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Every Stieltjes moment problem has a solution in Gel'fand-Shilov spaces

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Abstract. We prove that every Stieltjes problem has a solution in Gel'fand-Shilov spaces $\mathscr{S}^{\beta}$ for every $\beta > 1$. In other words, for an arbitrary sequence $\{\mu_n\}$ there exists a function φ in the Gel'fand-Shilov space $\mathscr{S}^{\beta}$ with support in the positive real line whose moment $\int_0^{\infty} x^n \varphi(x)\,dx = \mu_n$ for every nonnegative integer n.

This improves the result of A. J. Duran in 1989 very much who showed that every Stieltjes moment problem has a solution in the Schwartz space $\mathscr{S}$, since the Gel'fand-Shilov space is much a smaller subspace of the Schwartz space. Duran's result already improved the result of R. P. Boas in 1939 who showed that every Stieltjes moment problem has a solution in the class of functions of bounded variation. Our result is optimal in a sense that if $\beta \leq 1$ we cannot find a solution of the Stieltjes problem for a given sequence.

§1. Introduction.

If a sequence $\{\mu_n\}$ and a function ϕ can be written as

$$\int_0^{\infty} x^n \phi(x)\,dx = \mu_n, \quad \text{for } n = 0, 1, 2, \ldots, \tag{1.1}$$

then μ_n is called the *n-th moment* of ϕ, the sequence $\{\mu_n\}$ is called the *moment sequence* of ϕ and the function ϕ is called the *moment function* of the sequence $\{\mu_n\}$.

The moment problem asks mainly when a sequence $\{\mu_n\}$ is the moment sequence of a function ϕ defined on I. The Stieltjes moment problem which treats the case of $I = [0, \infty)$ was proposed and solved completely for the case of positive measures $d\mu(x)$ by Stieltjes in his classical paper "Recherches sur les fraction continue" in 1894–95, where the Stieltjes integral was introduced. In 1939 Boas [**B**] proved that every Stieltjes moment problem has a solution in the space of functions of bounded variation, i.e., for an arbitrarily given sequence $\{\mu_n\}$ of real numbers there exists a function ϕ of bounded variation satisfying (1.1). Quite recently, Duran [**D**] showed an improved result that every Stieltjes problem has a solution in the Schwartz space of rapidly decreasing functions. In 1994, Durán and Estrada [**DE**] made, as a part of their paper, a different proof based on the asymptotic analysis. It is natural to ask what is the smallest function space which ensures the existence of the moment function for an arbitrarily given sequence.

In this paper we will show that every Stieltjes problem has a solution in the Gel'fand-Shilov space $S^{\beta}(0, \infty)$, $\beta > 1$ which is a subspace of the Schwartz space. Furthermore, one cannot find a moment function in $S^{\beta}(0, \infty)$, $\beta \leq 1$ for some sequences. In this sense our result is optimal and $S^{\beta}(0, \infty)$, $\beta > 1$ is the smallest Gel'fand-Shilov spaces in which the Stieltjes problem has a solution.

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§2. Gel'fand-Shilov spaces on the right half line.

We introduce Gel'fand-Shilov spaces and their modifications for the Stieltjes moment problem and refer to [**GS**] for the definition of the Schwartz space and other function spaces in the distribution theory.

For $\alpha \geq 0$ and $\beta \geq 0$ we define function spaces S_α, S^β, and S_α^β by

$$S_\alpha = \left\{ \phi \in C^\infty(\boldsymbol{R}) \mid \sup_{x \in \boldsymbol{R}} |x^p \phi^{(q)}(x)| \leq C_q h^p p!^\alpha \ \exists C_q, h > 0 \right\},$$

$$S^\beta = \left\{ \phi \in C^\infty(\boldsymbol{R}) \mid \sup_{x \in \boldsymbol{R}} |x^p \phi^{(q)}(x)| \leq C_p h^q q!^\beta \ \exists C_p, h > 0 \right\},$$

$$S_\alpha^\beta = \left\{ \phi \in C^\infty(\boldsymbol{R}) \mid \sup_{x \in \boldsymbol{R}} |x^p \phi^{(q)}(x)| \leq C h^{p+q} p!^\alpha q!^\beta \ \exists C, h > 0 \right\}.$$

Modifying the above ordinary Gel'fand-Shilov spaces we introduce $S_\alpha(0,\infty), S^\beta(0,\infty)$, and $S_\alpha^\beta(0,\infty)$ by

$$S_\alpha(0,\infty) = \{\phi \in S_\alpha \mid \phi(x) = 0, \ \text{for} \ x \leq 0\},$$

$$S^\beta(0,\infty) = \{\phi \in S^\beta \mid \phi(x) = 0, \ \text{for} \ x \leq 0\},$$

$$S_\alpha^\beta(0,\infty) = \{\phi \in S_\alpha^\beta \mid \phi(x) = 0, \ \text{for} \ x \leq 0\}.$$

Throughout this paper the Fourier-Laplace transform $\hat{\phi}$ of a function ϕ is defined by

$$\hat{\phi}(\xi) = \int_{-\infty}^{\infty} \phi(x) e^{ix\xi}\, dx.$$

It is well known in [**GS**] that $\hat{S}_\alpha = S^\alpha$, $\hat{S}^\beta = S_\beta$ and $\hat{S}_\alpha^\beta = S_\beta^\alpha$ where $\hat{S}_\alpha, \hat{S}^\beta$ and $\hat{S}_\alpha^\beta$ are the Fourier transforms of functions of S_α, S^β and S_α^β respectively.

Now we state the Paley-Wiener type theorem for $S^\beta(0,\infty)$.

Proposition 2.1. *A function ψ defined on $\boldsymbol{R}$ is the Fourier transform of a function ϕ of class $S^\beta(0,\infty)$ if and only if*

(i) *ψ is of class S_β.*

(ii) *ψ can be extended to a continuous function Ψ in the closed upper half plane $\operatorname{Im} z \geq 0$ and analytic in $\operatorname{Im} z > 0$.*

(iii) *ψ vanishes as $z \to \infty$ in $\operatorname{Im} z \geq 0$.*

Proof. The facts (i) and (iii) are immediate from $\hat{S}^\beta = S_\beta$ and the Lebesgue dominated convergence theorem. To prove (ii) let $\psi(z) = \int_0^\infty e^{izt}\phi(t)\, dt$. Then $\psi(z)$ is well defined in the closed upper half plane $\operatorname{Im} z \geq 0$. By the simple inequality $|e^{iz_1} - e^{iz_2}| \leq |z_1 - z_2|$ in the closed upper half plane we can apply the Lebesgue dominated convergence theorem for the Newton quotient

$$\frac{\psi(z+h) - \psi(z)}{h} = \int_0^\infty \left(\frac{e^{i(z+h)t} - e^{izt}}{h} \right) \phi(t)\, dt \tag{2.1}$$

Letting $h \to 0$ in (2.1) we have

$$\psi'(z) = \int_0^\infty ite^{izt}\phi(t)\,dt$$

which proves (ii).

To prove the converse, in view of the Fourier inversion formula it suffices to show that $\phi(x) = (1/(2\pi))\int_{-\infty}^{\infty} e^{-ixt}\psi(t)\,dt$ belongs to $S^\beta(0,\infty)$. Since ϕ belongs to S^β by (i) it remains to prove that $\phi(x) = 0$ for $x < 0$. By the condition (ii) we can write

$$\phi(x) = \lim_{R\to\infty} \frac{1}{2\pi}\int_\Gamma e^{-ixz}\psi(z)\,dz$$

where Γ is the contour $\Gamma(t) = Re^{it}$, $0 \le t \le \pi$. By the condition (iii) and a usual calculation it is easy to see that $\phi(x) = 0$ for $x < 0$. $\square$

§3. Main Theorem.

Before proving the main theorem we need the following lemma based on Ritt's theorem. This lemma describes the existence of a bounded analytic function which has an arbitrarily given asymptotic power series whose proof can be found in [**DE**].

Lemma 3.1. *Let $\{a_n\}$ be an arbitrary sequence of complex numbers. Then there exists a bounded analytic function F defined in the sector $S = \{re^{i\theta}\,|\,r > 0, \theta_1 < \theta < \theta_2\}$ such that*

$$F(z) \sim a_0 + a_1 z + a_2 z^2 + \cdots$$

as $z \to 0$ in S.

In other words, $|F(z) - (a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n)| = O(z^{n+1})$ as $z \to 0$ in S.

We are now ready to state and prove the main theorem.

Theorem 3.2. *For any sequence $\{\mu_n\}$ of complex numbers there exists $\phi \in S^\beta(0,\infty)$ such that*

$$\mu_n = \int_0^\infty x^n\phi(x)\,dx \quad \textit{for } n = 0,1,2,\ldots. \tag{3.1}$$

Proof. Let ψ be the Fourier transform of the function ϕ in (3.1). Then the equation (3.1) is equivalent to the following Borel type theorem

$$\psi^{(n)}(0) = \int_0^\infty i^n x^n \phi(x)\,dx = i^n\mu_n. \tag{3.2}$$

Thus it suffices to find a function $\psi \in \hat{S}^\beta(0,\infty)$ satisfying (3.2) and then the moment function ϕ for the sequence $\{\mu_n\}$ can be obtained by

$$\phi(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \psi(\xi)e^{-ix\xi}\,d\xi.$$

To construct such a function we first employ the function

$$G_\beta(z) = \exp(e^{\pi i(1-1/(2\beta))}(z+i)^{1/\beta})$$

which is analytic in the region $\{z \mid -\pi/2 < \arg(z+i) < 3\pi/2\}$.

Since $1/G_\beta$ is analytic in the same region it can be written as

$$1/G_\beta(z) = a_0 + a_1 z + a_2 z^2 + \cdots$$

for $|z| < 1$.

Secondly, by Lemma 3.1 there exists $F(z)$ which is bounded analytic in the sector $S = \{z \in \boldsymbol{C} \mid -\pi/4 < \arg z < 5\pi/4\}$ with its asymptotic power series

$$F(z) \sim b_0 + b_1 z + b_2 z^2 + \cdots, \quad \text{as } z \to 0 \text{ in } S$$

where

$$b_n = \sum_{k=0}^{n} \frac{i^k}{k!} \mu_k a_{n-k}.$$

Now we want to show that the function $\psi(z) = F(z)G_\beta(z)$ satisfies the conditions (i), (ii) and (iii) in Proposition 2.1 and the equation (3.2), which will complete the proof in view of Proposition 2.1.

Note that by the construction of F, $\psi(z) = F(z)G_\beta(z)$ has an asymptotic power series

$$\psi(z) \sim \sum_{n=0}^{\infty} \frac{i^n \mu_n}{n!} z^n, \quad \text{as } z \to 0 \text{ in } S.$$

Therefore, if we define $\psi(0) = \mu_0$ we have

$$\left| \frac{\psi(z) - \psi(0)}{z} - i\mu_1 \right| = \frac{O(z^2)}{z} \quad \text{as } z \to 0 \text{ in } S$$

which means that $\psi'(0) = i\mu_1$. Also, since $\psi'(z)$ has the asymptotic expansion

$$\psi'(z) \sim \sum_{n=1}^{\infty} \frac{i^n \mu_n}{(n-1)!} z^{n-1}, \quad \text{as } z \to 0 \text{ in } S$$

we can repeat the same process and the equation (3.2) is satisfied.

The condition (ii) of Proposition 2.1 is obvious by defining $\psi(0) = \mu_0$. Now the image of the region $D := \{z \mid \operatorname{Im} z \geq -1\}$ under $h_\beta(z) := e^{\pi i(1-1/(2\beta))}(z+i)^{1/\beta}$ is contained in the sector $\pi - \pi/(2\beta) < \arg z < \pi + \pi/(2\beta)$. Thus it follows that $\cos(\arg h_\beta(z)) \leq -k_\beta$ where $k_\beta = \cos \pi/(2\beta) > 0$. From these properties we may decide the behavior of G_β at infinity in the domain D. Indeed, if $z \in D$ and $|z+i| \geq r$ we have

$$\begin{aligned} |G_\beta(z)| &= \exp \operatorname{Re}(e^{\pi i(1-1/(2\beta))}(z+i)^{1/\beta}) \\ &\leq \exp(-k_\beta r^{1/\beta}). \end{aligned}$$

This implies $G_\beta(z) \to 0$ as $z \to \infty$ in D.

Finally we prove $\psi(x) \in S^{\beta}$. If $|x| \le 1$, $\psi^{(q)}(x)$ is bounded by some constant which depends only on q. For $|x| > 1$, by the Cauchy integral formula we have

$$\begin{aligned}
|\psi^{(q)}(x)| &\le \frac{q!}{2\pi} \int_{|z-x|=1/2} \frac{|\psi(z)|}{|z-x|^{q+1}} |dz| \\
&\le 2^q q! \sup_{|z-x|=1/2} |\psi(z)| \\
&\le C 2^q q! \sup_{|z-x|=1/2} |G_\beta(z)| \\
&\le C 2^q q! \exp\left(-k_\beta \left(|x| - \frac{1}{2}\right)^{1/\beta}\right) \\
&\le C_q \exp(-k_\beta |x|^{1/\beta})
\end{aligned}$$

for some positive constants C_q which depend only on q and β. Multiplying $|x|^p$ on both sides of the above inequality and taking supremum for x we complete the proof. $\square$

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