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L^p-L^q estimates for damped wave equations and their applications to semi-linear problem

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Abstract. In this paper we study the Cauchy problem to the linear damped wave equation $u_{tt} - \Delta u + 2au_t = 0$ in $(0, \infty) \times \boldsymbol{R}^n$ $(n \geq 2)$. It has been asserted that the above equation has the diffusive structure as $t \to \infty$. We give the precise interpolation of the diffusive structure, which is shown by L^p-L^q estimates. We apply the above L^p-L^q estimates to the Cauchy problem for the semilinear damped wave equation $u_{tt} - \Delta u + 2au_t = |u|^\sigma u$ in $(0, \infty) \times \boldsymbol{R}^n$ $(2 \leq n \leq 5)$. If the power σ is larger than the critical exponent $2/n$ (Fujita critical exponent) and it satisfies $\sigma \leq 2/(n-2)$ when $n \geq 3$, then the time global existence of small solution is proved, and the decay estimates of several norms of the solution are derived.

1. Introduction.

Consider the Cauchy problem for the damped wave equation

$$\partial_t^2 u - \Delta u + 2a\partial_t u = 0, \quad u(0,x) = \varphi_0(x), \quad \partial_t u(0,x) = \varphi_1(x) \tag{1.1}$$

for $(t,x) \in (0,\infty) \times \boldsymbol{R}^n$, where a is a positive constant and Δ is the Laplace operator in $\boldsymbol{R}^n$. Here and after we denote $\partial_t = (\partial/\partial t)$, $\partial_x^\alpha = (\partial/\partial x_1)^{\alpha_1} \cdot \dots \cdot (\partial/\partial x_n)^{\alpha_n}$ for a multi-index of non-negative integers $\alpha = (\alpha_1, \dots, \alpha_n)$ and $|\alpha| = \alpha_1 + \cdots + \alpha_n$.

It has been indicated by several authors (Li [**15**], Bellout and Friedman [**1**]) that the damped wave equation has the diffusive structures as $t \to \infty$.

Recently, Nishihara [**24**] has shown the L^p-L^q estimates of the difference between the solution of (1.1) and the solution of the Cauchy problem to the corresponding heat equation

$$\partial_t \phi - \frac{1}{2a}\Delta\phi = 0, \quad \phi(0,x) = \frac{1}{2a}(2a\varphi_0(x) + \varphi_1(x)), \quad (t,x) \in (0,\infty) \times \boldsymbol{R}^n \tag{1.2}$$

for $1 \leq q \leq p \leq \infty$, when $n = 3$. In the case where $n = 1$, Marcati and Nishihara [**17**] have obtained the same kind of estimates. The problem (1.1) with $n = 1$ is related to the asymptotic behavior of solutions to the system of the compressible flow through porous media (see Hsiao and Liu [**9**] and Nishihara [**22**], [**23**]).

To obtain the results, they ([**17**] and [**24**]) have used the explicit formula of solutions for the damped wave equation (1.1). Because the explicit formula is rather complicated, it seems that their method does not work well when $n \neq 1, 3$. In this paper we apply Fourier analysis to avoid the above difficulties.

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The first aim in this paper is to give the precise interpretation of the diffusive structure of the problem (1.1), which is shown by the L^p-L^q estimates. We estimate separately the low frequency part and high frequency part of the solution to (1.1). To obtain the results, we use the method developed by Marshall, Strauss and Wainger [**18**], Levandosky [**14**] and Narazaki [**21**]. The second aim is to apply Theorems 1.1 and 1.2 to Cauchy problem for the damped wave equation with power nonlinear term.

We use standard function spaces $W^{m,p}$ $(L^p = W^{0,p})$ and H_p^s equipped with the norms

$$\|f\|_{W^{m,p}} = \sum_{k=0}^{m} \|D^k f\|_p, \quad \|f\|_{H_p^s} \equiv \|f\|_{s,p} \equiv \|\mathscr{F}^{-1}((1+|\xi|^2)^{s/2}\hat{f})\|_p,$$

respectively, where $\|f\|_p$ denotes the usual L^p-norm, $\|D^k f\|_p = \sum_{|\alpha|=k} \|\partial_x^\alpha f\|_p$, $\mathscr{F}f = \hat{f}$ denotes the Fourier transformation of f with respect to x:

$$\mathscr{F}f(\xi) \equiv \hat{f}(\xi) \equiv (2\pi)^{-n/2} \int e^{-ix\cdot\xi} f(x)\, dx$$

and $\mathscr{F}^{-1}$ denotes the inverse Fourier transformation:

$$\mathscr{F}^{-1}\hat{f}(x) = (2\pi)^{-n/2} \int e^{ix\cdot\xi} \hat{f}(\xi)\, d\xi.$$

A function $f \in L^1_{loc}(\boldsymbol{R}^n)$ will be said to belong to BMO if and only if

$$\|f\|_{\mathrm{BMO}} \equiv \sup_Q \int_Q |f(x) - f_Q|\, dx < \infty,$$

where supremum is taken over all balls in $\boldsymbol{R}^n$, $|Q|$ denotes the volume of the ball Q, and

$$f_Q \equiv \frac{1}{|Q|} \int_Q f(x)\, dx.$$

$\|f\|_{\mathrm{BMO}}$ is called BMO-norm of f, and it becomes a norm after dividing out the constant functions (see e.g. [**2**], [**3**]).

Our first aim is represented in the following theorems.

Theorem 1.1. *Let $n \geq 2$. Let $j \geq 0$ and $k \geq 0$ be integers and $1 \leq q \leq p \leq \infty$. Let $\chi(\xi)$ be a compact supported radial function of class C^∞, and $\chi = 1$ in a neighborhood of* 0. *Assume that $\varphi_i \in L^q$ for $i = 0, 1$. Then the solution u to* (1.1) *in the sense of distributions satisfies the following L^p-L^q estimate for any $\varepsilon > 0$,*

$$\begin{aligned} &\|\partial_t^j (-\Delta)^k \mathscr{F}^{-1}\{\chi(\xi)(\hat{u}(t,\xi) - \hat{\phi}(t,\xi))\}\|_p \\ &\quad \leq C(\varepsilon, p, q, j, k)(1+t)^{-(n/2)\cdot(1/q-1/p)-j-k-1+\varepsilon}(\|\varphi_0\|_q + \|\varphi_1\|_q) \end{aligned} \tag{1.3}$$

for some positive constant $C(\varepsilon, p, q, j, k)$, where ϕ is the solution to the Cauchy problem for the corresponding parabolic equation (1.2). *Moreover, in the case where $1 < q < p < \infty$, $p = q = 2$ or $p = \infty$, $q = 1$, we may take $\varepsilon = 0$ in the estimate* (1.3).

THEOREM 1.2. *Let* $n \geq 2$. *Let* $1 < q \leq p < \infty$. *Let* $\chi(\xi)$ *be a compact supported radial function of class* C^∞, *and* $\chi = 1$ *in a neighborhood of* 0. *Assume that* $\varphi_i \in L^q$ *for* $i = 0, 1$. *Then the solution* u *to* (1.1) *in the sense of distributions satisfies the following* L^p-L^q *estimate*

$$\begin{aligned} &\|\mathscr{F}^{-1}\{(1-\chi(\cdot))(\hat{u}(t,\cdot) - e^{-at}(M_0(t,\cdot)\hat{\varphi}_0(\cdot) + M_1(t,\cdot)\hat{\varphi}_1(\cdot)))\}\|_p \\ &\quad \leq C(p,q)e^{-\delta t}(\|\varphi_0\|_q + \|\varphi_1\|_q) \end{aligned} \tag{1.4}$$

for some constants $\delta > 0$ *and* $C(p,q) > 0$, *where*

$$\begin{aligned} M_1(t,\xi) = \frac{1}{\sqrt{|\xi|^2 - a^2}} \Bigg(&\sin t|\xi| \sum_{0 \leq k < (n-1)/4} \frac{(-1)^k}{(2k)!} t^{2k} \Theta(\xi)^{2k} \\ &- \cos t|\xi| \sum_{0 \leq k < (n-3)/4} \frac{(-1)^k}{(2k+1)!} t^{2k+1} \Theta(\xi)^{2k+1} \Bigg), \end{aligned}$$

$$\begin{aligned} M_0(t,\xi) = {} &\cos t|\xi| \sum_{0 \leq k < (n+1)/4} \frac{(-1)^k}{(2k)!} t^{2k} \Theta(\xi)^{2k} \\ &+ \sin t|\xi| \sum_{0 \leq k < (n-1)/4} \frac{(-1)^k}{(2k+1)!} t^{2k+1} \Theta(\xi)^{2k+1} + aM_1(t,\xi), \end{aligned}$$

and $\Theta(\xi) \equiv |\xi| - \sqrt{|\xi|^2 - a^2}$.

It is well-known that the solution ϕ to (1.2) satisfies

$$\|\partial_t^j D^k \phi(t,\cdot)\|_p \leq Ct^{-(n/2)\cdot(1/q-1/p)-j-k/2} \|2a\varphi_0 + \varphi_1\|_q \tag{1.5}$$

for $1 \leq q \leq p \leq \infty$ and non-negative integers j and k (see e.g. Ponce [**25**]). The estimate (1.5) implies that

$$\|\partial_t^j D^k \phi(t,\cdot)\|_p \leq C(b)(1+t)^{-(n/2)\cdot(1/q-1/p)-j-k/2} \|2a\varphi_0 + \varphi_1\|_q \tag{1.6}$$

for $1 \leq q \leq p \leq \infty$ and non-negative integers j and k, provided that $\mathrm{supp}(2a\hat{\varphi}_0 + \hat{\varphi}_1) \subset \{\xi : |\xi| \leq b\}$ for some constant $b > 0$. Here and after c, c_k, C, C_k etc. denote generic constants.

Theorems 1.1–1.2 imply the properties of the damped wave equation, which are summarized as follows.

(i) Let u be a solution of (1.1), let ϕ be a solution of (1.2) and let v be the solution of the following wave equation

$$\partial_t^2 v - \Delta v = 0, \quad v(0,x) = \varphi_0(x), \quad \partial_t v(0,x) = \varphi_1(x), \quad (t,x) \in (0,\infty) \times \boldsymbol{R}^n.$$

Then,

$$\hat{u}(t,\xi) \doteqdot \begin{cases} \hat{\phi}(t,\xi) & \text{for small } |\xi|, \\ e^{-at}\hat{v}(t,\xi) & \text{for large } |\xi|. \end{cases}$$

(ii) If the initial data are sufficiently smooth, then the damped wave equation may have the same properties as those to the heat equation.

(iii) If initial data have singularity, it propagates along the light cone, which is the wave property, though its strength decays exponentially.

When $n = 3$, Theorem 1.2, Proposition 4.4 (below) and the theory of Fourier multiplier (see [**2**], e.g.) imply that the estimate

$$\|\mathscr{F}^{-1}\{(1-\chi)(\hat{u}(t,\cdot) - e^{-at}\hat{w}(t,\cdot))\}\|_p \le Ce^{-\delta t}(\|\varphi_0\|_q + \|\varphi_1\|_q) \tag{1.7}$$

holds for $t \ge 0$, under the assumptions in Theorem 1.2, where

$$\hat{w}(t,\xi) = \left(\left(a + \frac{a^2}{2}t\right)\frac{\sin t|\xi|}{|\xi|} + \cos t|\xi|\right)\hat{\varphi}_0(\xi) + \frac{\sin t|\xi|}{|\xi|}\hat{\varphi}_1(\xi),$$

for $(t,\xi) \in (0,\infty) \times \boldsymbol{R}^3$. Theorem 1.1 and the estimates (1.5) and (1.7) show

$$\|u(t,\cdot) - \phi(t,\cdot) - e^{-at}w(t,\cdot)\|_p \le C(\varepsilon,p,q)t^{-3/2(1/q-1/p)-1+\varepsilon}(\|\varphi_0\|_q + \|\varphi_1\|_q) \tag{1.8}$$

for $t > 0$ with $1 < q \le p < \infty$ and $n = 3$, under the notations in Theorems 1.1–1.2. Moreover, in the estimate (1.8), we may set $\varepsilon = 0$ when $1 < q < p < \infty$ or $p = q = 2$. Because Hörmander's multiplier theorem on L^p holds only when $1 < p < \infty$ (see [**8**] and [**2**]), it seems difficult to prove the estimate (1.8) when $q = 1$ or $p = \infty$.

Nishihara [**24**] has shown the estimate (1.8) for $t \ge 1$ with $1 \le q \le p \le \infty$ and $\varepsilon = 0$. But Theorems 1.1 and 1.2 hold for any $n \ge 2$, and it seems that these theorems represent the diffusive structure of damped wave equation (1.1) precisely.

Recently Ikehata and Nishihara [**11**] have studied the problem (1.1) in abstract framework, and they have shown the estimate

$$\|u(t,\cdot) - \phi(t,\cdot)\|_2 \le C(1+t)^{-1}(\log(2+t))^{(1+\varepsilon)/2}(\|(1-\Delta)\varphi_0\|_2 + \|\sqrt{1-\Delta}\varphi_1\|_2)$$

for any $\varepsilon > 0$. But the estimates in Theorems 1.1 and 1.2 are sharper than theirs.

Remark 1.1. Let u be a solution of (1.1). Then, $\hat{u}(t,\xi)$ satisfies

$$\hat{u}(t,\xi) = e^{-at}\left(\cos t\sqrt{|\xi|^2 - a^2}\hat{\varphi}_0(\xi) + \frac{\sin t\sqrt{|\xi|^2 - a^2}}{\sqrt{|\xi|^2 - a^2}}(a\hat{\varphi}_0 + \hat{\varphi}_1)\right). \tag{1.9}$$

Assume that a function $\psi(\cdot) \in C_0^\infty(\boldsymbol{R}^n)$ satisfies $\operatorname{supp}\psi \subset \{\xi; |\xi| \ge b\}$ for some constant $b > 0$. Then the inequality

$$\|\partial_t^j(-\Delta)^k\mathscr{F}^{-1}(\psi(\cdot)\hat{u}(t,\cdot))\|_p \le Ce^{-\delta t}(\|\varphi_0\|_q + \|\varphi_1\|_q)$$

holds for $1 \le q \le p \le \infty$ and for integers $j \ge 0$ and $k \ge 0$, where $\delta > 0$ depends only on ψ.

Our second aim is to apply Theorems 1.1 and 1.2 to the Cauchy problem for the damped semilinear wave equation

$$\partial_t^2u - \Delta u + 2a\partial_t u = f(u), \quad u(0,x) = \varphi_0(x), \quad \partial_t u(0,x) = \varphi_1(x), \tag{1.10}$$

for $(t,x) \in (0,\infty) \times \boldsymbol{R}^n$, when $2 \le n \le 5$. The typical examples of nonlinear terms $f(u)$ are $|u|^\sigma u$ and $|u|^{1+\sigma}$. We study the time global existence of small solution to (1.10). Our interest is focused on the critical exponent $\sigma_c(n) = 2/n$.

For the Cauchy problem of the corresponding heat equation

$$\partial_t \phi - \frac{1}{2a} \Delta \phi = |\phi|^\sigma \phi, \quad \phi(0,x) = \frac{2a\varphi_0(x) + \varphi_1(x)}{2a}, \quad (t,x) \in (0,\infty) \times \boldsymbol{R}^n, \tag{1.11}$$

the critical exponent is $\sigma = 2/n$. In fact, if $0 < \sigma \le 2/n$, then the solution blows up in a finite time (Fujita [**4**], Hayakawa [**7**]) for certain small initial data, and if $\sigma > 2/n$, then time global solution exists when the initial data is small.

Matsumura [**19**] has shown the time global existence and the time decay estimate of the solution of the problem (1.10), provided that initial data is compact supported, sufficiently smooth and small, and the nonlinear term $f(u)$ is smooth and it satisfies

$$\left| \left(\frac{d}{du}\right)^k f(u) \right| \le C|u|^{\max(p-k,0)} \quad (0 \le k \le N),$$

where $p > 1 + 2/n$, $p \ge 2$ and N is a large integer. Many authors have studied the problem (1.10) ([**5**], [**6**], [**12**], [**13**], [**16**], [**20**]). Our methods are related ones in [**13**]. Gallay and Raugel [**5**], [**6**] have studied the large time behavior of solutions to the nonlinear damped wave equation introducing scaling variables.

Recently, Todorova and Yordanov [**30**] have studied the Cauchy problem for following damped wave equation

$$\partial_t^2 u - \Delta u + 2a\partial_t u = |u|^{1+\sigma}, \quad u(0,x) = \varphi_0(x), \quad \partial_t u(0,x) = \varphi_1(x), \tag{1.12}$$

for $(t,x) \in (0,\infty) \times \boldsymbol{R}^n$, and they have shown that the critical exponent is $\sigma_c(n) = 2/n$ for compactly supported initial data. More precisely, if $\sigma_c(n) < \sigma \le 2/(n-2)$, they have proved that the problem (1.12) admits a unique time-global solution when the compactly supported initial data (φ_0, φ_1) are small in $H_2^1 \times L^2$. If $0 < \sigma < \sigma_c(n)$, they have shown that the solutions of (1.12) do not exist globally for certain initial data, however small initial data are. Zhang [**31**] have studied the critical case $\sigma = \sigma_c(n)$ and he have shown the non-existence of time global solution to (1.12) for certain small initial data.

Ikehata, Miyaoka and Nakatake [**10**] have shown the global existence of the weak solution to (1.10) and its decay order when $2 < 1 + \sigma < n/(n-2)$ $(n = 1,2,3)$. Also they conjecture the global existence of the solution when $2 \ge 1 + \sigma > 1 + 2/n$ $(n = 3)$. In fact, our second goal is to give a positive answer to their conjecture in the case where $2 \le n \le 5$ without any assumptions on the support of initial data, under the following hypothesis.

Hypothesis H. Nonlinear term $f(u)$ is a function of class C^1 and it satisfies $|f(u)| \le A|u|^{1+\sigma}$, $|f'(u)| \le A|u|^\sigma$ for $u, v \in \boldsymbol{R}$, where A and σ are positive constants. Moreover it satisfies $|f'(u) - f'(v)| \le A|u - v|^\sigma$, if $\sigma < 1$.

Our second aim is represented in the following theorems.

Theorem 1.3. *Assume that Hypothesis H holds. Let* $4 \le n \le 5$, $2/n < \sigma \le 2/(n-2)$, $\sigma < 1$ *and*

$$(\varphi_0, \varphi_1) \in Z_1 \equiv (H_2^2 \cap H_{1+1/\sigma}^1 \cap H_{1+\sigma}^1 \cap L^1) \times (H_2^1 \cap L^{1+1/\sigma} \cap L^{1+\sigma} \cap L^1).$$

If $\|\varphi_0, \varphi_1\|_{Z_1}$ *is sufficiently small, then the solution* u *to* (1.10) *uniquely exists in*

$$C([0,\infty); H_2^2 \cap L^{1+1/\sigma} \cap L^{1+\sigma}) \cap C^1([0,\infty); H_2^1) \cap C^2([0,\infty); L^2),$$

and it satisfies the following estimates for $t \geq 0$:

$$\|u(t,\cdot)\|_p \leq C(1+t)^{-(n/2)\cdot(1-1/p)} \|\varphi_0, \varphi_1\|_{Z_1} \quad \textit{for } 1+\sigma \leq p \leq 1+1/\sigma,$$

$$\|\partial_t^j D^k u(t,\cdot)\|_2 \leq C(1+t)^{-n/4-j-k/2} \|\varphi_0, \varphi_1\|_{Z_1} \quad \textit{for } j+k \leq 2,\ j \leq 1, \quad \textit{and}$$

$$\|\partial_t^2 u(t,\cdot)\|_2 \leq C(1+t)^{-n/4-n\sigma/2} \|\varphi_0, \varphi_1\|_{Z_1}.$$

Theorem 1.4. *Assume that Hypothesis H holds. Let* $n = 3$, $2/3 < \sigma < 1$ *and*

$$(\varphi_0, \varphi_1) \in Z_2 \equiv (H_{1+1/\sigma}^1 \cap H_{1+\sigma}^1 \cap L^1) \times (L^{1+1/\sigma} \cap L^{1+\sigma} \cap L^1).$$

Then, if $\|\varphi_0, \varphi_1\|_{Z_2}$ *is sufficiently small, the solution* u *to* (1.10) *uniquely exists in*

$$C([0,\infty); H_2^1 \cap L^{1+1/\sigma} \cap L^{1+\sigma}) \cap C^1([0,\infty); L^2),$$

and it satisfies the following estimates:

$$\|u(t)\|_p \leq C(1+t)^{-(3/2)\cdot(1-1/p)} \|\varphi_0, \varphi_1\|_{Z_2} \quad \textit{for } 1+\sigma \leq p \leq 1+1/\sigma, \quad \textit{and}$$

$$\|\partial_t^j D^k u(t,\cdot)\|_2 \leq C(1+t)^{-3/4-j-k/2} \|\varphi_0, \varphi_1\|_{Z_2} \quad \textit{for } j+k \leq 1.$$

Theorem 1.5. *Assume that Hypothesis H holds. Let* $2 \leq n \leq 4$, $2/n < \sigma$, $1 \leq \sigma$ *and* $\sigma \leq 2/(n-2)$ *when* $n \geq 3$. *Let* $(\varphi_0, \varphi_1) \in Z_3 \equiv (H_2^1 \cap L^1) \times (L^2 \cap L^1)$. *Then, if* $\|\varphi_0, \varphi_1\|_{Z_3}$ *is sufficiently small, the solution* u *to* (1.10) *uniquely exists in* $C([0,\infty); H_2^1) \cap C^1([0,\infty); L^2)$, *and it satisfies the following estimates*:

$$\|\partial_t^j D^k u(t)\|_2 \leq C(1+t)^{-n/4-j-k/2} \|\varphi_0, \varphi_1\|_{Z_3} \quad \textit{for } j+k \leq 1. \tag{1.13}$$

Remark 1.2. Nishihara [**24**] has shown the global existence of the small data solution to (1.10) for initial data $(\varphi_0, \varphi_1) \in (W^{1,1} \cap W^{1,\infty}) \times (L^1 \cap L^\infty)$ when $n = 3$, and he has shown the estimates in Theorem 1.4 with $1 \leq p \leq \infty$.

Remark 1.3. The decay estimates in Theorems 1.3–1.5 imply the following energy estimate

$$|E(t)| \equiv \left| \frac{1}{2}(\|\partial_t u(t,\cdot)\|_2^2 + \|Du(t,\cdot)\|_2^2) - \int_{\boldsymbol{R}^n} F(u(t,\cdot))\,dx \right| \leq C(1+t)^{-n/2-1} \|\varphi_0, \varphi_1\|_{Z_i} \tag{1.14}$$

for $i = 1, 2, 3$, where $F'(s) = f(s)$ and $F(0) = 0$. Kawashima, Nakao and Ono [**13**] have treated the Cauchy problem (1.10) in $\boldsymbol{R}^n$ $(n \geq 1)$ with $|u|^\sigma u$ replaced by $-|u|^\sigma u$ and they have obtained the decay estimate (1.14) in the framework of higher power σ. Ikehata, Miyaoka and Nakatake [**10**] have treated the problem (1.10) with $f(u) = |u|^\sigma u$ under the assumptions in Theorem 1.5. They have obtained the decay estimate (1.14), and they have shown the following decay estimate $\|u(t)\|_2 \leq C(1+t)^{-n/4} \|\varphi_0, \varphi_1\|_{Z_1}$. But they have not obtained a sharp decay estimate of $\|\partial_t u(t)\|_2$. See (1.13).

2. Preliminaries.

In this section we state the preliminary results necessary for the proofs. $J_\mu(s)$ is the Bessel function of order μ. We shall denote $\tilde{J}_\mu(s) = J_\mu(s)/s^\mu$. Here and after we denote $g(s) = O(|s|^\sigma)$ when $|g(s)| \le C|s|^\sigma$ for a constant σ.

Lemma 2.1 (see e.g. [**14**]). *Assume that μ is not a negative integer. Then, the following equalities hold.*

(1) $s\tilde{J}_\mu'(s) = \tilde{J}_{\mu-1}(s) - 2\mu\tilde{J}_\mu(s)$.

(2) $\tilde{J}_\mu'(s) = -s\tilde{J}_{\mu+1}(s)$.

(3) $\tilde{J}_{-1/2}(s) = \sqrt{\dfrac{\pi}{2}}\cos s$.

(4) *If* $\operatorname{Re}\mu$ *is fixed, then* $|\tilde{J}_\mu(s)| \le Ce^{\pi|\operatorname{Im}\mu|}$, $(|s| \le 1)$,

$$J_\mu(s) = Cs^{-1/2}\cos\left(s - \frac{\mu}{2}\pi - \frac{\pi}{4}\right) + O(e^{2\pi|\operatorname{Im}\mu|}|s|^{-3/2}), \quad (|s| \ge 1).$$

(5) $r^2\rho\tilde{J}_{\mu+1}(r\rho) = -\dfrac{\partial}{\partial\rho}\tilde{J}_\mu(r\rho)$.

It is well-known that

Lemma 2.2 (see e.g. [**27**]). *Assume that $\hat{f}(\xi) \in L^p$ $(1 \le p \le 2)$ is a radial function. Then the equality $f(x) = \int_0^\infty g(\rho)\rho^{n-1}\tilde{J}_{n/2-1}(|x|\rho)\,d\rho$ holds, where $g(|\xi|) = \hat{f}(\xi)$.*

Lemma 2.3. *Let $1 \le q \le p \le \infty$ satisfy $1 - 1/r = 1/q - 1/p$, then the inequality $\|f * g\|_p \le C\|f\|_q\|g\|_r$ holds for any $f \in L^q$ and $g \in L^r$ (see e.g.* [**27**]).

Lemma 2.4 (Hardy-Littlewood-Sobolev). *Let $1 < q < p < \infty$ satisfy $1 - 1/r = 1/q - 1/p$. Assume that $|g(x)| \le A|x|^{-n/r}$, where A is a constant. Then the inequality $\|f * g\|_p \le C(p,q)A\|f\|_q$ holds for any $f \in L^p$ (see e.g.* [**26**]).

Lemma 2.5. *Assume that $p_0 \ne p_1$, $q_0 \ne q_1$ and that an operator T is bounded from L^{p_0} to L^{q_0} with norm M_0, and that the operator T is bounded from L^{p_1} to L^{q_1} with norm M_1. Then, the operator T is bounded from $L^{p(\theta)}$ to $L^{q(\theta)}$ with norm $M \le M_0^{1-\theta}M_1^\theta$, provided that $0 < \theta < 1$ and*

$$\frac{1}{p(\theta)} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad \frac{1}{q(\theta)} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$$

(see e.g. [**27**]).

Lemma 2.6. *Let $S = \{z = x + iy; 0 < x < 1, y \in \boldsymbol{R}\}$ be a strip and let T_z be an analytic family of linear operators satisfying*

$$\|T_{iy}h\|_{p_0} \le A_0N_0(y)\|h\|_{q_0}, \quad \|T_{1+iy}h\|_{p_1} \le A_1N_1(y)\|h\|_{q_1}, \quad N_0(0) = N_1(0) = 1$$

where $1 \le p_j, q_j \le \infty$ for $j = 0, 1$ and $\sup_{-\infty<y<\infty} e^{-b|y|}\log N_j(y) < \infty$ for some $b < \pi$. Then, if $0 < \theta < 1$, there is a constant $C(\theta, b)$ so that $\|T_\theta h\|_{p(\theta)} \le C(\theta, b)A_0^{1-\theta}A_1^\theta\|h\|_{q(\theta)}$ for

$$\frac{1}{p(\theta)} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad \frac{1}{q(\theta)} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

Furthermore we may replace $p_1 = \infty$ with BMO, provided that $p_0 \ne 1$ (see [**27**] *and* [**28**]).

Lemma 2.7 (Gagliardo-Nirenberg). *Let* $1 \le r < p \le \infty$, $1 \le q \le p$ *and* $m \ge 0$ *satisfy* $1/p = \theta(1/q - m/n) + (1-\theta)/r$, *then the inequality*

$$\|v\|_p \le C\|D^m v\|_q^{\theta}\|v\|_r^{(1-\theta)} \quad \textit{for } v \in H_q^m \cap L^r$$

holds with some $C > 0$, *provided that* $0 < \theta \le 1$ ($0 < \theta < 1$ *if* $1 < q < \infty$ *and* $m - n/q$ *is a non-negative integer*).

3. Proof of Theorem 1.1.

Choose and fix a radial function $0 \le \chi_1(\xi) \le 1$ of class C^∞ satisfying

$$\chi_1(\xi) = 1 \quad (|\xi| \le a/2), \quad \chi_1(\xi) = 0 \quad (|\xi| \ge 2a/3).$$

Occasionally, we write $\chi_1(\xi) = \chi_1(|\xi|)$. Remark 1.1 shows that Theorem 1.1 is a direct consequence of the next theorem.

Theorem 3.1. *Let* $v_1 \ge 0$ *and* $v_2 \ge 0$ *be integers and let* $1 \le q \le p \le \infty$. *Let* u *be the solution of* (1.1) *and let* ϕ *be the solution of* (1.2). *Assume that* $\varphi_0 \in L^q$, $\varphi_1 \in L^q$. *Then, for any* $0 < \varepsilon$, *there exits a constant* $C(\varepsilon, p, q, v_1, v_2)$ *satisfying*

$$\begin{aligned} &\|\partial_t^{v_1}(-\Delta)^{v_2}\mathscr{F}^{-1}\{\chi_1(\xi)(\hat{u}(t,\xi) - \hat{\phi}(t,\xi))\}\|_p \\ &\quad \le C(\varepsilon, p, q, v_1, v_2)(1+t)^{-(n/2)\cdot(1/q-1/p)-v_1-v_2-1+\varepsilon}(\|\varphi_0\|_q + \|\varphi_1\|_q). \end{aligned} \tag{3.1}$$

Moreover, when $1 < q < p < \infty$, $p = \infty$, $q = 1$ *or* $p = q = 2$, *we may take* $\varepsilon = 0$ *in the estimate* (3.1).

The solution u to (1.1) satisfies that

$$\begin{aligned} \chi_1(\xi)\hat{u}(t,\xi) = \chi_1(\xi)\Bigg\{ &\exp\left(-\frac{|\xi|^2 t}{2a}\right)\frac{2a\hat{\varphi}_0(\xi) + \hat{\varphi}_1(\xi)}{2a} + \frac{1}{2}g(t,|\xi|)\left(\hat{\varphi}_0(\xi) + \frac{a\hat{\varphi}_0(\xi) + \hat{\varphi}_1(\xi)}{\sqrt{a^2 - |\xi|^2}}\right) \\ &+ \exp\left(-\frac{|\xi|^2 t}{2a}\right)\frac{|\xi|^2}{\sqrt{a^2 - |\xi|^2}(a + \sqrt{a^2 - |\xi|^2})}\,\frac{a\hat{\varphi}_0(\xi) + \hat{\varphi}_1(\xi)}{2a} \\ &+ \frac{1}{2}\exp(-at - t\sqrt{a^2 - |\xi|^2})\left(\hat{\varphi}_0(\xi) - \frac{a\hat{\varphi}_0(\xi) + \hat{\varphi}_1(\xi)}{\sqrt{a^2 - |\xi|^2}}\right)\Bigg\} \\ \equiv \chi_1(\xi)\hat{\phi}(t,\xi) + \hat{V}_1(t,\xi) + \hat{V}_2(t,\xi) + \hat{V}_3(t,\xi), \end{aligned} \tag{3.2}$$

where

$$g(t,\rho) \equiv \exp(-at + t\sqrt{a^2 - \rho^2}) - \exp\left(-\frac{\rho^2 t}{2a}\right). \tag{3.3}$$

We begin with the estimates of $\|\partial_t^{v_1}(-\Delta)^{v_2}V_2(t,\cdot)\|_p$ and $\|\partial_t^{v_1}(-\Delta)^{v_2}V_3(t,\cdot)\|_p$.

LEMMA 3.1. *Let $\nu_1 \geq 0$ and $\nu_2 \geq 0$ be integers. For $1 \leq q \leq p \leq \infty$, the estimate*

$$\|\partial_t^{\nu_1}(-\Delta)^{\nu_2} V_2(t,\cdot)\|_p + \|\partial_t^{\nu_1}(-\Delta)^{\nu_2} V_3(t,\cdot)\|_p \leq C(1+t)^{-(n/2)\cdot(1/q-1/p)-\nu_1-\nu_2-1}(\|\varphi_0\|_q + \|\varphi_1\|_q).$$

holds.

PROOF. We begin with the estimate of $\|\partial_t^{\nu_1}(-\Delta)^{\nu_2} V_2(t,\cdot)\|_p$. We define the functions $\psi_2(x)$ and $\phi_2(t,x)$ by

$$\hat{\psi}_2(\xi) = \left(\frac{1}{2\pi}\right)^{n/2} \frac{\chi_1(\xi)}{\sqrt{a^2-|\xi|^2}(a+\sqrt{a^2-|\xi|^2})} \in \mathscr{S}(\boldsymbol{R}^n),$$

$$\hat{\phi}_2(t,\xi) = \frac{\chi_1(\xi)}{\sqrt{a^2-|\xi|^2}(a+\sqrt{a^2-|\xi|^2})} \exp\left(-\frac{|\xi|^2 t}{2a}\right)\left(\frac{a\hat{\varphi}_0(\xi)+\hat{\varphi}_1(\xi)}{2a}\right),$$

respectively. Since $\phi_2(t,x)$ is the smooth solution to the following problem

$$\partial_t\phi_2 - \frac{1}{2a}\Delta\phi_2 = 0, \quad \phi_2(0,\cdot) = \psi_2 * \left(\frac{a\varphi_0+\varphi_1}{2a}\right) \in W^{\infty,q},$$

(1.6) shows that

$$\begin{aligned}\|\partial_t^{\nu_1}(-\Delta)^{\nu_2} V_2(t,\cdot)\|_p &= \|\partial_t^{\nu_1}(-\Delta)^{\nu_2+1}\phi_2(t,\cdot)\|_p \\ &\leq C(1+t)^{-(n/2)\cdot(1/q-1/p)-\nu_1-\nu_2-1}\|\phi_2(0,\cdot)\|_q \\ &\leq C(1+t)^{-(n/2)\cdot(1/q-1/p)-\nu_1-\nu_2-1}(\|\varphi_0\|_q + \|\varphi_1\|_q).\end{aligned}$$

We have obtained the desired estimate of $\|V_2(t,\cdot)\|_p$.

Now we estimate $\|\partial_t^{\nu_1}(-\Delta)^{\nu_2} V_3(t,\cdot)\|_p$. Choose and fix a radial function χ_{11} of class C^∞ satisfying;

$$\chi_1(\xi)\chi_{11}(\xi) = \chi_1(\xi), \quad \operatorname{supp}\chi_{11} \subset \{\xi; |\xi| \leq 3a/4\}.$$

Then easy calculations show that

$$V_3(t,x) = e^{-at}\mathscr{F}^{-1}\left\{\chi_1(\xi)\exp(-t\sqrt{a^2-|\xi|^2})\frac{1}{2}\left(\hat{\varphi}_0(\xi) - \chi_{11}(\xi)\frac{a\hat{\varphi}_0(\xi)+\hat{\varphi}_1(\xi)}{\sqrt{a^2-|\xi|^2}}\right)\right\}.$$

Let $1-1/r = 1/q-1/p$. Easy calculations show that

$$\begin{aligned}&\|\mathscr{F}^{-1}\{\chi_1(\xi)|\xi|^{2\nu_2}\partial_t^k\exp(-t\sqrt{a^2-|\xi|^2})\}\|_{L^r} \\ &\quad\leq \|\mathscr{F}^{-1}\{\chi_1(\xi)|\xi|^{2\nu_2}\partial_t^k\exp(-t\sqrt{a^2-|\xi|^2})\}\|_{L^1\cap L^\infty} \\ &\quad\leq \sum_{j=1}^n \|x_j^{n+1}\mathscr{F}^{-1}\{\chi_1(\xi)|\xi|^{2\nu_2}\partial_t^k\exp(-t\sqrt{a^2-|\xi|^2})\}\|_{L^\infty} \\ &\qquad+ \|\mathscr{F}^{-1}\{\chi_1(\xi)|\xi|^{2\nu_2}\partial_t^k\exp(-t\sqrt{a^2-|\xi|^2})\}\|_{L^\infty} \leq C(1+t)^{n+1}\end{aligned}$$

for $0 \leq k \leq \nu_1$. Hence Lemma 2.3 shows that

$$\|\mathscr{F}^{-1}\{\chi_1(\xi)|\xi|^{2\nu_2}\partial_t^k \exp(-t\sqrt{a^2-|\xi|^2})\hat{h}\}\|_p \le C(1+t)^{n+1}\|h\|_q \tag{3.4}$$

for $1 \le q \le p \le \infty$. Similar calculations as (3.4) show that the operator B defined by

$$Bh = \mathscr{F}^{-1}\left(\frac{\chi_{11}(\xi)\hat{h}}{\sqrt{a^2-|\xi|^2}}\right)$$

is bounded from L^q to L^p for $1 \le q \le p \le \infty$. Therefore, we see that

$$\begin{aligned}
&\|\partial_t^{\nu_1}(-\Delta)^{\nu_2}V_3(t,\cdot)\|_p \\
&\quad \le Ce^{-at}\sum_{k=0}^{\nu_1}\|\mathscr{F}^{-1}\{\chi_1(\xi)|\xi|^{2\nu_2}\partial_t^k \exp(-t\sqrt{a^2-|\xi|^2})(\hat{\varphi}_0(\xi)-\mathscr{F}(B(a\varphi_0+\varphi_1)))\}\|_p \\
&\quad \le Ce^{-at}(1+t)^{n+1}(\|\varphi_0\|_q+\|\varphi_1\|_q) \le C(1+t)^{-(n/2)\cdot(1/q-1/p)-\nu_1-\nu_2-1}(\|\varphi_0\|_q+\|\varphi_1\|_q).
\end{aligned}$$

Thus we have obtained the desired estimate of $\|\partial_t^{\nu_1}(-\Delta)^{\nu_2}V_3(t,\cdot)\|_p$. □

For the estimate of $\|(-\Delta)^{\nu_2}V_1(t,\cdot)\|_p$ we introduce a function $I(t,x) \equiv \mathscr{F}^{-1}(\chi_1(\xi)g(t,|\xi|))$.

Then we see that

$$(-\Delta)^{\nu_2}V_1(t,x) = \frac{1}{2(2\pi)^{n/2}}((-\Delta)^{\nu_2}I(t,\cdot)) * (\varphi_0 + B(a\varphi_0+\varphi_1)). \tag{3.5}$$

Lemma 2.2 shows that

$$(-\Delta)^{\nu_2}I(t,x) = \int_0^\infty \chi_1(\rho)g(t,\rho)\rho^{n-1+2\nu_2}\tilde{J}_{(n/2)-1}(\rho|x|)\,d\rho. \tag{3.6}$$

The estimate of $\|(-\Delta)^{\nu_2}V_1\|_p$ is implied by the following proposition.

Proposition 3.1. *Let $\nu_2 \ge 0$ be an integer. Then, for any $t > 0$, the following estimates hold.*

(1) $\sup_x|(-\Delta)^{\nu_2}I(t,x)| \le C(1+t)^{-(n/2)-1-\nu_2}$.
(2) *When $n = 2m+1$,* $|(-\Delta)^{\nu_2}I(t,x)| \le C(1+|x|)^{-n-1}(1+t)^{-1/2-\nu_2}$.
(3) *When $n = 2m$,* $|(-\Delta)^{\nu_2}I(t,x)| \le C(1+|x|)^{-n-1/2}(1+t)^{-3/4-\nu_2}$.

Proof. Hereafter in the proof we may assume that $\rho < 2a/3$.

(1) The equality (3.3) shows that

$$|g(t,\rho)| \le C\rho^4 t \exp\left(-\frac{\rho^2 t}{2a}\right). \tag{3.7}$$

By Lemma 2.1(4), (3.6) and (3.7), we claim that

$$\begin{aligned}
|(-\Delta)^{\nu_2}I(t,x)| &\le C\int_0^{2a/3} t\rho^{n+3+2\nu_2}\exp\left(-\frac{\rho^2 t}{2a}\right)d\rho \\
&\le C\min\left(1, \int_0^\infty \left(\frac{s}{t}\right)^{n/2+1+\nu_2}\exp\left(-\frac{s}{2a}\right)ds\right) \\
&\le C(1+t)^{-n/2-1-\nu_2}.
\end{aligned} \tag{3.8}$$

In the integral (3.8) we have used the changes of the variables $s = \rho^2 t$.

(2) In the proof of Proposition 3.1(2)–(3), we may restrict ourselves to the case $|x| \geq 1$. Introduce the differential operator X by

$$Xv(t,\rho) = \frac{\partial}{\partial \rho}\left(\frac{1}{\rho} v(t,\rho)\right).$$

Then, the equality

$$X^k(\varphi(\rho)\rho^l) = \sum_{j=0}^{k} c_{jkl}\varphi^{(j)}(\rho)\rho^{l-2k+j} \tag{3.9}$$

holds for $\rho \neq 0$ and for integers $k \geq 0$ and $l \geq 0$. Lemma 2.1(5) and integration by parts show that

$$\begin{aligned}
&\int_0^\infty X^k(\chi_1(\rho)g(t,\rho)\rho^{n-1+2\nu_2})\tilde{J}_{m-k-1/2}(\rho|x|)\,d\rho \\
&\quad = -\frac{1}{|x|^2}\int_0^\infty X^k(\chi_1(\rho)g(t,\rho)\rho^{n-1+2\nu_2})\frac{1}{\rho}\frac{\partial}{\partial\rho}\tilde{J}_{m-k-3/2}(\rho|x|)\,d\rho \\
&\quad = -\frac{1}{|x|^2}X^k(\chi_1(\rho)g(t,\rho)\rho^{n-1+2\nu_2})\frac{1}{\rho}\tilde{J}_{m-k-3/2}(\rho|x|)\Big|_0^\infty \\
&\qquad + \frac{1}{|x|^2}\int_0^\infty X^{k+1}(\chi_1(\rho)g(t,\rho)\rho^{n-1+2\nu_2})\tilde{J}_{m-k-3/2}(\rho|x|)\,d\rho
\end{aligned}$$

for $0 \leq k \leq m-1$.

Lemma 2.1(4), (3.9) and the definition of the function χ_1 show that the term

$$\begin{aligned}
&\frac{1}{\rho}X^k(\chi_1(\rho)g(t,\rho)\rho^{n-1+2\nu_2})\tilde{J}_{m-k-3/2}(\rho|x|)\Big|_0^\infty \\
&\quad = \sum_{j+l\leq k} c_{jkl}\partial_\rho^j\chi_1(\rho)\partial_\rho^l g(t,\rho)\rho^{2m-2k+j+l-1+2\nu_2}\tilde{J}_{m-k-3/2}(\rho|x|)\Big|_0^\infty
\end{aligned}$$

vanishes for $0 \leq k \leq m-1$.

Therefore the equality

$$\begin{aligned}
&\int_0^\infty X^k(\chi_1(\rho)g(t,\rho)\rho^{n-1+2\nu_2})\tilde{J}_{m-k-1/2}(\rho|x|)\,d\rho \\
&\qquad = \frac{1}{|x|^2}\int_0^\infty X^{k+1}(\chi_1(\rho)g(t,\rho)\rho^{n-1+2\nu_2})\tilde{J}_{m-k-3/2}(\rho|x|)\,d\rho
\end{aligned} \tag{3.10}$$

holds for $0 \leq k \leq m-1$. Repeat the integration of (3.6) by parts m times, then the equality (3.10) gives

$$(-\Delta)^{\nu_2}I(t,x) = \sqrt{\frac{\pi}{2}}\frac{1}{|x|^{n-1}}\int_0^\infty X^m(\chi_1(\rho)g(t,\rho)\rho^{n-1+2\nu_2})\cos(\rho|x|)\,d\rho, \tag{3.11}$$

where we have used Lemma 2.1(3).

Since $g(t,0) = \partial_\rho g(t,0) = 0$ and $\chi_1(\rho) = 0$ for $\rho \ge 2a/3$, (3.7) and (3.9) show that

$$X^m(\chi_1(\rho)g(t,\rho)\rho^{n-1+2\nu_2})\Big|_0^\infty = \sum_{j+k\le m} c_{jk}\partial_\rho^j\chi_1(\rho)\partial_\rho^k g(t,\rho)\rho^{j+k+2\nu_2}\Big|_0^\infty = 0$$

and

$$\frac{\partial}{\partial\rho}X^m(\chi_1(\rho)g(t,\rho)\rho^{n-1+2\nu_2})\Big|_0^\infty = \sum_{j+k\le m+1} c_{jk}\partial_\rho^j\chi_1(\rho)\partial_\rho^k g(t,\rho)\rho^{j+k-1+2\nu_2}\Big|_0^\infty = 0, \quad (3.12)$$

where we have used $c_{00} = 0$ in (3.12) when $\nu_2 = 0$.

Since

$$\cos\rho|x| = -\frac{1}{|x|^2}\frac{\partial^2}{\partial\rho^2}\cos\rho|x|,$$

integration of (3.11) by parts and (3.9) show that

$$\begin{aligned}(-\Delta)^{\nu_2}I(t,x) &= -\sqrt{\frac{\pi}{2}}\frac{1}{|x|^{n+1}}\int_0^\infty\left(\frac{\partial}{\partial\rho}\right)^2 X^m(\chi_1(\rho)g(t,\rho)\rho^{n-1+2\nu_2})\cos(\rho|x|)\,d\rho \\ &= \sum_{j+k\le m+2}\frac{c_{j,k}}{|x|^{n+1}}\int_0^\infty \chi_1^{(j)}(\rho)\partial_\rho^k g(t,\rho)\rho^{j+k-2+2\nu_2}\cos(\rho|x|)\,d\rho \\ &\equiv \sum_{j+k\le m+2} I_{j,k}, \end{aligned} \quad (3.13)$$

where $I_{j,k} = 0$ for $j+k\le 1$ when $\nu_2 = 0$.

(3.3) gives $|\partial_\rho^k g(t,\rho)| \le C$ for $t\le 1$ and $\rho\le 2a/3$. Hence, the equality (3.13) shows that

$$|(-\Delta)^{\nu_2}I(t,x)| \le \frac{C}{|x|^{n+1}} \le \frac{C}{(1+|x|)^{n+1}} \quad (3.14)$$

for $0\le t\le 1$, where we have used the inequality $1/|x| \le 2/(1+|x|)$ for $|x|\ge 1$.

Hereafter in the proof of Proposition 3.1(2), we restrict ourselves to the case where $t\ge 1$. We define the function Θ_1 by

$$\Theta_1(\rho) = \frac{\rho^4}{2a(a+\sqrt{a^2-\rho^2})^2}$$

for $0\le\rho\le 2a/3$. Then,

$$g(t,\rho) = (\exp(-t\Theta_1(\rho)) - 1)\exp\left(-\frac{\rho^2 t}{2a}\right) \quad (3.15)$$

and

$$|\Theta_1^{(k)}(\rho)| \le C_k\rho^{4-k}.$$

Therefore, the inequality

$$\left|\left(\frac{\partial}{\partial\rho}\right)^k \exp(-t\Theta_1(\rho))\right| \le C \sum_{l\ge 1, j_1+\cdots+j_l=k} |t^l \Theta_1^{(j_1)}(\rho)\cdots\Theta_1^{(j_l)}\exp(-t\Theta_1(\rho))|$$
$$\le C\sum_{l=1}^{k} t^l \rho^{4l-k}\exp(-t\Theta_1(\rho)) \tag{3.16}$$

holds for $k \ge 1$. Easy calculations show

$$\left|\left(\frac{\partial}{\partial\rho}\right)^k \exp\left(-\frac{\rho^2 t}{2a}\right)\right| \le C\sum_{l=1}^{k} t^l\rho^{2l-k}\exp\left(-\frac{\rho^2 t}{2a}\right) \tag{3.17}$$

for $k \ge 1$. By (3.7) and (3.15)–(3.17), we see that, for $k \ge 1$,

$$\left|\left(\frac{\partial}{\partial\rho}\right)^k g(t,\rho)\right| \le C\sum_{l=1}^{k} t^l\rho^{4l-k}\sum_{i=0}^{k} t^i\rho^{2i}\exp\left(-\frac{\rho^2 t}{2a}\right). \tag{3.18}$$

By (3.13) and (3.18) we obtain the desired estimate as follows,

$$\begin{aligned}
|I_{j,k}| &\le \frac{C}{|x|^{n+1}}\int_0^{2a/3}\sum_{l=1}^{k} t^l\rho^{4l+j-2+2\nu_2}\sum_{i=0}^{k} t^i\rho^{2i}\exp\left(-\frac{\rho^2 t}{2a}\right)d\rho \quad (s=\rho^2 t)\\
&\le \frac{C}{|x|^{n+1}}\int_0^{\infty}\sum_{l=1}^{k}\frac{s^{2l+j/2-3/2+\nu_2}}{t^{l+j/2-1/2+\nu_2}}\sum_{i=0}^{k} s^i\exp\left(-\frac{s}{2a}\right)ds\\
&\le \frac{C}{|x|^{n+1}t^{1/2+\nu_2}}\\
&\le \frac{C}{(1+|x|)^{n+1}(1+t)^{1/2+\nu_2}}
\end{aligned}$$

for $k \ge 1$. When $k = 0$, the estimate (3.7) shows that

$$\begin{aligned}
|I_{j,0}| &\le \frac{C}{|x|^{n+1}}\int_0^{2a/3} t\rho^{2+j+2\nu_2}\exp\left(-\frac{\rho^2 t}{2a}\right)d\rho\\
&\le \frac{C}{|x|^{n+1}}\int_0^{\infty}\frac{s^{(1+j)/2+\nu_2}}{t^{(1+j)/2+\nu_2}}\exp\left(-\frac{s}{2a}\right)ds\\
&\le \frac{C}{|x|^{n+1}t^{1/2+\nu_2}} \le \frac{C}{(1+|x|)^{n+1}(1+t)^{1/2+\nu_2}}.
\end{aligned}$$

Thus we have obtained the desired estimate in the case where $n = 2m+1$.

(3) Now we consider the case where $n = 2m$. Repeat integration of (3.6) by parts $(m-1)$-times, then (3.10) shows that

$$(-\Delta)^{\nu_2} I(t,x) = \frac{1}{|x|^{n-2}}\int_0^{\infty} X^{m-1}(\chi_1(\rho)g(t,\rho)\rho^{n-1+2\nu_2})J_0(\rho|x|)\,d\rho. \tag{3.19}$$

(3.9) shows that

$$X^{m-1}(\chi_1(\rho)g(t,\rho)\rho^{n-1+2\nu_2}) = \sum_{k+l\le m-1} c_{kl}\rho^{k+l+1+2\nu_2}\chi_1^{(l)}(\rho)\left(\frac{\partial}{\partial\rho}\right)^k g(t,\rho). \tag{3.20}$$

Hence $\rho X^{m-1}(\chi_1(\rho)g(t,\rho)\rho^{n-1+2\nu_2})$ vanishes at $\rho = 0$ and $\rho \ge 2a/3$.

Since (see Lemma 2.1(1))

$$J_0(\rho|x|) = 2\tilde{J}_1(\rho|x|) + \rho\frac{\partial}{\partial\rho}\tilde{J}_1(\rho|x|), \tag{3.21}$$

the equality (3.19) shows that

$$\begin{aligned}(-\Delta)^{\nu_2}I(t,x) &= \frac{2}{|x|^{n-2}}\int_0^\infty X^{m-1}(\chi_1(\rho)g(t,\rho)\rho^{n-1+2\nu_2})\tilde{J}_1(\rho|x|)\,d\rho \\ &\quad + \frac{1}{|x|^{n-2}}\int_0^\infty X^{m-1}(\chi_1(\rho)g(t,\rho)\rho^{n-1+2\nu_2})\rho\frac{\partial}{\partial\rho}\tilde{J}_1(\rho|x|)\,d\rho \\ &= \frac{1}{|x|^{n-2}}\int_0^\infty X^{m-1}(\chi_1(\rho)g(t,\rho)\rho^{n-1+2\nu_2})\tilde{J}_1(\rho|x|)\,d\rho \\ &\quad - \frac{1}{|x|^{n-2}}\int_0^\infty \rho\frac{\partial}{\partial\rho}\{X^{m-1}(\chi_1(\rho)g(t,\rho)\rho^{n-1+2\nu_2})\}\tilde{J}_1(\rho|x|)\,d\rho \\ &\equiv I_1(t,x) + I_2(t,x). \end{aligned} \tag{3.22}$$

Lemma 2.1(4) shows

$$\begin{aligned} I_1(t,x) &= \frac{c}{|x|^{n-1/2}}\int_0^\infty X^{m-1}(\chi_1(\rho)g(t,\rho)\rho^{n-1+2\nu_2})\rho^{-3/2}\cos\left(\rho|x| - \frac{3}{4}\pi\right)d\rho \\ &\quad + \frac{c}{|x|^{n-2}}\int_0^\infty X^{m-1}(\chi_1(\rho)g(t,\rho)\rho^{n-1+2\nu_2})O((\rho|x|)^{-5/2})\,d\rho \\ &\equiv I_{11}(t,x) + I_{12}(t,x). \end{aligned} \tag{3.23}$$

(3.20) and the estimate (3.18) show that

$$|X^{m-1}(\chi_1(\rho)g(t,\rho)\rho^{n-1+2\nu_2})| \le C\sum_{k=0}^{m-1}\sum_{i=1}^{k} t^i\rho^{4i+1+2\nu_2}\sum_{j=0}^{k} t^j\rho^{2j}\exp\left(-\frac{\rho^2 t}{2a}\right). \tag{3.24}$$

By (3.23) and the estimate (3.24), we obtain the following estimates

$$|I_{12}(t,x)| \le \frac{C}{|x|^{n+1/2}} \le \frac{C}{(1+|x|)^{n+1/2}}$$

for $0 \le t \le 1$, and

$$
\begin{aligned}
|I_{12}(t,x)| &\le \frac{C}{|x|^{n+1/2}} \sum_{i=1}^{m-1} \sum_{j=0}^{m-1} \int_0^{2a/3} t^i \rho^{4i-3/2+2\nu_2} t^j \rho^{2j} \exp\left(-\frac{\rho^2 t}{2a}\right) d\rho \\
&\le \frac{C}{|x|^{n+1/2}} \sum_{i=1}^{m-1} \sum_{j=0}^{m-1} \int_0^{\infty} t^{-i+1/4-\nu_2} s^{2i-5/4+\nu_2} s^j \exp\left(-\frac{s}{2a}\right) ds \quad (s=\rho^2 t) \\
&\le \frac{C}{|x|^{n+1/2} t^{3/4+\nu_2}} \\
&\le \frac{C}{(1+|x|)^{n+1/2}(1+t)^{3/4+\nu_2}}
\end{aligned}
$$

for $t \ge 1$. Therefore we claim that

$$
|I_{12}(t,x)| \le \frac{C}{(1+|x|)^{n+1/2}} (1+t)^{-3/4-\nu_2}, \quad 0 \le t. \tag{3.25}
$$

Now we will estimate $I_{11}(t,x)$.

(3.7), (3.18) and (3.20) show that $X^{m-1}(\chi_1(\rho) g(t,\rho) \rho^{n-1+2\nu_2}) \rho^{-3/2}$ vanishes at $\rho = 0$ and $\rho \ge 2a/3$. Hence, integration of (3.23) by parts shows that

$$
\begin{aligned}
I_{11}(t,x) &= \frac{c}{|x|^{n+1/2}} \int_0^\infty X^{m-1}(\chi_1(\rho) g(t,\rho) \rho^{n-1+2\nu_2}) \rho^{-3/2} \frac{\partial}{\partial\rho} \sin\left(\rho|x| - \frac{3}{4}\pi\right) d\rho \\
&= \frac{c}{|x|^{n+1/2}} \int_0^\infty \frac{\partial}{\partial\rho} (X^{m-1}(\chi_1(\rho) g(t,\rho) \rho^{n-1+2\nu_2}) \rho^{-3/2}) \sin\left(\rho|x| - \frac{3}{4}\pi\right) d\rho \\
&= \frac{c}{|x|^{n+1/2}} \int_0^\infty \frac{\partial}{\partial\rho} (X^{m-1}(\chi_1(\rho) g(t,\rho) \rho^{n-1+2\nu_2})) \rho^{-3/2} \sin\left(\rho|x| - \frac{3}{4}\pi\right) d\rho \\
&\quad + \frac{c}{|x|^{n+1/2}} \int_0^\infty \rho^{-5/2} X^{m-1}(\chi_1(\rho) g(t,\rho) \rho^{n-1+2\nu_2}) \sin\left(\rho|x| - \frac{3}{4}\pi\right) d\rho \\
&\equiv I_{111} + I_{112}.
\end{aligned} \tag{3.26}
$$

By the same calculations as ones in (3.25) we see that

$$
|I_{112}(t,x)| \le C(1+|x|)^{-n-1/2}(1+t)^{-3/4-\nu_2}. \tag{3.27}
$$

By (3.7), (3.18) and (3.20) we see that

$$
\left| \frac{\partial}{\partial\rho} (X^{m-1}(\chi_1(\rho) g(t,\rho) \rho^{n-1+2\nu_2})) \right| \le C \sum_{j=1}^{m} t^j \rho^{4j+2\nu_2} \sum_{k=0}^{m} t^k \rho^{2k} \exp\left(-\frac{\rho^2 t}{2a}\right). \tag{3.28}
$$

Therefore, by (3.26) and (3.28), we obtain

$$
|I_{111}(t,x)| \le \frac{C}{(1+|x|)^{n+1/2}} (1+t)^{-3/4-\nu_2}. \tag{3.29}
$$

(3.26) and the estimates (3.27)–(3.29) show that

$$|I_{11}(t,x)| \le \frac{C}{(1+|x|)^{n+1/2}}(1+t)^{-3/4-\nu_2}. \tag{3.30}$$

By (3.23), (3.25) and (3.30), we obtain

$$|I_1(t,x)| \le \frac{C}{(1+|x|)^{n+1/2}}(1+t)^{-3/4-\nu_2}. \tag{3.31}$$

By the similar arguments as ones in the estimate of $I_1(t,x)$, we obtain the estimate of $I_2(t,x)$ as follows.

$$|I_2(t,x)| \le \frac{C}{(1+|x|)^{n+1/2}}(1+t)^{-3/4-\nu_2} \tag{3.32}$$

We omit the estimate of $I_2(t,x)$, because it's proof is very similar to ones in the previous part. By (3.22), (3.31) and (3.32), we have obtained the desired result. □

Proof of Theorem 3.1. First we consider the case where $\nu_1 = 0$. Proposition 3.1(1) shows

$$|(-\Delta)^{\nu_2} I(t,x)| \le C(1+t)^{-n/2-1-\nu_2}. \tag{3.33}$$

(3.5) and (3.33) show

$$\|(-\Delta)^{\nu_2} V_1(t,\cdot)\|_\infty \le C(1+t)^{-n/2-1-\nu_2}(\|\varphi_0\|_1 + \|\varphi_1\|_1). \tag{3.34}$$

When $p = q = 2$, (1.6), (3.2) and (3.7) show

$$\begin{aligned}\|(-\Delta)^{\nu_2} V_1(t,\cdot)\|_2 &\le C\left\| |\xi|^{2\nu_2+4}\chi_1(\xi)\left(\hat\varphi_0 + \frac{a\hat\varphi_0 + \hat\varphi_1}{\sqrt{a^2-|\xi|^2}}\right)\right\|_2 \\ &\le C(1+t)^{-\nu_2-1}(\|\varphi_0\|_2 + \|\varphi_1\|_2).\end{aligned} \tag{3.35}$$

Lemma 3.1 and the estimates (3.34)–(3.35) give the desired estimate with $p = \infty$, $q = 1$ or $p = q = 2$ when $\nu_1 = 0$.

Let $1 \le q \le p \le \infty$. Proposition 3.1(2), (3) shows

$$\|(-\Delta)^{\nu_2} I(t,\cdot)\|_1 \le C(1+t)^{-1-\nu_2+\varepsilon}\int (1+|x|)^{-n-2\varepsilon}\,dx \le C(\varepsilon)(1+t)^{-1-\nu_2+\varepsilon} \tag{3.36}$$

for $0 < \varepsilon \le 1/4$. By (3.33)–(3.36), we see that the constant $r \in [1,\infty]$ defined by $1 - 1/r = 1/q - 1/p$ satisfies

$$\begin{aligned}\|(-\Delta)^{\nu_2} I(t,\cdot)\|_r &\le \|(-\Delta)^{\nu_2} I(t,\cdot)\|_\infty^{1-1/r}\|(-\Delta)^{\nu_2} I(t,\cdot)\|_1^{1/r} \\ &\le C(1+t)^{-(n/2)\cdot(1/q-1/p)-1-\nu_2+\varepsilon}.\end{aligned} \tag{3.37}$$

The estimates (3.5), (3.37) and Lemma 2.3 show that $\|(-\Delta)^{\nu_2} V_1(t,\cdot)\|_p$ satisfies the desired estimate of Theorem 3.1. Hence, by Lemma 3.1, we have obtained the desired result of Theorem 3.1 with $\varepsilon > 0$, when $\nu_1 = 0$.

Let $1 < q < p < \infty$. Proposition 3.1(2) shows

$$|(-\Delta)^{\nu_2} I(t,x)| = |(-\Delta)^{\nu_2} I(t,x)|^{n/(n+1)} |(-\Delta)^{\nu_2} I(t,x)|^{1/(n+1)}$$
$$\leq C(1+|x|)^{-n}(1+t)^{-1-\nu_2}, \tag{3.38}$$

in the case where $n = 2m+1$. Replacing $n/(n+1)$ by $2n/(2n+1)$ in (3.38), we see that the estimate (3.38) is also valid in the case where $n = 2m$. The estimates (3.33)–(3.38) show that the positive constant r defined by $1 - 1/r = 1/q - 1/p$ satisfies

$$|(-\Delta)^{\nu_2} I(t,x)| = |(-\Delta)^{\nu_2} I(t,x)|^{1-1/r} |(-\Delta)^{\nu_2} I(t,x)|^{1/r}$$
$$\leq C(1+t)^{-(n/2)(1/q-1/p)-1-\nu_2}(1+|x|)^{-n/r}. \tag{3.39}$$

By the estimate (3.39), the equality (3.5) and Lemma 2.4, we see that $\|(-\Delta)^{\nu_2} V_1(t,\cdot)\|_p$ satisfies the desired estimate of Theorem 3.1 with $\varepsilon = 0$ and $1 < q < p < \infty$.

Hence, by Lemma 3.1, we have obtained the desired result of Theorem 3.1 with $\varepsilon = 0$ when $\nu_1 = 0$.

Now we consider the case where $\nu_1 \geq 1$. For a integer $\nu_1 \geq 1$, there exist functions $h_{\nu_1,1}(s)$, $h_{\nu_1,2}(s) \in C^\infty([0,a^2))$ satisfying

$$\partial_t^{\nu_1} g(t,\rho) = \rho^{2\nu_1} h_{\nu_1,1}(\rho^2) g(t,\rho) + \rho^{2\nu_1+2} h_{\nu_1,2}(\rho^2) \exp\left(-\frac{\rho^2 t}{2a}\right).$$

Hence, we see that

$$\partial_t^{\nu_1} |\xi|^{2\nu_2} \hat{I}(t,\xi) = h_{\nu_1,1}(|\xi|^2)|\xi|^{2\nu_1+2\nu_2} \hat{I}(t,\xi) + h_{\nu_1,2}(|\xi|^2)|\xi|^{2\nu_1+2\nu_2+2} \chi_1(\xi) \exp\left(-\frac{\rho^2 t}{2a}\right). \tag{3.40}$$

We define the operators B_1 and B_2 by

$$B_1 f = \mathscr{F}^{-1}(\chi_{11}(\xi) h_{\nu_1,1}(|\xi|^2) \hat{f}(\xi)), \quad B_2 f = \mathscr{F}^{-1}(\chi_1(\xi) h_{\nu_1,2}(|\xi|^2) \hat{f}(\xi))$$

respectively, then they are bounded from L^q to L^q $(1 \leq q \leq \infty)$. By (3.40), we see that

$$\partial_t^{\nu_1} (-\Delta)^{\nu_2} V_1(t,x) = \frac{1}{2}\left(\frac{1}{2\pi}\right)^{n/2} ((-\Delta)^{\nu_1+\nu_2} I(t,x) * B_1(\varphi_0 + B(a\varphi_0 + \varphi_1))$$
$$+ (-\Delta)^{\nu_1+\nu_2+1} \psi_{\nu_1}(t,x)),$$

where $\psi_{\nu_1}(t,x)$ is a smooth solution of the heat equation (1.2) with initial data $(a\varphi_0 + \varphi_1)/(2a)$ replaced by $B_2(\varphi_0 + B(a\varphi_0 + \varphi_1))$. Therefore, by (3.5) and the estimates (1.6) and (3.37)–(3.39), we see that

$$\|\partial_t^{\nu_1} (-\Delta)^{\nu_2} V_1(t,\cdot)\|_p \leq C(\varepsilon)(1+t)^{-(n/2)\cdot(1/q-1/p)-\nu_1-\nu_2-1+\varepsilon}(\|\varphi_0\|_q + \|\varphi_1\|_q) \tag{3.41}$$

for $0 < \varepsilon$ and $1 \leq q \leq p \leq \infty$, and

$$\|\partial_t^{\nu_1} (-\Delta)^{\nu_2} V_1(t,\cdot)\|_p \leq C(1+t)^{-(n/2)\cdot(1/q-1/p)-\nu_1-\nu_2-1}(\|\varphi_0\|_q + \|\varphi_1\|_q) \tag{3.42}$$

for $1 < q < p < \infty$ or $q = 1$, $p = \infty$.

The estimates (1.6), (3.35) and the equality (3.40) show that the estimate (3.42) also holds when $p = q = 2$. Lemma 3.1 and the estimates (3.41)–(3.42) give the desired estimate of Theorem 3.1. □

4. Proof of Theorem 1.2.

Choose and fix a radial function $0 \le \chi_2(\xi) \le 1$ of class C^∞ satisfying

$$\chi_2(\xi) = 0 \quad (|\xi| \le 2a), \quad \chi_2(\xi) = 1 \quad (|\xi| \ge 3a).$$

Sometimes we denote $\chi_2(|\xi|) = \chi_2(\xi)$. Remark 1.1 shows that Theorem 1.2 is a direct consequence of the next theorem.

Theorem 4.1. *Let $n \ge 2$, $1 < q \le p < \infty$. Assume that $\varphi_i \in L^q$ for $i = 0, 1$. Then, under the same notations on M_0 and M_1 in Theorem 1.2, the solution u to (1.1) in the sense of distributions satisfies the following L^p-L^q estimates.*

$$\begin{aligned}&\|\mathscr{F}^{-1}\{\chi_2(\cdot)(\hat{u}(t,\cdot) - e^{-at}(M_0(t,\cdot)\hat{\varphi}_0(\cdot) + M_1(t,\cdot)\hat{\varphi}_1(\cdot)))\}\|_p \\ &\le Ce^{-at}(1+t)^{3n+3}(\|\varphi_0\|_q + \|\varphi_1\|_q).\end{aligned}$$

In this section we will prove Theorem 4.1. The explicit formula (1.9) to the problem (1.1) gives

$$\hat{u}(t,\xi) = e^{-at}(\tilde{M}_0(t,\xi)\hat{\varphi}_0(\xi) + \tilde{M}_1(t,\xi)\hat{\varphi}_1(\xi)), \tag{4.1}$$

where

$$\begin{aligned}\tilde{M}_1(t,\xi) &= \frac{1}{\lambda}(\sin t|\xi| \cos t\Theta - \cos t|\xi| \sin t\Theta), \\ \tilde{M}_0(t,\xi) &= \cos t|\xi| \cos t\Theta + \sin t|\xi| \sin t\Theta + a\tilde{M}_1(t,\xi), \\ \lambda(\xi) &\equiv \sqrt{|\xi|^2 - a^2}\end{aligned}$$

and

$$\Theta(\xi) \equiv |\xi| - \lambda(\xi) = \frac{a^2}{|\xi| + \sqrt{|\xi|^2 - a^2}}.$$

For large $|\xi|$, $\Theta(\xi) \doteqdot a^2/(2|\xi|)$. Occasionally we denote $\Theta(|\xi|) = \Theta(\xi)$.

In this section we may assume that $\operatorname{supp}\hat{\varphi}_0 \cup \operatorname{supp}\hat{\varphi}_1 \subset \{\xi; |\xi| > 3a/2\}$.

We define the functions $h_c(y)$ and $h_s(y)$ by

$$h_c(y) = \cos y - \sum_{k=0}^{n} \frac{(-1)^k}{(2k)!} y^{2k}, \quad h_s(y) = \sin y - \sum_{k=0}^{n} \frac{(-1)^k}{(2k+1)!} y^{2k+1}$$

respectively. Then

$$|h_c^{(k)}(y)| \le C|y|^{2n+2-k}, \quad |h_s^{(k)}(y)| \le C|y|^{2n+3-k} \tag{4.2}$$

for $0 \le k \le 2(n+1)$. We define functions $II_c^{\pm}(t,x)$ and $II_s^{\pm}(t,x)$ by

$$II_c^{\pm}(t,x) = \mathscr{F}^{-1}(\chi_2(\xi)h_c(t\Theta(\xi))e^{\pm it|\xi|}), \quad II_s^{\pm}(t,x) = \mathscr{F}^{-1}(\chi_2(\xi)h_s(t\Theta(\xi))e^{\pm it|\xi|})$$

respectively. Then, Lemma 2.2 shows that

$$II_c^{\pm}(t,x) = \int_0^\infty \chi_2(\rho) h_c(t\Theta(\rho)) e^{\pm it\rho} \rho^{n-1} \tilde{J}_{n/2-1}(\rho|x|)\, d\rho,$$

$$II_s^{\pm}(t,x) = \int_0^\infty \chi_2(\rho) h_s(t\Theta(\rho)) e^{\pm it\rho} \rho^{n-1} \tilde{J}_{n/2-1}(\rho|x|)\, d\rho.$$

By (4.2) we see that the following estimates hold.

Lemma 4.1. (1) $\sup_x(|II_c^{\pm}(t,x)| + |II_s^{\pm}(t,x)|) \le C(1+t)^{2n+3}$.
(2) $\|II_c^{\pm}(t,\cdot)\|_1 + \|II_s^{\pm}(t,\cdot)\|_1 \le C(1+t)^{3n+3}$.

Proof. (1) Since $|\Theta(\rho)| \le a^2/\rho$, Lemma 2.1(4) and (4.2) show that

$$\begin{aligned} |II_c^{\pm}(t,x)| + |II_s^{\pm}(t,x)| &\le C\int_{2a}^\infty \rho^{n-1}(t^{2n+2}\Theta(\rho)^{2n+2} + t^{2n+3}\Theta(\rho)^{2n+3})\, d\rho \\ &\le C\int_{2a}^\infty \rho^{n-1}(t^{2n+2}\rho^{-(2n+2)} + t^{2n+3}\rho^{-(2n+3)})\, d\rho \\ &\le C(t^{2n+2} + t^{2n+3}) \le C(1+t)^{2n+3}. \end{aligned}$$

We have obtained the desired result.

(2) Here we may restrict ourselves to the case where $\rho \ge 3a/2$. Since

$$\left|\left(\frac{\partial}{\partial\rho}\right)^k \Theta(\rho)\right| \le C_k \rho^{-k-1},$$

the following estimates hold for $k \le 2(n+1)$.

$$\begin{aligned} \left|\left(\frac{\partial}{\partial\rho}\right)^k h_c(t\Theta(\rho))\right| &\le C\sum_{j_1+\cdots+j_l=k} |t^l \Theta^{(j_1)}(\rho)\cdots\Theta^{(j_l)}(\rho) h_c^{(l)}(t\Theta(\rho))| \\ &\le Ct^{2(n+1)}\rho^{-k-2(n+1)} \end{aligned} \tag{4.3}$$

and

$$\begin{aligned} \left|\left(\frac{\partial}{\partial\rho}\right)^k h_s(t\Theta(\rho))\right| &\le C\sum_{j_1+\cdots+j_l=k} |t^l \Theta^{(j_1)}(\rho)\cdots\Theta^{(j_l)}(\rho) h_c^{(l)}(t\Theta(\rho))| \\ &\le Ct^{2n+3}\rho^{-k-(2n+3)}. \end{aligned} \tag{4.4}$$

Hence, (3.9) and the estimates (4.3)–(4.4) show that the terms

$$\left(\frac{\partial}{\partial\rho}\right)^l X^k(\chi_2(\rho) h_c(t\Theta(\rho)) e^{\pm it\rho}\rho^{n-1})$$

and

$$\left(\frac{\partial}{\partial\rho}\right)^l X^k(\chi_2(\rho) h_s(t\Theta(\rho)) e^{\pm it\rho}\rho^{n-1})$$

vanish at $\rho = 0, \infty$ for $0 \le l \le 2$, $0 \le k \le m$, where $n = 2m+1$ or $n = 2m$.

We begin with the case where $n = 2m+1$. Lemma 2.1 and the similar calculations to ones in (3.9)–(3.13) give

$$\begin{aligned} II_c^{\pm}(t,x) &= \frac{c}{|x|^{n+1}} \int_0^{\infty} \frac{\partial^2}{\partial \rho^2} X^m(\chi_2(\rho) h_c(t\Theta(\rho)) e^{\pm it\rho} \rho^{n-1}) \cos \rho|x| \, d\rho \\ &= \frac{c}{|x|^{n+1}} \sum_{2 \le j+k+l \le m+2} c_{jkl} \int_0^{\infty} \chi_2^{(j)}(\rho) \frac{\partial^k}{\partial \rho^k} h_c(t\Theta(\rho)) (\pm it)^l e^{\pm it\rho} \\ &\quad \times \rho^{j+k+l-2} \cos \rho|x| \, d\rho. \end{aligned} \tag{4.5}$$

(4.3) and (4.5) show that

$$|II_c^{\pm}(t,x)| \le \frac{C}{|x|^{n+1}} \int_{2a}^{\infty} t^{2(n+1)}(1 + t^{m+2}) \rho^{-2} \, d\rho \le \frac{C}{|x|^{n+1}} (1+t)^{3n+3}. \tag{4.6}$$

(4.4) and the similar equality to (4.5) show that

$$|II_s^{\pm}(t,x)| \le \frac{C}{|x|^{n+1}} \int_{2a}^{\infty} t^{2n+3}(1 + t^{m+2}) \rho^{-2} \, d\rho \le \frac{C}{|x|^{n+1}} (1+t)^{3n+3}. \tag{4.7}$$

By the estimates (4.6)–(4.7) and the estimate in Lemma 4.1, we see that

$$|II_c^{\pm}(t,x)| + |II_s^{\pm}(t,x)| \le \frac{C}{(1+|x|)^{n+1}} (1+t)^{3n+3}.$$

We have obtained the desired estimate when $n = 2m+1$. Consider the case where $n = 2m$. Repeat the similar calculations to ones in (3.19)–(3.32) to obtain the following estimate

$$|II_c^{\pm}(t,x)| + |II_s^{\pm}(t,x)| \le \frac{C}{(1+|x|)^{n+1/2}} (1+t)^{3n+3},$$

when $n = 2m$. We have obtained the desired estimate. $\square$

Lemma 4.2. *For $1 < q \le p < \infty$, the estimate*

$$\|\mathscr{F}^{-1}(\chi_2(\xi)(\hat{u}(t,\xi) - e^{-at}\hat{v}(t,\xi)))\|_p \le Ce^{-at}(1+t)^{3n+3}(\|\varphi_0\|_q + \|\varphi_1\|_q)$$

holds, where

$$\hat{v}(t,\xi) = N_0(t,\xi)\hat{\varphi}_0(\xi) + N_1(t,\xi)\hat{\varphi}_1(\xi),$$

$$N_1(t,\xi) = \frac{1}{\lambda(\xi)} \left(\sin t|\xi| \sum_{k=0}^{n} \frac{(-1)^k}{(2k)!} t^{2k} \Theta(\xi)^{2k} - \cos t|\xi| \sum_{k=0}^{n} \frac{(-1)^k}{(2k+1)!} t^{2k+1} \Theta(\xi)^{2k+1} \right)$$

and

$$N_0(t,\xi) = \cos t|\xi| \sum_{k=0}^{n} \frac{(-1)^k}{(2k)!} t^{2k} \Theta(\xi)^{2k} + \sin t|\xi| \sum_{k=0}^{n} \frac{(-1)^k}{(2k+1)!} t^{2k+1} \Theta(\xi)^{2k+1} + aN_1(t,\xi).$$

Proof. (4.1) shows that

$$\chi_2(\xi)(\hat{u}(t,\xi) - e^{-at}\hat{v}(t,\xi))$$
$$= e^{-at}\chi_2(\xi)\{(\cos t|\xi| h_c(t\Theta(\xi)) + \sin t|\xi| h_s(t\Theta(\xi)) + \sin t|\xi| h_c(t\Theta(\xi))(a/\lambda)$$
$$- \cos t|\xi| h_s(t\Theta(\xi)(a/\lambda)))\hat{\varphi}_0 + (\sin t|\xi| h_c(t\Theta(\xi)) - \cos t|\xi| h_s(t\Theta(\xi)))(a/\lambda)\hat{\varphi}_0\}. \quad (4.8)$$

The operator B_2 defined by

$$B_2 f = \mathscr{F}^{-1}\left(\frac{1}{\sqrt{|\xi|^2 - a^2}}\hat{f}(\xi)\right)$$

satisfies $\|B_2 f\|_p \le C(p)\|f\|_p$, provided that $\operatorname{supp}\hat{f} \subset \{\xi; |\xi| \ge 3a/2\}$.

Since

$$\cos t\rho = \frac{e^{it\rho} + e^{-it\rho}}{2}, \quad \sin t\rho = \frac{e^{it\rho} - e^{-it\rho}}{2i},$$

Lemma 4.1 and (4.8) show the desired result. □

Here and after we denote a function that belongs to $L^\infty((0,\infty) \times \boldsymbol{R}^n)$ by $O(1)$, and $\log^+ x \equiv \max(\log x, 0)$. The function $L_\mu(t,x)$ defined by

$$L_\mu(t,x) = \int_{3a}^\infty \frac{1}{\rho}\tilde{J}_\mu(\rho|x|)\exp(it\rho)\,d\rho$$

satisfies the following equality.

Lemma 4.3. *Let* $-1/2 < \mu \le n/2 - 1$, *then,*

$$L_\mu(t,x) = \tilde{J}_\mu(0)\min\left(\log^+\left(\frac{1}{3at}\right), \log^+\left(\frac{1}{3a|x|}\right)\right) + O(1) \quad \textit{for} \ \ t > 0.$$

Proof. We begin with the case where $3a|x| \ge 1$. Lemma 2.1(4) shows that

$$|L_\mu(t,x)| \le C\int_{3a}^\infty \frac{1}{\rho^{\mu+3/2}}\frac{1}{|x|^{\mu+1/2}}\,d\rho \le \frac{C}{|x|^{\mu+1/2}} \le C.$$

Now we consider the case where $3a|x| < 1$. Lemma 2.1(4) also shows that

$$\left|\int_{1/|x|}^\infty \frac{1}{\rho}\exp(it\rho)\tilde{J}_\mu(\rho|x|)\,d\rho\right| = \left|\int_1^\infty \frac{1}{s}\exp\left(i\frac{ts}{|x|}\right)\tilde{J}_\mu(s)\,ds\right| \quad (s = \rho|x|)$$
$$\le C\int_1^\infty \frac{1}{s}\left(\frac{1}{s}\right)^{\mu+1/2} ds \le C.$$

Hence

$$L_\mu(t,x) = \int_{3a}^{1/|x|} \frac{1}{\rho}\exp(it\rho)\tilde{J}_\mu(\rho|x|)\,d\rho + O(1). \quad (4.9)$$

In the case where $0 < t \le |x|$, easy calculations show us that

$$\int_{3a}^{1/|x|} \frac{1}{\rho} \exp(it\rho)\tilde{J}_\mu(\rho|x|)\, d\rho$$

$$= \int_{3a|x|}^{1} \frac{1}{s} \exp\left(i\frac{ts}{|x|}\right)\tilde{J}_\mu(s)\, ds \quad (s = \rho|x|)$$

$$= \int_{3a|x|}^{1} \left(\frac{1}{s}\tilde{J}_\mu(0) + \frac{1}{s}(\tilde{J}_\mu(s) - \tilde{J}_\mu(0)) + \frac{1}{s}\left(\exp\left(\frac{its}{|x|}\right) - 1\right)\tilde{J}_\mu(s)\right) ds$$

$$= \tilde{J}_\mu(0) \log^+\left(\frac{1}{3a|x|}\right) + O(1). \tag{4.10}$$

The equalities (4.9) and (4.10) show that

$$L_\mu(t,x) = \tilde{J}_\mu(0) \log^+\left(\frac{1}{3a|x|}\right) + O(1), \quad \text{for } 0 < t \le |x|. \tag{4.11}$$

In the case where $|x| \le t$ and $3a|x| \le 1$, it follows that

$$\int_{3a}^{1/|x|} \frac{1}{\rho} \exp(it\rho)\tilde{J}_\mu(\rho|x|)\, d\rho$$

$$= \int_{3a}^{1/|x|} \frac{1}{\rho} \exp(it\rho)\tilde{J}_\mu(0)\, d\rho + \int_{3a}^{1/|x|} \frac{1}{\rho} \exp(it\rho)(\tilde{J}_\mu(\rho|x|) - \tilde{J}_\mu(0))\, d\rho$$

$$= \tilde{J}_\mu(0) \int_{3a}^{1/|x|} \frac{1}{\rho} \exp(it\rho)\, d\rho + O(1). \tag{4.12}$$

In the equality (4.12) we have used the estimate

$$\int_{3a}^{1/|x|} \frac{1}{\rho} \exp(it\rho)(\tilde{J}_\mu(\rho|x|) - \tilde{J}_\mu(0))\, d\rho = \int_{3a|x|}^{1} \frac{\tilde{J}_\mu(s) - \tilde{J}_\mu(0)}{s} \exp\left(it\frac{s}{|x|}\right) ds = O(1).$$

In the case where $3at \ge 1$, the first term of the right-hand side of (4.12) satisfies the following estimate

$$\left|\int_{3a}^{1/|x|} \frac{1}{\rho} \exp(it\rho)\, d\rho\right| = \left|\int_{3at}^{t/|x|} \frac{1}{s} \exp(is)\, ds\right| \le C.$$

In the case where $3at < 1$ and $|x| \le t$, the easy calculations show that

$$\begin{aligned}
\int_{3a}^{1/|x|} \frac{1}{\rho} \exp(it\rho)\, d\rho &= \int_{3at}^{t/|x|} \frac{1}{s} \exp(is)\, ds \\
&= \int_{3at}^{1} \frac{1}{s} \exp(is)\, ds + \int_{1}^{t/|x|} \frac{1}{s} \exp(is)\, ds \\
&= \int_{3at}^{1} \frac{1}{s} \exp(is)\, ds + O(1) \\
&= \log^+\left(\frac{1}{3at}\right) + O(1).
\end{aligned}$$

Hence we obtain

$$L_\mu(t,x) = \tilde{J}_\mu(0)\log^+\left(\frac{1}{3at}\right) + O(1), \quad \text{for } 0 < |x| \le t. \tag{4.13}$$

By (4.11) and (4.13), we have obtained the desired result. □

Lemma 2.2 shows that the function $K_\nu(t,x)$ defined by

$$K_\nu(t,x) \equiv \mathscr{F}^{-1}(\chi_2(\xi)|\xi|^{-\nu}e^{it|\xi|}) \tag{4.14}$$

satisfies

$$K_\nu(t,x) = \int_0^\infty \chi_2(\rho)\rho^{n-1-\nu}e^{it\rho}\tilde{J}_{n/2-1}(\rho|x|)\,d\rho. \tag{4.15}$$

Proposition 4.1. *Let $\nu \in [(n+1)/2, n]$ be an integer, and let K_ν be the function defined by* (4.14). *Then, the following equalities hold.*

(1) *If $(n+1)/2 < \nu \le n$, then,*

$$K_\nu(t,x) = \frac{c}{t^{n-\nu}}\left\{\min\left(\log^+\left(\frac{1}{3at}\right), \log^+\left(\frac{1}{3a|x|}\right)\right) + O(1)\right\}$$

for $t > 0$ and $x \in \boldsymbol{R}^n$, $|x| \ne 0$.

(2) *If $\nu = (n+1)/2$, then $n = 2m+1$ and $\nu = m+1$, and*

$$K_{m+1}(t,x) = \frac{c_1}{t^m}\left\{\log^+\left(\frac{1}{3a(t+|x|)}\right) + \log^+\left(\frac{1}{3a|t-|x||}\right)\right.$$
$$\left. + c_2\min\left(\log^+\left(\frac{1}{3at}\right), \log^+\left(\frac{1}{3a|x|}\right)\right) + O(1)\right\}$$

for $t > 0$ and $x \in \boldsymbol{R}^n$ with $|x| \ne 0$ and $t - |x| \ne 0$.

Proof. Lemma 2.1(2) shows that

$$\left(\frac{\partial}{\partial\rho}\right)^k(\rho^{n-1-\nu}\tilde{J}_{n/2-1}(\rho|x|)) = \sum_{j=0}^k c_{jk}\rho^{n-1-\nu+k-2j}|x|^{2(k-j)}\tilde{J}_{n/2-1+k-j}(\rho|x|).$$

Hence, if $\nu > n/2$,

$$\left(\frac{\partial}{\partial\rho}\right)^k(\chi_2(\rho)\rho^{n-1-\nu}\tilde{J}_{n/2-1}(\rho|x|))$$

vanishes at $\rho = 0$ and $\rho = \infty$ for $0 \le k \le n-\nu-1$. Since

$$\exp(it\rho) = \left(\frac{1}{it}\right)^{n-\nu}\frac{\partial^{n-\nu}}{\partial\rho^{n-\nu}}\exp(it\rho),$$

integration of (4.15) by parts gives

$$K_\nu(t,x) = \left(\frac{-1}{it}\right)^{n-\nu}\int_0^\infty\left(\frac{\partial}{\partial\rho}\right)^{n-\nu}(\chi_2(\rho)\rho^{n-1-\nu}\tilde{J}_{n/2-1}(\rho|x|))\exp(it\rho)\,d\rho. \tag{4.16}$$

Lemma 2.1(1) shows that

$$\frac{\partial}{\partial\rho}\tilde{J}_\nu(\rho|x|) = \rho^{-1}(\tilde{J}_{\nu-1}(\rho|x|) - 2\nu\tilde{J}_\nu(\rho|x|)).$$

Then we claim that

$$\left(\frac{\partial}{\partial\rho}\right)^k (\rho^{n-1-\nu}\tilde{J}_{n/2-1}(\rho|x|)) = \sum_{j=0}^{k} c_j\rho^{n-1-\nu-k}\tilde{J}_{n/2-1-j}(\rho|x|)$$

for $0 \le k \le (n-1)/2$. Therefore (4.16) shows that

$$\begin{aligned} K_\nu(t,x) &= \frac{1}{t^{n-\nu}}\sum_{k=0}^{n-\nu}\sum_{l=0}^{k} c_{k,l}\int_0^\infty \partial_\rho^{n-\nu-k}\chi_2(\rho)\rho^{n-1-\nu-k}\tilde{J}_{n/2-1-l}(\rho|x|)\exp(it\rho)\,d\rho \\ &= \frac{1}{t^{n-\nu}}\left(\sum_{l=0}^{n-\nu} c_{n-\nu,l}\int_{3a}^\infty \frac{1}{\rho}\tilde{J}_{n/2-1-l}(\rho|x|)\exp(it\rho)\,d\rho + O(1)\right). \qquad (4.17)\end{aligned}$$

In (4.17) we have used the inequalities

$$\int_{2a}^{3a}\left|\frac{1}{\rho}\chi_2(\rho)\tilde{J}_{n/2-1-l}(\rho|x|)\exp(it\rho)\right| d\rho \le C$$

and

$$\left|\int_0^\infty \partial_\rho^{n-\nu-k}\chi_2(\rho)\rho^{n-1-\nu-k}\tilde{J}_{n/2-1-l}(\rho|x|)\exp(it\rho)\,d\rho\right| \le C,$$

when $n-\nu-k > 0$.

(1) We begin with the case where $(n+1)/2 < \nu \le n$. By (4.17), we see that

$$K_\nu(t,x) = \frac{1}{t^{n-\nu}}\left(\sum_{l=0}^{n-\nu} c_{n-\nu,l}L_{n/2-1-l}(t,x) + O(1)\right). \qquad (4.18)$$

By (4.18) and Lemma 4.3, we have obtained the desired result.

(2) By (4.17), (4.18), Lemmas 2.1(3) and 4.3, we see that

$$\begin{aligned} K_{m+1}(t,x) &= \frac{c}{t^m}\left(\sum_{l=0}^{m-1} c_{m,l}L_{n/2-1-l}(t,x) + c_{m,m}\int_{3a}^\infty \frac{1}{\rho}\cos\rho|x|\exp it\rho\,d\rho + O(1)\right) \\ &= \frac{1}{t^m}\left\{c\min\left(\log^+\left(\frac{1}{3at}\right), \log^+\left(\frac{1}{3a|x|}\right)\right)\right. \\ &\qquad \left. + c_m\int_{3a}^\infty \frac{1}{\rho}\cos\rho|x|\exp it\rho\,d\rho + O(1)\right\}. \qquad (4.19)\end{aligned}$$

The equality

$$\int_{3a}^\infty \frac{\exp(i\sigma\rho)}{\rho}\,d\rho = \int_{3a|\sigma|}^\infty \frac{e^{is}}{s}\,d\rho = \log^+\left(\frac{1}{3a|\sigma|}\right) + O(1), \quad (s = \sigma\rho)$$

holds for a real constant $\sigma \neq 0$. Therefore we claim that

$$\begin{aligned}\int_{3a}^{\infty} \frac{1}{\rho} \cos \rho|x| e^{it\rho}\, d\rho &= \int_{3a}^{\infty} \frac{\exp(i\rho(t+|x|)) + \exp(i\rho(t-|x|))}{2\rho}\, d\rho \\ &= \frac{1}{2}\left\{\log\left(\frac{1}{3a(t+|x|)}\right) + \log\left(\frac{1}{3a|t-|x||}\right)\right\} + O(1). \qquad (4.20)\end{aligned}$$

By (4.19) and (4.20) we have obtained the desired estimates. □

It is well-known that the functions

$$\min\left(\log^+\left(\frac{1}{3at}\right), \log^+\left(\frac{1}{3a|x|}\right)\right), \quad \log^+\left(\frac{1}{3a(t+|x|)}\right) + \log^+\left(\frac{1}{3a|t-|x||}\right)$$

belong to BMO, and their BMO-norms are uniformly bounded for $t > 0$ (see e.g. [**28**]). Then we obtain the next proposition by Lemma 4.3.

Proposition 4.2. *Let $v \in [(n+1)/2, n]$ be an integer, then, $K_v(t,\cdot) \in BMO$ for $t > 0$, and it satisfies $\|K_v(t,\cdot)\|_{BMO} \leq Ct^{v-n}$.*

The next lemma is a direct consequence of Lemma 2.1(4).

Lemma 4.4. *Let $v \geq n+1$, then $K_v \in L^\infty((0,\infty) \times \boldsymbol{R}^n)$.*

The operators $T_v(t)$ defined by

$$T_v(t)h = K_v(t,\cdot) * h \qquad (4.21)$$

satisfies the following estimate.

Proposition 4.3. *When $v \geq m$, the operator $T_v(t)$ defined by* (4.21) *satisfies the estimate*

$$\|T_v(t)h\|_p \leq C(1+t^m)\|h\|_p \qquad (4.22)$$

for $1 < p < \infty$ and $t > 0$, where $n = 2m$ or $n = 2m+1$.

Proof. Consider the wave equation.

$$\partial_t^2 v - \Delta v = 0, \quad v(0,x) = f(x), \quad \partial_t v(0,x) = g(x), \quad t > 0,\ x \in \boldsymbol{R}^n. \qquad (4.23)$$

In the case where $n = 2m+1$, $m \geq 1$, the solution formula takes the form

$$\begin{aligned} v &= \sum_{k=0}^{m} a_k t^k \left(\frac{d}{dt}\right)^k \int_{|\omega|=1} f(x+t\omega)\, d\omega + \sum_{k=0}^{m-1} b_k t^{k+1} \left(\frac{d}{dt}\right)^k \int_{|\omega|=1} g(x+t\omega)\, d\omega \\ &\equiv T_c f + T_s g, \qquad (4.24)\end{aligned}$$

where $d\omega$ is the surface measure on the $(n-1)$-sphere and a_k, b_k are constants. The linear operators $T_c(t)$ and $T_s(t)$ defined by the equality (4.24) are bounded from $W^{m,p}$ to L^p and $W^{m-1,p}$ to L^p for $1 \leq p \leq \infty$ respectively, and they satisfy the following estimates.

$$\|T_c(t)h\|_p \leq C(1+t^m)\|h\|_{W_p^m}, \quad \|T_s(t)h\|_p \leq C(t+t^m)\|h\|_{W_p^{m-1}}. \qquad (4.25)$$

The estimates (4.25) hold also when $n = 2m$.

Hence, the operator

$$T_\nu(t)h = T_c(t)\mathscr{F}^{-1}\left(\frac{\chi_2(\xi)}{|\xi|^\nu}\hat{h}(\xi)\right) + iT_s(t)\mathscr{F}^{-1}\left(\frac{\chi_2(\xi)}{|\xi|^{\nu-1}}\hat{h}(\xi)\right)$$

is bounded from L^p to L^p and it satisfies the estimate

$$\|T_\nu(t)h\|_p \le C(1+t^m)\|h\|_p \tag{4.26}$$

for $\nu \ge m$, $1 < p < \infty$. In the estimate (4.26) we have used the well-known estimate;

$$\left\|\mathscr{F}^{-1}\left(\chi_2(\xi)\frac{(1+|\xi|^2)^{\mu/2}}{|\xi|^\nu}\hat{h}(\xi)\right)\right\|_p \le C\|h\|_p$$

for $1 < p < \infty$ and $\mu \le \nu$ (see e.g. [**26**]). Thus we have obtained the desired estimates. $\square$

Proposition 4.4. *Let $T_\nu(t)$ be the operator defined by* (4.21).

(1) *Let $\nu \in [n+1,\infty)$ be an integer. Then, the operator $T_\nu(t)$ is bounded from L^q to L^p for $1 < q \le p < \infty$ and $t > 0$, and it satisfies*

$$\|T_\nu(t)h\|_p \le C(1+t^m)^{(1+1/p-1/q)}\|h\|_q.$$

(2) *Let $\nu \in [(n+1)/2, n]$ be an integer. Then, the operator $T_\nu(t)$ is bounded from L^q to L^p for $1 < q \le p < \infty$ and $t > 0$, and it satisfies*

$$\|T_\nu(t)h\|_p \le C(1+t^m)t^{[2/q-1]_+(\nu-n)}\|h\|_q$$

where $[2/q-1]_+ \equiv \max(2/q-1,0)$.

Proof. (1) We may assume that $0 < 1/p < 1/q < 1$. Set $s \equiv 1/q - 1/p$ and $p_1 \equiv (1-s)p > 1$. Lemma 4.4 and Proposition 4.3 show that

$$\|T_\nu(t)h\|_\infty \le C\|h\|_1, \tag{4.27}$$

$$\|T_\nu(t)h\|_{p_1} \le C(1+t^m)\|h\|_{p_1}. \tag{4.28}$$

We use the interpolation theory (Lemma 2.6) between inequalities (4.27) and (4.28) to obtain that

$$T_\nu(t) \in \mathscr{L}(L^q, L^p), \quad \|T_\nu(t)h\|_p \le C(1+t^m)^{1-s}\|h\|_q,$$

where $1/p = (1-s)/p_1$, $1/q = s + (1-s)/p_1$. Thus we have obtained the desired estimate.

(2) For $0 \le \operatorname{Re} z \le 1$, we define the operator $T_\nu(t;z)$ by

$$T_\nu(t;z)h = T_\nu(t)\mathscr{F}^{-1}((1+|\xi|^2)^{\nu z/2}\hat{h}(\xi)).$$

For $\operatorname{Re} z = 1$, the inequality

$$\|T_\nu(t;z)h\|_2 \le C\|h\|_2 \tag{4.29}$$

holds. Proposition 4.2 shows that

$$\|T_\nu(t;0)h\|_{\mathrm{BMO}} = \|T_\nu(t)h\|_{\mathrm{BMO}} \le Ct^{\nu-n}\|h\|_1.$$

(see [**3**], [**18**]).

The operator M defined by $M(y)h = \mathscr{F}^{-1}((1+|\xi|^2)^{ivy/2}\hat{h}(\xi))$ is a bounded operator on BMO with norm $(1+|y|)^k$ where k is a smallest integer $> n/2$ ([**3**], [**18**]). Hence for complex z such that $\mathrm{Re}\, z = 0$, it follows that the operator $T_\nu(t;z)$ is bounded from L^1 to BMO and

$$\|T_\nu(t;z)h\|_{\mathrm{BMO}} \le C(1+|\mathrm{Im}\, z|)^k t^{\nu-n}\|h\|_1. \tag{4.30}$$

We use Stein interpolation theorem (Lemma 2.6) between (4.29) and (4.30) to obtain

$$\|T_\nu(t)h\|_{(1-\theta)\nu,p} \le C\|T_\nu(t;1-\theta)h\|_p \le Ct^{\theta(\nu-n)}\|h\|_q$$

where $1/p = (1-\theta)/2$, $1/q = (1+\theta)/2$ and $0 \le \theta < 1$. By Sobolev embedding theorem, $H_p^{(1-\theta)\nu} \subset L^{p'}$ for $p \le p' \le \infty$ with continuous injection (see [**26**], e.g.).

Therefore we obtain

$$\|T_\nu(t)h\|_p \le Ct^{(2/q-1)(\nu-n)}\|h\|_q \tag{4.31}$$

for $1 < q \le 2$, $0 \le 1/p \le 1 - 1/q$. By interpolation between the estimates (4.31) and (4.22), we see that $T_\nu(t) \in \mathscr{L}(L^q; L^p)$ for $1 < q \le p < \infty$ with norm

$$\|T_\nu(t)h\|_p \le Ct^{[2/q-1]_+(\nu-n)}(1+t^m)\|h\|_q. \tag{4.32}$$

By the estimate (4.32) we obtain the desired result. $\square$

Proof of Theorem 4.1. For $1 < q < \infty$, set $L_0^q = \{h \in L^q; \mathrm{supp}\, \hat{h} \subset [3a/2, \infty)\}$. It is well known that the operators U_1 and U_2 defined by

$$U_1 h = \mathscr{F}^{-1}(\Theta(\xi)|\xi|\hat{h}(\xi)), \quad U_2 h = \mathscr{F}^{-1}\left(\frac{|\xi|}{\lambda}\hat{h}(\xi)\right)$$

are bounded from L_0^q to L_0^q (see e.g. [**26**]). Now we may assume that $\mathrm{supp}\,\hat{\varphi}_j \subset [3a/2, \infty)$ for $j = 0, 1$.

For $0 \le k \le 2n+1$, $l = 0, 1$, $j = 0, 1$, the equality

$$\mathscr{F}^{-1}\left(\chi_2(\rho)\frac{t^k\Theta(\xi)^k}{\lambda^l}e^{it\rho}\hat{\varphi}_j\right) = t^k T_{k+l}(t)U_1^k U_2^l \varphi_j$$

holds. When $k + l \ge n+1$, $k \le 2n+1$, by Proposition 4.4(1), we see that

$$\left\|\mathscr{F}^{-1}\left(\chi_2(\rho)\frac{t^k\Theta(\xi)^k}{\lambda^l}e^{it\rho}\hat{\varphi}_j\right)\right\|_p \le C(1+t)^{2n+m+1}\|\varphi_j\|_q, \tag{4.33}$$

for $1 < q \le p < \infty$. When $(n+1)/2 \le k+l \le n$, $l = 0, 1$ and $j = 0, 1$, we see that

$$\left\|\mathscr{F}^{-1}\left(\chi_2(\rho)\frac{t^k\Theta(\xi)^k}{\lambda^l}e^{it\rho}\hat{\varphi}_j\right)\right\|_p \le C(1+t)^m t^{[2/q-1]_+(k+l-n)}t^k\|\varphi_j\|_q,$$

for $1 < q \le p < \infty$, and that

$$0 \le \left[\frac{2}{q}-1\right]_+(k+l-n)+k \le n.$$

Therefore we see that

$$\left\| \mathscr{F}^{-1}\left(\chi_2(\rho) \frac{t^k \Theta(\xi)^k}{\lambda^l} e^{it\rho} \hat{\varphi}_j \right) \right\|_p \le C(1+t)^{n+m} \|\varphi_j\|_q. \tag{4.34}$$

By (4.33)–(4.34), we see that

$$\begin{aligned} &\|\mathscr{F}^{-1}\{\chi_2(\xi)(M_0(t,\xi)\hat{\varphi}_0 + M_1(t,\xi)\hat{\varphi}_1 - \hat{v}(t,\xi))\}\|_p \\ &\qquad \le C(1+t)^{2n+m+1}(\|\varphi_0\|_q + \|\varphi_1\|_q). \end{aligned} \tag{4.35}$$

By Lemma 4.2 and the estimate (4.35), we obtain the desired estimate. □

For the application to the nonlinear problem (1.10), we will rewrite Theorems 1.1–1.2 (more precisely Theorems 3.1 and 4.1).

Choose and fix radial functions $0 \le \chi(\xi),\ \chi_3(\xi) \le 1$ of class C^∞ satisfying

$$\chi(\xi) = 1 \quad (|\xi| \le 3a), \quad \chi(\xi) = 0 \quad (|\xi| \ge 4a)$$

and

$$\chi_3(\xi) = 1 \quad (|\xi| \le 4a), \quad \chi_3(\xi) = 0 \quad (|\xi| \ge 5a).$$

Then the following equalities hold.

$$\chi(\xi)\chi_3(\xi) = \chi(\xi), \quad (1-\chi(\xi))\chi_2(\xi) = 1 - \chi(\xi),$$

where χ_2 is the cut-off function in Theorem 4.1. To derive a priori estimate of the solution to (1.10) we consider linear damped wave equation

$$\partial_t^2 w - \Delta w + 2a\partial_t w = g, \quad w(0,x) = \varphi_0(x), \quad \partial_t w(0,x) = \varphi_1(x) \tag{4.36}$$

for $(t,x) \in (0,\infty) \times \boldsymbol{R}^n$. The Duhanmel principle shows that the problem (4.36) is equivalent to the following integral equation

$$w(t) = (\partial_t S(t) + 2aS(t))\varphi_0 + S(t)\varphi_1 + \int_0^t S(t-s)g(s)\,ds,$$

where

$$S(t)h \equiv e^{-at}\mathscr{F}^{-1}\left(\frac{\sin t\sqrt{|\xi|^2 - a^2}}{\sqrt{|\xi|^2 - a^2}} \hat{h}(\xi) \right). \tag{4.37}$$

By Theorems 3.1 and 4.1, we see that following estimates hold.

Proposition 4.5. *The operator* $S(t)$ *defined by* (4.37) *satisfies the following estimates.*

(1) *Let* $1 \le q \le p \le \infty$ *and* j *and* k *be non-negative integers, then,*

$$\begin{aligned} &\|\partial_t^j(-\Delta)^k S(t)\mathscr{F}^{-1}(\chi\hat{h})\|_p \\ &\qquad \le C(p,q,j,k)(1+t)^{-(n/2)(1/q-1/p)-j-k}\|\mathscr{F}^{-1}(\chi\hat{h})\|_q. \end{aligned} \tag{4.38}$$

(2) *Assume that* $n = 2, 3$. *Then, the inequality*

$$\|S(t)\mathscr{F}^{-1}((1-\chi)\hat{h})\|_p \le C_p e^{-at}(1+t)^{3n+3}\|\mathscr{F}^{-1}((1-\chi)\hat{h})\|_p$$

holds for $1 < p < \infty$.

(3) *Assume that* $n = 4, 5$. *Then, the inequality*

$$\|S(t)\mathscr{F}^{-1}((1-\chi(\xi))\hat{h})\|_p \le C_p e^{-at}(1+t)^{3n+3}\|\mathscr{F}^{-1}((1-\chi(\xi))\hat{h})\|_p$$

holds for $4/3 < p < 4$.

Remark 4.1. Kawashima, Nakao and Ono [**13**] obtained the similar L^p-L^q estimates to ones in Proposition 4.5. Especially they obtained the estimate (4.38) with $1 \le q \le 2$, $2 \le p \le \infty$. In the next section we use the estimate (4.38) also in the case where $p = q = 1$.

For the proof of Proposition 4.5(3), we consider the following Cauchy problem

$$\partial_t^2 v - \Delta v = 0, \quad v(0,x) = 0, \quad v_t(0,x) = h(x), \quad t > 0,\ x \in \boldsymbol{R}^n \tag{4.39}$$

for $n = 4, 5$. By (4.25) we claim that the operator $T_s(t)$ is bounded from H_p^1 to L^p, and it satisfies

$$\|T_s(t)h\|_p \le C_p(t+t^2)\|h\|_{1,p} \quad \text{for } 1 < p < \infty.$$

The operator $\tilde{T}_s(t)$ defined by $\tilde{T}_s(t)h = T_s(t)\mathscr{F}^{-1}(\chi_2\hat{h})$ is bounded from H_p^1 to L^p for $1 < p < \infty$, and it satisfies $\tilde{T}_s(t)\mathscr{F}^{-1}((1-\chi)\hat{h}) = T_s(t)\mathscr{F}^{-1}((1-\chi)\hat{h})$ and

$$\|\tilde{T}_s(t)h\|_p \le C_p(t+t^2)\|h\|_{1,p}, \quad (1 < p < \infty), \tag{4.40}$$

because the operator defined by $h \mapsto \mathscr{F}^{-1}(\psi\hat{h})$ is bounded from L^p to L^p $(1 < p < \infty)$.

It is well-known that $(1+|\xi|^2)^{iy}$ is a Fourier multiplier on L^p $(1 < p < \infty)$ and it satisfies

$$\|\mathscr{F}^{-1}((1+|\xi|^2)^{iy}\hat{h})\|_p \le C(p)(1+y^2)^N\|h\|_p, \tag{4.41}$$

where N is a positive constant that is determined by space dimension n (see e.g. [**2**]). Introduce the analytic family of operators $U(z;t)$ as follows

$$U(z;t)h = \tilde{T}_s(t)\mathscr{F}^{-1}((1+|\xi|^2)^{z-1/2}\hat{h}).$$

Fix sufficiently small positive constant ε, then the estimates (4.40) and (4.41) show

$$\|U(iy;t)h\|_{1/\varepsilon} \le C_\varepsilon(1+y^2)^N(t+t^2)\|h\|_{1/\varepsilon}, \tag{4.42}$$

$$\|U(iy;t)h\|_{1/(1-\varepsilon)} \le C_\varepsilon(1+y^2)^N(t+t^2)\|h\|_{1/(1-\varepsilon)}. \tag{4.43}$$

On the other hand, $\tilde{T}_s(t)$ is a bounded operator from L^2 to H_2^1 and it satisfies

$$\|\tilde{T}_s(t)h\|_{1,2} \le \|h\|_2.$$

Therefore we see that

$$\|U(1+iy;t)h\|_2 \le \|h\|_2. \tag{4.44}$$

We may use interpolation theory (Lemma 2.6) between (4.42)–(4.43) and (4.44) with $\theta = 1/2$ to obtain that

$$\|\tilde{T}_s(t)h\|_p = \|U(1/2;t)h\|_p \le C(\varepsilon)(1+t^2)\|h\|_p, \tag{4.45}$$

where $1/p = (1+2\varepsilon)/4$ or $1/p = (3-2\varepsilon)/4$. Therefore we have obtained the next lemma.

Lemma 4.5. *Let $n = 4$ or 5, and let $1/4 < 1/p < 3/4$. Then, the operator $\tilde{T}_s(t)$ is bounded from L^p to L^p, and it satisfies*

$$\|\tilde{T}_s(t)h\|_p \le C(q)(1+t^2)\|h\|_p.$$

Moreover, the operator $\tilde{T}_c(t)$ defined by $\tilde{T}_c(t)h = T_c(t)\mathscr{F}^{-1}((1-\chi)\hat{h})$ is bounded from H_p^1 to L^p, and it satisfies

$$\|\tilde{T}_c(t)h\|_p \le C(p)(1+t^2)\|h\|_{1,p}.$$

Proof of Proposition 4.5. Let $u(t,\cdot) \equiv S(t)h$. Then $u(t,\cdot)$ is a solution of (4.36) with $g = 0$, $\varphi_0 = 0$ and $\varphi_1 = h$.

(1) By Theorem 1.1 with $\varepsilon = 1/4$ and $\chi = \chi_3$, we see that

$$\begin{aligned}\|\partial_t^j(-\varDelta)^k\mathscr{F}^{-1}\{\chi(\cdot)(\hat{u}(t,\cdot)-\hat{\phi}(t,\cdot))\}\|_p &= \|\partial_t^j(-\varDelta)^k\mathscr{F}^{-1}\{\chi_3(\cdot)\chi(\cdot)(\hat{u}(t,\cdot)-\hat{\phi}(t,\cdot))\}\|_p \\ &\le C(1+t)^{-n/2(1/q-1/p)-3/4-j-k}\|\mathscr{F}^{-1}(\chi\hat{h})\|_q\end{aligned}$$

for $1 \le q \le p \le \infty$, where ϕ is the solution of the heat equation

$$\partial_t\phi - \frac{1}{2a}\varDelta\phi = 0, \quad \phi(0,\cdot) = \frac{h}{2a}, \quad \text{in } (0,\infty)\times\boldsymbol{R}^n.$$

The estimate (1.6) shows that

$$\|\mathscr{F}^{-1}\partial_t^j(-\varDelta)^k(\chi_3(\cdot)\chi(\cdot)\hat{\phi}(t,\cdot))\|_p \le C(1+t)^{-n/2(1/q-1/p)-j-k}\|\mathscr{F}^{-1}(\chi\hat{h})\|_q$$

for $1 \le q \le p \le \infty$. Therefore we have obtained the desired result.

(2) When $n = 2, 3$, Theorem 1.2 shows that

$$M_1(t,\xi) = \frac{\sin t|\xi|}{|\xi|}\frac{|\xi|}{\sqrt{|\xi|^2 - a^2}}.$$

The function $v(t,\cdot)$ defined by

$$\hat{v}(t,\xi) = \chi_2(\xi)M_1(t,\xi)(1-\chi(\xi))\hat{h}(\xi)$$

is a solution to the wave equation (4.39) with replaced h by

$$\mathscr{F}^{-1}\left(\chi_2(\xi)(1-\chi(\xi))\frac{|\xi|}{\sqrt{|\xi|^2-a^2}}\hat{h}(\xi)\right) \in L^p.$$

Since

$$\chi_2(\xi)\frac{|\xi|}{\sqrt{|\xi|^2-a^2}}$$

is a Fourier-multiplier on L^p $(1 < p < \infty)$ (see e.g. [**2**]), Proposition 4.3 shows that

$$\|v(t,\cdot)\|_p \le C(1+t)\|\mathscr{F}^{-1}((1-\chi)\hat{h})\|_p$$

for $1 < p < \infty$. Theorem 4.1 with $\varphi_0 = 0$, $\varphi_1 = \mathscr{F}^{-1}((1-\chi)\hat{h})$ gives

$$\|S(t)\mathscr{F}^{-1}((1-\chi)\hat{h}) - e^{-at}v(t,\cdot)\|_p \le Ce^{-at}(1+t)^{3n+3}\|\mathscr{F}^{-1}((1-\chi)\hat{h})\|_p$$

for $1 < p < \infty$. Hence

$$\begin{aligned}\|S(t)\mathscr{F}^{-1}((1-\chi)\hat{h})\|_p &\le \|S(t)\mathscr{F}^{-1}((1-\chi)\hat{h}) - e^{-at}v(t,\cdot)\|_p + e^{-at}\|v(t,\cdot)\|_p \\ &\le Ce^{-at}(1+t)^{3n+3}\|\mathscr{F}^{-1}((1-\chi)\hat{h})\|_p\end{aligned}$$

for $1 < p < \infty$. Thus we have obtained the desired result.

(3) When $n = 4, 5$, Theorem 1.2 shows that

$$M_1(t,\xi) = \frac{\sin t|\xi|}{|\xi|}\frac{|\xi|}{\sqrt{|\xi|^2 - a^2}} - t\cos t|\xi|\frac{1}{|\xi|^2}\frac{|\xi|}{\sqrt{|\xi|^2 - a^2}}|\xi|\Theta(\xi).$$

Define the functions v_1 and v_2 by

$$\hat{v}_2(t,\xi) = \chi_2(\xi)\frac{\sin t|\xi|}{|\xi|}\frac{|\xi|}{\sqrt{|\xi|^2 - a^2}}(1-\chi(\xi))\hat{h}(\xi),$$

$$\hat{v}_1(t,\xi) = \chi_2(\xi)\cos t|\xi|\frac{1}{|\xi|^2}\frac{|\xi|}{\sqrt{|\xi|^2 - a^2}}|\xi|\Theta(\xi)(1-\chi(\xi))\hat{h}(\xi).$$

Then v_i $(i = 1, 2)$ are solutions of wave equation with initial data

$$v_1(0,\cdot) = 0, \quad \partial_t v_1(0,\cdot) = \mathscr{F}^{-1}\left(\chi_2(\xi)\frac{|\xi|}{\sqrt{|\xi|^2 - a^2}}(1-\chi(\xi))\hat{h}\right) \in L^p,$$

$$v_2(0,\cdot) = \mathscr{F}^{-1}\left(\chi_2(\xi)\frac{1}{|\xi|^2}\frac{|\xi|}{\sqrt{|\xi|^2 - a^2}}|\xi|\Theta(\xi)\right) \in H_p^2, \quad \partial_t v_2(0,\cdot) = 0.$$

Lemma 4.5 shows that

$$\|v_i(t)\|_p \le C_p(1+t^2)\|\mathscr{F}^{-1}((1-\chi)\hat{h})\|_p, \quad (4/3 < p < 4). \tag{4.46}$$

Theorem 4.1 with $\varphi_0 = 0$, $\varphi_1 = \mathscr{F}^{-1}((1-\chi)\hat{h})$ gives

$$\|S(t)\mathscr{F}^{-1}((1-\chi)\hat{h}) - e^{-at}(v_1(t,\cdot) - tv_2(t))\|_p \le Ce^{-at}(1+t)^{3n+3}\|\mathscr{F}^{-1}((1-\chi)\hat{h})\|_p \tag{4.47}$$

for $1 < p < \infty$. The estimates (4.46) and (4.47) show

$$\begin{aligned}&\|S(t)\mathscr{F}^{-1}((1-\chi)\hat{h})\|_p \\ &\quad \le \|S(t)\mathscr{F}^{-1}((1-\chi)\hat{h}) - e^{-at}(v_1(t,\cdot) - tv_2(t))\|_p + e^{-at}\|v_1(t,\cdot) - tv_2(t)\|_p \\ &\quad \le Ce^{-at}(1+t)^{3n+3}\|\mathscr{F}^{-1}((1-\chi)\hat{h})\|_p\end{aligned}$$

for $4/3 < p < 4$. Thus we have obtained the desired estimate. □

5. Application to the semi-linear problem.

We now apply Proposition 4.5 to the Cauchy problem for the semilinear damped wave equation (1.10). The local well-posedness of the problem is well known. The following classical result can be found in Strauss [**29**] for example.

Lemma 5.1. *Assume that Hypothesis H holds and assume that* $\sigma \le 2/(n-2)$ *when* $n \ge 3$. *Let* $\varphi_0 \in H_2^1$, $\varphi_1 \in L^2$. *Then the problem* (1.10) *possesses a unique local solution* u *such that*

$$u \in C^1([0,T); L^2) \cap C([0,T); H_2^1).$$

Here the solution can be continued beyond the interval $[0,T)$ *if* $\sup_{0 \le t < T} \|u(t)\|_{1,2} < \infty$. *Moreover, if* $\varphi_0 \in H_2^2$, $\varphi_1 \in H_2^1$, *then*

$$u \in C^2([0,T); L^2) \cap C^1([0,T); H_2^1) \cap C([0,T); H_2^2).$$

In view of this result, global existence of a solution follows from the boundedness of its H_2^1 norm.

Now we will prove Theorem 1.3. Here and after we fix $q = 1 + 1/\sigma$ and $q' = 1 + \sigma$. The assumptions in Theorem 1.3 imply that

$$\frac{4}{3} < q' < q < 4, \quad \frac{1}{q} + \frac{1}{q'} = 1.$$

We construct the approximate solutions $\{u_j\}_{j=0,1,\dots}$ to the Cauchy problem (1.10) as follows.

Let $u_{-1} = 0$, and let u_{j+1} be a solution of the following Cauchy problem

$$\partial_t^2 u_{j+1} - \Delta u_{j+1} + 2a\partial_t u_{j+1} = f(u_j), \quad u_{j+1}(0,x) = \varphi_0(x), \quad \partial_t u_{j+1}(0,x) = \varphi_1(x) \tag{5.1}$$

for $(t,x) \in (0,\infty) \times \boldsymbol{R}^n$ for $j \ge -1$. Then (5.1) is equivalent to the following system of the integral equations:

$$u_{j+1}^1(t,\cdot) = u_0^1(t,\cdot) + \int_0^t S(t-\tau) f^1(u_j(\tau,\cdot))\,d\tau, \tag{5.2}$$

$$u_{j+1}^2(t,\cdot) = u_0^2(t,\cdot) + \int_0^t S(t-\tau) f^2(u_j(t,\cdot))\,d\tau \tag{5.3}$$

for $j \ge 0$, where

$$u_{j+1}^1(t,\cdot) = \mathscr{F}^{-1}(\chi(\xi)\hat{u}_{j+1}(t,\cdot)), \quad u_{j+1}^2(t,\cdot) = \mathscr{F}^{-1}((1-\chi(\xi))\hat{u}_{j+1}(t,\cdot)),$$

$$f^1(u_j(t,\cdot)) = \mathscr{F}^{-1}(\chi(\xi)\hat{f}(u_j(t,\cdot))), \quad f^2(u_j(t,\cdot)) = \mathscr{F}^{-1}((1-\chi(\xi))\hat{f}(u_j(t,\cdot)))$$

and χ denotes the cut-off function in Proposition 4.5.

Proposition 5.1. *Assume that Hypothesis H holds. Let* $4 \le n \le 5$, $2/n < \sigma \le 2/(n-2)$, $\sigma < 1$ *and*

$$(\varphi_0, \varphi_1) \in Z_1 \equiv (H_2^2 \cap H_{1+1/\sigma}^1 \cap H_{1+\sigma}^1 \cap L^1) \times (H_2^1 \cap L^{1+1/\sigma} \cap L^{1+\sigma} \cap L^1).$$

If $\|\varphi_0, \varphi_1\|_{Z_1}$ *is sufficiently small, then there exists a small positive constant* η *such that*

$$C_1\|\varphi_0, \varphi_1\|_{Z_1} \leq \eta \leq C_2\|\varphi_0, \varphi_1\|_{Z_1},$$

and the approximate solutions $\{u_j\}_{j=0,1,\dots}$ *constructed in* (5.1) *exist for* $t \in [0, \infty)$ *and they satisfy that*

$$u_j^1 \in C([0, \infty); L^\infty \cap L^1), \quad u_j^2 \in C([0, \infty); H_2^2 \cap L^q \cap L^{q'}), \tag{5.4}$$

hence

$$u_j \in C([0, \infty); H_2^2) \cap C^1([0, \infty); H_2^1) \cap C^2([0, \infty); L^2) \tag{5.5}$$

for $j = 0, 1, \dots$. *Moreover, they satisfy the following estimates;*

(1) $\|u_j^1(t, \cdot)\|_\infty \leq 2\eta(1 + t)^{-n/2}$, $\|u_j^1(t, \cdot)\|_1 \leq 2\eta$.

(2) $\|u_j^2(t, \cdot)\|_q \leq 2\eta(1 + t)^{-\beta_1}$, $\|u_j^2(t, \cdot)\|_{q'} \leq 2\eta(1 + t)^{-\beta_2}$, *where*

$$\beta_1 = \frac{n}{2}\left(1 + \sigma - \frac{1}{q}\right), \quad \beta_2 = \frac{n}{2}\left(1 + \sigma - \frac{1}{q'}\right). \tag{5.6}$$

(3) $\|\partial_t^l D^k u_j^1(t, \cdot)\|_2 \leq 2\eta(1 + t)^{-\nu(k,l)}$ *for* $k + l \leq 2$, *where*

$$\nu(k, l) = \frac{n}{4} + \frac{k}{2} + \min\left(l, \frac{n\sigma}{2}\right).$$

(4) $\|\partial_t^l D^k u_j^2(t, \cdot)\|_2 \leq 2\eta(1 + t)^{-(n/2)\cdot(\sigma+1/2)-1/2}$ *for* $k + l \leq 2$.

The Proposition 5.1 is a consequence of the following lemmas.

The next lemma is a direct consequence of Proposition 4.5.

Lemma 5.2. *Under the same assumptions and notations as ones in Proposition* 5.1, *the solution* u_0 *of* (4.36) *with* $g = 0$ *satisfies*

$$u_0^1 \in C([0, \infty); L^1 \cap L^\infty), \quad u_0^2 \in C([0, \infty); L^q \cap L^{q'} \cap H_2^2),$$

hence

$$u_0 \in C([0, \infty); H_2^2) \cap C^1([0, \infty); H_2^1) \cap C^2([0, \infty); L^2).$$

Moreover, there exists a small positive constant η *such that*

$$C_1\|\varphi_0, \varphi_1\|_{Z_1} \leq \eta \leq C_2\|\varphi_0, \varphi_1\|_{Z_1}$$

and the following estimates hold.

(1) $\|u_0^1(t, \cdot)\|_\infty \leq \eta(1 + t)^{-n/2}$, $\|u_0^1(t, \cdot)\|_1 \leq \eta$.

(2) $\|u_0^2(t, \cdot)\|_q \leq \eta(1 + t)^{-\beta_1}$, $\|u_0^2(t, \cdot)\|_{q'} \leq \eta(1 + t)^{-\beta_2}$, *where* β_1 *and* β_2 *are defined by* (5.6).

(3) $\|\partial_t^l D^k u_0^1(t, \cdot)\|_2 \leq \eta(1 + t)^{-n/4-k/2-l}$, $\|\partial_t^l D^k u_0^2(t, \cdot)\|_{2,2} \leq \eta(1 + t)^{-(n/2)(\sigma+1/2)-1/2}$ *for* $k + l \leq 2$.

If $u_j^1 \in C([0, \infty); L^\infty \cap L^1)$ and $u_j^2 \in C([0, \infty); H_2^2 \cap L^q \cap L^{q'})$, then

$$u_j = u_j^1 + u_j^2 \in C([0, \infty); L^q \cap L^{q'} \cap H_2^2).$$

This implies that;

LEMMA 5.3. *Assume that* $u_j = u_j^1 + u_j^2$ $(j \geq 0)$ *satisfies* (5.4)–(5.5) *and decay estimates in Proposition* 5.1. *Then*,

$$f(u_j) \in C([0,\infty); H_2^1 \cap L^q \cap L^{q'} \cap L^1)$$

under the same assumptions and notations as ones in Proposition 5.1. *Moreover, the following estimates hold*;

(1) $\|f(u_j(t,\cdot))\|_q \leq C\eta^{1+\sigma}(1+t)^{-\beta_1}$.
(2) $\|f(u_j(t,\cdot))\|_{q'} \leq C\eta^{1+\sigma}(1+t)^{-\beta_2}$.
(3) $\|D^k f(u_j(t,\cdot))\|_2 \leq C\eta^{1+\sigma}(1+t)^{-(n/2)(\sigma+1/2)-k/2}$ *for* $k = 0, 1$.
(4) $\|f(u_j(t,\cdot))\|_1 \leq C\eta^{1+\sigma}(1+t)^{-n\sigma/2}$.

PROOF. Hypothesis H gives

$$\begin{aligned}
&\|f'(u_j(t_2,\cdot))Du_j(t_2,\cdot) - f'(u_j(t_1,\cdot))Du_j(t_1,\cdot)\|_2 \\
&\quad \leq A(\| |u_j(t_2,\cdot) - u_j(t_1,\cdot)|^\sigma Du_j(t_2,\cdot)\|_2 + \| |u_j(t_1,\cdot)|^\sigma (Du_j(t_2,\cdot) - Du_j(t_1,\cdot))\|_2) \\
&\quad \leq C(\|u_j(t_2,\cdot)\|_{2,2} + \|u_j(t_1,\cdot)\|_{2,2})\|u_j(t_2,\cdot) - u_j(t_1,\cdot)\|_{2,2}^\sigma. \qquad (5.7)
\end{aligned}$$

The estimate (5.7) shows that

$$f'(u_j(t,\cdot))Du_j(t,\cdot) \in C([0,\infty); L^2).$$

Similar calculations to ones in (5.7) show that

$$f(u_j(t,\cdot)) \in C([0,\infty); L^2 \cap L^q \cap L^1).$$

We will estimate several norms of $f(u_j(t,\cdot))$.

(1) By Hypothesis H and assumptions of the lemma, we see that

$$\begin{aligned}
\|f(u_j(t,\cdot))\|_q &\leq A\|u_j(t,\cdot)\|_{(1+\sigma)q}^{1+\sigma} \\
&\leq C(\|u_j^1(t,\cdot)\|_{(1+\sigma)q}^{1+\sigma} + \|u_j^2(t,\cdot)\|_{(1+\sigma)q}^{1+\sigma}) \\
&\leq C(\|u_j^1(t,\cdot)\|_\infty^{1+\sigma-1/q}\|u_j^1(t,\cdot)\|_1^{1/q} + \|u_j^2(t,\cdot)\|_{2,2}^{1+\sigma}) \\
&\leq C\eta^{1+\sigma}(1+t)^{-\beta_1},
\end{aligned}$$

where we have used Sobolev's lemma, $H_2^2 \subset L^{q(1+\sigma)}$, because

$$\frac{1}{2} - \frac{2}{n} \leq \frac{1}{q(1+\sigma)} \leq \frac{1}{2}$$

for $n \leq 5$. Thus we have obtained the desired estimate.

(2) By Hypothesis H and the assumptions of the lemma, we see that

$$\|f(u_j(t,\cdot))\|_{q'} \leq CA(\|u_j^1(t,\cdot)\|_\infty^{1+\sigma-1/(1+\sigma)}\|u_j^1(t,\cdot)\|_1^{1/(1+\sigma)} + \|u_j^2(t,\cdot)\|_{q'(1+\sigma)}), \qquad (5.8)$$

and

$$\|u_j^1(t,\cdot)\|_\infty^{1+\sigma-1/(1+\sigma)}\|u_j^1(t,\cdot)\|_1^{1/(1+\sigma)} \leq C\eta^{1+\sigma}(1+t)^{-\beta_2}. \qquad (5.9)$$

If $q'(1+\sigma) = (1+\sigma)^2 \geq 2$, then

$$\|u_j^2(t,\cdot)\|_{q'(1+\sigma)}^{1+\sigma} \le C\|u_j^2(t,\cdot)\|_{2,2}^{1+\sigma} \le C\eta^{1+\sigma}(1+t)^{-\beta_2}, \tag{5.10}$$

where we have used $H_2^2 \subset L^{q'(1+\sigma)}$, because

$$\frac{1}{2}-\frac{2}{n} \le \frac{1}{q'(1+\sigma)} \le \frac{1}{2}.$$

If $q'(1+\sigma) < 2$, Hölder's inequality shows that

$$\|u_j^2(t,\cdot)\|_{q'(1+\sigma)}^{1+\sigma} \le C\|u_j^2(t,\cdot)\|_2^{\theta_1(1+\sigma)}\|u_j^2(t,\cdot)\|_{q'}^{(1-\theta_1)(1+\sigma)} \le C\eta^{1+\sigma}(1+t)^{-\beta_2}, \tag{5.11}$$

where $\theta_1 \in (0,1)$ is determined by

$$\frac{\theta_1}{2}+\frac{1-\theta_1}{1+\sigma} = \frac{1}{q'(1+\sigma)}.$$

The estimates (5.8)–(5.11) show the desired result.

(3) Since

$$\frac{1}{2} = \frac{1/2}{q}+\frac{1/2}{q'},$$

the estimates in Lemma 5.1(1)–(2) and Hölder's inequality show that

$$\|f(u_j(t,\cdot))\|_2 \le C\|f(u_j(t,\cdot))\|_q^{1/2}\|f(u_j(t,\cdot))\|_{q'}^{1/2} \le C\eta^{1+\sigma}(1+t)^{-(n/2)(\sigma+1/2)}.$$

Hence it is sufficient to estimate $\|f'(u_j(t,\cdot))Du_j(t,\cdot)\|_2$ for the proof of Lemma 5.3(3) with $k=1$. By the estimates in Proposition 5.1, we see that

$$\|D^k u_j(t,\cdot)\|_2 \le C\eta(1+t)^{-n/4-k/2}$$

for $0 \le k \le 2$. Let $\theta_2 = n/2 - 1/\sigma \in (0,1]$. Then, Lemma 2.7 shows that

$$\|Du_j(t,\cdot)\|_{2n/(n-2)} \le C\|D^2 u_j(t,\cdot)\|_2, \tag{5.12}$$

$$\|u_j(t,\cdot)\|_{n\sigma} \le C\|Du_j(t,\cdot)\|_2^{\theta_2}\|u_j(t,\cdot)\|_2^{1-\theta_2}. \tag{5.13}$$

Hypothesis H, Hölder's inequality and the estimates (5.12)–(5.13) show that

$$\begin{aligned}\|f'(u_j(t,\cdot))Du_j(t,\cdot)\|_2 &\le CA\|u_j(t,\cdot)\|_{n\sigma}^{\sigma}\|Du_j(t,\cdot)\|_{2n/(n-2)} \\ &\le C\|u_j(t,\cdot)\|_2^{\sigma(1-\theta_2)}\|Du_j(t,\cdot)\|_2^{\sigma\theta_2}\|D^2u_j(t,\cdot)\|_2 \\ &\le C\eta^{1+\sigma}(1+t)^{-(n/2)(\sigma+1/2)-1/2}.\end{aligned}$$

Therefore we have obtained the desired result.

(4) Hypothesis H and the assumptions in the lemma give

$$\begin{aligned}\|f(u_j(t,\cdot))\|_1 &\le CA(\|u_j^1(t,\cdot)\|_{1+\sigma}^{1+\sigma} + \|u_j^2(t,\cdot)\|_{q'}^{1+\sigma}) \\ &\le C(\|u_j^1(t,\cdot)\|_\infty^{\sigma}\|u_j^1(t,\cdot)\|_1 + \|u_j^2(t,\cdot)\|_{q'}^{1+\sigma}) \\ &\le C\eta^{1+\sigma}(1+t)^{-n\sigma/2}.\end{aligned}$$

Thus we have obtained the desired estimate. □

PROOF OF PROPOSITION 5.1. We prove the proposition by induction. Lemma 5.2 shows that u_0^1 and u_0^2 satisfy the results of the proposition. Assume that u_j^1 and u_j^2 satisfy the results of the proposition for some $j \geq 0$. Then, Lemma 5.3 shows that

$$f(u_j) \in C([0,\infty); H_2^1 \cap L^q \cap L^{q'} \cap L^1).$$

Hence, by (5.2)–(5.3), we claim that

$$u_{j+1}^1 \in C([0,\infty); L^\infty \cap L^1), \quad u_{j+1}^2 \in C([0,\infty); H_2^2 \cap L^q \cap L^{q'}).$$

This implies that

$$u_{j+1} \in C([0,\infty); H_2^2) \cap C^1([0,\infty); H_2^1) \cap C^2([0,\infty); L^2).$$

Now we proceed with further estimates.

(1) The equation (5.2) gives

$$\|u_{j+1}^1(t,\cdot)\|_\infty \leq \|u_0^1(t,\cdot)\|_\infty + \int_0^t \|S(t-\tau)f^1(u_j(\tau,\cdot))\|_\infty \, d\tau. \tag{5.14}$$

Proposition 4.5(1) with $p = \infty$, $q = 1$ and Lemma 5.3(4) show that

$$\begin{aligned}
\int_0^{t/2} \|S(t-\tau)f^1(u_j(\tau,\cdot))\|_\infty \, d\tau &\leq C \int_0^{t/2} (1+t-\tau)^{-n/2} \|f(u_j(\tau))\|_1 \, d\tau \\
&\leq C\eta^{1+\sigma} \int_0^{t/2} (1+t-\tau)^{-n/2}(1+\tau)^{-n\sigma/2} \, d\tau \\
&\leq C\eta^{1+\sigma}(1+t)^{-n/2},
\end{aligned} \tag{5.15}$$

where we have used the inequalities $n\sigma > 2$ and $(1+t-\tau)^{-1} \leq 2(1+t)^{-1}$ for $0 < \tau < t/2$. Proposition 4.5(1) with $p = \infty$, $q = (1+\sigma)/\sigma$ and Lemma 5.3(1) show that

$$\begin{aligned}
&\int_{t/2}^t \|S(t-\tau)f^1(u_j(\tau,\cdot))\|_\infty \, d\tau \leq C \int_{t/2}^t (1+t-\tau)^{-n\sigma/(2(1+\sigma))} \|f(u_j(\tau,\cdot))\|_q \, d\tau \\
&\leq C\eta^{1+\sigma} \int_{t/2}^t (1+t-\tau)^{-n\sigma/(2(1+\sigma))}(1+\tau)^{-(n/2)\cdot(1+\sigma-\sigma/(1+\sigma))} \, d\tau \\
&\leq C\eta^{1+\sigma}(1+t)^{-n/2} \int_{t/2}^t (1+t-\tau)^{-n\sigma/(2(1+\sigma))}(1+\tau)^{-(n/2)\cdot(\sigma-\sigma/(1+\sigma))} \, d\tau \\
&\leq C\eta^{1+\sigma}(1+t)^{-n/2} \int_{t/2}^t (1+t-\tau)^{-n/2\sigma} \, d\tau \\
&\leq C\eta^{1+\sigma}(1+t)^{-n/2},
\end{aligned} \tag{5.16}$$

where we have used the inequalities $(1+\tau)^{-1} \leq 2(1+t)^{-1}$ and $(1+\tau)^{-1} \leq (1+t-\tau)^{-1}$ for $\tau \geq t/2$, and $n\sigma > 2$. By the estimates (5.14)–(5.16) and Lemma 5.2, we see that

$$\|u_{j+1}^1(t,\cdot)\|_\infty \leq (\eta + C\eta^{1+\sigma})(1+t)^{-n/2} \leq 2\eta(1+t)^{-n/2}$$

for sufficiently small constant $\eta > 0$.

Proposition 4.5(1) with $p = q = 1$ and Lemma 5.3(1) show that

$$\begin{aligned}\|u_{j+1}^1(t,\cdot)\|_1 &\le \|u_0^1(t,\cdot)\|_1 + \int_0^t \|S(t-\tau)f^1(u_j(\tau,\cdot))\|_1\,d\tau\\ &\le \eta + C\eta^{1+\sigma}\int_0^t (1+\tau)^{-n\sigma/2}\,d\tau\\ &\le 2\eta.\end{aligned}$$

Hence we have obtained the desired estimates.

(2) The equality (5.3), Proposition 4.5(3) and Lemma 5.3(1)–(2) show that

$$\begin{aligned}\|u_{j+1}^2(t,\cdot)\|_q &\le \|u_0^2(t,\cdot)\|_q + \int_0^t Ce^{-(a/2)(t-\tau)}\|f(u_j(\tau,\cdot))\|_q\,d\tau\\ &\le \eta(1+t)^{-\beta_1} + C\eta^{1+\sigma}\int_0^t e^{-(a/2)(t-\tau)}(1+\tau)^{-\beta_1}\,d\tau\\ &\le (\eta + C\eta^{1+\sigma})(1+t)^{-\beta_1}\\ &\le 2\eta(1+t)^{-\beta_1},\end{aligned}$$

and

$$\begin{aligned}\|u_{j+1}^2(t,\cdot)\|_{q'} &\le \|u_0^2(t,\cdot)\|_{q'} + \int_0^t Ce^{-(a/2)(t-\tau)}\|f(u_j(\tau,\cdot))\|_{q'}\,d\tau\\ &\le \eta(1+t)^{-\beta_2} + C\eta^{1+\sigma}\int_0^t e^{-(a/2)(t-\tau)}(1+\tau)^{-\beta_2}\,d\tau\\ &\le 2\eta(1+t)^{-\beta_2}.\end{aligned}$$

Hence we have obtained the desired estimates.

(3) Proposition 4.5(1) with $p=2$, $q=1$ and Lemma 5.3(4) show that

$$\begin{aligned}\int_0^{t/2} \|\partial_t^l D^k S(t-\tau)f^1(u_j(\tau,\cdot))\|_2\,d\tau &\le C\int_0^{t/2}(1+t-\tau)^{-n/4-k/2-l}\|f(u_j(\tau,\cdot))\|_1\,d\tau\\ &\le C\eta^{1+\sigma}\int_0^{t/2}(1+t-\tau)^{-n/4-k/2-l}(1+\tau)^{-n\sigma/2}\,d\tau\\ &\le C\eta^{1+\sigma}(1+t)^{-n/4-k/2-l}\end{aligned}\tag{5.17}$$

for $k+l\le 2$.

Proposition 4.5 with $p=q=2$ and Lemma 5.3(3) show that

$$\begin{aligned}\int_{t/2}^t \|\partial_t^l S(t-\tau)f^1(u_j(\tau,\cdot))\|_2\,d\tau &\le C\int_{t/2}^t (1+t-\tau)^{-l}\|f(u_j(\tau,\cdot))\|_2\,d\tau\\ &\le C\eta^{1+\sigma}\int_{t/2}^t (1+t-\tau)^{-l}(1+\tau)^{-(n/2)(\sigma+1/2)}\,d\tau\\ &= C\eta^{1+\sigma}\int_{t/2}^t (1+t-\tau)^{-l}(1+\tau)^{-n/4-l}(1+\tau)^{-n\sigma/2+l}\,d\tau\\ &\le C\eta^{1+\sigma}(1+t)^{-n/4-l}\end{aligned}\tag{5.18}$$

for $l \le 1$, and

$$\begin{aligned}
&\int_{t/2}^{t} \|\partial_t^l D^k S(t-\tau) f^1(u_j(\tau,\cdot))\|_2 \, d\tau \\
&\quad \le C \int_{t/2}^{t} (1+t-\tau)^{-l-(k-1)/2} \|Df(u_j(\tau,\cdot))\|_2 \, d\tau \\
&\quad \le C\eta^{1+\sigma} \int_{t/2}^{t} (1+t-\tau)^{-l-(k-1)/2} (1+\tau)^{-(n/2)(\sigma+1/2)-1/2} \, d\tau \\
&\quad = C\eta^{1+\sigma} \int_{t/2}^{t} (1+t-\tau)^{-l-(k-1)/2} (1+\tau)^{-n/4-l-k/2} (1+\tau)^{-n\sigma/2+l+(k-1)/2} \, d\tau \\
&\quad \le C\eta^{1+\sigma} (1+t)^{-n/4-l-k/2} \qquad (5.19)
\end{aligned}$$

for $k+l \le 2$ and $k \ge 1$. By the estimates (5.17)–(5.19) and Lemma 5.2, we see that

$$\|\partial_t^l D^k u_{j+1}^1(t,\cdot)\|_2 \le 2\eta(1+t)^{-n/4-l-k/2} \qquad (5.20)$$

for $k+l \le 2$, and $l \le 1$.

By (5.2), we see that

$$\partial_t^2 u_{j+1}^1(t,\cdot) = \partial_t^2 u_0^1(t,\cdot) + f^1(u_j(t,\cdot)) + \int_0^t \partial_t^2 S(t-\tau) f^1(u_j(\tau,\cdot)) \, d\tau,$$

then, by Proposition 4.5(1) with $p = q = 2$, Lemmas 5.2–5.3 and (5.17), we see that

$$\begin{aligned}
\|\partial_t^2 u_{j+1}^1(t,\cdot)\|_2 &\le \|\partial_t^2 u_0^1(t,\cdot)\|_2 + \|f(u_j(t,\cdot))\|_2 + \int_0^{t/2} \|\partial_t^2 S(t-\tau) f^1(u_j(\tau,\cdot))\|_2 \, d\tau \\
&\quad + \int_{t/2}^{t} \|\partial_t^2 S(t-\tau) f^1(u_j(\tau,\cdot))\|_2 \, d\tau \\
&\le \eta(1+t)^{-n/4-2} + C\eta^{1+\sigma}(1+t)^{-(n/2)\cdot(\sigma+1/2)} + C\eta^{1+\sigma}(1+t)^{-n/4-2} \\
&\quad + C\eta^{1+\sigma} \int_{t/2}^{t} (1+t-\tau)^{-2} (1+\tau)^{-(n/2)(\sigma+1/2)} \, d\tau \\
&\le 2\eta(1+t)^{-\nu(0,2)} \qquad (5.21)
\end{aligned}$$

for sufficiently small $\eta > 0$. By (5.20) and (5.21), we have obtained the desired estimate.

(4) Since

$$\frac{\sqrt{1+|\xi|^2}}{|\xi|}(1-\chi(\xi)) \in L^\infty,$$

Lemma 5.3(3) shows that

$$\|f^2(u_j(t,\cdot))\|_{1,2} \le C\|Df(u_j(t,\cdot))\|_2 \le C\eta^{1+\sigma}(1+t)^{-(n/2)\cdot(\sigma+1/2)-1/2}$$

for $j \ge 0$. (5.3) gives

$$\partial_t^l u_{j+1}^2(t,\cdot) = \partial_t^l u_0^2(t,\cdot) + \int_0^t \partial_t^l S(t-\tau) f^2(u_j(\tau,\cdot)) \, d\tau,$$

for $0 \le l \le 1$, and

$$\partial_t^2 u_{j+1}^2(t,\cdot) = \partial_t^2 u_0^2(t,\cdot) + f^2(u_j(t,\cdot)) + \int_0^t \partial_t^2 S(t-\tau) f^2(u_j(\tau,\cdot))\, d\tau.$$

Hence, easy calculation gives

$$\begin{aligned}\|\partial_t^l D^k u_{j+1}^2(t,\cdot)\|_2 &\le \|\partial_t^l D^k u_0^2(t,\cdot)\|_2 + C\|f^2(u_j(t,\cdot))\|_{1,2} + C\int_0^t e^{-a(t-\tau)} \|f^2(u_j(\tau,\cdot))\|_{1,2}\, d\tau \\ &\le 2\eta(1+t)^{-(n/2)\cdot(\sigma+1/2)-1/2}\end{aligned}$$

for $k+l \le 2$ and for sufficiently small $\eta > 0$. We have obtained the desired result by induction. $\square$

Lemma 5.4. *Under the notations and assumptions as in Proposition* 5.1, *the estimate*

$$\sup_{0\le t} \|u_{j+1}(t,\cdot) - u_j(t,\cdot)\|_{1,2} \le \frac{1}{2} \sup_{0\le t} \|u_j(t,\cdot) - u_{j-1}(t,\cdot)\|_{1,2}$$

holds for $j \ge 1$.

Proof. Proposition 5.1(4) shows that

$$\|u_j^2(t,\cdot)\|_{2,2} \le C\eta(1+t)^{-(n/2)\cdot(\sigma+1/2)-1/2}.$$

Hence, by easy calculations and the estimates in Proposition 5.1, we see that

$$\begin{aligned}\|f(u_j(t,\cdot)) - f(u_{j-1}(t,\cdot))\|_2 &\le C(\|u_j^1(t,\cdot)\|_\infty^\sigma + \|u_{j-1}^1(t,\cdot)\|_\infty^\sigma + \|u_j^2(t,\cdot)\|_{2,2}^\sigma + \|u_{j-1}^2(t,\cdot)\|_{2,2}^\sigma) \\ &\quad \times \|u_j(t,\cdot) - u_{j-1}(t,\cdot)\|_{1,2} \\ &\le C\eta^\sigma (1+t)^{-n\sigma/2} \|u_j(t,\cdot) - u_{j-1}(t,\cdot)\|_{1,2} \qquad (5.22)\end{aligned}$$

for $j \ge 1$. By the iteration scheme (5.1), $w \equiv u_{j+1} - u_j$ is the solution of the problem

$$\partial_t^2 w - \Delta w + 2a\partial_t w = f(u_j) - f(u_{j-1}), \quad w(0) = \partial_t w(0) = 0.$$

Hence, by Proposition 4.5 and (5.22), we see that

$$\begin{aligned}\sup_{0\le t\le T} \|u_{j+1}(t,\cdot) - u_j(t,\cdot)\|_{1,2} &\le C\int_0^T \|f(u_j(t,\cdot)) - f(u_{j-1}(t,\cdot))\|_2\, dt \\ &\le C\eta^\sigma \sup_{0\le t\le T} \|u_j(t,\cdot) - u_{j-1}(t,\cdot)\|_{1,2} \int_0^\infty (1+t)^{-n\sigma/2}\, dt \\ &\le C\eta^\sigma \sup_{0\le t\le T} \|u_j(t,\cdot) - u_{j-1}(t,\cdot)\|_{1,2} \\ &\le \frac{1}{2} \sup_{0\le t\le T} \|u_j(t,\cdot) - u_{j-1}(t,\cdot)\|_{1,2}\end{aligned}$$

for any $T > 0$, provided that $\eta > 0$ is sufficiently small. Therefore we have obtained the desired result. $\square$

Proof of Theorem 1.3. By Proposition 5.1 and Lemma 5.4, there is the function

$$u \in C([0,\infty); H_2^1) \cap L^\infty(0,\infty; L^q \cap L^{q'} \cap H_2^2)$$

satisfying

$$u_n \to u \quad \text{in } C([0,\infty); H_2^1), \tag{5.23}$$

$$u_n^1 \to u^1 \quad \text{weak}^* \text{ in } L^\infty(0,\infty; L^p), \quad (1 < p \le \infty)$$

$$u_n^2 \to u^2 \quad \text{weak}^* \text{ in } L^\infty(0,\infty; L^q \cap L^{q'} \cap H_2^2).$$

Moreover, the following estimates hold.

$$\|u^1(t,\cdot)\|_p \le \eta(1+t)^{-n/2(1-1/p)} \quad (1 \le p \le \infty),$$

$$\|u^2(t,\cdot)\|_q \le \eta(1+t)^{-(n/2)\cdot(1+\sigma-1/q)}, \quad \|u^2(t,\cdot)\|_{q'} \le \eta(1+t)^{-(n/2)\cdot(1+\sigma-1/q')}$$

and

$$\|\partial_t^l D^k u(t,\cdot)\| \le 2\eta(1+t)^{-\nu(k,l)}$$

for $k+l \le 2$.

Hence we see that $\|u(t,\cdot)\|_p \le \|u^1(t,\cdot)\|_p + \|u^2(t,\cdot)\|_p \le 4\eta(1+t)^{-(n/2)(1-1/p)}$ for $1+\sigma \le p \le 1+1/\sigma$. By (5.23) we see that $f(u_n) \to f(u)$ in $C([0,\infty); L^2)$.

Note that $u \in C(0,\infty; H_2^1)$ is a solution of (1.10). Lemma 5.1 shows that

$$u \in C([0,\infty); H_2^2) \cap C^1([0,\infty); H_2^1) \cap C([0,\infty); L^2).$$

Then we claim that

$$f(u) \in C([0,\infty); L^{1+1/\sigma} \cap L^{1+\sigma}).$$

Hence, Proposition 4.5 shows that

$$u \in C([0,\infty); L^{1+1/\sigma} \cap L^{1+\sigma}).$$

Thus we have obtained the desired result. □

To prove Theorems 1.4–1.5, we construct approximate solutions of the problem (1.10) by (5.1). For the proofs of these theorems, we need the following lemmas. Note that the proofs of the decay estimate to $\|\partial_t u(t,\cdot)\|_2$ and $\|Du(t,\cdot)\|_2$ are same as one of Theorem 1.3, replaced Z_1 by Z_2 or Z_3.

Lemma 5.5. *Under the assumptions and notations as ones in Theorem* 1.4, *the approximate solution u_j constructed by* (5.1) *inductively satisfies*

$$u_j^1 \in C([0,\infty); L^1 \cap L^\infty), \quad u_j^2 \in C([0,\infty); H_2^1 \cap L^q \cap L^{q'}).$$

Moreover, following estimates hold:

(1) $\|u_j^1(t,\cdot)\|_\infty \le 2\eta(1+t)^{-3/2}$, $\|u_j^1(t,\cdot)\|_1 \le 2\eta$,

(2) $\|u_j^2(t,\cdot)\|_q \le 2\eta(1+t)^{-\beta_1}$, $\|u_j^2(t,\cdot)\|_{q'} \le 2\eta(1+t)^{-\beta_2}$, *where β_1 and β_2 are defined by* (5.6).

(3) $\|\partial_t^l D^k u_j^1(t,\cdot)\|_2 \le 2\eta(1+t)^{-3/4-k/2-l}$, $\|\partial_t^l D^k u_j^2(t,\cdot)\|_{1,2} \le 2\eta(1+t)^{-3(\sigma+1/2)/2}$ *for* $k+l \le 1$.

Lemma 5.6. *Under the assumptions and notations as ones in Theorem* 1.5, *the approximate solution* u_j *constructed by* (5.1) *inductively satisfies*

$$u_j^1 \in C([0,\infty); L^1 \cap L^\infty), \quad u_j^2 \in C([0,\infty); H_2^1).$$

Moreover, the following estimates hold:

(1) $\|u_j^1(t,\cdot)\|_\infty \le 2\eta(1+t)^{-n/2}$, $\|u_j^1(t,\cdot)\|_1 \le 2\eta$,

(2) $\|\partial_t^l D^k u_j^1(t,\cdot)\|_2 \le 2\eta(1+t)^{-n/4-k/2-l}$, $\|\partial_t^l D^k u_j^2(t,\cdot)\|_{1,2} \le 2\eta(1+t)^{-n(\sigma+1/2)/2}$ *for* $k+l \le 1$.

Since the proof of Lemmas 5.5–5.6 and Theorems 1.3–1.4 is very similar to one of Theorem 1.2, we omit it.

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References

[1] H. Bellout and A. Friedman, Blow-up estimates for a nonlinear hyperbolic heat equation, SIAM J. Math. Anal., **20** (1989), pp. 354–366.

[2] J. Duoandikoetxea, Fourier Analysis, Grad. Stud. Math., **29**, Amer. Math. Soc., Providence, RI, 2001.

[3] C. Fefferman and E. M. Stein, H^p spaces of several variables, Acta Math., **129** (1974), pp. 137–193.

[4] H. Fujita, On the blowing up of solutions of the Cauchy problem for $u_t = \Delta u + u^{1+\alpha}$, J. Fac. Sci. Univ. Tokyo. Sec. I., **13** (1966), pp. 109–124.

[5] T. Gallay and G. Raugel, Stability of traveling waves for damped hyperbolic equation, Z. Angev. Math. Phys., **48** (1997), pp. 451–479.

[6] T. Gallay and G. Raugel, Scaling variables and asymptotic expansions in damped wave equations, J. Differential Equations, **150** (1998), pp. 42–97.

[7] K. Hayakawa, On nonexistence of global solutions of some semilinear parabolic equations, Proc. Japan Acad., **49** (1973), pp. 503–505.

[8] L. Hörmander, Estimates for translation invariant operators in L^p-spaces, Acta Math., **104** (1960), pp. 93–139.

[9] L. Hsiao and T.-P. Liu, Convergence to nonlinear diffusion waves for solutions of a system of hyperbolic conservation laws with damping, Comm. Math. Phys., **143** (1992), pp. 599–605.

[10] R. Ikehata, Y. Miyaoka and T. Nakatake, Decay estimates of solutions for dissipative wave equations in $\boldsymbol{R}^N$ with lower power nonlinearities, J. Math. Soc. Japan, **56** (2004), 365–373.

[11] R. Ikehata and K. Nishihara, Diffusion phenomena for second order linear evolution equations, Studia Math., **158** (2003), pp. 153–161.

[12] G. Karch, Selfsimilar profiles in large time asymptotic of solutions to damped wave equations, Studia Math., **143** (2000), pp. 175–197.

[13] S. Kawashima, M. Nakao and K. Ono, On the decay property of solutions to the Cauchy problem of the semilinear wave equation with dissipative term, J. Math. Soc. Japan, **47** (1995), pp. 617–653.

[14] S. P. Levandosky, Decay estimates for fourth order wave equations, J. Differential Equations, **143** (1998), pp. 360–413.

[15] T.-T. Li, Nonlinear heat conduction with finite speed of propagation, In: China-Japan Symposium on Reaction-Diffusion Equations and their Applications and Computational Aspect, Shanghai, 1994, (eds. T.-T. Li, M. Mimura, Y. Nishiura and Q.-X. Ye), World Sci. Publishing, River Edge, NJ, 1997, pp. 81–91.

[16] T.-T. Li and Y. Zhou, Breakdown of solutions to $\Box u + u_t = u^{1+\alpha}$, Discrete Contin. Dyn. Syst., **1** (1995), pp. 503–520.

[17] P. Marcati and K. Nishihara, The L^p-L^q estimates of solutions to one-dimensional damped wave equations and their applications to the compressible flow through porus media, J. Differential Equations, **191** (2003), pp. 445–469.

[18] B. Marshall, W. Strauss and S. Wainger, L^p-L^q estimates for the Klein-Gordon equations, J. Math. Pures Appl. (9), **59** (1980), pp. 417–440.
[19] A. Matsumura, On the asymptotic behavior of solutions of semi-linear wave equations, Publ. Res. Inst. Math. Sci., **12** (1976), pp. 169–189.
[20] M. Nakao and K. Ono, Global existence to the Cauchy problem for the semilinear dissipative wave equations, Math. Z., **214** (1993), pp. 325–342.
[21] T. Narazaki, $H^{m(s+1)}$-Solutions of Semilinear Beam Equations, Adv. Math. Sci. Appl., **13** (2003), pp. 1–41.
[22] K. Nishihara, Convergence rates to nonlinear diffusion waves for solutions of system of hyperbolic conservation laws with damping, J. Differential Equations, **131** (1996), pp. 171–188.
[23] K. Nishihara, Asymptotic behavior of solutions of quasilinear hyperbolic equations with linear damping, J. Differential Equations, **131** (1997), pp. 384–395.
[24] K. Nishihara, L^p-L^q Estimates of Solutions to the Damped Wave Equation in 3-dimensional Space and their Application, Math. Z., **244** (2003), pp. 631–649.
[25] G. Ponce, Global existence of small solutions to a class of nonlinear evolution equations, Nonlinear Anal., **9** (1985), pp. 399–418.
[26] C. D. Sogge, Fourier integrals in classical analysis, Cambridge Tracts in Math., **105**, Cambridge Univ. Press, Cambridge, 1993.
[27] E. M. Stein and G. Weiss, Introduction to Fourier Analysis on Euclidean Spaces, Princeton Math. Ser., **32**, Princeton Univ. Press, Princeton, NJ, 1971.
[28] E. M. Stein, Harmonic Analysis: real-variable methods, orthogonality, and oscillatory integrals, Princeton Math. Ser., **43**, Monographs in Harmonic Analysis, III, Princeton Univ. Press, Princeton, NJ, 1993.
[29] W. Strauss, Nonlinear Wave Equations, CBMS Reg. Conf. Ser. Math., **73**, Amer. Math. Soc., Providence, RI, 1989.
[30] G. Todorova and B. Yordanov, Critical exponent for a nonlinear wave equation with damping, J. Differential Equations, **174** (2000), pp. 464–489.
[31] Q. Zhang, A blow-up result for a nonlinear wave equation with damping: The critical case, C. R. Acad. Sci. Paris, **333** (2001), pp. 109–114.

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