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Markov or non-Markov property of $cM - X$ processes

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Abstract. For a Brownian motion with a constant drift X and its maximum process M, $M - X$ and $2M - X$ are diffusion processes by the extensions of Lévy's and Pitman's theorems. We show that $cM - X$ is not a Markov process if $c \in \boldsymbol{R} \setminus \{0, 1, 2\}$. We also give other elementary proofs of Lévy's and Pitman's theorems.

1. Introduction.

Let $B = \{B_t, t \geqq 0\}$ be a one-dimensional Brownian motion starting from 0 and, for $\mu \in \boldsymbol{R}$, define $B^{(\mu)} = \{B_t^{(\mu)}, t \geqq 0\}$, a Brownian motion with constant drift μ, by $B_t^{(\mu)} = B_t + \mu t$. We set

$$M_t = \max_{0 \leqq s \leqq t} B_s \quad \text{and} \quad M_t^{(\mu)} = \max_{0 \leqq s \leqq t} B_s^{(\mu)}.$$

It is well known that $M^{(\mu)} - B^{(\mu)} = \{M_t^{(\mu)} - B_t^{(\mu)}, t \geqq 0\}$ and $2M^{(\mu)} - B^{(\mu)} = \{2M_t^{(\mu)} - B_t^{(\mu)}, t \geqq 0\}$ are diffusion processes, both starting from 0. When $\mu = 0$, these facts are known as Lévy's and Pitman's theorems ([**12**], [**17**]) and, in this case, these processes are identical in law, respectively, with a reflecting Brownian motion on $[0, \infty)$ and with a three-dimensional Bessel process. See also Ikeda-Watanabe [**9**], Itô-McKean [**10**] and Revuz-Yor [**18**]. Also for the general case, some interesting properties, including the explicit forms of the transition densities, are known. For details, see Bingham [**2**], Fitzsimmons [**6**], Graversen-Shyriaev [**7**], Imhof [**8**], Rogers [**19**], Rogers-Pitman [**20**] and so on. Some of these results are further extended to other classes of stochastic processes. For example, see Bertoin [**1**], Matsumoto-Yor [**14**], [**15**], [**16**], Saisho-Tanemura [**21**].

In this paper we consider a general linear combination $Z_{(c)}^{(\mu)} = cM^{(\mu)} - B^{(\mu)}$, $c \in \boldsymbol{R}$, for a Brownian motion $B^{(\mu)}$ with constant drift. The main purpose of this paper is to show that $Z_{(c)}^{(\mu)}$'s are Markov processes only in trivial, Lévy's and Pitman's cases ($c = 0, 1, 2$, respectively) and are not in any other cases. That is, the main theorem is the following.

Theorem 1.1. *For any $c \in \boldsymbol{R} \setminus \{0, 1, 2\}$, $Z_{(c)}^{(\mu)}$ is not a Markov process.*

The assertion of Theorem 1.1 seems to have been a folklore to some people (see, e.g., Yor [**22**]). In fact, Jeulin [**11**] has shown that $Z_{(c)}^{(0)} = cM - B$ is not a time-

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homogeneous Markov process if $c > 1$ and $c \neq 2$. Theorem 1.1 gives a complete answer.

There is an important difference between Lévy's and Pitman's cases; the natural filtration of $M^{(\mu)} - B^{(\mu)}$ is identical to that of $B^{(\mu)}$ in the former case, while the natural filtration of $2M^{(\mu)} - B^{(\mu)}$ is strictly included in the latter case. It also seems to have been a folklore that, for the natural filtration of $Z_{(c)}^{(\mu)}$, the strict inclusion holds if and only if $c = 2$. In fact, Mansuy [**13**] recently answered this affirmatively. While we may prove Theorem 1.1 from his result, we do it in an elementary way.

Our method to prove Theorem 1.1 is based on the expression for the probability density of the joint distribution of $(M_t^{(\mu)}, B_t^{(\mu)})$ also due to Lévy and may be considered as an extension of that of Imhof [**8**], who has considered Pitman's case. Since some modifications of our method lead to elementary proofs of the Markov properties of $Z_{(1)}^{(\mu)}$ and $Z_{(2)}^{(\mu)}$, and only a sketch is given in [**8**], we also spend some part of this paper to them for completeness.

Finally, let $\{L_t^{(\mu)}, t \geqq 0\}$ be the local time of $B^{(\mu)}$ at 0. Then the process $\{cL_t^{(\mu)} + |B_t^{(\mu)}|, t \geqq 0\}$ has the same law as $Z_{(c+1)}^{(\mu)}$ and is called a perturbed Brownian motion. We should note that this process has also been studied extensively by many authors. See, e.g., Carmona-Petit-Yor [**3**], [**4**], Emery-Perkins [**5**] and so on. See also [**15**] for related topics.

This paper is organized as follows. In Section 2, we consider finite dimensional joint distributions of $(B^{(\mu)}, M^{(\mu)})$. The results play fundamental roles in the following sections. In Section 3, we show explicit forms of the probability densities of $Z_{(c),t}^{(\mu)}$ for fixed $t > 0$. In Sections 4 and 5, we consider $M^{(\mu)} - B^{(\mu)}$ and $2M^{(\mu)} - B^{(\mu)}$, respectively, and give another understanding of Lévy's and Pitman's theorems. Sections 6 and 7 are devoted to a proof of Theorem 1.1.

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2. Joint distributions of (X, M).

As is mentioned in Introduction, we are concerned with a Brownian motion $B^{(\mu)}$ with constant drift. Denote by $B^{(\mu),x} = \{B_t^{(\mu),x}, t \geqq 0\}$ the Brownian motion with constant drift μ defined on a usual filtered probability space $(\Omega, \mathscr{F}, P; (\mathscr{F}_t))$, where x stands for the starting point. We let $M^{(\mu),x} = \{M_t^{(\mu),x}, t \geqq 0\}$ be its maximum process, that is,

$$M_t^{(\mu),x} = \max_{0 \leqq s \leqq t} B_s^{(\mu),x}.$$

When $x = 0$, we simply denote as $B^{(\mu)}$ and $M^{(\mu)}$.

Throughout this paper, we set

$$\varphi(t, \xi) = \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{\xi^2}{2t}\right) \tag{2.1}$$

and

$$k(t, \xi) = \frac{\xi}{\sqrt{2\pi t^3}} \exp\left(-\frac{\xi^2}{2t}\right) = -\frac{\partial}{\partial \xi} \varphi(t, \xi). \tag{2.2}$$

First we show the following.

LEMMA 2.1. (i) *For* $x \in \boldsymbol{R}$ *and* $t > 0$, *the joint distribution of* $(B_t^{(\mu),x}, M_t^{(\mu),x})$ *is given by*

$$P(B_t^{(\mu),x} \in da, M_t^{(\mu),x} \in db) = 2e^{-\mu^2 t/2 + \mu(a-x)} k(t, 2b-a-x)\, da\, db \tag{2.3}$$

for $b \geqq a \vee x \equiv \max(a, x)$.

(ii) *One has*

$$P(B_t^{(\mu),x} \in da, M_t^{(\mu),x} \leqq b) = p_b^{(\mu)}(t, x, a)\, da, \tag{2.4}$$

for $b \geqq a \vee x$, *where*

$$p_b^{(\mu)}(t, x, a) = e^{-\mu^2 t/2 + \mu(a-x)} \{\varphi(t, a-x) - \varphi(t, 2b-a-x)\}. \tag{2.5}$$

PROOF. When $\mu = 0$, formula (2.3) is nothing else but the classical Lévy theorem on the joint distribution of (B_t^x, M_t^x) (cf. [**10**]). The general result is obtained from the Cameron-Martin theorem. Formula (2.4) is easily obtained by integrating both hand sides of (2.3) in b. □

Next, setting

$$M_{s,t}^{(\mu),x} = \max_{s \leqq u \leqq t} B_u^{(\mu),x} \quad \text{and} \quad M_{s,t}^{(\mu)} = \max_{s \leqq u \leqq t} B_u^{(\mu)}, \quad s > t, \tag{2.6}$$

we show the following.

PROPOSITION 2.2. *Consider a Brownian motion* $B^{(\mu),x}$ *with drift* μ *starting from* x. *Then, for any* $n = 1, 2, \ldots,$ *and* $\{t_i\}_{i=0}^n$ *with* $0 = t_0 < t_1 < \cdots < t_n$, *one has*

$$\begin{aligned} P(B_{t_1}^{(\mu),x} \in da_1, \ldots, B_{t_n}^{(\mu),x} \in da_n, M_{t_1}^{(\mu),x} \leqq b_1, \ldots, M_{t_{n-1},t_n}^{(\mu),x} \leqq b_n) \\ = \prod_{i=1}^n p_{b_i}^{(\mu)}(\tau_i, a_{i-1}, a_i) \cdot da_1 \cdots da_n \\ = e^{-\mu^2 t_n/2 + \mu(a_n - x)} \prod_{i=1}^n p_{b_i}(\tau_i, a_{i-1}, a_i) \cdot da_1 \cdots da_n, \end{aligned} \tag{2.7}$$

where $\tau_i = t_i - t_{i-1}$, $a_0 = x$ *and* $p_b(\tau, a, a') = p_b^{(0)}(\tau, a, a')$.

PROOF. When $n = 1$, we have shown the assertions in Lemma 2.1. Assume that (2.7), replaced n by $n-1$, holds. Then the Markov property of $B^{(\mu),x}$ implies

$$\begin{aligned} & P(B_{t_1}^{(\mu),x} \in da_1, \ldots, B_{t_n}^{(\mu),x} \in da_n, M_{t_1}^{(\mu),x} \leqq b_1, \ldots, M_{t_{n-1},t_n}^{(\mu),x} \leqq b_n) \\ & \quad = E[P(B_{t_1}^{(\mu),x} \in da_1, \ldots, B_{t_n}^{(\mu),x} \in da_n, M_{t_1}^{(\mu),x} \leqq b_1, \ldots, M_{t_{n-1},t_n}^{(\mu),x} \leqq b_n \,|\, \mathscr{F}_{t_{n-1}})] \\ & \quad = E[P(B_{\tau_n}^{(\mu),a_{n-1}} \in da_n, M_{0,\tau_n}^{(\mu),a_{n-1}} \leqq b_n) I_{\{B_{t_1}^{(\mu),x} \in da_1, \ldots, B_{t_{n-1}}^{(\mu),x} \in da_{n-1}, M_{t_1}^{(\mu),x} \leqq b_1, \ldots, M_{t_{n-2},t_{n-1}}^{(\mu),x} \leqq b_n\}}] \\ & \quad = P(B_{t_1}^{(\mu),x} \in da_1, \ldots, B_{t_{n-1}}^{(\mu),x} \in da_{n-1}, M_{t_1}^{(\mu),x} \leqq b_1, \ldots, M_{t_{n-2},t_{n-1}}^{(\mu),x} \leqq b_{n-1}) \\ & \qquad \times p_{b_n}^{(\mu)}(\tau_n, a_{n-1}, a_n)\, da_n, \end{aligned}$$

where we have used (2.4) for the last equality. Now formula (2.7) follows from the assumption of the induction. □

Noting

$$\frac{\partial}{\partial b} p_b(t, x, a) = 2k(t, 2b - a - x), \tag{2.8}$$

we obtain explicit expressions of the joint distribution of $(B_{t_1}^{(\mu)}, \dots, B_{t_n}^{(\mu)}, M_{t_1}^{(\mu)}, \dots, M_{t_{n-1}, t_n}^{(\mu)})$ by taking the differentials in b_i. For example, we obtain

$$\begin{aligned} &P(B_{t_1}^{(\mu)} \in da_1, B_{t_2}^{(\mu)} \in da_1, M_{t_1}^{(\mu)} \in db_1, M_{t_1, t_2}^{(\mu)} \leqq b_2) \\ &\qquad = 2e^{-\mu^2 t_2/2 + \mu a_2} k(t_1, 2b_1 - a_1) p_{b_2}(\tau_2, a_1, a_2)\, da_1 da_2 db_1, \end{aligned} \tag{2.9}$$

$$\begin{aligned} &P(B_{t_1}^{(\mu)} \in da_1, B_{t_2}^{(\mu)} \in da_2, M_{t_1}^{(\mu)} \in db_1, M_{t_1, t_2}^{(\mu)} \in db_2) \\ &\qquad = 4e^{-\mu^2 t_2/2 + \mu a_2} k(t_1, 2b_1 - a_1) k(\tau_2, 2b_2 - a_1 - a_2)\, da_1 da_2 db_1 db_2. \end{aligned} \tag{2.10}$$

These formulae will be used in the sequel.

3. One-dimensional distributions.

In this section we give explicit expressions for the probability densities of $Z_{(c),t}^{(\mu)}$ for fixed $t > 0$. We proceed separately in the four cases of $c = 1$, $c = 2$, $c > 1$ and $c \neq 2$, and $c < 1$. It is easy to see the existence of the density of $Z_{(c),t}^{(\mu),x} = cM_t^{(\mu),x} - B_t^{(\mu),x}$, $x \in \boldsymbol{R}$, $t > 0$, which we denote by $q_{(c),1}^{(\mu),x}(t, z)$.

By Lemma 2.1, we have

$$E[f(Z_{(c),t}^{(\mu),x})] = 2\int_x^\infty db \int_{-\infty}^b e^{-\mu^2 t/2 + \mu(a-x)} k(t, 2b - a - x) f(cb - a)\, da \tag{3.1}$$

for any non-negative Borel function f.

PROPOSITION 3.1. *When $c = 1$, one has for $z \geqq 0$*

$$q_{(1),1}^{(\mu),x}(t, z) = 2e^{-\mu^2 t/2 - \mu z} \int_0^\infty e^{\mu\beta} k(t, z + \beta)\, d\beta, \tag{3.2}$$

and, in particular,

$$q_{(1),1}^{(0),x}(t, z) = 2\varphi(t, z). \tag{3.3}$$

PROOF. On the right hand side of (3.1), we change the variables from a to z by $z = b - a$ and then the order of integrations. Then we obtain

$$\begin{aligned} E[f(M_t^{(\mu),x} - B_t^{(\mu),x})] &= 2e^{-\mu^2 t/2} \int_0^\infty e^{-\mu z} f(z)\, dz \int_x^\infty e^{\mu(b-x)} k(t, b + z - x)\, db \\ &= 2e^{-\mu^2 t/2} \int_0^\infty e^{-\mu z} f(z)\, dz \int_0^\infty e^{\mu\beta} k(t, z + \beta)\, d\beta \end{aligned}$$

and formula (3.2). Formula (3.3) follows from (3.2) since we can carry out the integration in ξ if $\mu = 0$. □

The next Proposition 3.2 has been given in Imhof [**8**] and is proven from (3.1). Propositions 3.3 and 3.4 below are also proven from (3.1) and we omit the proofs.

PROPOSITION 3.2. *When* $c = 2$, *one has for* $z \geqq x$

$$q_{(2),1}^{(\mu),x}(t,z) = 2e^{-\mu^2 t/2}\frac{\sinh \mu(z-x)}{\mu}k(t,z-x), \tag{3.4}$$

and

$$q_{(2),1}^{(0),x}(t,z) = 2(z-x)k(t,z-x). \tag{3.5}$$

PROOF. On the right hand side of (3.1), we change the variables by $z = 2b - a$ and then the order of integrations. Then we obtain

$$E[f(2M_t^{(\mu),x} - B_t^{(\mu),x})] = 2e^{-\mu^2 t/2}\int_x^\infty f(z)e^{-\mu(z+x)}k(t,z-x)\,dz\int_x^z e^{2\mu b}\,db,$$

from which formulae (3.4) and (3.5) follow. □

PROPOSITION 3.3. *When* $c > 1$ *and* $c \neq 2$, *one has for* $z \geqq (c-1)x$

$$q_{(c),1}^{(\mu),x}(t,z) = 2e^{-\mu^2 t/2-\mu(x+z)}\int_x^{z/(c-1)} e^{c\mu b}k(t,z-x+(2-c)b)\,db. \tag{3.6}$$

PROPOSITION 3.4. *When* $c < 1$, *one has for* $z \in \boldsymbol{R}$

$$q_{(c),1}^{(\mu),x}(t,z) = 2e^{-\mu^2 t/2-\mu(x+z)}\int_{x\vee(z/(c-1))}^{\infty} e^{c\mu b}k(t,z-x+(2-c)b)\,db. \tag{3.7}$$

4. Markov property of $M - X$.

By using the equality for the filtrations $\sigma\{Z_{(1),s}^{(\mu)}; s \leqq t\} = \sigma\{B_s; s \leqq t\} \equiv \mathscr{F}_t$ and repeating similar computations to those in this section, we can show, for any $s < t$ and $\Lambda \in \mathscr{F}_s$,

$$E[f(Z_{(1),t}^{(\mu)});\Lambda] = \int_0^\infty E[I_\Lambda; Z_{(1),s}^{(\mu)} \in dz_1]\int_0^\infty f(z_2)p_{(1)}^{(\mu)}(t-s,z_1,z_2)\,dz_2,$$

where the transition density $p_{(1)}^{(\mu)}(t,z_1,z_2)$ is given by

$$\begin{aligned} p_{(1)}^{(\mu)}(t,z_1,z_2) &= \exp\left(-\frac{1}{2}\mu^2 t - \mu(z_2-z_1)\right) \\ &\quad \times \left[2\int_0^\infty k(t,z_1+z_2+b)e^{\mu b}\,db + \varphi(t,z_1-z_2) - \varphi(t,z_1+z_2)\right]. \end{aligned} \tag{4.1}$$

Although we could obtain the desired results from this, we cannot apply this method for $Z_{(2)}^{(\mu)} = 2M^{(\mu)} - B^{(\mu)}$. So, we choose the following way, which may be applied to the study on $Z_{(2)}^{(\mu)}$ after some modification.

First we consider the two-dimensional distributions. For $t_1 < t_2$, we denote by $q_{(1),2}^{(\mu)}(t_1,z_1,t_2,z_2)$ the density of the joint distribution of $(Z_{(1),t_1}^{(\mu)}, Z_{(1),t_2}^{(\mu)})$ with respect to the Lebesgue measure:

$$P(Z_{(1),t_1}^{(\mu)} \in dz_1, Z_{(1),t_2}^{(\mu)} \in dz_2) = q_{(1),2}^{(\mu)}(t_1,z_1,t_2,z_2)\,dz_1 dz_2.$$

PROPOSITION 4.1. *For any* $t_1 < t_2$, $z_1, z_2 > 0$, *one has*

$$q^{(\mu)}_{(1),2}(t_1, z_1, t_2, z_2) = p^{(\mu)}_{(1)}(t_1, 0, z_1)p^{(\mu)}_{(1)}(\tau_2, z_1, z_2), \tag{4.2}$$

where $\tau_2 = t_2 - t_1$. *Moreover one has*

$$P(Z^{(\mu)}_{(1),t_2} \in dz_2 \,|\, Z^{(\mu)}_{(1),t_1} = z_1) = p^{(\mu)}_{(1)}(\tau_2, z_1, z_2)\, dz_2. \tag{4.3}$$

PROOF. For non-negative Borel functions f_1 and f_2 on $\boldsymbol{R}_+$, we write

$$E[f_1(Z^{(\mu)}_{(1),t_1})f_2(Z^{(\mu)}_{(1),t_2})] = E[f_1(Z^{(\mu)}_{(1),t_1})f_2(Z^{(\mu)}_{(1),t_2}); M^{(\mu)}_{t_1} \leqq M^{(\mu)}_{t_1,t_2}]$$
$$+ E[f_1(Z^{(\mu)}_{(1),t_1})f_2(Z^{(\mu)}_{(1),t_2}); M^{(\mu)}_{t_1} > M^{(\mu)}_{t_1,t_2}],$$

where $M^{(\mu)}_{s,t}$ is defined by (2.6). We compute the two terms on the right hand side separately.

Noting that $M^{(\mu)}_{t_2} = M^{(\mu)}_{t_1,t_2}$ on the set $\{M^{(\mu)}_{t_1} \leqq M^{(\mu)}_{t_1,t_2}\}$, we rewrite the first term with the help of (2.10):

$$E[f_1(Z^{(\mu)}_{(1),t_1})f_2(Z^{(\mu)}_{(1),t_2}); M^{(\mu)}_{t_1} \leqq M^{(\mu)}_{t_1,t_2}] = \int_0^\infty db_1 \int_{-\infty}^{b_1} da_1 \int_{b_1}^\infty db_2 \int_{-\infty}^{b_2} da_2$$
$$\times f_1(b_1 - a_1)f_2(b_2 - a_2) \cdot 4e^{-\mu^2 t_2/2 + \mu a_2} k(t_1, 2b_1 - a_1)k(\tau_2, 2b_2 - a_2 - a_1).$$

Change the variables by $z_1 = b_1 - a_1$ and $z_2 = b_2 - a_2$. Then, after changing the order of integrations, we obtain

$$\begin{aligned} &E[f_1(Z^{(\mu)}_{(1),t_1})f_2(Z^{(\mu)}_{(1),t_2}); M^{(\mu)}_{t_1} \leqq M^{(\mu)}_{t_1,t_2}] \\ &\quad = \int_0^\infty f_1(z_1)\, dz_1 \int_0^\infty f_2(z_2)\, dz_2 \cdot 4e^{-\mu^2 t_2/2 - \mu z_2} \\ &\quad\quad \times \int_0^\infty k(t_1, z_1 + b_1)\, db_1 \int_{b_1}^\infty e^{\mu b_2} k(\tau_2, z_1 + z_2 + b_2 - b_1)\, db_2 \\ &\quad = \int_0^\infty f_1(z_1)\, dz_1 \int_0^\infty f_2(z_2)\, dz_2 \cdot 4e^{-\mu^2 t_2/2 - \mu z_2} \\ &\quad\quad \times \int_0^\infty e^{\mu b_1} k(t_1, z_1 + b_1)\, db_1 \int_0^\infty e^{\mu\beta} k(\tau_2, z_1 + z_2 + \beta)\, d\beta. \end{aligned} \tag{4.4}$$

On the other hand, by using (2.9) and noting that $Z^{(\mu)}_{(1),t_2} = M^{(\mu)}_{t_1} - B^{(\mu)}_{t_2}$ if $M^{(\mu)}_{t_1} > M^{(\mu)}_{t_1,t_2}$, we rewrite the second term:

$$E[f_1(Z^{(\mu)}_{(1),t_1})f_2(Z^{(\mu)}_{(1),t_2}); M^{(\mu)}_{t_1} > M^{(\mu)}_{t_1,t_2}] = \int_0^\infty db_1 \int_{-\infty}^{b_1} da_1 \int_{-\infty}^{b_1} da_2$$
$$\times f_1(b_1 - a_1)f_2(b_1 - a_2) \cdot 2e^{-\mu^2 t_2/2 + \mu a_2} k(t_1, 2b_1 - a_1)p_{b_1}(\tau_2, a_1, a_2).$$

Now change the variables by $z_i = b_1 - a_i$, $i = 1, 2$, and the order of integrations. Then, using (2.5), we obtain

$$E[f_1(Z^{(\mu)}_{(1),t_1})f_2(Z^{(\mu)}_{(1),t_2}); M^{(\mu)}_{t_1} > M^{(\mu)}_{t_1,t_2}] = \int_0^\infty f_1(z_1)\,dz_1 \int_0^\infty f_2(z_2)\,dz_2$$

$$\times\, 2e^{-\mu^2 t_2/2-\mu z_2} \int_0^\infty e^{\mu b_1} k(t_1, b_1+z_1)\,db_1 \cdot [\varphi(\tau_2, z_1 - z_2) - \varphi(\tau_2, z_1+z_2)],$$

which, combined with (4.4), yields (4.2).

Formula (4.3) is a consequence of (3.2), (4.1) and (4.2). □

We proceed to a computation of the general finite dimensional distribution of $Z^{(\mu)}_{(1)}$. For this purpose fix a sequence $\Delta = \{t_i\}_{i=0}^n$ with $0 = t_0 < t_1 < \cdots < t_n$. We introduce the random numbers T_i by $T_0 = 0$, $T_1 = 1$ and

$$T_j = \min\{i > T_{j-1}; M^{(\mu)}_{t_{i-1}} < M^{(\mu)}_{t_{i-1},t_i}\}, \quad j \geqq 2,$$

and we set

$$\sigma = \sigma(\Delta) = \max\{j; T_j \leqq n\}. \tag{4.5}$$

Note that $\sigma = k$ means $M^{(\mu)}_{t_n} = M^{(\mu)}_{t_{T_k-1},t_{T_k}}$.

Then we show the following.

Proposition 4.2. (i) *Let f_i, $i = 1, \ldots, n$, be non-negative Borel functions on $\boldsymbol{R}_+ \times \boldsymbol{R}$. Then, under the notation above, one has*

$$E\left[\prod_{i=1}^n f_i(Z^{(\mu)}_{(1),t_i}, M^{(\mu)}_{t_i}); \sigma = 1\right]$$

$$= \int_0^\infty dz_1 \cdots \int_0^\infty dz_n \cdot e^{-\mu^2 t_n/2-\mu z_n} \prod_{i=2}^n [\varphi(\tau_i, z_{i-1}-z_i) - \varphi(\tau_i, z_{i-1}+z_i)]$$

$$\times \int_0^\infty \prod_{i=1}^n f_i(z_i, b_1) \cdot 2k(t_1, z_1+b_1)e^{\mu b_1}\,db_1.$$

(ii) *For any k with $2 \leqq k \leqq n$ and any sequence $1 = j_1 < j_2 < \cdots < j_k \leqq n$, one has*

$$E\left[\prod_{i=1}^n f_i(Z^{(\mu)}_{(1),t_i}, M^{(\mu)}_{t_i}); \sigma = k, T_i = j_i, 1 \leqq i \leqq k\right] \tag{4.6}$$

$$= \int_0^\infty dz_1 \cdots \int_0^\infty dz_n \cdot e^{-\mu^2 t_n/2-\mu z_n}$$

$$\times \prod_{i\in\{1,\ldots,n\}\setminus\{j_1,\ldots,j_k\}} [\varphi(\tau_i, z_{i-1}-z_i) - \varphi(\tau_i, z_{i-1}+z_i)]$$

$$\times \int_0^\infty \prod_{i=1}^{j_2-1} f_i(z_i, \beta_1) \cdot 2k(t_1, z_1+\beta_1)e^{\mu\beta_1}\,d\beta_1$$

$$\times \cdots \times \int_0^\infty \prod_{i=j_{k-1}}^{j_k-1} f_i\left(z_i, \sum_{\ell=1}^{k-1}\beta_\ell\right) \cdot 2k(\tau_{j_{k-1}}, z_{j_{k-1}-1} + z_{j_{k-1}} + \beta_{k-1})e^{\mu\beta_{k-1}}\,d\beta_{k-1}$$

$$\times \int_0^\infty \prod_{i=j_k}^n f_i\left(z_i, \sum_{\ell=1}^k \beta_\ell\right) \cdot 2k(\tau_{j_k}, z_{j_k-1} + z_{j_k} + \beta_k)e^{\mu\beta_k}\,d\beta_k.$$

Proof. (i) Since $\sigma = 1$ means $M^{(\mu)}_{t_{i-1},t_i} < M^{(\mu)}_{t_1}$ for every $i = 2, 3, \dots, n$, we deduce from Proposition 2.2

$$E\left[\prod_{i=1}^n f_i(Z^{(\mu)}_{(1),t_i}, M^{(\mu)}_{t_i}); \sigma = 1\right] = \int_0^\infty db_1 \int_{-\infty}^{b_1} da_1 \cdots \int_{-\infty}^{b_1} da_n \cdot \prod_{i=1}^n f_i(b_1 - a_i, b_1)$$

$$\times 2^{-\mu^2 t_n/2 + \mu a_n} 2k(t_1, 2b_1 - a_1) p_{b_1}(\tau_2, a_1, a_2) \cdots p_{b_1}(\tau_n, a_{n-1}, a_n).$$

Changing the variables by $z_i = b_1 - a_i$, $i = 1, 2, \dots, n$, and the order of integrations, we obtain the assertion.

(ii) We prove (4.6) by induction in k. When $k = 2$, it is easy to show

$$E\left[\prod_{i=1}^n f_i(Z^{(\mu)}_{(1),t_i}, M^{(\mu)}_{t_i}); \sigma = 2, T_2 = j_2\right]$$

$$= \int_0^\infty \int_{b_1}^\infty E\left[\prod_{i=1}^n f_i(Z^{(\mu)}_{(1),t_i}, M^{(\mu)}_{t_i}); M^{(\mu)}_{t_1} \in db_1, M^{(\mu)}_{t_1,t_{j_2-1}} \leqq b_1, M^{(\mu)}_{t_{j_2-1},t_{j_2}} \in db_2, M^{(\mu)}_{t_{j_2},t_n} \leqq b_2\right]$$

$$= \int_0^\infty db_1 \int_{-\infty}^{b_1} da_1 \cdots \int_{-\infty}^{b_1} da_{j_2-1} \int_{b_1}^\infty db_2 \int_{-\infty}^{b_2} da_{j_2} \cdots \int_{-\infty}^{b_2} da_n \cdot e^{-\mu^2 t_n/2 + \mu a_n}$$

$$\times 2k(t_1, 2b_1 - a_1) \prod_{i=2}^{j_2-1} p_{b_1}(\tau_i, a_{i-1}, a_i) \cdot \prod_{i=1}^{j_2-1} f_i(b_1 - a_i, b_1)$$

$$\times 2k(\tau_{j_2}, 2b_2 - a_{j_2-1} - a_{j_2}) \prod_{i=j_2+1}^{n} p_{b_2}(\tau_i, a_{i-1}, a_i) \cdot \prod_{i=j_2}^{n} f_i(b_2 - a_i, b_2).$$

Change the variables by $z_i = b_1 - a_i$ for $i = 1, \dots, j_2 - 1$ and $z_i = b_2 - a_i$ for $i = j_2, \dots, n$ and the order of integrations. Changing again the variables by $\beta_1 = b_1$ and $\beta_2 = b_2 - b_1$, we obtain (4.6) for $k = 2$.

Assume that (4.6) holds and consider the case where k is replaced by $k + 1$. Then the Markov property of $B^{(\mu)}$ yields

$$E\left[\prod_{i=1}^n f_i(Z^{(\mu)}_{(1),t_i}, M^{(\mu)}_{t_i}); \sigma(t_n) = k + 1, T_2 = j_2, \dots, T_{k+1} = j_{k+1}\right]$$

$$= E\left[\prod_{i=1}^{j_k} f_i(Z^{(\mu)}_{(1),t_i}, M^{(\mu)}_{t_i}) \cdot g(Z^{(\mu)}_{(1),t_{j_k}}, M^{(\mu)}_{t_i}); \sigma(t_{j_k}) = k, T_2 = j_2, \dots, T_k = j_k\right],$$

where, denoting by $E_a[\ ]$ the expectation with respect to a Brownian motion with drift μ starting from a,

$$g(z_{j_k}, b_k) = \int_{b_k}^\infty E_{b_k - z_{j_k}}\Bigg[\prod_{i=j_k+1}^n f_i(Z^{(\mu)}_{(1),t_i - t_{j_k}}, M^{(\mu)}_{t_i - t_{j_k}}); M^{(\mu)}_{t_{j_{k+1}-1} - t_{j_k}} \leqq b_k,$$

$$M^{(\mu)}_{t_{j_{k+1}-1} - t_{j_k}, t_{j_{k+1}} - t_{j_k}} \in db_{k+1}, M^{(\mu)}_{t_{j_{k+1}} - t_{j_k}, t_n - t_{j_k}} \leqq b_{k+1}\Bigg].$$

We can simplify $g(z_{j_k}, b_k)$ in a similar way to the case of $k = 2$ and we obtain

$$g(z_{j_k}, b_k) = \int_0^\infty dz_{j_{k+1}} \cdots \int_0^\infty dz_n \cdot e^{-\mu^2(t_n - t_{j_k})/2 - \mu(z_n - z_{j_k})}$$

$$\times \prod_{i=j_k+1}^{j_{k+1}-1} f_i(z_i, b_k) \cdot \prod_{i \in \{j_k+1,\dots,n\} \setminus \{j_{k+1}\}} [\varphi(\tau_i, z_{i-1} - z_i) - \varphi(\tau_i, z_{i-1} + z_i)]$$

$$\times \int_0^\infty \prod_{i=j_k+1}^{n} f_i(z_i, b_k + \beta_{k+1}) \cdot 2k(\tau_{j_{k+1}}, z_{j_{k+1}-1} + z_{j_{k+1}} + \beta_{k+1}) e^{\mu\beta_{k+1}} \, d\beta_{k+1}.$$

Now, using the assumption of the induction and inserting this integral representation for $g(z_{j_k}, b_k)$, we obtain (4.6) with k replaced by $k+1$. □

Taking simpler generic functions $f_i(z_i)$ which depend only on z_i in Proposition 4.2, we can sum up both hand sides of (4.6) in k and $\{j_1, \dots, j_k\}$ and obtain the following theorem. It shows that $Z^{(\mu)}_{(1)} = M^{(\mu)} - B^{(\mu)}$ is a Markov process whose transition probability density with respect to the Lebesgue measure is given by $p^{(\mu)}_{(1)}(t, z, z')$.

THEOREM 4.3. *For any sequence* $\{t_i\}_{i=0}^n$ *with* $0 = t_0 < t_1 < \cdots < t_n$ *and non-negative Borel functions* f_i *on* $\boldsymbol{R}_+$, *it holds that*

$$E\left[\prod_{i=1}^n f_i(Z^{(\mu)}_{(1),t_i})\right] = \int_0^\infty dz_1 \cdots \int_0^\infty dz_n \cdot \prod_{i=1}^n f_i(z_i) p^{(\mu)}_{(1)}(\tau_i, z_{i-1}, z_i),$$

with convention $z_0 = 0$.

5. Markov property of $2M - X$.

In this section we show the Markov property of $Z^{(\mu)}_{(2)} = 2M^{(\mu)} - B^{(\mu)}$ and an explicit expression of the transition probability. We proceed in a similar way to the previous section.

First we consider the two-dimensional distribution. For $t_1 < t_2$, we denote by $q^{(\mu)}_{(2),2}(t_1, z_1, t_2, z_2)$ the density of the joint distribution of $(Z^{(\mu)}_{(2),t_1}, Z^{(\mu)}_{(2),t_2})$ with respect to the Lebesgue measure:

$$P(Z^{(\mu)}_{(2),t_1} \in dz_1, Z^{(\mu)}_{(2),t_2} \in dz_2) = q^{(\mu)}_{(2),2}(t_1, z_1, t_2, z_2) \, dz_1 dz_2.$$

Then, by a similar way to that in the proof of Proposition 4.1, we can show the following:

PROPOSITION 5.1. *For any* $t_1 < t_2$, $z_1, z_2 > 0$, *one has*

$$q^{(\mu)}_{(2),2}(t_1, z_1, t_2, z_2) = p^{(\mu)}_{(2)}(t_1, 0, z_1) p^{(\mu)}_{(2)}(\tau_2, z_1, z_2), \tag{5.1}$$

where $p^{(\mu)}_{(2)}(t, z_1, z_2)$ *is given by*

$$p^{(\mu)}_{(2)}(t, z_1, z_2) = e^{-\mu^2 t/2} \frac{\sinh(\mu z_2)}{\sinh(\mu z_1)} [\varphi(t, z_1 - z_2) - \varphi(t, z_1 + z_2)] \tag{5.2}$$

and

$$p^{(\mu)}_{(2)}(t, 0, z_2) = 2e^{-\mu^2 t/2} \frac{\sinh(\mu z_2)}{\mu} k(t, z_2). \tag{5.3}$$

Moreover, one has

$$P(Z^{(\mu)}_{(2),t_2} \in dz_2 \mid Z^{(\mu)}_{(2),t_1} = z_1) = p^{(\mu)}_{(2)}(\tau_2, z_1, z_2)\, dz_2. \tag{5.4}$$

For a computation of the general dimensional distribution of $Z^{(\mu)}_{(2)}$, we show the following companion to Proposition 4.2. Since we can prove it by a similar induction, we omit the proof.

PROPOSITION 5.2. (i) *For any sequence $\{t_i\}_{i=0}^n$ with $0 = t_0 < t_1 < \cdots < t_n$ and any non-negative Borel functions f_i, $i = 1, \dots, n$, on $\boldsymbol{R}_+ \times \boldsymbol{R}_+$, one has*

$$E\left[\prod_{i=1}^n f_i(Z^{(\mu)}_{(2),t_i}, M^{(\mu)}_{t_i}); \sigma = 1\right] = \int_0^\infty dz_1 \cdots \int_0^\infty dz_n \cdot e^{-\mu^2 t_n/2} 2k(t_1, z_1)$$
$$\times \int_0^{z_1 \wedge \cdots \wedge z_n} e^{\mu(2b_1 - z_n)} \prod_{i=1}^n f_i(z_i, b_1) \cdot \prod_{i=2}^n p_{b_1}(\tau_i, z_{i-1}, z_i)\, db_1.$$

(ii) *For any k with $2 \leqq k \leqq n$ and any sequence $1 = j_1 < j_2 < \cdots < j_k \leqq n$, one has*

$$E\left[\prod_{i=1}^n f_i(Z^{(\mu)}_{(2),t_i}, M^{(\mu)}_{t_i}); \sigma = k, T_2 = j_2, \dots, T_k = j_k\right] \tag{5.5}$$
$$= \int_0^\infty dz_1 \cdots \int_0^\infty dz_n \cdot e^{-\mu^2 t_n/2}$$
$$\times 2k(t_1, z_1) \int_0^{z_1 \wedge \cdots \wedge z_n} \prod_{i=j_1+1}^{j_2-1} p_{b_1}(\tau_i, z_{i-1}, z_i) \prod_{i=j_1}^{j_2-1} f_i(z_i, b_1)\, db_1$$
$$\times \cdots$$
$$\times 2k(\tau_{j_k}, z_{j_{k-1}} + z_{j_k} - 2b_{k-1}) \int_{b_{k-1}}^{z_{j_k} \wedge \cdots \wedge z_n} \prod_{i=j_k+1}^n p_{b_k}(\tau_i, z_{i-1}, z_i)$$
$$\times e^{\mu(2b_k - z_n)} \prod_{i=j_k}^n f_i(z_i, b_k)\, db_k,$$

where $\prod_{i \in \varnothing} p_b(\tau_i, z_{i-1}, z_i) = 1$ by convention.

Next we show the following.

PROPOSITION 5.3. *For any sequence $1 = j_1 < j_2 < \cdots < j_k \leqq n$ and any non-negative Borel functions f_i, $i = 1, \dots, n$, on $\boldsymbol{R}_+$, one has*

$$\sum_{\ell=k}^n E\left[\prod_{i=1}^n f_i(Z^{(\mu)}_{(2),t_i}); \sigma = \ell, T_2 = j_2, \dots, T_k = j_k\right]$$
$$= \int_0^\infty dz_1 \cdots \int_0^\infty dz_n \prod_{i=1}^n f_i(z_i) \cdot e^{-\mu^2 t_n/2 - \mu z_n}\, 2k(t_1, z_1) \int_0^{z_1 \wedge \cdots \wedge z_n} \prod_{i=j_1+1}^{j_2-1} p_{b_1}(\tau_i, z_{i-1}, z_i)\, db_1$$
$$\times \cdots \times 2k(\tau_{j_{k-1}}, z_{j_{k-1}-1} + z_{j_{k-1}} - 2b_{k-2}) \int_{b_{k-2}}^{z_{j_{k-1}} \wedge \cdots \wedge z_n} \prod_{i \in \{j_{k-1}+1, \dots, n\} \setminus \{j_k\}} p_{b_{k-1}}(\tau_i, z_{i-1}, z_i)\, db_{k-1}$$
$$\times 2k(\tau_{j_k}, z_{j_k-1} + z_{j_k} - 2b_{k-1}) \int_{b_{k-1}}^{z_n} e^{2\mu b_k}\, db_k.$$

PROOF. Replace the functions $f_i(z_i, b_k)$ in (5.5) by $f_i(z_i)$. Then we have

$$
\begin{aligned}
(5.6)\quad & E\left[\prod_{i=1}^{n} f_i(Z^{(\mu)}_{(2),t_i}); \sigma = k, T_2 = j_2, \ldots, T_k = j_k\right] \\
&= \int_0^\infty dz_1 \cdots \int_0^\infty dz_n \prod_{i=1}^{n} f_i(z_i) \cdot e^{-\mu^2 t_n/2 - \mu z_n} \\
&\quad \times 2k(t_1, z_1) \int_0^{z_1 \wedge \cdots \wedge z_n} \prod_{i=j_1+1}^{j_2-1} p_{b_1}(\tau_i, z_{i-1}, z_i)\, db_1 \times \cdots \\
&\quad \times 2k(\tau_{j_{k-1}}, z_{j_{k-1}-1} + z_{j_{k-1}} - 2b_{k-2}) \int_{b_{k-2}}^{z_{j_{k-1}} \wedge \cdots \wedge z_n} \prod_{i=j_{k-1}+1}^{j_k-1} p_{b_{k-1}}(\tau_i, z_{i-1}, z_i)\, db_{k-1} \\
&\quad \times 2k(\tau_{j_k}, z_{j_k-1} + z_{j_k} - 2b_{k-1}) \int_{b_{k-1}}^{z_{j_k} \wedge \cdots \wedge z_n} e^{2\mu b_k} \prod_{i=j_k+1}^{n} p_{b_k}(\tau_i, z_{i-1}, z_i)\, db_k.
\end{aligned}
$$

Using this formula, we prove the assertion by a (reverse) induction in k. When $k = n$, every product in $p_b(\tau_i, z_{i-1}, z_i)$ is equal to 1 by our convention and the assertion is trivial.

Assume that we have the assertion for k. Note that

$$
\begin{aligned}
&\sum_{\ell=k-1}^{n} E\left[\prod_{i=1}^{n} f_i(Z^{(\mu)}_{(2),t_i}); \sigma = \ell, T_2 = j_2, \ldots, T_{k-1} = j_{k-1}\right] \\
&\quad = E\left[\prod_{i=1}^{n} f_i(Z^{(\mu)}_{(2),t_i}); \sigma = k-1, T_2 = j_2, \ldots, T_{k-1} = j_{k-1}\right] \\
&\qquad + \sum_{j_k = j_{k-1}+1}^{n} \sum_{\ell=k}^{n} E\left[\prod_{i=1}^{n} f_i(Z^{(\mu)}_{(2),t_i}); \sigma = \ell, T_2 = j_2, \ldots, T_k = j_k\right].
\end{aligned}
$$

We apply (5.6) to the first term on the right hand side and use the assumption of the induction for the second term, replacing Δ in (4.5) by the partition $\Delta' = \{t_i\}_{i=0}^{j_k}$. Then we obtain

$$
\begin{aligned}
&\sum_{\ell=k-1}^{n} E\left[\prod_{i=1}^{n} f_i(Z^{(\mu)}_{(2),t_i}); \sigma = \ell, T_2 = j_2, \ldots, T_{k-1} = j_{k-1}\right] \\
&\quad = \int_0^\infty dz_1 \cdots \int_0^\infty dz_n \prod_{i=1}^{n} f_i(z_i) \cdot e^{-\mu^2 t_n/2 - \mu z_n}\, 2k(t_1, z_1) \int_0^{z_1 \wedge \cdots \wedge z_n} \prod_{i=j_1+1}^{j_2-1} p_{b_1}(\tau_i, z_{i-1}, z_i)\, db_1 \\
&\quad \times \cdots \times 2k(\tau_{j_{k-2}}, z_{j_{k-2}-1}, z_{j_{k-2}} - 2b_{k-3}) \int_{b_{k-3}}^{z_{j_{k-2}} \wedge \cdots \wedge z_n} \prod_{i=j_{k-2}+1}^{j_{k-1}-1} p_{b_{k-2}}(\tau_i, z_{i-1}, z_i)\, db_{k-2} \\
&\quad \times 2k(\tau_{j_{k-1}}, z_{j_{k-1}-1} + z_{j_{k-1}} - 2b_{k-2}) \int_{b_{k-2}}^{z_{j_{k-1}} \wedge \cdots \wedge z_n} F(b_{k-1})\, db_{k-1},
\end{aligned}
$$

where $F(b_{k-1})$ is given by

$$F(b_{k-1}) = \prod_{i=j_{k-1}+1}^{n} p_{b_{k-1}}(\tau_i, z_{i-1}, z_i) \cdot e^{2\mu b_{k-1}}$$
$$+ \sum_{j_k=j_{k-1}+1}^{n} \prod_{i\in\{j_{k-1}+1,\dots,n\}\setminus\{j_k\}} p_{b_{k-1}}(\tau_i, z_{i-1}, z_i) \cdot 2k(\tau_{j_k}, z_{j_k-1}+z_{j_k}-2b_{k-1}) \int_{b_{k-1}}^{z_n} e^{2\mu b_k}\, db_k.$$

Moreover, in view of the formula

$$F(b_{k-1}) = -\frac{d}{db_{k-1}}\left\{\prod_{i=j_{k-1}}^{n} p_{b_{k-1}}(\tau_i, z_{i-1}, z_i)\int_{b_{k-1}}^{z_n} e^{2\mu b_k}\, db_k\right\}$$

and

$$\prod_{i=j_{k-1}}^{n} p_{z_{j_{k-1}}\wedge\cdots\wedge z_n}(\tau, z_{i-1}, z_i) = 0,$$

we obtain

$$\int_{b_{k-2}}^{z_{j_{k-1}}\wedge\cdots\wedge z_n} F(b_{k-1})\, db_{k-1} = \prod_{i=j_{k-1}+1}^{n} p_{b_{k-2}}(\tau_i, z_{i-1}, z_i) \cdot \int_{b_{k-2}}^{z_n} e^{2\mu b_k}\, db_k.$$

Now we have shown

$$\sum_{\ell=k-1}^{n} E\left[\prod_{i=1}^{n} f_i(Z^{(\mu)}_{(2),t_i}); \sigma=\ell, T_2=j_2,\dots,T_{k-1}=j_{k-1}\right]$$
$$= \int_0^\infty dz_1 \cdots \int_0^\infty dz_n \prod_{i=1}^{n} f_i(z_i)\cdot e^{-\mu^2 t_n/2-\mu z_n}\, 2k(t_1, z_1)\int_0^{z_1\wedge\cdots\wedge z_n} \prod_{i=j_1+1}^{j_2-1} p_{b_1}(\tau_i, z_{i-1}, z_i)\, db_1$$
$$\times\cdots\times 2k(\tau_{j_{k-2}}, z_{j_{k-2}-1}+z_{j_{k-2}}-2b_{k-3})\int_{b_{k-3}}^{z_{j_{k-2}}\wedge\cdots\wedge z_n} \prod_{i=j_{k-2}+1}^{j_{k-1}-1} p_{b_{k-2}}(\tau_i, z_{i-1}, z_i)\, db_{k-2}$$
$$\times 2k(\tau_{j_{k-1}}, z_{j_{k-1}-1}+z_{j_{k-1}}-2b_{k-2})\prod_{i=j_{k-1}+1}^{n} p_{b_{k-2}}(\tau_i, z_{i-1}, z_i)\int_{b_{k-2}}^{z_n} e^{2\mu b_k}\, db_k,$$

which implies the assertion of the proposition with k replaced by $k-1$. □

The following theorem shows that $Z^{(\mu)}_{(2)} = 2M^{(\mu)} - B^{(\mu)}$ is a Markov process with the transition probability density $p^{(\mu)}_{(2)}(t,z,z')$ in (5.2) and (5.3).

Theorem 5.4. *For any sequence $\{t_i\}_{i=0}^n$ with $0=t_0<t_1<\cdots<t_n$ and non-negative Borel functions f_i on $\boldsymbol{R}_+$, one has*

$$E\left[\prod_{i=1}^{n} f_i(Z^{(\mu)}_{(2),t_i})\right] = \int_0^\infty dz_1\cdots\int_0^\infty dz_n \cdot \prod_{i=1}^{n} f_i(z_i)\cdot\prod_{i=1}^{n} p^{(\mu)}_{(2)}(\tau_i, z_{i-1}, z_i),$$

where $\tau_i = t_i - t_{i-1}$, $i=1,\dots,n$ and $z_0=0$.

PROOF. Recall the simple relation

$$\frac{\partial}{\partial b} p_b(\tau, z, z') = -2k(\tau, z + z' - 2b)$$

and use Proposition 5.2(i) and Proposition 5.3. Then we obtain

$$\begin{aligned}
E\left[\prod_{i=1}^{n} f_i(Z^{(\mu)}_{(2),t_i})\right] &= E\left[\prod_{i=1}^{n} f_i(Z^{(\mu)}_{(2),t_i}); \sigma = 1\right] \\
&\quad + \sum_{j_2=2}^{n}\sum_{\ell=2}^{n} E\left[\prod_{i=1}^{n} f_i(Z^{(\mu)}_{(2),t_i}); \sigma = \ell, T_2 = j_2\right] \\
&= \int_0^\infty dz_1 \cdots \int_0^\infty dz_n \prod_{i=1}^{n} f_i(z_i) \cdot e^{-\mu^2 t_n/2} \\
&\quad \times 2k(t_1, z_1) \int_0^{z_1 \wedge \cdots \wedge z_n} \prod_{i=2}^{n} p_{b_1}(\tau_i, z_{i-1}, z_i) \cdot e^{\mu(2b_1 - z_n)}\, db_1 \\
&\quad + \sum_{j=2}^{n} \int_0^\infty dz_1 \cdots \int_0^\infty dz_n \prod_{i=1}^{n} f_i(z_i) \cdot e^{-\mu^2 t_n/2 - \mu z_n} \\
&\quad \times 2k(t_1, z_1) \int_0^{z_1 \wedge \cdots \wedge z_n} \prod_{i \in \{2,\dots,n\} \setminus \{j_2\}} p_{b_1}(\tau_i, z_{i-1}, z_i)\, db_1 \\
&\quad \times 2k(\tau_{j_2}, z_{j_2 - 1} + z_{j_2} - 2b_1) \int_{b_1}^{z_n} e^{2\mu b_2}\, db_2 \\
&= \int_0^\infty dz_1 \cdots \int_0^\infty dz_n \prod_{i=1}^{n} f_i(z_i) \cdot e^{-\mu^2 t_n - \mu z_n} 2k(t_1, z_1) \\
&\quad \times \int_0^{z_1 \wedge \cdots \wedge z_n} \left\{ -\frac{\partial}{\partial b_1}\left[\prod_{i=2}^{n} p_{b_1}(\tau_i, z_{i-1}, z_i) \int_{b_1}^{z_n} e^{2\mu b_2}\, db_2\right]\right\} db_1.
\end{aligned}$$

Now, noting that

$$\prod_{i=1}^{n} p_{z_1 \wedge \cdots \wedge z_n}(\tau_i, z_{i-1}, z_i) = 0,$$

we obtain the assertion of the theorem after some easy manipulations. □

6. Non-Markov property of $cM - X$ when $c > 1$ and $c \neq 2$.

In this section we prove Theorem 1.1 in the case of $1 < c < 2$. We can prove the assertion in the case of $c > 2$ by the same way and we leave the details to the reader. We first give explicit expressions of the probability densities of the two and three dimensional distributions of $Z^{(\mu)}_{(c)} = cM^{(\mu)} - B^{(\mu)}$.

PROPOSITION 6.1. *Assume $c > 1$ and $c \neq 2$. Then, for any $t_1 < t_2$, the distribution of $(Z^{(\mu)}_{(c),t_1}, Z^{(\mu)}_{(c),t_2})$ on $\boldsymbol{R}_+^2$ admits a density $q^{(\mu)}_{(c),2}(t_1,z_1,t_2,z_2)$ with respect to the Lebesgue measure which is given by*

$$q^{(\mu)}_{(c),2}(t_1,z_1,t_2,z_2) = 2e^{-\mu^2 t_2/2-\mu z_2} \int_0^{(z_1\wedge z_2)/(c-1)} k(t_1, z_1+(2-c)b_1)\,db_1$$
$$\times \left[2\int_{b_1}^{z_2/(c-1)} e^{\mu c b_2} k(\tau_2, z_1+z_2-cb_1+(2-c)b_2)\,db_2 \right.$$
$$\left. + e^{c\mu b_1}[\varphi(\tau_2, z_1-z_2) - \varphi(\tau_2, z_1+z_2+2(1-c)b_1)] \right],$$

where $\tau_2 = t_2 - t_1$.

We can prove this proposition in the same way as Propositions 4.1 and 5.1 and we omit it.

We next consider the three-dimensional distribution.

PROPOSITION 6.2. *Assume $c > 1$ and $c \neq 2$. Then, for any $t_1 < t_2 < t_3$, the distribution of $(Z^{(\mu)}_{(c),t_1}, Z^{(\mu)}_{(c),t_2}, Z^{(\mu)}_{(c),t_3})$ admits a density $q^{(\mu)}_{(c),3}(t_1,z_1,t_2,z_2,t_3,z_3)$ given by*

$$q^{(\mu)}_{(c),3}(t_1,z_1,t_2,z_2,t_3,z_3) = \int_0^{(z_1\wedge z_2\wedge z_3)/(c-1)} 2e^{-\mu^2 t_3/2-\mu z_3} k(t_1, z_1+(2-c)b_1)$$
$$\times f^{(\mu)}_{(c)}(b_1, z_1, \tau_2, z_2, \tau_3, z_3)\,db_1,$$

where $\tau_i = t_i - t_{i-1}$ and $f^{(\mu)}_{(c)} = \sum_{i=1}^4 f^{(\mu)}_{(c),i}$ with

$$f^{(\mu)}_{(c),1} = 4\int_{b_1}^{(z_2\wedge z_3)/(c-1)} k(\tau_2, z_1+z_2-cb_1+(2-c)b_2)\,db_2$$
$$\times \int_{b_2}^{z_3/(c-1)} e^{c\mu b_3} k(\tau_3, z_2+z_3-cb_2+(2-c)b_3)\,db_3,$$

$$f^{(\mu)}_{(c),2} = 2\int_{b_1}^{(z_2\wedge z_3)/(c-1)} e^{c\mu b_2} k(\tau_2, z_1+z_2-cb_1+(2-c)b_2)$$
$$\times [\varphi(\tau_3, z_2-z_3) - \varphi(\tau_3, z_2+z_3+2(1-c)b_2)]\,db_2,$$

$$f^{(\mu)}_{(c),3} = 2[\varphi(\tau_2, z_1-z_2) - \varphi(\tau_2, z_1+z_2+2(1-c)b_1)]$$
$$\times \int_{b_1}^{z_3/(c-1)} e^{c\mu b_3} k(\tau_3, z_2+z_3-cb_1+(2-c)b_3)\,db_3,$$

$$f^{(\mu)}_{(c),4} = e^{c\mu b_1}[\varphi(\tau_2, z_1-z_2) - \varphi(\tau_2, z_1+z_2+2(1-c)b_1)]$$
$$\times [\varphi(\tau_3, z_2-z_3) - \varphi(\tau_3, z_2+z_3+2(1-c)b_1)].$$

PROOF. For non-negative Borel functions f_1, f_2, f_3 on $\boldsymbol{R}_+$, we set

$$I_1 = E[f_1(Z^{(\mu)}_{(c),t_1})f_2(Z^{(\mu)}_{(c),t_2})f_3(Z^{(\mu)}_{(c),t_3}); M^{(\mu)}_{t_1} \leqq M^{(\mu)}_{t_1,t_2} \leqq M^{(\mu)}_{t_2,t_3}],$$

$$I_2 = E[f_1(Z^{(\mu)}_{(c),t_1})f_2(Z^{(\mu)}_{(c),t_2})f_3(Z^{(\mu)}_{(c),t_3}); M^{(\mu)}_{t_1} \leqq M^{(\mu)}_{t_1,t_2}, M^{(\mu)}_{t_2,t_3} < M^{(\mu)}_{t_1,t_2}],$$

$$I_3 = E[f_1(Z^{(\mu)}_{(c),t_1})f_2(Z^{(\mu)}_{(c),t_2})f_3(Z^{(\mu)}_{(c),t_3}); M^{(\mu)}_{t_1,t_2} < M^{(\mu)}_{t_1} \leqq M^{(\mu)}_{t_2,t_3}],$$

$$I_4 = E[f_1(Z^{(\mu)}_{(c),t_1})f_2(Z^{(\mu)}_{(c),t_2})f_3(Z^{(\mu)}_{(c),t_3}); M^{(\mu)}_{t_1,t_2} < M^{(\mu)}_{t_1}, M^{(\mu)}_{t_2,t_3} < M^{(\mu)}_{t_1}].$$

We compute each I_i using (2.7) and (2.8).

For I_1, we have

$$I_1 = \int_0^\infty db_1 \int_{-\infty}^{b_1} da_1 \int_{b_1}^\infty db_2 \int_{-\infty}^{b_2} da_2 \int_{b_2}^\infty db_3 \int_{-\infty}^{b_3} da_3 \cdot f_1(cb_1 - a_1)f_2(cb_2 - a_2)f_3(cb_3 - a_3)$$
$$\times 8e^{-\mu^2 t_3/2+\mu a_3} k(t_1, 2b_1 - a_1)k(\tau_2, 2b_2 - a_1 - a_2)k(\tau_3, 2b_3 - a_2 - a_3).$$

Changing the variables by $z_i = cb_i - a_i$, $i = 1,2,3$, and the order of integrations, we obtain

$$I_1 = \int_0^\infty f_1(z_1)\,dz_1 \int_0^\infty f_2(z_2)\,dz_2 \int_0^\infty f_3(z_3)\,dz_3 \cdot 2e^{-\mu^2 t_3/2-\mu z_3}$$
$$\times \int_0^{(z_1\wedge z_2\wedge z_3)/(c-1)} k(t_1, z_1 + (2-c)b_1) f^{(\mu)}_{(c),1}(b_1, z_1, \tau_2, z_2, \tau_3, z_3)\,db_1.$$

For I_2, from (2.7), we deduce

$$I_2 = \int_0^\infty db_1 \int_{-\infty}^{b_1} da_1 \int_{b_1}^\infty db_2 \int_{-\infty}^{b_2} da_2 \int_{-\infty}^{b_2} da_3 \cdot f_1(cb_1 - a_1)f_2(cb_2 - a_2)f_3(cb_3 - a_3)$$
$$\times 4e^{-\mu^2 t_3/2+\mu a_3} k(t_1, 2b_1 - a_1)k(\tau_2, 2b_2 - a_1 - a_2)p_{b_2}(\tau_3, a_2, a_3).$$

Change the variables by $z_1 = 2b_1 - a_1$ and $z_i = cb_2 - a_i$, $i = 2,3$, and the order of integrations. Then we obtain

$$I_2 = \int_0^\infty f_1(z_1)\,dz_1 \int_0^\infty f_2(z_2)\,dz_2 \int_0^\infty f_3(z_3)\,dz_3 \cdot 2e^{-\mu^2 t_3/2-\mu z_3}$$
$$\times \int_0^{(z_1\wedge z_2\wedge z_3)/(c-1)} k(t_1, z_1 + (2-c)b_1) f^{(\mu)}_{(c),2}(b_1, z_1, \tau_2, z_2, \tau_3, z_3)\,db_1.$$

We can modify I_3 and I_4 in the same way. □

For a proof of Theorem 1.1, we consider the conditional probability distribution of $Z^{(\mu)}_{(c),t_3}$ given $Z^{(\mu)}_{(c),t_1}$ and $Z^{(\mu)}_{(c),t_2}$:

$$P(Z^{(\mu)}_{(c),t_3} \in dz_3 \mid Z^{(\mu)}_{(c),t_1} = z_1, Z^{(\mu)}_{(c),t_2} = z_2) = \frac{q^{(\mu)}_{(c),3}(t_1, z_1, t_2, z_2, t_3, z_3)}{q^{(\mu)}_{(c),2}(t_1, z_1, t_2, z_2)} dz_3. \tag{6.1}$$

If $Z^{(\mu)}_{(c)}$ were a Markov process, then the density $q^{(\mu)}_{(c),3}/q^{(\mu)}_{(c),2}$ would not depend on t_1 and z_1. We show that it does depend on.

For this purpose, using the following lemma which is easily proved by an elementary Laplace method, we consider the asymptotic behavior as $t_1 \downarrow 0$ of the density on the right hand side of (6.1).

LEMMA 6.3. *Let φ and k be the functions defined by* (2.1) and (2.2), *respectively, and let* $z > 0$, $\delta > 0$, $\alpha > 0$. *Then, for any C^1 function g on* $[0,\delta]$, *one has*

$$\int_0^\delta k(t, z+\alpha b) g(b)\, db = \frac{1}{\alpha}\varphi(t,z)\left(g(0) + \frac{g'(0)}{\alpha z} t + o(t)\right) \tag{6.2}$$

as $t \downarrow 0$. Moreover, one also has (6.2) *for $\delta = \infty$, provided that the function g on $\boldsymbol{R}_+$ satisfies $|g(b)| + |g'(b)| \leqq C_1 \exp(C_2 b^2)$ for some $C_1, C_2 > 0$.*

Now we are in a position to give a proof of Theorem 1.1 in the case of $1 < c < 2$. We notice that we can prove Theorem 1.1 in the case of $c > 2$ by the same arguments, where we use a similar asymptotic formula for $\alpha < 0$ to (6.2).

PROOF OF THEOREM 1.1 IN THE CASE OF $1 < c < 2$. From Proposition 6.1 and Lemma 6.3 (here we consider only the leading term), we deduce

$$q_{(c),2}^{(\mu)}(t_1, z_1, t_2, z_2) = \frac{2}{2-c} e^{-\mu^2 t_2/2 - \mu z_2} \varphi(t_1, z_1) \phi_2(z_1, \tau_2, z_2) \cdot (1 + o(1))$$

for the density $q_{(c),2}^{(\mu)}$ of $(Z_{t_1}^{(c)}, Z_{t_2}^{(c)})$ as $t_1 \downarrow 0$, where

$$\phi_2(z_1, \tau_2, z_2) = 2\int_0^{z_2/(c-1)} e^{\mu c \beta} k(\tau_2, z_1 + z_2 + (2-c)\beta)\, d\beta$$
$$+ \varphi(\tau_2, z_1 - z_2) - \varphi(\tau_2, z_1 + z_2).$$

For the density $q_{(c),3}^{(\mu)}$ of the three-dimensional distribution, we deduce from Proposition 6.2 and Lemma 6.3

$$q_{(c),3}^{(\mu)}(t_1, z_1, t_2, z_2, t_3, z_3) = \frac{2}{2-c} e^{-\mu^2 t_3/2 - \mu z_3} f_{(c)}^{(\mu)}(0, z_1, \tau_2, z_2, \tau_3, z_3) \varphi(t_1, z_1) \cdot (1 + o(1))$$

as $t_1 \downarrow 0$. Hence we obtain

$$\lim_{t_1 \downarrow 0} \frac{q_{c,3}^{(\mu)}(t_1, z_1, t_2, z_2, t_3, z_3)}{q_{(c),2}^{(\mu)}(t_1, z_1, t_2, z_2)} = e^{-\mu^2 \tau_3/2 - \mu(z_3 - z_2)} r_{(c)}^{(\mu)}(z_1, \tau_2, z_2, \tau_3, z_3), \tag{6.3}$$

where

$$r_{(c)}^{(\mu)}(z_1, \tau_2, z_2, \tau_3, z_3) = \frac{f_{(c)}^{(\mu)}(0, z_1, \tau_2, z_2, \tau_3, z_3)}{\phi_2(z_1, \tau_2, z_2)}.$$

We set

$$\psi(z_1, \tau_2, z_2) = \varphi(\tau_2, z_1 - z_2) + \frac{c}{2-c}\varphi(\tau_2, z_1 + z_2).$$

It is not hard to see

$$f_{(c)}^{(\mu)}(0, z_1, \tau_2, z_2, \tau_3, z_3) = \psi(z_1, \tau_2, z_2) \phi_2(z_2, \tau_3, z_3) \cdot (1 + o(1))$$

and

$$\phi_2(z_1,\tau_2,z_2)=\psi(z_1,\tau_2,z_2)\cdot(1+o(1))$$

as $\tau_2\downarrow 0$. However these show that the limit of $r^{(\mu)}_{(c)}$ as $\tau_2\downarrow 0$ is independent of z_1. Thus we have to make a more precise inspection of $f^{(\mu)}_{(c)}$ and ϕ_2.

First we consider $\phi_2(z_1,\tau_2,z_2)$. With the help of (2.2), we obtain

$$\text{(6.4)}\qquad \begin{aligned}\phi_2(z_1,\tau_2,z_2)&=\psi(z_1,\tau_2,z_2)-\frac{2}{2-c}e^{\mu c z_2/(c-1)}\varphi\left(\tau_2,z_1+\frac{z_2}{c-1}\right)\\&\quad+\frac{2\mu c}{2-c}\int_0^{z_2/(c-1)}e^{\mu c\beta}\varphi(\tau_2,z_1+z_2+(2-c)\beta)\,d\beta\\&=\psi(z_1,\tau_2,z_2)+\frac{2\mu c}{(2-c)^2(z_1+z_2)}\tau_2\varphi(\tau_2,z_1+z_2)+o(\tau_2\varphi(\tau_2,z_1+z_2))\end{aligned}$$

as $\tau_2\downarrow 0$, where we have used Lemma 6.3 for the second equality.

Next we consider each $\tilde f^{(\mu)}_{(c),i}=f^{(\mu)}_{(c),i}(0,z_1,\tau_2,z_2,\tau_3,z_3)$, $i=1,\dots,4$. For $\tilde f^{(\mu)}_{(c),1}$, by using Proposition 6.2 and Lemma 6.3, we deduce

$$\begin{aligned}\tilde f^{(\mu)}_{(c),1}&=4\int_0^{(z_2\wedge z_3)/(c-1)}k(\tau_2,z_1+z_2+(2-c)b_2)\,db_2\\&\quad\times\int_{b_2}^{z_3/(c-1)}e^{c\mu b_3}k(\tau_3,z_2+z_3-cb_2+(2-c)b_3)\,db_3\\&=\frac{4}{2-c}\int_0^{z_3/(c-1)}e^{c\mu b_3}k(\tau_3,z_2+z_3+(2-c)b_3)\,db_3\cdot\varphi(\tau_2,z_1+z_2)-\frac{4}{(2-c)^2(z_1+z_2)}\\&\quad\times\left\{k(\tau_3,z_2+z_3)+c\int_0^{z_3/(c-1)}e^{c\mu b_3}k'(\tau_3,z_2+z_3+(2-c)b_3)\,db_3\right\}\tau_2\varphi(\tau_2,z_1+z_2)\\&\quad+o(\tau_2\varphi(\tau_2,z_1+z_2)),\end{aligned}$$

where $k'(t,\xi)=(\partial/\partial\xi)k(t,\xi)$.

For $\tilde f^{(\mu)}_{(c),2}$, by Lemma 6.3, we obtain

$$\begin{aligned}\tilde f^{(\mu)}_{(c),2}&=\frac{2}{2-c}[\varphi(\tau_3,z_2-z_3)-\varphi(\tau_3,z_2+z_3)]\varphi(\tau_2,z_1+z_2)\\&\quad+\frac{2}{(2-c)^2(z_1+z_2)}\{c\mu[\varphi(\tau_3,z_2-z_3)-\varphi(\tau_3,z_2+z_3)]\\&\quad-2(c-1)k(\tau_3,z_2+z_3)\}\tau_2\varphi(\tau_2,z_1+z_2)+o(\tau_2\varphi(\tau_2,z_1+z_2)).\end{aligned}$$

For $\tilde f^{(\mu)}_{(c),3}$ and $\tilde f^{(\mu)}_{(c),4}$, we have

$$\tilde f^{(\mu)}_{(c),3}=2\int_0^{z_3/(c-1)}e^{c\mu b_3}k(\tau_3,z_2+z_3+(2-c)b_3)\,db_3\cdot[\varphi(\tau_2,z_1-z_2)-\varphi(\tau_2,z_1+z_2)]$$

and

$$\tilde{f}^{(\mu)}_{(c),4} = [\varphi(\tau_3, z_2 - z_3) - \varphi(\tau_3, z_2 + z_3)][\varphi(\tau_2, z_1 - z_2) - \varphi(\tau_2, z_1 + z_2)].$$

Summing up the results for $\tilde{f}^{(\mu)}_{(c),i}$, we obtain

$$\begin{aligned} f^{(\mu)}_{(c)}(0, z_1, \tau_2, z_2, \tau_3, z_3) = \psi(z_1, \tau_2, z_2)\phi_2(z_2, \tau_3, z_3) + \frac{2}{(2-c)^2(z_1+z_2)} \\ \times \Big\{ c\mu[\varphi(\tau_3, z_2 - z_3) - \varphi(\tau_3, z_2 + z_3)] - 2ck(\tau_3, z_2 + z_3) \\ - 2c \int_0^{z_3/(c-1)} e^{c\mu b_3} k'(\tau_3, z_2 + z_3 + (2-c)b_3)\, db_3 \Big\} \tau_2 \varphi(\tau_2, z_1 + z_2) \\ + o(\tau_2 \varphi(\tau_2, z_1 + z_2)). \end{aligned}$$

Now, comparing this asymptotics as $\tau_2 \downarrow 0$ of $\tilde{f}^{(\mu)}_{(c)}$ with that of $\phi_2(z_1, \tau_2, z_2)$ given by (6.4), we obtain the desired dependence of $r^{(\mu)}_{(c)}(z_1, \tau_2, z_2, \tau_3, z_3)$ on the right hand side of (6.3) on z_1 if we show

$$\begin{aligned} \mu \int_0^{z_3/(c-1)} e^{c\mu b_3} k(\tau_3, z_2 + z_3 + (2-c)b_3)\, db_3 + k(\tau_3, z_2 + z_3) \\ + \int_0^{z_3/(c-1)} e^{c\mu b_3} k'(\tau_3, z_2 + z_3 + (2-c)b_3)\, db_3 \not\equiv 0. \end{aligned} \tag{6.5}$$

To show this, we note, by integration by parts, that the left hand side is equal to

$$\begin{aligned} -\frac{2(c-1)\mu}{2-c} \int_0^{z_3/(c-1)} e^{c\mu b_3} k(\tau_3, z_2 + z_3 + (2-c)b_3)\, db_3 \\ -\frac{c-1}{2-c} k(\tau_3, z_2 + z_3) + \frac{1}{2-c} e^{c\mu z_3/(c-1)} k\left(\tau_3, z_2 + \frac{z_3}{c-1}\right) \end{aligned}$$

and consider the asymptotics as $\tau_3 \downarrow 0$. Then, the first term is exactly of order $O(\varphi(\tau_3, z_2 + z_3))$ by Lemma 6.3 and the third term is exponentially smaller than the second term. Therefore the second term is the main term, which is strictly negative. Now we have shown (6.5) and the proof of Theorem 1.1 in the case of $1 < c < 2$ is completed. □

7. Non-Markov property of $cM - X$ when $c < 1$ and $c \neq 0$.

In this final section we prove Theorem 1.1 in the case of $c < 1$ and $c \neq 0$, which, together with the results in the previous section, completes our proof of Theorem 1.1. Note that $Z^{(\mu)}_{(c),t} = cM^{(\mu)}_t - B^{(\mu)}_t$ takes values in the whole $\boldsymbol{R}$ in this case. However, the method is the same as in the previous section and we show only a sketch of the proof.

At first we give explicit forms of the probability densities of the two and three dimensional distributions. We can prove the following two propositions in the same way as in the proofs of Propositions 6.1 and 6.2.

Proposition 7.1. *Assume $c < 1$ and $c \neq 0$. Then, for any $t_1 < t_2$, the distribution of $(Z^{(\mu)}_{(c),t_1}, Z^{(\mu)}_{(c),t_2})$ admits a density $q^{(\mu)}_{(c),2}(t_1, z_1, t_2, z_2)$ with respect to the Lebesgue measure which is given by*

$$\begin{aligned}
q^{(\mu)}_{(c),2}(t_1, z_1, t_2, z_2) = e^{-\mu^2 t_2/2 - \mu z_2}\Bigg[& 2\int_{(z_1 \wedge 0)/(c-1)}^{\infty} k(t_1, z_1 + (2-c)b_1)\, db_1 \\
& \times \int_{b_1 \vee (z_2/(c-1))}^{\infty} e^{c\mu b_2} k(\tau_2, z_1 + z_2 - cb_1 + (2-c)b_2)\, db_2 \\
& + \int_{(z_1 \wedge z_2 \wedge 0)/(c-1)}^{\infty} e^{\mu c b_1} k(t_1, z_1 + (2-c)b_1) \\
& \times [\varphi(\tau_2, z_1 - z_2) - \varphi(\tau_2, z_1 + z_2 + 2(1-c)b_1)]\, db_1 \Bigg],
\end{aligned}$$

where $\tau_2 = t_2 - t_1$, $z_1, z_2 \in \boldsymbol{R}$.

Proposition 7.2. *For any $t_1 < t_2 < t_3$, the distribution of $(Z^{(\mu)}_{(c),t_1}, Z^{(\mu)}_{(c),t_2}, Z^{(\mu)}_{(c),t_3})$ admits a density $q^{(\mu)}_{(c),3}(t_1, z_1, t_2, z_2, t_3, z_3)$ given by*

$$q^{(\mu)}_{(c),3}(t_1, z_1, t_2, z_2, t_3, z_3) = \sum_{i=1}^{4} q^{(\mu)}_{(c),3,i}(t_1, z_1, t_2, z_2, t_3, z_3),$$

where

$$\begin{aligned}
q^{(\mu)}_{(c),3,1} = 8e^{-\mu^2 t_3/2 - \mu z_3} & \int_{(z_1 \wedge 0)/(c-1)}^{\infty} k(t_1, z_1 + (2-c)b_1)\, db_1 \\
& \times \int_{b_1 \vee (z_2/(c-1))}^{\infty} k(\tau_2, z_1 + z_2 - cb_1 + (2-c)b_2)\, db_2 \\
& \times \int_{b_2 \vee (z_3/(c-1))}^{\infty} e^{c\mu b_3} k(\tau_3, z_2 + z_3 - cb_2 + (2-c)b_3)\, db_3, \\
q^{(\mu)}_{(c),3,2} = 4e^{-\mu^2 t_3/2 - \mu z_3} & \int_{(z_1 \wedge 0)/(c-1)}^{\infty} k(t_1, z_1 + (2-c)b_1)\, db_1 \\
& \times \int_{b_1 \vee ((z_2 \wedge z_3)/(c-1))}^{\infty} e^{c\mu b_2} k(\tau_2, z_1 + z_2 - cb_1 + (2-c)b_2) \\
& \times [\varphi(\tau_3, z_2 - z_3) - \varphi(\tau_3, z_2 + z_3 + 2(1-c)b_2)]\, db_2, \\
q^{(\mu)}_{(c),3,3} = 4e^{-\mu^2 t_3/2 - \mu z_3} & \int_{(z_1 \wedge z_2 \wedge 0)/(c-1)}^{\infty} k(t_1, z_1 + (2-c)b_1) \\
& \times [\varphi(\tau_2, z_1 - z_2) - \varphi(\tau_2, z_1 + z_2 + 2(1-c)b_1)]\, db_1 \\
& \times \int_{b_1 \vee (z_3/(c-1))}^{\infty} e^{c\mu b_3} k(\tau_3, z_2 + z_3 - cb_1 + (2-c)b_3)\, db_3,
\end{aligned}$$

$$q^{(\mu)}_{(c),3,4} = 2e^{-\mu^2 t_3/2-\mu z_3} \int_{(z_1\wedge z_2\wedge z_3\wedge 0)/(c-1)}^{\infty} e^{c\mu b_1} k(t_1, z_1+(2-c)b_1)$$
$$\times [\varphi(\tau_2, z_1-z_2) - \varphi(\tau_2, z_1+z_2+2(1-c)b_1)]$$
$$\times [\varphi(\tau_3, z_2-z_3) - \varphi(\tau_3, z_2+z_3+2(1-c)b_1)]\, db_1.$$

Using these propositions, we can complete our proof of Theorem 1.1 in the same way as in the previous section.

Proof of Theorem 1.1 when $c<1$ and $c\neq 0$. We recall that, for $t_1<t_2<t_3$,

$$P(Z^{(\mu)}_{(c),t_3}\in dz_3 \,|\, Z^{(\mu)}_{(c),t_1}=z_1, Z^{(\mu)}_{(c),t_2}=z_2) = \frac{q^{(\mu)}_{(c),3}(t_1,z_1,t_2,z_2,t_3,z_3)}{q^{(\mu)}_{(c),2}(t_1,z_1,t_2,z_2)} dz_3.$$

If $Z^{(\mu)}_{(c)}$ were a Markov process, then the density on the right hand side would not depend on t_1 and z_1.

We fix $z_1,z_2,z_3>0$. By Proposition 7.1 and Lemma 6.3, it is easy to show

$$q^{(\mu)}_{(c),2}(t_1,z_1,t_2,z_2) = \frac{2}{2-c} e^{-\mu^2 t_2/2-\mu z_2} \phi_3(z_1,\tau_2,z_2)\varphi(t_1,z_2)\cdot(1+o(1)),$$

as $t_1\downarrow 0$, where

$$\phi_3(z_1,\tau_2,z_2) = 2\int_0^\infty e^{c\mu b} k(\tau_2, z_1+z_2+(2-c)b)\, db + \varphi(\tau_2,z_1-z_2) - \varphi(\tau_2,z_1+z_2).$$

For the numerator $q^{(\mu)}_{(c),3}$, we also consider the asymptotics as $t_1\downarrow 0$. Then we can show that the limit

$$\lim_{t_1\downarrow 0} \frac{q^{(\mu)}_{(c),3}(t_1,z_1,t_2,z_2,t_3,z_3)}{q^{(\mu)}_{(c),2}(t_1,z_1,t_2,z_2)} = \tilde{r}^{(\mu)}_{(c)}(z_1,\tau_2,z_2,\tau_3,z_3)$$

exists and

$$\tilde{r}^{(\mu)}_{(c)}(z_1,\tau_2,z_2,\tau_3,z_3) = e^{-\mu^2\tau_3/2-\mu(z_3-z_2)} \frac{h^{(\mu)}_{(c)}(z_1,\tau_2,z_2,\tau_3,z_3)}{\phi_3(z_1,\tau_2,z_2)},$$

where $h^{(\mu)}_{(c)} = \sum_{i=1}^4 h^{(\mu)}_{(c),i}$ with

$$h^{(\mu)}_{(c),1} = 4\int_0^\infty k(\tau_2,z_1+z_2+(2-c)b_2)\, db_2 \int_{b_2}^\infty e^{c\mu b_3} k(\tau_3, z_2+z_3-cb_2+(2-c)b_3)\, db_3,$$

$$h^{(\mu)}_{(c),2} = 2\int_0^\infty e^{c\mu b_2} k(\tau_2,z_1+z_2+(2-c)b_2)[\varphi(\tau_3,z_2-z_3) - \varphi(\tau_3,z_2+z_3+2(1-c)b_2)]\, db_2,$$

$$h^{(\mu)}_{(c),3} = 2[\varphi(\tau_2,z_1-z_2)-\varphi(\tau_2,z_1+z_2)]\int_0^\infty e^{c\mu b_3} k(\tau_3,z_2+z_3+(2-c)b_3)\, db_3,$$

$$h^{(\mu)}_{(c),4} = [\varphi(\tau_2,z_1-z_2)-\varphi(\tau_2,z_1+z_2)][\varphi(\tau_3,z_2-z_3)-\varphi(\tau_3,z_2+z_3)].$$

Next we consider the asymptotic behavior of $\tilde{r}_{(c)}^{(\mu)}(z_1, \tau_2, z_2, \tau_3, z_3)$ as $\tau_2 \downarrow 0$. Again we will have the same leading and second terms in the denominator and the numerator. For the denominator, we obtain by Lemma 6.3

$$\phi_3(z_1, \tau_2, z_2) = \psi(z_1, \tau_2, z_2) + \frac{2c\mu}{(2-c)^2(z_1+z_2)}\tau_2\varphi(\tau_2, z_1+z_2)\cdot(1+o(1)).$$

For the numerator, some computations similar to those in the previous section yield

$$\begin{aligned} h_{(c)}^{(\mu)}(z_1, \tau_2, z_2, \tau_3, z_3) &= \phi_3(z_2, \tau_3, z_3)\psi(z_1, \tau_2, z_3) + \frac{2c}{(2-c)^2(z_1+z_2)} \\ &\quad \times \left[\mu[\varphi(\tau_3, z_2 - z_3) - \varphi(\tau_3, z_2+z_3)] - 2k(\tau_3, z_2+z_3)\right. \\ &\quad \left. - 2\int_0^\infty e^{c\mu b_3} k'(\tau_3, z_2+z_3+(2-c)b_3)\, db_3\right]\tau_2\varphi(\tau_2, z_1+z_2) \\ &\quad + o(\tau_2\varphi(\tau_2, z_1+z_2)). \end{aligned}$$

Finally, taking the asymptotics as $\tau_3 \downarrow 0$ into account, we obtain the dependence of $\tilde{r}_{(c)}^{(\mu)}(z_1, \tau_2, z_2, \tau_3, z_3)$ on z_1 and complete the proof of Theorem 1.1. □

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