $l_{a_1 a_2 \cdots a_k} = L_{a_1}(l_{a_2 \cdots a_k})$ inductively. Then $l_{a_1 a_2 \cdots a_k}$ is an arc joining x and a point of $f^{-k}(x)$. Let $x_{a_1 a_2 \cdots a_k} = \rho(l_{a_1 a_2 \cdots a_k})$.

Let $\gamma \in \tilde{U}(f, x)$. For $a_1 a_2 \cdots a_k \in W$, we define $\gamma_{\varnothing} = \gamma$ and $\gamma_{a_1 a_2 \cdots a_k} = L_{a_1}(\gamma_{a_2 \cdots a_k})$. Let $y_{a_1 a_2 \cdots a_k} = \rho(\gamma_{a_1 a_2 \cdots a_k})$. For $\underline{a} = a_1 a_2 \cdots \in \Sigma_d$, the limit $x_{\underline{a}} = \lim_{k \to \infty} y_{a_1 a_2 \cdots a_k}$ exists by the expandingness of f. Indeed,

$$d(y_{a_1 a_2 \cdots a_k}, y_{a_1 a_2 \cdots a_k a_{k+1}}) < C\lambda^k (M + (1 + \lambda)|\gamma|),$$

where $M = \max|l_i|$. Note that $x_{\underline{a}}$ is independent of γ.

Now we define a map $\pi : \Sigma_d \to S^2$ by $\underline{a} \mapsto x_{\underline{a}}$. The expandingness of f yields continuity of π. If we take $\gamma(1) \in J$, then each $x_{\underline{a}} \in J$. Thus the image of π is included in J. Conversely, take a point $a \in J$. Choose $\gamma(1)$ as a cluster point of $\{f^k(a) \mid k > 0\}$. Then for any $\varepsilon > 0$ and $K > 0$, there exist $k > K$ and $a_1 a_2 \cdots a_k \in W_k$ such that the distance between $y_{a_1 a_2 \cdots a_k}$ and a is less than ε. Therefore a belongs to $\pi(\Sigma_d)$, and so π is surjective.

For $\underline{a} = a_1 a_2 \cdots \in \Sigma_d$, we denote by $l_{\underline{a}}$ the limit of the arcs $l_{a_1 a_2 \cdots a_k}$. Then $l_{\underline{a}}$ is an arc between x and $\pi(\underline{a})$. It is easy to see that $L_{a_1}(l_{\sigma(\underline{a})}) = l_{\underline{a}}$. This implies the commutativity of the diagram. □

Definition 3.5. We call π the *coding map* of J with respect to r.

In the above proof, we have seen the following.

Corollary 3.6. *An expanding postcritically finite branched covering f has the unique Julia set, which is characterized as*

$$J_f = \bigcap_{n=1}^{\infty} \overline{\bigcup_{k=n}^{\infty} f^{-k}(y)},$$

where y is an arbitrary point in $S^2 - P_f^a$.

Set $P_f^e = \{x \in P_f^a \mid f^{-k}(x) \subset P_f^a$ for any $k > 0\}$. For $y \in S^2 - P_f^e$, the Julia set coincides with the cluster set of $\bigcup_{k=0}^{\infty} f^{-k}(y)$.

Corollary 3.7. *$P_f^r = P_f \cap J_f$ and $P_f^a = P_f \cap (S^2 - J_f)$.*

Corollary 3.8. *The following are equivalent.*

(1) *$x \in J_f$.*
(2) *For any neighborhood U of x, $S^2 - P_f^e \subset \bigcup_{k=1}^{\infty} f^k(U)$.*
(3) *For any neighborhood U of x, there exists $n > 0$ such that $J_f \subset f^n(U)$.*

Proof. The equivalence between (1) and (2) is a consequence of Corollary 3.6. (1) implies (3) because any open set $U \subset \Sigma_d$ satisfies the property $\sigma^n(U) = \Sigma_d$ for some $n > 0$. □

Definition 3.9. Two radials r, r' are said to be *homotopic* if there exists a homotopy $h : Q_d \times I \to S^2 - P_f$ such that $h(\cdot, 0) = r, h(\cdot, 1) = r'$ and $h(\cdot, t)$ is a radial for $0 \le t \le 1$.

Suppose r' is another radial with base point x', radial points x_k' and spokes l_k'. Let $h : Q_d \times [0,1] \to S^2 - P_f$ be a homotopy between r and r', and let L_k' be the lift with respect to r'. By α we denote the arc $\alpha(t) = h(0,t)$. Then $\rho(L_k(\alpha)) = x_k'$, and $L_k(\alpha) - l_k' - \alpha$ is trivial in $S^2 - P_f$. We define the map $T : \tilde{U}(f, x') \to \tilde{U}(f, x)$ by $T(\gamma) = \alpha + \gamma$. Then $T(L_k'(\gamma))$ and $L_k(T(\gamma))$ are homotopic in $S^2 - P_f$ with the endpoints fixed.

Let π' denote the coding map with respect to r'. Since $T(L_{a_1}' \circ L_{a_2}' \circ \cdots \circ L_{a_k}'(\gamma))$ and $L_{a_1} \circ L_{a_2} \circ \cdots \circ L_{a_k}(T(\gamma))$ are homotopic in $S^2 - P_f$ with the endpoints fixed,

Corollary 3.10. *If r and r' are homotopic, then $\pi = \pi'$.*

3.2. Topological properties of Julia sets.

In this subsection, we show several propositions on the topological properties of Julia sets and Fatou sets.

Definition 3.11. We say $S^2 - J_f$ is the *Fatou set* of f. A connected component of the Fatou set is called a *Fatou component*. For y in the Fatou set, we denote by $A_0(y)$ the Fatou component containing y. Then it is easily seen that $f : A_0(y) \to A_0(f(y))$ is a branched covering.

Lemma 3.12. *Let $y \in S^2 - J_f$, and let $x_1, x_2, \dots$ be a sequence which converges to y. Let $k_1, k_2, \dots$ be an unbounded sequence of positive integers. Then a cluster point of $\{f^{k_i}(x_i) \mid i = 1, 2, \dots\}$ is contained in P_f^a. In particular, a cluster point of $\{f^k(y) \mid k = 1, 2, \dots\}$ is contained in P_f^a.*

Proof. Suppose $\{f^{k_i}(x_i) \mid i = 1, 2, \dots\}$ has a cluster point x in $S^2 - P_f^a$. Then y lies in $\bigcap_{n=1}^{\infty} \overline{\bigcup_{k=n}^{\infty} f^{-k}(x)}$, and hence $y \in J_f$ by Corollary 3.6. □

Immediately,

Proposition 3.13. *A Fatou component W is eventually preperiodic, that is, $f^m(W) = f^{m+n}(W)$ for some $m \geq 0$, $n \geq 1$.*

Corollary 3.14. *The Fatou set $S^2 - J_f$ is characterized as*

$$S^2 - J_f = \{a \in S^2 \mid \{f^k(a)\} \text{ converge to some critical cycle}\}.$$

In particular, a Fatou component contains a backward image of a critical periodic point.

Corollary 3.15. *Let p be a critical periodic point of period n. Then $\lim_{k\to\infty} f^{kn}(x) = p$ for any $x \in A_0(p)$.*

Proof. It suffices to show that $A_0(p)$ has no other critical periodic point. Otherwise, take another periodic point $b \in A_0(p)$. Let γ be a path in $A_0(p)$ between a and b. Then there exists $z \in \gamma$ which belongs to the boundary of $\{x \mid \lim_{k\to\infty} f^{kn}(x) = p\}$. Since z must lie in J_f, we have a contradiction. □

Proposition 3.16. *The Julia set J_f has an interior point if and only if it is the entire sphere.*

Proof. Let U be an open set of S^2 included in J_f. Then $J_f = \overline{\bigcup_{k=1}^{\infty} f^k(U)} = S^2$ by Corollary 3.8. □

Proposition 3.17. *Let W be a Fatou component. Then W is simply connected and $\#(W \cap \bigcup_{k=0}^{\infty} f^{-k}(P_f)) = 1$. Consequently, the Julia set is connected.*

Proof. There exist $k \geq 0$, $n \geq 1$ such that $f^k(W) = f^{k+n}(W)$. Let p be the periodic point of period n in $f^k(W)$. Take a disc $D \subset A_0(p)$ such that $D \cap P_f = \{p\}$. Then each connected component of $f^{-k}(D)$ is simply connected and contains only one point of $f^{-k}(p)$.

Let γ be a closed curve in $W - f^{-k}(p)$. The image $f^{k+jn}(\gamma)$ is included in D for some large j. By deforming γ continuously in $W - f^{-k}(p)$, we can assume $\gamma \cap f^{-k-jn}(p) = \emptyset$, and then $f^{k+jn}(\gamma) \subset D - \{p\}$. If $f^{k+jn}(\gamma)$ is 0-homotopic in

$D - \{p\}$, then γ is also 0-homotopic in $W - f^{-k}(p)$. If $f^{k+jn}(\gamma)$ is non-trivial in $D - \{p\}$, then γ is homotopic in $W - f^{-k-jn}(p)$ to a connected component l of $f^{-k-jn}(\partial D)$. As mentioned above, l bounds a connected component $D' \subset f^{-k-jn}(D)$ that contains only one point of $f^{-k-jn}(p)$. Thus we have proved that a simple closed curve in $W - f^{-k}(p)$ is either 0-homotopic in $W - f^{-k}(p)$ or bounds a disc D' such that $D' \subset W$ and $\#(D' \cap f^{-k}(p)) = 1$. Therefore W is simply connected, and $\#(W \cap f^{-k}(p)) = 1$.

Suppose that there exist distinct points $a, b \in W \cap \bigcup_{m=1}^{\infty} f^{-m}(P_f)$. Then $f^k(a)$ and $f^k(b)$ are periodic points in P_f for some k. Since a Fatou component contains at most one periodic points (Corollary 3.15), we have $f^k(a) = f^k(b)$. That is a contradiction to the fact just proved. $\square$

Proposition 3.18. *There exists $m > 0$ such that $\#\pi^{-1}(p) < m$ for $p \in J_f$. In particular, $\pi^{-1}(p)$ consists of periodic sequences if p is periodic.*

Proof. Let p be a point in J_f. Then $\{l_{\underline{a}} \mid \underline{a} \in \pi^{-1}(p)\}$ can be divided into finite classes $B_1(p), B_2(p), \dots, B_{t(p)}(p)$, each consisting of elements mutually homotopic in $S^2 - P_f$ the endpoints fixed leaving. Indeed, suppose there exist infinite arcs $l_{\underline{a}^1}, l_{\underline{a}^2}, \dots$ which are not mutually homotopic with the endpoints fixed. We can assume that $\{\underline{a}^k\}$ converges to some $\underline{a}$. Then $l_{\underline{a}}$ is not an arc with finite length, and this is a contradiction. Moreover, it is similarly proved that there exists T such that $t(y) < T$ for any $y \in J_f$.

Note that $\sigma : \pi^{-1}(p) \to \pi^{-1}(f(p))$ is bijective if p is not a critical point, and d'-to-one if p is a critical point of degree d'. Since $p \in J_f$, there exists $c > 0$ such that $\sigma^k : \pi^{-1}(p) \to \pi^{-1}(f^k(p))$ is at most c-to-one for any $k > 0$. Suppose $\sigma^k : \pi^{-1}(p) \to \pi^{-1}(f^k(p))$ is d'-to-one. If $\underline{a}, \underline{b} \in \pi^{-1}(f^k(p))$ and if $l_{\underline{a}}, l_{\underline{b}}$ are homotopic, then by the homotopy lifting property, $w\underline{a} \in \pi^{-1}(p)$ $(w \in W_k)$ if and only if $w\underline{b} \in \pi^{-1}(p)$. Since $t(f^k(p)) < T$, we have $\#\{w \in W_k \mid S(w) \cap \pi^{-1}(p) \neq \emptyset\} < d'T \le cT$. That is true for any $k > 0$, and hence $\#\pi^{-1}(p) < cT$. $\square$

For a periodic $p \in J_f$, it is not necessarily true that the periods of sequences in $\pi^{-1}(p)$ are the same as the period of p. Indeed,

Example 3.19. Consider a polynomial $f(z) = z^3 - 3z$. The critical set $C_f = \{-1, 1, \infty\}$ and the postcritical set $P_f = \{-2, 2, \infty\}$. The dynamics on $C_f \cup P_f$ is $-1 \to 2 \to 2, 1 \to -2 \to -2, \infty \to \infty$. Take a radial as in Figure 1. Then $\pi(\overline{1}) = \pi(\overline{23}) = \pi(\overline{32}) = -2$.

3.3. Self-similar sets.

We can consider the Julia set as a quotient space of self-similar set, that is, there exists a self-similar set K and surjection $\rho : K \to J_f$. We use 'self-similar sets' in Hutchinson's sense as follows.

Fact 3.20. Let X be a complete metric space and $f_1, f_2 \cdots f_d$ be contractions on X (i.e. the Lipschitz constants of f_i's are less than one). Then there uniquely exists a non-empty compact set K such that $K = \bigcup_i f_i(K)$, which is called a self-similar set ([**3**], [**2**]). Moreover, there exists a continuous surjective map $\chi : \Sigma_d \to K$ such that $\chi \circ \sigma_i = f_i \circ \chi$ for i.

For $\gamma, \gamma' \in \tilde{U}(f, x)$, we say $\gamma \sim \gamma'$ if they are homotopic in $S^2 - P_f$ with the endpoints fixed. Let $U(f, x) = \tilde{U}(f, x)/\sim$. Then the projection $\rho^* : U(f, x) \to S^2 - P_f^a$ is naturally defined. Besides the mapping $L_k : \tilde{U}(f, x) \to \tilde{U}(f, x)$ induces a well-defined mapping $L_k^* : U(f, x) \to U(f, x)$. For $[\gamma]$ and $[\gamma']$ in $U(f, x)$, let

$$D([\gamma], [\gamma']) = \inf\{|l| : l \text{ is homotopic to } -\gamma + \gamma' \text{ in } S^2 - P_f \text{ with the endpoints fixed}\}.$$

The space $U(f, x)$ is a complete metric space in the distance $D(\cdot, \cdot)$. Then L_k^* is a contraction, and by Fact 3.20 there exists a self-similar set $K \subset U(f, x)$ and a surjection $\chi : \Sigma_d \to K$. It is evident that $J_f = \rho^*(K)$ and $\pi = \rho^* \circ \chi$.

4. Equivalence Relations on the symbolic dynamics.

In this section we treat the equivalence relations on the symbolic dynamics defined by the coding maps. In 4.1 we show that the equivalence relation is determined by the homotopical condition. Using that, we prove that if f and g are combinatorially equivalent, then they are topologically conjugate on the Julia sets. In 4.2 the conjugacy is extended on neighborhoods of Julia sets. This is a generalization of [**4**] Corollary 6.5.

4.1. Conjugacies on Julia sets.

THEOREM 4.1. *Let f be an expanding postcritically finite branched covering. Let r be a radial with base point x and $\pi : \Sigma_d \to J_f$ be the coding map with respect to r. Then for $\underline{a} = a_1 a_2 \cdots, \underline{b} = b_1 b_2 \cdots \in \Sigma_d$,*

$$\pi(\underline{a}) = \pi(\underline{b})$$

if and only if there exist curves $\alpha_k, \beta_k \in \tilde{U}(f, x)$ $(k = 0, 1, 2, \dots)$ such that $\alpha_k(0) = \beta_k(0) = x$, $\alpha_k(1) = \beta_k(1) \notin P_f$, $L_{a_{k+1}}(\alpha_{k+1})(1) = L_{b_{k+1}}(\beta_{k+1})(1)$, $|\alpha_k|, |\beta_k| < B < \infty$ and $\alpha_k - \beta_k$ is homotopic to $L_{a_{k+1}}(\alpha_{k+1}) - L_{b_{k+1}}(\beta_{k+1})$ in $S^2 - P_f$ with x fixed $(k = 0, 1, 2, \dots)$.

PROOF. We use the notation of Theorem 3.4.

Suppose $\pi(\underline{a}) = \pi(\underline{b})$. If $f^k(x_{\underline{a}}) = f^k(x_{\underline{b}})$ is not contained in P_f for any $k \geq 0$, then $\alpha_k = l_{\sigma^k(\underline{a})}$, $\beta_k = l_{\sigma^k(\underline{b})}$ satisfy the above condition. In the case where $f^m(x_{\underline{a}}) = f^m(x_{\underline{b}})$ is contained in P_f for some $m \geq 0$, setting $\beta_k = x$ (a constant map), we can take α_k in a neighborhood of $\gamma_k = l_{\sigma^k(\underline{a})} - l_{\sigma^k(\underline{b})}$. Indeed, there exists ε_1 such that for any curve γ in ε_1-neighborhood of γ_k, we can take γ' near γ_k which is homotopic to γ in $S^2 - P_f$ satisfying $||\gamma'| - |\gamma_k|| < \varepsilon$.

Conversely, suppose the existence of arcs α_k, β_k satisfying the condition. Let $\alpha_k' = L_{a_1} \circ L_{a_2} \circ \cdots \circ L_{a_k}(\alpha_k)$ and $\beta_k' = L_{b_1} \circ L_{b_2} \circ \cdots \circ L_{b_k}(\beta_k)$. Then $\alpha_k'(1) = \beta_k'(1)$ and $\alpha_k' = l_{a_1 a_2 \cdots a_k} + \tilde{\alpha}_k$, $\beta_k' = l_{b_1 b_2 \cdots b_k} + \tilde{\beta}_k$, where $\tilde{\alpha}_k$ (or $\tilde{\beta}_k$) is one component of $f^{-k}(\alpha_k)$ (or $f^{-k}(\beta_k)$). Therefore the distance between $x_{a_1 a_2 \cdots a_k}$ and $x_{b_1 b_2 \cdots b_k}$ is less than $2\lambda^k B$. Thus $\pi(\underline{a}) = \pi(\underline{b})$. □

DEFINITION 4.2. Let f and g be postcritically finite branched coverings. We say f and g are *equivalent*, $f \sim g$, if there exist two orientation-preserving homeomorphisms $\phi_1, \phi_2 : S^2 \to S^2$ such that $\phi_i(P_f) = P_g$, $i = 1, 2$, ϕ_1 and ϕ_2 are isotopic relative to P_f, and

$$\begin{array}{ccc} S^2 & \xrightarrow{f} & S^2 \\ {\scriptstyle \phi_1}\downarrow & & \downarrow{\scriptstyle \phi_2} \\ S^2 & \xrightarrow[g]{} & S^2 \end{array}$$

commutes.

LEMMA 4.3. *Suppose f and g are postcritically finite branched coverings. Let ϕ_1, ϕ_2 be homeomorphisms such that $\phi_i(P_f) = P_g$, $i = 1, 2$ and $\phi_2 \circ f = g \circ \phi_1$. If $H : S^2 \times I \to S^2$ is an isotopy such that $H(\cdot, 0) = \phi_2$ and $H(P_f, t) = P_g$ for $t \in I$, then there exist an isotopy $h : S^2 \times I \to S^2$ such that $h(\cdot, 0) = \phi_1$, $h(P_f, t) = P_g$ for $t \in I$, and $H(f(x), t) = g(h(x, t))$ for $t \in I$.*

PROOF. By applying the covering homotopy theorem to the covering $g : S^2 - g^{-1}(P_g) \to S^2 - P_g$ and the isotopy $H \circ (f \times \mathrm{id}) : (S^2 - f^{-1}(P_f)) \times I \to S^2 - P_g$, we obtain a homotopy $h : (S^2 - f^{-1}(P_f)) \times I \to S^2 - g^{-1}(P_g)$ such that $H \circ (f \times \mathrm{id}) = g \circ h$ and $h(\cdot, 0) = \phi_1$. Since for every $t \in T$, $h(\cdot, t)$ is a homeomorphism and is extended on S^2, we obtain the required isotopy. □

THEOREM 4.4. *Suppose that f and g are expanding and they are equivalent to each other. Then there exists a homeomorphism $\chi : J_f \to J_g$ such that $\chi \circ f = g \circ \chi$.*

PROOF. Since f and g are equivalent, there exist homeomorphisms $\phi_1, \phi_2 : (S^2, P_f) \to (S^2, P_g)$ such that $\phi_2 \circ f = g \circ \phi_1$ and there exists an isotopy $h : S^2 \times I \to S^2$ with $h(\cdot, 0) = \phi_2$, $h(\cdot, 1) = \phi_1$ and $h(P_f, t) = P_g$. Set $h_0 = h$. By Lemma 4.3, we have an isotopy $h_1 : S^2 \times I \to S^2$ such that $h_0 \circ (f \times \mathrm{id}) = g \circ h_1$, $h_1(\cdot, 0) = \phi_1$ and $h_1(P_f, t) = P_g$. Inductively we have isotopies $h_k : S^2 \times I \to S^2$ $(k = 1, 2, \ldots)$ such that

$$\begin{array}{ccc} S^2 \times I & \xrightarrow{f^k \times \mathrm{id}} & S^2 \times I \\ {\scriptstyle h_k}\downarrow & & \downarrow{\scriptstyle h} \\ S^2 & \xrightarrow[g^k]{} & S^2 \end{array}$$

commutes and $h_k(\cdot, 1) = h_{k+1}(\cdot, 0)$. Write $s_k = h_k(\cdot, 1)$. Note that $s_{k-1} \circ f = g \circ s_k$. Take points $x \in S^2 - P_f$ and $x' \in S^2 - P_g$ such that $\phi_2(x) = x'$. We define $\Phi_k : \tilde{U}(f, x) \to \tilde{U}(g, x')$ by $\Phi_0(\gamma) = \phi_2 \circ \gamma$ and $\Phi_{k+1}(\gamma) = h_0(x, \cdot) + h_1(x, \cdot) + \cdots + h_k(x, \cdot) + s_k \circ \gamma$ for $k = 0, 1, \ldots$

Let r be a radial for f with base point x. We denote by x_i the radial point of r and by l_i the spoke of r. Let $x_i' = \phi_1(x_i)$. Then $g^{-1}(x') = \{x_i' \mid i = 1, 2, \ldots, d\}$. Thus we can define a radial r' for g with spokes $l_i' = \Phi_1(l_i)$. We denote by $L_i' : \tilde{U}' \to \tilde{U}'$ the lift of g^{-1} with respect to r'. We can easily see that

$$\Phi_k \circ L_i(\gamma) \underset{\text{homotopic}}{\sim} L_i' \circ \Phi_{k-1}(\gamma). \tag{1}$$

Let $\alpha, \beta \in \tilde{U}(f, x)$ be arcs with common endpoints. Then it is clear that $\Phi_k(\alpha) - \Phi_k(\beta)$, $k \geq 0$ are homotopic to one another in $S^2 - P_g$ with x' fixed.

Let $\pi : \Sigma_d \to J_f$ be the coding map with respect to r and $\pi' : \Sigma_d \to J_g$ be the coding map with respect to r'. Suppose $\pi(\underline{a}) = \pi(\underline{b})$. Let α_k and β_k, $k = 0, 1, \dots$ be curves satisfying the condition in Theorem 4.1. Let $\alpha'_k = \Phi_k(\alpha_k)$, $\beta'_k = \Phi_k(\beta_k)$. From (1), we see that $\alpha'_k - \beta'_k$ is homotopic to $L'_{a_{k+1}}(\alpha'_{k+1}) - L'_{b_{k+1}}(\beta'_{k+1})$. Thus the curves α'_k, β'_k satisfy the condition of Theorem 4.1. Therefore $\pi'(\underline{a}) = \pi'(\underline{b})$. Consequently, the equivalence relations derived by π and π' are identical. Hence we have a homeomorphism between J_f and J_g. □

Remark 4.5. The homeomorphism between J_f and J_g is independent of the choice of r, and depends on the isotopy class of ϕ_1, ϕ_2 and the homotopy class of the isotopy h. Indeed, let

$$S_{f,x} = \{\{x_k\}_{k>0} \,|\, \text{a convergent sequence such that } f^k(x_k) = x\}.$$

Then $\alpha_f : \{x_k\}_{k>0} \mapsto \lim x_k$ is a surjection onto J_f. We set $b(\{x_k\}) = \{s_{k-1}(x_k)\}$. In Theorem 4.4, we have proved that $b : S_{f,x} \to S_{g,x'}$ and $\alpha_g \circ b \circ \alpha_f^{-1} : J_f \to J_g$ are well-defined. The mapping b depends on s_k, and is independent of r. □

4.2. Conjugacies on neighborhoods of Julia sets.

Lemma 4.6. *Suppose that f and g are expanding and they are equivalent to each other. Let $(p_1, p_2, \dots, p_n)$ be a critical cycle of f, and $(p'_1, p'_2, \dots, p'_n)$ the corresponding critical cycle of g. Then there exists a neighborhood W of J_f such that the conjugacy χ in Theorem 4.4 is extended on $J_f \cup (W \cap \bigcup_{j=1}^n A_0(p_j))$.*

Proof. For simplicity, we consider the case $n = 1$. Let p be a critical fixed point of f, and p' the corresponding critical fixed point of g. Suppose $f : A_0(p) \to A_0(p)$ is of degree d'. Take a small open disc $D \subset A_0(p)$ containing p such that $\overline{D} \subset f^{-1}(D)$. Let D_0 be the components of $f^{-1}(D)$ including D. Set $B_0 = D_0 - \overline{D}$ and $\gamma_0 = \partial D$, $\gamma_1 = \partial D_0$.

Fix a point $x \in \gamma_0$. Take a point $y_1 \in f^{-1}(x) \cap \gamma_1$ and an orientation-preserving homeomorphism $\zeta_0 : \boldsymbol{T} \to \gamma_0$ such that $\zeta_0(0) = x$, where $\boldsymbol{T} = \boldsymbol{R}/\boldsymbol{Z}$. Then there exists a homeomorphism $\zeta_1 : \boldsymbol{T} \to \gamma_1$ such that $\zeta_1(0) = y_1$, $f \circ \zeta_1(t) = \zeta_0(d't)$. Set $\gamma_k = (f \,|\, A_0(p))^{-1}(\gamma_{k-1})$ inductively. Take a path $c_0 \subset \overline{B_0}$ between x and y_1. We inductively define $c_k = (f \,|\, A_0(p))^{-1}(c_{k-1})$ that joins y_{k-1} and y_k. Then there exists a homeomorphism $\zeta_k : \boldsymbol{T} \to \gamma_k$ such that $\zeta_k(0) = y_k$, $f \circ \zeta_k(t) = \zeta_{k-1}(d't)$. By the expandingness of f, the sequence $\{\zeta_k : \boldsymbol{T} \to S^2\}$ uniformly converges to a continuous map $\zeta : \boldsymbol{T} \to S^2$ as $k \to \infty$. Indeed, take a homeomorphism $\xi : \boldsymbol{T} \times I \to \overline{B_0}$ such that $\xi(\cdot, i) = \zeta_i$, $i = 0, 1$ and $\xi(0, \cdot) = c_0$. Let

$$M = \sup_{t \in \boldsymbol{T}} \inf\{|c| : c \text{ is homotopic to } \xi(t, \cdot) \text{ with the endpoints fixed}\}.$$

Then we see $d(\zeta_k(t), \zeta_{k+1}(t)) \le C\lambda^k M$. Note that the image of ζ is equal to the boundary of $A_0(p)$.

Similarly, we take a small open disc $D' \subset A_0(p')$ containing p' such that $\overline{D}' \subset g^{-1}(D')$. Let ϕ_1, ϕ_2 be as in Theorem 4.4. Since $\phi_2(\gamma_0)$ and γ'_0 are homotopic, they are ambient isotopic (see [**11**] p. 11). From Lemma 4.3, changing ϕ_2 by an isotopy if necessary, we can assume that $\phi_2(\gamma_0) = \gamma'_0$. Set $x' = \phi_2(x)$ and $y'_1 = \phi_1(y_1)$. We then define B'_0, γ'_k, y'_k and ζ'_k similarly, and finally obtain a continuous map $\zeta' : \boldsymbol{T} \to \partial A_0(p')$.

By a method similar to Theorem 4.4, we can show that $\zeta(t) = \zeta(t') \Leftrightarrow \zeta'(t) = \zeta'(t')$ for $t, t' \in \boldsymbol{T}$. Therefore we have a homeomorphism $\chi' : \partial A_0(p) \to \partial A_0(p')$ such that $\chi' \circ \zeta = \zeta'$. Let s_k be the homeomorphisms in Theorem 4.4. Since $s_k(\gamma_{k+1}) = \gamma'_{k+1}$, the homeomorphism χ' is the restriction of χ in Theorem 4.4.

The conjugacy χ' is extended as follows. Take an orientation-preserving homeomorphism $\chi' : \overline{B_0} \to \overline{B'_0}$ such that $\chi'|\gamma_0 = \zeta'_0 \circ \zeta_0^{-1}$ and $\chi'|\gamma_1 = \zeta'_1 \circ \zeta_1^{-1}$. Set $B_k = (f \mid A_0(p))^{-1}(B_{k-1})$, $B'_k = (g \mid A_0(p))^{-1}(B'_{k-1})$ inductively. Then the homeomorphism χ' is uniquely extended from $\bigcup_{k=0}^{\infty} \overline{B_k}$ onto $\bigcup_{k=0}^{\infty} \overline{B'_k}$ so that $\chi' \circ f = g \circ \chi'$. From the expandingness of f, g, it follows that χ' is continuous on $\partial A_0(p)$. □

Theorem 4.7. *Suppose that f and g are expanding and they are equivalent to each other. Then f and g are topologically conjugate on neighborhoods of the Julia sets, that is, there exist neighborhoods W, W' of J_f, J_g and a homeomorphism $\chi : W \to W'$ such that $f^{-1}(W) \subset W$, $g^{-1}(W') \subset W'$ and $\chi \circ f = g \circ \chi$ on $f^{-1}(W)$.*

Proof. If $P_f^a = P_g^a = \emptyset$, then the theorem is proved in Theorem 4.4. Suppose that $P_f^e = \{x \in P_f \mid \bigcup_{k=0}^{\infty} f^{-k}(x) \subset P_f\}$ is equal to P_f^a. Then the Fatou set has only finitely many connected components, say $U_1, U_2, \dots, U_m$. Since $J_f = \bigcup_{k=1}^{m} \partial U_k$, the theorem is a consequence of Lemma 4.6.

If $P_f^a - P_f^e \neq \emptyset$, then the cluster set of $\bigcup_{k=0}^{\infty} f^{-k}(P_f^a)$ is equal to the Julia set. Let s_k be as in Theorem 4.4. Note that $s_{k-1} \mid f^{-k}(P_f) : f^{-k}(P_f) \to g^{-k}(P_g)$ is bijective and $s_{k-1} \mid f^{-k}(P_f) = s_k \mid f^{-k}(P_f)$. In view of Remark 4.5, we have a homeomorphism $s : \overline{\bigcup_{k=0}^{\infty} f^{-k}(P_f^a)} \to \overline{\bigcup_{k=0}^{\infty} g^{-k}(P_g^a)}$ such that $s \mid f^{-k}(P_f) = s_{k-1} \mid f^{-k}(P_f)$.

By Theorem 4.4, we have a homeomorphism $\chi : J_f \to J_g$ such that $\chi \circ f = g \circ \chi$. Let N be a small neighborhood of P_f^a such that $N \subset f^{-1}(N)$. By Lemma 4.6, the homeomorphism χ is extended on $S^2 - N$ so that $\chi = s$ on $\bigcup_{k=0}^{\infty} f^{-k}(P_f^a) - N$ and $\chi \circ f = g \circ \chi$ on $S^2 - N$. Since s is a homeomorphism, so is χ. □

5. Branch groups.

In this section we define groups from the homotopical condition of f and the radial r. We prove that the groups describe the equivalence relation, and restate Theorem 4.1.

Definition 5.1. Let A be a subset of the fundamental group $\pi(S^2 - P_f, x)$. We say $A = \{\alpha_i\}_i$ is *bounded* if there exist a family of closed curves $\{\bar{\alpha}_i\}_i$ and $B > 0$ such that $\bar{\alpha}_i$ is a representative of α_i with $|\bar{\alpha}_i| < B$.

We define $e : \pi(S^2 - P_f, x) \times \{1, 2, \dots, d\} \to \{1, 2, \dots, d\}$ by $j = e(\alpha, i)$ if $L_i(\bar{\alpha})(1) = x_j$ for $\bar{\alpha} \in \alpha \in \pi(S^2 - P_f, x)$. For given $\alpha \in \pi(S^2 - P_f, x)$,

$$e(\alpha, \cdot) : \{1, 2, \dots, d\} \to \{1, 2, \dots, d\}$$

is a permutation, and

$$e(\alpha, \cdot) \circ e(\beta, \cdot) = e(\beta\alpha, \cdot), \quad e(\alpha^{-1}, \cdot) = e(\alpha, \cdot)^{-1}. \tag{2}$$

We define a map $\tilde{L}_i : \pi(S^2 - P_f, x) \to \pi(S^2 - P_f, x)$ as $\tilde{L}_i(\alpha) = L_i(\bar{\alpha}) - l_{e(\alpha, i)}$. Then

$$(3) \qquad \tilde{L}_i(\alpha\beta) = \tilde{L}_i(\alpha)\tilde{L}_{e(\alpha,i)}(\beta), \quad \tilde{L}_i(\alpha^{-1}) = \tilde{L}_{e(\alpha,i)}(\alpha)^{-1}.$$

We write $\tilde{L}_{a_1} \circ \tilde{L}_{a_2} \cdots \circ \tilde{L}_{a_k} = \tilde{L}_{a_1 a_2 \cdots a_k}$.

Proposition 5.2. *Let f be an expanding postcritically finite branched covering. Let r be a radial with base point x and $\phi : \Sigma_d \to J_f$ be the coding map with respect to r. For $\underline{a} = a_1 a_2 \cdots, \underline{b} = b_1 b_2 \cdots \in \Sigma_d$,*

$$\phi(\underline{a}) = \phi(\underline{b})$$

if and only if there exist a bounded sequence $\{\alpha_k\}_{k=0,1,2,\ldots}$ of $\pi(S^2 - P_f, x)$ such that $b_{k+1} = e(\alpha_{k+1}, a_{k+1})$ and $\tilde{L}_{a_{k+1}}(\alpha_{k+1}) = \alpha_k$.

Proof. If $\{\alpha_i\}$ satisfies the above condition, then some family of curves $\{\bar{\alpha}_i\}$ consisting of representatives of α_i's and a family $\{\bar{\beta}_i\}$ with $\bar{\beta}_i(t) = x$ satisfy the condition in Theorem 4.1. Conversely, if $\{\bar{\alpha}_i\}$ and $\{\bar{\beta}_i\}$ satisfy the condition in Theorem 4.1, then $\{[\bar{\alpha}_i - \bar{\beta}_i]\} \subset \pi(S^2 - P_f, x)$ satisfies the above condition. □

We introduce groups, called *branch groups*, by which we can describe the equivalence relation on Σ_d. For a set X, we denote by $\Lambda(X)$ the set of bijections $X \to X$. Then $\Lambda(X)$ is a group with the product $hh' = h' \circ h$. Let $G(X) = \pi(S^2 - P_f, x)^X \times \Lambda(X)$. We consider a product on $G(X)$ as follows. We denote by p_1, p_2 the projections of $G(X)$ to the first entry and the second entry respectively. For $g, g' \in G(X)$, we define the product by $p_1(gg')(a) = p_1(g)(a)p_1(g')(p_2(g)(a))$ for $a \in X$ and $p_2(gg') = p_2(g)p_2(g')$. The unity $1 \in G(X)$ is defined by $p_1(1)(a) = 1 \in \pi(S^2 - P_f, x)$ and $p_2(1) = \mathrm{id}$. The inverse g^{-1} is defined by $p_1(g^{-1})(a) = p_1(g)(p_2(g)^{-1}(a))^{-1}$ and $p_2(g^{-1}) = p_2(g)^{-1}$. By easy calculation,

Proposition 5.3. *$G(X)$ is a group.*

Remark that p_2 is a homomorphism but p_1 is not a homomorphism.

We write $G(X) = G_k$ (or $G(X) = G_\infty$) for $X = W_k$ (or $X = \Sigma_d$). Since $W_0 = \{\varnothing\}$, $\Lambda(W_0)$ has only one element. Thus we can identify G_0 and $\pi(S^2 - P_f, x)$.

For $k = 1, 2, \ldots$ (or $k - 1 = k = \infty$), we define a map $F_k : G_{k-1} \to G_k$ as follows. For $\alpha \in G_0$, we set $g = (\tau, h) \in G_1$ by $\tau(i) = \tilde{L}_i(\alpha)$ for $i \in W_1 = \{1, 2, \ldots, d\}$ and $h = e(\alpha, \cdot)$. By (2) and (3), $F_1 : \alpha \mapsto g$ is a homomorphism. For $g = (\tau, h) \in G_{k-1}$, we set $g' = (\tau', h') \in G_k$ by $\tau'(\underline{a}) = \tilde{L}_{a_1}(\tau(\sigma(\underline{a})))$ and $h'(\underline{a}) = \sigma_{e(\tau(\sigma(\underline{a})), a_1)} h(\sigma(\underline{a}))$ for $\underline{a} = a_1 \cdots$ in W_k (or in Σ_d). By a straight calculation, we see that $F_k : g \mapsto g'$ is a homomorphism. We call F_k the *induced homomorphism* of f.

Remark 5.4. Branch groups and induced homomorphisms are described by means of universal coverings. See [**5**] for detail.

Definition 5.5. We say $g = (\tau, h) \in G_\infty$ is bounded if there exists $(\tau_k, h_k) \in G_\infty$ for $k = 1, 2, \ldots$ such that $F_\infty(\tau_1, h_1) = (\tau, h)$, $F_\infty(\tau_k, h_k) = (\tau_{k-1}, h_{k-1})$ and $\bigcup_{k=1}^{\infty} \tau_k(\Sigma_d) \cup \bigcup_{k=1}^{\infty} \tilde{\tau}_k(\Sigma_d)$ is bounded, where $F_\infty^k(\tau, h) = (\tilde{\tau}_k, \tilde{h}_k)$. Then $\mathrm{Bound}(G_\infty) = \{g \mid \text{bounded}\}$ is an F_∞-invariant subgroup.

Theorem 5.6. *Under the situation of Proposition 5.2, we have $\phi(\underline{a}) = \phi(\underline{b})$ if and only if there exists $(\tau, h) \in \mathrm{Bound}(G_\infty)$ such that $h(\underline{a}) = \underline{b}$.*

Proof. Suppose that $(\tau, h) \in G_\infty$ is bounded. Let $(\tau_k, h_k) \in G_\infty$ be as in the definition. Let $\underline{a} = a_1 a_2 \cdots \in \Sigma_d$ and $h(\underline{a}) = \underline{b} = b_1 b_2 \cdots$. Set $\alpha_k = \tau_k(\sigma^k(\underline{a}))$, $k = 0, 1, \dots$ Then $\tilde{L}_{a_k}(\alpha_k) = \alpha_{k-1}$ and $e(\alpha_k, a_k) = b_k$. Thus the condition of Proposition 5.2 is satisfied.

Suppose $\phi(\underline{a}) = \phi(\underline{b}) = y$. By Proposition 5.2, we obtain a bounded subset $\{\alpha_k\}$. In the case that $\underline{a}$ and $\underline{b}$ are not eventually periodic, we define (τ, h) as follows:

$$\begin{cases} \tau(\sigma_w(\sigma^j(\underline{a}))) = \tilde{L}_w(\alpha_j) & \text{for } w \in W_k, \\ \tau(\sigma_w(\sigma^j(\underline{b}))) = \tilde{L}_w(\alpha_j)^{-1} & \text{for } w \in W_k, \\ \tau(\underline{c}) = 1 & \text{otherwise,} \end{cases}$$

$$\begin{cases} h(\sigma_w(\sigma^j(\underline{a}))) = \sigma_{h_{k,j}(w)}(\sigma^j(\underline{b})) & \text{for } w \in W_k, \\ h(\sigma_w(\sigma^j(\underline{b}))) = \sigma_{h_{k,j}^{-1}(w)}(\sigma^j(\underline{a})) & \text{for } w \in W_k, \\ h(\underline{c}) = \underline{c} & \text{otherwise,} \end{cases}$$

where we denote $(\tau_{k,j}, h_{k,j}) = F_{k-1} \circ F_{k-2} \circ \cdots \circ F_0(\alpha_j)$. Then $\tau(\Sigma_d)$ is bounded and $F_\infty(\tau, h) = (\tau, h)$.

In the case that $\underline{a}$ and $\underline{b}$ are periodic, we define (τ_n, h_n) as follows:

$$\begin{cases} \tau_n(\sigma_w(\sigma^j(\underline{a}))) = \tilde{L}_w(\alpha_j) & \text{for } n \le j-k \le n+m-1, w \in W_k, \\ \tau_n(\sigma_w(\sigma^j(\underline{b}))) = \tilde{L}_w(\alpha_j)^{-1} & \text{for } n \le j-k \le n+m-1, w \in W_k, \\ & (\text{if } \sigma^N(\underline{a}) \ne \underline{b} \text{ for any } N) \\ \tau_n(\underline{c}) = 1 & \text{otherwise,} \end{cases}$$

$$\begin{cases} h_n(\sigma_w(\sigma^j(\underline{a}))) = \sigma_{h_{k,j}(w)}(\sigma^j(\underline{b})) & \text{for } n \le j-k \le n+m-1, w \in W_k, \\ h_n(\sigma_w(\sigma^j(\underline{b}))) = \sigma_{h_{k,j}(w)}(\sigma^j(\underline{b})) & \text{for } n \le j-k \le n+m-1, w \in W_k, \\ & (\text{if } \sigma^N(\underline{a}) \ne \underline{b} \text{ for any } N) \\ h_n(\underline{c}) = \underline{c} & \text{otherwise,} \end{cases}$$

where m is the minimum of the periods of $\underline{a}$ and $\underline{b}$, we denote $(\tau_{k,j}, h_{k,j}) = F_{k-1} \circ F_{k-2} \circ \cdots \circ F_0(\alpha_j)$. Then $F_\infty(\tau_n, h_n) = (\tau_{n-1}, h_{n-1})$ and (τ_0, h_0) is bounded.

If $\sigma^q(\underline{a})$ and $\sigma^q(\underline{b})$ are periodic for some integer $q > 0$, then we construct a bounded element $F_\infty^q(\tau_0, h_0)$ from the periodic case above. □

Example 5.7. Let us consider the polynomial $f(z) = z^2 - 2$. The critical set $C_f = \{0, \infty\}$ and the postcritical set $P_f = \{-2, 2\}$. The dynamics on $C_f \cup P_f$ is $\infty \mapsto \infty$, $0 \mapsto -2 \mapsto 2 \mapsto 2$. Take two radials r and r' as Figures 2.1 and 2.2, and generators A, B of $\pi(S^2 - P_f, x)$ as Figure 2.3.

In the case of r, see that $\phi(\underline{a}) = \phi(\underline{b})$ $(\underline{a} \ne \underline{b})$ if and only if $\{\underline{a}, \underline{b}\} = \{\sigma_w(12\bar{1}), \sigma_w(22\bar{1})\}$ for some $w \in W$. We obtain

$$\tilde{L}_1(A) = A, \quad \tilde{L}_2(A) = B, \quad \tilde{L}_1(B) = \tilde{L}_2(B) = 1$$

and

$$e(A,1) = 1, \quad e(A,2) = 2, \quad e(B,1) = 2, \quad e(B,2) = 1.$$

Define $g = (\tau, h) \in G_\infty$ by $\tau(\bar{1}) = A$, $\tau(2\bar{1}) = B$ and $\tau(\underline{a}) = 1$ otherwise, and $h(\sigma_w(22\bar{1})) = \sigma_w(12\bar{1})$, $h(\sigma_w(12\bar{1})) = \sigma_w(22\bar{1})$ and $h(\underline{a}) = \underline{a}$ otherwise. Then $F_\infty(g) = g$, and g is bounded.

In the case of r',

$$\pi^{-1}(2) = \{\bar{1}, \bar{2}\},$$
$$\tilde{L}_1(A) = A, \quad \tilde{L}_2(A) = B, \quad \tilde{L}_1(B) = A^{-1}B^{-1}, \quad \tilde{L}_2(B) = BA$$

and

$$e(A,1) = 1, \quad e(A,2) = 2, \quad e(B,1) = 2, \quad e(B,2) = 1.$$

Define $g_n = (\tau_n, h_n) \in G_\infty$, $n = 0, 1, \ldots$ by $\tau_n(\sigma_w \bar{1}) = \tilde{L}_w(A^{n+k-1}B^{-1})$, $\tau_n(\sigma_w \bar{2}) = \tilde{L}_w(BA^{-n-k+1})$ $(w \in W_k)$ and $\tau_n(\underline{a}) = 1$ otherwise, and $h_n(\sigma_w(\bar{1})) = \sigma_{w'}(\bar{2})$, $h_n(\sigma_w(\bar{2})) = \sigma_w(\bar{1})$ $(w \in W_k)$ and $h_n(\underline{a}) = \underline{a}$ otherwise, where w' is the word given by replacing 1's with 2's and 2's with 1's in w. Then $F_\infty(g_{n+1}) = g_n$, and g_0 is bounded. Now we show that there exists no bounded $g = (\tau, h)$ such that $F_\infty^m(g) = g$ for some $m > 0$ and $h(\bar{1}) = \bar{2}$. Suppose there exists such a $g = (\tau, h)$. Then $\tilde{L}_1^m(\tau(\bar{1})) = \tau(\bar{1})$ and $e(\tilde{L}^k(\tau(\bar{1})), 1) = 2$ $(k = 0, 1, \ldots)$. For $X = A^{n_0}B^{p_1}A^{n_1} \cdots B^{n_k}A^{n_k}B^{p_{k+1}} \in \pi(S^2 - P_f, x)$ (n_i, $p_i \neq 0$ for $i = 1, 2, \ldots, k$), we set $l(X) = k$. Note that $\tilde{L}_1(A^nB^{2p}) = A^n$, $\tilde{L}_1(A^nB^{2p+1}) = A^{n-1}B^{-1}$, $\tilde{L}_2(A^nB^{2p}) = B^n$, $\tilde{L}_2(A^nB^{2p+1}) = B^{n+1}A$. It is easily seen that $4l(\tilde{L}_1^2(X)) - 1 \leq l(X)$. Consequently, $l(\tau(\bar{1})) = 0$, so that we have $\tau(\bar{1}) = A^n$. Since $e(A^n, \cdot) = \mathrm{id}$, that is a contradiction. Therefore any g does not satisfy $F_\infty^m(g) = g$ and $h(\bar{1}) = \bar{2}$.

6. Branched coverings not equivalent to rational maps.

In this section, we give an example of expanding branched coverings which is not equivalent to any rational map. We then explain that under certain conditions, an expanding branched covering is always equivalent to some rational map.

Let S be a 2×2 integer matrix with determinant bigger than one. Recall that the quotient space $\boldsymbol{R}^2/\sim$ is considered as S^2, where we set $x \sim y$ if $x - y \in \boldsymbol{Z}^2$ or $x + y \in \boldsymbol{Z}^2$. Note that the projection $p : \boldsymbol{R}^2 \to S^2$ is a branched covering, and $Sx \sim Sy$ if $x \sim y$. Define $F : \boldsymbol{R}^2 \to \boldsymbol{R}^2$ by $F(x) = Sx + \beta$, where β is a 1/2-lattice point (i.e. $2\beta \in \boldsymbol{Z}^2$). Thus we have an orientation-preserving branched covering $f : S^2 \to S^2$ of degree $\det S$ such that $f \circ p = p \circ F$. If $x \in F^{-1}((1/2)\boldsymbol{Z}^2) - (1/2)\boldsymbol{Z}^2$, then $\deg_x(p \circ F) = 2$ (the local degree of $p \circ F$ at x) and $\deg_x(p) = 1$, so $\deg_{p(x)}(f) = 2$. If $x \in (1/2)\boldsymbol{Z}^2$, then $\deg_x(p \circ F) = 2$ and $\deg_x(p) = 2$, so $\deg_{p(x)}(f) = 1$. Thus $C_f = p(F^{-1}((1/2)\boldsymbol{Z}^2) - (1/2)\boldsymbol{Z}^2)$ and $P_f = p((1/2)\boldsymbol{Z}^2) = \{p(0,0), p(0,1/2), p(1/2,0), p(1/2,1/2)\}$. Define a metric on $S^2 - P_f$ by $\|v\| = \|dp^{-1}(v)\|$ for $v \in T_x(S^2), x \in S^2 - P_f$. Then we see that f is expanding if S has two eigenvalues with moduli bigger than one. By the Thurston theory ([**1**], Proposition 9.7), we know that f is equivalent to a rational map if and only if the two eigenvalues of S are complex conjugate or the same integer. Therefore, for example, if $S = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$ then f is expanding and not equivalent to a rational map. In the case $\det S = 2, 3, 4$, if f is expanding, then it is equivalent to a rational map.

As mentioned above, Thurston gives a topological condition for a postcritically finite branched covering to be equivalent to a rational map [**1**]. Suppose that f is a postcritically finite branched covering with hyperbolic orbifold. If we find a Thurston obstruction for f, then we know that f is not equivalent to a rational map. A

Levy cycle we define below is a special case of Thurston obstructions. In general, a Levy cycle implies that f is not equivalent to a rational map, and the nonexistence of Levy cycles does not imply the equivalence (for example, see [**12**]). However, under certain condition, for example, the degree-two case and the topological polynomial case, any Thurston obstruction includes a Levy cycle ([**9**], [**13**]). Thus, in these cases, if there exists no Levy cycle, then f is equivalent to a rational map. Since an expanding branched covering has no Levy cycle, an expanding branched covering is equivalent to a rational map.

Definition 6.1. Let f be a postcritically finite branched covering. A closed curve γ in $S^2 - P_f$ is called *peripheral* if one of discs bounded by γ contains at most one point of P_f.

Definition 6.2. A collection of disjoint simple closed curves $\{\gamma_1, \gamma_2, \dots, \gamma_n\}$ in $S^2 - P_f$ is called a *Levy cycle* if each γ_i is not peripheral and there exists a component $\gamma'_{i-1} \subset f^{-1}(\gamma_i)$ homotopic to γ_{i-1} in $S^2 - P_f$ and $f : \gamma'_{i-1} \to \gamma_i$ is of degree one for $i = 1, 2, \dots, n$, where $\gamma_n = \gamma_0$.

Proposition 6.3. *Suppose f be an expanding postcritically finite branched covering. Then there is no curve γ such that for each k, there exists a component $\gamma_k \subset f^{-k}(\gamma)$ satisfying that $f^k : \gamma_k \to \gamma$ is one-to-one and* $\min\{|\tilde{\gamma}| : \tilde{\gamma}$ *is homotopic to* $\gamma_k\} > 0$. *In particular, there is no Levy cycle.*

Proof. Assume that there exists such a curve γ. We can assume that γ has the length. Then $|\gamma_k| \le C\lambda^k|\gamma|$ and this is a contradiction. □

Definition 6.4. A postcritically finite branched covering $f : S^2 \to S^2$ is a *topological polynomial* if $f^{-1}(a) = \{a\}$ for some $a \in P_f$.

The following fact is known ([**9**], [**13**]).

Fact 6.5. Let f be a postcritically finite branched covering with hyperbolic orbifold which is either of degree two or a topological polynomial. If there exists no Levy cycle, then f is equivalent to a rational map.

The following fact is proved in [**4**] Theorem 5.15.

Fact 6.6. Let f be a postcritically finite branched covering with hyperbolic orbifold. Suppose that there exists a topological graph $L \subset S^2$ which satisfies the following: (1) $L \subset f^{-1}(L)$, (2) $f : L \to L$ is a homeomorphism, and (3) There exist continuous maps $F_1, F_2, \dots, F_d : S^2 - f^{-n}(L) \to S^2$ for some $n \ge 0$ such that $f \circ F_i = \mathrm{id}$. If there exists no Levy cycle, then f is equivalent to a rational map.

Corollary 6.7. *Let f be a postcritically finite branched covering with hyperbolic orbifold which either is of degree two, is a topological polynomial, or satisfies the condition in Fact* 6.6. *If f is expanding, then f is topologically conjugate to a rational map on neighborhoods of their Julia sets.*

Proof. This is clear by Facts 6.5, 6.6, and Theorem 4.7. □

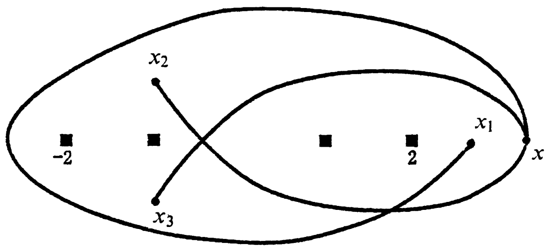

Figure 1. A radial r for $f(z) = z^3 - 3z$. The codes of -2 are $\bar{1}, \overline{23}, \overline{32}$.

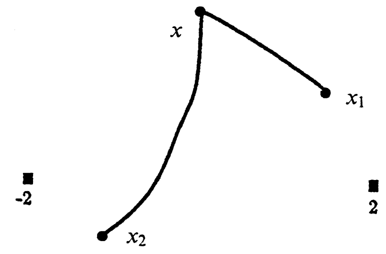

Figure 2.1. A radial r for $f(z) = z^2 - 2$.

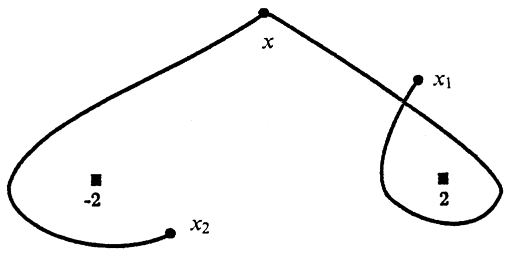

Figure 2.2. A radial r' for $f(z) = z^2 - 2$.

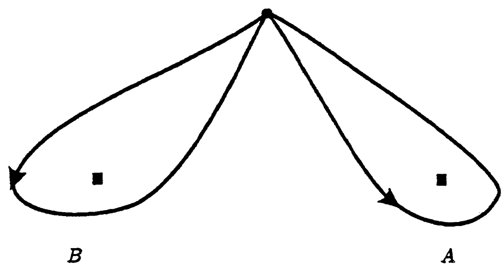

Figure 2.3. The closed curves A, B generate the fundamental group $\pi(S^2 - P_f, x)$.

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