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Well-behaved unbounded operator representations and unbounded C^*-seminorms

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Abstract. The first purpose is to characterize the existence of well-behaved $*$-representations of locally convex $*$-algebras by unbounded C^*-seminorms. The second is to define the notion of spectral $*$-representations and to characterize the existence of spectral well-behaved $*$-representations by unbounded C^*-seminorms.

1. Introduction.

Unbounded $*$-representations of $*$-algebras were considered for the first time in 1962, independently by H. J. Borchers [**9**] and A. Uhlmann [**33**] in the Wightman formulation of quantum field theory. A systematic study was undertaken only at the beginning of 1970, first by R. T. Powers [**28**] and G. Lassner [**21**], then by many mathematician, from the pure mathematical situations (operator theory, unbounded operator algebras, locally convex $*$-algebras, representations of Lie algebras, quantum groups etc.) and the physical applications (Wightman quantum field theory, unbounded CCR-algebras etc.). A survey of the theory of unbounded $*$-representations may be found in the monograph of K. Schmüdgen [**30**] and the lecture note of A. I. of us [**18**].

In the previous paper [**6**] two of us and H. Ogi have constructed unbounded $*$-representations of $*$-algebras on the basis of unbounded C^*-seminorms. In this context there has been investigated a class of well-behaved $*$-representations. Recently, Schmüdgen [**31**] has defined another (but related) notion of well-behaved $*$-representations. Those notions were considered in order to avoid pathologies which may appear for general $*$-representations and to select "nice" representations which may have a rich theory. In this paper we shall study the well-behavedness of unbounded $*$-reprensetations of *locally convex $*$-algebras* and characterize the existence of well-behaved $*$-representations of locally convex $*$-algebras by unbounded C^*-seminorms. Let $\mathscr{A}$ be a pseudo-complete locally convex $*$-algebra with identity 1 and let $\mathscr{A}_0$ be the Allan bounded part of $\mathscr{A}$ ([**1**]). In general, $\mathscr{A}_0$ is not even a subspace, and so we use the $*$-subalgebra $\mathscr{A}_b$ generated by the hermitian part of $\mathscr{A}_0$ as bounded $*$-subalgebra of $\mathscr{A}$. Let $\mathscr{I}_b$ be the largest left ideal of $\mathscr{A}$ contained in $\mathscr{A}_b$, that is, $\mathscr{I}_b = \{x \in \mathscr{A}_b; ax \in \mathscr{A}_b, {}^{\forall}a \in \mathscr{A}\}$. A $*$-representation π of $\mathscr{A}$ is said to be *uniformly nondegenerate* if $\pi(\mathscr{I}_b)\mathscr{D}(\pi)$ is total in the non-zero Hilbert space $\mathscr{H}_\pi$. A non-zero mapping p of a $*$-subalgebra $\mathscr{D}(p)$ of $\mathscr{A}$ into $\boldsymbol{R}^+ = [0, \infty)$ is said to be an *unbounded C^*-seminorm* on $\mathscr{A}$ if it is a C^*-seminorm on $\mathscr{D}(p)$. In [**6**] we have constructed a class $\{\pi_p\}$ of $*$-

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representations of $\mathscr{A}$ from an unbounded C^*-seminorm p on $\mathscr{A}$ as follows: $N_p \equiv \ker p$ is a $*$-ideal of $\mathscr{D}(p)$, and so the quotient $*$-algebra $\mathscr{D}(p)/N_p$ is a normed $*$-algebra with the C^*-norm $\|x + N_p\|_p \equiv p(x)$, $x \in \mathscr{D}(p)$. Let $\mathscr{A}_p$ denote the C^*-algebra obtained by the completion of $\mathscr{D}(p)/N_p$ and let Π_p be any faithful $*$-representation of $\mathscr{A}_p$ on a Hilbert space $\mathscr{H}_{\Pi_p}$. We define

$$\begin{cases} \mathscr{D}(\pi_p) = \text{the linear span of } \{\Pi_p(x+N_p)\xi; x \in \mathfrak{N}_p, \xi \in \mathscr{H}_{\Pi_p}\}, \\ \pi_p(a)\left(\sum_k \Pi_p(x_k + N_p)\xi_k\right) = \sum_k \Pi_p(ax_k + N_p)\xi_k \\ \qquad \text{for } a \in \mathscr{A},\ \{x_k\} \in \mathfrak{N}_p,\ \{\xi_k\} \in \mathscr{H}_{\Pi_p}, \end{cases}$$

where $\mathfrak{N}_p$ is a left ideal of $\mathscr{A}$ defined by

$$\mathfrak{N}_p = \{x \in \mathscr{D}(p); ax \in \mathscr{D}(p), {}^\forall a \in \mathscr{A}\}.$$

Then π_p is a $*$-representation of $\mathscr{A}$ on $\mathscr{H}_{\pi_p}$ (the closure of $\mathscr{D}(\pi_p)$ in $\mathscr{H}_{\Pi_p}$) such that $\|\overline{\pi_p(x)}\| = p(x)$, ${}^\forall x \in \mathfrak{N}_p$ and $\|\overline{\pi_p(x)}\| \le p(x)$, ${}^\forall x \in \mathscr{D}(p)$. The class of well-behaved $*$-representations is now selected as a subclass of the class $\{\pi_p\}$ of $*$-representations constructed before. If there exists a faithful $*$-representation Π_p of $\mathscr{A}_p$ on $\mathscr{H}_{\Pi_p}$ such that $\Pi_p((\mathfrak{N}_p \cap \mathscr{I}_b) + N_p)\mathscr{H}_{\Pi_p}$ is total in $\mathscr{H}_{\Pi_p}$, then p is said to be *topologically w-semifinite*, and the $*$-representation π_p of $\mathscr{A}$ constructed from such a Π_p is said to be *well-behaved*. Note that this implies $\|\overline{\pi_p(x)}\| = p(x)$, ${}^\forall x \in \mathscr{D}(p)$. The first purpose of this paper is to show that there exists a well-behaved $*$-representation of $\mathscr{A}$ if and only if there exists a uniformly nondegenerate $*$-representation of $\mathscr{A}$ and that this is the case if and only if there exists an unbounded C^*-seminorm p on $\mathscr{A}$ such that $\mathfrak{N}_p \cap \mathscr{I}_b \not\subset N_p$. Next we shall investigate the spectrality of $*$-representations. The spectrum $Sp_{\mathscr{A}_b}(x)$ and the spectral radius $r_{\mathscr{A}_b}(x)$ of $x \in \mathscr{A}$ are defined by

$$Sp_{\mathscr{A}_b}(x) = \{\lambda \in \boldsymbol{C}; {}^\nexists(\lambda 1 - x)^{-1} \text{ in } \mathscr{A}_b\},$$
$$r_{\mathscr{A}_b}(x) = \sup\{|\lambda|; \lambda \in Sp_{\mathscr{A}_b}(x)\}.$$

A $*$-representation π of $\mathscr{A}$ is said to be *spectral* if $Sp_{\mathscr{A}_b}(x) \subset Sp_{C_u^*(\pi)}(\overline{\pi(x)})$ for each $x \in \mathscr{A}_b$, where $C_u^*(\pi)$ is the C^*-algebra generated by $\overline{\pi(\mathscr{A}_b)}$. If $\pi \restriction \mathscr{B}$ is spectral for each unital closed $*$-subalgebra $\mathscr{B}$ of $\mathscr{A}$, then π is said to be *hereditary spectral*. The second purpose of this paper is to show that there exists a spectral well-behaved $*$-representation of $\mathscr{A}$ if and only if there exists a spectral uniformly nondegenerate $*$-representation of $\mathscr{A}$. The third purpose is to show that the existence of a hereditary spectral well-behaved $*$-representation of $\mathscr{A}$ implies a diration-property of $\mathscr{A}$. Speaking roughly, $\mathscr{A}$ is said to have diration-property if any closed $*$-representation of an arbitrary closed $*$-subalgebera $\mathscr{B} \subset \mathscr{A}$ may be extended in a certain sense to a closed $*$-representation of $\mathscr{A}$. The fourth purpose is the investigation of the relation between the concepts of well-behaved $*$-representations defined in [**6**] and [**31**] resp. By using multiplier algebras, it will be shown that both concepts are closely related to each other. Furthermore, there will be discussed a number of examples which illustrate the usability of concepts of well-behaved $*$-representations. These include the universal enveloping algebra $E(\mathscr{G})$ of the Lie algebra $\mathscr{G}$ of a Lie group G, locally convex $*$-algebras of distribution theory, generalized B^*-algebras of Allan [**2**] and Dixon [**12**], as well as their variants like pro-C^*-algebras, the multiplier algebra of the Pederson ideal of a C^*-algebra and the Moyal algebra of quantization [**16**] which turns out to be a well-behaved $*$-representation of the Moyal algebra.

2. Well-behaved $*$-representations of $*$-algebras.

In this section we shall characterize a well-behaved $*$-representation of a general $*$-algebra by an unbounded C^*-seminorm. We review the definition of $*$-representations. Throughout this section let $\mathscr{A}$ be a $*$-algebra with identity 1. Let $\mathscr{D}$ be a dense subspace in a Hilbert space $\mathscr{H}$ and let $\mathscr{L}^\dagger(\mathscr{D})$ denote the set of all linear operators X in $\mathscr{H}$ with the domain $\mathscr{D}$ for which $X\mathscr{D} \subset \mathscr{D}$, $\mathscr{D}(X^*) \supset \mathscr{D}$ and $X^*\mathscr{D} \subset \mathscr{D}$. Then $\mathscr{L}^\dagger(\mathscr{D})$ is a $*$-algebra with identity operator I under the usual linear operations and the involution $X \mapsto X^\dagger \equiv X^*\lceil\mathscr{D}$. A unital $*$-subalgebra of the $*$-algebra $\mathscr{L}^\dagger(\mathscr{D})$ is said to be an *O^*-algebra* on $\mathscr{D}$ in $\mathscr{H}$. A *$*$-representation* π of $\mathscr{A}$ on a Hilbert space $\mathscr{H}$ with a domain $\mathscr{D}$ is a $*$-homomorphism of $\mathscr{A}$ into $\mathscr{L}^\dagger(\mathscr{D})$ such that $\pi(1) = I$, and then we write $\mathscr{D}$ and $\mathscr{H}$ by $\mathscr{D}(\pi)$ and $\mathscr{H}_\pi$, respectively. Let π be a $*$-representation of $\mathscr{A}$. If $\mathscr{D}(\pi)$ is complete with respect to the graph topology t_π defined by the family of seminorms $\{\|\cdot\|_{\pi(x)} \equiv \|\cdot\| + \|\pi(x)\cdot\|; x \in \mathscr{A}\}$, then π is said to be *closed*. It is well-known that π is closed if and only if $\mathscr{D}(\pi) = \bigcap_{x\in\mathscr{A}} \mathscr{D}(\overline{\pi(x)})$. The closure $\tilde{\pi}$ of π is defined by

$$\mathscr{D}(\tilde{\pi}) = \bigcap_{x\in\mathscr{A}} \mathscr{D}(\overline{\pi(x)}) \quad \text{and} \quad \tilde{\pi}(x)\xi = \overline{\pi(x)}\xi \quad \text{for } x \in \mathscr{A},\ \xi \in \mathscr{D}(\tilde{\pi}).$$

Then $\tilde{\pi}$ is the smallest closed extension of π. We refer to [**17**], [**18**], [**21**], [**28**], [**30**] for more details on $*$-representations.

We define the notion of strongly nondegenerate $*$-representations of $\mathscr{A}$:

DEFINITION 2.1. A non-trivial $*$-representation π of $\mathscr{A}$ is said to be *strongly nondegenerate* if there exists a left ideal $\mathscr{I}$ of $\mathscr{A}$ such that $\mathscr{I} \subset \mathscr{A}_b^\pi \equiv \{x \in \mathscr{A}; \overline{\pi(x)} \in \mathscr{B}(\mathscr{H}_\pi)\}$ and $[\overline{\pi(\mathscr{I})\mathscr{H}_\pi}] = \mathscr{H}_\pi$, where $\mathscr{B}(\mathscr{H}_\pi)$ denotes the set of all bounded linear operators on $\mathscr{H}_\pi$ and $[\mathscr{K}]$ denotes the closed subspace in $\mathscr{H}_\pi$ generated by a subset $\mathscr{K}$ of $\mathscr{H}_\pi$.

First we consider when a strongly nondegenerate $*$-representation can be constructed from an unbounded C^*-seminorm on $\mathscr{A}$. Let p be a unbounded C^*-seminorm on $\mathscr{A}$. As shown in Section 1, we can construct a $*$-representation π_p of $\mathscr{A}$ from any faithful $*$-representation Π_p of the C^*-algebra $\mathscr{A}_p$, but π_p is not necessarily nontrivial, that is, the case $\mathscr{H}_{\pi_p} = \{0\}$ may arise (Example 6.18). Suppose that p satisfies the following condition (R):

(R) $\mathfrak{N}_p \not\subset N_p$.

Then π_p is a nontrivial $*$-representation of $\mathscr{A}$ on the non-zero Hilbert space $\mathscr{H}_{\pi_p}$ (the closure of $\mathscr{D}(\pi_p)$ in $\mathscr{H}_{\Pi_p}$) such that $\|\overline{\pi_p(b)}\| \le p(b)$, $^\forall b \in \mathscr{D}(p)$ and $\|\overline{\pi_p(x)}\| = p(x)$, $^\forall x \in \mathfrak{N}_p$ ([**6**]). Hence we call (R) the *representability condition*. Let p be an unbounded C^*-seminorm on $\mathscr{A}$ satisfying the representability condition (R). We denote by $\mathrm{Rep}(\mathscr{A}_p)$ the class of all faithful $*$-representations Π_p of the C^*-algebra $\mathscr{A}_p$ on Hilbert spaces $\mathscr{H}_{\Pi_p}$ and by $\mathrm{Rep}(\mathscr{A}, p)$ the set of all $*$-representations of $\mathscr{A}$ constructed as above by $(\mathscr{A}, p)$, that is,

$$\mathrm{Rep}(\mathscr{A}, p) = \{\pi_p; \Pi_p \in \mathrm{Rep}(\mathscr{A}_p)\}.$$

Here we show that π_p is always strongly nondegenerate.

Lemma 2.2. *Suppose that an unbounded C^*-seminorm p on $\mathscr{A}$ satisfies condition* (R). *Then every $\pi_p \in \mathrm{Rep}(\mathscr{A}, p)$ is strongly nondegenerate.*

Proof. Let $\Pi_p \in \mathrm{Rep}(\mathscr{A}_p)$. Since the $\| \ \|_p$-closure $\overline{\mathfrak{N}_p/N_p}^{\| \ \|_p}$ of $\{x + N_p; x \in \mathfrak{N}_p\}$ in $\mathscr{A}_p$ is a left ideal of the C^*-algebra $\mathscr{A}_p$, it follows that there exists a left approximate identity $\{E_\alpha\}$ in $\overline{\mathfrak{N}_p/N_p}^{\| \ \|_p}$, so that $\lim_\alpha \|(x + N_p)E_\alpha - (x + N_p)\|_p = 0$ for each $x \in \mathfrak{N}_p$. For any α, there exists a sequence $\{e_\alpha^{(n)}\}$ in $\mathfrak{N}_p$ such that $\lim_{n\to\infty} \|(e_\alpha^{(n)} + N_p) - E_\alpha\|_p = 0$. Take an arbitrary $\eta \in [\Pi_p(\mathfrak{N}_p + N_p)\mathscr{H}_{\Pi_p}] \cap [\pi_p(\mathfrak{N}_p)\Pi_p(\mathfrak{N}_p + N_p)\mathscr{H}_{\Pi_p}]^\perp$. Then we have

$$\begin{aligned}
(\Pi_p(x + N_p)\xi \mid \eta) &= \lim_\alpha (\Pi_p(x + N_p)\Pi_p(E_\alpha)\xi \mid \eta) \\
&= \lim_\alpha \lim_{n\to\infty} (\Pi_p(x + N_p)\Pi_p(e_\alpha^{(n)} + N_p)\xi \mid \eta) \\
&= \lim_\alpha \lim_{n\to\infty} (\pi_p(x)\Pi_p(e_\alpha^{(n)} + N_p)\xi \mid \eta) \\
&= 0
\end{aligned}$$

for each $x \in \mathfrak{N}_p$ and $\xi \in \mathscr{H}_{\Pi_p}$, which implies that $[\pi_p(\mathfrak{N}_p)\Pi_p(\mathfrak{N}_p + N_p)\mathscr{H}_{\Pi_p}] = [\Pi_p(\mathfrak{N}_p + N_p)\mathscr{H}_{\Pi_p}] = \mathscr{H}_{\pi_p}$. Hence π_p is strongly nondegenerate. □

Next we review well-behaved $*$-representations of $\mathscr{A}$ defined in [**6**] which play an important rule for the study of unbounded C^*-seminorms. Moreover, we investigate the relation of them and strongly nondegenerate $*$-representations. If

$$\mathrm{Rep}^{\mathrm{WB}}(\mathscr{A}, p) = \{\pi_p \in \mathrm{Rep}(\mathscr{A}, p); \mathscr{H}_{\pi_p} = \mathscr{H}_{\Pi_p}\} \neq \emptyset,$$

then p is said to be *weakly semifinite* (or abbreviated, *w-semifinite*), and an element π_p of $\mathrm{Rep}^{\mathrm{WB}}(\mathscr{A}, p)$ is said to be a *well-behaved $*$-representation* of $\mathscr{A}$. In a previous paper ([**6**, Proposition 2.5]) we have shown that if $\pi_p \in \mathrm{Rep}^{\mathrm{WB}}(\mathscr{A}, p)$ then π_p is strongly nondegenerate and $\|\overline{\pi_p(x)}\| = p(x)$, $^\forall x \in \mathscr{D}(p)$. By Lemma 2.2, π_p is always strongly nondegenerate. A strongly nondegenerate $*$-representation of $\mathscr{A}$ is not well-behaved in general, but it allows to construct a w-semifinite C^*-seminorm (and consequently also a well-behaved $*$-representation) as follows: Let r_π be the unbounded C^*-seminorm defined by

$$\begin{cases} \mathscr{D}(r_\pi) = \mathscr{A}_b^\pi, \\ r_\pi(x) = \|\overline{\pi(x)}\|, \quad x \in \mathscr{D}(r_\pi). \end{cases}$$

Lemma 2.3. *Suppose that π is a strongly nondegenerate $*$-representation of $\mathscr{A}$. Then r_π is a w-semifinite unbounded C^*-seminorm on $\mathscr{A}$.*

Proof. Since π is strongly nondegenerate, there exists a left ideal $\mathscr{I}$ of $\mathscr{A}$ such that $\mathscr{I} \subset \mathscr{A}_b^\pi$ and $[\pi(\mathscr{I})\mathscr{D}(\pi)] = \mathscr{H}_\pi$. Now we put

$$\Pi(x + N_{r_\pi}) = \overline{\pi(x)}, \quad x \in \mathscr{A}_b^\pi.$$

Since $\|\Pi(x + N_{r_\pi})\| = r_\pi(x) = \|x + N_{r_\pi}\|_{r_\pi}$ for each $x \in \mathscr{A}_b^\pi$, it follows that Π can be extended to the faithful $*$-representation $\Pi^N_{r_\pi}$ of $\mathscr{A}_{r_\pi}$ on the Hilbert space $\mathscr{H}_\pi$. We denote by $\pi^N_{r_\pi}$ the $*$-representation of $\mathscr{A}$ constructed from $\pi^N_{r_\pi}$. Then it follows from ([**6**, Proposition 4.1]) that $\pi^N_{r_\pi}$ is a well-behaved $*$-representation of $\mathscr{A}$, and hence r_π is weakly semifinite. $\square$

The following scheme may serve as a short sketch of the proofs of Lemma 2.2 and Lemma 2.3:

$$
\begin{array}{ccccc}
p & \dashrightarrow & \pi_p & \dashrightarrow & r_{\pi_p} \\
\text{unbounded } C^*\text{-seminorm} & & \textit{strongly nondegenerate} & & \textit{weakly semifinite} \\
\text{with condition (R)} & & *\text{-representation} & & \text{unbounded } C^*\text{-seminorm} \\
 & & & & \vdots \\
 & & \pi^N_{r_{\pi_p}} & \dashleftarrow & \dashleftarrow \\
 & & \textit{well-behaved} & & \\
 & & *\text{-representation.} & &
\end{array}
\qquad \text{(S1)}
$$

Here, the arrow $A \dashrightarrow B$ means that B is constructed from A.

We have the following

Proposition 2.4. *Let $\mathscr{A}$ be a $*$-algebra with identity 1. The following statements are equivalent*:

(i) *There exists a well-behaved $*$-representation of $\mathscr{A}$, that is, there exists a w-semifinite unbounded C^*-seminorm on $\mathscr{A}$.*

(ii) *There exists a strongly nondegenerate $*$-representation of $\mathscr{A}$.*

(iii) *There exists an unbounded C^*-seminorm on $\mathscr{A}$ satisfying condition* (R).

3. Well-behaved $*$-representations of locally convex $*$-algebras.

In this section we shall consider an extension of the results of Section 2 to the case of locally convex $*$-algebras. First, we review some notions of the theory of locally convex $*$-algebras. A *locally convex $*$-algebra* is a $*$-algebra which is also a Hausdorff locally convex space such that the multiplication is separately continuous and the involution is continuous. Let $\mathscr{A}$ be a locally convex $*$-algebra with identity 1. We denote by $\mathscr{B}$ the collection of all absolutely convex, bounded and closed subsets $\boldsymbol{B}$ of $\mathscr{A}$ such that $1 \in \boldsymbol{B}$ and $\boldsymbol{B}^2 \subset \boldsymbol{B}$. For any $\boldsymbol{B} \in \mathscr{B}$, let $\mathscr{A}[\boldsymbol{B}]$ denote the subspace of $\mathscr{A}$ generated by $\boldsymbol{B}$. Then $\mathscr{A}[\boldsymbol{B}] = \{\lambda x; \lambda \in \boldsymbol{C}, x \in \boldsymbol{B}\}$ and the equation: $\|x\|_{\boldsymbol{B}} = \inf\{\lambda > 0; x \in \lambda \boldsymbol{B}\}$ defines a norm on $\mathscr{A}[\boldsymbol{B}]$, which makes $\mathscr{A}[\boldsymbol{B}]$ a normed algebra. If $\mathscr{A}[\boldsymbol{B}]$ is complete for each $\boldsymbol{B} \in \mathscr{B}$, then $\mathscr{A}$ is said to be *pseudo-complete*. Note that $\mathscr{A}$ is pseudo-complete if it is sequentially complete. We refer to [**1**], [**2**], [**12**] for more details on locally convex $*$-algebras. Throughout this section $\mathscr{A}$ will denote a pseudo-complete locally convex $*$-algebra with identity 1. An element x of $\mathscr{A}$ is *bounded* if, for some non-zero $\lambda \in \boldsymbol{C}$, the set $\{(\lambda x)^n; n \in \boldsymbol{N}\}$ is bounded. The set of all bounded elements of $\mathscr{A}$ is denoted by $\mathscr{A}_0$. If $\mathscr{A}$ is commutative, then $\mathscr{A}_0$ is a $*$-subalgebra of $\mathscr{A}$, but it is not even a subspace of $\mathscr{A}$ in general. Hence we consider the $*$-subalgebra of $\mathscr{A}$ generated by $(\mathscr{A}_0)_h \equiv \{x \in \mathscr{A}_0; x^* = x\}$ as the *bounded $*$-subalgebra* of $\mathscr{A}$, and denote it by $\mathscr{A}_b$. In general, $(\mathscr{A}_0)_h \subset \mathscr{A}_b$ and $(\mathscr{A}_0)_h \subset \mathscr{A}_0$, but there is no definite relation between $\mathscr{A}_b$ and $\mathscr{A}_0$. Of course, $\mathscr{A}_b = \mathscr{A}_0$ if $\mathscr{A}$ is commutative. We put

$$\mathscr{I}_b = \{x \in \mathscr{A}_b; ax \in \mathscr{A}_b, {}^\forall a \in \mathscr{A}\}.$$

Then $\mathscr{I}_b$ is a left ideal of $\mathscr{A}$ which is the largest left ideal of $\mathscr{A}$ contained in $\mathscr{A}_b$. By ([**4**, Lemma 3.10]) we have the following

Lemma 3.1. *If π is a $*$-representation of $\mathscr{A}$, then $\mathscr{A}_b \subset \mathscr{A}_b^\pi$ and $\|\overline{\pi(x)}\| \leq \beta(x)$ for each $x \in (\mathscr{A}_0)_h$, where $\beta(x)$ is the radius of boundedness of x defined by $\beta(x) = \inf\{\lambda > 0; \{(\lambda^{-1}x)^n; n \in \boldsymbol{N}\}$ is bounded$\}$.*

Next we define the notion of uniform nondegenerateness of $*$-representations which is stronger than that of the strong nondegenerateness.

Definition 3.2. A non-trivial $*$-representation π of $\mathscr{A}$ is said to be *uniformly nondegenerate* if $[\pi(\mathscr{I}_b)\mathscr{D}(\pi)] = \mathscr{H}_\pi$.

To investigate the relation of uniformly nondegenerate $*$-representations and unbounded C^*-seminorms of a pseudo-complete locally convex $*$-algebra, we need a further notion:

Definition 3.3. An unbounded C^*-seminorm p on $\mathscr{A}$ is said to be *topologically w-semifinite* (abbreviated, *tw-semifinite*) if there exists an element $\Pi_p \in \mathrm{Rep}(\mathscr{A}_p)$ such that $[\Pi_p((\mathfrak{N}_p \cap \mathscr{I}_b) + N_p)\mathscr{H}_{\Pi_p}] = \mathscr{H}_{\Pi_p}$.

We denote by $\mathrm{Rep}^{\mathrm{UWB}}(\mathscr{A}, p)$ the set of all $*$-representations π_p of $\mathscr{A}$ constructed from $\Pi_p \in \mathrm{Rep}(\mathscr{A}_p)$ satisfying $[\Pi_p((\mathfrak{N}_p \cap \mathscr{I}_b) + N_p)\mathscr{H}_{\Pi_p}] = \mathscr{H}_{\Pi_p}$. It is clear that

$$\mathrm{Rep}^{\mathrm{UWB}}(\mathscr{A}, p) \subset \mathrm{Rep}^{\mathrm{WB}}(\mathscr{A}, p).$$

In case of *general* $*$-algebras an element π_p of $\mathrm{Rep}^{\mathrm{WB}}(\mathscr{A}, p)$ is said to be *well-behaved*, but in case of locally convex $*$-algebras an element π_p of $\mathrm{Rep}^{\mathrm{UWB}}(\mathscr{A}, p)$ is said to be *well-behaved*, and an element π_p of $\mathrm{Rep}^{\mathrm{WB}}(\mathscr{A}, p)$ is said to be *algebraically well-behaved.*

The existence of well-behaved $*$-representations of $\mathscr{A}$ may be characterized as follows:

Theorem 3.4. *Let $\mathscr{A}$ be a pseudo-complete locally convex $*$-algebra with identity 1. Then the following statements are equivalent*:

(i) *There exists a well-behaved $*$-representation of $\mathscr{A}$, that is, there exists a tw-semifinite unbounded C^*-seminorm on $\mathscr{A}$.*

(ii) *There exists a uniformly nondegenerate $*$-representation of $\mathscr{A}$.*

(iii) *There exists an unbounded C^*-seminorm p on $\mathscr{A}$ satisfying the representability condition* (UR):

(UR) $\mathfrak{N}_p \cap \mathscr{I}_b \not\subset N_p$.

Proof. (iii) $\Rightarrow$ (ii) Suppose that p is an unbounded C^*-seminorm on $\mathscr{A}$ satisfying condition (UR). By Lemma 2.2 and Lemma 2.3 r_{π_p} is a w-semifinite unbounded C^*-seminorm on $\mathscr{A}$. Then it follows from Lemma 3.1 that $\mathscr{D}(r_{\pi_p}) = \mathscr{A}_b^{\pi_p} \supset A_b$, and hence $\mathfrak{N}_{r_{\pi_p}} \supset \mathscr{I}_b$. Thus we define an unbounded C^*-seminorms $r_{\pi_p}^u$ on $\mathscr{A}$ by

$$\begin{cases} \mathscr{D}(r_{\pi_p}^u) = \mathscr{A}_b \\ r_{\pi_p}^u(x) = r_{\pi_p}(x) = \|\overline{\pi_p(x)}\|, \quad x \in \mathscr{D}(r_{\pi_p}^u). \end{cases}$$

Then we have $\mathfrak{N}_{r^u_{\pi_p}} = \mathscr{I}_b$, and so $r^u_{\pi_p}$ is an unbounded C^*-seminorm on $\mathscr{A}$ satisfying condition (UR). Hence an argument similar to the proof of Lemma 2.2 shows that

$$\begin{aligned}[][\pi_{r^u_{\pi_p}}(\mathscr{I}_b)\Pi_{r^u_{\pi_p}}(\mathfrak{N}_{r^u_{\pi_p}} + N_{r^u_{\pi_p}})\mathscr{H}_{\Pi_{r^u_{\pi_p}}}] &= [\pi_{r^u_{\pi_p}}(\mathfrak{N}_{r^u_{\pi_p}})\Pi_{r^u_{\pi_p}}(\mathfrak{N}_{r^u_{\pi_p}} + N_{r^u_{\pi_p}})\mathscr{H}_{\Pi_{r^u_{\pi_p}}}] \\ &= [\Pi_{r^u_{\pi_p}}(\mathfrak{N}_{r^u_{\pi_p}} + N_{r^u_{\pi_p}})\mathscr{H}_{\Pi_{r^u_{\pi_p}}}],\end{aligned}$$

which means that $\pi_{r^u_{\pi_p}}$ is uniformly nondegenerate.

(ii) $\Rightarrow$ (i) Suppose that π is a uniformly nondegenerate $*$-representation of $\mathscr{A}$. As shown above, the restriction r^u_π of r_π to $\mathscr{A}_b$ is an unbounded C^*-seminorm on $\mathscr{A}$ such that $\mathfrak{N}_{r^u_\pi} = \mathscr{I}_b$, and so it satisfies condition (UR). We denote by $\pi^N_{r^u_\pi}$ the natural $*$-representation of $\mathscr{A}$ constructed from a faithful $*$-representation $\Pi^N_{r^u_\pi}$ of the C^*-algebra $\mathscr{A}_{r^u_\pi}$ of $\mathscr{H}_\pi$. Then we have

$$[\Pi^N_{r^u_\pi}(\mathfrak{N}_{r^u_\pi} \cap \mathscr{I}_b + N_{r^u_\pi})\mathscr{H}_{\pi^N_{r^u_\pi}}] = \overline{[\pi(\mathscr{I}_b)\mathscr{H}_\pi]} = \mathscr{H}_\pi = \mathscr{H}_{\Pi^N_{r^u_\pi}},$$

which means that r^u_π is tw-semifinite and $\pi^N_{r^u_\pi}$ is a well-behaved $*$-representation of $\mathscr{A}$.

(i) $\Rightarrow$ (iii) This is trivial. This completes the proof. $\square$

The following schemes may serve as a sketch of the proof of Theorem 3.4:

π $\dashrightarrow$ r^u_π
uniformly nondegenerate $*$-representation — tw-semifinite unbounded C^*-seminorm

$\pi^N_{r^u_\pi}$ $\dashleftarrow$ (from r^u_π)
well-behaved $*$-representation (S2)

and

p $\dashrightarrow$ π_p $\dashrightarrow$ r_{π_p}
unbounded C^*-seminorm with condition (UR) — strongly nondegenerate $*$-representation — w-semifinite unbounded C^*-seminorm

$\downarrow$

$\pi_{r^u_{\pi_p}}$ $\dashleftarrow$ $r^u_{\pi_p} \equiv r_{\pi_p} \restriction \mathscr{A}_b$
uniformly nondegenerate $*$-representation

$\|$

$\dashrightarrow$ $r^u_{\pi_{r^u_{\pi_p}}}$
tw-semifnite unbounded C^*-seminorm

$\pi^N_{r^u_{\pi_p}}$ $\dashleftarrow$
well-behaved $*$-representation. (S3)

4. Spectral well-behaved $*$-representations of locally convex $*$-algebras.

In this section we shall define the notion of spectral $*$-representations of locally convex $*$-algebras and characterize them by unbounded C^*-seminorms.

Let $\mathscr{A}$ be a locally convex $*$-algebra. If $\mathscr{A}$ does not have an identity, then we may consider the locally convex $*$-algebra $\mathscr{A}_1$ obtained by adjoining an identity 1 to $\mathscr{A}$. The algebraic spectrum $Sp_{\mathscr{A}}(x)$ and the spectral radius $r_{\mathscr{A}}(x)$ of $x \in \mathscr{A}$ are defined by

$$Sp_{\mathscr{A}}(x) = \begin{cases} \{\lambda \in \boldsymbol{C}; \nexists (\lambda 1 - x)^{-1} \text{ in } \mathscr{A}\}, & \text{if } 1 \in \mathscr{A}; \\ \{\lambda \in \boldsymbol{C}; \nexists (\lambda 1 - x)^{-1} \text{ in } \mathscr{A}_1\} \cup \{0\}, & \text{if } 1 \notin \mathscr{A}; \end{cases}$$

and

$$r_{\mathscr{A}}(x) = \sup\{|\lambda|; \lambda \in Sp_{\mathscr{A}}(x)\}.$$

In topological cases, it is natural to consider also $Sp_{\mathscr{A}_b}(x)$ defined by

$$Sp_{\mathscr{A}_b}(x) = \begin{cases} \{\lambda \in \boldsymbol{C}; \nexists (\lambda 1 - x)^{-1} \text{ in } \mathscr{A}_b\}, & \text{if } 1 \in \mathscr{A}; \\ \{\lambda \in \boldsymbol{C}; \nexists (\lambda 1 - x)^{-1} \text{ in } (\mathscr{A}_b)_1\} \cup \{0\}, & \text{if } 1 \notin \mathscr{A}. \end{cases}$$

Throughout this section let $\mathscr{A}$ be a pseudo-complete locally convex $*$-algebra with identity 1.

We first define the notion of spectral $*$-representations of $\mathscr{A}$ as follows:

DEFINITION 4.1. A $*$-representation π of $\mathscr{A}$ is said to be *spectral* if $\underline{Sp_{\mathscr{A}_b}(x)} \subset Sp_{C_u^*(\pi)}(\overline{\pi(x)})$ for each $x \in \mathscr{A}_b$, where $C_u^*(\pi)$ is the C^*-algebra generated by $\overline{\pi(\mathscr{A}_b)}$. If $\pi \lceil \mathscr{B}$ is spectral for each unital closed $*$-subalgebra $\mathscr{B}$ of $\mathscr{A}$, then π is said to be *hereditary spectral*.

In order to characterize the existence of (hereditary) spectral uniformly nondegenerate $*$-representations of $\mathscr{A}$, we shall define and study the notion of spectrality of unbounded C^*-seminorms. Note that an element x of an arbitrary ($*$-)algebra $\mathscr{B}$ has the quasi-inverse $y \in \mathscr{B}$ if $x + y - xy = x + y - yx = 0$. In an unital algebra $\mathscr{B}$ (or in $\mathscr{B}_1$) this is equivalent to $(1 - y) = (1 - x)^{-1}$. An element $x \in \mathscr{B}$ is said to be quasi-regular if it has a quasi-inverse. In case of a topological $*$-algebra $\mathscr{B}$, $x \in \mathscr{B}$ is said to be quasi-invertible if it has a quasi-inverse belonging to $\mathscr{B}_b$. Let $\mathscr{B}^{qi}$ (resp. $\mathscr{B}^{qr}$) denote the set of all quasi-invertible (resp. quasi-regular) elements of $\mathscr{B}$.

DEFINITION 4.2. An unbounded C^*-seminorm p on $\mathscr{A}$ is said to be *spectral* if $\{x \in \mathscr{D}(p); p(x) < 1\} \subset \mathscr{D}(p)^{qi}$. If $p \lceil \mathscr{B}$ is spectral for each unital closed $*$-subalgebra $\mathscr{B}$ of $\mathscr{A}$, then p is said to be *hereditary spectral*.

REMARK 4.3. In [**6**] there has been defined the notion of spectrality of unboudned C^*-seminorms p on *general* $*$-algebras $\mathscr{A}$ as follows: p is said to be *spectral* if $\{x \in \mathscr{D}(p); p(x) < 1\} \subset \mathscr{D}(p)^{qr}$, and p is said to be *hereditary spectral* if $p\lceil\mathscr{B}$ is spectral for each $*$-subalgebra $\mathscr{B}$ of $\mathscr{A}$. When $\mathscr{A}$ is a locally convex $*$-algebra, such a p is said to be *algebraically* (*hereditary*) *spectral*. It is clear that if p is spectral, then it is algebraically spectral.

LEMMA 4.4. *Let p be an unbounded C^*-seminorm on $\mathscr{A}$. The following statements* (i) *and* (ii) *are equivalent*:

(i) *p is spectral.*
(ii) *$r_{\mathscr{D}(p)_b}(x) \leq p(x)$ for each $x \in \mathscr{D}(p)$.*
If this is true, then the following statements (iii) *and* (iv) *hold*:
(iii) *$r_{\mathscr{D}(p)}(x) = \lim_{n\to\infty} p(x^n)^{1/n} \leq r_{\mathscr{D}(p)_b}(x) \leq p(x)$ for each $x \in \mathscr{D}(p)$.*
In particular,

$$r_{\mathscr{D}(p)}(x) = r_{\mathscr{D}(p)_b}(x) = p(x) \quad \textit{for each } x^* = x \in \mathscr{D}(p).$$

(iv) $Sp_{\mathscr{D}(p)_b}(x) \cup \{0\} = Sp_{\mathscr{A}_p}(x + N_p) \cup \{0\}$,

$$Sp_{\mathscr{D}(p)_b}(x) = Sp_{\mathscr{D}(p)}(x),$$

and

$$r_{\mathscr{D}(p)_b}(x) = r_{\mathscr{D}(p)}(x) = \lim_{n\to\infty} p(x^n)^{1/n}$$

for each $x \in \mathscr{D}(p)_b$.

Proof. It is easily shown that (i) and (ii) are equivalent. Suppose that p is spectral. Since p is algebraically spectral, it follows from ([**24**, Theorem 3.1]) that (iii) holds. We show that (iv) holds. Let $\mathscr{A}_p^b$ be the C^*-subalgebra of the C^*-algebra $\mathscr{A}_p$ generated by $\{x + N_p; x \in \mathscr{D}(p)_b\}$. Suppose that $x \in \mathscr{D}(p)_b$ and that $\lambda \in \boldsymbol{C}\backslash(Sp_{\mathscr{A}_p^b}(x+N_p) \cup \{0\})$. Let $A \in \mathscr{A}_p^b$ be the quasi-inverse of $\lambda^{-1}(x+N_p)$. By the definitions of the quasi-inverse and of $\mathscr{A}_p^b$, we find $z \in \mathscr{D}(p)_b$ such that

$$p(\lambda^{-1}xz - \lambda^{-1}x - z) < 1, \quad p(\lambda^{-1}zx - \lambda^{-1}x - z) < 1.$$

By spectrality of p, the first of these inequalities implies that $-\lambda^{-1}xz + \lambda^{-1}x + z$ has a quasi-inverse $y \in \mathscr{D}(p)_b$. Hence

$$(-\lambda^{-1}xz + \lambda^{-1}x + z)y - (-\lambda^{-1}xz + \lambda^{-1}x + z) - y = 0,$$

$$\lambda^{-1}x(-zy + y + z) - \lambda^{-1}x + zy - y - z = 0,$$

which means that $\lambda^{-1}x$ has the right quasi-inverse $-zy + y + z \in \mathscr{D}(p)_b$. Similarly it can be shown, that it has a left quasi-inverse in $\mathscr{D}(p)_b$. Consequently,

$$Sp_{\mathscr{A}_p^b}(x + N_p) \cup \{0\} \supset Sp_{\mathscr{D}(p)_b}(x) \cup \{0\}. \tag{4.1}$$

Since the converse inclusion is trivial and $Sp_{\mathscr{A}_p^b}(x+N_p) \cup \{0\} = Sp_{\mathscr{A}_p}(x+N_p) \cup \{0\}$ ([**32**, Proposition 4.8]), the first equation in (iv) is established. Suppose now, that $\lambda^{-1}x \in \mathscr{D}(p)_b$ $(\lambda \in \boldsymbol{C}\backslash\{0\})$ has a quasi-inverse $y \in \mathscr{D}(p)$. Then $(1 - (\lambda^{-1}x + N_p))^{-1} = 1 - (y + N_p)$ is $\mathscr{A}_p$, i.e., $\lambda \notin Sp_{\mathscr{A}_p}(x+N_p)$. This implies $\lambda \notin Sp_{\mathscr{A}_p^b}(x+N_p)$. But then $\lambda \notin Sp_{\mathscr{D}(p)_b}(x) \cup \{0\}$ by (4.1). This proves

$$Sp_{\mathscr{D}(p)}(x) \cup \{0\} \supset Sp_{\mathscr{D}(p)_b}(x) \cup \{0\}, \quad x \in \mathscr{D}(p)_b.$$

Again, the converse inclusion is trivial, so that

$$Sp_{\mathscr{D}(p)}(x) \cup \{0\} = Sp_{\mathscr{D}(p)_b}(x) \cup \{0\}, \quad x \in \mathscr{D}(p)_b. \tag{4.2}$$

Consequently, the second equation in (iv) is satisfied if $\mathscr{D}(p)$ has no unit element. If $\mathscr{D}(p)$ (and consequently also $\mathscr{D}(p)_b$) has a unit $1_{\mathscr{D}(p)}$, (4.2) applied to $1_{\mathscr{D}(p)} - x$ instead of x shows that $x \in \mathscr{D}(p)_b$ is invertible in $\mathscr{D}(p)$ if and only if it is invertible in $\mathscr{D}(p)_b$, i.e., that $0 \in Sp_{\mathscr{D}(p)}(x)$ if and only if $0 \in Sp_{\mathscr{D}(p)_b}(x)$. This completes the proof of the second equation in (iv). Finally, the spectral radius formula in (iv) follows from (iii), which completes the proof. □

For spectral unbounded C^*-seminorms p on $\mathscr{A}$ whose domains $\mathscr{D}(p)$ contain $\mathscr{A}_b$ we have the following

LEMMA 4.5. *Suppose that p is a spectral unbounded C^*-seminorm on $\mathscr{A}$ such that $\mathscr{D}(p) \supset \mathscr{A}_b$. Then the following statements hold*:

(1) $p(x) = r_{\mathscr{A}_b}(x^*x)^{1/2}$ *for each* $x \in \mathscr{D}(p)$. *Hence, a spectral unbounded C^*-seminorm p on $\mathscr{A}$ whose domain contains $\mathscr{A}_b$ is uniquely determined in a sense that if q is a spectral unbounded C^*-seminorm on $\mathscr{A}$ whose domain contains $\mathscr{A}_b$ then $p(x) = q(x)$ for each $x \in \mathscr{D}(p) \cap \mathscr{D}(q)$. In particular, if p is a spectral C^*-seminorm on $\mathscr{A}$, then*

$$p(x) = r_{\mathscr{A}}(x^*x)^{1/2} = r_{\mathscr{A}_b}(x^*x)^{1/2}, \quad x \in \mathscr{A},$$

and so p is unique.

Suppose that q is an unbounded C^*-seminorm on $\mathscr{A}$ such that $\mathscr{D}(q) \supset \mathscr{A}_b$. Then the following (2)–(4) *hold*:

(2) *If $q \subset p$, then q is spectral.*

(3) *If $q \geq p$, then q is spectral and $q \subset p$.*

(4) $q(x) \leq p(x)$, $^\forall x \in \mathscr{D}(p) \cap \mathscr{D}(q)$.

PROOF. (1) Since $\mathscr{D}(p)_b = \mathscr{A}_b$, it follows from Lemma 4.4, (iii) that $p(h) = r_{\mathscr{A}_b}(h)$ for each $h \in \mathscr{D}(p)_h$, which implies that $p(x) = r_{\mathscr{A}_b}(x^*x)^{1/2}$ for each $x \in \mathscr{D}(p)$.

Suppose that q is an unbounded C^*-seminorm on $\mathscr{A}$ such that $\mathscr{D}(q) \supset \mathscr{A}_b$.

(2) This is trivial.

(3) Suppose that $q \geq p$. Then it is clear that q is spectral, and hence by (1) $q \subset p$.

(4) We put

$$\begin{cases} \mathscr{D}(r) = \mathscr{D}(p) \cap \mathscr{D}(q) \\ r(x) = \max(p(x), q(x)), \quad x \in \mathscr{D}(r). \end{cases}$$

Then r is an unbounded C^*-seminorm on $\mathscr{A}$ such that $\mathscr{D}(r) \supset \mathscr{A}_b$ and $r \geq p$. By (2) we have $r \subset p$, and hence $q(x) \leq p(x)$ for each $x \in \mathscr{D}(p) \cap \mathscr{D}(q)$. This completes the proof. □

We next define the notion of the spectral invariance of $\mathscr{A}$. We denote by $\operatorname{Rep} \mathscr{A}$ the class of all uniformly nondegenerate $*$-representations of $\mathscr{A}$. Suppose that $\operatorname{Rep} \mathscr{A} \neq \emptyset$. Then we can define the *unbounded Gelfand-Naimark C^*-seminorm* $|\ |$ on $\mathscr{A}$ as follows:

$$\begin{cases} \mathscr{D}(|\ |) = \{x \in \mathscr{A}; \sup_{\pi \in \operatorname{Rep} \mathscr{A}} \|\overline{\pi(x)}\| < \infty\}, \\ |x| = \sup_{\pi \in \operatorname{Rep} \mathscr{A}} \|\overline{\pi(x)}\|, \quad x \in \mathscr{D}(|\ |). \end{cases}$$

Then it follows from Lemma 3.1 that $\mathscr{D}(|\ |) \supset \mathscr{A}_b$. Hence we may define the unbounded C^*-seminorm $|\ |_u$ on $\mathscr{A}$ obtained by the restriction $|\ |$ to $\mathscr{A}_b$, that is,

$$\begin{cases} \mathscr{D}(|\ \ |_u) = \mathscr{A}_b, \\ |x|_u = |x|, \quad x \in \mathscr{D}(|\ \ |_u). \end{cases}$$

Lemma 4.6. *Suppose that* $\operatorname{Rep} \mathscr{A} \neq \emptyset$. *Then* $|\ \ |_u$ *is a tw-semifinite unbounded* C^*-*seminorm on* $\mathscr{A}$.

Proof. Let Γ be a set of uniformly nondegenerate $*$-representations of $\mathscr{A}$ such that for all $x \in \mathscr{A}$

$$\sup_{\pi \in \operatorname{Rep} \mathscr{A}} \|\pi(x)\| = \sup_{\pi \in \Gamma} \|\pi(x)\|.$$

Here $\|\pi(x)\|$ denotes the operator norm of the possibly non-closed operator $\pi(x)$ if $\pi(x)$ is bounded and $\|\pi(x)\| = \infty$ if $\pi(x)$ is unbounded. Define $\pi = \bigoplus_{\pi_\alpha \in \Gamma} \pi_\alpha$. Then π is uniformly nondegenerate. In fact, take an arbitrary $\xi = (\xi_\alpha) \in [\pi(\mathscr{I}_b)\mathscr{H}_\pi]^\perp$. Then $\xi_\alpha \in [\pi_\alpha(\mathscr{I}_b)\mathscr{H}_{\pi_\alpha}]^\perp$ for each α. Since π_α is uniformly nondegenerate, we have $\xi_\alpha = 0$, which implies that $[\pi(\mathscr{I}_b)\mathscr{H}_\pi] = \mathscr{H}_\pi$. Here we put

$$\Pi(x + N_{|\ |_u}) = \pi(x), \quad x \in \mathscr{A}_b.$$

Then Π can be extended to the faithful $*$-representation of the C^*-algebra $\mathscr{A}_{|\ |_u}$ on the Hilbert space $\mathscr{H}_\pi$ and the $*$-representation $\pi_{|\ |_u}$ of $\mathscr{A}$ constructed from Π is uniformly nondegenerate. Hence it follows from the proof of Theorem 3.4 that $|\ \ |_u$ is tw-semifinite. □

The C^*-algebra $\mathscr{A}_{|\ |_u}$ constructed from $|\ \ |_u$ is said to be the *enveloping* C^*-*algebra* of $\mathscr{A}$ and denoted by $\mathrm{EC}^*(\mathscr{A})$. The natural $*$-homomorphism:

$$x \in \mathscr{A}_b \mapsto x + N_{|\ |_u} \in \mathrm{EC}^*(\mathscr{A})$$

is denoted by j.

Definition 4.7. If $\operatorname{Rep} \mathscr{A} \neq \emptyset$ and $Sp_{\mathscr{A}_b}(x) = Sp_{\mathrm{EC}^*(\mathscr{A})}(j(x))$ for each $x \in \mathscr{A}_b$, then $\mathscr{A}$ is said to be *spectral invariant*.

Next, we characterize the existence of spectral well-behaved $*$-representations.

Theorem 4.8. *Let* $\mathscr{A}$ *be a pseudo-complete locally convex* $*$-*algebra with identity 1. The following statements are equivalent*:

(i) *There exists a spectral tw-semifinite unbounded* C^*-*seminorm on* $\mathscr{A}$ *whose domain contains* $\mathscr{A}_b$.

(ii) *There exists a spectral well-behaved* $*$-*representation of* $\mathscr{A}$.

(iii) *There exists a spectral uniformly nondegenerate* $*$-*representation of* $\mathscr{A}$.

(iv) $\mathscr{A}$ *is spectral invariant.*

In order to prove this theorem we shall prepare some lemmas.

Lemma 4.9. $|\ \ |_u$ *is spectral if and only if* $\mathscr{A}$ *is spectral invariant.*

Proof. This follows from Lemma 4.4. □

Lemma 4.10. *Let* π *be a* $*$-*representation of* $\mathscr{A}$. *Consider the following statements*:

(i) r_π^u *is hereditary spectral.*
(ii) π *is hereditary spectral.*
(iii) $r_{\pi \restriction \mathscr{B}}^u$ *is spectral for each closed* $*$*-subalgebra* $\mathscr{B}$ *of* $\mathscr{A}$ *with* 1.
(iv) r_π^u *is spectral.*
(v) π *is spectral.*

Then the following implications hold:

$$\begin{array}{ccccc} & & \text{(ii)} & & \text{(iv)} \\ \text{(i)} & \Longrightarrow & \Updownarrow & \Longrightarrow & \Updownarrow \\ & & \text{(iii)} & & \text{(v)}. \end{array}$$

Proof. We first show the equivalence of (iv) and (v). Since $\mathscr{D}(r_\pi^u) = \mathscr{A}_b$, it follows from Lemma 4.4 that r_π^u is spectral if and only if $\{x \in \mathscr{A}_b; r_\pi^u(x) < 1\} \subset \mathscr{A}^{qi}$. Suppose that π is spectral. Take an arbitrary $x \in \mathscr{A}_b$ such that $r_\pi^u(x) < 1$. Since $C_u^*(\pi) \equiv \overline{\pi(\mathscr{A}_b)}^{\|\ \|}$, it follows that $\overline{\pi(x)}$ is quasi-invertible in the C^*-algebra $C_u^*(\pi)$, and so $1 \notin Sp_{C_u^*(\pi)}(\overline{\pi(x)})$. Since π is spectral, we have $1 \notin Sp_{\mathscr{A}_b}(x)$, and so $x \in \mathscr{A}^{qi}$. Therefore r_π^u is spectral. Conversely suppose that r_π^u is spectral. Let $x \in \mathscr{A}_b$ and $\lambda \in \boldsymbol{C} \backslash Sp_{C_u^*(\pi)}(\overline{\pi(x)})$ be fixed. Since $\{\overline{\pi(y)}; y \in \mathscr{A}_b\}$ is dense in $C_u^*(\pi)$, we can find $y \in \mathscr{A}_b$ such that

$$r_\pi^u((\lambda 1 - x)y - 1) = \|(\lambda I - \overline{\pi(x)})\overline{\pi(y)} - I\| < 1,$$

$$r_\pi^u(y(\lambda 1 - x) - 1) = \|\overline{\pi(y)}(\lambda I - \overline{\pi(x)}) - I\| < 1.$$

Since r_π^u is spectral, $(\lambda 1 - x)y$ and $y(\lambda 1 - x)$ are invertible in $\mathscr{A}_b$ and so $\lambda \notin Sp_{\mathscr{A}_b}(x)$. Hence $Sp_{C_u^*(x)}(\overline{\pi(x)}) \supset Sp_{\mathscr{A}_b}(x)$. Thus π is spectral if and only if r_π^u is spectral.

Applying the equivalence of (iv) and (v) to $\pi \restriction \mathscr{B}$, where $\mathscr{B}$ is an arbitrary unital closed $*$-subalgebra of $\mathscr{A}$, we obtain the equivalence of (ii) and (iii). From the definitions of the unbounded C^*-seminorms $r_{\pi \restriction B}^u$ and $r_\pi^u \restriction \mathscr{B}$, it follows that

$$r_{\pi \restriction \mathscr{B}}^u \text{ is spectral if and only if } r_{\mathscr{B}_b}(x) \leq \|\overline{\pi(x)}\|, \quad {}^\forall x \in \mathscr{B}_b$$

and

$$r_\pi^u \restriction \mathscr{B} \text{ is spectral if and only if } r_{\mathscr{B}_b}(x) \leq \|\overline{\pi(x)}\|, \quad {}^\forall x \in \mathscr{A}_b \cap \mathscr{B},$$

so that the implication (i) $\Rightarrow$ (ii) holds since $\mathscr{B}_b \subset \mathscr{A}_b \cap \mathscr{B}$. This completes the proof. □

Lemma 4.11. *Suppose that* p *is a tw-semifinite unbounded* C^**-seminorm on* $\mathscr{A}$ *such that* $\mathscr{D}(p) \supset \mathscr{A}_b$. *Then the following statements hold*:

(1) *If* p *is spectral, then* $\operatorname{Rep} \mathscr{A} \neq \emptyset$ *and* $|\ |_u \subset p$.

(2) *If* p *is* (*hereditary*) *spectral, then every* $\pi_p \in \operatorname{Rep}^{\text{UWB}}(\mathscr{A}, p)$ *is* (*hereditary*) *spectral.*

Proof. (1) Suppose that p is spectral. By Lemma 4.5 (4), we have

$$|x|_u \leq p(x), \quad x \in \mathscr{A}_b.$$

On the other hand, since p is tw-semifinite, there exists an element π_p of $\operatorname{Rep}^{\text{UWB}}(\mathscr{A}, p)$

such that $\|\overline{\pi_p(x)}\| = p(x)$ for each $x \in \mathscr{D}(p)$, which implies that $\operatorname{Rep}\mathscr{A} \neq \emptyset$ and $p(x) \leq |x|_u$ for each $x \in \mathscr{A}_b$. Hence we have $|\ |_u \subset p$.

(2) Suppose that p is (hereditary) spectral, and $\pi_p \in \operatorname{Rep}^{\mathrm{UWB}}(\mathscr{A}, p)$. Then we have $r^u_{\pi_p} \subset p$, which implies by Lemma 4.5, (2) that $r^u_{\pi_p}$ is (hereditary) spectral. Hence it follows from Lemma 4.10 that π_p is (hereditary) spectral. □

Proof of Theorem 4.8.

(i) ⇒ (ii) This follows from Lemma 4.11, (2).

(ii) ⇒ (iii) This is trivial.

(iii) ⇒ (i) Let π be a spectral uniformly nondegenerate $*$-representation of $\mathscr{A}$. As seen in the proof of (ii) ⇒ (i) in Theorem 3.4, r^u_π is a tw-semifinite unbounded C^*-seminorm on $\mathscr{A}$. Furthermore, it follows from Lemma 4.10 that r^u_π is spectral.

(i) ⇒ (iv) Let p be a spectral tw-semifinite unbounded C^*-seminorm on $\mathscr{A}$ such that $\mathscr{D}(p) \supset \mathscr{A}_b$. By Lemma 4.11, (1), we have $|\ |_u \subset p$, which implies by Lemma 4.5, (2) that $|\ |_u$ is spectral. Hence it follows from Lemma 4.9 that $\mathscr{A}$ is spectral invariant.

(iv) ⇒ (i) Suppose that $\mathscr{A}$ is spectral invariant. By Lemma 4.9, $\operatorname{Rep}\mathscr{A} \neq \emptyset$ and $|\ |_u$ is spectral. Furthermore it follows from Lemma 4.6 that $|\ |_u$ is tw-semifinite. This completes the proof. □

Finally in this section, there are obtained several conditions for the existence of hereditary spectral well-behaved $*$-representations.

Proposition 4.12. *Let $\mathscr{A}$ be a pseudo-complete locally convex $*$-algebra with identity 1. Consider the following statements*:

(i) *There exists a hereditary spectral tw-semifinite unbounded C^*-seminorm on $\mathscr{A}$ whose domain contains $\mathscr{A}_b$.*

(ii) *There exists a hereditary spectral well-behaved $*$-representation of $\mathscr{A}$.*

(iii) *There exists a hereditary spectral uniformly nondegenerate $*$-representation of $\mathscr{A}$.*

Then the following implications (i) ⇒ (ii) ⇔ (iii) *hold.*

Proof. (i) ⇒ (ii) This follows from Lemma 4.11, (2).

(ii) ⇒ (iii) This is trivial.

(iii) ⇒ (ii) Let π be a hereditary spectral uniformly nondegenerate $*$-representation of $\mathscr{A}$. As shown in the proof (ii) ⇒ (i) of Theorem 3.4, r^u_π is a tw-semifinite unbounded C^*-seminorm on $\mathscr{A}$ and $\pi^N_{r^u_\pi}$ is a well-behaved $*$-representation of $\mathscr{A}$. Since π is hereditary spectral, it follows that for any closed $*$-subalgebra $\mathscr{B}$ of $\mathscr{A}$ containing 1,

$$\begin{aligned} Sp_{\mathscr{B}_b}(x) &\subset Sp_{\overline{\pi(\mathscr{B}_b)}^{\|\ \|}}(\overline{\pi(x)}) \\ &= Sp_{\overline{\pi^N_{r^u_\pi}(\mathscr{B}_b)}^{\|\ \|}}(\overline{\pi^N_{r^u_\pi}(x)}) \end{aligned}$$

for each $x \in \mathscr{B}_b$, which implies that $\pi^N_{r^u_\pi}$ is hereditary spectral. This completes the proof. □

5. Locally convex $*$-algebras with diration-property.

In [**6**] we have generalized the following diration-property of C^*-algebras:

Let $\mathscr{A}$ be a C^*-algebra and $\mathscr{B}$ any closed $*$-subalgebra of $\mathscr{A}$. For any $*$-representation π of $\mathscr{B}$ on a Hilbert space $\mathscr{H}_\pi$ there exists a $*$-representation ρ of $\mathscr{A}$ on a Hilbert space $\mathscr{H}_\rho$ such that $\mathscr{H}_\rho \supset \mathscr{H}_\pi$ as a closed subspace and $\pi(x) = \rho(x) \lceil \mathscr{H}_\pi$ for each $x \in \mathscr{B}$

to general $*$-algebras, and characterized it by the hereditary spectrality of unbounded C^*-seminorms. In this section we shall consider this diration-problem in case of locally convex $*$-algebras.

Definition 5.1. Let $\mathscr{A}$ be a pseudo-complete locally convex $*$-algebra with identity *1*. If for any unital *closed* $*$-subalgebra $\mathscr{B}$ of $\mathscr{A}$ and any closed $*$-representation π of $\mathscr{B}$ such that $[\pi(\mathscr{I}_b \cap \mathscr{B})\mathscr{D}(\pi)]^{t_\pi} = \mathscr{D}(\pi)$ there exists a closed $*$-representation ρ of $\mathscr{A}$ such that $[\rho(\mathscr{I}_b)\mathscr{D}(\rho)]^{t_\rho} = \mathscr{D}(\rho)$, $\mathscr{H}_\pi \subset \mathscr{H}_\rho$ and $\pi(x) \subset \widetilde{\rho \restriction \mathscr{B}}(x)$ for each $x \in \mathscr{B}$, then $\mathscr{A}$ is said to have *diration-property*. Here we denote by $[\mathscr{K}]^{t_\pi}$ the closed subspace of the complete locally convex space $\mathscr{D}(\pi)[t_\pi]$ generated by a subset $\mathscr{K}$ of $\mathscr{D}(\pi)$.

Theorem 5.2. *Let $\mathscr{A}$ be a pseudo-complete locally convex $*$-algebra with identity 1. Consider the following statements*:

(i) *There exists a hereditary spectral well-behaved $*$-representations of $\mathscr{A}$.*

(ii) *There exists a hereditary spectral uniformly nondegenerate $*$-representation of $\mathscr{A}$.*

(iii) *$\mathscr{A}$ has diration-property.*

Then the following implications (i) $\Leftrightarrow$ (ii) $\Rightarrow$ (iii) *hold.*

Proof. The equivalence of (i) and (ii) follows from Proposition 4.12.

(i) $\Rightarrow$ (iii) Let ρ_0 be a hereditary spectral well-behaved $*$-representation of $\mathscr{A}$. Let $\mathscr{B}$ be a unital closed $*$-subalgebra of $\mathscr{A}$ and π a closed $*$-representation of $\mathscr{B}$ such that $[\pi(\mathscr{I}_b \cap \mathscr{B})\mathscr{D}(\pi)]^{t_\pi} = \mathscr{D}(\pi)$. Then it follows from Lemma 4.10 that $r_{\mathscr{B}_b}(x) \le \|\overline{\rho_0(x)}\|$ for each $x \in \mathscr{B}_b$. Hence we have

$$\begin{aligned}\varlimsup_{n\to\infty} \|\overline{\pi(x)}^n\|^{1/n} = r_{C_u^*(\pi)}(\overline{\pi(x)}) &\le r_{\overline{\pi(\mathscr{B}_b)}}(\overline{\pi(x)}) \\ &\le r_{\mathscr{B}_b}(x) \\ &\le \|\overline{\rho_0(x)}\|, \quad {}^\forall x \in \mathscr{B}_b,\end{aligned}$$

which implies that

$$\|\overline{\pi(x)}\| \le \|\overline{\rho_0(x)}\|, \quad {}^\forall x \in \mathscr{B}_b. \tag{5.1}$$

We put

$$P_0(\overline{\rho_0(x)}) = \overline{\pi(x)}, \quad x \in \mathscr{B}_b.$$

By (5.1) P_0 can be extended to a $*$-representation of the C^*-algebra $C_u^*(\rho_0 \restriction \mathscr{B}) \equiv \overline{\rho_0(\mathscr{B}_b)}^{\|\ \|}$ on $\mathscr{H}_\pi$ and it is denoted by the same P_0. By the diration-property of C^*-algebras there exists a Hilbert space $\mathscr{H}_P$ containing $\mathscr{H}_\pi$ as a closed subspace and a $*$-representation P of the C^*-algebra $C_u^*(\rho_0) \lceil \mathscr{H}_\pi \equiv \overline{\rho_0(\mathscr{A}_b)}^{\|\ \|}$ on $\mathscr{H}_P$ such that $P(A) \restriction \mathscr{H}_\pi = P_0(A)$ for each $A \in C_u^*(\rho_0 \restriction \mathscr{B})$. We define a $*$-representation of $\mathscr{A}$ by

$$\begin{cases} \mathscr{D}(\rho) = \text{the linear span of } \{P(\overline{\rho_0(x)})\xi; x \in \mathscr{I}_b, \xi \in \mathscr{H}_P\} \\ \rho(a)P(\overline{\rho_0(x)})\xi = P(\overline{\rho_0(ax)})\xi, \quad a \in \mathscr{A},\ x \in \mathscr{I}_b,\ \xi \in \mathscr{H}_P, \end{cases}$$

and denote its closure by the same ρ. For any $b \in \mathscr{B}$, $x \in \mathscr{I}_b \cap \mathscr{B}$ and $\xi \in \mathscr{H}_\pi$ we have

$$\pi(x)\xi = P_0(\overline{\rho_0(x)})\xi = P(\overline{\rho_0(x)})\xi$$

and

$$\begin{aligned} \pi(b)\overline{\pi(x)}\xi = \overline{\pi(bx)}\xi &= P(\overline{\rho_0(bx)})\xi \\ &= \rho(b)P(\rho_0(x))\xi, \end{aligned}$$

which implies by $[\pi(\mathscr{I}_b \cap \mathscr{B})\mathscr{D}(\pi)]^{t_\pi} = \mathscr{D}(\pi)$ that $\mathscr{H}_\pi \subset \mathscr{H}_\rho$ and $\pi(x) \subset \widetilde{\rho \restriction \mathscr{B}}(x)$ for each $x \in \mathscr{B}$. This completes the proof. □

6. Special cases and examples.

The main purpose of this section is to give a number of examples of well-behaved $*$-representations of (locally convex) $*$-algebras and of corresponding unbounded C^*-seminorms satisfying conditions (R) or (UR). All these examples belong to one of the following special cases: Representations related to well-behaved $*$-representations in the sense of Schmüdgen [**31**], representations of pseudo-complete locally convex $*$-algebras satisfying $\mathscr{A} = \mathscr{A}_0$, and representations of GB^*-algebras.

6.1. Multiplier algebras and Schmüdgen's well-behaved $*$-representations.

Here we investigate the relation of two concepts of well-behaved $*$-representations using multiplier algebras.

Let $\mathfrak{X}$ be a $*$-algebra without unit such that $a = 0$ whenever $ax = 0$ for all $x \in \mathfrak{X}$. A *multiplier* on $\mathfrak{X}$ is a pair (l, r) of linear operators on $\mathfrak{X}$ such that $l(xy) = l(x)y$, $r(xy) = xr(y)$ and $xl(y) = r(x)y$ for each $x, y \in \mathfrak{X}$. Let $\Gamma(\mathfrak{X})$ be the collection of all multipliers on $\mathfrak{X}$. Then $\Gamma(\mathfrak{X})$ is a $*$-algebra with unit (ι, ι), where $\iota(x) = x$, $x \in \mathfrak{X}$, with pointwise linear operations, with multiplication defined by $(l_1, r_1)(l_2, r_2) = (l_1 l_2, r_2 r_1)$, and with the involution $(l, r)^* = (r^*, l^*)$, where $l^*(x) \equiv l(x^*)^*$ and $r^*(x) \equiv r(x^*)^*$, $x \in \mathfrak{X}$. For $x \in \mathfrak{X}$ we put

$$l_x(y) = xy \quad \text{and} \quad r_x(y) = yx, \quad y \in \mathfrak{X}.$$

Then the map $x \in \mathfrak{X} \mapsto (l_x, r_x) \in \Gamma(\mathfrak{X})$ embeds $\mathfrak{X}$ into a $*$-ideal of $\Gamma(\mathfrak{X})$. Let $\mathfrak{X}$ be a normed $*$-algebra with approximate identity. By an approximate identity for $\mathfrak{X}$, we mean that a net $\{e_\alpha\}$ in $\mathfrak{X}$, $\|e_\alpha\| \le 1$, such that $x = \lim_\alpha e_\alpha x = \lim_\alpha x e_\alpha$ for all $x \in \mathfrak{X}$. We denote by $\tilde{\mathfrak{X}}$ the Banach $*$-algebra obtained as completion of $\mathfrak{X}$. Then, for $(l, r) \in \Gamma(\mathfrak{X})$, since

$$\begin{aligned} \|r(a)\| &= \sup\{\|r(a)x\|; x \in \mathfrak{X} \text{ such that } \|x\| \le 1\} \\ &= \sup\{\|al(x)\|; x \in \mathfrak{X} \text{ such that } \|x\| \le 1\} \\ &\le \|a\| \sup\{\|l(x)\|; x \in \mathfrak{X} \text{ such that } \|x\| \le 1\} \end{aligned}$$

and similarly,

$$\|l(a)\| \leq \|a\| \sup\{\|r(x)\|; x \in \mathfrak{X} \text{ such that } \|x\| \leq 1\},$$

it follows that l is bounded if and only if r is bounded, and

$$\begin{aligned} \|l\| &= \sup\{\|l(a)\|; a \in \mathfrak{X} \text{ such that } \|a\| \leq 1\} \\ &= \sup\{\|r(a)\|; a \in \mathfrak{X} \text{ such that } \|a\| \leq 1\} \\ &= \|r\|, \end{aligned}$$

so that

$$\Gamma_c(\mathfrak{X}) \equiv \{(l, r) \in \Gamma(\mathfrak{X}); l \text{ is continuous}\}$$

is a normed $*$-algebra with the norm

$$\|(l, r)\| \equiv \sup\{\|l(a)\|; \|a\| \leq 1\}.$$

It is well-known that every element of $\Gamma(\tilde{\mathfrak{X}})$ is continuous and that $\Gamma(\tilde{\mathfrak{X}}) = \Gamma_c(\tilde{\mathfrak{X}})$ (denoted also by $M(\tilde{\mathfrak{X}})$) is a Banach $*$-algebra. We have the following diagram:

$$\begin{array}{ccccc} \tilde{\mathfrak{X}} & \hookrightarrow & M(\tilde{\mathfrak{X}}) & & \\ \uparrow & & \uparrow & & \\ \mathfrak{X} & \hookrightarrow & \Gamma_c(\mathfrak{X}) & \hookrightarrow & \Gamma(\mathfrak{X}). \end{array}$$

The map $\Gamma_c(\mathfrak{X}) \to M(\tilde{\mathfrak{X}})$ is defined by $(l, r) \to (\tilde{l}, \tilde{r})$ (where $\tilde{l}, \tilde{r}$ are the continuous extensions to $\tilde{\mathfrak{X}}$ of l and r, resp). If $\mathfrak{X}$ is a $*$-ideal of $\tilde{\mathfrak{X}}$, then

$$\begin{array}{ccccc} \tilde{\mathfrak{X}} & \hookrightarrow & \Gamma_c(\mathfrak{X}) & \hookrightarrow & M(\tilde{\mathfrak{X}}) \\ \Cup & & \Cup & & \\ x & & (l_x, r_x) & & \end{array}$$

and

$$\Gamma_c(\mathfrak{X}) = \{(l, r) \in M(\tilde{\mathfrak{X}}); l\mathfrak{X} \subset \mathfrak{X} \text{ and } r\mathfrak{X} \subset \mathfrak{X}\}.$$

By Proposition 2.4 we have the following

Proposition 6.1. *Let $\mathscr{A}$ be a $*$-algebra without identity such that $a = 0$ whenever $ax = 0$ for all $x \in \mathscr{A}$. Suppose that there exists a non-zero bounded, nondegenerate, closed $*$-representation π_0 of $\mathscr{A}$. Then there exists a well-behaved $*$-reprensetation π of the multiplier algebra $\Gamma(\mathscr{A})$ such that $\mathscr{H}_\pi = \mathscr{H}_{\pi_0}$ and $\pi((l_x, r_x)) = \pi_0(x) \restriction \mathscr{D}(\pi)$ for all $x \in \mathscr{A}$.*

Proof. Identifying $x \in \mathscr{A}$ with $(l_x, r_x) \in \Gamma(\mathscr{A})$, $\mathscr{A}$ becomes a subset (even an ideal) of $\Gamma(\mathscr{A})$. Now the unbounded C^*-seminorm p on $\Gamma(\mathscr{A})$ defined by

$$\begin{cases} D(p) = \mathscr{A}, \\ p(x) = \|\pi_0(x)\|, \quad x \in \mathscr{A} \end{cases}$$

satisfies condition (R) since $\mathfrak{N}_p = \mathscr{A} \not\subset N_p$. By Proposition 2.4, there exists a well-behaved $*$-representation of $\Gamma(\mathscr{A})$. But in the present situation we can define a representation Π_p of $\mathscr{A}_p$ on $\mathscr{H}_{\pi_0}$ such that $\Pi_p(x+N_p)=\pi_0(x)$ for all $x\in\mathscr{A}$. Since π_0 is nondegenerate,

$$\begin{aligned} D(\pi_p) &= \text{linear span of } \{\Pi_p(\mathscr{A}+N_p)\mathscr{H}_{\pi_0}\} \\ &= \text{linear span of } \{\pi_0(\mathscr{A})\mathscr{H}_{\pi_0}\} \end{aligned}$$

is dense in $\mathscr{H}_{\pi_0}$, i.e., π_p is well-behaved. □

Schmüdgen [**31**] has defined the notion of well-behaved $*$-representations of $*$-algebras. Here we shall introduce it and investigate the relation between his concept of well-behaved $*$-representations and that of well-behaved $*$-representations in our framework. Let $\mathscr{A}$ be a $*$-algebra with identity 1 and $\mathfrak{X}$ a normed $*$-algebra (without identity in general). The pair $(\mathscr{A},\mathfrak{X})$ is called a *compatible pair* if $\mathfrak{X}$ is a left $\mathscr{A}$-module with left action denoted by $\triangleright$, such that $(a\triangleright x)^*y=x^*(a^*\triangleright y)$ for all $x,y\in\mathfrak{X}$ and $a\in\mathscr{A}$. Then, Schmüdgen has shown that for any nondegenerate continuous bounded $*$-representation ρ of $\mathfrak{X}$ on $\mathscr{H}_\rho$ there exists a unique $*$-representation $\tilde{\rho}$ of $\mathscr{A}$ such that

$$\begin{cases} \mathscr{D}(\tilde{\rho}) \equiv \text{the linear span of } \rho(\mathfrak{X})\mathscr{H}_\rho, \\ \tilde{\rho}(a)(\rho(x)\xi)=\rho(a\triangleright x)\xi, \quad a\in\mathscr{A},\, x\in\mathfrak{X},\, \xi\in\mathscr{H}_\rho, \end{cases}$$

and he has called the closure ρ' of $\tilde{\rho}$ the *well-behaved $*$-representation of $\mathscr{A}$ associated with the compatible pair* $(\mathscr{A},\mathfrak{X})$. We consider this in our framework.

Theorem 6.2. *Let $(\mathscr{A},\mathfrak{X})$ be a compatible pair with left action $\triangleright$. Suppose that $\mathfrak{X}$ has an approximate identity. Then the map*: $x\in\mathfrak{X}\mapsto(l_x,r_x)\in\Gamma(\mathfrak{X})$ *embeds $\mathfrak{X}$ into a $*$-ideal of the multiplier algebra $\Gamma(\mathfrak{X})$. For any $a\in\mathscr{A}$ we put*

$$\begin{aligned} &l_{\tilde{a}}x=a\triangleright x, \\ &r_{\tilde{a}}x=(l_{\widetilde{a^*}}x^*)^*, \quad x\in\mathfrak{X}. \end{aligned}$$

Then, a $*$-homomorphism m of $\mathscr{A}$ into $\Gamma(\mathfrak{X})$ is defined by

$$m : a\in\mathscr{A}\mapsto(l_{\tilde{a}},r_{\tilde{a}})\in\Gamma(\mathfrak{X}).$$

Suppose that π is a non-zero $*$-representation of $\mathscr{A}$. Then the following statements are equivalent:

(i) *π coincides with the well-behaved* (*in the sense of* [**31**]) *$*$-representation ρ' of $\mathscr{A}$ associated with the compatible pair* $(\mathscr{A},\mathfrak{X})$.

(ii) *π is the closure of $\pi_r\circ m$, where π_r is the well-behaved* (*in the sense of* [**6**]) *$*$-representation π_r of $\Gamma(\mathfrak{X})$ constructed for some weakly semifinite unbounded C^*-seminorm r on $\Gamma(\mathfrak{X})$ such that $\mathscr{D}(r)=\{(l_x,r_x);x\in\mathfrak{X}\}$ and r is continuous on the normed $*$-algebra $\mathscr{D}(r)$.*

Proof. It is easily shown that m is a $*$-homomorphism of $\mathscr{A}$ into $\Gamma(\mathfrak{X})$.

(i) $\Rightarrow$ (ii) Let ρ be a nondegenerate continuous bounded $*$-representation of the normed $*$-algebra $\mathfrak{X}$ on $\mathscr{H}_\rho$ and set $\pi=\rho'$. We define an unbounded C^*-seminorm r_ρ on $\Gamma(\mathfrak{X})$ by

$$\begin{cases} \mathscr{D}(r_\rho)=\{(l_x,r_x);x\in\mathfrak{X}\}, \\ r_\rho((l_x,r_x))=\|\rho(x)\|, \quad x\in\mathfrak{X}. \end{cases}$$

By the continuity of ρ, r_ρ is continuous on the normed $*$-algebra $\mathscr{D}(r_\rho)$. Moreover, $\mathfrak{N}_{r_\rho} = \mathscr{D}(r_\rho)$ and $N_{r_\rho} \simeq \ker \rho$. Hence, a faithful $*$-representation Π_{r_ρ} of the C^*-algebra $\mathfrak{X}_{r_\rho}$ (the completion of $\mathscr{D}(r_\rho)/N_{r_\rho}$) on the Hilbert space $\mathscr{H}_\rho$ can be defined so that

$$\Pi_{r_\rho}((l_x, r_x) + N_{r_\rho}) = \rho(x), \quad x \in \mathfrak{X}.$$

As stated in the Introduction, the $*$-representation π_{r_ρ} of the multiplier algebra $\Gamma(\mathfrak{X})$ is constructed from Π_{r_ρ} as follows:

$$\begin{cases} \mathscr{D}(\pi_{r_\rho}) = \text{the linear span of } \Pi_{r_\rho}(\mathfrak{N}_{r_\rho} + N_{r_\rho})\mathscr{H}_\rho \\ \qquad\quad = \text{the linear span of } \rho(\mathfrak{X})\mathscr{H}_\rho, \\ \pi_{r_\rho}((l,r))\rho(x)\xi = \rho(l(x))\xi, \quad x \in \mathfrak{X},\ \xi \in \mathscr{H}_\rho. \end{cases}$$

Since ρ is nondegenerate, it follows that $\mathscr{H}_{\pi_\rho} = \mathscr{H}_\rho$, which implies that π_{r_ρ} is a well-behaved $*$-representation of $\Gamma(\mathfrak{X})$ constructed from the unbounded C^*-seminorm r_ρ. Furthermore, it follows that

$$\mathscr{D}(\pi_{r_\rho}) = \mathscr{D}(\tilde{\rho})$$

and

$$\begin{aligned} (\pi_{r_\rho} \circ m)(a)\rho(x)\xi = \pi_{r_\rho}((l_{\tilde{a}}, r_{\tilde{a}}))\rho(x)\xi &= \rho(l_{\tilde{a}}(x))\xi \\ &= \rho(a \triangleright x)\xi \\ &= \tilde{\rho}(a)\rho(x)\xi \end{aligned}$$

for all $a \in \mathscr{A}$, $x \in \mathfrak{X}$ and $\xi \in \mathscr{H}_\rho$, which implies (ii).

(ii) $\Rightarrow$ (i) Suppose π is the closure of $\pi_r \circ m$, where π_r is a well-behaved $*$-representation of $\Gamma(\mathfrak{X})$ constructed from an unbounded C^*-seminorm r on $\Gamma(\mathfrak{X})$ such that $\mathscr{D}(r) = \{(l_x, r_x); x \in \mathfrak{X}\}$ and r is continuous on the normed $*$-algebra $\mathscr{D}(r)$. Here we put

$$\rho(x) = \overline{\pi_r((l_x, r_x))}, \quad x \in \mathfrak{X}.$$

Then since π_r is well-behaved, it follows that ρ is a nondegenerate continuous bounded $*$-representation of the normed $*$-algebra $\mathfrak{X}$ on the Hilbert space $\mathscr{H}_\rho = \mathscr{H}_{\pi_r}$, and

$$\begin{aligned} \mathscr{D}(\tilde{\rho}) &= \text{linear span of } \rho(\mathfrak{X})\mathscr{H}_\rho \\ &= \text{linear span of } \{\overline{\pi_r((l_x, r_x))}\mathscr{H}_{\pi_r}; x \in \mathfrak{X}\} \\ &= \mathscr{D}(\pi_r). \end{aligned}$$

Moreover,

$$\begin{aligned} \tilde{\rho}(a)\rho(x)\xi &= \rho(a \triangleright x)\xi \\ &= \rho(l_{\tilde{a}}x)\xi \\ &= \overline{\pi_r((l_{\tilde{a}}l_x, r_x r_{\tilde{a}}))}\xi \\ &= \pi_r(m(a))\overline{\pi_r((l_x, r_x))}\xi \end{aligned}$$

for all $a \in \mathscr{A}$, $x \in \mathfrak{X}$ and $\xi \in \mathscr{H}_\rho$. Hence $\tilde{\rho} = \pi_r \circ m$, which implies (i). This completes the proof. □

REMARK 6.3. Let $(\mathscr{A}, \mathfrak{X})$ be a compatible pair. Suppose that $\mathfrak{X}$ is a $*$-ideal of the Banach $*$-algebra $\tilde{\mathfrak{X}}[\| \ \|]$ obtained by the completion of $\mathfrak{X}[\| \ \|]$. Then every C^*-seminorm r on $\mathfrak{X}$ is $\| \ \|$-continuous. Equivalently we show this for a bounded $*$-representation π of $\mathfrak{X}$. Since $\mathfrak{X}$ is a $*$-ideal of $\tilde{\mathfrak{X}}[\| \ \|]$, it follows that $\mathfrak{X}$ is quasi-inverse closed in $\tilde{\mathfrak{X}}[\| \ \|]$ and $\mathfrak{X}^{qi} = \mathfrak{X} \cap \tilde{\mathfrak{X}}[\| \ \|]^{qi}$. Hence, $\mathfrak{X}^{qi}$ is open in $\tilde{\mathfrak{X}}[\| \ \|]$, which implies

$$r_{\mathfrak{X}}(x) \leq \|x\|, \quad x \in \mathfrak{X},$$

where $r_{\mathfrak{X}}$ denotes the spectral radius of x in $\mathfrak{X}$, which implies that

$$\begin{aligned}
\|\pi(x)\|^2 = r_{\mathscr{B}(\mathscr{H}_\pi)}(\pi(x)^*\pi(x)) &\leq r_{\pi(\mathfrak{X})}(\pi(x)^*\pi(x)) \\
&\leq r_{\mathfrak{X}}(x^*x) \\
&\leq \|x^*x\| \\
&\leq \|x\|^2
\end{aligned}$$

for all $x \in \mathfrak{X}$. Thus, π is $\| \ \|$-continuous. Therefore we may take off the assumption of the continuity of the C^*-seminorm r on $\mathscr{D}(r)$ in Theorem 6.2, (ii).

As a consequence of Theorem 6.2 examples of well-behaved $*$-representations given by Schmüdgen in [**31**] are equivalent to well-behaved (in the sense of [**6**]) $*$-representations of the corresponding multiplier algebras. We discuss those examples in some detail (Examples 6.4–6.7). In the same way, the Moyal quantization is related to a well-behaved $*$-representation of the Moyal algebra (Example 6.8).

EXAMPLE 6.4. Let $\mathscr{A}$ be the $*$-algebra $\mathscr{P}(x_1, \ldots, x_n)$ of all polynomials with complex coefficients in n commuting hermitian elements $x_1, \ldots, x_n$, and let $\mathfrak{X}$ be the normed $*$-algebra $C_c(\boldsymbol{R}^n)$ of all compactly supported continuous functions on $\boldsymbol{R}^n$ with pointwise multiplication $(fg)(t) = f(t)g(t)$, the involution $f^*(t) = \overline{f(t)}$, and the norm $\|f\| = \sup_{t \in \boldsymbol{R}^n} |f(t)|$. It is clear that $(\mathscr{A}, \mathfrak{X})$ is a compatible pair with the left action

$$p \triangleright f = pf, \quad p \in \mathscr{A},\ f \in \mathfrak{X}.$$

Let π be a closed $*$-representation of $\mathscr{A}$. Then, according to Theorem 6.2 and [**31**], the following statements are equivalent:

(i) π is integrable, that is, $\pi(a)^* = \overline{\pi(a^*)}$ for all $a \in \mathscr{A}$.

(ii) π is a well-behaved (in the sense of [**31**]) $*$-representation of $\mathscr{A}$ associated with the compatible pair $(\mathscr{A}, \mathfrak{X})$.

(iii) π is the closure of $\pi_r \circ m$, where π_r is a well-behaved (in the sense of [**6**]) $*$-representation of the multiplier algebra $\Gamma(C_c(\boldsymbol{R}^n))$ constructed from an unbounded C^*-seminorm r whose domain is $\{(l_f, r_f); f \in C_c(\boldsymbol{R}^n)\}$.

EXAMPLE 6.5. Let G be a finite dimensional real Lie group with the left Haar measure μ, $\mathscr{G}$ the Lie algebra of G and $E(\mathscr{G})$ the complex universal enveloping algebra of $\mathscr{G}$. The algebra $E(\mathscr{G})$ is a $*$-algebra with the involution $x^* = -x$, $x \in \mathscr{G}$ [**30**], [**31**].

The space $C_c^\infty(G)$ of C^∞-functions on G with compact supports is a normed $*$-algebra with the convolution multiplication

$$(f * g)(v) = \int_G f(u) g(u^{-1}v)\, d\mu(u),$$

the involution

$$f^*(v) = \delta(v)^{-1}\overline{f(v^{-1})},$$

where δ denotes the modular function on G, and the L^1-norm

$$\|f\|_1 = \int_G |f(v)|\, d\mu(v).$$

The completion of $C_c^\infty(G)$ is nothing but the Banach $*$-algebra $L^1(G)$, and $C_c^\infty(G)$ contains a bounded approximate identity for $L^1(G)$. Furthermore, $(E(\mathscr{G}), C_c^\infty(G))$ is a compatible pair with the left action $\triangleright$:

$$(x \triangleright f)(u) = (\tilde{x} f)(u) \equiv \left.\frac{d}{dt}\right|_{t=0} f(e^{-tx}u), \quad x \in E(\mathscr{G}),\ f \in C_c^\infty(G).$$

Let π be a closed $*$-representation of the $*$-algebra $E(\mathscr{G})$. Then the following statements are equivalent by [**31**], Section 3 and Theorem 6.2.

(i) π is G-integrable, that is, it is of form $\pi = dU$ for some strongly continuous unitary representation U of G on a Hilbert space $\mathscr{H}$, where dU is a $*$-representation of $E(\mathscr{G})$ defined by

$$\begin{cases} \mathscr{D}(dU) = \mathscr{D}^\infty(U) \equiv \text{the space of } C^\infty\text{-vectors in } \mathscr{H} \text{ for } U, \\ dU(x)\varphi = \left.\dfrac{d}{dt}\right|_{t=0} U(e^{tx})\varphi, \quad \varphi \in \mathscr{D}(dU). \end{cases}$$

(ii) π is a well-behaved (in the sense of [**31**]) $*$-representation of $E(\mathscr{G})$ associated with the compatible pair $(E(\mathscr{G}), C_c^\infty(G))$.

(iii) π is the closure of $\pi_r \circ m$, where π_r is a well-behaved (in the sense of [**6**]) $*$-representation of $\Gamma(C_c^\infty(G))$ defined by an unbounded C^*-seminorm r whose domain is $\{(l_f, r_f); f \in C_c^\infty(G)\}$ and which is continuous with respect to the L^1-norm $\|\ \|_1$ of $C_c^\infty(G)$.

EXAMPLE 6.6. Let $\mathscr{A}$ be the $*$-algebra generated by unit 1 and two hermitian generators p and q satisfying the commutation relation $pq - qp = -i1$, and let π_S be the Schrödinger representation of $\mathscr{A}$ on the Hilbert space $L^2(\boldsymbol{R})$ with domain $\mathscr{D}(\pi_S) = \mathscr{S}(\boldsymbol{R})$, that is, it is a $*$-representation of $\mathscr{A}$ defined by

$$(\pi_S(p)f)(t) = -i\frac{d}{dt}f,$$

$$(\pi_S(q)f)(t) = tf(t), \quad f \in \mathscr{S}(\boldsymbol{R}).$$

Let P and Q be the self-adjoint operators and let $W(s,t)$ be the unitary operator on $L^2(\boldsymbol{R})$ defined by

$$P = \overline{\pi_S(p)}, \quad Q = \overline{\pi_S(q)}, \quad W(s,t) = e^{2\pi i(sQ+tP)}, \quad s,t \in \boldsymbol{R}.$$

To any $f \in \mathscr{S}(\boldsymbol{R}^2)$ the Weyl calculus assigns a bounded operator $W(f)$ on the Hilbert space $L^2(\boldsymbol{R})$ by

$$W(f) = \iint \hat{f}(s,t) W(s,t)\, dsdt,$$

where $\hat{f}$ is the Fourier transform of f. The Schwartz space $\mathscr{S}(\boldsymbol{R}^2)$ is a normed $*$-algebra with the multiplication $f\#g$, the involution f^* and the norm $\| \ \|$:

$$\begin{aligned}
&(f\#g)(t_1,t_2)\\
&\quad = \iiiint f(u_1,u_2)g(v_1,v_2)e^{4\pi i[(t_1-u_1)(t_2-v_2)-(t_1-v_1)(t_2-u_2)]}\, du_1du_2dv_1dv_2,\\
&f^*(t_1,t_2) = \overline{f(t_1,t_2)},\\
&\|f\| = \|W(f)\|,
\end{aligned}$$

and

$$W(f)W(g) = W(f\#g) \quad \text{and} \quad W(f)^* = W(f^*), \quad f,g \in \mathscr{S}(\boldsymbol{R}^2).$$

$(\mathscr{A}, \mathscr{S}(\boldsymbol{R}^2))$ is a compatible pair with the following action:

$$p \triangleright f = \left(\frac{1}{2i}\frac{\partial}{\partial t_1} + 2\pi t_2\right)f, \quad q \triangleright f = \left(t_1 - \frac{1}{4\pi i}\frac{\partial}{\partial t_2}\right)f, \quad f \in \mathscr{S}(\boldsymbol{R}^2).$$

By [**31**, Section 4] and Theorem 6.2 we have the following:

Let π be a closed $*$-representation of $\mathscr{A}$. The following statements are equivalent:

(i) π is standard, [**28**], that is, it is unitarily equivalent to the direct sum of the Schrödinger representation of $\mathscr{A}$.

(ii) π is a well-behaved $*$-representation of $\mathscr{A}$ associated with the compatible pair $(\mathscr{A}, \mathscr{S}(\boldsymbol{R}^2))$.

(iii) $\pi = \pi_r \circ m$ for a well-behaved $*$-representation π_r of $\Gamma(\mathscr{S}(\boldsymbol{R}^n))$ constructed from an unbounded C^*-norm r on $\Gamma(\mathscr{S}(\boldsymbol{R}^n))$ such that $\mathscr{D}(r) = \{(l_f, r_f); f \in \mathscr{S}(\boldsymbol{R}^n)\}$ and r is continuous on the normed $*$-algebra $\mathscr{D}(r)$.

Example 6.7. Schmüdgen [**31**] has given the examples of well-behaved $*$-representations of the coordinate $*$-algebra $O(\boldsymbol{R}_q^2)$ of the real quantum plane and the coordinate $*$-algebra $O(SU_q(1,1))$ of the quantum group $SU_q(1,1)$.

We consider well-behaved $*$-representations of the Moyal algebra. For the Moyal algebra we refer to [**16**].

Example 6.8. Consider a system having n degrees of freedom and the configuration space $\boldsymbol{R}^n$. Then the phase space is identified with the cotangent boundle $T^*\boldsymbol{R}^n = \boldsymbol{R}^n \times \boldsymbol{R}^n$, (q,p) being the canonical variables. On the Hilbert space $\mathscr{H} = L^2(\boldsymbol{R}^n)$, the *Moyal quantizer* is given by the operators

$$(\Omega^h(q,p)f)(x) = 2^n \exp\left(\frac{2i}{h}p(x-q)\right)f(2q-x).$$

For a symbol a of the Schwartz space $\mathscr{S}(T^*\boldsymbol{R}^n)$, the Bochner integral

$$(Q_h(a)f)(x) = \frac{1}{(2\pi h)^n}\int_{T^*\boldsymbol{R}^n} a(q,p)(\Omega^h(q,p)f)(x)\,d^nqd^np$$

defines the *Moyal quantization* map. Then $Q_h(a)$ is a trace class operator on $L^2(\boldsymbol{R}^n)$ such that $a(q,p) = \mathrm{Tr}[Q_h(a)\Omega^h(q,p)]$. The *Moyal product*

$$a \times_h b(u) = \iint \mathrm{Tr}[\Omega^h(u)\Omega^h(v)\Omega^h(w)]a(v)b(w)\,d^nvd^nw$$

converts $\mathscr{S}(T^*\boldsymbol{R}^n)$ into a non-commutative $*$-algebra $\mathfrak{X}$ such that $Q_h(a\times_h b) = Q_h(a)Q_h(b)$. It is a normed $*$-algebra $\mathfrak{X}$ with Hilbert-Schmidt operator norm induced by Q_h. The Moyal product extends to large classes of distributions as follows: For $T \in \mathscr{S}'(\boldsymbol{R}^{2n})$ and $a \in \mathscr{S}(\boldsymbol{R}^{2n})$, we can define $T\times_h a$ and $a\times_h T$ in $\mathscr{S}'(\boldsymbol{R}^{2n})$ by

$$\langle T\times_h a, b\rangle = \langle T, a\times_h b\rangle,$$
$$\langle a\times_h T, b\rangle = \langle T, b\times_h a\rangle, \quad b \in \mathscr{S}(\boldsymbol{R}^{2n}).$$

The multiplier algebra of $\mathfrak{X}$ in the space of tempered distributions defined by

$$\mathscr{M} = \{T \in \mathscr{S}'(\boldsymbol{R}^{2n}); T\times_h a, a\times_h T \text{ are in } \mathscr{S}(\boldsymbol{R}^n) \text{ for all } a \in \mathscr{S}(\boldsymbol{R}^{2n})\}$$

is a $*$-algebra with Moyal product and complex conjugation containing $\mathfrak{X}$ as a $*$-ideal. This $\mathscr{M}$ is called the *Moyal algebra*. The Moyal quantizer Q_h extends as a $*$-representation [denoted by Q_h also] of $\mathscr{M}$ into unbounded operators on $L^2(\boldsymbol{R}^n)$ such that $Q_h(q) =$ multiplication by x and $Q_h(p) = -ih(\partial/\partial x)$. Thus $(\mathscr{M},\mathfrak{X})$ is a compatible pair and $\mathscr{M} \subset \Gamma(\mathfrak{X})$. Theorem 6.2 immediately gives the following:

Let π be a $*$-representation of $\mathscr{M}$. Then the following are equivalent:

(i) π is a well-behaved $*$-representation of $\mathscr{M}$ associated with the compatible pair $(\mathscr{M},\mathfrak{X})$.

(ii) $\pi = \pi_r \circ m$ for a well-behaved $*$-representation π_r of $\Gamma(\mathfrak{X})$ defined by an unbounded C^*-seminorm r whose domain is $\{(l_a, r_a); a \in \mathfrak{X}\}$. In particular, the Moyal quantization map Q_h is a well-behaved $*$-representation of the Moyal algebra $\mathscr{M}$.

6.2. Pseudo-complete locally convex $*$-algebras $\mathscr{A}$ with $\mathscr{A} = \mathscr{A}_0$.

Let $\mathscr{A}$ be a pseudo-complete locally convex $*$-algebra with $\mathscr{A} = \mathscr{A}_0$. Then $\mathscr{A}_b = \mathscr{I}_b = \mathscr{A}$. When $\mathscr{A}$ does not have identity, we will consider the pseudo-complete locally convex $*$-algebra $\mathscr{A}_1$ obtained by adjoining an identity *1*. By Lemma 3.1 and Theorem 4.8 we have the following:

Corollary 6.9. *Let $\mathscr{A}$ be a pseudo-complete locally convex $*$-algebra with $\mathscr{A} = \mathscr{A}_0$. Then the following statements are equivalent*:

(i) *There exists a spectral well-behaved bounded $*$-representation of $\mathscr{A}$.*

(ii) *There exists a spectral C^*-seminorm on $\mathscr{A}$.*
(iii) *$\mathscr{A}$ is spectral invariant.*

Example 6.10. Let $\mathscr{A}$ be a Banach $*$-algebra. Then there exists a spectral unbounded C^*-seminorm on $\mathscr{A}$ if and only if $\mathscr{A}$ is hermitian, that is, $Sp_{\mathscr{A}}(x) \subset \boldsymbol{R}$ for each $x \in \mathscr{A}_h$. Let $\mathscr{D} = \{\alpha \in \boldsymbol{C}; |\alpha| \leq 1\}$ and $\mathscr{A} = \{f \in C(\mathscr{D}); f$ is analytic in the interior of $\mathscr{D}\}$. Then the disc algebra $\mathscr{A}$ is a Banach $*$-algebra which is not hermitian under the usual operations, the involution $f^*(\alpha) = \overline{f(\bar{\alpha})}$ and the uniform norm. Hence there is no spectral unbounded C^*-seminorm on the disc algebra.

Example 6.11. Let $\mathscr{S}(\boldsymbol{R}^n)$ be the Schwartz space of rapidly decreasing infinitely differentiable functions on $\boldsymbol{R}^n$ equipped with the topology defined by the seminorms $\{\|\ \|_{m,k}; m,k = 0,1,\ldots\}$, where

$$\|f\|_{m,k} = \sup_{|p| \leq m} \left\{ \sup_{x \in \boldsymbol{R}^n} \left\{ (1+|x|)^k \left| \left(\frac{\partial}{\partial x}\right)^p f(x) \right| \right\} \right\}.$$

(1) $\mathscr{S}(\boldsymbol{R}^n)$ is a Fréchet $*$-algebra with $\mathscr{S}(\boldsymbol{R}^n) = \mathscr{S}(\boldsymbol{R}^n)_0$ under the pointwise multiplication $fg : (fg)(x) \equiv f(x)g(x)$ and the involution $f^* : f^*(x) = \overline{f(x)}$, and $\|f\|_\infty = \sup_{x \in \boldsymbol{R}^n} |f(x)|$ is a spectral C^*-norm on $\mathscr{S}(\boldsymbol{R}^n)$.

(2) $\mathscr{S}(\boldsymbol{R}^n)$ is a Fréchet $*$-algebra with the convolution multiplication

$$(f * g)(x) = \frac{1}{(2\pi)^{n/2}} \int_{\boldsymbol{R}^n} f(x-y)g(y)\,dy$$

and the involution

$$f^*(x) = \overline{f(-x)}.$$

The Fourier transform $f \mapsto \hat{f}$ establishes an isomorphism between $\mathscr{S}(\boldsymbol{R}^n)$ with convolution multiplication and $\mathscr{S}(\boldsymbol{R}^n)$ with pointwise multiplication. It follows that $\|f\| \equiv \|\hat{f}\|_\infty \equiv \sup_{x \in \boldsymbol{R}^n} |\hat{f}(x)|$ is a spectral C^*-norm on the convolution algebra $\mathscr{S}(\boldsymbol{R}^n)$. In fact, $\|\ \|$ is the Gelfand-Naimark pseudo norm and $E(\mathscr{S}(\boldsymbol{R}^n)) \simeq C^*(\boldsymbol{R}^n)$ (the group C^*-algebra of $\boldsymbol{R}^n$) $\simeq C_0(\boldsymbol{R}^n) \equiv \{f \in C(\boldsymbol{R}^n); \lim_{|x| \to \infty} f(x) = 0\}$.

Example 6.12. Let $\mathscr{D}(\boldsymbol{R}^n) (= C_c^\infty(\boldsymbol{R}^n))$ be the space of C^∞-functions on $\boldsymbol{R}^n$ with compact supports. Let $\mathscr{K}$ be the set of all compact subsets of $\boldsymbol{R}^n$ and let

$$\mathscr{D}_K(\boldsymbol{R}^n) = \{f \in \mathscr{D}(\boldsymbol{R}^n); \operatorname{supp} f \subset K\}, \quad K \in \mathscr{K}.$$

Then $\mathscr{D}_K(\boldsymbol{R}^n)$ is a Fréchet space with the topology of uniform convergence on K of functions as well as all their derivatives, and $\mathscr{D}(\boldsymbol{R}^n) = \bigcup\{\mathscr{D}_K(\boldsymbol{R}^n); K \in \mathscr{K}\} = \varinjlim \mathscr{D}_K(\boldsymbol{R}^n)$ with the usual inductive limit topology.

(1) $\mathscr{D}(\boldsymbol{R}^n)$ with pointwise multiplication is a complete locally convex $*$-algebra which is a LF Q-algebra (that is, a LF-space which is a Q-algebra). The norm $\|\ \|_\infty$ is a spectral C^*-norm and $E(\mathscr{D}(\boldsymbol{R}^n)) \simeq C_0(\boldsymbol{R}^n)$.

(2) $\mathscr{D}(\boldsymbol{R}^n)$ is also a complete locally convex $*$-algebra with convolution multiplication [**23**] which is an ideal of $L^1(\boldsymbol{R}^n)$, and $\|f\| \equiv \|\hat{f}\|_\infty$ is a spectral C^*-norm on $\mathscr{D}(\boldsymbol{R}^n)$.

Example 6.13. The Schwarz space $\mathscr{S}(\boldsymbol{R}^n \times \boldsymbol{R}^n)$ equipped with the Volterra convolution and the involution:

$$(f \circ g)(x, y) = \int_{\boldsymbol{R}^n} f(x, z) g(z, y)\, dz,$$
$$f^*(x, y) = \overline{f(y, x)}$$

is a complete locally convex $*$-algebra with a spectral C^*-seminorm. In fact, let $f \in \mathscr{S}(\boldsymbol{R}^n \times \boldsymbol{R}^n)$. We put

$$[\pi_0(f)\varphi](x) = \int_{\boldsymbol{R}^n} f(x, y)\varphi(y)\, dy, \quad \varphi \in \mathscr{S}(\boldsymbol{R}^n).$$

Then we can show that $\pi_0(f)$ can be extended to a bounded linear operator $\pi(f)$ on $L^2(\boldsymbol{R}^n)$ and π is a continuous bounded $*$-representation of $\mathscr{S}(\boldsymbol{R}^n \times \boldsymbol{R}^n)$ on $L^2(\boldsymbol{R}^n)$. By the simple calculation we have

$$f^{[n]} \equiv \overbrace{f \circ \cdots \circ f}^{n} = \left(\int_{\boldsymbol{R}^n} f(x, x)\, dx \right)^{n-1} f, \quad n \in \boldsymbol{N}$$

and

$$\left| \int_{\boldsymbol{R}^n} f(x, x)\, dx \right| < 1 \quad \text{if } r_\pi(f) < 1,$$

which implies that the C^*-seminorm r_π on $\mathscr{S}(\boldsymbol{R}^n \times \boldsymbol{R}^n)$ is spectral. Similarly, $\mathscr{D}(\boldsymbol{R}^n \times \boldsymbol{R}^n)$ has a spectral C^*-seminorm.

6.3. GB^*-algebras.

G. R. Allan [**2**] and P. G. Dixon [**12**] defined the notion of GB^*-algebras which is a generalization of C^*-algebra: A pseudo-complete locally convex $*$-algebra $\mathscr{A}$ is said to be a *GB^*-algebra* over $\boldsymbol{B}_0$ if $\boldsymbol{B}_0$ is the greatest member in $\mathscr{B}^* \equiv \{\boldsymbol{B} \in \mathscr{B}; \boldsymbol{B}^* = \boldsymbol{B}\}$ and $(1 + x^*x)^{-1} \in \mathscr{A}[\boldsymbol{B}_0]$ for every $x \in \mathscr{A}$. Let $\mathscr{A}$ be a GB^*-algebra over $\boldsymbol{B}_0$. Then the unbounded C^*-norm $p_{\boldsymbol{B}_0}$ on $\mathscr{A}$ is defined by

$$\begin{cases} \mathscr{D}(p_{\boldsymbol{B}_0}) = \mathscr{A}[\boldsymbol{B}_0] \\ p_{\boldsymbol{B}_0}(x) = \|x\|_{\boldsymbol{B}_0}, \quad x \in \mathscr{D}(p_{\boldsymbol{B}_0}) \end{cases}$$

and it has the following properties:

(1) $p_{\boldsymbol{B}_0}$ is spectral and $\mathscr{A}_b = \mathscr{A}[\boldsymbol{B}_0] = \mathscr{D}(p_{\boldsymbol{B}_0})$. In fact, since $\boldsymbol{B}_0$ is absolutely convex, it follows that $(x + x^*)/2, (x - x^*)/(2i) \in (\boldsymbol{B}_0)_h \subset (\mathscr{A}_0)_h$ for each $x \in \boldsymbol{B}_0$, which implies that $x \in \mathscr{A}_b$. Hence $\mathscr{A}[\boldsymbol{B}_0] \subset \mathscr{A}_b$. Conversely, since $(\mathscr{A}_0)_h \subset \mathscr{A}[\boldsymbol{B}_0]$, we have $\mathscr{A}_b \subset \mathscr{A}[\boldsymbol{B}_0]$. Therefore, $\mathscr{A}_b = \mathscr{A}[\boldsymbol{B}_0]$. Since $\mathscr{A}_b$ is a C^*-algebra, it follows that $p_{\boldsymbol{B}_0}$ is spectral. By (1) we have the following

(2) $p_{\boldsymbol{B}_0}$ satisfies condition (UR) if and only if $\mathscr{I}_b \neq \{0\}$.

Here we give an example of a GB^*-algebra with $\mathscr{I}_b \neq \{0\}$.

Example 6.14. $\Gamma^n = \{z = (z_1, \ldots, z_n) \in \boldsymbol{C}^n; |z_i| = 1, i = 1, \ldots, n\}$ and $C^\infty(\Gamma^n)$ the Fréchet space of all C^∞-functions on Γ^n with the topology defined by the seminorms

$$\|f\|_N = \max_{|\alpha|\le N} \sup_{z\in\Gamma^n} |D^\alpha f(z)|, \quad N = 0, 1, \ldots.$$

Let $C^\infty(\Gamma^n)'$ be the dual space of $C^\infty(\Gamma^n)$. Then, with the weak topology $\sigma = \sigma(C^\infty(\Gamma^n)', C^\infty(\Gamma^n))$, $(C^\infty(\Gamma^n)', \sigma)$ is a sequentially complete convolution algebra which is a GB^*-algebra with $A(\boldsymbol{B}_0) = p_M(\Gamma^n)$ (the C^*-algebra of all pseudo-measures on Γ^n). Since the Fourier-Stiltjes transform $\mu \mapsto \hat{\mu}$ is a $*$-isomorphism of $C^\infty(\Gamma^n)'$ onto the $*$-algebra $\mathfrak{s}'$ of tempered sequences:

$$\mathfrak{s}' = \{\sigma = (\sigma_p)_{p\in\boldsymbol{Z}^n}; \sigma_p \in \boldsymbol{C}, {}^\forall p \text{ and } {}^\exists k > 0 \text{ such that } \{(1+|p|)^{-k}\sigma_p\} \in l^\infty(\boldsymbol{Z}^n)\},$$

it follows that

$$\|\mu\| \equiv \sup\{|\hat{\mu}(k)|; k \in \boldsymbol{Z}^n\}$$

is an unbounded C^*-norm on $C^\infty(\Gamma^n)'$ satisfying

$$\begin{aligned}\mathscr{I}_b &= \mathfrak{N}_{\|\ \|} \\ &\simeq \{(x_k)_{k\in\boldsymbol{Z}^n} \in l^\infty(\boldsymbol{Z}^n); \{\hat{a}(k)\hat{x}(k)\}_{k\in\boldsymbol{Z}^n} \in l^\infty(\boldsymbol{Z}^n), {}^\forall a \in \mathfrak{s}'\} \\ &\supset \{(x_k); x_k \neq 0 \text{ for only finite terms}\}.\end{aligned}$$

We consider the cases of pro-C^*-algebras and C^*-like locally convex $*$-algebras which are important in GB^*-algebras.

A complete locally convex $*$-algebra $\mathscr{A}[\tau]$ is said to be *pro-C^*-algebra* if the topology τ is determined by a directed family $\Gamma = \{p_\lambda\}_{\lambda\in\Lambda}$ of C^*-seminorms. Then, any C^*-seminorm p_λ satisfies condition (R), but it does not necessarily satisfy condition (UR). We put

$$\begin{cases} \mathscr{D}(p_\Gamma) = \left\{x \in \mathscr{A}; \sup_{\lambda\in\Lambda} p_\lambda(x) < \infty\right\}, \\ p_\Gamma(x) = \sup_{\lambda\in\Lambda} p_\lambda(x), \quad x \in \mathscr{D}(p_\Gamma). \end{cases}$$

Then $\mathscr{A}$ is a GB^*-algebra over $\boldsymbol{B}_0 = \mathscr{U}(p_\Gamma) \equiv \{x \in \mathscr{D}(p_\Gamma); p_\Gamma(x) \le 1\}$ and $p_\Gamma = p_{\boldsymbol{B}_0}$, and so p_Γ is a spectral unbounded C^*-norm on $\mathscr{A}$ such that

(1) $\mathscr{A}_b = \mathscr{D}(p_\Gamma)$, and so $\mathscr{I}_b = \mathfrak{N}_{p_\Gamma}$;

(2) p_Γ satisfies condition (UR) if and only if $\mathscr{I}_b \neq \{0\}$ if and only if p_λ is a C^*-seminorm with condition (UR) for some $\lambda \in \Lambda$.

EXAMPLE 6.15. Let $\mathscr{A}$ be a C^*-algebra without identity and $\mathscr{K}_{\mathscr{A}}$ the Pedersen ideal of $\mathscr{A}$, that is, a minimal dense hereditary ideal of $\mathscr{A}$ [**22**], [**27**]. For $a \in \mathscr{A}$ we denote by L_a the closed left ideal $\overline{\mathscr{A}a}$ generated by a, and R_a the closed right ideal generated by a. We denote by M_a the C^*-algebra of all pairs (l, r) consisting of linear maps $l : L_a \to L_a$ and $r : R_{a^*} \to R_{a^*}$ such that $yl(x) = r(y)x$ for each $x \in L_a$ and $y \in R_{a^*}$. Note that l and r are automatically bounded, and that M_a is a C^*-algebra. Furthermore, if $a, b \in \mathscr{A}$ with $0 \le a \le b$, then $L_a \subset L_b$, $R_a \subset R_b$ and the restriction map $(l, r) \to (l\lceil L_a, r\lceil R_a)$ defines a $*$-homomorphism from M_b to M_a. It is shown by Phillips [**27**] that the multiplier algebra $\Gamma(\mathscr{K}_{\mathscr{A}})$ of $\mathscr{K}_{\mathscr{A}}$ is isomorphic to the pro-C^*-algebra $\varprojlim_{a \in (\mathscr{K}_{\mathscr{A}})_+} M_a$ such that

$$\begin{aligned}\Gamma(\mathscr{K}_{\mathscr{A}})_b = \Gamma_c(\mathscr{K}_{\mathscr{A}}) &\equiv \{(l, r) \in \Gamma(\mathscr{K}_{\mathscr{A}}); l(\text{or } r) \text{ is bounded}\} \\ &\simeq M(\mathscr{A}) \quad (\text{the multiplier algebra of the } C^*\text{-algebra } \mathscr{A}, \text{ in fact, } \Gamma(\mathscr{A})).\end{aligned}$$

Hence, $\mathscr{K}_{\mathscr{A}} \subset \mathscr{I}_b$, and so there exists a spectral well-behaved $*$-representation of $\Gamma(\mathscr{K}_{\mathscr{A}})$. If $\mathscr{A}$ is a unital pro-C^*-algebra such that $\mathscr{A}_b = M(\mathscr{B})$ for a non-unital C^*-algebra $\mathscr{B}$ in $\mathscr{A}_b$, and if $\mathscr{A} \subset \Gamma(\mathscr{K}_{\mathscr{B}})$, then $\mathscr{K}_{\mathscr{B}} \subset \mathscr{I}_b \neq \{0\}$ and p_Γ satisfies (UR). This holds if $\mathscr{A}$ is commutative.

(1) Let ω be the pro-C^*-algebra of all complex sequences equipped with the usual operations, the involution and the topology of the sequences $\Gamma = \{p_k\}$ of C^*-seminorms: $p_k(\{x_n\}) = |x_k|$, and denote by $\mathscr{A}$ the C^*-algebra $c_0 \equiv \{\{x_n\} \in \omega; \lim_{n\to\infty} x_n = 0\}$ with the C^*-norm $\|\{x_n\}\| = \sup_n |x_n|$. Then we have

$$\begin{aligned}\mathscr{K}_{\mathscr{A}} &= c_{00} \equiv \{\{x_n\} \in c_0; x_n \neq 0 \text{ for only finite numbers } n\}, \\ M(\mathscr{A}) &= c \equiv \{\{x_n\} \in \omega; \{x_n\} \text{ is bounded}\}, \\ \Gamma(\mathscr{K}_{\mathscr{A}}) &= \omega.\end{aligned}$$

(2) Let $C(\boldsymbol{R}^n)$ be the pro-C^*-algebra of all complex-valued continuous functions on $\boldsymbol{R}^n$ with the compact open topology defined by the sequence $\Gamma = \{p_k\}$ of C^*-seminorms: $p_k(f) \equiv \sup\{|f(x)|; x \in \boldsymbol{R}^n \text{ and } |x| \leq k\}$, and denote by $\mathscr{A}$ the C^*-algebra $C_0(\boldsymbol{R}^n)$. Then we have

$$\begin{aligned}\mathscr{K}_{\mathscr{A}} &= C_c(\boldsymbol{R}^n) \equiv \{f \in C(\boldsymbol{R}^n); \operatorname{supp} f \text{ is compact}\}, \\ M(\mathscr{A}) &= C_b(\boldsymbol{R}^n) \equiv \{f \in C(\boldsymbol{R}^n); f \text{ is bounded}\}, \\ \Gamma(\mathscr{K}_{\mathscr{A}}) &= C(\boldsymbol{R}^n).\end{aligned}$$

Example 6.16. Let $L^\infty_{\text{loc}}(\boldsymbol{R}^n)$ be the pro-C^*-algebra of all Lebesgue measurable functions on $\boldsymbol{R}^n$ which are essentially bounded on every compact subset of $\boldsymbol{R}^n$ equipped with the topology defined by the sequence $\Gamma = \{p_k\}$ of C^*-seminorms: $p_k(f) \equiv \text{ess.sup}\{|f(x)|; |x| \leq k\}$. Then we have

$$\begin{cases}\mathscr{D}(p_\Gamma) = L^\infty(\boldsymbol{R}^n), \\ p_\Gamma(f) = \operatorname*{ess.sup}_{x \in \boldsymbol{R}^n} |f(x)|, \quad f \in \mathscr{D}(p_\Gamma)\end{cases}$$

and

$$\mathscr{I}_b = \mathfrak{N}_{p_\Gamma} \supset L^\infty_c(\boldsymbol{R}^n) \equiv \{f \in L^\infty_{\text{loc}}(\boldsymbol{R}^n); \operatorname{supp} f \text{ is compact}\}.$$

Hence p_Γ is a spectral unbounded C^*-norm on $L^\infty_{\text{loc}}(\boldsymbol{R}^n)$ with condition (UR).

Example 6.17. Let $\{\mathscr{A}_\lambda\}_{\lambda \in \Lambda}$ be a family of C^*-algebras $\mathscr{A}_\lambda$ with C^*-norms p_λ. Then the product space $\prod_{\lambda \in \Lambda} \mathscr{A}_\lambda$ is a pro-C^*-algebra equipped with the multiplication: $(x_\lambda)(y_\lambda) \equiv (x_\lambda y_\lambda)$, the involution: $(x_\lambda)^* = (x_\lambda^*)$ and the topology defined by the family $\Gamma = \{p_\lambda\}_{\lambda \in \Lambda}$ of C^*-norms. Then we have

$$\begin{cases} \mathscr{D}(p_\Gamma) = \{(x_\lambda) \in \prod_{\lambda \in \Lambda} \mathscr{A}_\lambda; \sup_{\lambda \in \Lambda} p_\lambda(x_\lambda) < \infty\}, \\ p_\Gamma((x_\lambda)) = \sup_{\lambda \in \Lambda} p_\lambda(x_\lambda), \quad (x_\lambda) \in \mathscr{D}(p_\Gamma) \end{cases}$$

and

$$\mathscr{I}_b = \mathfrak{N}_{p_\Gamma} \supset \left\{ (x_\lambda) \in \prod_{\lambda \in \Lambda} \mathscr{A}_\lambda; \{\lambda \in \Lambda; x_\lambda \neq 0\} \text{ is finite} \right\}.$$

Hence p_Γ is a spectral unbounded C^*-norm on $\prod_{\lambda \in \Lambda} \mathscr{A}_\lambda$ with condition (UR), and any p_λ is an unbounded C^*-seminorm on $\prod_{\lambda \in \Lambda} \mathscr{A}_\lambda$ with condition (UR) but it is not spectral.

Next we consider C^*-like locally convex $*$-algebras. Let $\mathscr{A}[\tau]$ be a locally convex $*$-algebra. A directed family $\Gamma = \{p_\lambda\}_{\lambda \in \Lambda}$ of seminorms determining the topology τ is said to be *C^*-like* if for any $\lambda \in \Lambda$ there exists a $\lambda' \in \Lambda$ such that $p_\lambda(xy) \leq p_{\lambda'}(x) p_{\lambda'}(y)$, $p_\lambda(x^*) \leq p_{\lambda'}(x)$ and $p_\lambda(x)^2 \leq p_{\lambda'}(x^*x)$ for each $x, y \in \mathscr{A}$. Then any p_λ is not necessarily submultiplicative, but the unbounded C^*-norm p_Γ on $\mathscr{A}$ is defined by

$$\begin{cases} \mathscr{D}(p_\Gamma) = \{x \in \mathscr{A}; \sup_{\lambda \in \Lambda} p_\lambda(x) < \infty\}, \\ p_\Gamma(x) = \sup_{\lambda \in \Lambda} p_\lambda(x), \quad x \in \mathscr{D}(p_\Gamma). \end{cases}$$

A complete locally convex $*$-algebra $\mathscr{A}[\tau]$ is said to be *C^*-like* if there exists a C^*-like family $\Gamma = \{p_\lambda\}_{\lambda \in \Lambda}$ of seminorms determining the topology τ such that $\mathscr{D}(p_\Gamma)$ is τ-dense in $\mathscr{A}$. Then it follows from [**20**, Theorem 2.1] that $\mathscr{A}$ is a GB^*-algebra over $\boldsymbol{B}_0 = \mathscr{U}(p_\Gamma)$ and $p_\Gamma = p_{\boldsymbol{B}_0}$. Hence p_Γ is a spectral unbounded C^*-norm on $\mathscr{A}$ with $\mathscr{A}_b = \mathscr{D}(p_\Gamma)$.

Example 6.18. The Arens algebra $L^\omega[0,1] \equiv \bigcap_{1 \leq p < \infty} L^p[0,1]$ is a C^*-like locally convex $*$-algebra with the C^*-like family of norms $\Gamma = \{\| \ \|_p; 1 \leq p < \infty\}$, and

$$\begin{cases} \mathscr{D}(p_\Gamma) = L^\infty[0,1], \\ p_\Gamma(f) = \|f\|_\infty, \quad f \in \mathscr{D}(p_\Gamma) \end{cases}$$

and $\mathscr{I}_b = \mathfrak{N}_{p_\Gamma} = \{0\}$. Hence p_Γ is a spectral unbounded C^*-norm on $L^\infty[0,1]$ which does not satisfy condition (UR).

Example 6.19. We consider a $*$-subalgebra $\mathscr{A}$ of the Arens algebra $L^\omega[0,1]$ defined by

$$\mathscr{A} = \{f \in L^\omega[0,1]; f\lceil_{[0,1/2]} \in C[0,1/2]\}.$$

Then $\mathscr{A}$ is a C^*-like locally convex $*$-algebra with the C^*-like family $\Gamma = \{\| \cdot \|_{\infty,p}; 1 \leq p < \infty\}$ of seminorms:

$$\|f\|_{\infty,p} \equiv \max\left\{ \sup_{0 \leq t \leq 1/2} |f(t)|, \|f\lceil_{[1/2,1]}\|_p \right\},$$

and $\mathscr{I}_b = \{f \in \mathscr{A}; f\lceil_{[1/2,1]} = 0 \text{ a.e.}\}$. Hence p_Γ is a spectral unbounded C^*-norm on $\mathscr{A}$ with condition (UR).

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