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The degree of symmetry of certain compact smooth manifolds

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Abstract. In this paper, we estimate the degree of symmetry and the semi-simple degree of symmetry of certain fiber bundles by virtue of the rigidity theorem with respect to the harmonic map due to Schoen and Yau. As a corollary of this estimate, we compute the degree of symmetry and the semi-simple degree of symmetry of certain product manifolds. In addition, by Albanese map, we estimate the degree of symmetry and the semi-simple degree of symmetry of a compact smooth manifold under some topological assumptions.

1. Introduction.

Let M^n be a compact connected smooth n-manifold and $N(M^n)$ the *degree of symmetry* of M^n, that is, the maximum of the dimensions of the isometry groups of all possible Riemannian metrics on M^n. (All the manifolds of this paper are to be compact and smooth.) Of course, $N(M)$ is the maximum of the dimensions of the compact Lie groups which can act effectively and smoothly on M. The following is well known:

$$N(M^n) \leq n(n+1)/2. \tag{1}$$

In addition, if the equality holds, then M^n is diffeomorphic to the standard sphere S^n or the real projective space $\boldsymbol{RP}^n$. In [**10**] H. T. Ku, L. N. Mann, J. L. Sicks and J. C. Su obtained similar results on a product manifold $M^n = M_1^{n_1} \times M_2^{n_2}$ $(n \geq 19)$ where M_i is a compact connected smooth manifold of dimension n_i: they showed that

$$N(M) \leq n_1(n_1+1)/2 + n_2(n_2+1)/2, \tag{2}$$

and that if the equality holds, then M^n is a product of two spheres, two real projective spaces or a sphere and a real projective space. A preliminary lemma for the proof of Ku-Mann-Sicks-Su's results claims that if M^n $(n \geq 19)$ is a compact connected smooth n-manifold which is not diffeomorphic to the complex projective space $\boldsymbol{CP}^m$ $(n = 2m)$, then

$$N(M^n) \leq k(k+1)/2 + (n-k)(n-k+1)/2 \tag{3}$$

holds for each $k \in \boldsymbol{N}$ such that the k-th Betti number b_k of M is nonzero. Then we see that when a compact oriented smooth manifold M of dimension $4m \geq 20$ has nonzero signature $\sigma(M)$, then the following holds:

$$N(M) \leq N(\boldsymbol{CP}^{2m}) = 4m(m+1). \tag{4}$$

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The equality in (4) was also showed by Ku-Mann-Sicks-Su in [**10**]. The *semi-simple degree of symmetry* $N_s(M)$ [**4**] is defined similarly, where we consider only actions of semi-simple compact Lie groups on M. It is clear that (1) also holds for the semi-simple degree of symmetry if the manifold has dimension ≥ 2. Moreover, since the semi-simple compact Lie group $SU(2m+1)$ acts naturally on $\boldsymbol{CP}^{2m}$, we can see that

$$N_s(\boldsymbol{CP}^{2m}) = 4m(m+1)$$

holds provided $4m \geq 20$. D. Burghelea and R. Schultz [**4**] showed $N_s(M^n) = 0$ if there exist $\alpha_1, \dots, \alpha_n$ in $H^1(M^n; \boldsymbol{R})$ with $\alpha_1 \cup \cdots \cup \alpha_n \neq 0$.

In Ku-Mann-Sicks-Su's estimates (2), (3) and Burghelea-Schultz's result they mainly assumed conditions on the topology of M^n. When considering if there exists a non-trivial S^1-action or S^3-*action* on a manifold, we often meet obstructions from its differential structure. Here a nontrivial S^3-action [**11**] on a manifold means an effective and smooth S^3-action or $SO(3)$-action on it. Let us see some examples as follows:

A *spin manifold* [**11**] is an oriented Riemannian manifold with a spin structure on its tangent bundle. A famous theorem of M. Atiyah and F. Hirzebruch [**2**] claims that a spin manifold has degree of symmetry 0 if the index of the Dirac operator on it, or equivalently, its $\hat{A}$-genus, is nonzero. Let X be a spin manifold of dimension $8q+1$ (resp. $8q+2$), Atiyah and Singer [**3**] showed that the real dimension (resp. complex dimension) (mod 2) of the space of harmonic spinors on X can be identified with a certain *KO*-characteristic number $\alpha(X)$ of the spin-cobordism class of X. Let Θ_n be the group of homotopy n-spheres. This *KO*-characteristic number was shown by Milnor and Adams [**1**] [**15**] to give a nontrivial homomorphism $\alpha : \Theta_n \to \boldsymbol{Z}_2$ for $n = 8q+1$ or $8q+2$. Since for $n = 8q+1$ or $8q+2$, the homotopy n-spheres which bound spin manifolds form a subgroup $BSpin_n$ of index 2 in Θ_n, we see that $\mathrm{Ker}\, \alpha = BSpin_n$. For the α-invariant is additive with respect to connected sums of manifolds, it is always possible to change the differentiable structure of a spin manifold X, in dimension $8q+1$ or $8q+2$, to make $\alpha(X)$ nonzero. It follows from Lawson and Yau [**13**] that if $\alpha(X)$ is nonzero, then there exists no nontrivial smooth effective S^3 action on X, or equivalently, the only compact, connected effective transformation groups on X are tori, from which the followings hold:

$$N(X) \leq \dim X, \quad N_s(X) = 0. \tag{5}$$

Definition 1.1. We call a manifold *significant* if and only if it is oriented and has nonzero signature. A manifold is said to be $\hat{A}$-*nontrivial* if and only if it is spin and has nonzero $\hat{A}$-genus. A manifold X is said to be α-*nontrivial* if and only if it is spin, of dimension $8q+1$ or $8q+2$, and $\alpha(X) \neq 0$, where q may be zero.

Definition 1.2. We call a manifold S^3-*trivial* if and only if there exists no smooth and effective S^3-action on it, or equivalently, its semisimple degree of symmetry is zero.

Remark 1.1. Both $\hat{A}$-nontrivial manifolds and α-nontrivial manifolds are S^3-trivial. Lawson and Yau [**13**] showed that if a compact manifold does not admit a Riemannian metric of positive scalar curvature, then it is S^3-trivial.

One of the purposes of this paper is to make some estimate for certain nontrivial compact fiber bundles and generalize partially Ku-Mann-Sicks-Su's estimates (2). In particular, we obtain their bundle versions for (4) and the results by Atiyah-Hirzebruch and Lawson-Yau when taking the special fibers in Definition 1.1.

Theorem 1.1. *Let V be a compact manifold which can be equipped with a real analytic metric of nonpositive curvature and E a compact smooth fiber bundle over V such that the fiber F of E is connected. Then the followings hold*:

$$N(E) \leq \dim F(\dim F+1)/2 + N(V), \quad N_s(E) \leq \dim F(\dim F+1)/2. \tag{6}$$

Particularly,

(i) *suppose E is oriented and F is a significant manifold of dimension ≥ 20. Then the following holds*:

$$N(E) \leq \dim F(\dim F+4)/4 + N(V), \quad N_s(E) \leq \dim F(\dim F+4)/4. \tag{7}$$

(ii) *Suppose E is spin and F is an $\hat{A}$-nontrivial manifold. Then E is S^3-trivial and the following holds*:

$$N(E) \leq N(V). \tag{8}$$

(iii) *Suppose E is spin and F is an α-nontrivial manifold. Then E is S^3-trivial and the following holds*:

$$N(E) \leq \dim F + N(V). \tag{9}$$

(iv) *Suppose Σ^n is an exotic n-sphere which does not bound a spin manifold and V is spin. Then $\Sigma^n \times V$ is not diffeomorphic to $S^n \times V$.*

Remark 1.2. By a result in [**12**] we know the dimension of isometry group of V is rank of the center of $\pi_1(V)$. On the other hand from [**5**] we know that if a compact connected Lie group acting smoothly and effectively on a compact aspherical manifold A, then it is a torus of dimension $\leq$ rank of the center of $\pi_1(A)$. Combining these two results, we immediately see the degree of symmetry of V is equal to rank of the center of $\pi_1(V)$.

Remark 1.3. In Theorem 1.1, V cannot be replaced by an arbitrary compact manifold because the Hopf bundle $S^1 \to S^3 \to S^2$ forms a counterexample.

Remark 1.4. Let T^2 be a two dimensional torus and K a Klein bottle. Then $N(T^2) = 2$ and $N(K) = 1$ hold. Therefore we see that the connectivity of fiber F is necessary for the first inequality in (6) in Theorem 1.1.

By the definition of degree of symmetry, it is easy to see that for a product manifold $M_1 \times M_2$, where M_i is a compact connected smooth manifold, the following holds:

$$N(M_1 \times M_2) \geq N(M_1) + N(M_2). \tag{10}$$

Combining (6), (7) and (8) with (10), we immediately obtain the following.

Corollary 1.1. *Let V be a compact manifold which can be equipped with a real analytic metric of nonpositive curvature. Then the followings hold*:

$$N(S^n \times V) = N(S^n) + N(V), \quad N_s(S^n \times V) = N_s(S^n).$$

Suppose V is oriented. Then the equalities

$$N(\boldsymbol{CP}^{2m} \times V) = N(\boldsymbol{CP}^{2m}) + N(V), \quad N_s(\boldsymbol{CP}^{2m} \times V) = N(\boldsymbol{CP}^{2m})$$

hold provided $4m \geq 20$. *Moreover if V is spin and X is $\hat{A}$-nontrivial, then the following holds*:

$$N(X \times V) = N(V).$$

Remark 1.5. Corollary 1.1 shows the estimates (6), (7) and (8) in Theorem 1.1 are sharp for bundles with fibers as sphere S^n, complex projective space $\boldsymbol{CP}^{2m}$ $(m \geq 5)$ and $\hat{A}$-nontrivial manifold respectively.

From (3) we see that if M^n $(n \geq 19)$ is a compact connected smooth n-manifold with nonzero first Betti number, then the following holds:

$$N(M^n) \leq n(n-1)/2 + 1.$$

The other of the purposes of this paper is to refine this inequality and Burghelea-Schultz's result:

Theorem 1.2.* *Let M be an n-dimensional compact smooth manifold with nonzero first Betti number b_1.*

(i) *Suppose that there exist k one-dimensional real cohomology classes $\alpha_1, \dots, \alpha_k$ of M such that their cup product $\alpha_1 \cup \cdots \cup \alpha_k$ does not vanish in the k-dimensional real cohomology group $H^k(M; \boldsymbol{R})$. Then the followings hold*:

$$N(M) \leq (n-k+1)(n-k)/2 + k, \quad N_s(M) \leq (n-k+1)(n-k)/2.$$

(ii) *Let i be* 1 *or* 2. *If $b_1 \geq i$, then the followings hold*:

$$N(M) \leq (n-i+1)(n-i)/2 + i, \quad N_s(M) \leq (n-i+1)(n-i)/2.$$

Particularly, if $n - i = 1$, then $N_s(M) = 0$.

(iii) *Suppose $b_1 \geq 3$. Then the following holds*:

$$N(M) \leq \begin{cases} (n-2)(n-1)/2, & \text{if } n \geq 5, \\ 4, & \text{if } n = 4. \end{cases} \tag{11}$$

This paper is organized as follows. In Section 2, we prepare for the following sections. Particularly, if there exists a nontrivial harmonic map from a compact Riemannian manifold M to a compact manifold of nonpositive curvature, we can estimate the dimension of isometry group of M from above (cf. Lemma 2.2 and Lemma 2.3). In Section 3, from the assumptions in Theorem 1.1, we show the nontriviality of the harmonic map homotopic the fibration map from E to V and prove Theorem 1.1 by the cobordism theory. In Section 4, we show the nontriviality of the Albanese map from M to a $b_1(M)$-dimensional flat torus and prove Theorem 1.2.

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*The author has announced Theorem 1.2 in [**20**].

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2. Preliminaries.

2.1. The isometry group of a Riemannian manifold and the harmonic map due to Schoen and Yau.

For a compact Riemannian manifold M let $I(M)$, $I^0(M)$ be the isometry group of M and its identity component respectively. The following is known:

Proposition 2.1 (cf. Theorem 4 in [**19**]). *Suppose M, N are compact real analytic Riemannian manifolds and N has nonpositive sectional curvatures. Suppose $h: M \to N$ is a surjective harmonic map and its induced map $h_*: \pi_1(M) \to \pi_1(N)$ is also surjective. Then the space of surjective harmonic maps homotopic to h is represented by $\{\beta \circ h \mid \beta \in I^0(N)\}$.*

Lemma 2.1. *Let M^m be a connected Riemannian manifold and f a smooth map from it to a smooth manifold N^n. Suppose $y \in N$ is a regular point of f and F is a connected component of the submanifold $f^{-1}(y)$. If an isometry α of M satisfies that $h \circ \alpha = h$ and that*

$$\alpha(x) = x \quad \textit{for any } x \in F,$$

then α is the identity map of M.

Proof. Restricting the differential $d\alpha$ of the isometry α on the tangent bundle TF of F, we obtain the identity map. $h \circ \alpha = h$ implies that the restriction of $d\alpha$ on the normal bundle of TF in the tangent bundle TM of M is also the identity map. Hence the map $d\alpha: TM|_F \to TM|_F$ is the identity. By the connectivity of M, we can see that the isometry α is the identity map of M. $\square$

Lemma 2.2. *Under the hypotheses of Proposition 2.1 it follows that*

$$\dim I^0(M) \leq (m-n+1)(m-n)/2 + \dim I^0(N),$$

where $m = \dim M$ and $n = \dim N$.

Proof. Taking an element $\alpha \in I^0(M)$, we obtain a surjective harmonic map $h \circ \alpha$ homotopic to h. By Proposition 2.1 and the surjectivity of h, there exists a unique $\rho(\alpha) \in I^0(N)$ such that $h \circ \alpha = \rho(\alpha) \circ h$. We see that $\rho: I^0(M) \to I^0(N)$ is a homomorphism. The proof is completed if we can show that $\operatorname{Ker}\rho$, which acts smoothly and effectively on M, has dimension $\leq (m-n+1)(m-n)/2$. Taking a regular value $y \in V$ of h, since h is surjective we see that $h^{-1}(y)$ is an $(m-n)$-dimensional submanifold of M. Let F be an arbitrary connected component of $h^{-1}(y)$. By the definition of ρ and Lemma 2.1, $\operatorname{Ker}\rho$ acts effectively on F and then has dimension $\leq (m-n+1)(m-n)/2$.

2.2. The isometry group of a Riemannian manifold and the Albanese map.

For a compact oriented Riemannian manifold M with nonzero first Betti number $b_1(M)$, let $\mathscr{H}$ be the real vector space of all harmonic 1-forms on M and v the natural projecion from the universal covering $\tilde{M}$ of M. For $x_0 \in \tilde{M}$, set $p_0 = v(x_0)$. We define a smooth map $\tilde{a} : \tilde{M} \to \mathscr{H}^*$ from $\tilde{M}$ to the dual space $\mathscr{H}^*$ of $\mathscr{H}$ by a line integral

$$\tilde{a}(x)(\omega) = \int_{x_0}^{x} v^*\omega.$$

For $\sigma \in \pi_1(M)$

$$\tilde{a}(\sigma x) = \tilde{a}(x) + \psi(\sigma)$$

holds, where $\psi(\sigma)(\omega) = \int_{x_0}^{\sigma x_0} v^*\omega$, so that ψ is a homomorphism from $\pi_1(M)$ into $\mathscr{H}^*$ as an additive group. It is a fact that $\Delta = \psi(\pi_1(M))$ is a lattice in the vector space $\mathscr{H}^*$, and clearly this vector space has a natural Euclidean metric from the global inner prduct of forms on M. With the quotient metric, we call the torus $A(M) = \mathscr{H}^*/\Delta$ the *Albanese torus* of Riemannian manifold M. By the above relation between $\tilde{a}$ and ψ, we obtain a map $a : M \to A(M)$ satisfying $\tilde{a}(x) \in a \circ v(x)$ for any $x \in \tilde{M}$. We call the map a the *Albanese map*. From the very construction of a, we see that the map it induces on fundamental groups

$$a_* : \pi_1(M) \to \pi_1(A(M))$$

is surjective and that a^* maps the space of harmonic 1-forms on $A(M)$ isomorphically onto $\mathscr{H}$. By Corollary 1 in [**17**], the Albanese map is harmonic. We shall prove

Lemma 2.3. *Let M be an n-dimensional oriented compact Riemannian manifold and $a : M \to A(M)$ its Albanese map. Let da denote the differential of a and set*

$$r_a := \max\{\operatorname{rank} da(p) \mid p \in M\}.$$

Then $\dim I^0(M) \le (n - r_a + 1)(n - r_a)/2 + r_a$.

Proof. For any $\gamma \in I^0(M)$, $a \circ \gamma$ is also a harmonic mapping from M to the Albanese torus $A(M)$ and homotopic to a. By Lemma 3 in [**17**] there is a unique translation $\rho(\gamma)$ of the torus $A(M)$ such that

$$a \circ \gamma = \rho(\gamma) \circ a.$$

Then we have a homomorphism $\rho : I^0(M) \to T^{b_1}$, where the torus T^{b_1} is the translation group of Albanese torus $A(M)$. Similarly to the proof of Lemma 2.2, we can show that $\operatorname{Ker}\rho$ has dimension $\le (n - r_a + 1)(n - r_a)/2$. As a subgroup of the translation group of $A(M)$, $\operatorname{Im}\rho$ acts freely on the image of a so that $\operatorname{Im}\rho$ has dimension $\le \dim a(M) = r_a$. The proof is completed. $\square$

3. Proof of Theorem 1.1.

We firstly prove a topological result on fiber bundles.

Proposition 3.1. *Let $p_0 : E \to B$ be a fiber bundle over a compact connected smooth manifold B such that the fiber of E is also connected. Suppose $p_1 : E \to B$ is a continuous map homotopic to p_0. Then p_1 is surjective.*

Proof. The proof is an application of the Serre spectral sequence. Suppose there exists a point $x \in B$ such that the image of $p_1 : E \to B$ lies in the space $B' := B - \{x\}$. Then the composition of $p_1 : E \to B$ with the inclusion $i : B' \to B$ is homotopic to the projection $p_0 : E \to B$. It is known that there exists a fibration (in the sense of Serre) $p_2 : E' \to B'$ and a map $f : E' \to E$ such that f is a homotopy equivalence and the compostion $p_1 \circ f$ is homotopic to p_2. Let F and F' be the homotopy fibers and $\mathscr{H}_a^*$, $\mathscr{H}_b^*$ the Serre local systems of the fiber bundles $p_0 : E \to B$ and $p_2 : E' \to B'$ respectively. Then for these two fiber bundles we have two spectral sequences

$$\text{(a):}\quad E_2^{p,q} = H^p(B; \mathscr{H}_a^q) \to H^*(E)$$

and

$$\text{(b):}\quad E_2^{p,q} = H^p(B'; \mathscr{H}_b^q) \to H^*(E')$$

respectively, where we use cohomology groups with coefficients $\boldsymbol{Z}/2\boldsymbol{Z}$. Since $f : E' \to E$ is a fiber bundle map in the sense of homotopy over the map $i : B' \to B$, we also have a natural map between the spectral sequences $\phi : \text{(a)} \to \text{(b)}$. For $n = \dim B$ and $k = \dim F$, we compare the $E_\infty^{n,k}$-terms of (a) and (b) by the map ϕ. The $E_\infty^{n,k}$-term of (a) is $\boldsymbol{Z}/2\boldsymbol{Z}$, which is naturally isomorphic to $H^{n+k}(E)$. The $E_\infty^{n,k}$-term of (b) is 0, whose proof is put in the next paragragh. Hence the map ϕ on the $E_\infty^{n,k}$-term is 0, which implies that the natural map $f_* : H^{n+k}(E) \to H^{n+k}(E')$ is trivial. This contradicts that $f : E \to E'$ is a homotopy equivalence.

Now we prove that the $E_\infty^{n,k}$-term of (b) is 0. We have only to show that the $E_2^{n,k}$-term of (b) is 0, which is just a special case of the equality

$$H^n(B'; \mathscr{S}) = 0$$

for any local system $\mathscr{S}$ with coefficients $\boldsymbol{Z}/2\boldsymbol{Z}$ over B'. Since B is a smooth compact connected manifold of dimension n, we claim that B has a cell structure with only one n-cell. In fact if taking a decomposition of B by polyhedra such that the number of the n-cells is minimum, then we see that the number of the n-cell is one. Otherwise, since B is connected, then there exist two n-cells which have a common $(n-1)$-dimenional cell. Deleting the common $(n-1)$-cell, we obtain another decomposition of B by polyhedra with less number of n-cells. Contradictions! It is clear that $B' = B - \{x\}$ has the homotopy type of the $(n-1)$-skeleton B^{n-1} of this decomposition. Calculating $H^*(B^{n-1}; \mathscr{S})$ by this cell decompostion, we immediately see $H^n(B^{n-1}; \mathscr{S}) = 0$ and complete the proof. □

Lemma 3.1. *If a real projective space $\boldsymbol{RP}^n$ is spin, then it bounds a spin manifold.*

Proof. The total Stiefel-Whitney class of $\boldsymbol{KP}^n$ is $w = 1 + w_1 + w_2 + \cdots = (1+g)^n$ where g, the generator of the cohomology ring of $\boldsymbol{KP}^n$, has dimension 1 and 2 for $\boldsymbol{K} = \boldsymbol{R}$ and $\boldsymbol{C}$ respectively. When $\boldsymbol{K} = \boldsymbol{R}$, the condition $w_1 = 0$, $w_2 = 0$ is equivalent to $(n+1) \equiv n(n+1)/2 \equiv 0 \pmod 2$. That is to say that $\boldsymbol{RP}^n$ is spin if and only if $n \equiv 3 \pmod 4$. It is obvious that $\boldsymbol{CP}^{\text{odd}}$ is spin. Let n be $4k+3$ and L the tautological line bundle on CP^{2k+1}. $L \otimes L$ is spin since its first Chern class is the twice of that of L. Then we see the total space of the disk bundle of L^2 is spin and has the boundary $\boldsymbol{RP}^n$. □

Proof of Theorem 1.1. We remark that if a compact Lie group G acts smoothly on a compact manifold M, then there exists an analytic structure on M such that G acts on it analytically. In fact by Theorem 4.7.1 in [**9**] there exists an analytic embedding of M to Euclidean space. The remark follows by that there exists an equivariantly analytic embedding of M to a representation space of G, which can be proved similar to Theorem 4.7.1 in [**9**]. Hence the degree of symmetry of E is also equal to the maximum of the dimensions of the isometry groups of all real analytic Riemannian metrics on E.

For the proof of the first inequality in (6), we have only to show that for any real analytic Riemannian metric on E, the inequality

$$\dim I^0(E) \leq \dim F(\dim F + 1)/2 + N(V)$$

holds. The projection $p : E \to V$ of the fiber bundle induces a surjective map from $\pi_1(E)$ to $\pi_1(V)$ since the fiber F is connected. Using a well-known result by J. Eells and J. Sampson [**7**], we see that there exist harmonic maps homotopic to the projection $p : E \to V$. By Proposition 3.1, we see that each of them is surjective. Combining Remark 1.2 and Lemma 2.2, we obtain the above inequality. For the proof of the second inequality in (6), we have only to show that for a semi-simple compact Lie group G which acts isometrically on the analytic Riemannian manifold E, the following estimate holds:

$$\dim G \leq \dim F(\dim F + 1)/2. \tag{12}$$

Using a harmonic map homotopic to the projection $p : E \to V$, we can construct a homomorphism $\rho : G \to I^0(V)$ by the same way as the proof of Lemma 2.2. Since any Lie group homomorphism from G to a torus is trivial, it is followed that

$$\dim G = \dim \operatorname{Ker} \rho.$$

We obtain (12) by the estimate of $\dim \operatorname{Ker} \rho$ in the proof of Lemma 2.2.

For the proof of (i), (ii) and (iii) of Theorem 1.1, we also have only to prove corresponding results for the isometry group with respect to any analytic Riemannian metric on E. Let h be a harmonic map homotopic to the projection $p : E \to V$. We recall the homomorhism $\rho : I^0(E) \to I^0(V)$ constructed from the harmonic map h. Taking a regular point $y \in V$ of h, and a connected component F' of $h^{-1}(y)$, we have showed in the proof of Lemma 2.2 that $\operatorname{Ker} \rho$ acts effectively on F'.

We claim that there exists a cobordism in E between F' and F. In fact, choosing generic smooth homotopy $P : E \times [0,1] \to V$ between h and p such that y is also the regular value of P, we can see that $P^{-1}(y) = \mathscr{F}$ is a submanifold of $E \times [0,1]$ with boundary F' and F. Since the normal bundle of $\mathscr{F}$ in $E \times [0,1]$ is trivial, $\mathscr{F}$ is oriented if E is oriented and it is spin if E is spin. It implies that F and F' are oriented cobordant if E is oriented and they are spin cobordant if E is spin. Since signature, $\hat{A}$-genus and KO-characteristic number are invariants of oriented cobordism, spin cobordism respectively, if F is significant, $\hat{A}$-nontrivial or α-nontrivial, so is F'. Although F' may be not connected, we see that there exists a connected component F^* of F' which has the significant, $\hat{A}$-nontrivial or α-nontrivial property if F' does. For $\operatorname{Ker} \rho$ acts effectively on F^*, if F^* is significant, we can estimate its dimension by (4) and then obtain the first inequality of (7). The second inequality of (7) follows similarly to (6). If F^* is $\hat{A}$-nontrivial, by Atiyah-Hirzebruch's theorem we see that $\operatorname{Ker} \rho$ is trivial and (ii) of Theorem 1.1 follows. The proof of (iii) of Theorem 1.1 is completed by Lawson-Yau's result (cf. (5)).

Finally in the above argument taking the product manifold $\Sigma^n \times V$ as a trivial bundle on V, we can find a connected component F' of $h^{-1}(y)$ which is spin and does not bound a spin manifold. Lemma 3.1 tells us that if a real projective space is spin, then it must bound a spin manifold. Then we see that F' is diffeomorphic to neither a sphere S^n nor a real projective space $\boldsymbol{RP}^n$. Since $\operatorname{Ker}\rho$ acts effectively on F', it follows by (1) that

$$\dim \operatorname{Ker}\rho \leq N(F') < N(S^n) = n(n+1)/2.$$

Hence we obtain that $N(\Sigma^n \times V) < N(S^n \times V)$, which completes the proof of (iv). $\square$

4. Proof of Theorem 1.2.

In order to prove Theorem 1.2, we need lemmas.

LEMMA 4.1. *Let $\pi : M' \to M$ be a finite covering between compact smooth manifolds. Then we have $N(M) \leq N(M')$.*

PROOF. We can assume that $\pi : \tilde{M} \to M$ is a Riemannian covering. It is enough to show $\dim I(M) \leq \dim I(M')$. Since $I(M)$ and $I(M')$ are Lie groups of finite dimension, we only need to compare the dimensions of their Lie algebras. Given a Killing vector field V on M, the pullback of V by π is also a Killing field on M' so that the Lie algebra of $I(M)$ is a subalgebra of that of $I(M')$. $\square$

Let $\pi : \tilde{X} \to X$ be an n-sheeted covering space defined by an action of group Γ on $\tilde{X}$. Then (cf. [**8**], Proposition 3H.1) with coefficients in a field F whose characteristic is 0 or a prime not dividing n, the map $\pi^* : H^k(X;F) \to H^k(\tilde{X};F)$ is injective with image the subgroup $H^k(\tilde{X};F)^{\Gamma}$ consisting of classes α such that $\gamma^*(\alpha) = \alpha$ for all $\gamma \in \Gamma$. In particular, we see

LEMMA 4.2. *Let M be a non-orientable compact manifold, $\pi : M' \to M$ its orientable double covering. Then*

(1) $b_1(M) \leq b_1(M')$;

(2) *If M has the property that there exist k one dimensional real cohomology classes $\alpha_1, \ldots, \alpha_k$ of M such that $\alpha_1 \cup \cdots \cup \alpha_k$ is nonzero in $H^k(M;\boldsymbol{R})$, then so does M'.*

LEMMA 4.3. *Let M be an n-dimensional oriented compact Riemannian manifold with nonzero first Betti number b_1. Let $a : M \to A(M)$ be its Albanese map.*

(1) *Suppose there exist k integral one dimensional real cohomology classes $\alpha_1, \ldots, \alpha_k$ such that $\alpha_1 \cup \cdots \cup \alpha_k$ does not vanish in $H^k(M;\boldsymbol{R})$. Then $r_a \geq k$ holds.*

(2) *Let r be 1 or 2. If $b_1 \geq r$, then $r_a \geq r$ holds.*

PROOF. (1) By the assumption, the Albanese map a of M induces a non-trivial homomorphism $a^* : H^k(A(M);\boldsymbol{R}) \to H^k(M;\boldsymbol{R})$, which implies there exists a nonzero k-form ω on $A(M)$ such that its pullback $a^*(\omega)$ is also a nonzero k-form. Since r_a is equal to

$$\max\{j \mid a^* : \Omega^j(A(M)) \to \Omega^j(M) \text{ is not identically zero}\},$$

we see $r_a \geq k$.

(2) In case of $r=1$ this statement is obvious. When $b_1 \geq 2$, it is implied by the general unique continuation property of harmonic mappings (cf. Theorem 3 in [**18**]). In fact, if the maximal rank of da is 1, a maps M onto a closed geodesic of $A(M)$ since $a: M \to A(M)$ is harmonic. This contradicts surjectivity of the homomorphism $a_*: \pi_1(M) \to \pi_1(A(M)) \cong \boldsymbol{Z}^{b_1}$. □

Finally we arrive at the proof of Theorem 1.2.

Proof of Theorem 1.2. By Lemma 4.1 together with Lemma 4.2, we may assume M is an oriented Riemannian manifold and let $a: M \to a(M)$ denote its Albanese map. We omit the proof of the estimates for semi-simple degree of symmetry here since it is similar to that of the second inequality in (6).

From Lemma 2.3 together with Lemma 4.3, we see that one of the upper bounds of $\dim I(M)$ is

$$\max\left\{\frac{1}{2}(n-j+1)(n-j)+j \,|\, j=k,k+1,\dots,n\right\},$$

which is equal to $(n-k+1)(n-k)/2+k$. Hence we obtain (i) of Theorem 1.2.

By Lemma 2.3 and Lemma 4.3, we obtain (ii) of Theorem 1.2.

For the proof of (iii) of Theorem 1.2, we have only to consider the analytic Riemannian metric on M. Since the first Betti number b_1 is not less than 3, we see by Lemma 4.3 that $r_a \geq 2$ holds. If $r_a \geq 3$, then from Lemma 2.3 we know

$$\dim I(M) \leq \frac{1}{2}(n-2)(n-3)+3.$$

Suppose $r=2$. We recall the homomorphism ρ from $I^0(M)$ to the translation group T^{b_1} of $A(M)$ constructed in the proof of Lemma 2.3. We claim that the homomorphism ρ is trivial so that

$$\dim I^0(M) = \dim \operatorname{Ker} \rho \leq \frac{1}{2}(n-1)(n-2).$$

Otherwise, there is a translation group S^1 acting freely and isometrically on the image of a. Since both M and $A(M)$ are real analytic, a theorem of Morrey [**16**] shows that the harmonic mapping a is in fact real analytic. By well-known theorems in real analytic geometry [**14**] we know that both M and $A(M)$ can be triangulated so that $a(M)$ is a 2-dimensional compact connected simplicial subcomplex of $A(M)$. We write the orbit space of the free and isometric S^1 actions on $A(M)$ and $a(M)$ by $A(M)/S^1$ and $a(M)/S^1$ respectively, in which the former is in fact also a flat torus of dimension b_1-1. Since the natural projection map $\pi: A(M) \to A(M)/S^1$ is totally geodesic, we see that by a result in [**6**] the composition map $\pi \circ a: M \to A(M)/S^1$ is a harmonic map, whose image is $a(M)/S^1$, the orbit space of the free S^1 action on the two dimensional simplicial subcomplex $a(M)$ of $A(M)$. Hence $a(M)/S^1$, the image of $\pi \circ a$ in $A(M)/S^1$ has dimensional 1 so that the differential of harmonic map $\pi \circ a$ has rank ≤ 1 at any point of M. By Theorem 3 in [**18**], we see that $\pi \circ a$ maps M onto a closed geodesic of $A(M)/S^1$, which means that $a(M)$ is a 2-dimensional torus.

This contradicts the surjectivity of the homomorphism $a_* : \pi_1(M) \to \pi_1(A(M)) \cong \boldsymbol{Z}^{b_1}$ ($b_1 \geq 3$). Hence we obtain

$$\dim I(M) \leq \max\left\{\frac{1}{2}(n-3)(n-2)+3, \frac{1}{2}(n-1)(n-2)\right\},$$

which implies (iii) of Theorem 1.2. □

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