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Massera's theorem for almost periodic solutions of functional differential equations

Dedicated to Professor Yoshiyuki Hino on his sixtieth birthday

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Abstract. The Massera Theorem for almost periodic solutions of linear periodic ordinary differential equations of the form $(*)$ $x' = A(t)x + f(t)$, where f is almost periodic, is stated and proved. Furthermore, it is extended to abstract functional differential equations $(**)$ $x' = Ax + F(t)x_t + f(t)$, where A is the generator of a compact semigroup, F is periodic and f is almost periodic. The main techniques used in the proofs involve a new variation of constants formula in the phase space and a decomposition theorem for almost periodic solutions.

1. Introduction and preliminaries.

Let us consider the following linear ordinary differential equation

$$\dot{x}(t) = A(t)x(t) + f(t), \quad t \geq 0,\ x(t) \in \boldsymbol{C}^n, \tag{1}$$

where $A(t)$ is a continuous matrix function which is periodic in t, f is an almost periodic function. As is well known, a theorem of Massera's [**24**] (which one often calls *the Massera Theorem*) says that if f is periodic with the same period as A, then Equation (1) has a periodic solution with the same period as f and A if and only if it has a bounded solution on the positive half line $[0, +\infty)$. This classical theorem has been extended to various kinds of evolution equations (see, e.g, [**4**], [**15**], [**38**], [**29**], [**27**], ...). However, for the case of almost periodic A and f the Massera Theorem fails (see e.g. [**6**], [**19**]). Recent studies show that for many classes of equations, namely for A periodic and f almost periodic, if we assume a stronger assumption on the existence of a bounded solution on the whole line, then Equation (1) has an almost periodic solution with the same structure of spectrum as f (see [**29**], [**7**]). Technically, this assumption is necessary for carrying the so-called "decomposition technique" of the bounded solution. Furthermore, this technique can be directly applied to the infinite dimensional case. Meanwhile, the Massera Theorem requires only the existence of a bounded solution on the positive half line which appears to be a substantially weaker assumption. Hence, the Massera Theorem in full for almost periodic solutions of Equation (1) is still open. This paper is an attempt to resolve completely this problem. Moreover, we will extend

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it to several larger classes of equations in the infinite dimensional case including abstract functional differential equations

$$\frac{du(t)}{dt} = Au(t) + F(t)u_t + f(t), \tag{2}$$

where A is the generator of a compact semigroup of linear operators, $F(t)$ is a bounded linear operator from a phase space $\mathscr{B}$, which satisfies several axioms listed below, which depends strongly continuously and periodically on t, and u_t is an element of $\mathscr{B}$ which is defined as $u_t(\theta) = u(t+\theta)$ for $\theta \le 0$. To this end, we will make use of a new variation of constants formula in the phase space [**17**] combined with the decomposition technique developed in [**29**], [**7**]. We emphasize that the variation of constants formula in the phase space is established in Theorem 4.2 without assuming the Riesz representation for $F(t)$ (cf. [**17**]).

We now give a brief outline of this paper. In the rest of this section we will summarize several well known notions and results on almost periodic functions, spectrum of a function as well as the naturally associated evolution semigroup of a given strongly continuous semigroup and the relations between them. Section 2 is devoted to the proof of the classical Massera Theorem for almost periodic solutions of linear ordinary differential equations (Theorem 2.1). A similar result holds for difference equations which will be the key tool to study the abstract functional differential equations with infinite delay in Section 4. The main part of the paper is Section 4 which begins by recalling the notion of a uniform fading memory phase space and a variation of constants formula in such phase spaces. The main results of the papers are stated for almost periodic solutions in Theorem 4.4 and for quasi periodic solutions in Theorem 4.7.

1.1. Spectrum of a function.

Recall that the *Beurling spectrum* of a $\boldsymbol{X}$-bounded uniformly continuous function u is defined to be

$$sp(u) := \{\xi \in \boldsymbol{R} : \forall \varepsilon > 0 \ \exists f \in L^1(\boldsymbol{R}), \operatorname{supp} \tilde{f} \subset (\xi - \varepsilon, \xi + \varepsilon), f * u \neq 0\}$$

where $\tilde{f}(s) := \int_{-\infty}^{+\infty} e^{-ist} f(t)\,dt$, $f * u(s) := \int_{-\infty}^{+\infty} f(s-t)u(t)\,dt$. The following theorem will list some main properties of the spectrum of a bounded uniformly continuous function.

Theorem 1.1. *Let $f, g_n \in BUC(\boldsymbol{R}, \boldsymbol{X})$, $n \in \boldsymbol{N}$ such that $g_n \to f$ as $n \to \infty$. Then*

i) *$sp(f)$ is closed,*
ii) *$sp(f(\cdot + h)) = sp(f)$,*
iii) *If $\alpha \in \boldsymbol{C} \backslash \{0\}$ $sp(\alpha f) = sp(f)$,*
iv) *If $sp(g_n) \subset \Lambda$ for all $n \in \boldsymbol{N}$ then $sp(f) \subset \bar{\Lambda}$,*
v) *If A is a closed operator, $f(t) \in D(A)$ $\forall t \in \boldsymbol{R}$ and $Af(\cdot) \in BUC(\boldsymbol{R}, \boldsymbol{X})$, then, $sp(\mathscr{A}f) \subset sp(f)$,*
vi) *$sp(\psi * f) \subset sp(f) \cap \operatorname{supp} \tilde{\psi}$, $\forall \psi \in L^1(\boldsymbol{R})$.*

For the proof we refer the reader to [**33**, p. 20–21]. In this paper by almost periodic functions we mean the almost periodic functions in the sense of Bohr (we refer the reader to [**21**] for the definition and basic properties of such almost periodic functions). If f is an almost periodic function, the following limit

$$a(\lambda, f) := \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} f(t)e^{-i\lambda t}\,dt, \quad \forall\lambda \in \boldsymbol{R} \tag{3}$$

exists and is called *Bohr transform* of f. As is known, there is an at most countable set of reals λ such that the above limit differs from zero. This set will be denoted by $\sigma_b(f)$ and called *Bohr spectrum* of f. We will need the following lemma in the sequel.

Lemma 1.2. *Let $x(\cdot) \in AP(\boldsymbol{X})$, $Q(t) \in L(\boldsymbol{X})$ be strongly continuous and* 1*-periodic in $t \in \boldsymbol{R}$. Then, letting $y(t) := Q(t)x(t)$, $t \in \boldsymbol{R}$ we have that $y(\cdot) \in AP(\boldsymbol{X})$ and*

$$e^{i\sigma_b(y)} \subset e^{i\sigma_b(x)}. \tag{4}$$

Proof. By the uniform boundedness principle we have

$$\sup_{t\in\boldsymbol{R}} \|Q(t)\| < \infty.$$

Next, using the Approximation Theorem for almost periodic functions we can easily show that $y(\cdot)$ is approximated by a sequence of trigonometric polynomials of the form

$$y_n(t) := Q_n(t)x_n(t), \quad \text{where}$$

$$x_n(t) = \sum_{k=1}^{N(n)} a_{k,n}e^{i\lambda_{k,n}t}, \quad a_{k,n} \in \boldsymbol{X},\ \lambda_{k,n} \in \sigma_b(x)$$

$$Q_n(t) = \sum_{k=1}^{M(n)} B_{k,n}e^{i\mu_{k,n}t}, \quad B_{k,n} \in L(\boldsymbol{X}),\ \mu_{k,n} \in 2\pi\boldsymbol{Z},$$

where $x_n(\cdot)$ approximates $x(\cdot)$. Let $\lambda \in \boldsymbol{R}$ such that $e^{i\lambda} \notin e^{i\sigma_b(x)}$. We will show that $e^{i\lambda} \notin e^{i\sigma_b(y)}$. If this is the case, the above spectral estimate holds. For any positive ε there is (sufficiently large) $N \in \boldsymbol{N}$ such that $\sup_{t\in\boldsymbol{R}}\|y_n(t) - y(t)\| < \varepsilon$ for all $n \geq N$. On the other hand, for any $m \in \boldsymbol{Z}$, letting $\lambda_m := \lambda + 2m\pi$, we have

$$\begin{aligned}
\|a(\lambda_m, y)\| &= \left\|\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} e^{-i(\lambda+2m\pi)t}y(t)\,dt\right\| \\
&\leq \left\|\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} e^{-i(\lambda+2m\pi)t}Q_n(t)x_n(t)\,dt\right\| \\
&\quad + \left\|\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} e^{-i(\lambda+2m\pi)t}(y(t) - Q_n(t)x_n(t))\,dt\right\| \\
&= \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \|e^{-i(\lambda+2m\pi)t}(y(t) - Q_n(t)x_n(t))\|\,dt \\
&\leq \varepsilon.
\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, this yields that $a(\lambda_m, y) = 0$. Hence, $(\lambda + 2\pi\boldsymbol{Z}) \cap \sigma_b(y) = \varnothing$, i.e., the above spectral estimate holds. $\square$

Throughout the paper we will use the relation $sp(f) = \overline{\sigma_b(f)}$. The space of all $\boldsymbol{X}$-valued almost periodic functions will be denoted by $AP(\boldsymbol{X})$. The set $sp(f)$ can characterize the behavior of the function f. For example, f is τ-periodic if and only if $e^{\tau sp(f)} = \{1\}$, f is anti τ-periodic if and only if $e^{\tau sp(f)} = \{-1\}$. If $sp(f)$ is countable and $\boldsymbol{X}$ does not contain any subspaces isomorphic to c_0 (the space of numerical sequences converging to 0), then f is almost periodic. An almost periodic function f is called *quasi periodic* if $\sigma_b(f)$ has an integer and finite basis (see [**21**, pp. 47–48]). If $sp(f) = \overline{\sigma_b(f)}$ has an integer and finite basis, then, f is quasi periodic. Hence, if the spectrum of f is good enough one can have relevant conclusions on its behavior. In the rest of this paper we will prove the existence of an almost periodic solution with spectrum similar to the one of the forcing term f. In this way, we can extend the Massera Theorem to almost periodic solutions. For the sake of simplicity of notation we will assume that the period of $A(\cdot)$ and $F(\cdot)$ is 1, and would like to emphasize that this assumption does not constitute any restrictions on the obtained results.

Unless otherwise stated, we will use the usual notation. For instance, $\boldsymbol{N}$, $\boldsymbol{R}$, $\boldsymbol{C}$ denote the set of natural, real, complex numbers, respectively. Γ will stand for the unit circle in $\boldsymbol{C}$, i.e., $\Gamma := \{z \in \boldsymbol{C} : |z| = 1\}$. As usual, $BUC(\boldsymbol{R}, \boldsymbol{X})$, $BC(\boldsymbol{R}, \boldsymbol{X})$, $C(\boldsymbol{R}, \boldsymbol{X})$ denote the spaces of all $\boldsymbol{X}$-valued bounded uniformly continuous functions, bounded continuous functions, continuous functions on $\boldsymbol{R}$, respectively.

1.2. Evolution semigroups and decomposition theorems.

In this subsection, we will summarize several notions and results concerned with evolution semigroups and decomposition theorems.

Definition 1.3. The following formal semigroup associated with a given strongly continuous semigroup $(T(t))_{t \geq 0}$

$$(T^h u)(t) := T(h)u(t-h), \quad \forall t \in \boldsymbol{R}, \tag{5}$$

where u is an element of some function space, is called *evolution semigroup* associated with the semigroup $(T(t))_{t \geq 0}$.

Below we are going to discuss the relation between this evolution semigroup and the following inhomogeneous equation

$$x(t) = T(t-s)x(s) + \int_s^t T(t-\xi)f(\xi)\,d\xi, \quad \forall t \geq s \tag{6}$$

associated with a strongly continuous $(T(t))_{t \geq 0}$. Let us define the operator $\mathscr{L} : D(\mathscr{L}) \subset BUC(\boldsymbol{R}, \boldsymbol{X}) \to BUC(\boldsymbol{R}, \boldsymbol{X})$, where $D(\mathscr{L})$ consists of all solutions of Equation (6) $u(\cdot) \in BUC(\boldsymbol{R}, \boldsymbol{X})$ with some $f \in BUC(\boldsymbol{R}, \boldsymbol{X})$, and in this case $\mathscr{L}u(\cdot) := f$. This operator $\mathscr{L}$ is well defined as a single-valued operator and is obviously an extension of the differential operator $d/dt - A$ (see e.g. [**25**]). Below, by abuse of notation, we will use the same notation $\mathscr{L}$ to designate its restriction to closed subspaces of $BUC(\boldsymbol{R}, \boldsymbol{X})$ if this does not make any confusion.

Lemma 1.4. *Let $(T(t))_{t \geq 0}$ be a strongly continuous semigroup. Then its associated evolution semigroup $(T^h)_{h \geq 0}$ is strongly continuous at every bounded uniformly continuous solution of Equation* (6) *with almost periodic f, in particular at every element of $AP(\boldsymbol{X})$.*

Moreover, the infinitesimal generator of $(T^h)_{h\geq 0}$ *in the space* $\mathscr{S}$ *of all elements of* $BUC(\boldsymbol{R}, \boldsymbol{X})$ *at which* $(T^h)_{h\geq 0}$ *is strongly continuous, is the restriction of the operator* $\mathscr{L}$ *to* $\mathscr{S}$.

Lemma 1.5. *If* $u \in BUC(\boldsymbol{R}, \boldsymbol{X})$ *is a mild solution to Equation* (2), *then the evolution semigroup* $(T^h)_{h\geq 0}$, *associated with the semigroup* $(T(t))_{t\geq 0}$, *is strongly continuous at u.*

We refer the reader to [**3**] and [**31**] and the references therein for more information on the history and applications of evolution semigroups to the study of the stability and exponential dichotomy of dynamical systems. In [**18**] the reader can find a systematic presentation of new applications of evolution semigroups to the study of almost periodic solutions of differential equations in Banach spaces.

Assume that $\{g|_{\boldsymbol{R}^-} : g \in AP(\boldsymbol{X})\} \subset \mathscr{B}$, where $\boldsymbol{R}^- = (-\infty, 0]$. Obviously, in this case $F(\cdot)v.$ will be almost periodic for every given almost periodic v. We will denote by $\mathscr{F}$ the operator acting on $AP(\boldsymbol{X})$ defined by the formula

$$\mathscr{F}v(\xi) := F(\xi)v_\xi, \quad \forall v \in AP(\boldsymbol{X}).$$

Note that from the 1-periodicity of $F(\cdot)$

$$\mathscr{F}S(1) = S(1)\mathscr{F},$$

where one denotes by $(S(t))_{t\in\boldsymbol{R}}$ the translation group on $AP(\boldsymbol{X})$, i.e., $S(t)v(s) = v(t+s)$, $\forall t, s \in \boldsymbol{R}$. For an almost periodic function $x(\cdot)$ the following characterization is very useful:

Theorem 1.6. $x(\cdot)$ *is an almost periodic mild solution of Equation* (2) *with almost periodic* f *if and only if* $(\mathscr{L} - \mathscr{F})x(\cdot) = f$.

Let us consider the subspace $\mathscr{M} \subset AP(\boldsymbol{X})$ consisting of all functions $v \in AP(\boldsymbol{X})$ such that $\overline{e^{isp(v)}} =: \sigma(v) \subset S_1 \cup S_2$, where $S_1, S_2 \subset \boldsymbol{S}^1$ are disjoint closed subsets of the unit circle.

Theorem 1.7. *Under the above notations and assumptions the function space* $\mathscr{M}$ *can be split into a direct sum* $\mathscr{M} = \mathscr{M}_1 \oplus \mathscr{M}_2$ *such that* $v \in \mathscr{M}_i$ *if and only if* $\sigma(v) \subset S_i$ *for* $i = 1, 2$. *Moreover, the above defined linear operator* $\mathscr{F}$ *in* $AP(\boldsymbol{X})$ *leaves invariant* $\mathscr{M}$ *as well as* $\mathscr{M}_j$, $j = 1, 2$.

Proof. In view of [**29**] it suffices to show only the last assertion. To this end, we can show that

$$e^{isp\mathscr{F}v} \subset e^{isp(v)}, \quad \forall v \in AP(\boldsymbol{X}). \tag{7}$$

In fact, since $\mathscr{F}$ is a bounded linear operator, by the Approximation Theorem of Almost Periodic Functions, it suffices to prove the above estimate for trigonometric polynomials. Suppose that

$$v = \sum_{k=1}^{N} a_k e^{i\lambda_k t}, \quad \lambda_k \in \boldsymbol{R},\ a_k \in \boldsymbol{X},\ t \in \boldsymbol{R}.$$

Then

$$\begin{aligned}\mathscr{F}v(t) &= F(t)v_t \\ &= \sum_{k=1}^{N} F(t)a_k e^{i\lambda_k \cdot} e^{i\lambda_k t} \\ &= \sum_{k=1}^{N} b_k(t) e^{i\lambda_k t},\end{aligned}$$

where $b_k(t) := F(t)a_k e^{i\lambda_k \cdot}$, $e^{i\lambda_k \cdot}$ is a function defined on $(-\infty, 0]$. Since $F(\cdot)$ is 1 periodic, $b_k(\cdot)$ is also 1-periodic. Thus, it can be seen that every term in the sum has its spectrum satisfying the estimate (7). Finally, $\mathscr{F}v$ satisfies (7), too. □

COROLLARY 1.8. *Let u be an almost periodic mild solution of Equation* (2) *such that $e^{isp(u)} \subset \overline{e^{isp(f)}} \cup K$, where K has finitely many elements. Then, Equation* (2) *has an almost periodic mild solution w such that $e^{isp(w)} \subset \overline{e^{isp(f)}}$.*

PROOF. It suffices to take $S_1 := \overline{e^{isp(f)}}$, $S_2 := K \backslash \overline{e^{isp(f)}}$. Then Theorems 1.6 and 1.7 apply. □

2. Massera theorem for almost periodic solutions of linear ode.

In this section we will prove the Massera Theorem for almost periodic solutions of linear periodic ordinary differential equations. Namely, we will prove below Theorem 2.1 which extends the Massera Theorem (see [**24**]) to almost periodic solutions, and improves [**6**, Theorem 5.8, p. 86]. The proof will be carried out in an elementary manner, but the obtained results seem to be as sharpest as possible.

THEOREM 2.1. *Assume that the matrix function $A(t)$ is continuous and* 1*-periodic, and f is almost periodic. Then,* (1) *has an almost periodic solution $u(\cdot)$ with $e^{i\sigma_b(u)} \subset e^{i\sigma_b(f)}$ if and only if it has a bounded solution $x(\cdot)$ on the positive half line $\boldsymbol{R}^+$. In particular if $A(t)$ is independent of t, then the existence of a solution bounded on $\boldsymbol{R}^+$ yields the existence of an almost periodic solution $u(\cdot)$ such that $\sigma_b(u) \subset \sigma_b(f)$, and hence if f is quasi periodic, then u is quasi periodic.*

PROOF. Suppose first that $A(t)$ is independent of t and that $x(\cdot)$ is a given solution which is bounded on $\boldsymbol{R}^+$. As in [**15**] we will construct a solution bounded on the whole line of Equation (1). Namely, let $x_n(\cdot) := x(n + \cdot)$ which is defined on $[-n, +\infty)$ for every $n \in \boldsymbol{N}$ as a solution of the equation

$$\frac{dx(t)}{dt} = Ax(t) + f(n+t), \quad t \in [-n, +\infty).$$

Since $x_n(0)$, $n \in \boldsymbol{N}$ is a bounded sequence in $\boldsymbol{C}^n$ it contains a subsequence $x_{n_p}(0)$ which is convergent to $z \in \boldsymbol{C}^n$. As the function f is almost periodic the sequence f_{n_p} should contain a subsequence f_{m_k} which converges uniformly to an almost periodic function f_∞. Let us consider the solution $y(t)$ of the equation

$$\frac{dy(t)}{dt} = Ay(t) + f_\infty(t) \tag{8}$$

with $y(0) = z$. We will show that for every fixed $N \in \boldsymbol{N}$ the sequence $\{x_{n_k}\}$ is convergent uniformly to $y(\cdot)$ on $[-N, N]$. In fact, by the variation of constants formula, for $n \geq N$ we have

$$\begin{aligned}\sup_{t\in[-N,N]} \|x_{n_k}(t) - y(t)\| &\leq \sup_{t\in[-N,N]} \|e^{At}\| \, \|x_{n_k}(0) - z\| \\ &\quad + N \sup_{t\in[-N,N]} \|e^{At}\| \sup_{t\in[-N,N]} \|f_{n_k}(t) - f_\infty(t)\| \\ &\leq C_1\|x_{n_k}(0) - z\| + C_2 \sup_{t\in\boldsymbol{R}} \|f_{n_k}(t) - f_\infty(t)\|,\end{aligned}$$

where C_1, C_2 are positive constants independent of n. As a consequence of this we have $\sup_{t\in[-N,N]}\|y(t)\| \leq \sup_{t\in[0,+\infty)}\|x(t)\|$, so, the solution $y(\cdot)$ is bounded on the whole line. Since f_∞ is almost periodic, the solution $y(\cdot)$ should be almost periodic (see [**6**, Theorem 5.8]). On the other hand, since $f_\infty(-n_k + \cdot)$ is uniformly convergent to f, by the same argument as above we can choose a subsequence n_p such that $y(-n_p + \cdot)$ is convergent to an almost periodic solution $w(t)$, $t \in \boldsymbol{R}$ of Equation (1), i.e., Equation (1) has an almost periodic solution $w(t)$ on the whole line. Taking Bohr transform of $a(\lambda, w) := \lim_{T\to\infty}(1/(2T)) \int_{-T}^{T} e^{-i\lambda t} w(t)\, dt$, $\lambda \in \boldsymbol{R}$ we have

$$a(\lambda, \dot{w}) = i\lambda a(\lambda, w) = Aa(\lambda, w) + a(\lambda, f).$$

Hence,

$$(i\lambda - A)a(\lambda, w) = a(\lambda, f). \tag{9}$$

It follows from this fact that

$$\sigma_b(f) \subset \sigma_b(w) \subset \sigma_b(f) \cup \sigma_i(A), \tag{10}$$

where $\sigma_i(A) := \{\xi \in \boldsymbol{R} : i\xi \in \sigma(A)\}$, and $\sigma(A)$ denotes the set of eigenvalues of the matrix A. We set $u(t) = w(t) - \sum_{\lambda\in L} a(\lambda, w)e^{i\lambda t}$, where $L := \sigma_i(A)\backslash\sigma_b(f)$. We will show that u is an almost periodic solution of (1) with the required properties. In fact, by the definition of L and (9), for every $\lambda \in L$, $a(\lambda, f) = 0$, so, $(i\lambda - A)a(\lambda, w) = 0$. Consequently,

$$\begin{aligned}\dot{u} &= \dot{w}(t) - \sum_{\lambda\in L} i\lambda a(\lambda, w)e^{i\lambda t} \\ &= Aw(t) + f(t) - A\sum_{\lambda\in L} a(\lambda, w)e^{i\lambda t} \\ &= A\left(w(t) - \sum_{\lambda\in L} a(\lambda, w)e^{i\lambda t}\right) + f(t) \\ &= Au(t) + f(t).\end{aligned}$$

Obviously, in view of (10) $\sigma_b(u) \subset \sigma_b(f)$.

If $A(t)$ depends on t, then we can transform the equation in question into an autonomous one by a Floquet transformation, i.e., there is a 1-periodic continuous nonsingular matrix $Q(t)$ such that by the change of variable $y(t) = Q(t)x(t)$ the equation

(1) is transformed into the autonomous equation $\dot{y}(t) = By(t) + g(t)$, where $g(t) = Q(t)f(t)$. Thus, by the above argument, there is an almost periodic solution v to this equation such that $\sigma_b(v) \subset \sigma_b(g)$. It can be shown easily that the function $u(t) := Q^{-1}(t)v(t)$ is an almost periodic solution of equation (1) with desired properties as $e^{i\sigma_b(Q^{-1}(\cdot)v(\cdot))} \subset e^{i\sigma_b(f)}$. □

Remark 2.2. i) In [**26**], [**25**], [**37**], in case $A(t)$ depends 1-periodically on t, the existence of a solution u with spectral estimate $e^{isp(u)} \subset \overline{e^{isp(f)}}$ has been proved (of course, for the infinite dimensional equations). Since $\overline{\sigma_b(f)} = sp(f)$, if f is $\sqrt{2}$-periodic, then it is expected that possibly, $\overline{e^{isp(f)}}$ fills the whole unit circle $\Gamma := \{z \in \boldsymbol{C} : |z| = 1\}$. Hence, in this case one may get nothing new if $sp(f)$ is a bit complicated. Meanwhile, our condition in the above theorem still gives information on the solution u as a solution with minimal spectrum.

ii) If the equation in question is considered in the real space $\boldsymbol{R}^n$ rather than in $\boldsymbol{C}^n$, then the Floquet transform will have the double period (see e.g. [**1**]). In general, the above results are valid with appropriate modifications of statements.

iii) As shown in Example 5.1 in general, without additional conditions the Massera Theorem fails in the infinite dimensional case. In the above proof an essential assumption is the compactness of bounded closed sets in the finite dimensional Banach space. Hence, in the infinite dimensional case the above result can be directly extended to the following equation

$$\frac{du(t)}{dt} = Au(t) + f(t), \quad u(t) \in \boldsymbol{X},\ t \in \boldsymbol{R}, \tag{11}$$

where A is the generator of a compact semigroup of linear operators $T(t)$ on a given Bach space $\boldsymbol{X}$. Using the integral

$$P := \frac{1}{2\pi i}\int_\gamma R(\lambda, T(1))\,d\lambda, \tag{12}$$

where $\gamma \subset \rho(T(1))$ is a contour encirling the origin in $\boldsymbol{C}$ we can decompose the phase space $\boldsymbol{X}$ into the direct sum $\boldsymbol{X} = \boldsymbol{X}_1 \oplus \boldsymbol{X}_2$, where $\boldsymbol{X}_1 := \operatorname{Im} P$, $\boldsymbol{X}_2 := \operatorname{Ker} P$. Obviously, $\dim \boldsymbol{X}_1 < \infty$. On the other hand, on $\boldsymbol{X}_2$ the semigroup $(I-P)T(t)(I-P)$, $t \in \boldsymbol{R}$ is exponentially stable. Thus, the problem is reduced to the consideration of the almost periodic solutions of the other component on $\boldsymbol{X}_1$ which is of finite dimension. We will discuss more general equations having such properties in Section 4.

3. Almost periodic solutions of difference equations.

In this section we will prove the Massera Theorem for almost periodic solutions of difference equations as a preparatory step for functional differential equations which will be considered in the next section. Let us consider the difference equation

$$x(n+1) = Bx(n) + c(n), \quad n \in \boldsymbol{Z}, \tag{13}$$

where $B \in L(\boldsymbol{X})$ is a bounded linear operator such that $r_e(B) < 1$ ($r_e(B)$ denotes the essential spectral radius of B) and $c(n)$ is an almost periodic sequence in $\boldsymbol{X}$. Let us denote $\sigma_\Gamma(B) := \Gamma \cap \sigma(B)$. The reader is referred to the Appendix at the end of this paper for analogs of notion of spectrum of a sequence and related results. We will prove the Massera Theorem for Equation (13), i.e., the following

Theorem 3.1. *Let Equation* (13) *have a bounded solution* $x(n)$, $n \in \boldsymbol{N}$. *Then it has an almost periodic solution* y *on* $\boldsymbol{Z}$ *such that* $\sigma_b(y) \subset \sigma_b(c)$.

Proof. We will use the reduction principle to prove the theorem. In fact, let $r_e(B) < \rho < 1$ such that the circle of radius ρ, centered at the origin does not contain any point of $\sigma(B)$. This is possible in view of our assumption that $r_e(B) < 1$. Hence, the integral

$$J := \frac{1}{2\pi i} \int_{C_\rho} R(\lambda, B)\, d\lambda$$

is the projection. Moreover, note that $\dim \operatorname{Ker}(J)$ is finite. Thus, the Banach space $\boldsymbol{X}$ can be split into a direct sum $\boldsymbol{X}_1 \oplus \boldsymbol{X}_2$ where $\boldsymbol{X}_1 = \operatorname{Ker}(J)$, $\boldsymbol{X}_2 = \operatorname{Im}(J)$. Using this decomposition, the problem of finding almost periodic solutions to Equation (13) is trivially reduced to finding almost periodic solutions to the following equation

$$x_1(n) = B_1 x_1(n-1) + c_1(n-1), \tag{14}$$

where $x_1(n) := (I - J)x(n)$, $B_1 := (I - J)B(I - J)$, $c_1(n) := (I - J)c(n)$. In fact, for the other component, in view of the exponential stability of the equation in $\boldsymbol{X}_2$, the existence and uniqueness of an almost periodic solution is well-known. Indeed, let us denote by $S(k)$ the translation $[S(k)x](n) := x(k+n)$ and by $\mathscr{B}_2$ the multiplication by $B_2 := JBJ$ in the space of all almost periodic two-sided sequences. Then, obviously, $r_\sigma(\mathscr{B}_2 S(-1)) < 1$ because of the exponential stability. Hence, the unique almost periodic solution is $x_2 = (I - \mathscr{B}_2 S(-1))^{-1} S(-1)c_2$. Thus, it suffices now to deal with the first component equation. We consider the sequence $x_1(p + \cdot)$, where $p \in \boldsymbol{N}$. Every term of this sequence is a bounded solution to the Equation (13) with the right hand side $c_1(p + \cdot)$. Since the sequence c_1 is an almost periodic two-sided sequence, there exists a sequence p_k such that $c_1(p_k + \cdot)$ is convergent uniformly to $\bar{c}$ on $\boldsymbol{Z}$. On the other hand, since $x_1(p_k)$ is a bounded sequence in a finite dimensional space, it contains a convergent subsequence. Thus, without loss of generality, we can assume that this sequence is convergent, itself. This procedure leads to the existence of a bounded two-sided sequence $\bar{y}_1$ which is a solution of the equation

$$y(n) = B_1 y(n-1) + \bar{c}(n-1), \quad n \in \boldsymbol{Z}.$$

In the same way as in [**6**, Theorem 5.8], we can show that $\bar{y}_1$ is an almost periodic sequence. Therefore (14) has an almost periodic solution y_1 which is the limit of some subsequence of $\bar{y}_1(-p_k + \cdot)$. Now using the elementary decomposition technique developed in the above section we can decompose from the two sided sequence y_1 (which is obviously almost periodic) an almost periodic component d with spectrum $\sigma_b(d) \subset \sigma_b(c)$. This proves the theorem. □

4. Abstract functional differential equations.

We consider in this section the abstract functional differential equation

$$\frac{du(t)}{dt} = Au(t) + F(t)u_t + f(t), \tag{15}$$

where A is the generator of a semigroup of linear operators on a Banach space $\boldsymbol{X}$, $F(t)$ is a bounded linear operator from $\mathscr{B}$ into $\boldsymbol{X}$ which is periodic in t with period 1, where $\mathscr{B}$ is a uniform fading memory phase space of Equation (2) with infinite delay satisfying the axioms listed below and $f \in BC(\boldsymbol{R}, \boldsymbol{X})$. We will impose conditions on Equation (2) so that the monodromy operator, which we will denote by B, satisfies the assumptions listed in Theorem 3.1.

4.1. Uniform fading memory phase spaces.

We will give a precise definition of the notion of uniform fading memory space for Equation (2) in this subsection. Let us denote the norm of $\boldsymbol{X}$ by $\|\cdot\|_{\boldsymbol{X}}$. For any function $x : (-\infty, a) \to \boldsymbol{X}$ and $t < a$, we define a function $x_t : \boldsymbol{R}^- := (-\infty, 0] \to \boldsymbol{X}$ by $x_t(s) = x(t+s)$ for $s \in \boldsymbol{R}^-$. Let $(\mathscr{B}, \|\cdot\|_{\mathscr{B}})$ be a Banach space, consisting of functions $\psi : (-\infty, 0] \to \boldsymbol{X}$ such that

(A1) There exist a positive constant N and locally bounded functions $K(\cdot)$ and $M(\cdot)$ on $\boldsymbol{R}^+$ with the property that if $x : (-\infty, a) \mapsto \boldsymbol{X}$ is continuous on $[\sigma, a)$ with $x_\sigma \in \mathscr{B}$ for some $\sigma < a$, then for all $t \in [\sigma, a)$,
  (i) $x_t \in \mathscr{B}$,
  (ii) x_t is continuous in t (w.r.t. $\|\cdot\|_{\mathscr{B}}$),
  (iii) $N\|x(t)\|_{\boldsymbol{X}} \le \|x_t\|_{\mathscr{B}} \le K(t-\sigma)\sup_{\sigma \le s \le t}\|x(s)\|_{\boldsymbol{X}} + M(t-\sigma)\|x_\sigma\|_{\mathscr{B}}$,

(A2) If $\{\phi^k\}$, $\phi^k \in \mathscr{B}$, converges to ϕ uniformly on any compact set in $\boldsymbol{R}^-$ and if $\{\phi^k\}$ is a Cauchy sequence in $\mathscr{B}$, then $\phi \in \mathscr{B}$ and $\phi^k \to \phi$ in $\mathscr{B}$.

The space $\mathscr{B}$ is called a *uniform fading memory space*, if it satisfies (A1) and (A2) with $K(\cdot) \equiv K$ (a constant) and $M(\beta) \to 0$ as $\beta \to \infty$ in (A1). A typical example of uniform fading memory spaces is the following one:

$$C_\gamma := C_\gamma(\boldsymbol{X}) = \left\{\phi \in C(\boldsymbol{R}^-; \boldsymbol{X}) : \lim_{\theta \to -\infty} \frac{\|\phi(\theta)\|_{\boldsymbol{X}}}{e^{\gamma\theta}} = 0\right\}$$

which is equipped with norm $\|\phi\|_{C_\gamma} = \sup_{\theta \le 0}\|\phi(\theta)\|_{\boldsymbol{X}}/e^{\gamma\theta}$, where γ is a negative constant.

It is known [**11**, Lemma 3.2] that if $\mathscr{B}$ is a uniform fading memory space, then $BC := BC(\boldsymbol{R}^-; \boldsymbol{X}) \subset \mathscr{B}$ and the inclusion map from BC into $\mathscr{B}$ is continuous. For other properties of uniform fading memory spaces, we refer the reader to the book [**16**]. In connection with the almost periodic functions taking values in a uniform fading memory space $\mathscr{B}$ we have the following.

Lemma 4.1. *Let $\mathscr{B}$ be a uniform fading memory space and u is an almost periodic function taking values in $\boldsymbol{X}$. Denoting $v(t) := u_t$ we have*
 i) *v is a $\mathscr{B}$-valued almost periodic function,*
 ii) *$\sigma_b(u) = \sigma_b(v)$.*

Proof. By the Approximation Theorem of almost periodic functions, there is a sequence of trigonometric polynomials $P_n(t) = \sum_{k=1}^{N(n)} a_{n,k}e^{i\lambda_{n,k}t}$ with $\lambda_{n,k} \in \sigma_b(u)$, which is convergent uniformly to $u(t)$. Now using the axiom (A1) (iii) of the uniform fading memory space, we can easily construct a sequence of trigonometric polynomials in $\mathscr{B}$ with Bohr exponents in $\sigma_b(u)$ which uniformly approximates $v(t) := u_t$. Obviously, v is almost periodic. Now we prove the spectral estimate by using again the axiom A1 (iii). In fact, under the standing assumption there are positive constants N, K (which are independent of u) such that

$$N\|u(t)\|_X \le \|u_t\|_{\mathscr{B}} \le K \sup_{t \in \boldsymbol{R}} \|u(t)\|_X. \tag{16}$$

Hence, for $\lambda \in \boldsymbol{R}$ the following

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} e^{-i\lambda t} u(t)\, dt = 0$$

holds if

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} e^{-i\lambda t} u_t\, dt = 0,$$

i.e., $\sigma_b(u) \subset \sigma_b(v)$. Conversely, by the basic properties of almost periodic functions (see e.g. [**21**, pp. 22–23]) if $\lambda \in \boldsymbol{R} \backslash \sigma_b(u)$ the limit

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T+a}^{T+a} e^{-i\lambda t} u(t)\, dt = 0$$

exists uniformly in $a \in \boldsymbol{R}$. Thus using (16) we can conclude that $\lambda \in \boldsymbol{R} \backslash \sigma_b(v)$, i.e., $\sigma_b(v) \subset \sigma_b(u)$, proving the lemma. □

4.2. A variation of constants formula for FDE.

We consider now the abstract functional differential equation (2) with the uniform fading memory phase space $\mathscr{B}$. Throughout the paper we shall assume that $F(t)\phi$ is continuous in $(t, \phi) \in \boldsymbol{R} \times \mathscr{B}$ and linear in $\phi \in \mathscr{B}$, and it is periodic in t with period 1. For any $(\sigma, \phi) \in \boldsymbol{R} \times \mathscr{B}$, there exists a (unique) function $u : \boldsymbol{R} \mapsto \boldsymbol{X}$ such that $u_\sigma = \phi$, u is continuous on $[\sigma, \infty)$ and the following relation holds:

$$u(t) = T(t-\sigma)\phi(0) + \int_\sigma^t T(t-s)\{F(s)u_s + f(s)\}\, ds, \quad t \ge \sigma,$$

(cf. [**13**, Theorem 1]). The function u is called a (*mild*) *solution* of (2) through (σ, ϕ) on $[\sigma, \infty)$, and denoted by $u(\cdot, \sigma, \phi; f)$. Also, a function $v \in C(\boldsymbol{R}, \boldsymbol{X})$ is called a (*mild*) *solution of Equation* (2) *on* $\boldsymbol{R}$, if $v_t \in \mathscr{B}$ for all $t \in \boldsymbol{R}$ and it satisfies $u(t, \sigma, v_\sigma; f) = v(t)$ for all t and σ with $t \ge \sigma$. For any $t \ge s$, we define an operator $U(t,s)$ on $\mathscr{B}$ by

$$U(t,s)\phi = u_t(s, \phi; 0), \quad \phi \in \mathscr{B}.$$

We can easily see that under the assumption on the strong continuity and periodicity of $F(t)$, the two-parameter family $(U(t,s))_{t\ge s}$ is a strongly continuous evolutionary process on $\mathscr{B}$, which is called the *solution process* of (2). By a strongly continuous evolutionary process in a Banach space $\boldsymbol{Y}$ we mean a two-parameter family of bounded linear operators $(V(t,s))_{t\ge s}$, $(-\infty < s \le t < \infty)$ from $\boldsymbol{Y}$ to $\boldsymbol{Y}$ such that the following conditions are satisfied:

i) $V(t,t) = I, \ \forall t \in \boldsymbol{R}$,
ii) $V(t,s)V(s,r) = V(t,r), \ \forall t \ge s \ge r$,
iii) For every fixed $y \in \boldsymbol{Y}$ the following map is continuous:

$$\{(\eta,\xi) \in \boldsymbol{R}^2 : \eta \ge \xi\} \ni (t,s) \to V(t,s)y,$$

iv) There exist positive constants N, ω such that

$$\|V(t,s)\| \le Ne^{\omega(t-s)}, \quad \forall t \ge s, \quad t,s \in \boldsymbol{R}.$$

By the principle of superposition, we get the relation

$$\begin{aligned} u_t(\sigma,\phi;f) &= u_t(\sigma,\phi;0) + u_t(\sigma,0;f) \\ &= U(t,\sigma)\phi + u_t(\sigma,0;f). \end{aligned} \tag{17}$$

In what follows, we shall give a representation of $u_t(\sigma,0;f)$ in terms of f and the solution process $(U(t,s))_{t\ge s}$. To this end, we introduce a function Γ^n defined by

$$\Gamma^n(\theta) = \begin{cases} (n\theta+1)I, & -1/n \le \theta \le 0 \\ 0, & \theta < -1/n, \end{cases}$$

where n is any positive integer and I is the identity operator on $\boldsymbol{X}$. It follows from (A1) that if $x \in \boldsymbol{X}$, then $\Gamma^n x \in \mathscr{B}$ with $\|\Gamma^n x\|_{\mathscr{B}} \le K(1)\|x\|_{\boldsymbol{X}}$. Moreover, since the process $(U(t,s))_{t\ge s}$ is strongly continuous, the $\mathscr{B}$-valued function $U(t,s)\Gamma^n f(s)$ is continuous in $s \in (-\infty,t]$ whenever $f \in BC(\boldsymbol{R},\boldsymbol{X})$.

The following theorem yields a representation formula for solutions of (2) in the phase space:

Theorem 4.2. *The segment $u_t(\sigma,\phi;f)$ of solution $u(\cdot,\sigma,\phi,f)$ of* (2) *satisfies the following relation in $\mathscr{B}$*:

$$u_t(\sigma,\phi;f) = U(t,\sigma)\phi + \lim_{n\to\infty}\int_\sigma^t U(t,s)\Gamma^n f(s)\,ds, \quad t \ge \sigma. \tag{18}$$

Moreover, the above limit exists uniformly for bounded $|t-\sigma|$.

Proof. It suffices to show that the following limit

$$\lim_{n\to\infty}\left\|\int_\sigma^t U(t,s)\Gamma^n f(s)\,ds - u_t(\sigma,0;f)\right\|_{\mathscr{B}} = 0$$

exists uniformly in (t,σ) such that $t \ge \sigma$ and $t-\sigma$ is bounded. The integral $\int_\sigma^t U(t,s)\Gamma^n f(s)\,ds$ is understood as the limit of a Riemann sum of the form $\phi^\Delta := \sum_k U(t,s_k)\Gamma^n f(s_k)\Delta s_k$ in $\mathscr{B}$. Observe that $\phi^\Delta(\theta) = \sum_k u(t+\theta,s_k,\Gamma^n f(s_k);0)\Delta s_k$ is a Riemann sum of the integral

$$\int_\sigma^t u(t+\theta,s,\Gamma^n f(s);0)\,ds =: \xi^n(t,\sigma)(\theta),$$

and that it converges to the above integral uniformly on any compact set in $\boldsymbol{R}^-$ because of the uniform continuity of $u(t+\theta,s,\Gamma^n f(s);0)$ as a function of (θ,s) on $\boldsymbol{R}^- \times [\sigma,t]$. Since $\xi^n(t,\sigma)(\theta)$ is continuous in $\theta \le 0$ with $\xi^n(t,\sigma)(\theta) = 0$ for $\theta \le \sigma - t - 1/n$, it follows from (A1)-(i) that $\xi^n(t,\sigma) \in \mathscr{B}$. Moreover, we get

$$\|\xi^n(t,\sigma) - \phi^\Delta\|_{\mathscr{B}} \le K_1 \cdot \sup_{\sigma-t-1/n\le\theta\le 0} \|\xi^n(t,\sigma)(\theta) - \phi^\Delta(\theta)\|_X$$

by (A1)-(iii), where $K_1 = K(t-\sigma+1)$. Thus ϕ^Δ converges to $\xi^n(t,\sigma)$ in $\mathscr{B}$, and hence

$$\left\|\int_\sigma^t U(t,s)\Gamma^n f(s)\,ds - \xi^n(t,\sigma)\right\|_{\mathscr{B}} = 0.$$

Given $n \in \boldsymbol{N}$, and $\sigma \le s \le t$, let $u_n^s := u(\cdot, s, \Gamma^n f(s); 0)$. Invoking the definitions of Γ^n, u_n^s and $\xi^n(t,\sigma)$, and denoting $J(t) := \min(t, t+\theta+1/n)$ by a simple computation one can show that for $-(t-\sigma) < \theta \le 0$

$$\begin{aligned}
\xi^n(t,\sigma)(\theta) &= \int_{\theta+t}^{J(t)} \{(t+\theta-s)n+1\} f(s)\,ds + \int_\sigma^{t+\theta} T(t+\theta-s) f(s)\,ds \\
&\quad + \int_\sigma^{t+\theta}\left[\int_s^{t+\theta} T(t+\theta-\rho)F(\rho)(u_n^s)_\rho\,d\rho\right] ds \\
&= \int_{\theta+t}^{J(t)} \{(t+\theta-s)n+1\} f(s)\,ds + \int_\sigma^{t+\theta} T(t+\theta-s) f(s)\,ds \\
&\quad + \int_\sigma^{t+\theta}\left[\int_\sigma^{\rho} T(t+\theta-\rho)F(\rho)(u_n^s)_\rho\,ds\right] d\rho \\
&= \int_{\theta+t}^{J(t)} \{(t+\theta-s)n+1\} f(s)\,ds + \int_\sigma^{t+\theta} T(t+\theta-s) f(s)\,ds \\
&\quad + \int_\sigma^{t+\theta} T(t+\theta-\rho)F(\rho)\left[\int_\sigma^{\rho} U(\rho,s)\Gamma^n f(s)\,ds\right] d\rho \\
&= \int_{\theta+t}^{J(t)} \{(t+\theta-s)n+1\} f(s)\,ds + \int_\sigma^{t+\theta} T(t+\theta-s) f(s)\,ds \\
&\quad + \int_\sigma^{t+\theta} T(t+\theta-s)F(s)\xi^n(s,\sigma)\,ds.
\end{aligned}$$

Hence, one has the following formula:

$$\xi^n(t,\sigma)(\theta) = \begin{cases}
0 & \text{if } \theta < -(t-\sigma) - \dfrac{1}{n}; \\
\displaystyle\int_\sigma^{J(t)} \{(t+\theta-s)n+1\} f(s)\,ds & \text{if } -(t-\sigma) - \dfrac{1}{n} \le \theta \le -(t-\sigma); \\
\displaystyle\int_{\theta+t}^{J(t)} \{(t+\theta-s)n+1\} f(s)\,ds & \\
\quad\displaystyle + \int_\sigma^{t+\theta} T(t+\theta-s) f(s)\,ds & \\
\quad\displaystyle + \int_\sigma^{t+\theta} T(t+\theta-s)F(s)\xi^n(s,\sigma)\,ds & \text{if } -(t-\sigma) < \theta \le 0.
\end{cases}$$

We now consider $\|\xi^n(t,\sigma)(\theta) - \xi^m(t,\sigma)(\theta)\|$. Recall that $\mathscr{B}$ is a uniform fading memory space. Thus, the above calculations lead to the inequality

$$\|\xi^n(t,\sigma) - \xi^m(t,\sigma)\|_{\mathscr{B}} \le K\left(\frac{1}{n} + \frac{1}{m}\right)\left(\sup_{\sigma \le s \le t} \|f(s)\|_X\right) + MK\int_\sigma^t e^{|\omega|(t-s)}\|F(s)\|\,\|\xi^n(s,\sigma) - \xi^m(s,\sigma)\|_{\mathscr{B}}\,ds.$$

(Here, $M > 0$ and $\omega \in \boldsymbol{R}$ are such that $\|T(r)\| \le Me^{\omega r}$.) An application of Gronwall's Lemma leads to

$$\|\xi^n(t,\sigma) - \xi^m(t,\sigma)\|_{\mathscr{B}} \le C(\sigma,t)K\left(\frac{1}{n} + \frac{1}{m}\right),$$

with a constant $C(\sigma,t) > 0$ depending only on the length $(t-\sigma)$. This shows that $\{\xi^n(t,\sigma)\}_n$ is a Cauchy sequence in $\mathscr{B}$, uniformly over $t \ge \sigma$ with length $(t-\sigma)$ bounded.

Given $t \ge \sigma$, let $\xi(t,\sigma)$ be the limit of $\xi^n(t,\sigma)$ in $\mathscr{B}$ as $n \to \infty$. Then, by the above calculations for $\xi^n(t,\sigma)(\theta)$, we see that $\xi^n(t,\sigma)(\theta)$ converges to the following function

$$h(\theta) = \begin{cases} 0, & \text{for } \theta \in (-\infty, -(t-\sigma)] \\ \int_\sigma^{t+\theta} T(t+\theta-s)\{f(s) + F(s)\xi(s,\sigma)\}\,ds & \text{for } \theta \in [-(t-\sigma), 0], \end{cases}$$

uniformly for θ in each compact subset of $\boldsymbol{R}^-$. Hence it follows from (A2) that $\xi(t,\sigma)(\theta) = h(\theta)$. Thus, for $\sigma \in \boldsymbol{R}$, if we define $u^\sigma : \boldsymbol{R} \mapsto \boldsymbol{X}$ by $u^\sigma(t) = 0$ for $t \le \sigma$, and $u^\sigma(t) = \xi(t,\sigma)(0)$ for $t \ge \sigma$, then we have $(u^\sigma)_\sigma = 0$ and $(u^\sigma)_s = \xi(s,\sigma)$ for $s \ge \sigma$. Noting that for $t \ge \sigma$,

$$\begin{aligned} u^\sigma(t) &= \xi(t,\sigma)(0) \\ &= \int_\sigma^t T(t-s)\{f(s) + F(s)\xi(s,\sigma)\}\,ds \\ &= \int_\sigma^t T(t-s)\{f(s) + F(s)(u^\sigma)_s\}\,ds, \end{aligned}$$

we finally conclude that $u^\sigma = u(\cdot\,,\sigma,0;f)$. Thus

$$\lim_{n\to\infty}\int_\sigma^t U(t,s)\Gamma^n f(s)\,ds = \lim_{n\to\infty}\xi^n(t,\sigma) = \xi(t,\sigma) = (u^\sigma)_t = u_t(\sigma,0;f)$$

uniformly over $t \ge \sigma$ with length $t - \sigma$ bounded. This completes the proof of the theorem. □

Throughout this paper we will make as a *standing assumption* that $\mathscr{B}$ is a uniform fading memory space, A is the generator of a compact semigroup $(T(t))_{t\ge 0}$ and $F(t)\phi$ is continuous in $(t,\phi) \in \boldsymbol{R} \times \mathscr{B}$, linear in $\phi \in \mathscr{B}$ and periodic in t with period 1.

Under the standing assumption the following assertion holds:

Theorem 4.3. *Let B be the monodromy operator of the corresponding homogeneous equation of* (2), *i.e.* $B = U(1,0)$. *Then* $r_e(B) < 1$.

Proof. For the proof we refer the reader to [**38**, Theorem 4.8]. □

We now consider the following discrete equation in the phase space $\mathscr{B}$ associated with Equation (2)

$$u_{n+1} = Bu_n + g_n, \quad n \in \boldsymbol{Z} \tag{19}$$

where $g_n := u_{n+1}(0, n, f)$. Under the standing assumption we have $r_e(B) < 1$. Hence, if we assume that there is a function u defined on $\boldsymbol{R}$ which is a mild solution of (2) bounded on the half line $\boldsymbol{R}^+$, then Theorem 3.1 is applicable. As a result we have shown that in the phase space $\mathscr{B}$ there exists an almost periodic two sided sequence z_n, $n \in \boldsymbol{Z}$ which is a solution of (19). However, our aim is to prove the existence of an almost periodic solution on the whole line w of Equation (2) such that $e^{isp(w)} \subset \overline{e^{isp(f)}}$. This can be done by using the variation of constants formula presented in Theorem 4.2. The main result of this section will be the following.

Theorem 4.4. *Let the standing assumption be met. Moreover, let Equation* (2) *have a mild solution bounded on* $\boldsymbol{R}^+$. *Then there exists an almost periodic mild solution* w *to Equation* (2) *on the whole line* $\boldsymbol{R}$ *such that* $e^{i\sigma_b(w)} \subset \overline{e^{isp(f)}}$.

Proof. The proof is divided into several steps.

a) *Existence of an almost periodic solution on the whole line*. First of all, by the above argument there exists an almost periodic solution $x(n)$, $n \in \boldsymbol{Z}$ to the discrete equation (19). We now construct an almost periodic solution to Equation (2) by solving the Cauchy problem on every interval $[n-1, n)$, $n \in \boldsymbol{Z}$, i.e., the following equation

$$\begin{cases} \dot{u}(t) = Au(t) + Fu_t + f(t), & t \in (n-1, n) \\ u_{n-1} = x(n-1). \end{cases} \tag{20}$$

Since $x(n)$, $n \in \boldsymbol{Z}$ is a solution of Equation (19) the solution u defined above is well defined on the whole line $\boldsymbol{R}$ and is a bounded continuous mild solution of Equation (2). Now, as in [**6**, Chapter 9] we will show that the solution u is an almost periodic mild function. To this end, let us extend the sequence $x(n)$, $n \in \boldsymbol{Z}$ to the whole real line as follows $x(t) := ([t] + 1 - t)x([t]) + (t - [t])x([t] + 1)$, $t \in \boldsymbol{R}$, where $[t]$ denotes the integer p such that $p \le t < p + 1$. As is well-known (see e.g. [**6**, pp. 163–164]), the function $x(\cdot)$, defined in this way, is almost periodic. As $x(\cdot)$ and f are almost periodic, so is the function $g : \boldsymbol{R} \ni t \mapsto (x(t), f(t)) \in \mathscr{B} \times \boldsymbol{X}$ (see [**21**, p. 6]). Obviously, the sequence $\{g(n)\} = \{(x(n), f(n))\}$ is almost periodic. Hence, for every positive ε the following set is relatively dense (see [**6**, p. 163–164])

$$T := \boldsymbol{Z} \cap T(g, \varepsilon), \tag{21}$$

where

$$T(g, \varepsilon) := \left\{ \tau \in \boldsymbol{R} : \sup_{t \in \boldsymbol{R}} \|g(t + \tau) - g(t)\| < \varepsilon \right\},$$

i.e., the set of ε periods of g. Hence, for every $m \in T$ we have

$$\|f(t + m) - f(t)\|_{\boldsymbol{X}} < \varepsilon, \quad \forall t \in \boldsymbol{R}, \tag{22}$$

$$\|x(n + m) - x(n)\|_{\mathscr{B}} < \varepsilon, \quad \forall n \in \boldsymbol{Z}. \tag{23}$$

By the uniform boundedness principle obviously $\sup_{t \in [0,1]} \|F(t)\| < \infty$. Since u is a mild solution to Equation (2), for $0 \le s < 1$ and all $n \in \boldsymbol{N}$, we have

$$\begin{aligned}
&\|u(n+m+s) - u(n+s)\|_X \\
&\quad \le \|T(s)\| \, \|x(n+m) - x(n)\|_{\mathscr{B}} + \int_0^s \|T(s-\xi)\| \\
&\qquad \times \left(\sup_{0 \le \eta \le 1} \|F(\eta)\| \, \|u_{n+m+\xi} - u_{n+\xi}\|_{\mathscr{B}} + \|f(n+m+\xi) - f(n+\xi)\|_X \right) d\xi \\
&\quad \le Ce^{\omega} \|x(n+m) - x(n)\|_{\mathscr{B}} + Ce^{\omega} \int_0^s \left(\sup_{0 \le \eta \le 1} \|F(\eta)\| \right. \\
&\qquad \left. \times \|u_{n+m+\xi} - u_{n+\xi}\|_{\mathscr{B}} + \|f(n+m+\xi) - f(n+\xi)\|_X \right) d\xi,
\end{aligned}$$

where C and ω are constants satisfying $\|T(t)\| \le Ce^{\omega t}$. Hence

$$\begin{aligned}
\|u_{n+m+s} - u_{n+s}\|_{\mathscr{B}} \le Ce^{\omega} \|x(n+m) - x(n)\|_{\mathscr{B}} + Ce^{\omega} \sup_{0 \le \eta \le 1} \|F(\eta)\| \int_0^s (\|u_{n+m+\xi} - u_{n+\xi}\|_{\mathscr{B}} \\
+ \|f(n+m+\xi) - f(n+\xi)\|_X) \, d\xi.
\end{aligned}$$

Using the Gronwall inequality we can show that

$$N \|u(n+m+s) - u(n+s)\|_X \le \|u_{n+m+s} - u_{n+s}\|_{\mathscr{B}} \le \varepsilon M, \tag{24}$$

where M is a constant which depends only on $\sup_{0 \le \eta \le 1} \|F(\eta)\|, C, \omega$. This shows that m is a $(\varepsilon M/N)$-period of the function $u(\cdot)$. Finally, since T is relatively dense for every ε, we see that $u(\cdot)$ is an almost periodic mild solution of Equation (2).

b) *Spectral estimate of the solution u.* We now prove the following estimate

Lemma 4.5.

$$e^{i\sigma_b(u)} \subset \sigma_\Gamma(B) \cup e^{i\sigma_b(f)}. \tag{25}$$

Proof. First, it may be noted that for every fixed $n \in \boldsymbol{N}$ the function $h : t \mapsto \Gamma^n f(t)$ is almost periodic and $\sigma_b(h) \subset \sigma_b(f)$. Next, let us consider the function

$$p_n(t) := \int_{t-1}^t U(t,s) \Gamma^n f(s) \, ds.$$

We will show that p_n is almost periodic and $e^{i\sigma_b(p_n)} \subset e^{i\sigma_b(f)}$. In fact, since h is almost periodic with $\sigma_b(h) \subset \sigma_b(f)$, there is a sequence of trigonometric polynomials $h_m(t) = \sum_{k=1}^{N(m)} a_{k,m} e^{i\lambda_{k,m} t}$ with $\lambda_{k,m} \in \sigma_b(f)$ approximating h. Thus, for every $\lambda \in \boldsymbol{R}$ such that $e^{i\lambda} \notin e^{i\sigma_b(f)}$, since

$$e^{-i\lambda_{k,m} t} \int_{t-1}^t U(t,s) a_{k,m} e^{i\lambda_{k,m} s} \, ds$$

is a function of t with period one, we have

$$
\begin{aligned}
\|a(\lambda, p_n)\|_{\mathscr{B}} &:= \left\| \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} e^{-i\lambda t} p_n(t)\,dt \right\|_{\mathscr{B}} \\
&= \left\| \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} e^{-i\lambda t} \int_{t-1}^{t} U(t,s)h(s)\,dsdt \right\|_{\mathscr{B}} \\
&\le \left\| \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} e^{-i\lambda t} \int_{t-1}^{t} U(t,s)h_m(s)\,dsdt \right\|_{\mathscr{B}} \\
&\quad + \left\| \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} e^{-i\lambda t} \int_{t-1}^{t} U(t,s)(h(s)-h_m(s))\,dsdt \right\|_{\mathscr{B}} \\
&\le \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \left\| \int_{t-1}^{t} U(t,s)(h(s)-h_m(s))\,ds \right\|_{\mathscr{B}} dt \\
&\le \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} Ne^{\omega} \sup_{s\in \boldsymbol{R}} \|h(s)-h_m(s)\|_{\mathscr{B}}\,dt \\
&\le Ne^{\omega} \sup_{s\in \boldsymbol{R}} \|h(s)-h_m(s)\|_{\mathscr{B}},
\end{aligned}
$$

where N, ω are determined from the growth bound of the process $(U(t,s))_{t\ge s}$. Hence when h_m approaches h we can see that $a(\lambda, p_n) = 0$, and hence, $e^{i\sigma_b(p_n)} \subset e^{i\sigma_b(f)}$. Similarly, since the limit

$$
p(t) := \lim_{n\to\infty} \int_{t-1}^{t} U(t,s)\Gamma^n f(s)\,ds
$$

is uniform in t we can show that p is almost periodic and $e^{i\sigma_b(p)} \subset e^{i\sigma_b(f)}$. Since the function u, constructed as above, is almost periodic, by Lemma 4.1 the map $t \mapsto u_t$ is also almost periodic, $\sigma_b(u) = \sigma_b(u.)$, and satisfies the equation

$$
u_t = U(t,t-1)u_{t-1} + p(t), \quad t \in \boldsymbol{R}. \tag{26}
$$

Using the 1-periodicity of $U(t,t-1)$, almost periodicity of u_t and $p(t)$, and this equation, we are going to prove the spectral estimate of the lemma. For simplicity, put $v(t) := u_t$, $P(t) := U(t,t-1)$ and

$$
\Gamma_{B,f} := \{\mu \in \Gamma : \mu \notin (\sigma_\Gamma(B) \cup e^{i\sigma_b(f)})\}.
$$

To prove (25) it suffices to show that $\Gamma_{B,f} \cap e^{i\sigma_b(u)} = \varnothing$. Now suppose that $\lambda \in \boldsymbol{R}$ such that $e^{i\lambda} \in \Gamma_{B,f}$. Then $\lambda_m := \lambda + 2m\pi \notin \sigma_b(f)$ and hence $\lambda_m \notin \sigma_b(p)$ for any $m \in \boldsymbol{Z}$. Taking the Bohr transforms of both sides of Equation (26) we arrive at

$$
\begin{aligned}
a(\lambda_m, v) &= \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} e^{-i\lambda_m t} P(t)v(t-1)\,dt + a(\lambda_m, p) \\
&= e^{-i\lambda_m} \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} e^{-i\lambda_m t} P(t)v(t)\,dt \\
&= e^{-i\lambda} \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} e^{-i\lambda_m t} P(t)v(t)\,dt. \qquad (27)
\end{aligned}
$$

Thus,

$$0 = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} e^{-i\lambda_m t}(I - e^{-i\lambda} P(t))v(t)\, dt. \tag{28}$$

Applying Lemma 1.2 to the functions $v(t)$ and $w(t) := Q(t)v(t)$, where $Q(t) := (e^{i\lambda} - P(t))$, and since $(e^{i\lambda} - P(t))^{-1}$ exists and is strongly continuous in t (see e.g. [**26**, Lemma 1] or the references therein), we get $e^{i\sigma_b(v)} = e^{i\sigma_b(w)}$. On the other hand, (28) yields $a(\lambda_m, w) = 0$ for any $m \in \boldsymbol{Z}$, and so, $e^{i\lambda} \notin e^{i\sigma_b(w)} = e^{i\sigma_b(v)}$. Thus, $\Gamma_{B,f} \cap e^{i\sigma_b(v)} = \emptyset$, and so by Lemma 4.1 $\Gamma_{B,f} \cap e^{i\sigma_b(u)} = \emptyset$. This finishes the proof of the lemma. □

c) *Decomposition of an almost periodic solution.* From Lemma 4.5 it follows in particular that $\overline{e^{isp(u)}} \subset \sigma_\Gamma(B) \cup \overline{e^{isp(f)}}$. Obviously, $\sigma_\Gamma(B)$ is finite. Thus, setting $\Lambda_1 := \overline{e^{isp(f)}}$, $\Lambda_2 := \sigma_\Gamma(B) \backslash \overline{e^{isp(f)}}$ we are in a position to apply Corollary 1.8 to finish the proof of the theorem. □

4.3. Autonomous case and quasi periodic solutions.

In the case where $F(t)$ does not depend on t we can refine the above technique to prove the existence of quasi periodic solutions. Let us consider the autonomous equation

$$u'(t) = Au(t) + Fu_t + f(t), \quad t \in \boldsymbol{R}. \tag{29}$$

LEMMA 4.6. *Let the standing assumption be fulfilled. Then for any almost periodic mild solution u of Equation* (29) *the following spectral estimate holds*

$$\sigma_b(u) \subset \sigma_i(\Delta) \cup \sigma_b(f), \tag{30}$$

where

$$\sigma_i(\Delta) := \{\xi \in \boldsymbol{R} : i\xi - A - Fe^{i\xi\cdot} \text{ is not invertible in } L(\boldsymbol{X})\}.$$

PROOF. As is known (see e.g. [**17**]), under the standing assumption, $\sigma_i(\Delta)$ coincides with $\sigma_i(\mathscr{G}) := \{\xi \in \boldsymbol{R} : i\xi \in \sigma(\mathscr{G})\}$, where $\mathscr{G}$ denotes the generator of the solution semigroup associated with (29) on the phase space $\mathscr{B}$. By [**17**, Proposition 4.2 and Theorem 4.3] the above estimate is reduced to that of an ordinary differential equation. Hence, as in the Section 3 this estimate holds. □

Thus (30) can be used to study the existence of quasi periodic mild solutions to Equation (29). Finally, we have

THEOREM 4.7. *Under the standing assumption, if Equation* (29) *has a bounded mild solution on the positive half line and f is quasi periodic function, then there exists a quasi periodic mild solution on the whole line w to Equation* (29) *such that* $\sigma_b(w) \subset \sigma_b(f)$.

PROOF. The assumption yields in particular the existence of an almost periodic mild solution u of Equation (29) on the whole line. Since $\sigma_i(\Delta)$ consists of finitely elements, the following function is well defined

$$w(t) := u(t) - \sum_{\lambda \in \sigma_i(\Delta)} a(\lambda, u)e^{i\lambda t}. \tag{31}$$

We now show that w is the desired mild solution. We again use the decomposition of the variation of constants formula in the phase space (see [**17**, Proposition 4.2 and Theorem 4.3]) to reduce the problem to the finite dimensional case. The next step of the proof can be taken from the one of Theorem 2.1. Using (30) one can decompose the above-mentioned almost periodic mild solution to get a component which is again an almost periodic mild solution w of (29) satisfying $\sigma_b(w) \subset \sigma_b(f)$. Since f is quasi periodic if and only if the spectrum $\sigma_b(f)$ has an integer and finite basis. This characterization of quasi periodicity of f is inherited by w. □

5. Discussion and examples.

In this paper the assumption on the compactness of the semigroup generated by the operator A in Equation (2) is essential which guarantees the existence of a bounded mild solution on the whole line to Equation (2) as well as the one of an almost periodic solution if we know beforehand the existence of a bounded mild solution on the positive half line. Without the above mentioned assumption the Massera Theorem fails even for the simplest equation as in the example below.

Example 5.1. In the infinite dimensional case even the following simplest equation

$$\dot{u}(t) = f(t) \tag{32}$$

may have a bounded solution, but does not accept any almost periodic solution. In fact in the space c_0 of numerical sequences converging to 0, let us consider the function $f(t) := \{(1/n)\cos(t/n)\}$. The function f is almost periodic, but its integral $F(t) := \int_0^t f(\xi)\,d\xi = \{\sin(t/n)\}$, which is bounded, is not almost periodic. Of course, every solution of the equation (32) is of the form $c + F(t)$. Hence, all solutions of this equation are not almost periodic. This is because of the geometric structure of the Banach space $\boldsymbol{X}$ on which the equation is defined (see a counter-example in [**21**, Chapter 6] on the condition that $\boldsymbol{X}$ does not contain c_0).

In this paper we have used the compactness assumption to get Theorem 4.3 and the fact that $\sigma_i(\Delta)$ coincides with $\sigma_i(\mathscr{G})$. Without the compactness assumption, further assumptions should be made, for instance, the assumption that a bounded uniformly continuous solution on the whole line exists, $\sigma_i(\Delta)\backslash sp(f)$ is closed, $sp(f)$ is countable, and $\boldsymbol{X}$ does not contain c_0. For more details in this direction we refer the reader to [**7**] with notice that for the uniform fading memory space $\mathscr{B}$ similar computations can be made for the infinite delay case.

6. Appendix: Almost periodic two sided sequences.

In this appendix we will state several results which are discrete analogs of well-known results on almost periodic functions. We will denote by $B(\boldsymbol{Z}, \boldsymbol{X})$ the space of all bounded two-sided sequences $x : \boldsymbol{Z} \to \boldsymbol{X}$, where $\boldsymbol{X}$ is a Banach space. In this space we consider the translation operators $S(k)$, $k \in \boldsymbol{Z}$, defined as

$$[S(k)x](n) := x(k+n), \quad \forall n \in \boldsymbol{Z},\ x \in B(\boldsymbol{Z}, \boldsymbol{X}).$$

Definition 6.1. A bounded two-sided sequence $\{x_n, n \in \boldsymbol{Z}\}$ is said to be *almost periodic* if the orbit $\{S(k)x, k \in \boldsymbol{Z}\}$ is relatively compact in $B(\boldsymbol{Z}, \boldsymbol{X})$.

Next, we define the Bohr transform of an almost periodic two-sided sequence x by the formula

$$\hat{x}_\lambda := \lim_{N \to \infty} \frac{1}{2N} \sum_{k=-N}^{N} \lambda^{-k} x(k), \quad \lambda \in \Gamma, \tag{33}$$

where $\Gamma := \{z \in \boldsymbol{C} : |z| = 1\}$. As for almost periodic functions, the Bohr transform of a two-sided sequence exists and the set $\sigma_b(x) := \{\lambda \in \Gamma : \hat{x}_\lambda \neq 0\}$ is countable. In this appendix we will state the following result, the proof of which is a straightforward verification of [**6**, Theorem 5.8]. Let us consider the equation

$$x(k+1) = Ax(k) + g(k), \quad x_k \in \boldsymbol{C}^N,\ k \in \boldsymbol{Z}, \tag{34}$$

where $\{g(k), k \in \boldsymbol{Z}\}$ is an almost periodic two-sided sequence.

Theorem 6.2. *If x is a bounded solution of* (34), *then x is almost periodic.*

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