J. Math. Soc. Japan
Vol. 56, No. 1, 2004

Derived category of squarefree modules and local cohomology with monomial ideal support

By Kohji YANAGAWA

(Received Oct. 16, 2000)
(Revised Sept. 9, 2002)

Abstract. A *squarefree module* over a polynomial ring $S = k[x_1, \dots, x_n]$ is a generalization of a Stanley-Reisner ring, and allows us to apply homological methods to the study of monomial ideals more systematically.

The category **Sq** of squarefree modules is equivalent to the category of finitely generated left Λ-modules, where Λ is the incidence algebra of the Boolean lattice $2^{\{1,\dots,n\}}$. The derived category $D^b(\mathbf{Sq})$ has two duality functors $\boldsymbol{D}$ and $\boldsymbol{A}$. The functor $\boldsymbol{D}$ is a common one with $H^i(\boldsymbol{D}(M^\bullet)) = \operatorname{Ext}_S^{n+i}(M^\bullet, \omega_S)$, while the *Alexander duality functor* $\boldsymbol{A}$ is rather combinatorial. We have a strange relation $\boldsymbol{D} \circ \boldsymbol{A} \circ \boldsymbol{D} \circ \boldsymbol{A} \circ \boldsymbol{D} \circ \boldsymbol{A} \cong \boldsymbol{T}^{2n}$, where $\boldsymbol{T}$ is the translation functor. The functors $\boldsymbol{A} \circ \boldsymbol{D}$ and $\boldsymbol{D} \circ \boldsymbol{A}$ give a non-trivial autoequivalence of $D^b(\mathbf{Sq})$. This equivalence corresponds to the Koszul duality for Λ, which is a Koszul algebra with $\Lambda^! \cong \Lambda$. Our $\boldsymbol{D}$ and $\boldsymbol{A}$ are also related to the Bernstein-Gel'fand-Gel'fand correspondence.

The local cohomology $H^i_{I_\Delta}(S)$ at a Stanley-Reisner ideal I_Δ can be constructed from the squarefree module $\operatorname{Ext}_S^i(S/I_\Delta, \omega_S)$. We see that Hochster's formula on the $\boldsymbol{Z}^n$-graded Hilbert function of $H^i_{\mathfrak{m}}(S/I_\Delta)$ is also a formula on the characteristic cycle of $H^{n-i}_{I_\Delta}(S)$ as a module over the Weyl algebra $A = k\langle x_1, \dots, x_n, \partial_1, \dots, \partial_n \rangle$ (if $\operatorname{char}(k) = 0$).

1. Introduction.

The Stanley-Reisner ring of an abstract simplicial complex $\Delta \subset 2^{\{1,\dots,n\}}$, which is the quotient of a polynomial ring $S = k[x_1, \dots, x_n]$ by the squarefree monomial ideal I_Δ, is a central concept of combinatorial commutative algebra (see [**7**], [**22**]). In [**24**], the author defined a *squarefree* $\boldsymbol{N}^n$-graded S-module. A Stanley-Reisner ring S/I_Δ, its syzygy module $\operatorname{Syz}_i(S/I_\Delta)$, the canonical module ω_S, and $\operatorname{Ext}_S^i(S/I_\Delta, \omega_S)$ are always squarefree. Using this notion, we can treat Stanley-Reisner rings and related objects in a categorical way, see [**9**], [**15**], [**18**], [**20**], [**21**], [**25**], [**26**], [**27**]. In the present paper, we will study the derived category of squarefree modules.

Let $\mathbf{Sq}_S$ (or simply $\mathbf{Sq}$) be the category of squarefree S-modules and their degree preserving maps. Then $\mathbf{Sq}$ is equivalent to the category of finitely generated left Λ-modules, where Λ is the incidence algebra of the Boolean lattice $2^{\{1,\dots,n\}}$.

Let $D^b(\mathbf{Sq})$ be the derived category of bounded complexes in $\mathbf{Sq}$. We have contravariant functors $\boldsymbol{D}$ and $\boldsymbol{A}$ from $D^b(\mathbf{Sq})$ to itself satisfying $\boldsymbol{D}^2 \cong \boldsymbol{A}^2 \cong \mathrm{Id}_{D^b(\mathbf{Sq})}$. If M is a squarefree module, so is $\operatorname{Ext}_S^i(M, \omega_S)$. Moreover, for $M^\bullet \in D^b(\mathbf{Sq})$, we can define $\boldsymbol{D}(M^\bullet) \in D^b(\mathbf{Sq})$ with $H^i(\boldsymbol{D}(M^\bullet)) = \operatorname{Ext}_S^{n+i}(M^\bullet, \omega_S)$ in a natural way, see

2000 *Mathematics Subject Classification.* Primary 13D25; Secondary 13D02, 13D45, 13F55, 13N10, 18E30.

Key Words and Phrases. Stanley-Reisner ring, local duality, Alexander duality, local cohomology, Bernstein-Gel'fand-Gel'fand correspondence, Koszul duality.

Proposition 3.2. On the other hand, extending an idea of Eagon-Reiner [**8**], Miller [**15**] and Römer [**20**] constructed the *Alexander duality functor* $\boldsymbol{A}$ on $\mathbf{Sq}$. Since $\boldsymbol{A}$ is exact, we can regard it as a duality functor on $D^b(\mathbf{Sq})$.

Using $D^b(\mathbf{Sq})$, we can get simple and systematic proofs of many results in [**15**], [**20**], [**21**], [**24**], [**25**]. Moreover, we prove a strange natural equivalence

$$\boldsymbol{D}\circ\boldsymbol{A}\circ\boldsymbol{D}\circ\boldsymbol{A}\circ\boldsymbol{D}\circ\boldsymbol{A}\cong\boldsymbol{T}^{2n},$$

where $\boldsymbol{T}$ is the translation functor on $D^b(\mathbf{Sq})$.

Let $E=\bigwedge S_1^*$ be the exterior algebra. A squarefree module over E, which was defined by Römer [**20**], is also a natural concept. The category $\mathbf{Sq}_E$ of squarefree E-modules is equivalent to $\mathbf{Sq}_S$ in a natural way. A famous theorem of Bernstein-Gel'fand-Gel'fand [**4**] states that the bounded derived category of finitely generated $\boldsymbol{Z}$-graded S-modules is equivalent to the bounded derived category of finitely generated $\boldsymbol{Z}$-graded left E-modules. The functors defining this equivalence preserve the squarefreeness, and coincide with $\boldsymbol{A}\circ\boldsymbol{D}$ and $\boldsymbol{D}\circ\boldsymbol{A}$ in the squarefree case under the equivalence $\mathbf{Sq}_S\cong\mathbf{Sq}_E$. We have another relation to Koszul duality. The incidence algebra Λ of $2^{\{1,\dots,n\}}$ is a Koszul algebra whose quadratic dual $\Lambda^!$ is isomorphic to Λ itself. The functors $\boldsymbol{A}\circ\boldsymbol{D}$ and $\boldsymbol{D}\circ\boldsymbol{A}$ give a non-trivial autoequivalence of $D^b(\mathbf{Sq})$. This equivalence corresponds to the Koszul duality $D^b(\mathbf{mod}_\Lambda)\cong D^b(\mathbf{mod}_{\Lambda^!})$.

In the last section, under the assumption that $\mathrm{char}(k)=0$, we study modules over the Weyl algebra $k\langle x_1,\dots,x_n,\partial_1,\dots,\partial_n\rangle$ associated to squarefree modules (e.g., the local cohomology module $H^i_{I_\Delta}(S)$). Especially, we give the formula for their characteristic cycles.

After I received the referee's report for the first version, I widely revised the paper. Among other things, Proposition 4.6 is a new result of the second version which was submitted in September 2001. The present version is the fourth one, in which some proofs and expositions are revised.

2. Preliminaries.

Let $S=k[x_1,\dots,x_n]$ be a polynomial ring over a field k. Consider an $\boldsymbol{N}^n$-grading $S=\bigoplus_{\boldsymbol{a}\in\boldsymbol{N}^n}S_{\boldsymbol{a}}=\bigoplus_{\boldsymbol{a}\in\boldsymbol{N}^n}kx^{\boldsymbol{a}}$, where $x^{\boldsymbol{a}}=\prod_{i=1}^n x_i^{a_i}$ is the monomial with the exponent $\boldsymbol{a}=(a_1,\dots,a_n)$. We denote the graded maximal ideal $(x_1,\dots,x_n)$ by $\mathfrak{m}$. For a $\boldsymbol{Z}^n$-graded module M and $\boldsymbol{a}\in\boldsymbol{Z}^n$, $M_{\boldsymbol{a}}$ means the degree $\boldsymbol{a}$ component of M, and $M(\boldsymbol{a})$ denotes the shifted module with $M(\boldsymbol{a})_{\boldsymbol{b}}=M_{\boldsymbol{a}+\boldsymbol{b}}$. We denote the category of S-modules by **Mod**, and the category of $\boldsymbol{Z}^n$-graded S-modules by ***Mod**. Here a morphism f in ***Mod** is an S-homomorphism $f:M\to N$ with $f(M_{\boldsymbol{a}})\subset N_{\boldsymbol{a}}$ for all $\boldsymbol{a}\in\boldsymbol{Z}^n$. See [**12**] for information on ***Mod**.

For $M,N\in{}^*\mathbf{Mod}$ and $\boldsymbol{a}\in\boldsymbol{Z}^n$, set ${}^*\mathrm{Hom}_S(M,N)_{\boldsymbol{a}}:=\mathrm{Hom}_{{}^*\mathbf{Mod}}(M,N(\boldsymbol{a}))$. Then

$${}^*\mathrm{Hom}_S(M,N):=\bigoplus_{\boldsymbol{a}\in\boldsymbol{Z}^n}{}^*\mathrm{Hom}_S(M,N)_{\boldsymbol{a}}$$

has a natural $\boldsymbol{Z}^n$-graded S-module structure. If M is finitely generated, then ${}^*\mathrm{Hom}_S(M,N)$ is isomorphic to the usual $\mathrm{Hom}_S(M,N)$ as the underlying S-module. Thus, we simply denote ${}^*\mathrm{Hom}_S(M,N)$ by $\mathrm{Hom}_S(M,N)$ in this case. In the same situation, $\mathrm{Ext}^i_S(M,N)$ also has a $\boldsymbol{Z}^n$-grading with $\mathrm{Ext}^i_S(M,N)_{\boldsymbol{a}}=\mathrm{Ext}^i_{{}^*\mathbf{Mod}}(M,N(\boldsymbol{a}))$.

For $\boldsymbol{a} \in \boldsymbol{Z}^n$, set $\operatorname{supp}_+(\boldsymbol{a}) := \{i \mid a_i > 0\} \subset [n] := \{1, \dots, n\}$. We say $\boldsymbol{a} \in \boldsymbol{Z}^n$ is *squarefree* if $a_i = 0, 1$ for all $i \in [n]$. When $\boldsymbol{a} \in \boldsymbol{Z}^n$ is squarefree, we sometimes identify $\boldsymbol{a}$ with $\operatorname{supp}_+(\boldsymbol{a})$. Let $\Delta \subset 2^{[n]}$ be a simplicial complex (i.e., $\Delta \neq \emptyset$, and $F \in \Delta$ and $G \subset F$ imply $G \in \Delta$). The *Stanley-Reisner ideal* of Δ is the squarefree monomial ideal $I_\Delta := (x^F \mid F \notin \Delta)$ of S. Any squarefree monomial ideal is the Stanley-Reisner ideal I_Δ for some Δ. We say S/I_Δ is the *Stanley-Reisner ring* of Δ.

Definition 2.1 ([**24**]). We say a $\boldsymbol{Z}^n$-graded S-module M is *squarefree*, if the following conditions are satisfied.

(a) M is $\boldsymbol{N}^n$-graded (i.e., $M_{\boldsymbol{a}} = 0$ if $\boldsymbol{a} \notin \boldsymbol{N}^n$), and $\dim_k M_{\boldsymbol{a}} < \infty$ for all $\boldsymbol{a} \in \boldsymbol{N}^n$.

(b) The multiplication map $M_{\boldsymbol{a}} \ni y \mapsto x^{\boldsymbol{b}} y \in M_{\boldsymbol{a}+\boldsymbol{b}}$ is bijective for all $\boldsymbol{a}, \boldsymbol{b} \in \boldsymbol{N}^n$ with $\operatorname{supp}_+(\boldsymbol{a}+\boldsymbol{b}) = \operatorname{supp}_+(\boldsymbol{a})$.

A squarefree module M is generated by its squarefree part $\bigcup_{F \subset [n]} M_F$. Thus it is finitely generated. For a simplicial complex $\Delta \subset 2^{[n]}$, I_Δ and S/I_Δ are squarefree modules. A free module $S(-F)$, $F \subset [n]$, is also squarefree. In particular, the $\boldsymbol{Z}^n$-graded canonical module $\omega_S = S(-\mathbf{1})$ of S is squarefree, where $\mathbf{1} = (1, \dots, 1)$.

Denote by $\mathbf{Sq}_S$ (or simply $\mathbf{Sq}$) the full subcategory of ${}^*\mathbf{Mod}$ consisting of all the squarefree modules. In ${}^*\mathbf{Mod}, \mathbf{Sq}$ is closed under kernels, cokernels and extensions ([**24**, Lemma 2.3]). For the study of $\mathbf{Sq}$, the incidence algebra of a finite partially ordered set (*poset*, for short) is very useful, as shown in [**18**], [**27**]. In Section 4 of the present paper, we will use further properties of the incidence algebra (of a Boolean lattice). So we now recall basic properties of an incidence algebra for the reader's convenience. See [**2**, §III. 1] for undefined terminology.

Let P be a finite poset. The incidence algebra $\Lambda = I(P, k)$ of P over k is the k-vector space with a basis $\{e_{x,y} \mid x, y \in P \text{ with } x \geq y\}$. The k-bilinear multiplication defined by $e_{x,y} e_{z,w} = \delta_{y,z} e_{x,w}$ makes Λ a finite dimensional associative k-algebra. (The usual definition is the opposite ring of our Λ. But we use the above definition for the convenience in a later section.) Set $e_x := e_{x,x}$. Then $1 = \sum_{x \in P} e_x$ and $e_x e_y = \delta_{x,y} e_x$. We have $\Lambda \cong \bigoplus_{x \in P} \Lambda e_x$ as a left Λ-module, and each Λe_x is indecomposable.

An incidence algebra Λ is the algebra associated with a *quiver with relations*. For a poset P, we consider the quiver $\Gamma = \{\Gamma_0, \Gamma_1\}$ with $\Gamma_0 = P$ and

$$\Gamma_1 = \{x\cdot \leftarrow \cdot y \mid x, y \in P, x > y, \text{ but there is } no \ z \in P \text{ with } x > z > y\}.$$

So Γ is (essentially) the Hasse diagram of P. Set

$$\rho := \{p_1 - p_2 \mid p_1 \text{ and } p_2 \text{ are paths of } \Gamma \text{ with } s(p_1) = s(p_2) \text{ and } e(p_1) = e(p_2)\},$$

where $s(p_i)$ and $e(p_i)$ represent the initial vertex and the final vertex of p_i respectively. Let $k(\Gamma, \rho)$ be the algebra associated with (Γ, ρ). Then we have an isomorphism $\psi : k(\Gamma, \rho) \xrightarrow{\cong} \Lambda$. Here, if $[p]$ is the residue class containing a path p of Γ, we have $\psi([p]) = e_{x,y}$, where $x = e(p)$ and $y = s(p)$.

Denote the category of finitely generated left Λ-modules by $\mathbf{mod}_\Lambda$. If $N \in \mathbf{mod}_\Lambda$, we have $N = \bigoplus_{x \in P} N_x$ as a k-vector space, where $N_x := e_x N$. Note that $e_{x,y} N_y \subset N_x$ and $e_{x,y} N_z = 0$ for $y \neq z$. If $f : N \to N'$ is a morphism in $\mathbf{mod}_\Lambda$, then $f(N_x) \subset N'_x$. Under the isomorphism $\Lambda \cong k(\Gamma, \rho)$, $\mathbf{mod}_\Lambda$ is equivalent to the category $\mathbf{Rep}(\Gamma, \rho)$ of representations of (Γ, ρ) by [**2**, III, Proposition 1.7]. If $(V, f) \in \mathbf{Rep}(\Gamma, \rho)$ corresponds to $N \in \mathbf{mod}_\Lambda$, then $N_x = V(x)$ for $x \in P$. We have explicit descriptions of simple objects, indecomposable projectives, and indecomposable injectives in $\mathbf{mod}_\Lambda \cong \mathbf{Rep}(\Gamma, \rho)$, see [**2**, §III. 1].

Let $2^{[n]}$ be the Boolean lattice (i.e., we regard the power set $2^{[n]}$ of $[n]$ as a poset by inclusions), and $\Lambda = I(2^{[n]}, k)$ its incidence algebra. For $M \in \mathbf{Sq}$, set $\Phi(M) := N = \bigoplus_{F \subset [n]} N_F$ to be a k-vector space with $M_F \cong N_F$. Then N has a left Λ-module structure such that the multiplication map $N_F \ni y \mapsto e_{G,F}\, y \in N_G$ for $G \supset F$ is induced by $M_F \ni y \mapsto x^{(G \setminus F)} y \in M_G$. It is easy to see that Φ gives a covariant functor $\mathbf{Sq} \to \mathbf{mod}_\Lambda$. Recall that $\Lambda \cong k(\Gamma, \rho)$, where Γ is a quiver whose set of vertices is $2^{[n]}$, and $\mathbf{mod}_\Lambda \cong \mathbf{Rep}(\Gamma, \rho)$. If M is a squarefree module, $\Phi(M)$ corresponds to the representation $(V, f) \in \mathbf{Rep}(\Gamma, \rho)$ with $V(F) = M_F$ and $f_{F \cup \{i\}, F} : V(F) = M_F \ni y \mapsto x_i y \in M_{F \cup \{i\}} = V(F \cup \{i\})$ for $F \subset [n]$ and $i \in [n] \setminus F$. In [**26**], the author used *sheaves on a poset* to understand squarefree modules. But this notion is equivalent to that of representations of (Γ, ρ) in our context.

PROPOSITION 2.2 ([**26**], [**27**]). *Let $\Lambda = I(2^{[n]}, k)$ be the incidence algebra. The functor Φ constructed above gives an equivalence $\mathbf{Sq} \cong \mathbf{mod}_\Lambda$.*

For a subset $F \subset [n]$, P_F denotes the monomial prime ideal $(x_i \mid i \notin F)$ of S. The next result follows from Proposition 2.2 and [**2**, §III. 1].

COROLLARY 2.3 ([**25**]). $\mathbf{Sq}$ *is an abelian category, and has enough projectives and injectives. An indecomposable projective (resp. injective) object in* $\mathbf{Sq}$ *is isomorphic to $S(-F)$ (resp. S/P_F) for some $F \subset [n]$. For any squarefree module M, both* $\operatorname{proj.dim}_{\mathbf{Sq}} M$ *and* $\operatorname{inj.dim}_{\mathbf{Sq}} M$ *are at most n.*

Many invariants of squarefree modules are naturally described in terms of Λ. For example, if M is a squarefree module with $N := \Phi(M)$, $\dim_S M = \max\{|F| \mid N_F \neq 0\} = n - \min\{i \mid \operatorname{Ext}^i_\Lambda(N, \Lambda) \neq 0\}$ and $\operatorname{proj.dim}_S M = \operatorname{proj.dim}_\Lambda N = \max\{i \mid \operatorname{Ext}^i_\Lambda(N, \Lambda) \neq 0\}$. See Remark 3.3 below for information on $\operatorname{Ext}^i_\Lambda(N, \Lambda)$.

We also remark that $\mathbf{Sq}$ admits the Jordan-Hölder theorem and the Krull-Schmidt theorem and a simple object in $\mathbf{Sq}$ (i.e., a non-zero squarefree module without non-trivial squarefree submodule) is isomorphic to $(S/P_F)(-F)$ for some F.

DEFINITION 2.4 ([**25**]). A $\boldsymbol{Z}^n$-graded S-module $M = \bigoplus_{\boldsymbol{a} \in \boldsymbol{Z}^n} M_{\boldsymbol{a}}$ is called *straight*, if the following two conditions are satisfied.

(a) $\dim_k M_{\boldsymbol{a}} < \infty$ for all $\boldsymbol{a} \in \boldsymbol{Z}^n$.
(b) The multiplication map $M_{\boldsymbol{a}} \ni y \mapsto x^{\boldsymbol{b}} y \in M_{\boldsymbol{a}+\boldsymbol{b}}$ is bijective for all $\boldsymbol{a} \in \boldsymbol{Z}^n$ and $\boldsymbol{b} \in \boldsymbol{N}^n$ with $\operatorname{supp}_+(\boldsymbol{a} + \boldsymbol{b}) = \operatorname{supp}_+(\boldsymbol{a})$.

For a $\boldsymbol{Z}^n$-graded S-module $M = \bigoplus_{\boldsymbol{a} \in \boldsymbol{Z}^n} M_{\boldsymbol{a}}$, we call the submodule $\bigoplus_{\boldsymbol{a} \in \boldsymbol{N}^n} M_{\boldsymbol{a}}$ the $\boldsymbol{N}^n$-*graded part* of M, and denote it by $\mathcal{N}(M)$. If M is straight then $\mathcal{N}(M)$ is squarefree. Conversely, for any squarefree module N, there is a unique (up to isomorphism) straight module $\mathcal{Z}(N)$ whose $\boldsymbol{N}^n$-graded part is isomorphic to N. For example, $\mathcal{Z}(S/P_F) \cong {}^*E(S/P_F)$, where ${}^*E(S/P_F)$ is the injective envelope of S/P_F in ${}^*\mathbf{Mod}$. Denote by $\mathbf{Str}_S$ (or simply $\mathbf{Str}$) the full subcategory of ${}^*\mathbf{Mod}$ consisting of all the straight S-modules.

Proposition 2.5 ([**25**, Proposition 2.7]). *The functors* $\mathcal{N} : \mathbf{Str} \to \mathbf{Sq}$ *and* $\mathcal{Z} : \mathbf{Sq} \to \mathbf{Str}$ *give an equivalence* $\mathbf{Sq} \cong \mathbf{Str}$.

Let $\mathrm{Com}^b(\mathbf{Sq})$ be the category of bounded cochain complexes of squarefree modules, and $D^b(\mathbf{Sq})$ the bounded derived category of $\mathbf{Sq}$. A squarefree module M can be regarded as a complex $\cdots \to 0 \to M \to 0 \to \cdots$ with M at the 0^{th} place. For a complex $M^\bullet$ and an integer p, let $M^\bullet[p]$ be the p^{th} *translation* of $M^\bullet$. That is, $M^\bullet[p]$ is a complex with $M^i[p] = M^{i+p}$ and $d_{M[p]} = (-1)^p d_M$.

A complex $M^\bullet \in \mathrm{Com}^b(\mathbf{Sq})$ has a projective resolution $P^\bullet \in \mathrm{Com}^b(\mathbf{Sq})$. That is, there is a quasi-isomorphism $P^\bullet \to M^\bullet$ and each P^i is projective in $\mathbf{Sq}$. We say $P^\bullet$ is *minimal* if $d_P(P^{i-1}) \subset \mathfrak{m}P^i$ for all i. A minimal projective resolution of $M^\bullet \in \mathrm{Com}^b(\mathbf{Sq})$ in $\mathbf{Sq}$ is a $\boldsymbol{Z}^n$-graded minimal S-free resolution of $M^\bullet$. Under the same notation as Proposition 2.2, a projective resolution $P^\bullet \in \mathrm{Com}^b(\mathbf{Sq})$ is minimal if and only if so is $Q^\bullet := \Phi(P^\bullet) \in \mathrm{Com}^b(\mathbf{mod}_\Lambda)$, that is, $d_Q(Q^{i-1}) \subset \mathfrak{r}Q^i$ for all i. Here $\mathfrak{r} = \langle e_{F,G} \mid F \supsetneq G \rangle$ is the Jacobson radical of Λ. Hence every $M^\bullet \in \mathrm{Com}^b(\mathbf{Sq})$ has a unique minimal projective resolution, and any projective resolution is a direct sum of a minimal one and an exact complex. Let $P^\bullet$ be a minimal projective resolution of $M^\bullet \in \mathrm{Com}^b(\mathbf{Sq})$. We define $\beta_i(F, M^\bullet) \in \boldsymbol{N}$ so that

$$P^{-i} \cong \bigoplus_{F \subset [n]} S(-F)^{\beta_i(F, M^\bullet)}.$$

Similarly, every $M^\bullet \in \mathrm{Com}^b(\mathbf{Sq})$ has an injective resolution $I^\bullet \in \mathrm{Com}^b(\mathbf{Sq})$. That is, there is a quasi-isomorphism $M^\bullet \to I^\bullet$ and each I^i is injective in $\mathbf{Sq}$. We say $I^\bullet$ is *minimal* if I^i is a *essential extension of $\ker(d_I^i)$ for all i (i.e., $L \cap \ker(d_I^i) \neq \{0\}$ for any non-zero $\boldsymbol{Z}^n$-graded submodule L of I^i). As projective resolutions, $I^\bullet$ is minimal if and only if so is $J^\bullet := \Phi(I^\bullet)$ (i.e., each J^i is an essential extension of $\ker(d_J^i)$). Thus every $M^\bullet \in \mathrm{Com}^b(\mathbf{Sq})$ has a unique minimal injective resolution, and any injective resolution is a direct sum of a minimal one and an exact complex. For $M^\bullet \in \mathrm{Com}^b(\mathbf{Sq})$ and $F \subset [n]$, we define natural numbers $\bar{\mu}^i(F, M^\bullet)$ so that

$$I^i \cong \bigoplus_{F \subset [n]} (S/P_F)^{\bar{\mu}^i(F, M^\bullet)},$$

where $I^\bullet \in \mathrm{Com}^b(\mathbf{Sq})$ is a minimal injective resolution of $M^\bullet$. If $I^\bullet \in \mathrm{Com}^b(\mathbf{Sq})$ is a (minimal) injective resolution of $M^\bullet \in \mathrm{Com}^b(\mathbf{Sq})$, then $\mathcal{Z}(I^\bullet)$ is a (minimal) injective resolution of $\mathcal{Z}(M^\bullet)$ in ${}^*\mathbf{Mod}$. Hence

$$\bar{\mu}^i(F, M^\bullet) = \mu^i(P_F, \mathcal{Z}(M^\bullet)),$$

where $\mu^i(-)$ is the usual Bass number of a complex (cf. [**19**]).

Note that $\beta_i(F, -)$ and $\bar{\mu}^i(F, -)$ are invariants of isomorphic classes in $D^b(\mathbf{Sq})$.

For $M^\bullet$ and $N^\bullet$, we define a complex $\mathrm{Hom}_S^\bullet(M^\bullet, N^\bullet)$ by $\mathrm{Hom}_S^i(M^\bullet, N^\bullet) = \prod_{j \in \boldsymbol{Z}} \mathrm{Hom}_S(M^j, N^{i+j})$ and the differential $d^i(f) = ((-1)^i f_{j+1} d_M^j + d_N^{i+j} f_j)_{j \in \boldsymbol{Z}}$ for $f = (f_j)_{j \in \boldsymbol{Z}} \in \mathrm{Hom}_S^i(M^\bullet, N^\bullet)$. Note that if $M^\bullet, N^\bullet \in \mathrm{Com}^b({}^*\mathbf{Mod})$ and each M^i is finitely generated then $\mathrm{Hom}_S^\bullet(M^\bullet, N^\bullet) \in \mathrm{Com}^b({}^*\mathbf{Mod})$.

Lemma 2.6. *Let* $I^\bullet$ *be a* (*not necessarily minimal*) *injective resolution of* $M^\bullet \in \mathrm{Com}^b(\mathbf{Sq})$. *For* $F \subset [n]$, *we have*

$$\bar{\mu}^i(F, M^\bullet) = \dim_k[H^i(\mathrm{Hom}^\bullet_S(S/P_F, I^\bullet))]_F.$$

Proof. If $E^\bullet \in \mathrm{Com}^b(\mathbf{Sq})$ is an exact complex consisting of injective objects, then $E^\bullet$ splits and $\mathrm{Hom}^\bullet_S(S/P_F, E^\bullet)$ is clearly exact. So we may assume that $I^\bullet$ is minimal. Note that $\mathrm{Hom}_S(S/P_F, S/P_G)$ is isomorphic to S/P_G if $F \supset G$, and 0 otherwise. Hence we have $[\mathrm{Hom}^i_S(S/P_F, I^\bullet)]_F = k^{\bar{\mu}^i(F, M^\bullet)}$ and the differentials of $[\mathrm{Hom}^\bullet_S(S/P_F, I^\bullet)]_F$ are 0. So we are done. $\square$

3. Functors on the derived category of squarefree modules.

Let Λ be the incidence algebra of $2^{[n]}$. If $N \in \mathbf{mod}_\Lambda$, then $\mathrm{Hom}_k(N, k)$ has a right Λ-module (i.e., a left Λ^{op}-module) structure such that $(f\lambda)(a) = f(\lambda a)$ for $\lambda \in \Lambda$ and $a \in N$, see [**2**, §II. 3]. But the opposite ring Λ^{op} of Λ is isomorphic to Λ itself by $\Lambda^{\mathrm{op}} \ni e_{F,G} \mapsto e_{G^{\mathrm{c}}, F^{\mathrm{c}}} \in \Lambda$, where $F^{\mathrm{c}} := [n] \setminus F$. Thus $\mathrm{Hom}_k(-, k)$ gives a contravariant functor from $\mathbf{mod}_\Lambda$ to itself. By the equivalence $\mathbf{Sq} \cong \mathbf{mod}_\Lambda$ of Proposition 2.2, we have an exact contravariant functor from $\mathbf{Sq}$ to itself. We call this functor the *Alexander duality functor*, and denote it by $\boldsymbol{A}$. We have $\boldsymbol{A} \circ \boldsymbol{A} \cong \mathrm{Id}_{\mathbf{Sq}}$, see [**2**, II, Theorem 3.3].

The functor $\boldsymbol{A}$ was defined independently by Miller [**15**] and Römer [**20**] extending an idea of Eagon-Reiner [**8**]. But their constructions of $\boldsymbol{A}$ are different from the above one. Römer's definition is similar to ours, but it uses squarefree modules over an exterior algebra. Miller's definition uses straight modules and the Matlis duality. In fact, we have $\boldsymbol{A}(M) \cong \mathscr{N}({}^*\mathrm{Hom}_S(\mathscr{L}(M), {}^*E(k))(-\mathbf{1}))$.

It is easy to see that $\boldsymbol{A}(M)_F$ is the k-dual of $M_{F^{\mathrm{c}}}$, and the multiplication $\boldsymbol{A}(M)_F \ni y \mapsto x_i y \in \boldsymbol{A}(M)_{F \cup \{i\}}$ for $i \notin F$ is the k-dual of $M_{F^{\mathrm{c}} \setminus \{i\}} \ni y \mapsto x_i y \in M_{F^{\mathrm{c}}}$. For example, $\boldsymbol{A}(S(-F)) = S/P_{F^{\mathrm{c}}}$ and $\boldsymbol{A}(S/I_\Delta) = I_{\Delta^*}$, where $\Delta^* := \{F \subset [n] \mid F^{\mathrm{c}} \notin \Delta\}$ is (*Eagon-Reiner's*) *Alexander dual complex* ([**8**]) of Δ.

A complex $I^\bullet \in \mathrm{Com}^b(\mathbf{Sq})$ is a (minimal) injective resolution of $M^\bullet$ if and only if the Alexander dual $\boldsymbol{A}(I^\bullet)$ is a (minimal) projective resolution of $\boldsymbol{A}(M^\bullet)$. Hence we have $\bar{\mu}^i(F, M^\bullet) = \beta_i(F^{\mathrm{c}}, \boldsymbol{A}(M^\bullet))$.

The following is a key lemma of this section.

Lemma 3.1 ([**25**, Lemma 3.20]). *For a squarefree module* M *and a subset* $F \subset [n]$, $\mathscr{N}(\mathrm{Hom}_S(M, {}^*E(S/P_F)))$ *is isomorphic to* $(M_F)^* \otimes_k (S/P_F)$. *Here* $(M_F)^*$ *is the dual* k*-vector space of* M_F, *but we set the degree of* $(M_F)^*$ *to be* 0 (*since it is essentially* $\mathrm{Hom}_k(M_F, [S/P_F]_F)$). *In particular,* $\mathscr{N}(\mathrm{Hom}_S(M, {}^*E(S/P_F)))$ *is squarefree.*

Let $\omega^\bullet$ be a minimal injective resolution of $\omega_S[n]$ in ${}^*\mathbf{Mod}$ (according to the usual convention on dualizing complexes, we use $\omega_S[n]$ instead of ω_S itself). The complex $\omega^\bullet$ is of the form

$$\omega^\bullet : 0 \to \omega^{-n} \to \omega^{-n+1} \to \cdots \to \omega^0 \to 0, \tag{1}$$

$$\omega^i = \bigoplus_{\substack{F \subset [n] \\ |F| = -i}} {}^*E(S/P_F),$$

and the differential is composed of $(-1)^{\alpha(j,F)} \cdot \mathrm{nat} : {}^*E(S/P_F) \to {}^*E(S/P_{F \setminus \{j\}})$ for $j \in F$, where $\mathrm{nat}: {}^*E(S/P_F) \to {}^*E(S/P_{F\setminus\{j\}})$ is induced by the natural surjection $S/P_F \to S/P_{F\setminus\{j\}}$, and $\alpha(j,F) := \#\{i \in F \mid i < j\}$. See [**7**, §5.7].

Proposition 3.2. *Let $M^\bullet \in \mathrm{Com}^b(\mathbf{Sq})$, and $P^\bullet \in \mathrm{Com}^b(\mathbf{Sq})$ its projective resolution. Then $\mathcal{N}(\mathrm{Hom}_S^\bullet(M^\bullet, \omega^\bullet)), \mathcal{N}(\mathrm{Hom}_S^\bullet(P^\bullet, \omega^\bullet))$ and $\mathrm{Hom}_S^\bullet(P^\bullet, \omega_S[n])$ belong to $\mathrm{Com}^b(\mathbf{Sq})$, and are isomorphic in $D^b(\mathbf{Sq})$.*

Proof. By Lemma 3.1, $\mathcal{N}(\mathrm{Hom}_S^\bullet(M^\bullet, \omega^\bullet))$ and $\mathcal{N}(\mathrm{Hom}_S^\bullet(P^\bullet, \omega^\bullet))$ are in $\mathrm{Com}^b(\mathbf{Sq})$. Since $\mathrm{Hom}_S(S(-F), \omega_S) \cong S(-F^{\mathrm{c}})$, $\mathrm{Hom}_S^\bullet(P^\bullet, \omega_S[n])$ also belongs to $\mathrm{Com}^b(\mathbf{Sq})$. Applying [**11**, Exercise III 5.1] to ${}^*\mathbf{Mod}$, we have $\boldsymbol{Z}^n$-graded quasi-isomorphisms $\mathrm{Hom}_S^\bullet(M^\bullet, \omega^\bullet) \to \mathrm{Hom}_S^\bullet(P^\bullet, \omega^\bullet)$ and $\mathrm{Hom}_S^\bullet(P^\bullet, \omega_S[n]) \to \mathrm{Hom}_S^\bullet(P^\bullet, \omega^\bullet)$. Hence we have quasi-isomorphisms

$$\mathcal{N}(\mathrm{Hom}_S^\bullet(M^\bullet, \omega^\bullet)) \to \mathcal{N}(\mathrm{Hom}_S^\bullet(P^\bullet, \omega^\bullet))$$

and

$$\mathrm{Hom}_S^\bullet(P^\bullet, \omega_S[n]) = \mathcal{N}(\mathrm{Hom}_S^\bullet(P^\bullet, \omega_S[n])) \to \mathcal{N}(\mathrm{Hom}_S^\bullet(P^\bullet, \omega^\bullet)). \qquad \square$$

It is easy to see that $\boldsymbol{D}: M^\bullet \mapsto \mathcal{N}(\mathrm{Hom}_S^\bullet(M^\bullet, \omega^\bullet))$ defines a contravariant functor from $D^b(\mathbf{Sq})$ to itself. If $P^\bullet$ is a projective resolution of $M^\bullet, \mathrm{Hom}_S^\bullet(P^\bullet, \omega_S[n])$ and $\mathcal{N}(\mathrm{Hom}_S^\bullet(P^\bullet, \omega^\bullet))$ are isomorphic to $\boldsymbol{D}(M^\bullet)$ in $D^b(\mathbf{Sq})$ by Proposition 3.2. Hence $H^i(\boldsymbol{D}(M^\bullet)) = \mathrm{Ext}_S^{n+i}(M^\bullet, \omega_S)$ and $\boldsymbol{D} \circ \boldsymbol{D} \cong \mathrm{Id}_{D^b(\mathbf{Sq})}$.

Remark 3.3. Let Λ be the incidence algebra of $2^{[n]}$. For $N \in \mathbf{mod}_\Lambda$, the right Λ-module $\mathrm{Hom}_\Lambda(N, \Lambda)$ can be seen as a left Λ-module by the isomorphism $\Lambda^{\mathrm{op}} \cong \Lambda$ given in the beginning of this section. Similarly, $\mathrm{Ext}_\Lambda^i(N, \Lambda) \in \mathbf{mod}_\Lambda$. Let $\Phi: \mathbf{Sq} \to \mathbf{mod}_\Lambda$ be the functor of Proposition 2.2, and let $\boldsymbol{P}_{\mathbf{Sq}}$ (resp. $\boldsymbol{P}_\Lambda$) be the full subcategory of $\mathbf{Sq}$ (resp. $\mathbf{mod}_\Lambda$) consisting of projective objects. Then the homotopic categories $K^b(\boldsymbol{P}_{\mathbf{Sq}})$ and $K^b(\boldsymbol{P}_\Lambda)$ are equivalent to $D^b(\mathbf{Sq})$ and $D^b(\mathbf{mod}_\Lambda)$ respectively. Both $\mathrm{Hom}_\Lambda^\bullet(\Phi(-), \Lambda)$ and $\Phi \circ \mathrm{Hom}_S^\bullet(-, \omega_S)$ define functors from $K^b(\mathbf{Sq})(\cong D^b(\mathbf{Sq}))$ to $K^b(\boldsymbol{P}_\Lambda)(\cong \boldsymbol{D^b}(\mathbf{mod}_\Lambda))$ and the isomorphisms

$$\mathrm{Hom}_\Lambda(\Phi(S(-F)), \Lambda) \cong \mathrm{Hom}_\Lambda(\Lambda e_F, \Lambda) \cong \Lambda e_{F^{\mathrm{c}}} \cong \Phi(\mathrm{Hom}_S(S(-F), \omega_S))$$

give a natural equivalence $\mathrm{Hom}_\Lambda^\bullet(\Phi(-), \Lambda) \cong \Phi \circ \mathrm{Hom}_S^\bullet(-, \omega_S)$. Hence if M is a squarefree module, then $\mathrm{Ext}_\Lambda^i(\Phi(M), \Lambda) \cong \Phi(\mathrm{Ext}_S^i(M, \omega_S))$. Moreover $\boldsymbol{D}$ corresponds to the right derived functor $\boldsymbol{R}\mathrm{Hom}_\Lambda^\bullet(-, \Lambda)$ up to translation.

For $N \in \mathbf{mod}_\Lambda$, we have $\mathrm{Hom}_k(N, k) \cong \mathrm{Hom}_\Lambda(N, \bar{E})$ as left Λ^{op} $(\cong \Lambda)$-modules, where $\bar{E}$ is the injective envelope of $\Lambda/\mathfrak{r}$ as a left Λ-module, see [**2**, §II. 3]. So $\boldsymbol{A}$ is a representable functor too.

Let $I^\bullet$ be a minimal injective resolution of $M^\bullet \in \mathrm{Com}^b({}^*\mathbf{Mod})$ in ${}^*\mathbf{Mod}$. For $\boldsymbol{a} \in \boldsymbol{Z}^n$ and $i \in \boldsymbol{N}$, let $\mu^i(\mathfrak{m}, M^\bullet)_{\boldsymbol{a}}$ be the number of copies of ${}^*E(S/\mathfrak{m})(\boldsymbol{a})$ which appear in the Krull-Schmidt decomposition of I^i.

Proposition 3.4. *If $M^\bullet \in D^b(\mathbf{Sq})$, then $\mu^i(\mathfrak{m}, M^\bullet)_{\boldsymbol{a}} \neq 0$ implies $\boldsymbol{a}$ is squarefree. Moreover $\mu^i(\mathfrak{m}, M^\bullet)_F = \beta_i(F, \boldsymbol{D}(M^\bullet))$ for all $F \subset [n]$.*

Proof. Since $\mathrm{Hom}_S(S(-\boldsymbol{a}), {}^*E(k)) = {}^*E(k)(\boldsymbol{a})$, the argument of [**19**, Theorem 3.6] also works here. $\square$

For a squarefree module M, we can describe $\boldsymbol{D}(M) = \mathcal{N}(\mathrm{Hom}^\bullet(M, \omega^\bullet))$ explicitly. By Lemma 3.1, we have

$$\text{(2)} \qquad \boldsymbol{D}(M) : 0 \to \boldsymbol{D}^{-n}(M) \to \boldsymbol{D}^{-n+1}(M) \to \cdots \to \boldsymbol{D}^0(M) \to 0,$$

$$\boldsymbol{D}^i(M) = \bigoplus_{\substack{F \subset [n] \\ |F| = -i}} (M_F)^* \otimes_k (S/P_F).$$

As in the lemma, the degree of $(M_F)^*$ is $0 \in \boldsymbol{Z}^n$. The differential is composed of the maps

$$(-1)^{\alpha(j,F)} \cdot (v_j)^* \otimes_k \mathrm{nat} : (M_F)^* \otimes_k S/P_F \to (M_{F \setminus \{j\}})^* \otimes_k S/P_{F \setminus \{j\}}$$

for $j \in F$. Here $(v_j)^*$ is the k-dual of the multiplication map $v_j : M_{F \setminus \{j\}} \ni y \mapsto x_j y \in M_F$ and "nat" is the natural surjection $S/P_F \to S/P_{F \setminus \{j\}}$. Note that $\boldsymbol{D}(M)$ is a complex of injective objects in **Sq** and it is minimal. Thus we have

$$\bar{\mu}^i(F, \boldsymbol{D}(M)) = \begin{cases} \dim_k M_F & \text{if } i = -|F|, \\ 0 & \text{otherwise.} \end{cases}$$

For a complex $M^\bullet = \{M^i, \delta^i\} \in \mathrm{Com}^b(\mathbf{Sq})$, we can also describe the complex $\boldsymbol{D}(M^\bullet)$ in a similar way. In fact,

$$\boldsymbol{D}^t(M^\bullet) = \bigoplus_{i-j=t} \boldsymbol{D}^i(M^j) = \bigoplus_{-|F|-j=t} (M_F^j)^* \otimes_k (S/P_F),$$

and the differential is given by

$$\boldsymbol{D}^t(M^\bullet) \supset (M_F^j)^* \otimes_k (S/P_F) \ni x \otimes y \mapsto d_{\boldsymbol{D}(M^j)}(x \otimes y) + (-1)^t (\delta^*(x) \otimes y) \in \boldsymbol{D}^{t+1}(M^\bullet),$$

where $\delta^* : (M_F^j)^* \to (M_F^{j-1})^*$ is the k-dual of $\delta_F^{j-1} : M_F^{j-1} \to M_F^j$, and $d_{\boldsymbol{D}(M^j)}$ is the $-|F|^{\text{th}}$ differential of $\boldsymbol{D}(M^j)$. The complex $\boldsymbol{D}(M^\bullet)$ is a complex of injective objects, but it is not minimal in general.

Proposition 3.5 (cf. [**25**, Proposition 3.8]). *If* $M^\bullet \in \mathrm{Com}^b(\mathbf{Sq})$, *then*

$$\bar{\mu}^i(F, M^\bullet) = \mu^i(P_F, \mathscr{Z}(M^\bullet)) = \dim_k [\mathrm{Ext}_S^{n-|F|-i}(M^\bullet, \omega_S)]_F.$$

Proof. Since $\boldsymbol{D}^2 \cong \mathrm{Id}_{\mathbf{Sq}}$, it suffices to show $\bar{\mu}^i(F, \boldsymbol{D}(M^\bullet)) = \dim_k [H^{-|F|-i}(M^\bullet)]_F$. To see this, we use Lemma 2.6. The differential $d_{\boldsymbol{D}(M^j)}$ induces the zero map on $[\mathrm{Hom}_S^\bullet(S/P_F, \boldsymbol{D}(M^\bullet))]_F$. Thus the complex $[\mathrm{Hom}_S^\bullet(S/P_F, \boldsymbol{D}(M^\bullet))]_F$ of k-vector spaces is isomorphic to the complex $(M_F^\bullet)^*[|F|]$. So we are done. $\square$

The next result was proved in [**21**, Theorem 2.6] for the module case.

Corollary 3.6. *If* $M^\bullet \in \mathrm{Com}^b(\mathbf{Sq})$, *then*

$$\beta_i(F, M^\bullet) = \dim_k [\mathrm{Ext}_S^{|F|-i}(\boldsymbol{A}(M^\bullet), \omega_S)]_{F^{\mathsf{c}}}.$$

Proof. We have $\beta_i(F, M^\bullet) = \bar{\mu}^i(F^{\mathsf{c}}, \boldsymbol{A}(M^\bullet)) = \dim_k [H^{-|F^{\mathsf{c}}|-i}(\boldsymbol{D} \circ \boldsymbol{A}(M^\bullet))]_{F^{\mathsf{c}}}$. $\square$

Let M be a squarefree module. Next we will describe the complex

$$\mathscr{F}(M) := \boldsymbol{A} \circ \boldsymbol{D}(M) = \boldsymbol{A}(\mathscr{N}(\mathrm{Hom}^\bullet(M, \omega^\bullet))).$$

For each $F \subset [n]$, $(M_F)^\circ$ denotes a k-vector space with a bijection $\psi_F : M_F \to (M_F)^\circ$. We denote $\psi_F(y) \in (M_F)^\circ$ by y°, and set $\deg(y^\circ) = 0$. (The essential meaning of M_F° is the k-dual of $\mathrm{Hom}_k(M_F, (S/P_F)_F)$.) Then

$$\mathscr{F}^i(M) = \bigoplus_{|F|=i} (M_F)^\circ \otimes_k S(-F^{\mathsf{c}})$$

and the differential map is given by

$$d(y^\circ \otimes s) = \sum_{j \notin F} (-1)^{\alpha(j,F)} (x_j y)^\circ \otimes x_j s.$$

Since $\boldsymbol{A}$ is faithful and exact, we have the following.

Corollary 3.7 (cf. [**24**, Theorem 2.10]). *For all $i \in \boldsymbol{Z}$ and all $M \in \mathbf{Sq}$, we have $H^i(\mathscr{F}(M)) = \boldsymbol{A}(\mathrm{Ext}_S^{n-i}(M, \omega_S))$. In particular, $H^i(\mathscr{F}(M)) = 0$ for all $i \neq d$ if and only if M is a Cohen-Macaulay module of dimension d or $M = 0$.*

For a complex $M^\bullet = \{M^i, \delta^i\} \in \mathrm{Com}^b(\mathbf{Sq})$, we can also describe $\mathscr{F}(M^\bullet) = \boldsymbol{A} \circ \boldsymbol{D}(M^\bullet)$ in the following way:

$$\mathscr{F}^t(M^\bullet) = \bigoplus_{i+j=t} \mathscr{F}^i(M^j) = \bigoplus_{|F|+j=t} (M_F^j)^\circ \otimes_k S(-F^{\mathsf{c}}),$$

and the differential is given by

$$\mathscr{F}^t(M^\bullet) \supset (M_F^j)^\circ \otimes_k S(-F^{\mathsf{c}}) \ni y^\circ \otimes s \mapsto d_{\mathscr{F}(M^j)}(y^\circ \otimes s) + (-1)^t \bar{\delta}^j(y^\circ) \otimes s \in \mathscr{F}^{t+1}(M^\bullet).$$

Here $d_{\mathscr{F}(M^j)}$ is the $|F|^{\text{th}}$ differential of $\mathscr{F}(M^j)$, and $\bar{\delta}^j : (M_F^j)^\circ \to (M_F^{j+1})^\circ$ is induced by $\delta^j : M^j \to M^{j+1}$. Note that $\mathscr{F}(M^\bullet)$ is a complex of projective objects, but not minimal in general.

Let $P^\bullet$ be a *minimal* projective resolution of $M^\bullet \in \mathrm{Com}^b(\mathbf{Sq})$. Thus

$$P^j = \bigoplus_{F \subset [n]} S(-F)^{\beta_{-j}(F, M^\bullet)}.$$

For an integer i, the *i-linear strand* $P^\bullet_{\langle i \rangle}$ is defined to be the complex such that

$$P^j_{\langle i \rangle} = \bigoplus_{|F|=i-j} S(-F)^{\beta_{-j}(F, M^\bullet)}$$

is a direct summand of P^j and the differential $P^j_{\langle i \rangle} \to P^{j+1}_{\langle i \rangle}$ is the corresponding component of the differential $P^j \to P^{j+1}$ of $P^\bullet$ (so this map is represented by a matrix of linear forms). The next result generalizes [**24**, Theorem 4.1].

Theorem 3.8. *If $M^\bullet \in D^b(\mathbf{Sq})$, the i-linear strand $P^\bullet_{\langle i \rangle}$ of $M^\bullet$ is isomorphic to $\mathscr{F}(\mathrm{Ext}_S^i(\boldsymbol{A}(M^\bullet), \omega_S))[n-i]$.*

The following is immediate from Corollary 3.7 and Theorem 3.8.

Corollary 3.9 (Römer [**21**]). *Let M be a squarefree module. Then M is componentwise linear (i.e., the i-linear strand $P^\bullet_{\langle i \rangle}$ is acyclic for any i) if and only if $\boldsymbol{A}(M)$*

is sequentially Cohen-Macaulay (*i.e.*, $\mathrm{Ext}_S^i(\boldsymbol{A}(M), \omega_S)$ *is a Cohen-Macaulay module of dimension* $n - i$ *for all* i).

To prove Theorem 3.8, we reconstruct $P^\bullet_{\langle i \rangle}$ using the spectral sequence. Let $Q^\bullet$ be a (not necessarily minimal) projective resolution of $M^\bullet \in \mathrm{Com}^b(\mathbf{Sq})$. Consider the $\mathfrak{m}$-adic filtration $Q^\bullet = F_0 Q^\bullet \supset F_1 Q^\bullet \supset \cdots$ of $Q^\bullet$ with $F_i Q^\bullet = \mathfrak{m}^i Q^\bullet$. Set $\mathrm{gr}_\mathfrak{m}(M) := \bigoplus_{i \geq 0} \mathfrak{m}^i M / \mathfrak{m}^{i+1} M$ for an S-module M, and regard it as a module over $\mathrm{gr}_\mathfrak{m} S = \bigoplus_{i \geq 0} \mathfrak{m}^i / \mathfrak{m}^{i+1} \cong S$. Since Q^t is a free S-module, $Q_0^t := \bigoplus_{p+q=t} E_0^{p,q} = \bigoplus_{p \geq 0} \mathfrak{m}^p Q^t / \mathfrak{m}^{p+1} Q^t = \mathrm{gr}_\mathfrak{m} Q^t$ is isomorphic to Q^t (if we identify $\mathrm{gr}_\mathfrak{m} S$ with S). The maps $d_0^{p,q} : E_0^{p,q} \to E_0^{p,q+1}$ make $Q_0^\bullet$ a cochain complex of free $\mathrm{gr}_\mathfrak{m}(S)$-modules. Consider the decomposition $Q^\bullet = P^\bullet \oplus C^\bullet$, where $P^\bullet$ is minimal and $C^\bullet$ is exact. If we identify Q_0^t with $Q^t = P^t \oplus C^t$, the differential d_0 of $Q_0^\bullet$ is given by $(0, d_C)$. Hence we have $Q_1^t = \bigoplus_{p+q=t} E_1^{p,q} \cong P^t$. The maps $d_1^{p,q} : E_1^{p,q} = \mathfrak{m}^p P^t / \mathfrak{m}^{p+1} P^t \to E_1^{p+1,q} = \mathfrak{m}^{p+1} P^{t+1} / \mathfrak{m}^{p+2} P^{t+1}$ make $Q_1^\bullet$ a cochain complex of free $\mathrm{gr}_\mathfrak{m}(S) (\cong S)$-modules whose differential is the "linear term" of the differential d_P of $P^\bullet$. Thus, under the identification $Q_1^t = P^t$, the complex $Q_1^\bullet$ is isomorphic to $\bigoplus_{i \in \boldsymbol{Z}} P^\bullet_{\langle i \rangle}$.

Proof of Theorem 3.8. Since $\boldsymbol{A} \circ \boldsymbol{D} \circ \boldsymbol{D} \circ \boldsymbol{A} \cong \mathrm{Id}_{D^b(\mathbf{Sq})}$, it suffices to prove the i-linear strand of $\boldsymbol{A} \circ \boldsymbol{D}(M^\bullet)$ is isomorphic to $\mathscr{F}(H^{-n+i}(M^\bullet))[n-i]$. Recall that $\mathscr{F}(M^\bullet) = \boldsymbol{A} \circ \boldsymbol{D}(M^\bullet)$ is a complex of projective objects. Set $Q^\bullet = \mathscr{F}(M^\bullet)$, and consider the $\mathfrak{m}$-adic filtration $F_i Q^\bullet = \mathfrak{m}^i Q^\bullet$ of $Q^\bullet$. Under the above notation, the differential $d_0^t : Q_0^t \cong \mathscr{F}^t(M^\bullet) \to Q_0^{t+1} \cong \mathscr{F}^{t+1}(M^\bullet)$ is given by $(-1)^t \delta$. Thus

$$Q_1^t \cong \bigoplus_{|F|+j=t} H^j(M^\bullet)_F \otimes_k S(-F^\mathsf{c}) = \bigoplus_{l+j=t} \mathscr{F}^l(H^j(M^\bullet)),$$

and the differential of $Q_1^\bullet$ is induced by that of $\mathscr{F}(M^j)$. Hence we can easily check that $Q_1^\bullet$ is isomorphic to $\bigoplus_{j \in \boldsymbol{Z}} \mathscr{F}(H^j(M^\bullet))[-j]$. By the remark before this proof, the i-linear strand of $\boldsymbol{A} \circ \boldsymbol{D}(M^\bullet)$ is isomorphic to $\mathscr{F}(H^{-n+i}(M^\bullet))[n-i]$. $\square$

Theorem 3.10. *We have a natural equivalence* $\boldsymbol{D} \circ \boldsymbol{A} \circ \boldsymbol{D} \circ \boldsymbol{A} \circ \boldsymbol{D} \circ \boldsymbol{A} \cong \boldsymbol{T}^{2n}$ *in* $D^b(\mathbf{Sq})$, *where* $\boldsymbol{T}$ *is the translation functor* (*i.e.*, $\boldsymbol{T}^{2n} : M^\bullet \mapsto M^\bullet[2n]$).

Proof. For $M^\bullet = \{M^i, \delta^i\} \in \mathrm{Com}^b(\mathbf{Sq})$, the complex $\mathrm{Hom}_S^\bullet(\mathscr{F}(M^\bullet), \omega_S[n])$ is isomorphic to $\boldsymbol{D} \circ \boldsymbol{A} \circ \boldsymbol{D}(M^\bullet)$ in $D^b(\mathbf{Sq})$. We have

$$\begin{aligned} \mathrm{Hom}_S^i(\mathscr{F}(M^\bullet), \omega_S[n]) &= \mathrm{Hom}_S\left(\bigoplus_{-i-n=|F|+j} (M_F^j)^\circ \otimes_k S(-F^\mathsf{c}), \omega_S \right) \\ &= \bigoplus_{-i-n=|F|+j} (M_F^j)^* \otimes_k S(-F) \\ &= \bigoplus_{i=-n-|F|+j} (M_F^{-j})^* \otimes_k S(-F). \end{aligned}$$

Here we simply denote the dual vector space of $(M_F^{-j})^\circ$ by $(M_F^{-j})^*$, since $(M_F^{-j})^\circ \cong M_F^{-j}$ as k-vector spaces (only the degrees are different). Also here $\deg(M_F^{-j})^* = 0 \in \boldsymbol{Z}^n$. The differential of $\mathrm{Hom}_S^\bullet(\mathscr{F}(M^\bullet), \omega_S[n])$ is given by

$$(M_F^{-j})^* \otimes_k S(-F) \ni y \otimes s \mapsto \sum_{l \in F} (-1)^{\alpha(l,F)+n+|F|-j} v_l^*(y) \otimes x_l s + (-1)^{n-1} \delta^*(y) \otimes s,$$

where $v_l^* : (M_F^{-j})^* \to (M_{F \setminus \{l\}}^{-j})^*$ is the k-dual of $v_l : M_{F\setminus\{l\}}^{-j} \ni z \mapsto x_l z \in M_F^{-j}$, and $\delta^* : (M_F^{-j})^* \to (M_F^{-j-1})^*$ is the k-dual of $\delta^{-j-1} : M_F^{-j-1} \to M_F^{-j}$.

Similarly, $\mathscr{F}(\boldsymbol{A}(M^\bullet))$ represents $\boldsymbol{A} \circ \boldsymbol{D} \circ \boldsymbol{A}(M^\bullet)$ in $D^b(\mathbf{Sq})$, and we have

$$\begin{aligned}\mathscr{F}^i(\boldsymbol{A}(M^\bullet)) &= \bigoplus_{i=|F|+j} (\boldsymbol{A}^j(M)_F)^\circ \otimes_k S(-F^{\mathsf{c}}) \\ &= \bigoplus_{i=|F|+j} (M_{F^{\mathsf{c}}}^{-j})^* \otimes_k S(-F^{\mathsf{c}}) \\ &= \bigoplus_{i=n-|F|+j} (M_F^{-j})^* \otimes_k S(-F).\end{aligned}$$

Also here, we simply denote $(\boldsymbol{A}(M^{-j})_F)^\circ = ((M_{F^{\mathsf{c}}}^{-j})^*)^\circ$ by $(M_{F^{\mathsf{c}}}^{-j})^*$. The differential of the above complex is given by

$$(M_F^{-j})^* \otimes_k S(-F) \ni y \otimes s \mapsto \sum_{l \in F} (-1)^{\alpha(l,F^{\mathsf{c}})} v_l^*(y) \otimes x_l s + (-1)^{|F^{\mathsf{c}}|+j} \delta^*(y) \otimes s.$$

For an integer $l \in \boldsymbol{Z}$, set $\beta(l) := 1$ if $l \equiv 1, 2 \pmod 4$, and $\beta(l) := 0$ if $l \equiv 3, 0 \pmod 4$. We also set $\alpha(A,B) := \#\{(a,b) \mid a > b, a \in A, b \in B\}$ for $A, B \subset [n]$. Then the multiplication by $(-1)^{\alpha(F,[n])+\beta(|F|-j)+|F|n+j}$ on $(M_F^{-j})^* \otimes_k S(-F)$, which can be regarded as a submodule of both $\mathrm{Hom}_S^{-n-|F|+j}(\mathscr{F}(M^\bullet), \omega_S[n])$ and $\mathscr{F}^{n-|F|+j}(\boldsymbol{A}(M^\bullet))$, induces quasi-isomorphism between $\mathrm{Hom}_S^\bullet(\mathscr{F}(M^\bullet), \omega_S[n])$ and $\boldsymbol{T}^{2n} \circ \mathscr{F}(\boldsymbol{A}(M^\bullet))$. So $\boldsymbol{D} \circ \boldsymbol{A} \circ \boldsymbol{D} \cong \boldsymbol{T}^{2n} \circ \boldsymbol{A} \circ \boldsymbol{D} \circ \boldsymbol{A}$ as a functor on $D^b(\mathbf{Sq})$. Since $(\boldsymbol{A} \circ \boldsymbol{D} \circ \boldsymbol{A}) \circ (\boldsymbol{A} \circ \boldsymbol{D} \circ \boldsymbol{A}) \cong \mathrm{Id}_{D^b(\mathbf{Sq})}$, we get the assertion. □

EXAMPLE 3.11. For $F \subset [n]$, we have the following.

$$\begin{aligned}\boldsymbol{D} \circ \boldsymbol{A} \circ \boldsymbol{D} \circ \boldsymbol{A} \circ \boldsymbol{D} \circ \boldsymbol{A}(S(-F)) &= \boldsymbol{D} \circ \boldsymbol{A} \circ \boldsymbol{D} \circ \boldsymbol{A} \circ \boldsymbol{D}(S/P_{F^{\mathsf{c}}}) \\ &= \boldsymbol{D} \circ \boldsymbol{A} \circ \boldsymbol{D} \circ \boldsymbol{A}((S/P_{F^{\mathsf{c}}})(-F^{\mathsf{c}})[-|F|+n]) \\ &= \boldsymbol{D} \circ \boldsymbol{A} \circ \boldsymbol{D}((S/P_F)(-F)[|F|-n]) \\ &= \boldsymbol{D} \circ \boldsymbol{A}\,((S/P_F)[n]) \\ &= \boldsymbol{D}\,(S(-F^{\mathsf{c}})[-n]) \\ &= S(-F)[2n].\end{aligned}$$

4. Relation to Koszul duality.

Let $S = k[x_1, \dots, x_n]$ be a polynomial ring as in the previous sections, and $E := \bigwedge S_1^* = k\langle e_1, \dots, e_n \rangle$ an exterior algebra. E is a $\boldsymbol{Z}^n$-graded ring with $\deg(e_i) = (0, \dots, 0, -1, 0, \dots, 0) = -\deg(x_i)$ where -1 is in the i^{th} position. When we regard S and E as $\boldsymbol{Z}$-graded rings, we set $\deg(x_i) = 1$ and $\deg(e_i) = -1$ for all i. In this paper, E-modules are *left* E-modules unless otherwise specified. For a $\boldsymbol{Z}^n$-graded E-module M and

$\boldsymbol{a} \in \boldsymbol{Z}^n$, $M_{\boldsymbol{a}}$ means the degree $\boldsymbol{a}$ component of M, and $M(\boldsymbol{a})$ is the shifted module with $M(\boldsymbol{a})_{\boldsymbol{b}} = M_{\boldsymbol{a}+\boldsymbol{b}}$ as in the polynomial ring case.

Denote the category of finitely generated $\boldsymbol{Z}$-graded S-modules (resp. E-modules) by $\mathbf{mod}_S$ (resp. $\mathbf{mod}_E$). Although $\mathbf{mod}_S$ and $\mathbf{mod}_E$ are far from equivalent, a famous theorem of Bernstein-Gel'fand-Gel'fand [**4**] states that $D^b(\mathbf{mod}_S) \cong D^b(\mathbf{mod}_E)$ as triangulated categories. First, we will see that this equivalence also holds in the $\boldsymbol{Z}^n$-graded context. Denote the category of finitely generated $\boldsymbol{Z}^n$-graded S-modules (resp. E-modules) by ${}^*\mathbf{mod}_S$ (resp. ${}^*\mathbf{mod}_E$).

There are several papers concerning the Bernstein-Gel'fand-Gel'fand correspondence. But their conventions are not quite the same. In this paper, we basically follow [**10**], which is well suited for our purpose. Here we give functors defining $D^b({}^*\mathbf{mod}_S) \cong D^b({}^*\mathbf{mod}_E)$. For $M \in {}^*\mathbf{mod}_S$, we define $\mathscr{R}(M) = \mathrm{Hom}_k(E(-\mathbf{1}), M)$ to be a $\boldsymbol{Z}^n$-graded cochain complex of free E-modules as follows. (The original definition is $\mathscr{R}(M) = \mathrm{Hom}_k(E, M)$, but we use this grading. We will also shift the grading of $\mathscr{L}(N)$ defined below.) We can define a $\boldsymbol{Z}^n$-graded left E-module structure on $\mathrm{Hom}_k(E(-\mathbf{1}), M_{\boldsymbol{a}})$ by $(af)(e) = f(ea)$. Then $\mathrm{Hom}_k(E(-\mathbf{1}), M_{\boldsymbol{a}}) \cong E(-\boldsymbol{a})^{\oplus \dim_k M_{\boldsymbol{a}}}$. Set the cohomological degree of $\mathrm{Hom}_k(E(-\mathbf{1}), M_{\boldsymbol{a}})$ to be $\|\boldsymbol{a}\| := \sum_{j \in [n]} a_j$. The differential of $\mathscr{R}(M)$ is defined by

$$\mathrm{Hom}_k(E(-\mathbf{1}), M_{\boldsymbol{a}}) \ni f \mapsto \left[e \mapsto \sum_{i \in [n]} x_i f(e_i e) \right] \in \bigoplus_{i \in [n]} \mathrm{Hom}_k(E(-\mathbf{1}), M_{\boldsymbol{a}+\varepsilon_i}),$$

where $\varepsilon_i \in \boldsymbol{N}^n$ is the squarefree vector whose support is $\{i\}$. We also define the complex $\mathscr{R}(M^\bullet) = \bigoplus_{j \in \boldsymbol{Z}} \mathrm{Hom}_k(E(-\mathbf{1}), M^j)$ for a complex $M^\bullet = \{M^j, \delta^j\}$ in ${}^*\mathbf{mod}_S$. The cohomological degree i component of $\mathscr{R}(M^\bullet)$ is $\bigoplus_{i=j+\|\boldsymbol{a}\|} \mathrm{Hom}_k(E(-\mathbf{1}), M^j_{\boldsymbol{a}})$ and the differential is given by

$$\mathscr{R}^i(M^\bullet) \supset \mathrm{Hom}_k(E(-\mathbf{1}), M^j_{\boldsymbol{a}}) \ni f \mapsto d_{\mathscr{R}(M^j)}(f) + (-1)^i(\delta^j \circ f) \in \mathscr{R}^{i+1}(M^\bullet),$$

where $d_{\mathscr{R}(M^j)}$ is the $\|\boldsymbol{a}\|^{\text{th}}$ differential of $\mathscr{R}(M^j)$. We can apply $\mathscr{R}$ to a $\boldsymbol{Z}$-graded complex $M^\bullet \in \mathrm{Com}^b(\mathbf{mod}_S)$ (in this case, we replace $E(-\mathbf{1})$ by $E(-n)$). Then $\mathscr{R}$ is equivalent to the functor given in [**4**], [**3**], [**10**] up to degree shifting. For $M^\bullet \in \mathrm{Com}^b({}^*\mathbf{mod}_S)$, $\mathscr{R}(M^\bullet)$ has only finitely many non-vanishing cohomologies. And $\mathscr{R}$ induces a covariant functor from $D^b({}^*\mathbf{mod}_S)$ to $D^b({}^*\mathbf{mod}_E)$, which is also denoted by $\mathscr{R}$.

Next, we will define the functor $\mathscr{L} : \mathrm{Com}^b({}^*\mathbf{mod}_E) \to \mathrm{Com}^b({}^*\mathbf{mod}_S)$. Set $\mathscr{L}(N^\bullet) = \bigoplus_{i \in \boldsymbol{Z}} S(-\mathbf{1}) \otimes_k N^i$ for a complex $N^\bullet = \{N^i, \delta^i\}$ in ${}^*\mathbf{mod}_E$. The cohomological degree of $\mathscr{L}(N^\bullet)$ is given by $\mathscr{L}^i(N^\bullet) = \bigoplus_{i=j-\|\boldsymbol{a}\|} S(-\mathbf{1}) \otimes_k N^j_{\boldsymbol{a}}$. And the differential is defined by

$$\mathscr{L}^i(N^\bullet) \supset S(-\mathbf{1}) \otimes_k N^j_{\boldsymbol{a}} \ni s \otimes y \mapsto \sum_{l \in [n]} x_l s \otimes e_l y + (-1)^i (s \otimes \delta^j(y)) \in \mathscr{L}^{i+1}(N^\bullet).$$

If we apply $\mathscr{L}$ to $\boldsymbol{Z}$-graded complexes, it is equivalent to the functor given in [**4**], [**3**], [**10**] up to degree shifting. If $N^\bullet$ is bounded, so is $\mathscr{L}(N^\bullet)$. And $\mathscr{L}$ induces a covariant functor from $D^b({}^*\mathbf{mod}_E)$ to $D^b({}^*\mathbf{mod}_S)$, which is also denoted by $\mathscr{L}$.

In the $\boldsymbol{Z}$-graded case, Bernstein-Gel'fand-Gel'fand [**4**] (see also [**3**], [**10**]) states that

$\mathscr{L} : \mathrm{Com}^b(\mathbf{mod}_E) \to \mathrm{Com}^b(\mathbf{mod}_S)$ is a left adjoint to $\mathscr{R} : \mathrm{Com}^b(\mathbf{mod}_S) \to \mathrm{Com}^b(\mathbf{mod}_E)$, that is, we have a natural isomorphism

$$\varphi : \mathrm{Hom}_{\mathrm{Com}^b(\mathbf{mod}_S)}(\mathscr{L}(N^\bullet), M^\bullet) \xrightarrow{\cong} \mathrm{Hom}_{\mathrm{Com}^b(\mathbf{mod}_E)}(N^\bullet, \mathscr{R}(M^\bullet))$$

for $M^\bullet \in \mathrm{Com}^b(\mathbf{mod}_S)$ and $N^\bullet \in \mathrm{Com}^b(\mathbf{mod}_E)$. Moreover, the map $\mathscr{L} \circ \mathscr{R}(M^\bullet) \to M^\bullet$ associated to the identity map $\mathscr{R}(M^\bullet) \to \mathscr{R}(M^\bullet)$ is a quasi-isomorphism. Similarly, the map $N^\bullet \to \mathscr{R} \circ \mathscr{L}(N^\bullet)$ associated to the identity map $\mathscr{L}(N^\bullet) \to \mathscr{L}(N^\bullet)$ is a quasi-isomorphism. Hence $\mathscr{R}$ and $\mathscr{L}$ define an equivalence $D^b(\mathbf{mod}_S) \cong D^b(\mathbf{mod}_E)$.

We can regard a $\boldsymbol{Z}^n$-graded module as a $\boldsymbol{Z}$-graded module by $M_i = \bigoplus_{\|\boldsymbol{a}\|=i} M_{\boldsymbol{a}}$. In this sense, $\mathrm{Com}^b({}^*\mathbf{mod}_S)$ and $\mathrm{Com}^b({}^*\mathbf{mod}_E)$ are (non-full) subcategories of $\mathrm{Com}^b(\mathbf{mod}_S)$ and $\mathrm{Com}^b(\mathbf{mod}_E)$ respectively. If $M^\bullet \in \mathrm{Com}^b({}^*\mathbf{mod}_S)$ and $N^\bullet \in \mathrm{Com}^b \cdot ({}^*\mathbf{mod}_E)$, then the restriction of φ gives the isomorphism

$$\mathrm{Hom}_{\mathrm{Com}^b({}^*\mathbf{mod}_S)}(\mathscr{L}(N^\bullet), M^\bullet) \cong \mathrm{Hom}_{\mathrm{Com}^b({}^*\mathbf{mod}_E)}(N^\bullet, \mathscr{R}(M^\bullet)).$$

Thus the quasi-isomorphisms $\mathscr{L} \circ \mathscr{R}(M^\bullet) \to M^\bullet$ and $N^\bullet \to \mathscr{R} \circ \mathscr{L}(N^\bullet)$ are $\boldsymbol{Z}^n$-graded. Hence we have the following.

Theorem 4.1 (BGG correspondence ($\boldsymbol{Z}^n$-graded version)). *The functors $\mathscr{R}$ and $\mathscr{L}$ define an equivalence of triangulated categories $D^b({}^*\mathbf{mod}_S) \cong D^b({}^*\mathbf{mod}_E)$.*

The functors $\mathscr{R}$ and $\mathscr{L}$ are closely related to $\boldsymbol{D}$ and $\boldsymbol{A}$ of the previous section. To see this, we recall the definition of a squarefree module over E.

Definition 4.2 (Römer [**20**]). A $\boldsymbol{Z}^n$-graded E-module $N = \bigoplus_{\boldsymbol{a} \in \boldsymbol{Z}^n} N_{\boldsymbol{a}}$ is *squarefree* if N is finitely generated and $N = \bigoplus_{F \subset [n]} N_{-F}$.

For example, a monomial ideal of E is always squarefree. We denote the full subcategory of ${}^*\mathbf{mod}_E$ consisting of all the squarefree E-modules by $\mathbf{Sq}_E$. We have the functors $\mathscr{S} : \mathbf{Sq}_E \to \mathbf{Sq}_S$ and $\mathscr{E} : \mathbf{Sq}_S \to \mathbf{Sq}_E$ giving an equivalence $\mathbf{Sq}_S \cong \mathbf{Sq}_E$. Here $\mathscr{S}(N)_F = N_{-F}$ for $N \in \mathbf{Sq}_E$, and the multiplication map $\mathscr{S}(N)_F \ni y \mapsto x_i y \in \mathscr{S}(N)_{F \cup \{i\}}$ for $i \notin F$ is given by $\mathscr{S}(N)_F = N_{-F} \ni z \mapsto (-1)^{\alpha(i,F)} e_i z \in N_{-(F \cup \{i\})} = \mathscr{S}(N)_{F \cup \{i\}}$. See [**20**] for further information.

We have $D^b(\mathbf{Sq}_S) \cong D^b_{\mathbf{Sq}_S}({}^*\mathbf{mod}_S)$ by [**11**, Exercises III 2.2] (but use projective resolutions instead of injective resolutions). So $D^b(\mathbf{Sq}_S)$ can be seen as a full subcategory of $D^b({}^*\mathbf{mod}_S)$. On the other hand, for $N \in {}^*\mathbf{mod}_E$, set

$$N' := \bigoplus_{\boldsymbol{a} \in \boldsymbol{N}^n} N_{-\boldsymbol{a}} \quad \text{and} \quad N'' := \bigoplus_{\substack{\boldsymbol{a} \in \boldsymbol{N}^n \text{ and } \boldsymbol{a} \text{ is} \\ \textit{not} \text{ squarefree}}} N_{-\boldsymbol{a}} \subset N'.$$

Note that N' and N'' are E-modules, and $\mathscr{N}(N) := N'/N''$ is squarefree. If all cohomologies of $N^\bullet \in \mathrm{Com}^b({}^*\mathbf{mod}_E)$ are squarefree, then $\mathscr{N}(N^\bullet)$ and $N^\bullet$ are isomorphic in $D^b({}^*\mathbf{mod}_E)$. Hence we have $D^b_{\mathbf{Sq}_E}({}^*\mathbf{mod}_E) \cong D^b(\mathbf{Sq}_E)$.

Comparing $\mathscr{L}$ and $\mathscr{F} = \boldsymbol{A} \circ \boldsymbol{D}$ defined in the last section, we have the following.

Proposition 4.3. *If $N^\bullet$ is a* (*bounded*) *complex of squarefree E-modules, then $\mathscr{L}(N^\bullet) = S(-\mathbf{1}) \otimes_k N^\bullet$ is a* (*bounded*) *complex of squarefree S-modules. Hence $\mathscr{L}$ gives a functor from $D^b(\mathbf{Sq}_E)$ to $D^b(\mathbf{Sq}_S)$. Moreover, for $M^\bullet \in \mathrm{Com}^b(\mathbf{Sq}_S)$, we have $\mathscr{L} \circ \mathscr{E}(M^\bullet) = \boldsymbol{A} \circ \boldsymbol{D}(M^\bullet)$.*

On the other hand, $\mathscr{R}(M)$ is not a complex of squarefree E-modules. In fact, a free E-module $E(-\boldsymbol{a})$ is not squarefree unless $\boldsymbol{a} = 0$. But we have the following.

PROPOSITION 4.4. *If* $M^\bullet \in D^b(\mathbf{Sq}_S)$, *then* $\mathscr{R}(M^\bullet) \in D^b_{\mathbf{Sq}_E}({}^*\mathbf{mod}_E) \cong D^b(\mathbf{Sq}_E)$. *Moreover, we have a natural equivalence* $\mathscr{S} \circ \mathscr{R} \cong \boldsymbol{D} \circ \boldsymbol{A}$.

PROOF. We have $M^\bullet \cong \boldsymbol{A} \circ \boldsymbol{D} \circ \boldsymbol{D} \circ \boldsymbol{A}(M^\bullet) = \mathscr{L} \circ \mathscr{E} \circ \boldsymbol{D} \circ \boldsymbol{A}(M^\bullet)$ in $D^b(\mathbf{Sq}_S)$ (and in $D^b({}^*\mathbf{mod}_S)$) by Proposition 4.3. From Theorem 4.1, $\mathscr{R}(M^\bullet) \cong \mathscr{R} \circ \mathscr{L} \circ \mathscr{E} \circ \boldsymbol{D} \circ \boldsymbol{A}(M^\bullet) \cong \mathscr{E} \circ \boldsymbol{D} \circ \boldsymbol{A}(M^\bullet) \in D^b_{\mathbf{Sq}_E}({}^*\mathbf{mod}_E)$. Since $\mathscr{S} \circ \mathscr{E} \cong \mathrm{Id}_{D^b(\mathbf{Sq}_S)}$, we are done. $\square$

Let $R = \bigoplus_{i \geq 0} R_i$ be an $\boldsymbol{N}$-graded associative k-algebra such that $\dim_k R_i < \infty$ for all i and $R_0 \cong k^m$ for some $m \in \boldsymbol{N}$ as an algebra. Then $\mathfrak{r} := \bigoplus_{i>0} R_i$ is the graded Jacobson radical. We say R is *Koszul*, if a left R-module $R/\mathfrak{r}$ admits a graded projective resolution

$$\cdots \to P^{-2} \to P^{-1} \to P^0 \to R/\mathfrak{r} \to 0$$

such that P^{-i} is generated by its degree i component, that is, $P^{-i} = RP^{-i}_i$ (we say such a resolution is a *linear* resolution). If R is Koszul, it is a quadratic ring, and its *quadratic dual ring* $R^!$ (see [**3**, Definition 2.8.1]) is Koszul again, and isomorphic to the opposite ring of the Yoneda algebra $E(R) := \bigoplus_{i \geq 0} \mathrm{Ext}^i_R(R/\mathfrak{r}, R/\mathfrak{r})$.

Let $\mathbf{gr.mod}_R$ be the category of finitely generated $\boldsymbol{Z}$-graded left R-modules. If R is a Koszul algebra with $R_i = 0$ for $i \gg 0$, and $R^!$ is left noetherian, we have functors

$$DF : D^b(\mathbf{gr.mod}_R) \ni N^\bullet \mapsto R^! \otimes_{R_0} N^\bullet \in D^b(\mathbf{gr.mod}_{R^!})$$

and

$$DG : D^b(\mathbf{gr.mod}_{R^!}) \ni M^\bullet \mapsto \mathrm{Hom}_{R_0}(R, M^\bullet) \in D^b(\mathbf{gr.mod}_R)$$

giving the equivalence $D^b(\mathbf{gr.mod}_R) \cong D^b(\mathbf{gr.mod}_{R^!})$ called *Koszul duality*, see [**3**, Theorem 2.12.6]. The exterior algebra E is a Koszul algebra with $E^! \cong S$. Thus the Bernstein-Gel'fand-Gel'fand correspondence is a classical example of Koszul duality.

Let Λ be the incidence algebra of $2^{[n]}$ over k. Then Λ has an $\boldsymbol{N}$-grading with $\deg(e_{F,G}) = |F \backslash G|$. Note that $\Lambda_0 = \bigoplus_{F \subset [n]} ke_F \cong k^{2^n}$. For each $i \in \boldsymbol{Z}$, let $\mathbf{gr.mod}_\Lambda(i)$ be the full subcategory of $\mathbf{gr.mod}_\Lambda$ consisting of $N \in \mathbf{gr.mod}_\Lambda$ such that $N_j = \bigoplus_{|F|=i+j} N_F$ for all $j \in \boldsymbol{Z}$. The forgetful functor gives an equivalence $\mathbf{gr.mod}_\Lambda(i) \cong \mathbf{mod}_\Lambda$ for all $i \in \boldsymbol{Z}$, and $D^b(\mathbf{gr.mod}_\Lambda(i)) \cong D^b_{\mathbf{gr.mod}_\Lambda(i)}(\mathbf{gr.mod}_\Lambda)$ is a full subcategory of $D^b(\mathbf{gr.mod}_\Lambda)$. Since $\Lambda/\mathfrak{r}\Lambda \cong \bigoplus_{F \subset [n]} ke_F$ as left Λ-modules, and each ke_F has a linear projective resolution

$$\cdots \to \bigoplus_{\substack{G \supset F \\ |G|=|F|+2}} \Lambda e_G \to \bigoplus_{\substack{G \supset F \\ |G|=|F|+1}} \Lambda e_G \to \Lambda e_F \to ke_F \to 0$$

(here we regard ke_F and Λe_G as objects in $\mathbf{gr.mod}_\Lambda(|F|)$), Λ is Koszul. To see this we can use Proposition 2.2. In fact, a minimal free resolution of a squarefree module $(S/P_F)(-F)$, which corresponds to ke_F, is given by the Koszul complex with respect to $\{x_i \mid i \notin F\}$.

LEMMA 4.5. *The quadratic dual ring* $\Lambda^!$ *of* Λ *is isomorphic to* Λ *itself.*

One might think $\Lambda^!$ should be a "negatively graded ring", since $\Lambda^!$ is generated by $\mathrm{Hom}_{\Lambda_0}(\Lambda_1, \Lambda_0)$ as a $\Lambda_0^!$-algebra. But we use the same convention as [**3**] here, so we regard $\Lambda^!$ as a positively graded ring with $\Lambda_1^! = \mathrm{Hom}_{\Lambda_0}(\Lambda_1, \Lambda_0)$.

Proof. Let $T := T_{\Lambda_0}\Lambda_1 = \Lambda_0 \oplus \Lambda_1 \oplus (\Lambda_1 \otimes_{\Lambda_0} \Lambda_1) \oplus \cdots = \bigoplus_{i \geq 0} \Lambda_1^{\otimes i}$ be the tensor ring of $\Lambda_1 = \langle e_{F\cup\{i\},F} \,|\, F \subset [n], i \notin F \rangle$. (See [**3**, §2.7] for the linear algebra over a semi-simple algebra Λ_0 used here.) Then $\Lambda \cong T/I$, where

$$I = (e_{F\cup\{i,j\},F\cup\{i\}} \otimes e_{F\cup\{i\},F} - e_{F\cup\{i,j\},F\cup\{j\}} \otimes e_{F\cup\{j\},F} \,|\, F \subset [n], i, j \notin F)$$

is a two sided ideal. Let $\Lambda_1^* := \mathrm{Hom}_{\Lambda_0}(\Lambda_1, \Lambda_0)$ be the dual of the *left* Λ_0-module Λ_1. Then Λ_1^* has a right Λ_0-module structure such that $(fa)(v) = (f(v))a$, and a left Λ_0-module structure such that $(af)(v) = f(va)$, where $a \in \Lambda_0$, $f \in \Lambda_1^*$, $v \in \Lambda_1$. As a left (or right) Λ_0-module, Λ_1^* is generated by $\{e^*_{F,F\cup\{i\}} \,|\, F \subset [n], i \notin F\}$, where

$$e^*_{F,F\cup\{i\}}(e_{G\cup\{j\},G}) = \delta_{F,G}\delta_{i,j} e_{F\cup\{i\}}.$$

Let $T^* = T_{\Lambda_0}\Lambda_1^*$ be the tensor ring of Λ_1^*. Note that $e^*_{F,F\cup\{i\}} \otimes e^*_{G,G\cup\{j\}} \neq 0$ if and only if $F \cup \{i\} = G$. We have that $(\Lambda_1^* \otimes_{\Lambda_0} \Lambda_1^*)$ is isomorphic to $(\Lambda_1 \otimes_{\Lambda_0} \Lambda_1)^* = \mathrm{Hom}_{\Lambda_0}(\Lambda_1 \otimes_{\Lambda_0} \Lambda_1, \Lambda_0)$ via $(f \otimes g)(v \otimes w) = g(vf(w))$, where $f, g \in \Lambda_1^*$ and $v, w \in \Lambda_1$. In particular,

$$(e^*_{F,F\cup\{i\}} \otimes e^*_{F\cup\{i\},F\cup\{i,j\}})(e_{F\cup\{i,j\},F\cup\{i\}} \otimes e_{F\cup\{i\},F}) = e_{F\cup\{i,j\}}.$$

Easy computation shows that the quadratic dual ideal

$$I^\perp = (f \in \Lambda_1^* \otimes \Lambda_1^* \,|\, f(v) = 0 \quad \text{for all } v \in I_2 \subset \Lambda_1 \otimes \Lambda_1 = T_2) \subset T^*$$

of I is equal to

$$(e^*_{F,F\cup\{i\}} \otimes e^*_{F\cup\{i\},F\cup\{i,j\}} + e^*_{F,F\cup\{j\}} \otimes e^*_{F\cup\{j\},F\cup\{i,j\}} \,|\, F \subset [n], i, j \notin F, i \neq j).$$

The k-algebra homomorphism defined by

$$\Lambda_0 \ni e_F \mapsto e_{F^{\mathrm{c}}} \in \Lambda_0^!(= \Lambda_0) \quad \text{and} \quad \Lambda_1 \ni e_{F\cup\{i\},F} \mapsto (-1)^{\alpha(i,F)} e^*_{(F\cup\{i\})^{\mathrm{c}},F^{\mathrm{c}}} \in \Lambda_1^!$$

gives a graded isomorphism $\Lambda \cong \Lambda^!$. □

Since $\Lambda(\cong \Lambda^!)$ is an artinian algebra, we have the functors DF and DG defining $D^b(\mathbf{gr.mod}_\Lambda) \cong D^b(\mathbf{gr.mod}_{\Lambda^!})$. In the next result, we will denote the contravariant functors from $D^b(\mathbf{mod}_\Lambda)$ to itself induced by $\boldsymbol{D}$ and $\boldsymbol{A}$ for $D^b(\mathbf{Sq})$ (under the equivalence $\mathbf{Sq} \cong \mathbf{mod}_\Lambda$ of Proposition 2.2) also by $\boldsymbol{D}$ and $\boldsymbol{A}$.

Theorem 4.6. *Let the notation be as above. If* $N^\bullet \in D^b(\mathbf{gr.mod}_\Lambda(0))$, *then we have* $DF(N^\bullet) \in D^b(\mathbf{gr.mod}_\Lambda(n))$ *and* $DG(N^\bullet) \in D^b(\mathbf{gr.mod}_\Lambda(-n))$ *under the isomorphism* $\Lambda^! \cong \Lambda$ *of Lemma* 4.5. *By the equivalence* $\mathbf{gr.mod}_\Lambda(j) \cong \mathbf{mod}_\Lambda$, *$DF$ and DG give endofunctors of* $D^b(\mathbf{mod}_\Lambda)$. *Then* $DF \cong \boldsymbol{A} \circ \boldsymbol{D}$ *and* $DG \cong \boldsymbol{D} \circ \boldsymbol{A}$ *as endofunctors of* $D^b(\mathbf{mod}_\Lambda)$.

Proof. First, we recall the construction of $DF : D^b(\mathbf{gr.mod}_\Lambda) \ni N^\bullet \mapsto \Lambda^! \otimes_{\Lambda_0} N^\bullet \in D^b(\mathbf{gr.mod}_{\Lambda^!})$ under the same notation as the proof of the previous lemma. Note that $\Lambda_0^! = \Lambda_0 = \bigoplus_{F \subset [n]} k e_F$. The component $(DF)^t(N^\bullet)$ of cohomological degree t is $\bigoplus_{t=i+j} \Lambda^! \otimes_{\Lambda_0} N^i_j$. For $N \in \mathbf{gr.mod}_\Lambda$, a left $\Lambda^!$-module $\Lambda^! \otimes_{\Lambda_0} N = \bigoplus_{F \subset [n]} \Lambda^! e_F \otimes_k N_F$

is generated by $\{e_F \otimes n_F \mid F \subset [n] \text{ and } n_F \in N_F\}$. If $N \in \mathbf{gr.mod}_\Lambda(0)$, the degree of $e_F \otimes n_F$ is $\deg(e_F) - \deg(n_F) = -|F|$. For $N^\bullet = \{N^i, \delta^i\} \in \mathrm{Com}^b(\mathbf{gr.mod}_\Lambda)$, the differential of $DF(N^\bullet)$ is given by

$$(DF)^t(N^\bullet) \ni e_F \otimes n_F \mapsto (-1)^t \sum_{l \notin F} e^*_{F, F\cup\{l\}} \otimes (e_{F\cup\{l\},F} \cdot n_F) + e_F \otimes \delta(n_F),$$

see [**3**, Theorem 2.12.1].

The graded isomorphism $\Lambda \xrightarrow{\cong} \Lambda^!$ makes $M \in \mathbf{gr.mod}_{\Lambda^!}$ a graded left Λ-module (without changing the grading of M), and gives an equivalence $\mathbf{gr.mod}_{\Lambda^!} \cong \mathbf{gr.mod}_\Lambda$. From now on, we regard DF as an endofunctor of $D^b(\mathbf{gr.mod}_\Lambda)$ by the equivalence $\mathbf{gr.mod}_{\Lambda^!} \cong \mathbf{gr.mod}_\Lambda$. So we have $DF(N) = \bigoplus_{F\subset[n]} \Lambda e_{F^c} \otimes_k N_F$ for $N \in \mathbf{gr.mod}_\Lambda$. If $N \in \mathbf{gr.mod}_\Lambda(0)$, then the degree of $e_{F^c} \otimes n_F \in \Lambda e_{F^c} \otimes_k N_F \subset DF(N)$ is $-|F| = |F^c| - n$. Thus $DF(N) \in \mathbf{gr.mod}_\Lambda(n)$. For $N^\bullet \in D^b(\mathbf{gr.mod}_\Lambda(0))$, the cohomological degree of $DF(N^\bullet)$ is given by $(DF)^t(N^\bullet) = \bigoplus_{t=j+|F|} \Lambda e_{F^c} \otimes_k N^j_F$, and the differential sends $e_{F^c} \otimes n_F \in (DF)^t(N^\bullet)$ to

$$\sum_{l\notin F} (-1)^{t+\alpha(l,F)} e_{F^c,(F\cup\{l\})^c} \otimes (e_{F\cup\{l\},F} \cdot n_F) + e_{F^c} \otimes \delta(n_F).$$

In Section 3, we study the endofunctor $\mathscr{F} = \boldsymbol{A} \circ \boldsymbol{D}$ on $D^b(\mathbf{Sq})$. Under the equivalence $\mathbf{Sq} \cong \mathbf{mod}_\Lambda$ of Proposition 2.2, this functor induces an endofunctor of $D^b(\mathbf{mod}_\Lambda)$. We also denote it by $\boldsymbol{A} \circ \boldsymbol{D}$. Then for $N^\bullet \in D^b(\mathbf{mod}_\Lambda)$, the component $(\boldsymbol{A} \circ \boldsymbol{D})^t(N^\bullet)$ of cohomological degree t is $\bigoplus_{t=j+|F|} \Lambda e_{F^c} \otimes_k N^j_F$, and an element $e_{F^c} \otimes n_F \in \Lambda e_{F^c} \otimes_k N_F \subset (\boldsymbol{A} \circ \boldsymbol{D})^t(N^\bullet)$ is sent to

$$\sum_{l\notin F} (-1)^{\alpha(l,F)} e_{F^c,(F\cup\{l\})^c} \otimes (e_{F\cup\{l\},F} \cdot n_F) + (-1)^t e_{F^c} \otimes \delta(n_F)$$

by the differential. A quasi-isomorphism $(DF)^t(N^\bullet) \ni x \mapsto (-1)^{\beta(t-1)} x \in (\boldsymbol{A} \circ \boldsymbol{D})^t(N^\bullet)$ gives a natural equivalence $DF \cong \boldsymbol{A} \circ \boldsymbol{D}$, where $\beta(-)$ is the function defined in the proof of Theorem 3.10. The natural equivalence $DG \cong \boldsymbol{D} \circ \boldsymbol{A}$ can be proved in a similar way. □

5. Local cohomology modules as holonomic D-modules.

In this section, we study a local cohomology module $H^i_{I_\Delta}(S)$. The following result was essentially obtained by Mustaţă [**17**] and Terai [**23**], and can be proved by the same argument as the proof of [**25**, Theorem 2.11].

Theorem 5.1 (cf. [**17**], [**23**], [**25**]). *Let Γ_{I_Δ} be the local cohomology functor with supports in I_Δ. Then $\Gamma_{I_\Delta}(\omega^\bullet) \in D^b(\mathbf{Str})$ and $\mathscr{L} \circ \boldsymbol{D}(S/I_\Delta) \cong \Gamma_{I_\Delta}(\omega^\bullet)$. In particular, $H^i_{I_\Delta}(S)(-\mathbf{1}) \cong H^i_{I_\Delta}(\omega_S) \cong \mathscr{L}(\mathrm{Ext}^i_S(S/I_\Delta, \omega_S))$.*

See [**16**], [**27**] for further results on minimal flat resolutions of $\Gamma_{I_\Delta}(\omega^\bullet)$.

In the rest of this section, we assume that $\mathrm{char}(k) = 0$. Let

$$A := A_n(k) = k\langle x_1, \ldots, x_n, \partial_1, \ldots, \partial_n\rangle$$

be the Weyl algebra acting on S, and let $\{F_i\}_{i\geq 0}$ with $F_i = \langle x^{\boldsymbol{a}}\partial^{\boldsymbol{b}} \mid |\boldsymbol{a}| + |\boldsymbol{b}| \leq i \rangle$ be the Bernstein filtration of A. Here $|\boldsymbol{a}| = \sum_{i=1}^{n} |a_i|$ for $\boldsymbol{a} = (a_1, \dots, a_n)$. Then the associated graded ring $\operatorname{gr} A := \bigoplus_{i\geq 0} F_i/F_{i-1}$ is isomorphic to the polynomial ring $k[\bar{x}_1, \dots, \bar{x}_n, \bar{\partial}_1, \dots, \bar{\partial}_n]$ of $2n$ variables. See, for example, [**5**].

In [**25**], the author pointed out that a straight S-module M has a holonomic A-module structure. But if we consider the $\boldsymbol{Z}^n$-grading, the *left* A-module structure given in [**25**] is somewhat unnatural. So we will give a more natural treatment here.

Let M be a left A-module. Set

$$M_{\mathrm{rat}} := \bigoplus_{\boldsymbol{a}\in \boldsymbol{Z}^n} M_{\boldsymbol{a}}, \quad \text{where} \quad M_{\boldsymbol{a}} = \{ y \in M \mid (x_i\partial_i)y = a_i y \ \text{for all} \ i\}. \tag{3}$$

Then M_{rat} is an A-submodule with $x_i M_{\boldsymbol{a}} \subset M_{\boldsymbol{a}+\varepsilon_i}$ and $\partial_i M_{\boldsymbol{a}} \subset M_{\boldsymbol{a}-\varepsilon_i}$. In particular, M_{rat} is a $\boldsymbol{Z}^n$-graded S-module. For example, $S_{\mathrm{rat}} = S$ and the $\boldsymbol{Z}^n$-grading given by (3) coincides with the usual one. If $a_i \neq -1$, the map $M_{\boldsymbol{a}} \ni y \mapsto x_i y \in M_{\boldsymbol{a}+\varepsilon_i}$ is bijective. In fact, its inverse is $(1/(a_i+1))\partial_i : M_{\boldsymbol{a}+\varepsilon_i} \to M_{\boldsymbol{a}}$. If M is a finitely generated left A-module, then $\dim_k M_{\boldsymbol{a}} < \infty$ for all $\boldsymbol{a} \in \boldsymbol{Z}^n$. (In fact, if $V \subset M_{\boldsymbol{a}}$ is a k-vector subspace and $M' := AV \subset M$ is the submodule generated by V, then $M' \cap M_{\boldsymbol{a}} = V$ by the construction. Since M is a noetherian A-module, $M_{\boldsymbol{a}}$ is finite dimensional.) Hence $M_{\mathrm{rat}}(-\mathbf{1})$ is a straight S-module in this case.

While $M_{\mathrm{rat}} = 0$ in many cases, we have the following.

Proposition 5.2. *Let* $\mathbf{mod}_A$ *be the category of finitely generated left* A*-modules. Then* $(-)_{\mathrm{rat}}(-\mathbf{1}) : \mathbf{mod}_A \to \mathbf{Str}$ *is a dense functor.*

Proof. If $f : M \to N$ is an A-homomorphism, we have $f(M_{\boldsymbol{a}}) \subset N_{\boldsymbol{a}}$ for all $\boldsymbol{a} \in \boldsymbol{Z}^n$. So $(-)_{\mathrm{rat}}(-\mathbf{1})$ gives a functor. Next we prove the density. Let M be a $\boldsymbol{Z}^n$-graded S-module with $M(-\mathbf{1}) \in \mathbf{Str}$. We will define $\partial_i y$ for $y \in M_{\boldsymbol{a}}$ as follows.

$(*)$ If $a_i \neq 0$, the map $M_{\boldsymbol{a}-\varepsilon_i} \ni z \mapsto x_i z \in M_{\boldsymbol{a}}$ is bijective, hence there is a unique element $y' \in M_{\boldsymbol{a}-\varepsilon_i}$ such that $x_i y' = y$. Set $\partial_i y := a_i y'$. If $a_i = 0$, we set $\partial_i y = 0$.

It is easy to check that $(*)$ makes M a left A-module with $M = M_{\mathrm{rat}}$. $\square$

In the situation of the proof of Proposition 5.2, $(*)$ is *not* a unique way to make M an A-module. Consider the case $n = 1$ (i.e., $S = k[x]$). Set $M := A/Ax\partial$. Then M has a k-basis $\{1, x, x^2, \dots, \partial, \partial^2, \dots\}$. So $M = M_{\mathrm{rat}}$ and $M(-\mathbf{1}) \cong \omega_S \oplus {}^*E(k)$ as S-modules. Since $\partial M_0 \neq 0$, the A-module structure of M is not given by $(*)$.

We say a finitely generated left A-module M is a straight A-module if $M = M_{\mathrm{rat}}$ and its A-module structure is given by $(*)$. If M and N are straight A-modules, then an A-homomorphism $f : M \to N$ is nothing other than a $\boldsymbol{Z}^n$-graded S-homomorphism. Thus the category $\mathbf{Str}_A$ of straight A-modules is equivalent to $\mathbf{Str}_S$.

A local cohomology module $H_I^i(S)$ has a natural A-module structure for any ideal I (cf. [**14**]). In the monomial ideal case, we have the following.

Proposition 5.3. *Let* I_Δ *be a squarefree monomial ideal. Then* $H_{I_\Delta}^i(S)$ *is a straight* A*-module* (*i.e., the* A*-module structure is given by* $(*)$).

Proof. Recall that $H_{I_\Delta}^i(S)$ is the i^{th} cohomology of the Čech complex $C^\bullet$ with respect to monomial generators of I_Δ. Each term of $C^\bullet$ is a direct sum of copies of the localizations S_{x^F} of S at $\{x^F, x^{2F}, \dots\}$. Note that $S_{x^F}(-\mathbf{1}) \cong \mathscr{X}(S(-F^{\mathrm{c}}))$ is a straight S-module, and its A-module structure as a localization of S is give by $(*)$. Thus $C^\bullet$ is a complex of straight A-modules. The natural A-module structure of $H^i_{I_\Delta}(S) \cong H^i(C^\bullet)$ is given in this way. So we are done. □

Usually, the canonical module ω_S, which is a straight S-module, is regarded as a *right* A-module using Lie differentials. So it seems that a straight S-module M itself (not the shifted module $M(\mathbf{1})$) should be a right A-module.

For a right A-module M, consider an A-submodule

$$M_{\mathrm{rat}} := \bigoplus_{\boldsymbol{a} \in \boldsymbol{Z}^n} M_{\boldsymbol{a}}, \quad \text{where} \quad M_{\boldsymbol{a}} = \{y \in M \mid y(x_i\partial_i) = -a_i y \text{ for all } i\}. \tag{4}$$

If M is finitely generated, M_{rat} is a straight S-module by the same argument as left A-modules. Conversely, any straight S-module can be a right A-module with $M_{\mathrm{rat}} = M$ as Proposition 5.2. The right A-module ω_S satisfies $\omega_S = (\omega_S)_{\mathrm{rat}}$, and the $\boldsymbol{Z}^n$-grading given by (4) coincides with the one given by $\omega_S \cong S(-\mathbf{1})$.

It is well-known that a left A-module M can be viewed as a right A-module, if we set $yx_i = x_i y$ and $y\partial_i = -\partial_i y$ for $y \in M$. When we regard M as a right A-module in this way, we denote it by M_A. Then $(M_A)_{\mathrm{rat}} \cong M_{\mathrm{rat}}(-\mathbf{1})$ as S-modules. So the degree shifting by $-\mathbf{1}$ also appears here. It is also noteworthy that, for a $\boldsymbol{Z}^n$-graded S-module M, the shifted module $M(-\mathbf{1})$ is straight if and only if so is the graded Matlis dual ${}^*\mathrm{Hom}_S(M, {}^*E(k))$. Related arguments for straight modules over a normal semigroup ring are found in §6 of [**27**].

Proposition 5.4. *If M is a finitely generated left A-module, then M_{rat} is holonomic.*

Proof. We may assume that $M = M_{\mathrm{rat}}$. Consider the filtration $\Gamma_0 \subset \Gamma_1 \subset \cdots$ with $\Gamma_i := \sum_{|\boldsymbol{a}| \le i} M_{\boldsymbol{a}}$ of M. Then $F_i\Gamma_j \subset \Gamma_{i+j}$ for all $i, j \ge 0$, where $\{F_i\}$ is the Bernstein filtration of A. Hence $\mathrm{gr}\, M$ has a $\mathrm{gr}\, A$-module structure. Moreover, $\mathrm{gr}\, M$ is a $\boldsymbol{Z}^{2n}$-graded $\mathrm{gr}\, A = k[\bar{x}_1, \dots, \bar{x}_n, \bar{\partial}_1, \dots, \bar{\partial}_n]$-module such that the degree of the image of $y \in M_{\boldsymbol{a}}$, $\boldsymbol{a} \in \boldsymbol{Z}^n$, in $\mathrm{gr}\, M$ is $\boldsymbol{b} \in \boldsymbol{Z}^{2n}$, where

$$b_i = \begin{cases} a_i & \text{if } i \le n \text{ and } a_i \ge 0, \\ -a_{i-n} & \text{if } i > n \text{ and } a_{i-n} < 0, \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see that $\mathrm{gr}\, M$ is a squarefree $\mathrm{gr}\, A$-module, in particular, finitely generated. If $[\mathrm{gr}\, M]_I \ne 0$ for some $I \subset [2n]$, then $|I| \le n$. Thus $\dim_{\mathrm{gr}\, A}(\mathrm{gr}\, M) \le n$ (if $M \ne 0$, $\dim_{\mathrm{gr}\, A}(\mathrm{gr}\, M) = n$), that is, M is holonomic. □

Let M be a finitely generated left A-module. Then M admits a good filtration $\{\Gamma_i\}_{i \ge 0}$, that is, the associated graded module $\mathrm{gr}\, M := \bigoplus_{i \ge 0} \Gamma_i / \Gamma_{i-1}$ is a finitely generated $\mathrm{gr}\, A$-module (cf. [**5**]). We denote the set of minimal associated primes of $\mathrm{gr}\, M$ as a $\mathrm{gr}\, A$-module by $\mathrm{SS}(M)$. For $Q \in \mathrm{SS}(M)$, we denote the multiplicity of the $\mathrm{gr}\, A$-module $\mathrm{gr}\, M$ at Q by $e_Q(M)$ (cf. [**6**, A.3]). It is known that $\mathrm{SS}(M)$ and $e_Q(M)$ do not depend on the particular choice of a good filtration of M.

For $F \subset [n]$, we denote the monomial prime ideal $(\bar{x}_i \mid i \notin F) + (\bar{\partial}_j \mid j \in F)$ of $\mathrm{gr}\, A = k[\bar{x}_1, \dots, \bar{x}_n, \bar{\partial}_1, \dots, \bar{\partial}_n]$ by Q_F. It is easy to see that Q_F is an involutive ideal (i.e., closed under the Poisson product, see [**6**, A.3]) of dimension n. Conversely, every involutive monomial prime ideal of dimension n is of the form Q_F for some F.

Proposition 5.5. *Let M be a finitely generated left A-module. Then*

$$\mathrm{SS}(M_{\mathrm{rat}}) = \{Q_F \mid M_{F-\mathbf{1}} \neq 0\} \quad \textit{and} \quad e_{Q_F}(M_{\mathrm{rat}}) = \dim_k M_{F-\mathbf{1}}.$$

Here F represents the squarefree vector whose support is $F \subset [n]$.

We need the following lemma.

Lemma 5.6. *Let M be a finitely generated left A-module with $M = M_{\mathrm{rat}}$. If M is not a straight A-module, then M is not simple as an A-module.*

Proof. Since M is not straight, there are some $\boldsymbol{a} \in \boldsymbol{Z}^n$, $i \in [n]$, and $y \in M_{\boldsymbol{a}}$ such that $a_i = 0$ and $\partial_i y \neq 0$. Let $N := A(\partial_i y)$ be the submodule of M. Since $x_i(\partial_i y) = a_i y = 0$, we have $y \notin N$. Hence $N \neq 0, M$. $\square$

Proof of Proposition 5.5. We may assume that $M = M_{\mathrm{rat}}$. Note that the submodule N constructed in the proof of Lemma 5.6 satisfies $N = N_{\mathrm{rat}}$. (More generally, if $M = M_{\mathrm{rat}}$, any submodule M' of M satisfies $M' = (M')_{\mathrm{rat}}$.) By Lemma 5.6, we have a filtration $0 = M_0 \subset M_1 \subset \cdots \subset M_t = M$ such that $M_i = (M_i)_{\mathrm{rat}}$ and M_i/M_{i-1} is straight for each i. Recall that $\mathbf{Str}_A \cong \mathbf{Str}_S \cong \mathbf{Sq}_S$ and a simple object in $\mathbf{Sq}$ is isomorphic to $(S/P_F)(-F)$ for some $F \subset [n]$. So we may assume that $M_i/M_{i-1} \cong \mathscr{L}(S/P_F(-F))(\mathbf{1}) =: L[F]$. Take the filtration Γ of $L[F]$ given in the proof of Proposition 5.4. Then we have $\mathrm{gr}(L[F]) = (\mathrm{gr}\, A)/Q_F$. Hence $\mathrm{SS}(L[F]) = \{Q_F\}$ and $e_{Q_F}(L[F]) = 1$. On the other hand, we have $\dim_k L[F]_{F'-\mathbf{1}} = \delta_{F,F'}$ for all $F' \subset [n]$. Since $e_{Q_F}(-)$ is additive, we are done. $\square$

The characteristic cycle of $H^i_{I_\Delta}(S)$ as an A-module (i.e., $e_{Q_F}(H^i_{I_\Delta}(S))$ for $Q_F \in \mathrm{SS}(H^i_{I_\Delta}(S))$) was studied in [**1**], but we will give another approach here. The next corollary shows that Hochster's formula ([**22**, Theorem II.4.1]) on the Hilbert function of $H^i_{\mathfrak{m}}(S/I_\Delta)$ is also a formula on the characteristic cycle of $H^j_{I_\Delta}(S)$.

Corollary 5.7. *Let I_Δ be the Stanley-Reisner ideal of a simplicial complex $\Delta \subset 2^{[n]}$. For all $F \subset [n]$ and all $i \geq 0$, we have*

$$e_{Q_F}(H^i_{I_\Delta}(S)) = \dim_k \tilde{H}_{n-i+|F|+1}(\mathrm{lk}_\Delta F; k),$$

where $\mathrm{lk}_\Delta F = \{G \subset [n] \mid G \cap F = \varnothing$ and $F \cup G \in \Delta\}$.

Proof. By Propositions 5.3 and 5.5, we have $e_{Q_F}(H^i_{I_\Delta}(S)) = \dim_k [H^i_{I_\Delta}(\omega_S)]_F$. But $\dim_k [H^i_{I_\Delta}(\omega_S)]_F = \dim_k \tilde{H}_{n-i+|F|+1}(\mathrm{lk}_\Delta F; k)$ by Terai's formula ([**23**]). $\square$

Remark 5.8. The relation between Hochster's formula and Terai's formula is explained by the isomorphisms $\mathcal{N}(H^i_{I_\Delta}(\omega_S)) \cong \mathrm{Ext}^i_S(S/I_\Delta, \omega_S) \cong H^{n-i}_{\mathfrak{m}}(S/I_\Delta)^*$, where $(-)^*$ means the graded Matlis dual.

Acknowledgments. Main parts of this research were done during my stay at Mathematical Science Research Institute. I am grateful to this institute for warm hospitality. Special thank are due to Professor David Eisenbud for valuable suggestions concerning the Bernstein-Gel'fand-Gel'fand correspondence. I also thank Professors Mitsuyasu Hashimoto, Ezra Miller, Yuji Yoshino, and the referee for helpful comments.

References

[1] J. Alvarez Montaner, Characteristic cycles of local cohomology modules of monomial ideals, J. Pure Appl. Algebra, **150** (2000), 1–25.
[2] M. Auslander, I. Reiten and S. O. Smalø, Representation theory of Artin algebras, corrected reprint of the 1995 original, Cambridge Stud. Adv. Math., **36**, Cambridge Univ. Press, Cambridge, 1997.
[3] A. Beilinson, V. Ginzburg and W. Soergel, Koszul duality patterns in representation theory, J. Amer. Math. Soc., **9** (1996), 473–527.
[4] I. N. Bernstein, I. M. Gel'fand and S. I. Gel'fand, Algebraic vector bundles on $\boldsymbol{P}^n$ and problems of linear algebra, (Russian), Funktsional. Anal. i Prilozhen., **12** (1978), 66–67.
[5] J.-E. Björk, Rings of differential operators, North-Holland Math. Library, **21**, North-Holland, 1979.
[6] J.-E. Björk, Analytic $\mathscr{D}$-modules and applications, Math. Appl., **247**, Kluwer Academic Publishers, 1993.
[7] W. Bruns and J. Herzog, Cohen-Macaulay rings, revised edition, Cambridge Stud. Adv. Math., **39**, Cambridge Univ. Press, 1998.
[8] J. A. Eagon and V. Reiner, Resolution of Stanley-Reisner rings and Alexander duality, J. Pure Appl. Algebra, **130** (1998), 265–275.
[9] D. Eisenbud, S. Popescu and S. Yuzvinsky, Hyperplane arrangement cohomology and monomials in the exterior algebra, Trans. Amer. Math. Soc., **355** (2003), 4365–4383.
[10] D. Eisenbud, G. Fløystad and F.-O. Schreyer, Sheaf cohomology and free resolutions over exterior algebra, Trans. Amer. Math. Soc., **355** (2003), 4397–4426.
[11] S. I. Gel'fand and Y. I. Manin, Methods of homological algebra, Springer-Verlag, 1996.
[12] S. Goto and K. Watanabe, On graded rings, II (Z^n-graded rings), Tokyo J. Math., **1** (1978), 237–261.
[13] E. L. Green and R. Martinez-Villa, Koszul and Yoneda algebras II, in Algebras and modules, II (Geiranger, 1996), CMS Conf. Proc., **24**, Amer. Math. Soc., 1998, 227–244.
[14] G. Lyubeznik, Finiteness properties of local cohomology modules (an application of D-modules to Commutative Algebra), Invent. Math., **113** (1993), 41–55.
[15] E. Miller, The Alexander duality functors and local duality with monomial support, J. Algebra, **231** (2000), 180–234.
[16] E. Miller, Resolutions and duality for monomial ideals, Ph.D. thesis, UC Berkeley, 2000.
[17] M. Mustaţă, Local cohomology at monomial ideals, J. Symbolic Comput., **29** (2000), 709–720.
[18] V. Reiner, V. Welker and K. Yanagawa, Local Cohomology of Stanley-Reisner Rings with supports in General Monomial Ideals, J. Algebra, **244** (2001), 706–736.
[19] P. Roberts, Homological invariants of modules over commutative rings, Presses de l'Université de Montréal, Sem. Math. Sup., **72** (1980).
[20] T. Römer, Generalized Alexander duality and applications, Osaka J. Math., **38** (2001), 469–485.
[21] T. Römer, Cohen-Macaulayness and squarefree modules, Manuscripta Math., **104** (2001), 39–48.
[22] R. Stanley, Combinatorics and commutative algebra, 2nd ed., Progr. Math., **41**, Birkhäuser, 1996.
[23] N. Terai, Local cohomology modules with respect to monomial ideals, preprint.
[24] K. Yanagawa, Alexander duality for Stanley-Reisner rings and squarefree $\boldsymbol{N}^n$-graded modules, J. Algebra, **225** (2000), 630–645.
[25] K. Yanagawa, Bass Numbers of Local cohomology modules with supports in monomial ideals, Math. Proc. Cambridge Philos. Soc., **131** (2001), 45–60.
[26] K. Yanagawa, Sheaves on finite posets and modules over normal semigroup rings, J. Pure Appl. Algebra, **161** (2001), 341–366.
[27] K. Yanagawa, Squarefree modules and local cohomology modules at monomial ideals, In: Local Cohomology and Its Applications (ed. G. Lyubeznik), Lecture Notes in Pure and Appl. Math., **226**, Dekker, 2002, pp. 207–231.

Kohji Yanagawa
Department of Mathematics
Graduate School of Science
Osaka University, Toyonaka, Osaka 560-0043
Japan
E-mail: yanagawa@math.sci.osaka-u.ac.jp