Meromorphic functions sharing three values

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Abstract. In this paper, we prove a result on uniqueness of meromorphic functions sharing three values counting multiplicity. As applications of this, many known results can be improved. Examples are provided to show that the results in this paper are best possible.

1. Introduction and main results.

In this paper, by meromorphic function we shall always mean a meromorphic function in the complex plane. We adopt the standard notations in the Nevanlinna theory of meromorphic functions as explained in [**1**]. For any nonconstant meromorphic function $f(z)$, we denote by $S(r, f)$ any quantity satisfying $S(r, f) = o(T(r, f))$ for $r \to \infty$ except possibly a set of r of finite linear measure. Let k be a positive integer, we denote by $N_{k)}(r, f)$ the counting function of poles of f with multiplicity $\leq k$. We further define (see [**2**])

$$N_{(2}(r, f) = N(r, f) - N_{1)}(r, f),$$

$$\overline{N}_{(2}(r, f) = \overline{N}(r, f) - N_{1)}(r, f),$$

$$N_{(3}(r, f) = N(r, f) - N_{2)}(r, f).$$

Let f and g be two nonconstant meromorphic functions and let a be a finite complex number. If f and g have the same a-points with the same multiplicities, we say that f and g share the value a CM (counting multiplicity) (see [**2**]). If $1/f$ and $1/g$ share the value 0 CM, we say that f and g share ∞ CM.

M. Ozawa [**3**], H. Ueda [**4**], G. Brosch [**5**], H. Yi [**6**], [**7**], [**8**], S. Ye [**9**], P. Li [**10**], Q. Zhang [**11**] and other authors (see [**2**]) dealt with the problem of uniqueness of meromorphic functions that share three distinct values. In 1995, H. Yi proved the following result, which is an improvement of some theorems given by H. Ueda [**4**], H. Yi [**6**] and S. Ye [**9**].

THEOREM A (see [**8**, Theorem 4]). *Let f and g be two distinct nonconstant meromorphic functions sharing $0, 1$ and ∞ CM, and let a $(\neq 0, 1)$ be a finite complex number. If*

$$N\left(r, \frac{1}{f - a}\right) \neq T(r, f) + S(r, f), \tag{1.1}$$

then a is a Picard value of f, f is a fractional linear transformation of g and one of the following three cases will hold:

(i) ∞ *is a Picard value of* f, $1-a$ *and* ∞ *are Picard values of* g, *and* $(f-a) \cdot (g+a-1) \equiv a(1-a)$;

(ii) 0 *is a Picard value of* f, $a/(a-1)$ *and* 0 *are Picard values of* g, *and* $f + (a-1)g \equiv a$;

(iii) 1 *is a Picard value of* f, $1/a$ *and* 1 *are Picard values of* g, *and* $f \equiv ag$.

In this paper, we improve the above theorem and obtain the following result.

THEOREM 1.1.   *Let f and g be two distinct nonconstant meromorphic functions sharing $0, 1$ and ∞ CM.   If there exists a finite complex number a $(\neq 0, 1)$ such that a is not a Picard value of f, and*

$$N_{1)}\left(r, \frac{1}{f-a}\right) \neq T(r, f) + S(r, f), \tag{1.2}$$

then

$$N_{1)}\left(r, \frac{1}{f-a}\right) = \frac{k-2}{k} T(r, f) + S(r, f), \tag{1.3}$$

and one of the following cases will hold:

(i) $f = \dfrac{e^{(k+1)\gamma} - 1}{e^{s\gamma} - 1}$, $\quad g = \dfrac{e^{-(k+1)\gamma} - 1}{e^{-s\gamma} - 1}$, $\quad$ *with* $\quad \dfrac{(a-1)^{k+1-s}}{a^{k+1}} = \dfrac{s^s(k+1-s)^{k+1-s}}{(k+1)^{k+1}}$ *and* $a \neq \dfrac{k+1}{s}$;

(ii) $f = \dfrac{e^{s\gamma} - 1}{e^{(k+1)\gamma} - 1}$, $\quad g = \dfrac{e^{-s\gamma} - 1}{e^{-(k+1)\gamma} - 1}$, $\quad$ *with* $\quad a^s(1-a)^{k+1-s} = \dfrac{s^s(k+1-s)^{k+1-s}}{(k+1)^{k+1}}$ *and* $a \neq \dfrac{s}{k+1}$;

(iii) $f = \dfrac{e^{s\gamma} - 1}{e^{-(k+1-s)\gamma} - 1}$, $\quad g = \dfrac{e^{-s\gamma} - 1}{e^{(k+1-s)\gamma} - 1}$, $\quad$ *with* $\quad \dfrac{(-a)^s}{(1-a)^{k+1}} = \dfrac{s^s(k+1-s)^{k+1-s}}{(k+1)^{k+1}}$ *and* $a \neq -\dfrac{s}{k+1-s}$;

(iv) $f = \dfrac{e^{k\gamma} - 1}{\lambda e^{s\gamma} - 1}$, $g = \dfrac{e^{-k\gamma} - 1}{(1/\lambda)e^{-s\gamma} - 1}$, *with* $\lambda^k \neq 0, 1$ *and* $\dfrac{(a-1)^{k-s}}{\lambda^k a^k} = \dfrac{s^s(k-s)^{k-s}}{k^k}$;

(v) $f = \dfrac{e^{s\gamma} - 1}{\lambda e^{k\gamma} - 1}$, $g = \dfrac{e^{-s\gamma} - 1}{(1/\lambda)e^{-k\gamma} - 1}$, *with* $\lambda^s \neq 0, 1$ *and* $\lambda^s a^s(1-a)^{k-s} = \dfrac{s^s(k-s)^{k-s}}{k^k}$;

(vi) $f = \dfrac{e^{s\gamma} - 1}{\lambda e^{-(k-s)\gamma} - 1}$, $g = \dfrac{e^{-s\gamma} - 1}{(1/\lambda)e^{(k-s)\gamma} - 1}$, *with* $\lambda^s \neq 0, 1$ *and* $\dfrac{(-\lambda a)^s}{(1-a)^k} = \dfrac{s^s(k-s)^{k-s}}{k^k}$;

where γ is a nonconstant entire function, s and k (≥ 2) are positive integers such that s and $k+1$ are mutually prime and $1 \leq s \leq k$ in (i), (ii) and (iii), s and k are mutually prime and $1 \leq s \leq k-1$ in (iv), (v) and (vi).

From Theorem 1.1, we immediately obtain the following corollary:

COROLLARY 1.1.   *Let f and g be two nonconstant meromorphic functions sharing $0, 1$ and ∞ CM, and let a $(\neq 0, 1)$ be a finite complex number such that a is not a Picard value of f, and $N_{1)}(r, 1/(f-a)) \neq T(r, f) + S(r, f)$.   If for any positive integer k (≥ 2),*

$$N_{1)}\left(r,\frac{1}{f-a}\right) \neq \frac{k-2}{k}T(r,f) + S(r,f),$$

then $f \equiv g$.

2. Some lemmas.

LEMMA 2.1. *Let* f *be a nonconstant meromorphic function, and let* a_1 *and* a_2 *be two distinct values in the extended complex plane, and* a_3 *be a meromorphic function satisfying* $T(r,a_3) = S(r,f)$ *and* $a_3 \not\equiv a_j$ *for* $j = 1,2$. *If*

$$\overline{N}\left(r,\frac{1}{f-a_1}\right) + \overline{N}\left(r,\frac{1}{f-a_2}\right) = S(r,f), \tag{2.1}$$

then

$$N_{1)}\left(r,\frac{1}{f-a_3}\right) = T(r,f) + S(r,f). \tag{2.2}$$

PROOF. Using (2.1), by the second fundamental theorem for small functions we have

$$T(r,f) \leq \overline{N}\left(r,\frac{1}{f-a_3}\right) + S(r,f). \tag{2.3}$$

Thus,

$$\overline{N}\left(r,\frac{1}{f-a_3}\right) = T(r,f) + S(r,f). \tag{2.4}$$

Obviously,

$$\overline{N}\left(r,\frac{1}{f-a_3}\right) + \frac{1}{2}N_{(2}\left(r,\frac{1}{f-a_3}\right) \leq N\left(r,\frac{1}{f-a_3}\right) \leq T(r,f) + S(r,f). \tag{2.5}$$

From (2.4) and (2.5) we obtain

$$N_{(2}\left(r,\frac{1}{f-a_3}\right) = S(r,f). \tag{2.6}$$

Again from (2.4) and (2.6) we get (2.2). $\qquad\square$

LEMMA 2.2. *Let* f *and* g *be two distinct nonconstant meromorphic functions sharing* $0,1$ *and* ∞ *CM. If* f *is a fractional linear transformation of* g, *then for any finite complex number* a ($\neq 0,1$), *either* a *is a Picard value of* f, *or*

$$N_{1)}\left(r,\frac{1}{f-a}\right) = T(r,f) + S(r,f).$$

PROOF. By assumption, there is a fractional linear transformation $w = L(u)$ such that $f = L(g)$. Assume that a is not a Picard value of f. By virtue of Lemma 2.1, it suffices to show that f have two distinct Picard values. Assume that f has at most one Picard values. Then, two of the values $0, 1$ and ∞ are not Picard values of f, and the

rest b of them is a Picard value, because the values among $0, 1, \infty$ which are not Picard values of f are fixed points of $L(u)$ and $L(u)$ ($\not\equiv u$) has at most two fixed points. Set $c := L(b)$. Obviously, $c \neq b$ and c is a Picard value of f, because b is a Picard value of g too. This is a contradiction. This completes the proof of Lemma 2.2. $\qquad\square$

LEMMA 2.3 (see [**2**, Lemma 4.5] or [**13**, Lemma 5]). *Let f and g be two distinct nonconstant meromorphic functions sharing $0, 1$ and ∞ CM, and let a ($\neq 0, 1$) be a finite complex constant. Then*

$$N_{(3}\left(r, \frac{1}{f-a}\right) + N_{(3}\left(r, \frac{1}{g-a}\right) = S(r, f).$$

Let f and g be two distinct nonconstant meromorphic functions sharing $0, 1$ and ∞ CM. We use $N_0(r)$ to denote the counting function of the zeros of $f - g$ that are not zeros of f, $f - 1$ and $1/f$ (see [**8**] or [**11**]).

The following lemma is essentially due to Q. Zhang.

LEMMA 2.4 (see [**11**, Proof of Theorem 1 and Theorem 2]). *Let f and g be two distinct nonconstant meromorphic functions sharing $0, 1$ and ∞ CM, and let $N_0(r) \neq S(r, f)$. If f is a fractional linear transformation of g, then*

$$N_0(r) = T(r, f) + S(r, f).$$

If f is not any fractional linear transformation of g, then

$$N_0(r) \leq \frac{1}{2} T(r, f) + S(r, f),$$

and f and g assume one of the following relations:

(i) $\quad f \equiv \dfrac{e^{(k+1)\gamma} - 1}{e^{s\gamma} - 1}, \quad g \equiv \dfrac{e^{-(k+1)\gamma} - 1}{e^{-s\gamma} - 1}$;

(ii) $\quad f \equiv \dfrac{e^{s\gamma} - 1}{e^{(k+1)\gamma} - 1}, \quad g \equiv \dfrac{e^{-s\gamma} - 1}{e^{-(k+1)\gamma} - 1}$;

(iii) $\quad f \equiv \dfrac{e^{s\gamma} - 1}{e^{-(k+1-s)\gamma} - 1}, \quad g \equiv \dfrac{e^{-s\gamma} - 1}{e^{(k+1-s)\gamma} - 1}$;

where γ is a nonconstant entire function, s and k (≥ 2) are positive integers such that s and $k+1$ are mutually prime and $1 \leq s \leq k$.

REMARK. Let f be a nonconstant meromorphic function. By the definition of $S(r, f)$, there is a set E of r of finite linear measure such that

$$S(r, f) = o(T(r, f)) \quad (r \to \infty, r \notin E). \tag{2.7}$$

In [**11**], Q. Zhang first proved the conclusion of Lemma 2.4. Using the conclusion of Lemma 2.4, Q. Zhang proved the following theorems:

Theorem 1 in [**11**]. Let f and g be two distinct nonconstant meromorphic functions sharing $0, 1$ and ∞ CM. If

$$\varlimsup_{\substack{r \to \infty \\ r \notin E}} \frac{N_0(r)}{T(r, f)} > \frac{1}{2},$$

where E is a set of r of finite linear measure with (2.7), then f is a fractional linear transformation of g.

Theorem 2 in [**11**]. *Let f and g be two distinct nonconstant meromorphic functions sharing $0, 1$ and ∞ CM. If*

$$0 < \varlimsup_{\substack{r \to \infty \\ r \notin E}} \frac{N_0(r)}{T(r, f)} \leq \frac{1}{2},$$

where E is a set of r of finite linear measure with (2.7), then f is not any fractional linear transformation of g, and f and g assume one of the three relations in Lemma 2.4.

LEMMA 2.5 (see [**14**]). *Let $s\ (>0)$ and t are mutually prime integers, and let c be a finite complex number such that $c^s = 1$, then there exists one and only one common zero of $\omega^s - 1$ and $\omega^t - c$.*

LEMMA 2.6 (see [**15**]). *Let f be a nonconstant meromorphic function, and let*

$$F = \sum_{k=0}^{p} a_k f^k \bigg/ \sum_{j=0}^{q} b_j f^j$$

be an irreducible rational function in f with constant coefficients $\{a_k\}$ and $\{b_j\}$, where $a_p \neq 0$ and $b_q \neq 0$. Then

$$T(r, F) = dT(r, f) + S(r, f),$$

where $d = \max\{p, q\}$.

LEMMA 2.7 (see [**16**]). *Let*

$$P(\omega) = \omega^n + a\omega^m + b, \tag{2.8}$$

where m and n are positive integers such that $n > m$, a and b are finite nonzero complex numbers.
(i) *The algebraic equation $P(\omega) = 0$ has no roots with multiplicity ≥ 3;*
(ii) *If*

$$\frac{b^{n-m}}{a^n} \neq \frac{(-1)^n m^m (n-m)^{n-m}}{n^n}, \tag{2.9}$$

the algebraic equation $P(\omega) = 0$ has n distinct simple roots, no multiple roots;
(iii) *If n and m are mutually prime and*

$$\frac{b^{n-m}}{a^n} = \frac{(-1)^n m^m (n-m)^{n-m}}{n^n}, \tag{2.10}$$

the algebraic equation $P(\omega) = 0$ has $n - 1$ distinct roots, where $n - 2$ roots are simple, one is double.

PROOF. (i) The conclusion is obvious, we now omit it.
(ii) Let

$$P(\omega) = \omega^n + a\omega^m + b, \tag{2.11}$$

then

$$P'(\omega) = n\omega^{n-1} + am\omega^{m-1}. \tag{2.12}$$

If ω_0 is a double root of $P(\omega) = 0$, then if and only if

$$P(\omega) = P'(\omega) = 0.$$

Combining (2.11) and (2.12) we can easily get

$$\omega_0^m = -\frac{nb}{a(n-m)}, \quad \omega_0^n = \frac{bm}{n-m}. \tag{2.13}$$

Since $(\omega_0^m)^n = (\omega_0^n)^m$, from (2.13) we have

$$\left(-\frac{nb}{a(n-m)}\right)^n = \left(\frac{bm}{n-m}\right)^m, \tag{2.14}$$

which can be rewritten as

$$\frac{b^{n-m}}{a^n} = \frac{(-1)^n m^m (n-m)^{n-m}}{n^n}, \tag{2.15}$$

accordingly, if $P(\omega) = 0$ has n distinct simple roots, then $P(\omega) = 0$ has no any multiple root, if and only if (2.9) holds.

(iii)   Let ω_0 be a double root of $P(\omega) = 0$, using proceeding as in (ii), we can get (2.13) and (2.14).   On the other hand, since n and m are mutually prime, there exist one and only one pair of integers s and t such that

$$ns - mt = 1 \quad (0 < s < m, 0 < t < n). \tag{2.16}$$

From (2.13) and (2.16) we can easily have

$$\omega_0 = \omega_0^{ns-mt} = \left(\frac{bm}{n-m}\right)^s \cdot \left(-\frac{nb}{a(n-m)}\right)^{-t},$$

which implies that $P(\omega) = 0$ has one and only one double root. $\square$

LEMMA 2.8 (see [**8**, Lemma 1]).   *Let f and g be two distinct nonconstant meromorphic functions sharing $0, 1$ and ∞ CM, then there exist two entire functions α and β such that*

$$f \equiv \frac{e^\alpha - 1}{e^\beta - 1}, \quad g \equiv \frac{e^{-\alpha} - 1}{e^{-\beta} - 1}, \tag{2.17}$$

where $e^\beta \not\equiv 1$, $e^\alpha \not\equiv 1$ and $e^{\beta-\alpha} \not\equiv 1$, and

$$T(r, g) + T(r, e^\alpha) + T(r, e^\beta) = O(T(r, f)) \quad (r \notin E), \tag{2.18}$$

where E is a set of r of finite linear measure.

LEMMA 2.9 (see [**8**, Lemma 3]).   *Let α be a nonconstant entire function, then*

$$T(r, \alpha') = S(r, e^\alpha). \tag{2.19}$$

LEMMA 2.10 (see [**11**, Lemma 6]). *Let f_1 and f_2 be two nonconstant meromorphic functions satisfying*

$$\overline{N}(r, f_j) + \overline{N}\left(r, \frac{1}{f_j}\right) = S(r) \quad (j = 1, 2).$$

Then either

$$N_0(r, 1; f_1, f_2) = S(r)$$

or there exist two integers p and q $(|p| + |q| > 0)$ such that

$$f_1^p \cdot f_2^q \equiv 1,$$

where $N_0(r, 1; f_1, f_2)$ denotes the reduced counting function of the common 1-points of f_1 and f_2, and $T(r) = T(r, f_1) + T(r, f_2)$, $S(r) = o(T(r))$ $(r \to \infty, r \notin E)$, E is a set of r of finite linear measure.

LEMMA 2.11 (see [**8**, Lemma 4]). *Let f and g be two nonconstant meromorphic functions sharing $0, 1$ and ∞ CM. If $f \not\equiv g$, then*

$$N_{(2}(r, f) + N_{(2}\left(r, \frac{1}{f}\right) + N_{(2}\left(r, \frac{1}{f - 1}\right) = S(r, f).$$

3. Proof of Theorem 1.1.

If f is a fractional linear transformation of g, by Lemma 2.2 we have that either a is a Picard value of f, or $N_{1)}(r, 1/(f - a)) = T(r, f) + S(r, f)$, which contradicts the assumption of Theorem 1.1. Thus, f is not a fractional linear transformation of g. By Theorem A we have

$$N\left(r, \frac{1}{f - a}\right) = T(r, f) + S(r, f). \tag{3.1}$$

From (1.2) and (3.1) we obtain

$$N_{(2}\left(r, \frac{1}{f - a}\right) \neq S(r, f). \tag{3.2}$$

By Lemma 2.3,

$$N_{(3}\left(r, \frac{1}{f - a}\right) = S(r, f). \tag{3.3}$$

Combining (3.2) and (3.3) we get

$$\overline{N}_{(2}\left(r, \frac{1}{f - a}\right) \neq S(r, f). \tag{3.4}$$

We discuss the following two cases.

CASE 1. Suppose that

$$N_0(r) \neq S(r, f).$$

By Lemma 2.4 we know that f and g assume one of the three relations in Lemma 2.4. We discuss the following three subcases.

Subcase 1.1. Suppose that f and g assume the form (i) in Lemma 2.4. Thus,

$$f = \frac{e^{(k+1)\gamma} - 1}{e^{s\gamma} - 1}, \quad g = \frac{e^{-(k+1)\gamma} - 1}{e^{-s\gamma} - 1}, \tag{3.5}$$

which assume the form (i) in Theorem 1.1. By Lemma 2.5, we know that there exists one and only one common zero of $\omega^{k+1} - 1$ and $\omega^s - 1$. By Lemma 2.6, we have from (3.5)

$$T(r,f) = kT(r,e^\gamma) + S(r,f). \tag{3.6}$$

From (3.5) we have

$$f - a = \frac{e^{(k+1)\gamma} - ae^{s\gamma} + (a-1)}{e^{s\gamma} - 1}. \tag{3.7}$$

Let

$$P(\omega) = \omega^{k+1} - a\omega^s + (a-1), \tag{3.8}$$

$$Q(\omega) = \frac{\omega^{k+1} - a\omega^s + (a-1)}{\omega^s - 1}. \tag{3.9}$$

If $a = (k+1)/s$, from (3.8) we know that $\omega = 1$ is a double root of $P(\omega) = 0$. Again by Lemma 2.7, the equation $P(\omega) = 0$ has $k-1$ distinct simple roots. From (3.9) we know that $Q(\omega) = 0$ has k distinct simple roots. From (3.6) and (3.7),

$$N_{1)}\left(r, \frac{1}{f-a}\right) = kT(r,e^\gamma) + S(r,f) = T(r,f) + S(r,f),$$

which contradicts (1.2). Thus, $a \neq (k+1)/s$ and $\omega = 1$ is a simple root of $P(\omega) = 0$. If

$$\frac{(a-1)^{k+1-s}}{a^{k+1}} \neq \frac{s^s(k+1-s)^{k+1-s}}{(k+1)^{k+1}},$$

by Lemma 2.7, we know that $Q(\omega) = 0$ has k distinct simple roots, which is also a contradiction. Thus,

$$\frac{(a-1)^{k+1-s}}{a^{k+1}} = \frac{s^s(k+1-s)^{k+1-s}}{(k+1)^{k+1}}.$$

By Lemma 2.7, we know that $Q(\omega) = 0$ has $k-2$ distinct simple roots and one double root. From (3.6), (3.7) and (3.9) we obtain

$$N_{(2}\left(r, \frac{1}{f-a}\right) = \frac{2}{k}T(r,f) + S(r,f). \tag{3.10}$$

Combining (3.1) and (3.10) we get (1.3).

SUBCASE 1.2. Suppose that f and g assume the form (ii) in Lemma 2.4. Thus,

$$f = \frac{e^{s\gamma} - 1}{e^{(k+1)\gamma} - 1}, \quad g = \frac{e^{-s\gamma} - 1}{e^{-(k+1)\gamma} - 1}, \tag{3.11}$$

which assume the form (ii) in Theorem 1.1. By Lemma 2.6, we have from (3.11)

$$T(r, f) = kT(r, e^{\gamma}) + S(r, f). \tag{3.12}$$

From (3.11) we have

$$f - a = -\frac{a(e^{(k+1)\gamma} - (1/a)e^{s\gamma} - (a-1)/a)}{e^{(k+1)\gamma} - 1}. \tag{3.13}$$

In the same manner as Subcase 1.1, we have $a \neq s/(k+1)$ and $a^s(1-a)^{k+1-s} = s^s(k+1-s)^{k+1-s}/(k+1)^{k+1}$, and can obtain (1.3).

SUBCASE 1.3. Suppose that f and g assume the form (iii) in Lemma 2.4. Thus,

$$f = \frac{e^{s\gamma} - 1}{e^{-(k+1-s)\gamma} - 1}, \quad g = \frac{e^{-s\gamma} - 1}{e^{(k+1-s)\gamma} - 1}, \tag{3.14}$$

which assume the form (iii) in Theorem 1.1. From (3.14) we have

$$f - a = \frac{e^{(k+1)\gamma} + (a-1)e^{(k+1-s)\gamma} - a}{1 - e^{(k+1-s)\gamma}}. \tag{3.15}$$

In the same manner as Subcase 1.1, we have $a \neq -s/(k+1-s)$ and $(-a)^s/(1-a)^{k+1} = s^s(k+1-s)^{k+1-s}/(k+1)^{k+1}$, and can obtain (1.3).

CASE 2. Suppose that

$$N_0(r) = S(r, f). \tag{3.16}$$

Noting f and g share $0, 1$ and ∞ CM, by Lemma 2.8 we have (2.17) and (2.18). From (2.17) we have

$$T(r, f) \leq T(r, e^{\alpha}) + T(r, e^{\beta}) + O(1). \tag{3.17}$$

From (2.18) and Lemma 2.9 we have

$$T(r, \alpha') + T(r, \beta') = S(r, f). \tag{3.18}$$

Again from (2.17) we get

$$f - a = \frac{e^{\alpha} - ae^{\beta} + (a-1)}{e^{\beta} - 1}. \tag{3.19}$$

Assume that $T(r, e^{\beta}) = S(r, f)$. Noting 0 and ∞ are Picard values of e^{α}, by Lemma 2.1 we have from (2.17) and (3.18)

$$N_{1)}\left(r, \frac{1}{f-a}\right) = T(r, f) + S(r, f),$$

which is a contradiction. Thus, $T(r, e^\beta) \neq S(r, f)$. Similarly, we have $T(r, e^\alpha) \neq S(r, f)$ and $T(r, e^{\alpha-\beta}) \neq S(r, f)$. Particularly, none of e^α, e^β and $e^{\alpha-\beta}$ are constants. From (2.17) we obtain

$$f - g = \frac{(e^\alpha - 1)(1 - e^{\beta-\alpha})}{e^\beta - 1}. \tag{3.20}$$

We use $N_0^*(r)$ to denote the counting function of the common zeros of $e^\alpha - 1$ and $e^\beta - 1$. From (3.20), the following formula is obviously

$$N_0(r) = N_0^*(r) + S(r, f).$$

From this and (3.16),

$$N_0^*(r) = S(r, f). \tag{3.21}$$

Let z_0 be a multiple zero of $f - a$, but not a zero of α', β' and $\beta' - \alpha'$. From (3.19) we obtain

$$e^{\alpha(z_0)} - a e^{\beta(z_0)} + a - 1 = 0 \tag{3.22}$$

and

$$\alpha'(z_0) e^{\alpha(z_0)} - a\beta'(z_0) e^{\beta(z_0)} = 0. \tag{3.23}$$

From (3.22) and (3.23) we have

$$e^{\alpha(z_0)} = \frac{(1-a)\beta'(z_0)}{\beta'(z_0) - \alpha'(z_0)}, \quad e^{\beta(z_0)} = \frac{(1-a)\alpha'(z_0)}{a(\beta'(z_0) - \alpha'(z_0))}. \tag{3.24}$$

Let

$$f_1 = \frac{(\beta' - \alpha')e^\alpha}{(1-a)\beta'}, \quad f_2 = \frac{a(\beta' - \alpha')e^\beta}{(1-a)\alpha'}. \tag{3.25}$$

Set

$$T(r) = T(r, f_1) + T(r, f_2), \quad S(r) = o(T(r)) \quad (r \to \infty, r \notin E), \tag{3.26}$$

E is a set of r of finite linear measure. From (3.17), (3.18), (3.25) and (3.26) we get

$$S(r, f) = S(r). \tag{3.27}$$

From (2.19), (3.18), (3.25) and (3.27) we have

$$\overline{N}(r, f_j) + \overline{N}\left(r, \frac{1}{f_j}\right) = S(r) \quad (j = 1, 2). \tag{3.28}$$

From (3.24) and (3.25), we have $f_1(z_0) = 1$, $f_2(z_0) = 1$. Thus,

$$\overline{N}_{(2}\left(r, \frac{1}{f - a}\right) \leq N_0(r, 1; f_1, f_2) + S(r, f), \tag{3.29}$$

where $N_0(r, 1; f_1, f_2)$ denotes the reduced counting function of the common 1-points of f_1 and f_2. From (3.4), (3.27) and (3.29) we obtain

$$N_0(r, 1; f_1, f_2) \neq S(r). \tag{3.30}$$

Noting (3.28) and (3.30) and using Lemma 2.10, we know that there exist two integers p and q $(|p| + |q| > 0)$ such that

$$f_1^p \cdot f_2^q \equiv 1. \tag{3.31}$$

Noting $T(r, e^\alpha) \neq S(r, f)$ and $T(r, e^\beta) \neq S(r, f)$, from (3.25) and (3.31) we have $p \neq 0$ and $q \neq 0$. From (3.25) and (3.31), we obtain

$$e^{p\alpha + q\beta} \equiv \left(\frac{(1-a)\beta'}{\beta' - \alpha'}\right)^p \left(\frac{(1-a)\alpha'}{a(\beta' - \alpha')}\right)^q, \tag{3.32}$$

by logarithmic differentiation, we can get

$$p\alpha' + q\beta' = \frac{q + p\alpha'/\beta'}{(\alpha'/\beta')(1 - \alpha'/\beta')}\left(\frac{\alpha'}{\beta'}\right)'. \tag{3.33}$$

If $\alpha'/\beta' \not\equiv -q/p$, from (3.33) we have

$$\alpha' \equiv \frac{(\alpha'/\beta')'}{1 - (\alpha'/\beta')}.$$

By integration, we obtain

$$e^\alpha\left(1 - \frac{\alpha'}{\beta'}\right) \equiv c_1, \tag{3.34}$$

where c_1 is a nonzero constant. From (3.34) we get

$$\beta' \equiv \frac{\alpha' e^\alpha}{e^\alpha - c_1}.$$

Again by integration, we have

$$e^\beta \equiv c_2(e^\alpha - c_1), \tag{3.35}$$

where c_2 is also a nonzero constant. From (3.35) we know that c_1 is a Picard value of e^α, which is impossible. Thus,

$$\frac{\alpha'}{\beta'} \equiv -\frac{q}{p} \tag{3.36}$$

and hence

$$p\alpha' + q\beta' \equiv 0.$$

By integration, we obtain

$$p\alpha + q\beta \equiv c_0, \tag{3.37}$$

where c_0 is a finite constant. Noting $e^{\alpha - \beta}$ is not a constant, from (3.37) we know that $p \neq -q$. Without loss of generality, from (3.36) we may assume that p and q are two integers such that p and q are mutually prime and $q > 0$. Let $\gamma = \alpha/q$. From this, (2.17) and (3.37) we have

$$f = \frac{e^{q\gamma} - 1}{\lambda e^{-p\gamma} - 1}, \quad g = \frac{e^{-q\gamma} - 1}{(1/\lambda)e^{p\gamma} - 1}, \tag{3.38}$$

where $\lambda = e^{c_0/q}$ is a nonzero constant. Obviously, if and only if $\lambda^q = 1$, $\omega^q - 1 = 0$ and $\lambda\omega^{-p} - 1 = 0$ have a common root. Noting $N_0^*(r) = S(r, f)$, from (3.38) we get

$$\lambda^q \neq 0, 1. \tag{3.39}$$

Noting $q > 0$ and $p \neq -q$, we discuss the following three subcases.

SUBCASE 2.1.   Suppose that $q > -p > 0$. Setting $k = q$ and $s = -p$, from (3.38) we get

$$f = \frac{e^{k\gamma} - 1}{\lambda e^{s\gamma} - 1}, \quad g = \frac{e^{-k\gamma} - 1}{(1/\lambda)e^{-s\gamma} - 1}, \tag{3.40}$$

which assume the form (iv) in Theorem 1.1. From (3.39) we have

$$\lambda^k \neq 0, 1. \tag{3.41}$$

By Lemma 2.6, we have from (3.40)

$$T(r, f) = kT(r, e^{\gamma}) + S(r, f). \tag{3.42}$$

From (3.40) we have

$$f - a = \frac{e^{k\gamma} - a\lambda e^{s\gamma} + (a - 1)}{\lambda e^{s\gamma} - 1}. \tag{3.43}$$

Let

$$R(\omega) = \frac{\omega^k - a\lambda\omega^s + (a - 1)}{\lambda\omega^s - 1}. \tag{3.44}$$

If

$$\frac{(a - 1)^{k-s}}{\lambda^k a^k} \neq \frac{s^s(k - s)^{k-s}}{k^k},$$

by Lemma 2.7, we know that $R(\omega) = 0$ has k distinct simple roots, which is a contradiction. Thus,

$$\frac{(a - 1)^{k-s}}{\lambda^k a^k} = \frac{s^s(k - s)^{k-s}}{k^k}. \tag{3.45}$$

By Lemma 2.7, we know that $Q(\omega) = 0$ has $k - 2$ distinct simple roots and one double root. From (3.42), (3.43) and (3.44) we obtain

$$N_{(2}\left(r, \frac{1}{f - a}\right) = \frac{2}{k}T(r, f) + S(r, f). \tag{3.46}$$

Combining (3.1) and (3.46) we get (1.3).

SUBCASE 2.2.   Suppose that $-p > q > 0$. Setting $k = -p$ and $s = q$, from (3.38) we get

$$f = \frac{e^{s\gamma} - 1}{\lambda e^{k\gamma} - 1}, \quad g = \frac{e^{-s\gamma} - 1}{(1/\lambda)e^{-k\gamma} - 1}, \tag{3.47}$$

which assume the form (v) in Theorem 1.1. From (3.39) we have

$$\lambda^s \neq 0, 1. \tag{3.48}$$

By Lemma 2.6, we have from (3.47)

$$T(r, f) = kT(r, e^{\gamma}) + S(r, f). \tag{3.49}$$

From (3.47) we have

$$f - a = \frac{-a\lambda(e^{k\gamma} - (1/(a\lambda))e^{s\gamma} + (1 - a)/(a\lambda))}{\lambda e^{k\gamma} - 1}. \tag{3.50}$$

In the same manner as Subcase 2.1, we have

$$\lambda^s a^s (1 - a)^{k-s} = \frac{s^s (k - s)^{k-s}}{k^k}, \tag{3.51}$$

and can obtain (1.3).

SUBCASE 2.3. Suppose that $p > 0$. Setting $k = p + q$ and $s = q$, from (3.38) we get

$$f = \frac{e^{s\gamma} - 1}{\lambda e^{-(k-s)\gamma} - 1}, \quad g = \frac{e^{-s\gamma} - 1}{(1/\lambda)e^{(k-s)\gamma} - 1}, \tag{3.52}$$

which assume the form (vi) in Theorem 1.1. From (3.39) we have

$$\lambda^s \neq 0, 1. \tag{3.53}$$

From (3.52) we have

$$f - a = \frac{e^{k\gamma} - (1 - a)e^{(k-s)\gamma} - a\lambda}{\lambda - e^{(k-s)\gamma}}. \tag{3.54}$$

In the same manner as Subcase 2.1, we have

$$\frac{(-\lambda a)^s}{(1 - a)^k} = \frac{s^s (k - s)^{k-s}}{k^k}, \tag{3.55}$$

and can obtain (1.3).

Theorem 1.1 is thus completely proved.

4. On two results of P. Li.

In 1998, P. Li proved the following result:

THEOREM B (see [**10**, Theorem 1]). *Let f and g be two distinct nonconstant meromorphic functions sharing $0, 1$ and ∞ CM. Suppose additionally that f is not a fractional linear transformation of g and that there exists a finite complex number a ($\neq 0, 1$) such that*

$$T(r,f) \le c\overline{N}_{(2}\left(r,\frac{1}{f-a}\right) + S(r,f), \tag{4.1}$$

here c (> 0) is a constant, then there exist a nonconstant entire function γ, a nonzero constant λ and two integers t (> 0), s which are mutually prime, such that

$$f = \frac{e^{t\gamma}-1}{\lambda e^{-s\gamma}-1}, \quad g = \frac{e^{-t\gamma}-1}{(1/\lambda)e^{s\gamma}-1}, \tag{4.2}$$

$$\frac{(1-a)^{s+t}}{a^t} = \lambda^t \frac{(1-\theta)^{s+t}}{\theta^t}, \tag{4.3}$$

with $\theta = -t/s \ne 1, a$.

From Theorem 1.1, we can obtain the following result, which is an improvement and supplement of Theorem B.

THEOREM 4.1. *Let f and g be two distinct nonconstant meromorphic functions sharing 0, 1 and ∞ CM. If there exists a finite complex number a (≠ 0, 1) such that*

$$\overline{N}_{(2}\left(r,\frac{1}{f-a}\right) \ne S(r,f), \tag{4.4}$$

then the conclusions of Theorem 1.1 hold, and

$$\overline{N}_{(2}\left(r,\frac{1}{f-a}\right) = \frac{1}{k}T(r,f) + S(r,f). \tag{4.5}$$

PROOF. From (4.4) we know that a is not a Picard value of f, and $N_{1)}(r, 1/(f-a)) \ne T(r,f) + S(r,f)$. By Theorem 1.1, we immediately obtain the conclusion of Theorem 4.1. □

In 1998, P. Li proved the following result:

THEOREM C (see [**10**, Theorem 2]). *Let f and g be two distinct nonconstant meromorphic functions sharing 0, 1 and ∞ CM. Suppose additionally that f is not a fractional linear transformation of g and that there exists a finite complex number a (≠ 0, 1) such that*

$$N_{1)}\left(r,\frac{1}{f-a}\right) = S(r,f), \tag{4.6}$$

then f and g assume one of the following forms:

(i) $f = \dfrac{e^{3\gamma}-1}{e^{\gamma}-1}$, $g = \dfrac{e^{-3\gamma}-1}{e^{-\gamma}-1}$, *with* $a = \dfrac{3}{4}$;

(ii) $f = \dfrac{e^{3\gamma}-1}{\lambda e^{2\gamma}-1}$, $g = \dfrac{e^{-3\gamma}-1}{(1/\lambda)e^{-2\gamma}-1}$, *with* $a = -3$ *and* $\lambda^3 = 1$;

(iii) $f = \dfrac{e^{\gamma}-1}{e^{3\gamma}-1}$, $g = \dfrac{e^{-\gamma}-1}{e^{-3\gamma}-1}$, *with* $a = \dfrac{4}{3}$;

(iv) $f = \dfrac{e^{2\gamma}-1}{\lambda e^{3\gamma}-1}$, $g = \dfrac{e^{-2\gamma}-1}{(1/\lambda)e^{-3\gamma}-1}$, *with* $a = -\dfrac{1}{3}$ *and* $\lambda^2 = 1$;

(v) $\quad f = \dfrac{e^{2\gamma} - 1}{e^{-\gamma} - 1}, \quad g = \dfrac{e^{-2\gamma} - 1}{e^{\gamma} - 1}, \quad \text{with } a = \dfrac{1}{4};$

(vi) $\quad f = \dfrac{e^{\gamma} - 1}{e^{-2\gamma} - 1}, \quad g = \dfrac{e^{-\gamma} - 1}{e^{2\gamma} - 1}, \quad \text{with } a = 4;$

(vii) $\quad f = \dfrac{e^{2\gamma} - 1}{\lambda e^{\gamma} - 1}, \quad g = \dfrac{e^{-2\gamma} - 1}{(1/\lambda)e^{-\gamma} - 1}, \quad \text{with } \lambda^2 \neq 1 \text{ and } a^2\lambda^2 = 4(a - 1);$

(viii) $\quad f = \dfrac{e^{\gamma} - 1}{\lambda e^{2\gamma} - 1}, \quad g = \dfrac{e^{-\gamma} - 1}{(1/\lambda)e^{-2\gamma} - 1}, \quad \text{with } \lambda \neq 1 \text{ and } 4a(1 - a)\lambda = 1;$

(ix) $\quad f = \dfrac{e^{\gamma} - 1}{\lambda e^{-\gamma} - 1}, \quad g = \dfrac{e^{-\gamma} - 1}{(1/\lambda)e^{\gamma} - 1}, \quad \text{with } \lambda \neq \dfrac{1 - a}{2} \text{ and } (1 - a)^2 + 4a\lambda = 0;$

where γ is a nonconstant entire function.

From Theorem 1.1, we can obtain the following result.

THEOREM 4.2. *Let f and g be two distinct nonconstant meromorphic functions sharing $0, 1$ and ∞ CM. If there exists a finite complex number a $(\neq 0, 1)$ such that a is not a Picard value of f, and*

$$N_{1)}\left(r, \frac{1}{f - a}\right) \leq uT(r, f) + S(r, f), \tag{4.7}$$

where $u < 1/3$, then

$$N_{1)}\left(r, \frac{1}{f - a}\right) = 0, \tag{4.8}$$

and f and g assume one of the following forms:

(i) $\quad f = \dfrac{e^{3\gamma} - 1}{e^{\gamma} - 1}, \quad g = \dfrac{e^{-3\gamma} - 1}{e^{-\gamma} - 1}, \quad \text{with } a = \dfrac{3}{4};$

(ii) $\quad f = \dfrac{e^{3\gamma} - 1}{e^{2\gamma} - 1}, \quad g = \dfrac{e^{-3\gamma} - 1}{e^{-2\gamma} - 1}, \quad \text{with } a = -3;$

(iii) $\quad f = \dfrac{e^{\gamma} - 1}{e^{3\gamma} - 1}, \quad g = \dfrac{e^{-\gamma} - 1}{e^{-3\gamma} - 1}, \quad \text{with } a = \dfrac{4}{3};$

(iv) $\quad f = \dfrac{e^{2\gamma} - 1}{e^{3\gamma} - 1}, \quad g = \dfrac{e^{-2\gamma} - 1}{e^{-3\gamma} - 1}, \quad \text{with } a = -\dfrac{1}{3};$

(v) $\quad f = \dfrac{e^{2\gamma} - 1}{e^{-\gamma} - 1}, \quad g = \dfrac{e^{-2\gamma} - 1}{e^{\gamma} - 1}, \quad \text{with } a = \dfrac{1}{4};$

(vi) $\quad f = \dfrac{e^{\gamma} - 1}{e^{-2\gamma} - 1}, \quad g = \dfrac{e^{-\gamma} - 1}{e^{2\gamma} - 1}, \quad \text{with } a = 4;$

(vii) $\quad f = \dfrac{e^{2\gamma} - 1}{\lambda e^{\gamma} - 1}, \quad g = \dfrac{e^{-2\gamma} - 1}{(1/\lambda)e^{-\gamma} - 1}, \quad \text{with } \lambda^2 \neq 1 \text{ and } a^2\lambda^2 = 4(a - 1);$

(viii) $\quad f = \dfrac{e^{\gamma} - 1}{\lambda e^{2\gamma} - 1}, \quad g = \dfrac{e^{-\gamma} - 1}{(1/\lambda)e^{-2\gamma} - 1}, \quad \text{with } \lambda \neq 1 \text{ and } 4a(1 - a)\lambda = 1;$

(ix) $\quad f = \dfrac{e^{\gamma} - 1}{\lambda e^{-\gamma} - 1}, \quad g = \dfrac{e^{-\gamma} - 1}{(1/\lambda)e^{\gamma} - 1}, \quad \text{with } \lambda \neq 1 \text{ and } (1 - a)^2 + 4a\lambda = 0;$

where γ is a nonconstant entire function.

PROOF. By Theorem 1.1 and (4.7), we know that the conclusions of Theorem 1.1 hold, where $k = 2$. From this, we immediately obtain the conclusion of Theorem 4.2. $\qquad\square$

REMARK. Obviously, Theorem 4.2 is an improvement of Theorem C. It is easy to show that (ix) in Theorem C and (ix) in Theorem 4.2 are equivalent to each other. We next prove that (iv) in Theorem C and (iv) in Theorem 4.2 are equivalent to each other. In fact, in (iv) of Theorem C, $\lambda^2 = 1$. From this we obtain $\lambda = 1$ or $\lambda = -1$. When $\lambda = 1$, from (iv) in Theorem C we obtain (iv) in Theorem 4.2. When $\lambda = -1$, using $\gamma + \pi i$ in place of γ in (iv) of Theorem C, we obtain (iv) in Theorem 4.2. Similarly, we can prove that (ii) in Theorem C and (ii) in Theorem 4.2 are equivalent to each other.

EXAMPLE 4.1. Let $f(z) = e^{3z} + e^{2z} + e^z + 1$, $g(z) = e^{-3z} + e^{-2z} + e^{-z} + 1$ and $a = (20 + 4\sqrt{2}i)/27$. Then it is easily verified that f and g share $0, 1$ and ∞ CM, and

$$N_{1)}\left(r, \frac{1}{f-a}\right) = \frac{1}{3}T(r,f) + S(r,f).$$

Moreover, f and g do not assume one of the forms in Theorem 4.2. This illustrates that the assumption $u < 1/3$ in Theorem 4.2 is best possible.

5. Some result of entire functions.

In 1995, H. Yi proved the following result.

THEOREM D (see [**8**, Theorem 1]). *Let f and g be two distinct nonconstant meromorphic functions sharing $0, 1$ and ∞ CM, and let a ($\neq 0, 1$) be a finite complex number. If*

$$N\left(r, \frac{1}{f-a}\right) \neq T(r,f) + S(r,f) \tag{5.1}$$

and

$$N(r,f) \neq T(r,f) + S(r,f), \tag{5.2}$$

then a and $1 - a$ are Picard values of f and g respectively, and also ∞ is so, and

$$(f-a)(g+a-1) \equiv a(1-a). \tag{5.3}$$

From Theorem 1.1, we immediately obtain the following result.

THEOREM 5.1. *If, in addition to the assumptions of Theorem 1.1,*

$$\overline{N}(r,f) = S(r,f), \tag{5.4}$$

then

$$N_{1)}\left(r, \frac{1}{f-a}\right) = \frac{k-2}{k}T(r,f) + S(r,f), \tag{5.5}$$

and one of the following two cases will hold:

(i) $f = e^{k\gamma} + e^{(k-1)\gamma} + \cdots + 1$, $g = e^{-k\gamma} + e^{-(k-1)\gamma} + \cdots + 1$, with $(a-1)^k/a^{k+1} = k^k/(k+1)^{k+1}$ *and* $a \neq k+1$;

(ii) $f = -e^{k\gamma} - e^{(k-1)\gamma} - \cdots - e^{\gamma}$, $g = -e^{-k\gamma} - e^{-(k-1)\gamma} - \cdots - e^{-\gamma}$, *with* $(-a)^k/(1-a)^{k+1} = k^k/(k+1)^{k+1}$ *and* $a \neq -k$;

where γ is a nonconstant entire function, k (≥ 2) is a positive integer.

From Theorem 4.1 we immediately obtain the following result.

THEOREM 5.2.   *If, in addition to the assumptions of Theorem* 4.1,

$$\overline{N}(r, f) = S(r, f), \tag{5.6}$$

then the conclusions of Theorem 5.1 *hold.*

From Theorem 4.2 we immediately obtain the following result.

THEOREM 5.3.   *If, in addition to the assumptions of Theorem* 4.2,

$$\overline{N}(r, f) \leq v T(r, f) + S(r, f), \tag{5.7}$$

where $v < 1/2$, then

$$N_{1)}\left(r, \frac{1}{f-a}\right) = 0, \quad \overline{N}(r, f) = 0, \tag{5.8}$$

and one of the following two cases will hold:

(i) $f = e^{2\gamma} + e^{\gamma} + 1$, $g = e^{-2\gamma} + e^{-\gamma} + 1$, *with* $a = 3/4$;

(ii) $f = -e^{2\gamma} - e^{\gamma}$, $g = -e^{-2\gamma} - e^{-\gamma}$, *with* $a = 1/4$;

where γ is a nonconstant entire function.

EXAMPLE 5.1.   Let $f(z) = (e^{3z} - 1)/(e^{2z} - 1)$, $g(z) = (e^{-3z} - 1)/(e^{-2z} - 1)$ and $a = -3$. Then it is easily verified that f and g share $0, 1$ and ∞ CM, $N_{1)}(r, 1/(f - a)) = 0$ and

$$\overline{N}(r, f) = \frac{1}{2} T(r, f) + S(r, f).$$

Moreover, f and g do not assume one of the forms in Theorem 5.3.   This illustrates that the assumption $v < 1/2$ in Theorem 5.3 is best possible.

THEOREM 5.4.   *Let f and g be two distinct nonconstant meromorphic functions sharing $0, 1$ and ∞ CM. If there exists a finite complex number a $(\neq 0, 1)$ such that*

$$N_{1)}\left(r, \frac{1}{f-a}\right) \leq u T(r, f) + S(r, f), \tag{5.9}$$

$$\overline{N}(r, f) \leq v T(r, f) + S(r, f), \tag{5.10}$$

and

$$N_{1)}\left(r, \frac{1}{g-a}\right) \neq T(r, g) + S(r, g), \tag{5.11}$$

where $u < 1/3$ and $v < 1/2$, then

$$N_{1)}\left(r, \frac{1}{f-a}\right) = 0, \tag{5.12}$$

and one of the following three cases will hold:

(i) $\left(f - \frac{1}{2}\right)\left(g - \frac{1}{2}\right) \equiv \frac{1}{4}$, *with* $a = \frac{1}{2}$;

(ii) $f = e^{2\gamma} + e^{\gamma} + 1$, $g = e^{-2\gamma} + e^{-\gamma} + 1$, *with* $a = \frac{3}{4}$;

(iii) $f = -e^{2\gamma} - e^{\gamma}$, $g = -e^{-2\gamma} - e^{-\gamma}$, *with* $a = \frac{1}{4}$;

where γ *is a nonconstant entire function.*

PROOF.   We discuss the following two cases.

CASE 1.   Suppose that a is a Picard value of f.   By Theorem A, we know that f and g assume one of the three relations in Theorem A.   We discuss the following three subcases.

SUBCASE 1.1.   Suppose that f and g assume the relation (i) in Theorem A.   From this we obtain,

$$(f - a)(g + a - 1) \equiv a(1 - a), \tag{5.13}$$

and $1 - a$ and ∞ are Picard values of g.   If $a \neq 1 - a$, by Lemma 2.2 we have

$$N_{1)}\left(r, \frac{1}{g-a}\right) = T(r, g) + S(r, g),$$

which contradicts (5.11).   Thus $a = 1 - a$, and hence $a = 1/2$.   From this we obtain the form (i) in Theorem 5.4.

SUBCASE 1.2.   Suppose that f and g assume the relations (ii) in Theorem A.   From this we obtain, 0 and a are Picard values of f.   By Lemma 2.2 we have

$$N_{1)}(r, f) = T(r, f) + S(r, f),$$

which contradicts (5.10).

SUBCASE 1.3.   Suppose that f and g assume the relations (iii) in Theorem A.   From this we obtain, 1 and a are Picard values of f.   By Lemma 2.2 we have

$$N_{1)}(r, f) = T(r, f) + S(r, f),$$

which contradicts (5.10).

CASE 2.   Suppose that a is not a Picard value of f.   Using Theorem 5.3, we obtain the forms (ii) and (iii) in Theorem 5.4.   □

REMARK 5.1.   It is clear that the conclusions of Theorem 5.1, Theorem 5.2, Theorem 5.3 and Theorem 5.4 hold when f and g be two distinct nonconstant entire functions.

EXAMPLE 5.2.   Let $f(z) = 2(e^z + 1)$, $g(z) = -(e^{-z} + 1)$ and $a = 2$.   Then it is

easily verified that f and g share $0, 1$ and ∞ CM, $N_{1)}(r, 1/(f-a)) = 0$, $\overline{N}(r, f) = 0$, and

$$N_{1)}\left(r, \frac{1}{g-a}\right) = T(r, g) + S(r, g).$$

Moreover, f and g do not assume one of the forms in Theorem 5.4. This illustrates that the assumption (5.11) in Theorem 5.4 is best possible.

6. An application of the results in this paper.

Let h be a nonconstant meromorphic function, and let S is a subset of distinct elements in extended complex plane. Define

$$E_h(S) = \bigcup_{a \in S} \{z \mid h(z) - a = 0\},$$

where each zero of $h(z) - a = 0$ with multiplicity m is repeated m times in $E_h(S)$ (see [17]).

In 1982, F. Gross and C. Yang [18] asked whether there exist two sets $S_1 = \{a_1, a_2\}$ and $S_2 = \{b_1, b_2\}$ such that for any two nonconstant entire functions f and g the conditions $E_f(S_j) = E_g(S_j)$ $(j = 1, 2)$ imply $f \equiv g$ or not. F. Gross and C. Yang (see [18]) studied the question for the case $a_1 + a_2 = b_1 + b_2$. In 1990, H. Yi (see [19]) proved the following Theorem which is an extension and correction of the result of Gross and Yang.

THEOREM E (see [19]). *Let $S_1 = \{a_1, a_2\}$ and $S_2 = \{b_1, b_2\}$ be two pairs of distinct elements with $a_1 + a_2 = b_1 + b_2 = c$ but $a_1 a_2 \neq b_1 b_2$. Suppose that there are two nonconstant entire functions f and g of finite order such that $E_f(S_j) = E_g(S_j)$ for $j = 1, 2$. Then f and g must satisfy exactly one of the following relations:*
   (i) $f \equiv g$,
   (ii) $f + g \equiv a_1 + a_2$,
   (iii) $(f - c/2)(g - c/2) \equiv \pm((a_1 - a_2)/2)^2$, *where* $c = a_1 + a_2$. *This occurs only for* $(a_1 - a_2)^2 + (b_1 - b_2)^2 = 0$.
   (iv) $(f - a_j)(g - a_k) \equiv (-1)^{j+k}(a_1 - a_2)^2$ *for* $j, k = 1, 2$. *This occurs only for* $3(a_1 - a_2)^2 + (b_1 - b_2)^2 = 0$.
   (v) $(f - b_j)(g - b_k) \equiv (-1)^{j+k}(b_1 - b_2)^2$ *for* $j, k = 1, 2$. *This occurs only for* $(a_1 - a_2)^2 + 3(b_1 - b_2)^2 = 0$.

In 1998, Y. H. Li and C. T. Zhou [20] and independently P. Li [11] proved the following theorem, which is an improvement and extension of Theorem E.

THEOREM F. *Let $S_1 = \{a_1, a_2\}$ and $S_2 = \{b_1, b_2\}$ be two pairs of distinct elements with $a_1 + a_2 = b_1 + b_2$ but $a_1 a_2 \neq b_1 b_2$, and let $S_3 = \{\infty\}$. Suppose that f and g are two nonconstant meromorphic functions satisfying $E_f(S_j) = E_g(S_j)$ for $j = 1, 2, 3$. Then the conclusions of Theorem E hold.*

The proofs of Theorem F are long in [11] and [20]. Now we give a simple proof of Theorem F.

Let

$$F = \frac{(f - c/2)^2 - ((a_1 - a_2)/2)^2}{((b_1 - b_2)/2)^2 - ((a_1 - a_2)/2)^2}, \quad G = \frac{(g - c/2)^2 - ((a_1 - a_2)/2)^2}{((b_1 - b_2)/2)^2 - ((a_1 - a_2)/2)^2}, \quad (6.1)$$

where $c = a_1 + a_2 = b_1 + b_2$. If $F \equiv G$, from (6.1) we have

$$f \equiv g \quad \text{or} \quad f + g \equiv a_1 + a_2, \tag{6.2}$$

which assume the forms (i) and (ii) in Theorem F. Next, suppose that $F \not\equiv G$. From $E_f(S_j) = E_g(S_j)$ $(j = 1, 2, 3)$ we know that F and G share $0, 1$ and ∞ CM. From (6.1), we have $N_{1)}(r, F) = 0$. Again by Lemma 2.13 we obtain

$$N(r, F) = S(r, F). \tag{6.3}$$

Set

$$a = -\frac{((a_1 - a_2)/2)^2}{((b_1 - b_2)/2)^2 - ((a_1 - a_2)/2)^2}. \tag{6.4}$$

From (6.1) we have

$$F - a = \frac{(f - c/2)^2}{((b_1 - b_2)/2)^2 - ((a_1 - a_2)/2)^2}, \quad G - a = \frac{(g - c/2)^2}{((b_1 - b_2)/2)^2 - ((a_1 - a_2)/2)^2}. \tag{6.5}$$

From (6.5) we obtain

$$N_{1)}\left(r, \frac{1}{F - a}\right) = 0, \quad N_{1)}\left(r, \frac{1}{G - a}\right) = 0. \tag{6.6}$$

Noting (6.3) and (6.6), by Theorem 5.4 we know that one of the three cases in Theorem 5.4 holds. From this we obtain the form (iii), (iv) and (v) in Theorem F.

This completes the proof of Theorem F.

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