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# Delta link homotopy for two component links, III

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**Abstract.** In this note, we will study Delta link homotopy, which is an equivalence relation of ordered and oriented link types. Previously, a necessary condition was given by a pair of numerical invariants derived from the Conway polynomials for two link types to be Delta link homotopic. In this note, we will show that, for two component links, if their pairs of numerical invariants coincide then the two links are Delta link homotopic.

## 1. Introduction.

For two link diagrams $K$ and $L$ which differ only in one place as in Figure 1.1.1, a local move between $K$ and $L$ is called a $\Delta$ move. Furthermore, for two links $\kappa$ and $\lambda$ represented by $K$ and $L$, $\kappa$ and $\lambda$ are said to be transformed into each other by a $\Delta$ move.

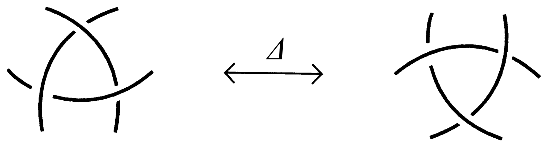

Figure 1.1.1.

It is known the following result by Matveev [**M**] and by Murakami and the first author [**MN**].

PROPOSITION 1. *Two knots* (*or links*) *can be transformed into each other by a finite sequence of $\Delta$ moves if and only if the two knots* (*or links*) *have the same number of components*, *and*, *for properly chosen orders and orientations*, *they have the same linking numbers between the corresponding components*.

In the case that all arcs illustrated in the Figure 1.1.1 are contained in the same component, the above move is called a *self $\Delta$ move* (cf. [**S**]). For two links $\kappa$ and $\lambda$, if $\kappa$ can be transformed into $\lambda$ by a finite sequence of self $\Delta$ moves, $\kappa$ and $\lambda$ are said to be *$\Delta$ link homotopic* (or *self $\Delta$-equivalent*). For a $\mu$-component link $\kappa = \kappa_1 \cup \cdots \cup \kappa_\mu$, let $\delta_1(\kappa) = a_{\mu-1}(\kappa)$, and $\delta_2(\kappa) = a_{\mu+1}(\kappa) - a_{\mu-1}(\kappa) \times (\sum_{i=1}^{\mu} a_2(\kappa_i))$, for the coefficient $a_i(\kappa)$ of the term $z^i$ in the Conway polynomial of $\kappa$. It is known the following result by the preceding work [**N**], which is a generalization of a work of Okada in [**O**].

PROPOSITION 2. *If two links are $\Delta$ link homotopic*, *then their $\delta_2$ coincide*.

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*Key Words and Phrases.* Delta link homotopy, Conway polynomial, Vassiliev invariant.

From Propositions 1 and 2, we have the following Corollary.

Corollary. *If two 2-component links are $\Delta$ link homotopic, then their pairs of $\delta_1$ and $\delta_2$ coincide.*

From the Corollary, we have classified all prime 2-component links with seven crossings or less in [**N**] up to $\Delta$ link homotopy. There the following question is raised:

Question. Is a pair of $\delta_1$ and $\delta_2$ a faithful invariant of $\Delta$ link homotopy for 2-component links?

For the Question above, we will give an affirmative answer as follows:

Theorem 3. *Two 2-component links are $\Delta$ link homotopic if and only if their pairs of $\delta_1$ and $\delta_2$ coincide.*

The proof of Theorem 3 will be given in Section 3. For $\mu$-component links, it is still open whether the set of $\delta_1$ and $\delta_2$ of all sublinks is a faithful invariant of $\Delta$ link homotopy or not.

## 2. $C_n$-move and $C_n$-link.

In this section, we will recall the techniques of Murakami and the first author in [**MN**] and of Taniyama and Yasuhara in [**TY**] and [**TY2**]. Habiro introduced a sequence of his $C_n$-moves ($n \geq 1$) in [**H2**] to show that two oriented knots have the same Vassiliev invariants of order less than or equal to $n$ if and only if they can be transformed into each other by a finite sequence of $C_{n+1}$-moves. We remark that an ordinary unknotting operation is a $C_1$-move, and that a $\Delta$ move is a $C_2$-move. A $C_3$-move is formerly called a clasp-pass move in [**H**]. Taniyama and Yasuhara introduced their $C_n$-links ($n \geq 1$) to show that a $C_n$-move can be realized by the result of fusion with a $C_n$-link (cf. [**TYO**]). Furthermore, they showed that a finite sequence of $C_{n+1}$-moves can unlink, unknot, and untwist fusion-bands with a $C_n$-link to be trivial fusion-bands (possibly with half-twists). And a finite sequence of $C_{n+1}$-moves can make a root of a fusion-band which connects a $C_n$-link slide along the original link and pass through a root of a fusion-band which connects another $C_m$-link. We remark that a $C_1$-link is a Hopf-link, and that a $C_2$-link is Borromean rings. In this note, we use the mark in figures of fusion-bands to mean a half-twist as in Figure 2.0.1, where the circle with a slash in (1) means a right-handed half-twist of a band as in (2), and the circle with a back slash in (3) means a left-handed half-twist of a band as in (4).

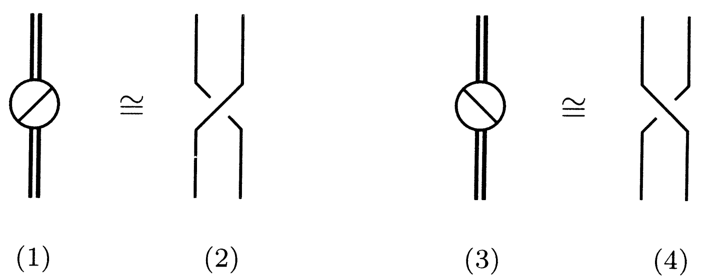



Figure 2.0.1.

### 2.1. $\varDelta$ move and Borromean rings.

We recall a $\varDelta$ move (or $C_2$-move) as in Figure 1.1.1. The following claims are known in [**MN**] and [**TY**]. Another aspect of a $\varDelta$ move can be seen in Figure 2.1.1, which illustrates Claim 1.1 in [**MN**]: "A clasp can leap over a hurdle."

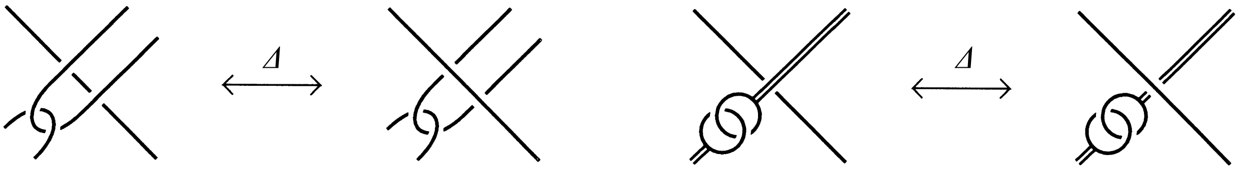
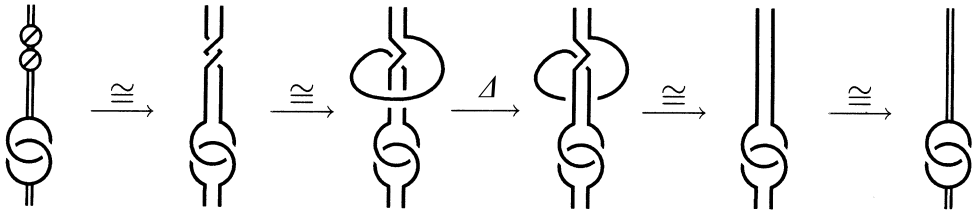



Figure 2.1.1.

Claim 2.1.1. *A local move as in Figure* 2.1.1 *can be realized by a single $\varDelta$ move.*

As a special case of Claim 2.1.1, we have the following Claim 2.1.2.

Claim 2.1.2. *The number of half-twists of a fusion-band with a Hopf-link can be changed by* $\pm 2$ *by a single $\varDelta$ move as* (1) *to* (6) *in Figure* 2.1.2.




(1) (2) (3) (4) (5) (6)

Figure 2.1.2.

Claim 2.1.3. *A $\varDelta$ move can be realized by the result of fusion with Borromean rings as in Figure* 2.1.3.


Figure 2.1.3.

### 2.2. Clasp-pass move and $C_3$-link.

A clasp-pass move (or $C_3$-move) is shown in Figure 2.2.1, which is a replacement of 4-string trivial tangles.

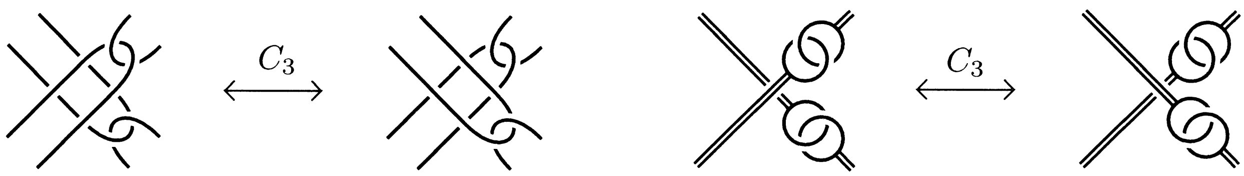



Figure 2.2.1.

The following claims are known in [**TY2**]. Another aspect of a $C_3$-move can be seen in Figure 2.2.2, which illustrates Claim 2.2.1.

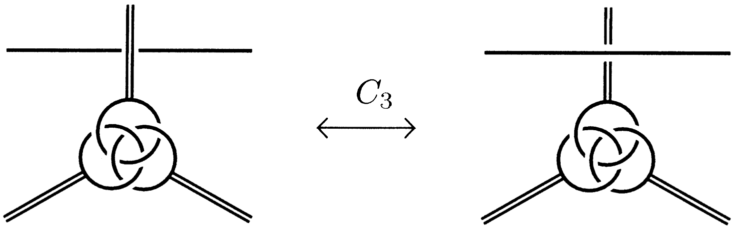



Figure 2.2.2.

Claim 2.2.1. *A fusion-band with Borromean rings can "leap" over a sub-arc of the link by a single clasp-pass move as in Figure* 2.2.2.

As special cases of Claim 2.2.1, we have the following Claims 2.2.2 and 2.2.3.

Claim 2.2.2. *The number of half-twists of a fusion-band with Borromean rings can be changed by* $\pm 2$ *by a single clasp-pass move as in Figure* 2.2.3.

Claim 2.2.3. *A half-twist of a fusion-band with Borromean rings can be moved to an adjacent fusion-band with the same Borromean rings by twice clasp-pass moves as in Figure* 2.2.4.

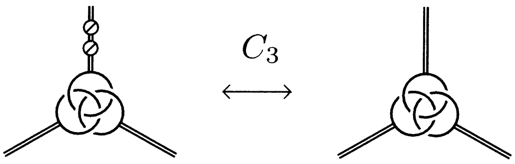



Figure 2.2.3.

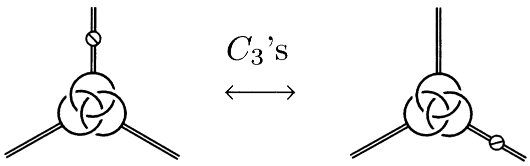



Figure 2.2.4.

Claim 2.2.4. *A clasp-pass move can be realized by the result of fusion with a* $C_3$-*link as in Figure* 2.2.5.

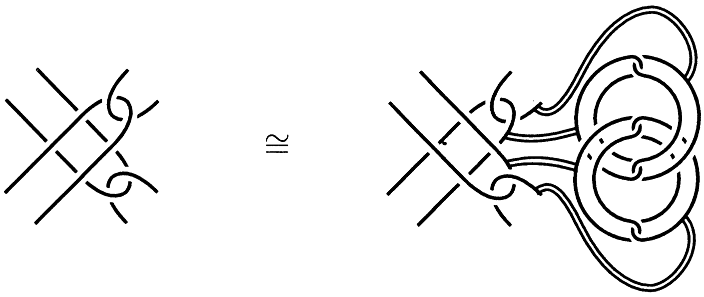



Figure 2.2.5.

## 2.3. $C_4$-move.

A $C_4$-move is shown in Figure 2.3.1, which is a replacement of 5-string trivial tangles. Another aspect of a $C_4$-move can be seen in Figure 2.3.2.

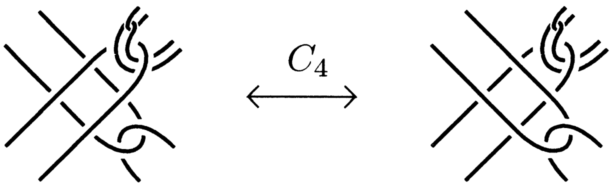



Figure 2.3.1.

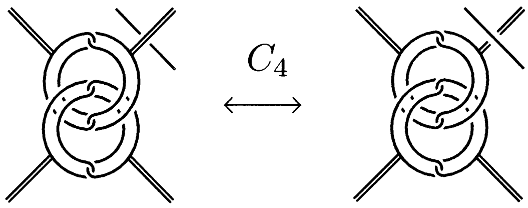



Figure 2.3.2.

Claim 2.3.1. *A fusion-band with a $C_3$-link can "leap" over a sub-arc of the link by a single $C_4$-move as in Figure* 2.3.2.

As a special case of Claim 2.3.1, we have the following Claim 2.3.2.

Claim 2.3.2. *The number of half-twists of a fusion-band with a $C_3$-link can be changed by $\pm 2$ by a single $C_4$-move as in Figure* 2.3.3.

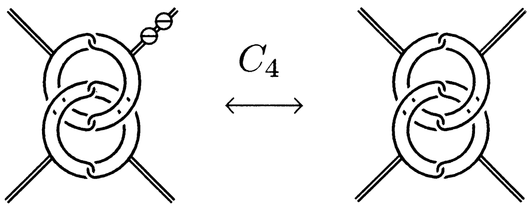



Figure 2.3.3.

A $C_4$-move can be realized by the result of fusion with a $C_4$-link, which is a Brunnian link with five components.

**2.4. $C_n$-move and self $\Delta$ move.**

The definition of self $\Delta$ move implies the following Claim 2.4.1.

Claim 2.4.1. *If all of the three fusion-bands with the same Borromean rings are connected to the same component, the fusion-bands can be removed by a single self $\Delta$ move.*

A clasp-pass move is a local move to replace 4-string trivial tangles as in Figure 2.2.1, and it can be realized by twice $\Delta$ moves for arbitrary three strings. (Cf. Figures 2.1.1 and 2.2.1.) That implies the following Claim 2.4.2.

Claim 2.4.2. *If three of the four fusion-bands with the same $C_3$-link are connected to the same component, the fusion-bands can be removed by twice self $\Delta$ moves.*

A $C_4$-move is a local move to replace 5-string trivial tangles as in Figure 2.3.1, and it can be realized by twice $C_3$-moves for arbitrary four strings. (Cf. Figures 2.2.1 and 2.3.1.) Furthermore, it can be realized by at most four times $\Delta$ moves for arbitrary three strings. If three of the five fusion-bands with the same $C_4$-link are connected to the same component, the fusion-bands with a $C_4$-link can be removed by at most four times self $\Delta$ move. In the case of a $C_4$-move for a 2-component link, three of the five fusion-bands with the same $C_4$-link are connected to the same component. That implies the following Claim 2.4.3.

Claim 2.4.3. *The fusion-bands with the same $C_4$-link for a 2-component link can be removed by a finite sequence of self $\Delta$ move.*

The observation above is based on Habiro's observation, and some part can be seen in [**OTY**].

**2.5. Generating Borromean rings.**

A $(2, 2\delta_1)$-torus link can be realized by the result of fusions with $\delta_1$ Hopf-links as in the upper-left of Figure 2.5.1. Grasp and twist once the right part, and we have the upper-right. By clasp-pass moves at suitable $(1/2)\delta_1(\delta_1 - 1)$ crossings of fusion-bands, we have the lower-left. By $\Delta$ moves at the $\delta_1$ fusion-bands (Claim 2.1.2), we have the lower-right, which is the same as the original. We remark that none of the last $\delta_1$ times $\Delta$ moves is self $\Delta$ move. This observation can be seen in the proof of Theorem 1.5 in [**TY2**].

Claim 2.5.1. *A $(2, 2\delta_1)$-torus link can be realized by the result of fusions for a $(2, 2\delta_1)$-torus link with $\delta_1$ sets of Borromean rings (each triple of whose fusion-bands is not connected to the same component) and with a finite number of $C_3$-links.*

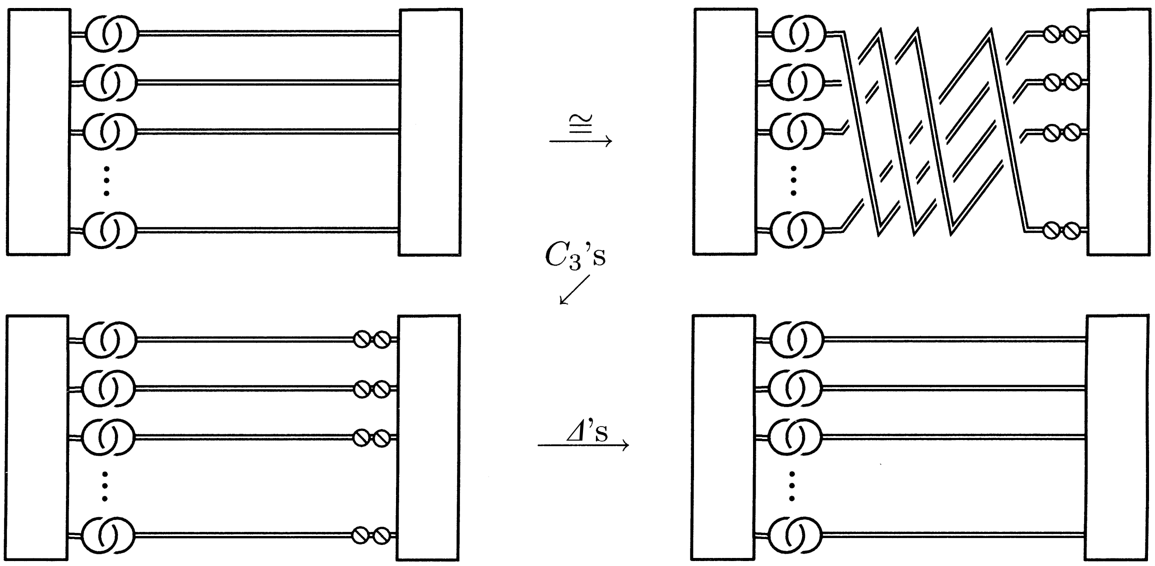



Figure 2.5.1.

## 3. Proof of Theorem 3.

In this section, by a finite sequence of self $\Delta$ moves, we will transform a given 2-component link into the result of fusion for the $(2, 2\delta_1)$-torus link with at most one set of Borromean rings and with a finite number of $C_3$-links. From the numbers of sets of Borromean rings and $C_3$-links, we will calculate their $\delta_2$. The coincidence of their $\delta_2$

implies the coincidence of the numbers of sets of Borromean rings and $C_3$-links. This implies the proof of Theorem 3.

**3.1. Torus link with Borromean rings.**

We recall Proposition 1. If $\delta_1$ of two 2-component links coincide, then they can be transformed into each other by a finite sequence of $\Delta$ moves. In other words, an arbitrary 2-component link can be realized by the result of fusion for the $(2, 2\delta_1)$-torus link with a finite number of Borromean rings as in Figure 3.1.1. If all of the three fusion-bands with the same Borromean rings are connected to the same component, a single self $\Delta$ move removes it by Claim 2.4.1. Now we count the number of sets of fusion-bands with Borromean rings each one of which is connected to distinct components. If the number is odd, and if the $\delta_1$ is odd, the number can be changed into even by Claim 2.5.1. Therefore, in the case that $\delta_1$ is odd, the number is assumed to be even. In the case that $\delta_1$ is even, the number is even or odd.

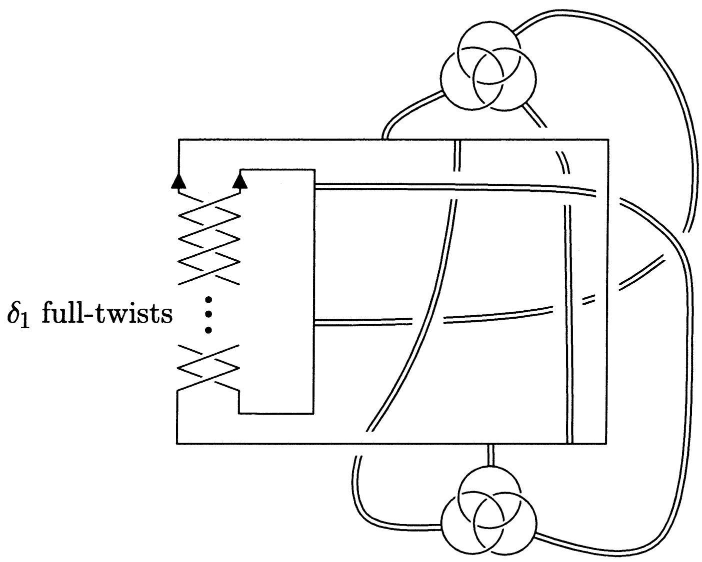



Figure 3.1.1.

**3.2. Sliding a fusion-band with Borromean rings by clasp-pass moves.**

The following Claim 3.2.1 can be seen essentially in [**TY2**].

Claim 3.2.1. *A root of a fusion-band with Borromean rings can pass through a root of a fusion-band with another Borromean rings or $C_3$-link by a finite sequence of clasp-pass moves.*

Proof of Claim 3.2.1. Slide the root of the left fusion-band with Borromean rings along the original link and another fusion-band from the upper-left to the upper-right in Figure 3.2.1. By Claim 2.1.1, the crossings of the fusion-band and a sub-arc of Borromean rings can be changed by twice clasp-pass moves, from the upper-right to the lower-left. Pulling the fusion-band by an isotopy, we have the lower-right from the lower-left. In the case of $C_3$-link, the parallel argument can be applied. The proof of Claim 3.2.1 is complete. □

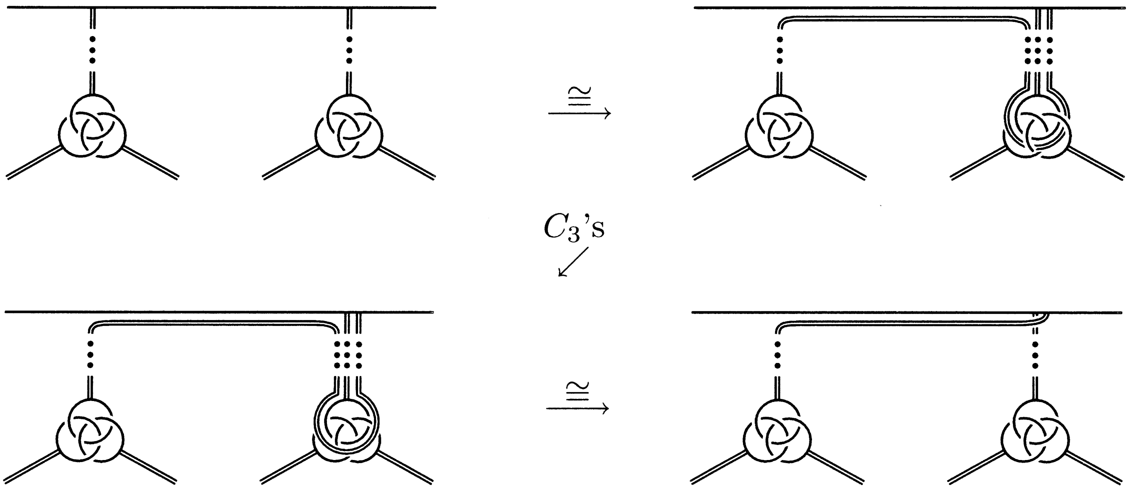



Figure 3.2.1.

A crossing-change of fusion-bands can be realized by twice clasp-pass moves from Claim 2.2.1. Therefore, a finite sequence of clasp-pass moves can unlink, unknot, and untwist fusion-bands with Borromean rings to be trivial fusion-bands (possibly with half-twists). We can transform all the sets of fusion-bands with Borromean rings by a finite sequence of clasp-pass moves so that the 2-component link is realized by the result of fusion for the $(2, 2\delta_1)$-torus link with a finite number of Borromean rings and $C_3$-links as in Figure 3.2.2, where all the fusion-bands with Borromean rings are suitably arranged.

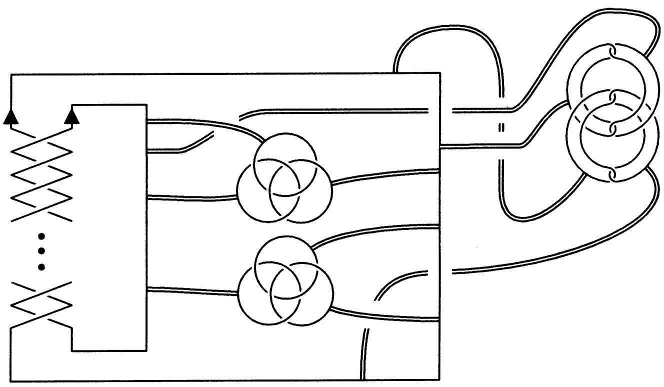


Figure 3.2.2.

**3.3. Removing fusion-bands with Borromean rings.**

From the argument above, all the fusion-bands with Borromean rings are assumed to be suitably arranged. We can add a single half-twist on each one of fusion-bands with the same Borromean rings by the following operation, which can be seen in the proof of Theorem 1.4 in [**TY2**]. We consider that as in Figure 3.3.1 (1). We perform the $\pi$-rotation for the part of the Borromean rings by the horizontal axis to deform (1) to (2) in Figure 3.3.1. The left lower fusion-band can pass over fusion-bands with another Borromean rings and $C_3$-links downwards along the left component to come back to the top by a finite sequence of clasp-pass moves as in (2) to (3). Then we can add a single half-twist on each one of the fusion-bands. Moreover, the sum of the numbers of half-twists of fusion-bands with the same Borromean rings can be reduced by two from Claims 2.2.2 and 2.2.3 by a finite sequence of clasp-pass moves.

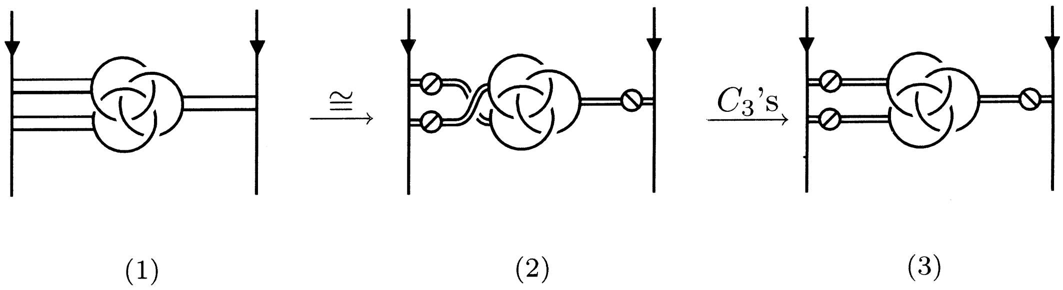



Figure 3.3.1.

Claim 3.3.1. *Two sets of suitably arranged fusion-bands with Borromean rings can be removed by a finite sequence of clasp-pass moves.*

Proof of Claim 3.3.1. Suppose that the link is realized by the result of fusion for the $(2, 2\delta_1)$-torus link with two or more sets of Borromean rings and with a finite number of $C_3$-links, where all the fusion-bands with Borromean rings are suitably arranged. We remark that any two sets of suitably arranged fusion-bands with Borromean rings are isotopic regardless of connection of fusion-bands for the $(2, 2\delta_1)$-torus link, as in Figure 3.3.2.

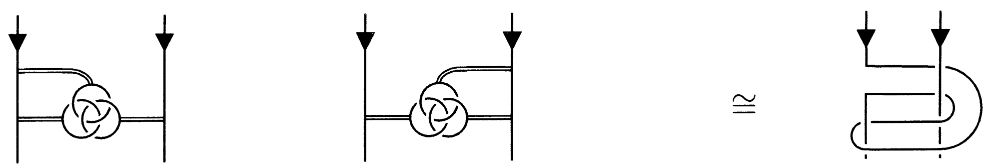



Figure 3.3.2.

From the transformation above, we can assumed that, for the adjacent two sets of suitably arranged fusion-bands with Borromean rings, the numbers of half-twists of fusion-bands are 0 and 1 as in Figure 3.3.3 (1), where the pair of fusion-bands with Borromean rings is removed by twice self $\Delta$ moves, as (1) to (6) in Figure 3.3.3. The proof is complete. □

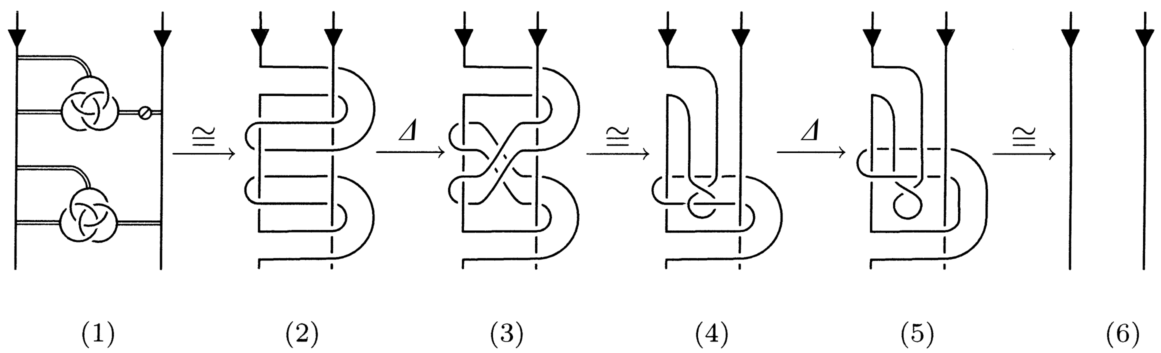



Figure 3.3.3.

Hence, the number of sets of suitably arranged fusion-bands with Borromean rings can be reduced by two by a finite sequence of clasp-pass moves. In the case that $\delta_1$ is odd, the number can be reduced to 0. In the case that $\delta_1$ is even, the number can be reduced to 1 or 0.

**3.4. Sliding a fusion-band with a $C_3$-link by $C_4$-moves.**

The following Claim 3.4.1 can be seen essentially in [**TY2**] as well as that of Claim 3.2.1.

Claim 3.4.1. *A root of a fusion-band with a $C_3$-link can pass through a root of a fusion-band with another Borromean rings or $C_3$-link by a finite sequence of $C_4$-moves* (*and so by a finite sequence of self $\Delta$ moves*).

Proof of Claim 3.4.1 is given by the parallel argument to that of Claim 3.2.1.

From Claim 2.3.1, a crossing-change of fusion-bands can be realized by twice $C_4$-moves (and so by a finite sequence of self $\Delta$ moves). Therefore, a finite sequence of $C_4$-moves (and so a finite sequence of self $\Delta$ moves) can unlink, unknot, and untwist fusion-bands with a $C_3$-link to be trivial fusion-bands (possibly with half-twists). We can transform all the sets of fusion-bands with a $C_3$-link by a finite sequence of $C_4$-moves (and so by a finite sequence of self $\Delta$ moves) so that the 2-component link is realized by the result of fusion for the $(2, 2\delta_1)$-torus link with at most one set of Borromean rings and a finite number of $C_3$-links as in Figure 3.4.1, where all the fusion-bands with Borromean rings and with a finite number of $C_3$-links are suitably arranged.

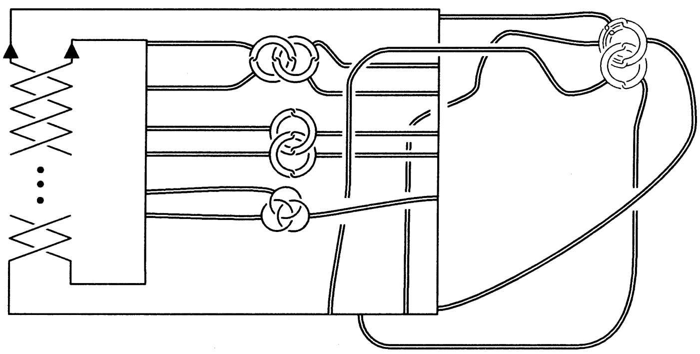


Figure 3.4.1.

By Claim 2.4.2, two of the four fusion-bands with the same $C_3$-link are assumed to be connected to one component and the other two to the other component. If a fusion-band with a $C_3$-link is half-twisted as in Figure 3.4.2 (1), the fusion-band can be deformed by an isotopy through the adjacent fusion-band into that in Figure 3.4.2 (2). A $C_4$-move (and so a finite sequence of self $\Delta$ moves) and an isotopy transform (2) into (3), where the crossings of $C_3$-link are changed. Therefore, we can remove half-twists on all fusion-bands with $C_3$-links by a finite sequence of $C_4$-moves (and so by a finite sequence of self $\Delta$ moves). In other words, we have the following Claim 3.4.2.

Claim 3.4.2. *The difference of hooking can be realized by a single half-twist of an incident fusion-band by a finite sequence of $C_4$-move* (*and so by a finite sequence of self $\Delta$ moves*).

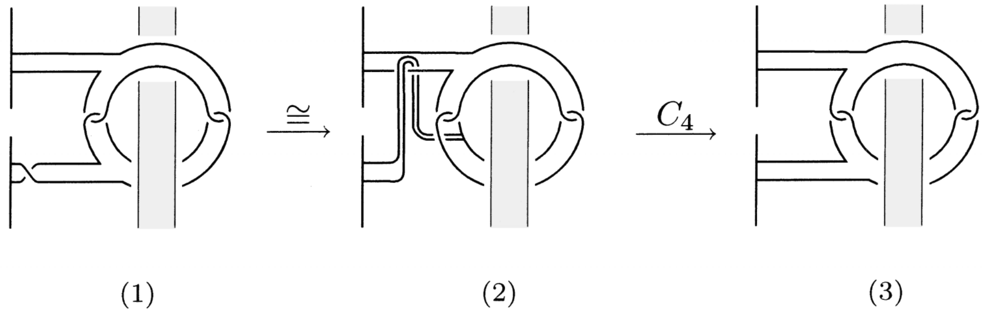



Figure 3.4.2.

**3.5. Removing fusion-bands with $C_3$-links.**

Any suitably arranged set of fusion-bands with $C_3$-links can be deformed and/or transformed into two types of suitably arranged set of fusion-bands with $C_3$-links as (A) and (B) in Figure 3.5.1, where the crossings of $C_3$-links have several possibilities. That is because the flip for the right part as in Figure 3.5.2 can assure that two pairs of fusion-bands which are connected to the same components are adjacent.

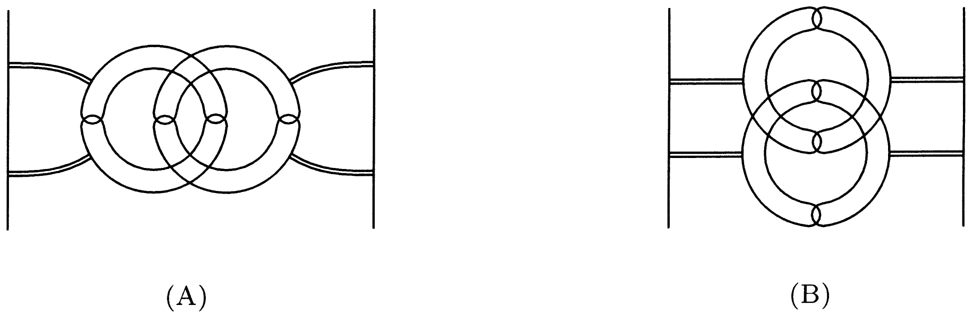



Figure 3.5.1.

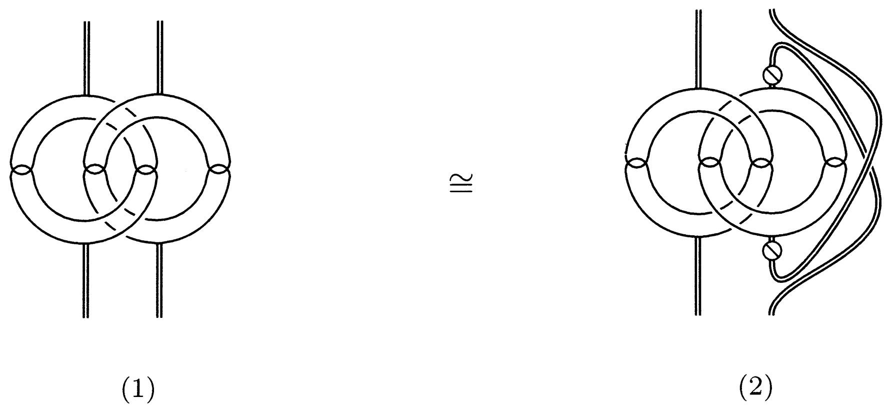



Figure 3.5.2.

For the type (A), a suitably arranged set of fusion-bands with a $C_3$-link can be removed by a finite sequence of self $\Delta$ moves. The left (or right) part can be deformed by an isotopy into a kind of Whitehead double like as in Figure 3.5.3 (1). Therefore, the set can be deformed by an isotopy into a kind of iterated Whitehead double as in Figure 3.5.3 (2), which can be removed by a finite sequence of self $\Delta$ moves.

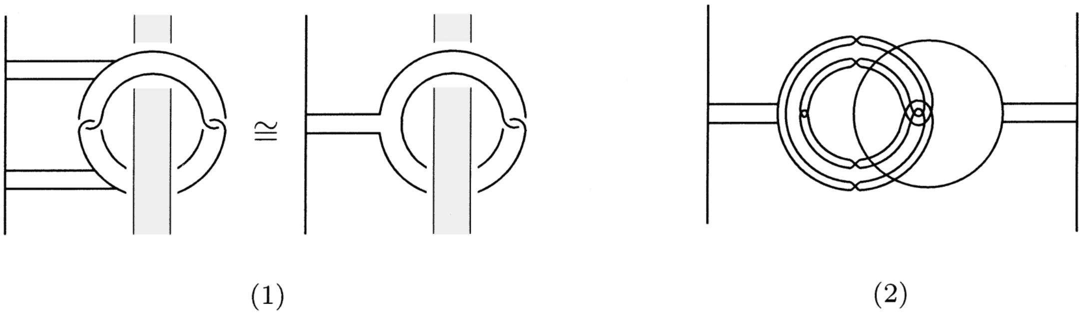



Figure 3.5.3.

For the type (**B**), there are eight patterns of crossings of $C_3$-links (see Figure 2.8 in [**NO**]). The eight patterns can be transformed into two patterns by a finite sequence of self $\varDelta$ moves. For example, the pattern (a) in Figure 3.5.4 is transformed into the

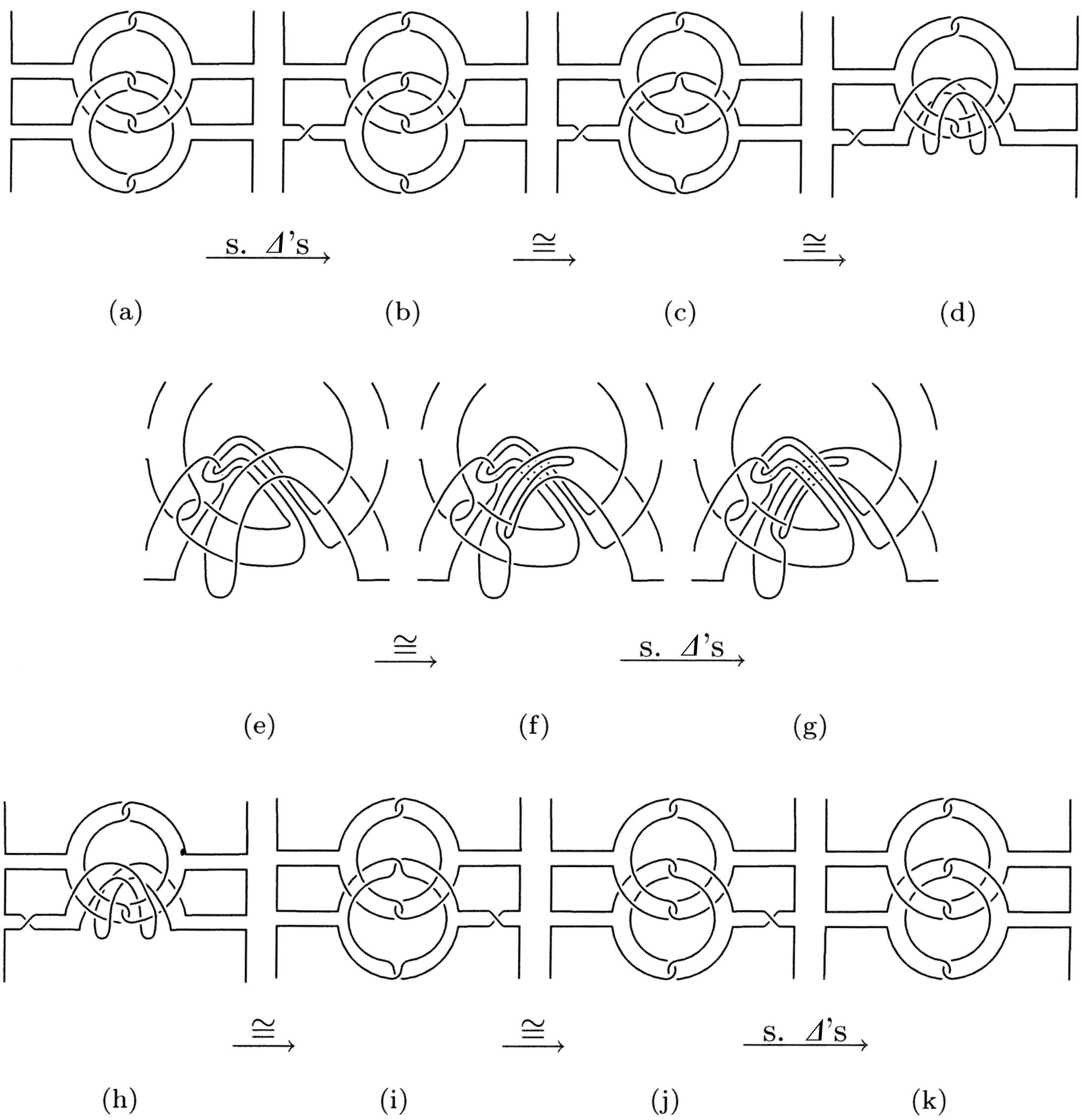
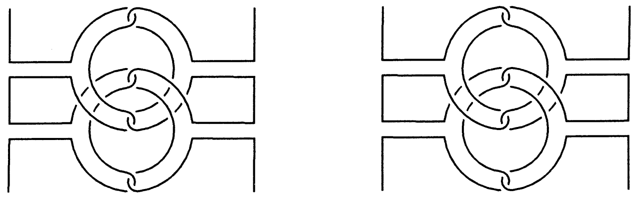



Figure 3.5.4

pattern (k) in Figure 3.5.4. By a finite sequence of self $\Delta$ moves, (a) is transformed into (b) by Claim 3.4.2. (b) is deformed into (c) and into (d) by an isotopy. The central part of (d) is deformed into (e) and into (f) by an isotopy. (f) is transformed into (g) by a finite sequence of self $\Delta$ moves, and so we have (h). (h) is deformed into (i) and into (j) by an isotopy. (j) is transformed into (k) by a finite sequence of self $\Delta$ moves by Claim 3.4.2.

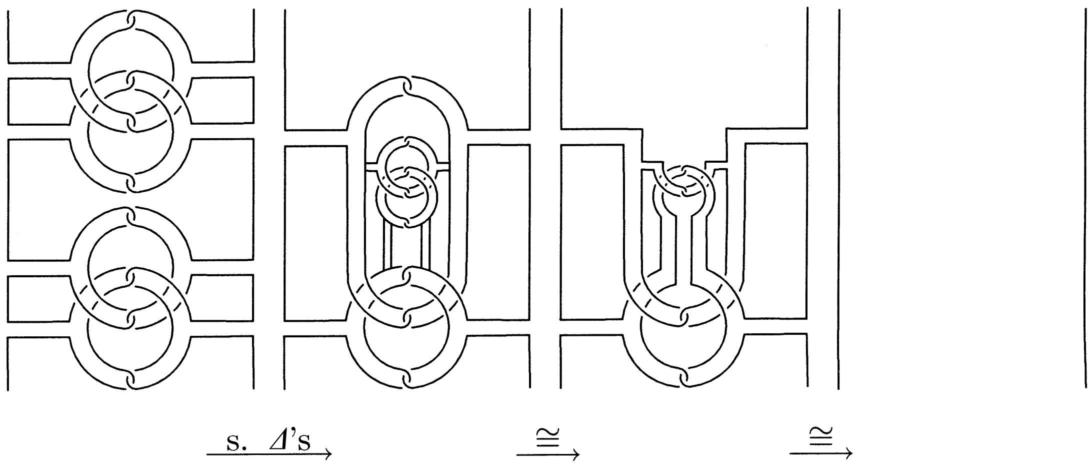
By the parallel argument, the eight patterns can be transformed into the two patterns in Figure 3.5.5 by a finite sequence of self $\Delta$ moves.



(1) (2)

Figure 3.5.5.

A pair of both patterns can be removed by a finite sequence of $C_4$-moves (and so by a finite sequence of self $\Delta$ moves) as (1) to (4) in Figure 3.5.6.




(1) (2) (3) (4)

Figure 3.5.6.

Therefore, an arbitrary 2-component link can be transformed into the result of fusion for the $(2, 2\delta_1)$-torus link with at most one set of Borromean rings and with a finite number of $C_3$-links, where all the fusion-bands with at most one set of Borromean rings and with a finite number of $C_3$-links of patterns in Figure 3.5.5 are suitably arranged.

### 3.6. Effect on $\delta_2$.

In the case that $\delta_1$ is even, a suitable arranged set of fusion-bands with Borromean rings changes $\delta_2$ of the link by $\pm(\delta_1 + 1)$. We remark that $\pm(\delta_1 + 1)$ is an odd integer.

Adding a suitable arranged set of fusion-bands with a $C_3$-link of pattern (1) (or (2), respectively) in Figure 3.5.5 changes $\delta_2$ of the link by $-2$ (or 2, respectively). And $\delta_2$ depends only on the numbers of the set of fusion-bands with Borromean rings and the sets of fusion-bands with $C_3$-links. If the pairs of $\delta_1$ and $\delta_2$ coincide, the pairs of the above numbers should coincide. Therefore, the two 2-component links can be transformed into each other by a finite sequence of self $\Delta$ moves. The proof of Theorem 3 is complete. □

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