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Tits alternatives and low dimensional topology

By Jonathan A. Hillman

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Abstract. We use cohomological group theory and properties of L^2-Betti numbers to determine the solvable groups with presentations of deficiency 1, to give a new proof of the "Tits alternative" for subgroups of Haken 3-manifold groups, and to study the fundamental groups of closed 4-manifolds with Euler characteristic 0 and, in particular, 2-knot groups.

A well-known theorem of Tits asserts that every finitely generated linear group is either virtually solvable or contains a nonabelian free subgroup [**Ti72**]. We shall establish a similar result for groups G which are locally virtually indicable, virtually of finite cohomological dimension and such that $\boldsymbol{Z}[G]$ is coherent. We shall then examine in more detail several low-dimensional situations: groups of cohomological dimension 2, PD_3-groups, the fundamental groups of closed 4-manifolds with Euler characteristic 0 and 2-knot groups.

The results on groups of finite cohomological dimension are given in §1. These are based on the detailed study of Mayer-Vietoris sequences for HNN extensions in [**BG85**]. We give also some conditions under which $H^s(G;\boldsymbol{Z}[G]) = 0$, which shall be used later to show that certain 4-manifolds are aspherical. In §2 we show that if $G \cong H*_\phi$ is an ascending HNN extension then the first L^2 Betti number $\beta_1^{(2)}(G)$ is 0, and we use this to give several alternative characterizations of solvable groups of cohomological dimension two. In §3 we show that a locally virtually indicable PD_3-group G which has no nonabelian free subgroup and such that every finitely generated subgroup is almost finitely presentable must be virtually poly-Z. Since subgroups of the fundamental groups of Haken 3-manifolds satisfy these conditions this gives a new proof of the result of [**EJ73**]. The most substantial argument is in §4, where we show that if M is a closed 4-manifold such that $\chi(M) = 0$, $\pi = \pi_1(M)$ is locally virtually indicable and has no nonabelian free subgroup, and $\boldsymbol{Z}[\pi]$ is coherent then either π is solvable and c.d.$\pi = 2$ or π is virtually Z^2 or π has two ends or M is homeomorphic to an infrasolvmanifold. The remaining three sections are devoted to 2-knot groups. The brief §5 gives several characterizations of the 2-knot group $\Phi = Z*_2$. In §6 we show that if a 2-knot group π has an abelian normal subgroup A of rank ≥ 1 then either π' is finite or $\pi \cong \Phi$ or π is a PD_4^+-group or A has rank 1 and π/A has infinitely many ends. In the final section we extend this tetrachotomy under mild coherence hypotheses on the knot group, and we complete Yoshikawa's determination of the 2-knot groups with abelian HNN bases.

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§1. Groups of finite cohomological dimension.

If G is a group let G' and $\sqrt{G}$ denote the commutator subgroup and Hirsch-Plotkin radical of G, respectively. The group G is *indicable* if $\mathrm{Hom}(G, Z) \neq 0$ or $G = 1$. (We allow the latter alternative for brevity in the statements of our results below). Thus it is locally virtually indicable if every finitely generated infinite subgroup has a subgroup of finite index which maps onto Z. If G is elementary amenable let $h(G)$ denote its Hirsch length, and let $\alpha(G) = \min\{\alpha \mid G \in X_\alpha\}$ be its level. Elementary amenable groups are locally virtually indicable, and elementary amenable groups of finite cohomological dimension are virtually solvable. (See Chapter I of [**Hi94**].)

A group G is FP_n if the augmentation $\boldsymbol{Z}[G]$-module Z has a projective resolution which is finitely generated in degrees $\leq n$. It is FP if it has finite cohomological dimension and is FP_n for $n = \mathrm{c.d.}G$. "Finitely generated" is equivalent to FP_1, while "finitely presentable" implies FP_2. Groups which are FP_2 are also said to be *almost finitely presentable*. (There are FP groups which are not finitely presentable [**BB97**].)

A group is *coherent* if its finitely generated subgroups are all finitely presentable, while a ring R is *coherent* if all finitely presentable R-modules are FP_∞.

Lemma 1. *Let G be a group such that $\boldsymbol{Z}[G]$ is coherent. Then every finitely generated subgroup of G is FP_∞.*

Proof. Let H be a finitely generated subgroup of G. Then the augmentation $\boldsymbol{Z}[H]$-module Z is FP_2 over $\boldsymbol{Z}[H]$. Hence the induced module $\boldsymbol{Z}[G] \otimes_H Z$ is FP_2 over $\boldsymbol{Z}[G]$, and so is FP_∞, as $\boldsymbol{Z}[G]$ is coherent. Since the inclusion of $\boldsymbol{Z}[H]$ into $\boldsymbol{Z}[G]$ is a faithfully flat ring extension it follows that Z is FP_∞ as a $\boldsymbol{Z}[H]$-module, i.e., that H is FP_∞. □

This lemma suggests a common generalization of the notions of "coherent group" and "coherent group ring". We shall say that a group is *almost coherent* if every finitely generated subgroup is FP_2 (almost finitely presentable). In [**BS79**] it is shown that if G is finitely generated and solvable then the conditions "G is coherent", "$\boldsymbol{Z}[G]$ is coherent", "G is almost coherent" and "G is virtually an ascending HNN extension with polycyclic base" are equivalent. (See also [**Gr78**].) Otherwise not much is known about the relations between the various notions of coherence.

Theorem 2. *Let H be an FP_∞ group and $\phi : H \to H$ a monomorphism. Let $G = H*_\phi$. Then* $\mathrm{c.d.}G = \mathrm{c.d.}H + 1$.

Proof. We may assume that $n = \mathrm{c.d.}H < \infty$, since in all cases, $\mathrm{c.d.}H \leq \mathrm{c.d.}G \leq \mathrm{c.d.}H + 1$, by Proposition 6.2 of [**Bi76**]. There is an exact sequence

$$0 \to H^n(H; \boldsymbol{Z}[G]) \to H^n(H; \boldsymbol{Z}[G]) \to H^{n+1}(G; \boldsymbol{Z}[G]) \to 0,$$

by Theorem 0.1 of [**BG85**]. Now $H^n(H; \boldsymbol{Z}[G]) \cong H^n(H; \boldsymbol{Z}[H]) \otimes_H \boldsymbol{Z}[G] \neq 0$, since H is FP_n and $\mathrm{c.d.}H = n$. The monomorphism on the left is not onto, by Lemma 3.4 and the subsequent Remark 3.5 of [**BG85**]. Hence $H^{n+1}(G; \boldsymbol{Z}[G]) \neq 0$ and so $\mathrm{c.d.}G = n + 1$. □

We shall say that a group is *restrained* if it has no nonabelian free subgroup.

Corollary. *Let G be a finitely generated, locally virtually indicable, restrained group such that* v.c.d.$G = n < \infty$ *and every finitely generated subgroup of G is FP_n. Then G is virtually solvable.*

Proof. The result is clearly true if v.c.d.$G \leq 1$. Suppose that it holds for all groups H with v.c.d.$H <$ v.c.d.G. We may assume that G is torsion free and indicable. Hence G is an HNN extension with finitely generated base and associated subgroups, by Theorem A of [**BS78**]. Since G has no nonabelian free subgroup the extension is ascending, so $G \cong H*_\phi$ for some finitely generated subgroup H and monomorphism $\phi : H \to H$. Since H is FP_n and c.d.$H \leq n$ it is FP_∞. Hence c.d.$H <$ c.d.G, by the Theorem, and so H is virtually solvable. Since G is an ascending HNN extension with virtually solvable base it is elementary amenable. As c.d.$G < \infty$ it must be virtually solvable, by Theorem I.3 of [**Hi94**]. □

Can the hypothesis that "every finitely generated subgroup is FP_n" be relaxed to "G is almost coherent"? If we assume instead that G is finitely generated, almost coherent, elementary amenable and $\alpha(G) < \infty$ an easy induction on $\alpha(G)$ shows that G is constructible and virtually solvable, hence virtually torsion free [**BB76**].

A group is a *PD_n-group* if it is a Poincaré duality group of formal dimension n. Virtually solvable PD_n-groups are virtually poly-Z and of Hirsch length n, by Theorem 9.23 of [**Bi76**].

Theorem 3. *Let H be a group and $\phi : H \to H$ a monomorphism. If H is FP_{n-1} and the ascending HNN extension $G = H*_\phi$ is a PD_n-group then ϕ is an isomorphism. Hence H is normal in G and so is a PD_{n-1}-group.*

Proof. If G is a PD_n-group then c.d.$H = n-1$ [**St77**]. (Therefore the FP_{n-1} group H is in fact FP.) Let $X \cong R$ be the tree associated to the HNN extension. Let $\mathscr{D}^{n-1}$ be the local coefficient system on X which associates to each vertex or edge the group $H^{n-1}(H; \boldsymbol{Z}[H])$ and to the inclusions of the initial and terminal vertices of an edge the identity homomorphism and the restriction induced by ϕ, respectively. Then $H^n(G; \boldsymbol{Z}[G]) \cong H^1_c(X; \mathscr{D}^{n-1})$, by Theorem 3.2 of [**BG85**].

This is in turn a direct limit $\varinjlim M_k$, indexed by Z, where the map from M_k to M_{k+1} is equivalent to the direct sum of $[H : \phi(H)]$ copies of the restriction map from $H^{n-1}(H; \boldsymbol{Z}[H])$ to $H^{n-1}(\phi(H); \boldsymbol{Z}[H])$. (See Remark 3.5 of [**BG85**].) Thus this direct limit can only be infinite cyclic if $[H : \phi(H)] = 1$. □

Induction on n gives a sharper analogue of the Corollary to Theorem 2.

Corollary. *A group G is virtually poly-Z of Hirsch length n if and only if it is virtually a PD_n-group, is locally virtually indicable and restrained and every finitely generated subgroup is FP_{n-1}.*

This corollary also follows from the Corollary to Theorem 2 together with Theorem 9.23 of [**Bi76**].

A clearly necessary condition for a closed 4-manifold M with fundamental group π to be aspherical is that $H^s(\pi; \boldsymbol{Z}[\pi]) = 0$ for $s \leq 2$. In conjunction with a suitable "counting" argument, this condition is often sufficient (cf. [**Hi89**], [**Hi94**], [**Hi97**]). We shall next give two conditions under which such cohomology groups vanish.

Theorem 4. *If G has an almost coherent, locally virtually indicable, restrained normal subgroup E with a finitely generated, one-ended subgroup then either $H^s(G;\boldsymbol{Z}[G]) = 0$ for $s \leq 2$ or G is elementary amenable and $h(G) = 2$.*

Proof. We may assume that $E = \bigcup E_n$ where $\{E_n\}_{n\geq 1}$ is an increasing sequence of finitely generated, one-ended subgroups. Since E is locally virtually indicable there are subgroups $F_n \leq E_n$ such that $[E_n : F_n] < \infty$ and which map onto Z. Since E is almost coherent these subgroups are FP_2, and since they are restrained they are ascending HNN extensions over FP_2 bases H_n. Since E_n has one end H_n has one or two ends.

If H_n has two ends then E_n is elementary amenable and $h(E_n) = 2$. It follows easily that if H_n has two ends for all n then $[E_{n+1} : E_n] < \infty$ and E is elementary amenable and $h(E) = 2$. If $[G : E] < \infty$ then G is elementary amenable and $h(G) = 2$, so we may assume that $[G : E] = \infty$. If E is finitely generated then it is FP_2 and so $H^s(G;\boldsymbol{Z}[G]) = 0$ for $s \leq 2$, by an LHSSS argument. This is also the case if E is not finitely generated, for then $H^s(E;\boldsymbol{Z}[G]) = 0$ for $s \leq 2$, by Theorem 3.3 of [**GS81**], and we may again apply an LHSSS argument.

Otherwise we may assume that H_n has one end, for all $n \geq 1$. In this case $H^s(F_n;\boldsymbol{Z}[F_n]) = 0$ for $s \leq 2$, by Theorem 0.1 of [**BG85**]. Therefore $H^s(\pi;\boldsymbol{Z}[\pi]) = 0$ for $s \leq 2$, by Theorem I.9 of [**Hi94**]. □

The theorem applies if $E = \sqrt{G}$ is large enough, since finitely generated nilpotent groups are virtually poly-Z. A similar argument shows that if $h(\sqrt{G}) \geq r$ then $H^s(G;\boldsymbol{Z}[G]) = 0$ for $s < r$. If moreover $[G : \sqrt{G}] = \infty$ then $H^r(G;\boldsymbol{Z}[G]) = 0$ also.

Are the hypotheses that E be almost coherent and locally virtually indicable necessary? Is it sufficient that E be restrained and be an increasing union of finitely generated, one-ended subgroups?

Theorem 5. *Let $G = B*_\phi$ be an HNN extension with FP_2 base B and associated subgroups I and $\phi(I) = J$, and which has a restrained normal subgroup $N \leq \langle\langle B\rangle\rangle$. If either N is locally virtually Z and G/N has one end or N has a finitely generated, one-ended subgroup then $H^s(G;\boldsymbol{Z}[G]) = 0$ for $s \leq 2$.*

Proof. Let t be the stable letter, so that $tit^{-1} = \phi(i)$, for all $i \in I$. Suppose that $N \cap J \neq N \cap B$, and let $b \in N \cap B - J$. Then $b^t = t^{-1}bt$ is in N, since N is normal in G. Let a be any element of $N \cap B$. Since N has no nonabelian free subgroup there is a word $w \in F(2)$ such that $w(a, b^t) = 1$ in G. It follows from Britton's Lemma that a must be in I and so $N \cap B = N \cap I$. In particular, N is the increasing union of copies of $N \cap B$.

Hence G/N is an HNN extension with base $B/N \cap B$ and associated subgroups $I/N \cap I$ and $J/N \cap J$. Therefore if G/N has one end the latter groups are infinite, and so B, I and J each have one end. If N is virtually Z then $H^s(G;\boldsymbol{Z}[G]) = 0$ for $s \leq 2$, by an LHSSS argument. If N is locally virtually Z but is not finitely generated then it is the increasing union of a sequence of two-ended subgroups and $H^s(N;\boldsymbol{Z}[G]) = 0$ for $s \leq 1$, by Theorem 3.3 of [**GS81**]. Since $H^2(B;\boldsymbol{Z}[G]) \cong H^0(B;H^2(N \cap B;\boldsymbol{Z}[G]))$ and $H^2(I;\boldsymbol{Z}[G]) \cong H^0(I;H^2(N \cap I;\boldsymbol{Z}[G]))$, the restriction map from $H^2(B;\boldsymbol{Z}[G])$ to $H^2(I;\boldsymbol{Z}[G])$ is injective. If N has a finitely generated, one-ended subgroup N_1, we may assume that $N_1 \leq N \cap B$, and so B, I and J also have one end. Moreover $H^s(N \cap B; \boldsymbol{Z}[G]) = 0$ for $s \leq 1$, by Theorem I.9 of [**Hi94**]. We again see that the restriction map from $H^2(B; \boldsymbol{Z}[G])$ to $H^2(I; \boldsymbol{Z}[G])$ is injective. The result now follows in these cases from Theorem 0.1 of [**BG85**]. $\square$

§2. Groups of cohomological dimension 2.

In this section we shall examine more closely the case of groups of cohomological dimension 2. In particular, we use a simple vanishing criterion for the first L^2-Betti number of a group to show that solvable groups with presentations of deficiency 1 have cohomological dimension ≤ 2.

Let X be a finite complex with fundamental group π. The von Neumann algebra $A(\pi)$ is the algebra of bounded operators on the Hilbert space completion $\ell^2(\pi)$ of $\boldsymbol{C}[\pi]$ which commute with every $\boldsymbol{C}[\pi]$-linear operator (i.e., the bicommutant). The *L^2-Betti numbers* of X are defined by $\beta_i^{(2)}(X) = \dim_{A(\pi)}(\overline{H}_2^{(2)}(X))$ where the L^2-homology $\overline{H}_i^{(2)}(X) = \overline{H}_i(C_*^{(2)})$ is the homology of the Hilbert $A(\pi)$-complex $C_*^{(2)} = \ell^2 \otimes C_*(\tilde{X})$ of square summable chains on $\tilde{X}$. ([**At76**]. See also [**Ec94**], [**Lü94**].) They are multiplicative in finite covers, and for $i = 0$ or 1 depend only on $\pi_1(X)$. (In particular, $\beta_0^{(2)}(\pi) = 0$ if π is infinite). The alternating sum of the L^2-Betti numbers is the Euler characteristic $\chi(X)$ [**At76**]. The usual Betti numbers of a space or group (with coefficients $\boldsymbol{Q}$) shall be denoted by $\beta_i(X)$.

If π is the fundamental group of a finite aspherical 2-complex we say that it has *geometric dimension at most* 2, written g.d.$\pi \leq 2$. For each $m \neq 0 \in Z$ let $Z*_m$ be the ascending HNN extension with presentation $\langle a, t \mid tat^{-1} = a^m \rangle$. The 2-complexes corresponding to these presentations are aspherical. Let $D = (Z/2Z) * (Z/2Z)$ be the infinite dihedral group.

Theorem 6. *Let $G = H*_\phi$ be a finitely presentable group which is an ascending HNN extension with finitely generated base H. Then $\beta_1^{(2)}(G) = 0$.*

Proof. Let t be the stable letter and let H_n be the subgroup generated by H and t^n, and suppose that H is generated by g elements. Then $[G : H_n] = n$, so $\beta_1^{(2)}(H_n) = n\beta_1^{(2)}(G)$. But each H_n is also finitely presentable and generated by $g+1$ elements. Hence $\beta_1^{(2)}(H_n) \leq g+1$, and so $\beta_1^{(2)}(G) = 0$. $\square$

This result extends part of Theorem 2.1 of [**Lü94**], which considered extensions of Z by finitely presentable normal subgroups.

Corollary. *Let G be a finitely presentable group which is an ascending HNN extension with finitely generated base H. Then $def(G) \leq 1$, and $def(G) = 1$ if and only if* g.d.$G \leq 2$ *and $\beta_2(G) = \beta_1(G) - 1$.*

Proof. If X is a finite 2-complex corresponding to a presentation P for G then $\chi(X) = 1 - def(P)$. Now $\chi(X) = \beta_2^{(2)}(X)$ by Theorem 6. Hence $def(P) \leq 1$, and if $def(P) = 1$ then X is aspherical, by Theorem 1 of [**Hi97**]. In that case g.d.$\pi \leq 2$ and $\beta_2(G) = \beta_1(G) - 1$. Conversely, if X is a finite aspherical 2-complex with $\pi_1(X) \cong G$ and $\beta_2(G) = \beta_1(G) - 1$ then $\chi(X) = 0$. After collapsing a maximal tree in X we may assume it has a single 0-cell, and then the presentation read off the 1- and 2-cells has deficiency 1. □

Theorem 7. *Let G be a finitely generated group such that* c.d.$G = 2$. *Then $G \cong Z*_m$ for some $m \neq 0$ if and only if it is almost coherent, restrained and G/G' is infinite.*

Proof. The conditions are easily seen to be necessary. Conversely, if they hold G is FP_2 and so is an HNN extension with finitely generated base H, by Theorem A of [**BS78**]. The HNN extension must be ascending, as G has no nonabelian free subgroup, and the base H is FP_2. Hence $H^2(G;\boldsymbol{Z}[G])$ is a quotient of $H^1(H;\boldsymbol{Z}[G]) \cong H^1(H;\boldsymbol{Z}[H]) \otimes \boldsymbol{Z}[G/H]$, by Theorem 0.1 of [**BG85**]. Now $H^2(G;\boldsymbol{Z}[G]) \neq 0$, since c.d.$G = 2$, and so $H^1(H;\boldsymbol{Z}[H]) \neq 0$. Since H has no nonabelian free subgroup it must have two ends, so $H \cong Z$ and $G \cong Z*_m$ for some $m \neq 0$. □

Corollary. *Let G be an almost finitely presentable group. Then the following are equivalent*:

(i) *$G \cong Z$ or $Z*_m$ for some $m \neq 0$;*
(ii) *G is torsion free, virtually solvable and $h(G) \leq 2$.*
(iii) *G is virtually solvable and* c.d.$G \leq 2$;
(iv) *G is elementary amenable and $def(G) = 1$; and*
(v) *G is almost coherent, restrained and $def(G) = 1$.*

Proof. Condition (i) clearly implies the others. Suppose (ii) holds. We may assume that $h(G) = 2$ and $h(\sqrt{G}) = 1$ (for otherwise $G \cong Z$, $Z^2 = Z*_1$ or $Z*_{-1}$). Hence $h(G/\sqrt{G}) = 1$, and so $G/\sqrt{G}$ is an extension of Z or D by a finite normal subgroup. If $G/\sqrt{G}$ maps onto D then $G \cong A *_C B$, where $[A:C] = [B:C] = 2$ and $h(A) = h(B) = h(C) = 1$, and so $G \cong Z*_{-1}$. But then $h(\sqrt{G}) = 2$. Hence we may assume that G maps onto Z, and so G is an ascending HNN extension with finitely generated base H. Since H is torsion free, solvable and $h(H) = 1$ it must be infinite cyclic and so (ii) implies (i). If $def(G) = 1$ then G is an ascending HNN extension with finitely generated base, so $\beta_1^{(2)}(G) = 0$, by Theorem 6. Hence (iv) and (v) each imply (iii), by Theorem 7 and the Corollary to Theorem 6. Finally (iii) implies (ii). □

See [**Wi96**] for a more group-theoretic determination of the solvable groups of deficiency 1. These are in fact the finitely *generated* solvable groups of cohomological dimension at most 2 [**Gi79**]. Note also that if $def(G) > 1$ then G contains nonabelian free subgroups [**Ro77**].

Theorem 8. *Let G be a finitely generated group such that* c.d.$G = 2$. *If G has a nontrivial normal subgroup E which is either almost coherent, locally virtually indicable and restrained or is elementary amenable then either $E \cong Z$ or G is metabelian.*

Proof. Let E' be a finitely generated subgroup of E. Then E' is metabelian, by Theorem 7 and its corollary, and so all words in E of the form $[[g,h],[g',h']]$ are trivial. Hence E is metabelian also. Therefore $A = \sqrt{E}$ is nontrivial, and as A is characteristic in E it is normal in π. Since A is the union of its finitely generated subgroups, which are torsion free nilpotent groups of Hirsch length ≤ 2, it is abelian.

If $A \cong Z$ then $[G : C_G(A)] \le 2$. Moreover $C_G(A)'$ is free, by Theorem 8.8 of [**Bi76**]. If $C_G(A)'$ is cyclic then $\pi \cong Z^2$ or $Z \tilde{\times} Z$; if $C_G(A)'$ is nonabelian then $E = A \cong Z$. Otherwise c.d.A = c.d.$C_G(A) = 2$ and so $C_G(A) = A$, by Theorem 8.8 of [**Bi76**]. If A has rank 1 then Aut(A) is abelian, so $G' \le C_G(A)$ and G is metabelian. If $A \cong Z^2$ then G/A is isomorphic to a subgroup of $GL(2, \boldsymbol{Z})$, and so is virtually free. As A together with an element $t \in G$ of infinite order modulo A would generate a subgroup of cohomological dimension 3, which is impossible, the quotient G/A must be finite. Hence $G \cong Z^2$ or $Z \times_{-1} Z$. □

Can either of the hypotheses that E be almost coherent and locally virtually indicable be removed from this theorem?

In higher dimensions FP_∞ solvable groups need not be almost coherent. For instance, the group G with presentation $\langle a, x, y \mid xy = yx, xax^{-1} = a^2, yay^{-1} = a^3 \rangle$ is solvable and has a finite 3-dimensional Eilenberg-Mac Lane space, since it is an iterated HNN extension. Hence c.d.$G = 3 = h(G)$ and G is FP_∞. However the subgroup generated by $\{a, xy^{-1}\}$ is not isomorphic to one of the groups allowed by the Corollary to Theorem 5 and so cannot be FP_2.

§3. PD_3-groups.

By the Corollary to Theorem 3 a PD_3-group is virtually poly-Z if and only if it is almost coherent, virtually indicable and restrained. The results of §2 enable us to give a corresponding Tits alternative for suitable subgroups of PD_3-groups.

Theorem 9. *Let G be a PD_3-group. Then every almost coherent, locally virtually indicable subgroup of G is either virtually solvable or contains a nonabelian free subgroup.*

Proof. Let S be a locally virtually indicable subgroup of G which contains no nonabelian free subgroup. If $[G : S] < \infty$ then S is again a PD_3-group and so is virtually poly-Z, by the Corollary to Theorem 3.

If $[G : S] = \infty$ then c.d.$S \le 2$, by [**St77**]. The finitely generated subgroups of S are virtually indicable, and hence are metabelian, by Theorem 7 and its Corollary. Hence S is metabelian also. □

As the fundamental groups of virtually Haken 3-manifolds are coherent and locally virtually indicable, this implies the "Tits alternative" for such groups [**EJ73**]. In fact solvable subgroups of infinite index in 3-manifold groups are virtually abelian. This remains true if $K(G, 1)$ is a finite PD_3-complex, by Corollary 1.4 of [**KK99**]. Does this hold for all PD_3-groups?

A slight modification of the argument gives the following corollary.

Corollary. *A PD_3-group G is virtually poly-Z if and only if it is coherent, restrained and has a subgroup of finite index with infinite abelianization.*

If the PD_3-group G has a subgroup S of finite index such that $\beta_1(S) \ge 2$ and is restrained then the hypothesis of coherence is redundant, for there is then an epimorphism $p : S \to Z$ with finitely generated kernel, by [**BNS87**], and Ker p is a PD_2-group, by Theorem 1 of [**Hi98**].

The argument of Theorem 9 and its corollary extend to show by induction on m that a PD_m-group is virtually poly-Z if and only if it is restrained and every finitely generated subgroup is FP_{m-1} and virtually indicable.

§4. 4-manifolds with Euler characteristic 0.

In this section we shall consider the 4-dimensional analogue of the above question, imposing the further restriction that the Euler characteristic of the manifolds arising be 0. We shall determine almost completely the coherent elementary amenable groups which arise in this way.

LEMMA 10. *Let M be a PD_4^+-complex with $\chi(M) = 0$ and such that $\pi = \pi_1(M)$ is an extension of $Z*_m$ by a finite normal subgroup F, for some $m \neq 0$. Then the finite abelian subgroups of F are cyclic. If $F \neq 1$ then π has a subgroup of finite index which is a central extension of $Z*_n$ by a nontrivial finite cyclic group, where n is a power of m.*

PROOF. Let $\hat{M}$ be the infinite cyclic covering space corresponding to the subgroup $I(\pi) = \{g \in \pi \mid \exists n > 0, g^n \in \pi'\}$. Since M is compact and $\Lambda = \boldsymbol{Z}[Z]$ is noetherian the groups $H_i(\hat{M}; \boldsymbol{Z}) = H_i(M; \Lambda)$ are finitely generated as Λ-modules. Since M is orientable, $\chi(M) = 0$ and $H_1(M; \boldsymbol{Z})$ has rank 1 they are Λ-torsion modules, by the Wang sequence for the projection of $\hat{Y}$ onto Y. Now $H_2(\hat{M}; \boldsymbol{Z}) \cong \overline{\mathrm{Ext}^1_\Lambda(I(\pi)/I(\pi)', \Lambda)}$, by Poincaré duality. There is an exact sequence $0 \to T \to I(\pi)/I(\pi)' \to I(Z*_m) \cong \Lambda/(t-m) \to 0$, where T is a finite Λ-module. Therefore $\mathrm{Ext}^1_\Lambda(I(\pi)/I(\pi)', \Lambda) \cong \Lambda/(t-m)$ and so $H_2(I(\pi); \boldsymbol{Z})$ is a quotient of $\Lambda/(mt-1)$, which is isomorphic to $Z[1/m]$ as an abelian group. Now $I(\pi)/\mathrm{Ker}\, f \cong Z[1/m]$ also, and $H_2(Z[1/m]; \boldsymbol{Z}) \cong Z[1/m] \wedge Z[1/m] = 0$ (see page 334 of [**Ro82**]). Hence $H_2(I(\pi); \boldsymbol{Z})$ is finite, by an LHS spectral sequence argument, and so is cyclic, of order relatively prime to m.

Let t in π generate $\pi/I(\pi) \cong Z$. Let A be a maximal abelian subgroup of F and let $C = C_\pi(A)$. Then $q = [\pi : C]$ is finite, since F is finite and normal in π. In particular, t^q is in C and C maps onto Z, with kernel J, say. Since J is an extension of $Z[1/m]$ by a finite normal subgroup its centre ζJ has finite index in J. Therefore the subgroup G generated by ζJ and t^q has finite index in π, and there is an epimorphism f from G onto $Z*_{m^q}$, with kernel A. Moreover $I(G) = f^{-1}(I(Z*_{m^q}))$ is abelian, and is an extension of $Z[1/m]$ by the finite abelian group A. Hence it is isomorphic to $A \oplus Z[1/m]$ (see page 106 of [**Ro82**]). Now $H_2(I(G); \boldsymbol{Z})$ is cyclic of order prime to m. On the other hand $H_2(I(G); \boldsymbol{Z}) \cong (A \wedge A) \oplus (A \otimes Z[1/m])$ and so A must be cyclic.

If $F \neq 1$ then A is cyclic, nontrivial, central in G and $G/A \cong Z*_{m^s}$. $\square$

A ring R is *weakly finite* if every onto endomorphism of R^n is an isomorphism, for all $n \geq 0$. (In [**Hi94**] the term "SIBN ring" was used instead.) Finitely generated stably free modules over weakly finite rings have well defined ranks, and the rank is strictly positive if the module is nonzero. Skew fields are weakly finite, as are subrings of weakly finite rings.

LEMMA 11. *Let M be a finite PD_4-complex with fundamental group π. Suppose that π has a nontrivial finite cyclic central subgroup F with quotient $G = \pi/F$ such that* g.d.$G = 2$*, G has one end and def$(G) = 1$. Then $\chi(M) \geq 0$. If $\chi(M) = 0$ and $\boldsymbol{F}_p[G]$ is a weakly finite ring for some prime p dividing $|F|$ then π is virtually Z^2.*

Proof. Let $\hat{M}$ be the covering space of M with group F, and let $\Xi = \boldsymbol{F}_p[G]$. Let $C_* = C_*(M;\Xi) = \boldsymbol{F}_p \otimes C_*(M)$ be the equivariant cellular chain complex of $\hat{M}$ with coefficients $\boldsymbol{F}_p$, and let c_q be the number of q-cells of M, for $q \geq 0$. Let $H_p = H_p(M;\Xi) = H_p(\hat{M};\boldsymbol{F}_p)$. For any left Ξ-module H let $e^qH = \mathrm{Ext}^q_{\Xi}(H,\Xi)$.

Suppose first that M is orientable. Since $\hat{M}$ is a connected open 4-manifold $H_0 = \boldsymbol{F}_p$ and $H_4 = 0$, while $H_1 \cong \boldsymbol{F}_p$ also. Since G has one end Poincaré duality and the UCSS give $H_3 = 0$ and $e^2H_2 \cong \boldsymbol{F}_p$, and an exact sequence

$$0 \to e^2\boldsymbol{F}_p \to \overline{H}_2 \to e^0H_2 \to e^2H_1 \to \overline{H}_1 \to e^1H_2 \to 0.$$

In particular, $e^1H_2 \cong \boldsymbol{F}_p$ or is 0. Since g.d.$G = 2$ and $def(G) = 1$ the augmentation module has a resolution $0 \to \Xi^r \to \Xi^{r+1} \to \Xi \to \boldsymbol{F}_p \to 0$. The chain complex C_* gives four exact sequences $0 \to Z_1 \to C_1 \to C_0 \to \boldsymbol{F}_p \to 0$, $0 \to Z_2 \to C_2 \to Z_1 \to \boldsymbol{F}_p \to 0$, $0 \to B_2 \to Z_2 \to H_2 \to 0$ and $0 \to C_4 \to C_3 \to B_2 \to 0$. Using Schanuel's Lemma several times we find that the cycle submodules Z_1 and Z_2 are stably free, of stable ranks $c_1 - c_0$ and $c_2 - c_1 + c_0$, respectively. Dualizing the last two sequences gives two new sequences $0 \to e^0B_2 \to e^0C_3 \to e^0C_4 \to e^1B_2 \to 0$ and $0 \to e^0H_2 \to e^0Z_2 \to e^0B_2 \to e^1H_2 \to 0$, and an isomorphism $e^1B_2 \cong e^2H_2 \cong \boldsymbol{F}_p$. Further applications of Schanuel's Lemma show that e^0B_2 is stably free of rank $c_3 - c_4$, and hence that e^0H_2 is stably free of rank $c_2 - c_1 + c_0 - (c_3 - c_4) = \chi(M)$. (Note that we do not need to know whether $e^1H_2 \cong \boldsymbol{F}_p$ or is 0, at this point). Since Ξ maps onto the field $\boldsymbol{F}_p$ the rank must be nonnegative, and so $\chi(M) \geq 0$.

If $\chi(M) = 0$ and $\Xi = \boldsymbol{F}_p[G]$ is a weakly finite ring then $e^0H_2 = 0$ and so $e^2\boldsymbol{F}_p = e^2H_1$ is a submodule of $\boldsymbol{F}_p \cong \overline{H}_1$. Moreover it cannot be 0, for otherwise the UCSS would give $H_2 = 0$ and then $H_1 = 0$, which is impossible. Therefore $e^2\boldsymbol{F}_p \cong \boldsymbol{F}_p$.

If M is nonorientable and $p > 2$ the above argument applies to the orientation cover, since p divides $|\mathrm{Ker}(w_1(M)|_F)|$, and Euler characteristic is multiplicative in finite covers. If $p = 2$ a similar argument applies directly without assuming that M is orientable.

Since G is torsion free and indicable it must be a PD_2-group, by Theorem V.12.2 of [**DD**]. Since $def(G) = 1$ it follows that G is virtually Z^2, and hence that π is also virtually Z^2. □

We may now give the main result of this section.

Theorem 12. *Let M be a finite PD_4-complex whose fundamental group π is an ascending HNN extension with finitely generated base B. Then $\chi(M) \geq 0$, and hence $q(\pi) \geq 0$. If $\chi(M) = 0$ and B is FP_2 and finitely ended then either π has two ends or has a subgroup of finite index which is isomorphic to Z^2 or $\pi \cong Z*_m$ or $Z*_m \tilde{\times} (Z/2Z)$ for some $m \neq 0$ or ± 1 or M is aspherical.*

Proof. The L^2 Euler characteristic formula gives $\chi(M) = \beta_2^{(2)}(M) \geq 0$, since $\beta_i^{(2)}(M) = \beta_i^{(2)}(\pi) = 0$ for $i = 0$ or 1, by Theorem 6.

Let $\phi : B \to B$ be the monomorphism determining $\pi \cong B*_\phi$. If B is finite then ϕ is an automorphism and so π has two ends. If B is FP_2 and has one end then $H^s(\pi;\boldsymbol{Z}[\pi]) = 0$ for $s \leq 2$, by Theorem 0.1 of [**BG85**]. If moreover $\chi(M) = 0$ then M is aspherical, by Corollary 3.5 of [**Ec94**], since $\beta^{(2)}(\pi) = 0$. If B has two ends then it is an extension of Z or D by a finite normal subgroup F. As ϕ must map F isomorphically to itself, F is normal in π, and is the maximal finite normal subgroup of π. Moreover $\pi/F \cong Z*_m$, for some $m \neq 0$, if $B/F \cong Z$, and is a semidirect product $Z*_m \tilde{\times} (Z/2Z)$, with a presentation $\langle a, t, u \mid tat^{-1} = a^m, tut^{-1} = ua^r, u^2 = 1, uau = a^{-1} \rangle$, for some $m \neq 0$ and some $r \in Z$, if $B/F \cong D$. (We may in fact assume that $r = 0$ or 1).

Suppose first that M is orientable, and that $F \neq 1$. Then π has a subgroup σ of finite index which is a central extension of $Z*_{m^q}$ by a finite cyclic group, for some $q \geq 1$, by Lemma 10. Let p be a prime dividing q. Since $Z*_{m^q}$ is a torsion free solvable group the ring $\Xi = \boldsymbol{F}_p[Z*_{m^q}]$ has a skew field of fractions L, which as a right Ξ-module is the direct limit of the system $\{\Xi_\theta \mid 0 \neq \theta \in \Xi\}$, where each $\Xi_\theta = \Xi$, the index set is ordered by right divisibility $(\theta \leq \phi\theta)$ and the map from Ξ_θ to $\Xi_{\phi\theta}$ sends ξ to $\phi\xi$ [**KLM88**]. In particular, Ξ is a weakly finite ring and so σ is torsion free, by Lemma 11. Therefore $F = 1$.

If M is nonorientable then $w_1(M)|_F$ must be injective, and so another application of Lemma 11 (with $p = 2$) shows again that $F = 1$. □

Is M still aspherical if B is assumed only finitely generated and one ended?

Corollary A. *Let M be a closed 4-manifold with $\chi(M) = 0$ and such that $\pi = \pi_1(M)$ is locally virtually indicable, restrained and its finitely generated subgroups are FP_3. Then either π has two ends or is virtually Z^2 or $\pi \cong Z*_m$ for some $m \neq 0$ or ± 1 or M is homeomorphic to an infrasolvmanifold.*

Proof. Let M^+ be the orientation cover of M. Then $\chi(M^+) = 0$ and so $\beta_1(M^+) \geq 1$, by Poincaré duality. Thus after passing to M^+ if necessary we may assume that π maps onto Z. Hence π is an ascending HNN extension $\pi \cong B*_\phi$, where the base B is finitely generated and so FP_3, and $\phi: B \to B$ is a monomorphism. Since π has no nonabelian free subgroup B has at most two ends. Hence Lemma 11 and Theorem 12 apply. If B has one end then it must be a PD_3-group and ϕ an isomorphism, by Theorem 5. Therefore B is virtually poly-Z of Hirsch length 3, by the Corollary to Theorem 3, and so M is homeomorphic to an infrasolvmanifold, by Theorem VI.2 of [**Hi94**]. □

Although the two ended groups realized by PD_4-complexes with $\chi = 0$ are essentially understood, the question of which are realized by closed 4-manifolds with $\chi = 0$ seems rather difficult.

There are nine groups which are virtually Z^2 and are fundamental groups of PD_4-complexes with Euler characteristic 0. All may be realized by closed $\boldsymbol{S}^2 \times \boldsymbol{E}^2$-manifolds. Moreover if $\chi(M) = 0$ and π is virtually Z^2 then M is finitely covered by $S^2 \times T$. (See Chapter VII of [**Hi94**].)

Let $w: Z*_m \to Z^\times$ be a homomorphism. Attaching a 2-handle along a suitable loop to $S^1 \times D^3 \natural S^1 \times D^3$ or $S^1 \times D^3 \natural S^1 \tilde{\times} D^3$ gives a bounded 4-manifold Y with $\pi_1(Y) \cong Z*_m$, $w_1(Y) = w$ and $\chi(Y) = 0$. The double $M = Y \cup_{\partial Y} Y$ is then a closed 4-manifold with $\chi(M) = 0$ and $(\pi_1(M), w_1(M)) \cong (Z*_m, w)$. It may be shown that homotopy equivalent closed 4-manifolds with fundamental group $Z*_m$ are homeomorphic; however adequately characterizing the possible homotopy types remains a problem.

Corollary B. *A closed* 4*-manifold with* $\chi(M) = 0$ *is homeomorphic to an infrasolvmanifold if and only if* $\pi = \pi_1(M)$ *has no nonabelian free subgroup, its finitely generated subgroups are* FP_3 *and it has a subgroup* σ *of finite index which is an extension of a poly-Z group of Hirsch length at least* 2 *by an infinite normal subgroup.*

Each 4-dimensional infrasolvmanifold admits exactly one of the geometries $\boldsymbol{E}^4$, $\boldsymbol{Nil}^3 \times \boldsymbol{E}^1$, $\boldsymbol{Nil}^4$, $\boldsymbol{Sol}^4_{m,n}$, $\boldsymbol{Sol}^4_0$ or $\boldsymbol{Sol}^4_1$.

§5. 2-Knots—the group $\boldsymbol{\Phi}$.

A 2-knot is a locally flat embedding $K : S^2 \to S^4$. The closed 4-manifold $M(K)$ obtained by surgery on the 2-knot K is orientable, $\chi(M(K)) = 0$ and $\pi_1(M(K)) \cong \pi K = \pi_1(S^4 - K(S^2))$. Thus we may apply the results of §4 to the study of such 2-knot groups πK. Let $M(K)'$ denote the covering space associated to the commutator subgroup $(\pi K)'$.

In order to clarify the statements and arguments in §6, we shall give here several characterizations of the group $\Phi = Z*_2$, which is a 2-knot group [**Fo62**] that plays a somewhat exceptional role.

Theorem 13. *Let* π *be a* 2*-knot group such that* $\mathrm{c.d.}\pi = 2$ *and* π *has a nontrivial normal subgroup* E *which is either elementary amenable or almost coherent, locally virtually indicable and restrained. Then either* $\pi \cong \Phi$ *or* E *is central and* π' *is free; in either case* $def(\pi) = 1$.

Proof. If π is solvable then $\pi \cong Z*_m$, for some $m \neq 0$, by the corollary to Theorem 7. Since $\pi/\pi' \cong Z$ we must have $m = 2$ and so $\pi \cong \Phi$. Otherwise $E \cong Z$, by Theorem 8. Then $[\pi : C_\pi(E)] \leq 2$ and $C_\pi(E)'$ is free, by Theorem 8.8 of [**Bi76**]. This free subgroup must be nonabelian for otherwise G would be solvable. Hence $E \cap C_\pi(E)' = 1$ and so E maps injectively to $H = G/C_G(E)'$. As H has an abelian normal subgroup of index at most 2 and $H/H' \cong Z$ we must in fact have $H \cong Z$. It follows easily that $C_\pi(E) = E$, and so π' is free. The final observation follows readily. □

The following alternative characterizations of Φ shall be useful.

Theorem 14. *Let* π *be a* 2*-knot group with maximal locally finite normal subgroup* T. *Then* $\pi/T \cong \Phi$ *if and only if* π *is elementary amenable and* $h(\pi) = 2$. *Moreover the following are equivalent*:

(i) π *has an abelian normal subgroup* A *of rank* 1 *such that* π/A *has two ends*;
(ii) π *is elementary amenable*, $h(\pi) = 2$ *and* π *has an abelian normal subgroup* A *of rank* 1;
(iii) π *is almost coherent*, *elementary amenable and* $h(\pi) = 2$;
(iv) $\pi \cong \Phi$.

Proof. Since π is finitely presentable and has infinite cyclic abelianization it is an HNN extension $\pi \cong H*_\phi$ with base H a finitely generated subgroup of π' by Theorem A of [**BS78**]. Since π is elementary amenable the extension must be ascending. Since $h(\pi'/T) = 1$ and π'/T has no nontrivial locally-finite normal subgroup $[\pi'/T : \sqrt{\pi'/T}] \leq 2$. The meridianal automorphism of π' induces a meridianal automorphism on $(\pi'/T)/\sqrt{\pi'/T}$ and so $\pi'/T = \sqrt{\pi'/T}$. Hence π'/T is a torsion free rank 1 abelian group. Let $J = H/H \cap T$. Then $h(J) = 1$ and $J \leq \pi'/T$ so $J \cong Z$. Now ϕ induces a monomorphism $\psi : J \to J$ and $\pi/T \cong J*_\psi$. Since $\pi/\pi' \cong Z$ we must have $J*_\psi \cong \Phi$.

If (i) holds then π is elementary amenable and $h(\pi) = 2$. Suppose (ii) holds. We may assume without loss of generality that A is the normal closure of an element of infinite order, and so π/A is finitely presentable. Since π/A is elementary amenable and $h(\pi/A) = 1$ it is virtually Z. Therefore π is virtually an HNN extension with base a finitely generated subgroup of A, and so is coherent. If (iii) holds then $\pi \cong \Phi$, by Corollary A of Theorem 12. Since Φ clearly satisfies conditions (i)–(iii) this proves the theorem. □

COROLLARY. *If $\pi/T \cong \Phi$ then T is either trivial or infinite.*

In fact we believe that in this case T must be trivial, but have not yet been able to prove this.

§6. Abelian normal subgroups.

In this section we shall consider 2-knot groups with infinite abelian normal subgroups. The class with rank 1 abelian normal subgroups includes the groups of torus knots and twist spins, the group Φ, and all 2-knot groups with finite commutator subgroup. If there is such a subgroup of rank > 1 the knot manifold is aspherical.

THEOREM 15. *Let K be a 2-knot such that $\pi = \pi K$ has an almost coherent, locally virtually indicable, restrained normal subgroup E which is not locally finite. Then either π' is finite or $\pi \cong \Phi$ or $M(K)$ is aspherical or E is abelian of rank 1 and π/E has infinitely many ends or E is elementary amenable, $h(E) = 1$ and π/E has one or infinitely many ends.*

PROOF. Let F be a finitely generated subgroup of E. Since F is FP_2 and virtually indicable it has a subgroup of finite index which is an HNN extension over a finitely generated base [**BS79**]. Since F is restrained the HNN extension is ascending, and so $\beta_1^{(2)}(F) = 0$, by Theorem 6. Hence $\beta_1^{(2)}(E) = 0$ and so $\beta_1^{(2)}(\pi) = 0$, by Theorem 3.3 of [**Lü98**].

If E is almost coherent, locally virtually indicable and has a finitely generated, one-ended subgroup and π is not elementary amenable of Hirsch length 2 then $H^2(\pi; \boldsymbol{Z}[\pi]) = 0$, by Theorem 4. Hence $M(K)$ is aspherical, by Eckmann's Theorem [**Hi97**].

Otherwise every finitely generated infinite subgroup of E has two ends, so E is elementary amenable and $h(E) = 1$. If π/E is finite then π' is finite. If π/E has two ends then $\pi \cong \Phi$, by Theorem 14. If E is abelian and π/E has one end π is 1-connected at ∞, by Theorem 1 of [**Mi87**], and so $H^s(\pi; \boldsymbol{Z}[\pi]) = 0$ for $s \leq 2$, by [**GM86**]. Hence $M(K)$ is again aspherical, by Eckmann's Theorem [**Hi97**].

The remaining possibilities are that either π/E has infinitely many ends or that E is nonabelian and π/E has one end. □

Does this theorem hold without any coherence hypothesis? Note that the other hypotheses hold if E is elementary amenable and $h(E) \geq 2$. If E is elementary amenable, $h(E) = 1$ and π/E has one end is $H^2(\pi; \boldsymbol{Z}[\pi]) = 0$?

If E is locally FP_3 and M is aspherical then E is virtually solvable, and then π has nontrivial torsion free abelian normal subgroups (for instance, $\zeta\sqrt{E}$).

Corollary. *If $\sqrt{\pi}$ is not locally finite then either π' is finite or $\pi \cong \Phi$ or $M(K)$ is aspherical or $h(\sqrt{\pi}) = 1$ and $\pi/\sqrt{\pi}$ has one or infinitely many ends.*

Proof. Finitely generated nilpotent groups are polycyclic. □

Theorem 16. *Let K be a 2-knot whose group $\pi = \pi K$ has an infinite abelian normal subgroup A, of rank $r \geq 1$. Then $r \leq 4$ and*

(i) *if $r = 1$ either π' is finite or $\pi \cong \Phi$ or $M(K)$ is aspherical or $e(\pi/A) = \infty$;*
(ii) *if $r = 1$, $e(\pi/A) = \infty$ and $\pi' \leq C_\pi(A)$ then A and $\sqrt{\pi}$ are virtually Z;*
(iii) *if $r = 1$ and $A \not\leq \pi'$ then $M(K)'$ is a PD_3^+-complex, and is aspherical if and only if π' is a PD_3^+-group if and only if $e(\pi') = 1$;*
(iv) *if $r = 2$ then $A \cong Z^2$ and $M(K)$ is aspherical;*
(v) *if $r = 3$ then $A \cong Z^3$, $A \leq \pi'$ and $M(K)$ is homeomorphic to an infra-solvmanifold;*
(vi) *if $r = 4$ then $A \cong Z^4$ and $M(K)$ is homeomorphic to a flat 4-manifold.*

Proof. The four possibilities in case (i) correspond to whether π/A is finite or has one, two or infinitely many ends, by Theorem 15. These possibilities are mutually exclusive; if $e(\pi/A) = \infty$ then a Mayer-Vietoris argument implies that π cannot be a PD_4-group.

Suppose that $r = 1$, and $A \leq \zeta\pi'$. Then A is a module over $\boldsymbol{Z}[\pi/\pi'] \cong \Lambda$. On replacing A by a subgroup, if necessary, we may assume that A is cyclic as a Λ-module and is $\boldsymbol{Z}$-torsion free. If moreover $e(\pi/A) = \infty$ then $\sqrt{\pi}/A$ must be finite and $N = \pi'/A$ is not finitely generated. We may write N as an increasing union of finitely generated subgroups $N = \bigcup_{n \geq 1} N_n$. Let S be an infinite cyclic subgroup of A and let $G = \pi'/S$. Then G is an extension of N by A/S, and so is an increasing union $G = \bigcup G_n$, where G_n is an extension of N_n by A/S. If A is not finitely generated then A/S is an infinite abelian normal subgroup. Therefore if some G_n is finitely generated then it has one end, and so $H^1(G_n; F) = 0$ for any free $\boldsymbol{Z}[G_n]$-module F. Otherwise we may write G_n as an increasing union of finitely generated subgroups $G_n = \bigcup_{m \geq 1} G_{nm}$, where G_{nm} is an extension of N_n by a finite cyclic group Z/d_mZ, d_m divides d_{m+1} for all $m \geq 1$, and $A/S = \bigcup Z/d_mZ$. Let u be a generator of the subgroup Z/d_1Z, and let $\overline{G}_n = G_n/\langle u \rangle$ and $\overline{G}_{nm} = G_{nm}/\langle u \rangle$ for all $m \geq 1$. Then $\overline{G}_{n1} \cong N_n$, and so $\overline{G}_n \cong N_n \times (A/d_1^{-1}S)$. Since N_n is finitely generated and $A/d_1^{-1}S$ is infinite we again find that $H^1(\overline{G}_n; F) = 0$ for any free $\boldsymbol{Z}[\overline{G}]$-module F. Therefore $H^1(\overline{G}; F) = 0$ for any free $\boldsymbol{Z}[\overline{G}]$-module F, by Theorem I.9 of [**Hi94**]. An application of the LHSSS for π' as an extension of $\overline{G}$ by the normal subgroup $d_1^{-1}S \cong Z$ then gives $H^s(\pi'; \boldsymbol{Z}[\pi]) = 0$ for $s \leq 2$. Another LHSSS argument then gives $H^s(\pi; \boldsymbol{Z}[\pi]) = 0$ for $s \leq 2$ and so $M(K)$ is aspherical. As observed above, this contradicts the hypothesis $e(\pi/A) = \infty$.

Suppose next that $r = 1$ and A is not contained in π'. Let $x_1, \dots x_n$ be a set of generators for π and let s be an element of A which is not in π'. As each commutator $[s, x_i]$ is in $\pi' \cap A$ it has finite order, e_i say. Let $e = \Pi e_i$. Then $[s^e, x] = s^e(xs^{-1}x^{-1})^e = (sxs^{-1}x^{-1})^e$, so s^e commutes with all the generators. The subgroup generated by $\{s^e\} \cup \pi'$ has finite index in π and is isomorphic to $Z \times \pi'$, so π' is finitely presentable. Hence $M(K)'$ is an orientable PD_3-complex, by Theorem III.3 of [**Hi94**], and $M(K)$ is aspherical if and only if π' has one end, by Poincaré duality in $M(K)'$. (In particular, A is finitely generated.)

If $r = 2$ then $M(K)$ is aspherical by Theorem VI.11 of [**Hi94**], and has an abelian normal subgroup $A_1 \cong Z^2$. An LHSSS argument shows that $H^2(\pi/A_1; \boldsymbol{Z}[\pi/A_1]) \cong Z$, and so π/A_1 is virtually a surface group [**Bo99**]. Hence we also must have $A \cong Z^2$. If $r > 2$ then $r \leq 4$, $A \cong Z^r$ and $M(K)$ is homeomorphic to an infrasolvmanifold by Theorem VI.2 of [**Hi94**]. In particular, π is virtually poly-Z and $h(\pi) = 4$. If $r = 3$ then $A \leq \pi'$, for otherwise $h(\pi/\pi' \cap A) = 2$, which is impossible for a group with abelianization Z. If $r = 4$ then $[\pi : A] < \infty$ and so $M(K)$ is homeomorphic to a flat 4-manifold. □

The 2-knot groups with finite commutator subgroup are listed in Chapter V of [**Hi89**]. The group Φ is the group of Fox's Examples 10 and 11 [**Fo62**]. If K is an r-twist spin of a classical knot for some $r > 2$ then $\zeta\pi K \neq 1$ and $M(K)$ is aspherical. The final possibility of (i) is represented by Artin spins of torus knots. (The corresponding knot manifolds are not aspherical.) If $r = h(\sqrt{\pi}) = 2$ it can be shown that $M(K)$ is s-cobordant to a $\widetilde{\boldsymbol{SL}} \times \boldsymbol{E}^1$-manifold.

It remains an open question whether abelian normal subgroups of PD_n groups must be finitely generated. If this is so, Φ is the only 2-knot group with an abelian normal subgroup of positive rank which is not finitely generated.

The argument goes through with A a nilpotent normal subgroup. Can it be extended to the Hirsch-Plotkin radical? The difficulties are when $h(\sqrt{\pi}) = 1$ and $\pi/\sqrt{\pi}$ has one or infinitely many ends.

Corollary A. *If A has rank 1 its torsion subgroup T is finite, and if moreover π' is infinite and π'/A is finitely generated $T = 1$.*

The evidence suggests that if π' is finitely generated and infinite then A is free abelian. Little is known about the rank 0 case. All the other possibilities allowed by this theorem occur. In particular, if π is torsion free and $\pi' \cap A = 1$ then π' is a free product of PD_3^+-groups and free groups, and the various possibilities (π' finite, or having one or infinitely many ends) are realized by twists spins of classical knots. Is every 2-knot K such that $\zeta\pi \not\leq \pi'$ and π is torsion free s-concordant to a fibred knot?

Corollary B. *If π' is finitely generated then either π' is finite or $\pi' \cap A = 1$ or $M(K)$ is aspherical. If moreover $\pi' \cap A$ has rank 1 then $\zeta\pi' \neq 1$.*

Proof. As $\pi' \cap A$ is torsion free $\mathrm{Aut}(\pi' \cap A)$ is abelian. Hence $\pi' \cap A \leq \zeta\pi'$. □

If π' is FP_2 and $\pi' \cap A$ is infinite then π' is the fundamental group of an aspherical Seifert fibred 3-manifold. There are no known examples of 2-knot groups π with π' finitely generated but not finitely presentable.

We may construct examples of 2-knots π with $\zeta\pi' \cong Z$ as follows. Let N be a closed 3-manifold such that $v = \pi_1(N)$ has weight 1 and $v/v' \cong Z$, and let $w = w_1(N)$. Then $H^2(N; Z^w) \cong Z$. Let M_e be the total space of the S^1-bundle over N with Euler class $e \in H^2(N; Z^w)$. Then M_e is orientable, and $\pi_1(M_e)$ has weight 1 if $e = \pm 1$ or if $w \neq 0$ and e is odd. In such cases surgery on a weight class in M_e gives S^4, so $M_e \cong M(K)$ for some 2-knot K.

In particular, we may take N to be the result of 0-framed surgery on a classical knot. If the classical knot is 3_1 or 4_1 (i.e., is fibred of genus 1) then the resulting 2-knot group has commutator subgroup the nilpotent group with presentation $\langle x, y \mid [x, [x, y]] = [y, [x, y]] = 1\rangle$. For examples with $w \neq 0$ we may take one of the nonorientable surface bundles with group $\langle t, a_i, b_i \ (1 \leq i \leq n) \mid \Pi[a_i, b_i] = 1, ta_it^{-1} = b_i, tb_it^{-1} = a_ib_i \ (1 \leq i \leq n)\rangle$, where n is odd.

§7. 2-Knot groups with special HNN bases.

Since knot groups are finitely presentable and have infinite cyclic abelianization they are HNN extensions with finitely generated base and associated subgroups, and a knot group is restrained if and only if the HNN extension is ascending and the base is restrained. In all known cases a 2-knot K has a *minimal Seifert hypersurface*, i.e., there is an orientable submanifold V of S^4 such that $\partial V = K$ and all small normal displacements of V into $S^4 - V$ induce monomorphisms on fundamental groups. The knot group is then an HNN extension with finitely presentable base and associated subgroups.

Theorem 17. *Let K be a 2-knot with a minimal Seifert hypersurface, and such that $\pi = \pi K$ has an abelian normal subgroup A. Then $\pi' \cap A$ is finite cyclic or is torsion free, and $\zeta\pi$ is finitely generated.*

Proof. By assumption, $\pi = HNN(H; \phi : I \cong J)$ for some finitely presentable group H and isomorphism of ϕ of subgroups I and J, where $I \cong J \cong \pi_1(V)$ for some Seifert hypersurface V. Let $t \in \pi$ be the stable letter. Either $H \cap A = I \cap A$ or $H \cap A = J \cap A$ (by Britton's Lemma). Hence $\pi' \cap A = \bigcup_{n \in Z} t^n(I \cap A)t^{-n}$ is a monotone union. Since $I \cap A$ is an abelian normal subgroup of a 3-manifold group it is finitely generated [**Ga92**], and since V is orientable $I \cap A$ is torsion free or finite. If $I \cap A$ is finite cyclic or is central in π then $I \cap A = t^n(I \cap A)t^{-n}$, for all n, and so $\pi' \cap A = I \cap A$. (In particular, $\zeta\pi$ is finitely generated.) Otherwise $\pi' \cap A$ is torsion free. □

This argument derives from [**Yo92**], [**Yo97**], where it was shown that if A is a finitely generated abelian normal subgroup then $\pi' \cap A \leq I \cap J$.

Corollary. *Let K be a 2-knot with a minimal Seifert hypersurface. Then $\zeta\pi \cong 1$, $Z/2Z$, Z, $Z \oplus (Z/2Z)$ or Z^2.*

A 2-knot with a minimal Seifert hypersurface and such that $\zeta\pi = Z/2Z$ is constructed in [**Yo82**]. This paper also gives an example with $\zeta\pi \cong Z$, $\zeta\pi < \pi'$ and such that $\pi/\zeta\pi$ has infinitely many ends. The other possibilities are realized by fibred 2-knots (Artin spins and twist spins of the trefoil and figure-eight knots).

It is plausible that if K is a 2-knot such that $\pi = \pi K$ has an infinite restrained normal subgroup N then either π' is finite or $\pi \cong \Phi$ or $M(K)$ is aspherical and $\sqrt{\pi} \neq 1$ or N is virtually Z and π/N has infinitely many ends. This tetrachotomy was established in Theorems 15 and 16 under mild coherence hypotheses on N. We shall show it holds also under hypotheses only slightly stronger than requiring that K be an HNN extension with finitely presentable base and associated subgroups.

THEOREM 18. *Let K be a 2-knot whose group $\pi = \pi K$ is an ascending HNN extension over an FP_2 base H with finitely many ends. Then either π' is finite or $\pi \cong \Phi$ or $M(K)$ is aspherical.*

PROOF. Most of this theorem follows from Theorem 12. If π is virtually torsion free solvable of Hirsch length 2 then $\pi \cong \Phi$, by Theorem 14. □

COROLLARY. *If H is FP_3 and has one end then π' is a PD_3^+-group.*

PROOF. This follows from Theorem 3. □

Does this remain true if we assume only that H is FP_2 and has one end?

The class of groups considered in the next result probably includes all restrained 2-knot groups.

THEOREM 19. *Let π be a 2-knot group. Then the following are equivalent*:

(i) *π is restrained, locally FP_3 and locally virtually indicable*;

(ii) *π is an ascending HNN extension $H*_\phi$ where H is FP_3, restrained and virtually indicable*;

(iii) *π is elementary amenable and has an abelian normal subgroup of* rank > 0;

(iv) *π is elementary amenable and is an ascending HNN extension $H*_\phi$ where H is FP_2*;

(v) *π' is finite or $\pi \cong \Phi$ or π is torsion free virtually poly-Z and $h(\pi) = 4$.*

PROOF. Condition (i) implies (ii) by Corollary A of Theorem 12. If (ii) holds and H has one end then $\pi' = H$ and is a PD_3-group, by the Corollary to Theorem 18. Since H is virtually indicable and admits a meridianal automorphism, it must have a subgroup of finite index which maps onto Z^2. Hence H is virtually poly-Z, by the Corollary to Theorem 3. Hence (ii) implies (v). Conditions (iii) and (iv) imply (v) by Theorems 16 and 18, respectively. On the other hand (v) implies (i)–(iv). □

This is a first approximation to a "Tits alternative" for 2-knot groups.

THEOREM 20. *Let K be a 2-knot whose group $\pi = \pi K$ is an HNN extension with FP_2 base H and associated subgroups I and $\phi(I) = J$, and which has an infinite restrained normal subgroup N. Suppose either that the HNN extension is ascending or that $\beta_1^{(2)}(\pi) = 0$ and N is not locally finite. Then either π' is finite or $\pi \cong \Phi$ or $M(K)$ is aspherical or N is locally virtually Z and π/N has infinitely many ends.*

PROOF. Suppose first that the HNN extension is ascending. If H is finite or $N \cap H$ is infinite then H has finitely many ends (cf. the Corollary to Theorem I.9 of [**Hi94**]) and Theorem 18 applies. Therefore we may assume that H has infinitely many ends and $N \cap H$ is finite. But then $N \not\leq \pi'$, so π is virtually $\pi' \times Z$. Hence $\pi' = H$ and the infinite cyclic cover $M(K)'$ is a PD_3-complex. In particular $\pi' \cap N = 1$ and π/N has infinitely many ends.

Suppose next that $\pi' \cap N$ is locally finite, and let $G = \pi' N$. Then $[\pi : G] < \infty$, and G also is an HNN extension with FP_2 base and associated subgroups. Since $\pi' \cap N$ is locally finite this must be an ascending HNN extension, and so either π' is finite or $\pi \cong \Phi$ or $M(K)$ is aspherical, by Theorem 18. If $\pi' \cap N$ is locally virtually Z and $\pi/\pi' \cap N$ has two ends then π is elementary amenable and $h(\pi) = 2$, so $\pi \cong \Phi$. Otherwise we may assume that either $\pi/\pi' \cap N$ has one end or $\pi' \cap N$ has a finitely generated, one-ended subgroup. In either case $H^s(\pi; \boldsymbol{Z}[\pi]) = 0$ for $s \leq 2$, by Theorem 5, and so $M(K)$ is aspherical, by Eckmann's Theorem (see [**Hi97**]). $\square$

Note that $\beta_1^{(2)}(\pi) = 0$ if N is amenable.

We may also complete Yoshikawa's study of 2-knot groups which are HNN extensions with abelian base. (The first two paragraphs of the following proof outline the arguments of [**Yo86**], [**Yo92**].)

Theorem 21. *Let π be a 2-knot group which is an HNN extension with abelian base. Then either π is metabelian or it has a deficiency 1 presentation $\langle t, x \mid t x^n t^{-1} = x^{n+1} \rangle$ for some $n > 1$.*

Proof. Suppose that $\pi = HNN(A; \phi : B \to C)$, where A is abelian. Since π is finitely presentable and $\pi/\pi' \cong Z$ it is also an HNN extension with finitely generated base and associated subgroups [**BS78**]. Moreover we may assume the base is a subgroup of A. If A is not finitely generated considerations of normal forms with respect to the latter HNN structure imply that it must be ascending, and so π is metabelian [**Yo92**].

Suppose that A is finitely generated. Let j be the inclusion of B into A. Then $\phi - j : B \to A$ is an isomorphism, by the Mayer-Vietoris sequence for homology with coefficients $\boldsymbol{Z}$ for the HNN extension. Hence $\operatorname{rank}(A) = \operatorname{rank}(B) = r$, say, and the torsion subgroups of A, B and C coincide. In particular the image of the torsion subgroup of A in π is a finite normal subgroup N, and π/N is torsion free. Consider now the Mayer-Vietoris sequence for homology with coefficients Λ. Then $t\phi - j$ is injective and $\pi'/\pi'' \cong H_1(\pi; \Lambda)$ has rank r as an abelian group. Now $H_2(A; \boldsymbol{Z}) \cong A \wedge A$ (see page 334 of [**Ro82**]) and so $H_2(\pi; \Lambda) \cong \operatorname{Cok}(t \wedge_2 \phi - \wedge_2 j)$. Therefore $H_2(\pi; \Lambda)$ has rank $\binom{r}{2}$, and so $\binom{r}{2} \leq r$, since $H_2(\pi; \Lambda)$ is a quotient of $\operatorname{Ext}^1_\Lambda(H_1(\pi; \Lambda), \Lambda)$, by Hopf's Theorem and Poincaré duality. If $r = 0$ then clearly $B = A$ and so π is metabelian. Yoshikawa goes on to show that if $r = 3$ then $B = A$ also, while $r = 2$ is impossible. A similar argument using coefficients $\boldsymbol{F}_p\Lambda$ instead he shows that if $r = 1$ then N must be cyclic of odd order.

Thus we may assume that $A \cong Z \oplus N$ and $N \cong Z/\beta Z$ for some $\beta \geq 1$. The group π/N then has a presentation $\langle t, x \mid t x^n t^{-1} = x^{n+1} \rangle$ (with $n \geq 1$), and has one end. Let p be a prime. There is an isomorphism of the subfields $\boldsymbol{F}_p(X^n)$ and $\boldsymbol{F}_p(X^{n+1})$ of the rational function field $\boldsymbol{F}_p(X)$ which carries X^n to X^{n+1}. Therefore $\boldsymbol{F}_p(X)$ embeds in a skew field L containing an element t such that $t X^n t^{-1} = X^{n+1}$, by Theorem 5.5.1 of [**Co95**]. It is clear from the argument of this theorem that the group ring $\boldsymbol{F}_p[\pi/N]$ embeds as a subring of L, and so this group ring is weakly finite.

As $C_\pi(N)$ has finite index in π the associated covering space M_C is a closed 4-manifold, while $C_\pi(N)/N$ has one end and $\mathrm{g.d.}C_\pi(N)/N = 2$. As $\boldsymbol{F}_p[C_\pi(N)/N]$ is a subring of $\boldsymbol{F}_p[\pi/N]$ it is also weakly finite. Therefore we may apply Lemma 11 to conclude that N must be trivial, since a knot group cannot be virtually Z^2. Since π is metabelian if $n = 1$ this completes the proof. $\square$

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Jonathan A. Hillman
School of Mathematics and Statistics
The University of Sydney
Sydney, NSW 2006
Australia
E-mail: jonh@maths.usyd.edu.au