J. Math. Soc. Japan
Vol. 56, No. 3, 2004

On a necessary condition for analytic wellposedness of the Cauchy problem for parabolic equation

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(Received Dec. 17, 2001)
(Revised Mar. 18, 2003)

Abstract. In this paper, we give a necessary condition for the Cauchy problem for parabolic equation in order to be uniquely solvable in the analytic class.

1. Introduction.

We are concerned with the Cauchy problem for the following parabolic equation with the coefficient depending only on x

$$\begin{cases} \partial_t u(t,x) = a(x,D_x)u(t,x) + b(x,D_x)u(t,x), & (t,x) \in [0,T] \times \boldsymbol{R}^l, \\ u(0,x) = u_0(x), & x \in \boldsymbol{R}^l, \end{cases} \tag{1.1}$$

where

$$a(x,D_x)u = \sum_{i,j=1}^{l} a_{ij}(x) D_i D_j u, \quad b(x,D_x)u = \sum_{i=1}^{l} b_i(x) D_i u + c(x) u$$

and $D_j = -i\partial_{x_j}$. Here, we assume that the coefficients $a_{ij}(x), b_i(x)$ and $c(x)$ are real analytic in the sense that there are constants $c_a > 0$ and $\rho_a > 0$ such that

$$|D_x^\alpha a(x)| \le c_a \rho_a^{-|\alpha|} |\alpha|! \tag{1.2}$$

for $x \in \boldsymbol{R}^l$, $\alpha = (\alpha_1, \ldots, \alpha_l) \in \boldsymbol{N}^l$, where $D_x = -i\partial_x$ and $\boldsymbol{N} = \{0,1,2,\ldots\}$.

We call that the Cauchy problem (1.1) is H^∞-wellposed for $t \ge 0$, if for any $u_0 \in H^\infty$ there exists a unique solution $u(t,x) \in H^\infty(\boldsymbol{R}^l)$, $t \ge 0$. It is known that the condition

$$\operatorname{Re} a(x,\xi) \le 0 \quad \text{for all } (x,\xi) \in \boldsymbol{R}^l \times \boldsymbol{R}^l \tag{1.3}$$

is necessary in order for (1.1) to be H^∞-wellposed. Historically, the condition (1.3) was proved by Petrowsky [**7**] in the case where the coefficients depend only on t, and Mizohata [**5**] proved it in the case of variable coefficients. Sadamatsu [**8**] considered the necessary condition in order for (1.1) to be L^2-wellposed. D'Ancona and Spagnolo [**1**] considered the following equation

$$\partial_t u(t,x) = (1 - \cos x)\partial_x^2 u(t,x), \quad x \in \boldsymbol{R}. \tag{1.4}$$

2000 *Mathematics Subject Classification.* Primary 35K10; Secondary 35A20, 35K15.
Key Words and Phrases. parabolic equation, initial value problem, Cauchy problem, wellposedness.

They constructed a solution of (1.4) with an analytic initial datum which is not analytic with respect to x for $t > 0$.

In this paper, we shall try to generalize the result of D'Ancona and Spagnolo. Denote by L^2_ρ the set of functions with radius of convergence $\rho > 0$ which is defined by

$$L^2_\rho(\boldsymbol{R}^l) = \{u(x) \in L^2(\boldsymbol{R}^l); e^{\rho\langle\xi\rangle}\hat{u}(\xi) \in L^2(\boldsymbol{R}^l_\xi)\}, \tag{1.5}$$

where $\langle\xi\rangle = \sqrt{1+|\xi|^2}$, $|\xi| = \sqrt{\xi_1^2 + \cdots + \xi_l^2}$ and $\hat{u}$ means the Fourier transform of u. In particular, we should mention that the function belonging to L^2_ρ in (1.5) gives to be a real analytic function with radius of convergence $\rho > 0$. Then, the wellposedness in the analytic class should be defined by the following

Definition 1.1. We call that the Cauchy problem (1.1) is analytically wellposed in $[0, T]$, if there are $\rho_0 > 0$ and a monotone increasing function $\mu(t)$ continuously defined in $[0, T]$ and $\mu(0) = 0$ such that for any $u_0 \in \bigcap_{\rho>0} L^2_\rho$ there is a solution $u(t,x) \in C^1([0,T]; L^2_{\rho_0})$ of the Cauchy problem (1.1) satisfying

$$\|e^{\rho\langle D_x\rangle}u(t,\cdot)\|_{L^2} \le C\|e^{(\rho+\mu(t))\langle D_x\rangle}u_0(\cdot)\|_{L^2} \tag{1.6}$$

for $0 \le t \le T$ and any ρ with $0 < \rho < \rho_0$.

Concerning the wellposedness in the analytic class, we should mention that if there is a constant δ_0 such that

$$\operatorname{Re} a(x,\xi) \le -\delta_0|\xi|^2 < 0 \quad \text{for all } (x,\xi) \in \boldsymbol{R}^l \times \boldsymbol{R}^l$$

then for a function $\rho(t)$ with $0 < \rho(t) < \delta_0$ the solution $u(t,x)$ of the Cauchy problem (1.1) satisfies

$$\|e^{\rho(t)\langle D_x\rangle}u(t,\cdot)\|_{L^2} \le C\|e^{\rho(0)\langle D_x\rangle}u_0(\cdot)\|_{L^2} \tag{1.7}$$

for $0 \le t \le T$. In particular, (1.6) with $\mu(t) = 0$ follows when $\rho(t)$ is constant in (1.7). Our main result is the following

Theorem 1.2. *In order that* (1.1) *is analytically wellposed, the condition* (1.3) *is necessary, moreover, if there exist* x_0 *and* ξ_0 *such that* $\operatorname{Re} a(x_0,\xi_0) = 0$ *then* $\operatorname{Re} a(x,\xi_0) = 0$ *for all* $x \in \boldsymbol{R}^l$.

2. Proof of Theorem 1.2.

In this section we shall prove Theorem 1.2 by reduction to absurdity. The argument is carried out in the following way. We assume that there exist x_0 and ξ_0 such that

$$\operatorname{Re} a(x_0,\xi_0) \ge 0 \tag{2.1}$$

(see Step 1), and that (1.1) is analytically wellposed. Then, from these two assumptions, we deduce two inequalities which are not compatible by choosing an initial data suitably (see Step 5). Mainly, we use the micro-local energy method devised by Mizohata [**6**] (see Step 2 and Step 4).

Step 1. Exponential map.

The assumption (2.1) in contraposition to Theorem 1.2 is divided into two cases,

(i) there are x_0 and ξ_0 such that $\operatorname{Re} a(x_0, \xi_0) = 0$, and there exists a x_1 satisfying $\operatorname{Re} a(x_1, \xi_0) \neq 0$ (2.2)

and

(ii) there exist x_0 and ξ_0 such that $\operatorname{Re} a(x_0, \xi_0) > 0$. (2.3)

We transform $u(t,x)$ in (1.1) into $v(t,x)$ by

$$v(t,x) = e^{\rho\omega\cdot D_x} u(t,x) = (2\pi)^{-l} \int_{\boldsymbol{R}^l} e^{ix\cdot\xi + \rho\omega\cdot\xi} \hat{u}(t,\xi)\, d\xi \tag{2.4}$$

for $\rho > 0$ and $\omega \in \boldsymbol{C}^l$ with $|\omega| = 1$ as Kajitani [**3**]. Then, since

$$\begin{aligned} \partial_t v &= e^{\rho\omega\cdot D_x} \partial_t u \\ &= e^{\rho\omega\cdot D_x} (a(x, D_x) + b(x, D_x)) u \\ &= e^{\rho\omega\cdot D_x} a(x, D_x) e^{-\rho\omega\cdot D_x} v + e^{\rho\omega\cdot D_x} b(x, D_x) e^{-\rho\omega\cdot D_x} v, \end{aligned}$$

(1.1) is transformed into the Cauchy problem below

$$\begin{cases} \partial_t v(t,x) = A(x, D_x) v(t,x), & (t,x) \in [0,T] \times \boldsymbol{R}^l, \\ v(0,x) = e^{\rho\omega\cdot D_x} u_0(x), & x \in \boldsymbol{R}^l, \end{cases} \tag{2.5}$$

where $A(x, D_x) v = e^{\rho\omega\cdot D_x} a(x, D_x) e^{-\rho\omega\cdot D_x} v + e^{\rho\omega\cdot D_x} b(x, D_x) e^{-\rho\omega\cdot D_x} v$.

Here, let us define the pseudo-differential operator on L^2 by

$$a_\Lambda(x, D_x) u = e^{\rho\omega\cdot D_x} a(x, D_x) e^{-\rho\omega\cdot D_x} u \tag{2.6}$$

$$= (2\pi)^{-l} \int_{\boldsymbol{R}^l} e^{ix\cdot\xi} a(x - i\rho\omega, \xi) \hat{u}(\xi)\, d\xi. \tag{2.7}$$

(2.7) can be calculated, since $a(x, \xi)$ is a real analytic function with respect to x. Therefore, $a_\Lambda(x, D_x)$ in (2.6) is a pseudo-differential operator whose symbol has the representation

$$a_\Lambda(x, \xi) = a(x - i\rho\omega, \xi). \tag{2.8}$$

In the case (i), we get the following lemma.

Lemma 2.1. *Let $a(x, \xi)$ is a real analytic function with convergence radius ρ_a with respect to x. If there are x_0 and ξ_0 such that* $\operatorname{Re} a(x_0, \xi_0) = 0$, *and* $\operatorname{Re} a(x, \xi_0) = 0$ *does not identically vanish in $\boldsymbol{R}^l$, then for any $\rho \in (0, \rho_a)$ there exists $\omega \in \boldsymbol{C}^l$ satisfying $|\omega| = 1$ such that*

$$\operatorname{Re} a(x_0 - i\rho\omega, \xi_0) > 0. \tag{2.9}$$

Proof. $a(x, \xi_0)$ can be extended to a holomorphic function $a(z, \xi_0)$ by analytic continuation, since $a(x, \xi_0)$ is a real analytic function with respect to x. If we assume that there exists $\omega \in \boldsymbol{C}^l$ such that $|\omega| = 1$ and $\operatorname{Re} a(x - i\rho\omega, \xi_0) \leq 0$ for all $x \in \boldsymbol{R}^l$, then $\max_{z \in \boldsymbol{C}^l} \operatorname{Re} a(z, \xi_0) = \operatorname{Re} a(x_0, \xi_0) = 0$ in contradict to the maxmin principle. Therefore, we can get (2.9). □

Hereafter, we consider only the case (i). We assume (2.9) for the Cauchy problem (2.5) transformed by the exponential mapping defined in (2.4). On the other hand, in the case (ii), we can also carry the same argumemt below without the transformation by exponential mapping in terms of $\rho = 0$ in (2.4).

Step 2. Micro-localizer.

Following Mizohata [**6**], we give here the definition of micro-localizers. For a some positive number r_0, we take the sequence $\{\beta_N(x)\}_{N=1,2,\dots}$ of functions in C_0^∞ possessing following properties,

$$\begin{aligned}
&\text{(i)} \quad 0 \le \beta_N(x) \le 1,\\
&\text{(ii)} \quad \beta_N(x) = \begin{cases} 1 & \text{for } |x - x_0| \le \dfrac{1}{2} r_0, \\ 0 & \text{for } |x - x_0| \ge r_0, \end{cases}\\
&\text{(iii)} \quad |\beta_{N(q)}(x)| \le K^{1+N}|q|!, \quad |q| \le N = 1, 2, \dots.
\end{aligned} \tag{2.10}$$

In the same way, we take the sequence $\{\alpha_N(\xi)\}_{N=1,2,\dots}$ of functions in C_0^∞ such that

$$\begin{aligned}
&\text{(i)} \quad 0 \le \alpha_N(\xi) \le 1,\\
&\text{(ii)} \quad \alpha_N(\xi) = \begin{cases} 1 & \text{for } |\xi - \xi_0| \le \dfrac{1}{2} r_0, \\ 0 & \text{for } |\xi - \xi_0| \ge r_0, \end{cases}\\
&\text{(iii)} \quad |\alpha_N^{(p)}(\xi)| \le K^{1+N}|p|!, \quad |p| \le N = 1, 2, \dots.
\end{aligned}$$

For such a sequence of functions in C_0^∞ as to satisfy the property (iii), refer to Hörmander [**2**]. Let

$$\alpha_{Nn}(\xi) = \alpha_N\left(\frac{\xi}{n}\right),$$

which is particularly estimated by

$$|\alpha_{Nn}^{(p)}(\xi)| \le K^{1+N}|p|! n^{-|p|}, \quad |p| \le N = 1, 2, \dots, \tag{2.11}$$

where n is a large parameter. $\alpha_{Nn}(D_x)v(x)$ is defined by Fourier transform as follows

$$(\widehat{\alpha_{Nn} v})(\xi) = \alpha_{Nn}(\xi)\hat{v}(\xi).$$

We consider the function $\alpha_{Nn}(D_x)\beta_N(x)v(t,x)$, instead of the solution $v(t,x)$ of (2.5) and put

$$L[v] = \partial_t v - A(x, D_x)v.$$

Operating the micro-localizer $\alpha_{Nn}(D)\beta_N(x)$ from the left, that is,

$$\begin{aligned}
0 &= \alpha_{Nn}(D_x)\beta_N(x)L[v]\\
&= L[\alpha_{Nn}(D_x)\beta_N(x)v] - [\alpha_{Nn}(D_x)\beta_N(x), A(x, D_x)]v,
\end{aligned}$$

we have

$$L[\alpha_{Nn}(D_x)\beta_N(x)v] = F,$$

where $F = [\alpha_{Nn}(D_x)\beta_N(x), A(x, D_x)]v$ and $[\,,]$ stands for a commutator. Similarly, operating the micro-localizer $\alpha_{Nn}^{(p)}(D_x)\beta_{N(q)}(x)$ with $0 \le |p+q| \le N-1$ from the left on $L[v]$, we have

$$L[\alpha_{Nn}^{(p)}(D_x)\beta_{N(q)}(x)v] = F_{pq},$$

where

$$F_{pq} = [\alpha_{Nn}^{(p)}(D_x)\beta_{N(q)}(x), A(x, D_x)]v. \tag{2.12}$$

Thus, by putting

$$v_{p,q}(t,x) = \alpha_{Nn}^{(p)}(D_x)\beta_{N(q)}(x)v(t,x),$$

we have the micro-localized Cauchy problem

$$\begin{cases} \partial_t v_{p,q}(t,x) = A(x, D_x)v_{p,q}(t,x) + F_{pq}(t,x), \\ v_{p,q}(0,x) = \alpha_{Nn}^{(p)}(D_x)\beta_{N(q)}(x)e^{\rho\omega\cdot D_x}u_0(x). \end{cases} \tag{2.13}$$

Here, the symbol of $A(x, D_x) = e^{\rho\omega\cdot D_x}a(x, D_x)e^{-\rho\omega\cdot D_x} + e^{\rho\omega\cdot D_x}b(x, D_x)e^{-\rho\omega\cdot D_x}$, by (2.8), can be written by

$$A(x,\xi) = a_A(x,\xi) + b_A(x,\xi). \tag{2.14}$$

Step 3. Asymptotic expression.

In our argument, the commutator $[\alpha_{Nn}^{(p)}(D_x)\beta_{N(q)}(x), A(x, D_x)]$ defined in (2.12) plays a crucial role. We shall give here its asymptotic expression. We take the sequence $\{\tilde{\alpha}_N(\xi)\}_{N=1,2,\dots}$ of functions in C_0^∞ such that

(i) $0 \le \tilde{\alpha}_N(\xi) \le 1$,

(ii) $\tilde{\alpha}_N(\xi) = \begin{cases} 1 & \text{for } |\xi - \xi_0| \le \dfrac{3}{2}r_0, \\ 0 & \text{for } |\xi - \xi_0| \ge 2r_0, \end{cases}$

(iii) $|\tilde{\alpha}_N^{(p)}(\xi)| \le K^{1+N}|p|!, \quad |p| \le N = 1, 2, \dots,$

and let

$$\tilde{\alpha}_{Nn}(\xi) = \tilde{\alpha}_N\left(\frac{\xi}{n}\right)$$

whose support contains the support of $\alpha_{Nn}(\xi)$, where n is a large parameter. Then

$$\begin{aligned} &[\alpha_{Nn}^{(p)}(D_x)\beta_{N(q)}(x), A(x, D_x)] \\ &\quad = [\alpha_{Nn}^{(p)}(D_x)\beta_{N(q)}(x), A(x, D_x)\tilde{\alpha}_{Nn}(D_x)] \\ &\qquad + [\alpha_{Nn}^{(p)}(D_x)\beta_{N(q)}(x), A(x, D_x)(1 - \tilde{\alpha}_{Nn}(D_x))]. \end{aligned} \tag{2.15}$$

In order to show the asymptotic expression of $[\alpha_{Nn}^{(p)}(D_x)\beta_{N(q)}(x), \tilde{A}(x, D_x)]$, where $\tilde{A}(x, D_x) = A(x, D_x)\tilde{\alpha}_{Nn}(D_x)$, we consider the asymptotic expression in the case of $|p| = 0$, $|q| = 0$ hereafter, that is, $[\alpha_{Nn}(D_x)\beta_N(x), \tilde{A}(x, D_x)]$. The symbol of $\alpha_{Nn}(D_x)\beta_N(x)$ is expressed by

$$(2\pi)^{-l} \iint e^{-iy\eta}\alpha_{Nn}(\xi + \eta)\beta_N(x + y)\, dyd\eta.$$

Moreover, the symbol of $\alpha_{Nn}(D_x)\beta_N(x)\tilde{A}(x, D_x)$ is expressed by

$$(2\pi)^{-2l} \iiiint e^{-iy\eta - iy'\eta'}\alpha_{Nn}(\xi + \eta + \eta')\beta_N(x + y)\tilde{A}(x + y', \xi)\, dyd\eta dy'd\eta'. \tag{2.16}$$

In the same way, the symbol of $\tilde{A}(x, D_x)\alpha_{Nn}(D_x)\beta_N(x)$ is expressed by

$$(2\pi)^{-2l} \iiiint e^{-iy\eta - iy'\eta'}\tilde{A}(x, \xi + \eta')\alpha_{Nn}(\xi + \eta)\beta_N(x + y + y')\, dyd\eta dy'd\eta'. \tag{2.17}$$

By the change of variables $y = w'$, $y' = w - w'$ and $\eta = z + z'$, $\eta' = z$ in (2.17), noting

$$-iy\eta - iy'\eta' = -iwz - iw'z'$$

and

$$\tilde{A}(x, \xi + \eta')\alpha_{Nn}(\xi + \eta)\beta_N(x + y + y') = \tilde{A}(x, \xi + z)\alpha_{Nn}(\xi + z + z')\beta_N(x + w),$$

we have

$$(2.17) = (2\pi)^{-2l} \iiiint e^{-iwz - iw'z'}\tilde{A}(x, \xi + z)\alpha_{Nn}(\xi + z + z')\beta_N(x + w)\, dwdzdw'dz' \tag{2.18}$$

$$= (2\pi)^{-2l} \iiiint e^{-iy\eta - iy'\eta'}\alpha_{Nn}(\xi + \eta + \eta')\beta_N(x + y)\tilde{A}(x, \xi + \eta)\, dyd\eta dy'd\eta' \tag{2.19}$$

by replacing $w = y$, $w' = y'$ and $z = \eta$, $z' = \eta'$ in (2.18). Therefore, the symbol of

$$[\alpha_{Nn}(D_x)\beta_N(x), \tilde{A}(x, D_x)] = \alpha_{Nn}(D_x)\beta_N(x)\tilde{A}(x, D_x) - \tilde{A}(x, D_x)\alpha_{Nn}(D_x)\beta_N(x)$$

which is the difference between (2.16) and (2.19) can be expressed by

$$(2\pi)^{-2l} \iiiint e^{-iy\eta - iy'\eta'}\alpha_{Nn}(\xi + \eta + \eta')\beta_N(x + y) \times \{\tilde{A}(x + y', \xi) - \tilde{A}(x, \xi + \eta)\}\, dyd\eta dy'd\eta'. \tag{2.20}$$

Now, by Taylor expansion to $\tilde{A}(x + y', \xi) - \tilde{A}(x, \xi + \eta)$ with respect to y' and $-\eta$, it follows

$$\begin{aligned}
&\tilde{A}(x + y', \xi) - \tilde{A}(x, \xi + \eta) \\
&\quad = \sum_{1 \le |\mu + \nu| \le N'} \mu!^{-1}\nu!^{-1}y'^{\mu}(-1)^{|\nu|}\eta^{\nu}\partial_x^{\mu}\partial_\xi^{\nu}\tilde{A}(x, \xi + \eta) \\
&\qquad + (N' + 1)\int_0^1 (1 - \theta)^{N'} \sum_{|\mu + \nu| = N' + 1} \mu!^{-1}\nu!^{-1}y'^{\mu}(-1)^{|\nu|}\eta^{\nu}\partial_x^{\mu}\partial_\xi^{\nu}\tilde{A}(x + \theta y', \xi + \eta - \theta\eta)\, d\theta.
\end{aligned}$$

Then, we can write

$$(2.20) = (2\pi)^{-2l} \sum_{1 \le |\mu+\nu| \le N'} \iiiint e^{-iy\eta - iy'\eta'} \alpha_{Nn}(\xi + \eta + \eta')\beta_N(x+y)$$

$$\times \mu!^{-1}\nu!^{-1}y'^{\mu}(-1)^{|\nu|}\eta^{\nu}\partial_x^{\mu}\partial_{\xi}^{\nu}\tilde{A}(x, \xi + \eta)\, dyd\eta dy'd\eta'$$

$$+ (N'+1)\int_0^1 (1-\theta)^{N'} \sum_{|\mu+\nu| = N'+1} (2\pi)^{-2l} \iiiint e^{-iy\eta - iy'\eta'} \alpha_{Nn}(\xi + \eta + \eta')\beta_N(x+y)$$

$$\times \mu!^{-1}\nu!^{-1}y'^{\mu}(-1)^{|\nu|}\eta^{\nu}\partial_x^{\mu}\partial_{\xi}^{\nu}\tilde{A}(x + \theta y', \xi + \eta - \theta\eta)\, d\theta dyd\eta dy'd\eta'. \tag{2.21}$$

Similarly, in general case for p and q with $0 \le |p+q| \le N-1$, the asymptotic expression of $[\alpha_{Nn}^{(p)}(D_x)\beta_{N(q)}(x), \tilde{A}(x, D_x)]$ can be written by

$$(2\pi)^{-2l} \sum_{1 \le |\mu+\nu| \le N'} \iiiint e^{-iy\eta - iy'\eta'} \alpha_{Nn}^{(p)}(\xi + \eta + \eta')\beta_{N(q)}(x+y)$$

$$\times \mu!^{-1}\nu!^{-1}y'^{\mu}(-1)^{|\nu|}\eta^{\nu}\partial_x^{\mu}\partial_{\xi}^{\nu}\tilde{A}(x, \xi + \eta)\, dyd\eta dy'd\eta'$$

$$+ (N'+1)\int_0^1 (1-\theta)^{N'} \sum_{|\mu+\nu| = N'+1} (2\pi)^{-2l} \iiiint e^{-iy\eta - iy'\eta'} \alpha_{Nn}^{(p)}(\xi + \eta + \eta')\beta_{N(q)}(x+y)$$

$$\times \mu!^{-1}\nu!^{-1}y'^{\mu}(-1)^{|\nu|}\eta^{\nu}\partial_x^{\mu}\partial_{\xi}^{\nu}\tilde{A}(x + \theta y', \xi + \eta - \theta\eta)\, d\theta dyd\eta dy'd\eta'$$

$$= \tilde{f}_{N'pq}(x, \xi) + \tilde{r}_{N'pq}(x, \xi). \tag{2.22}$$

$\tilde{f}_{N'pq}(x, \xi)$ and $\tilde{r}_{N'pq}(x, \xi)$ shall be estimated in Lemma 2.4 and Lemma 2.5 after in Step 4 respectively.

Here, let us state a well known fact on pseudo-differential operators.

LEMMA 2.2. *Let $a(x, \xi) \in S^m$. Then, there exists a positive constant C such that*

$$\|a(x, D_x)u\| \le C|a|_{l_0}^{(m)}\|u\|_{H^m} \tag{2.23}$$

with semi-norm

$$|a|_{l_0}^{(m)} = \max_{|\alpha|+|\beta| \le l_0} \sup_{x, \xi \in \mathbf{R}^n} \{|a_{(\beta)}^{(\alpha)}(x, \xi)|\langle\xi\rangle^{-m+|\alpha|}\}$$

for $u \in H^m$, where $\|u(\cdot)\|_{H^m}$ means $\|\langle D\rangle^m u(\cdot)\|_{L^2}$.

PROOF. For the proof, refer to [**4**] for example. □

Next, we consider the remaining term in (2.15), $[\alpha_{Nn}^{(p)}(D_x)\beta_{N(q)}(x), \bar{A}(x, D_x)]$, where $\bar{A}(x, D_x) = A(x, D_x)(1 - \tilde{\alpha}_{Nn}(D_x))$. The symbol of $\alpha_{Nn}^{(p)}(D_x)\beta_{N(q)}(x)$ is calculated by

$$(2\pi)^{-l} \iint e^{-iy\eta}\alpha_{Nn}^{(p)}(\xi + \eta)\beta_{N(q)}(x+y)\, dyd\eta = a_{N'}(x, \xi) + r_{N'}(x, \xi),$$

where

$$a_{N'}(x, \xi) = \sum_{|\gamma| \le N'} \gamma!^{-1}\alpha_{Nn}^{(p+\gamma)}(\xi)\beta_{N(q+\gamma)}(x)$$

and

$$r_{N'}(x,\xi)$$

$$= (N'+1)\sum_{|\gamma|=N'+1}\int_0^1 \gamma!^{-1}(1-\theta)^{N'}(2\pi)^{-l}\iint e^{-iy\eta}\alpha_{Nn}^{(p+\gamma)}(\xi+\eta)\beta_{N(q+\gamma)}(x+\theta y)\,d\theta dy d\eta$$

which is estimated by

$$|r_{N'}{}^{(\alpha)}_{(\beta)}(x,\xi)| \le CK^{2(1+N)}n^{-(N'+1)-|p|-|\alpha|}(N'+1)!|p+q|!|\alpha+\beta|!.$$

Then

$$[\alpha_{Nn}^{(p)}(D_x)\beta_{N(q)}(x),\bar{A}(x,D_x)] = [a_{N'}(x,D_x),\bar{A}(x,D_x)] + [r_{N'}(x,D_x),\bar{A}(x,D_x)]. \quad (2.24)$$

We get the symbol of $[a_{N'},\bar{A}]$,

$$\sum_{0\le|\tau|\le N'}\tau!^{-1}\{a_{N'}^{(\tau)}(x,\xi)\bar{A}_{(\tau)}(x,\xi)-\bar{A}^{(\tau)}(x,\xi)a_{N'(\tau)}(x,\xi)\}+\bar{r}_{N'}(x,\xi), \quad (2.25)$$

where

$$\bar{r}_{N'}(x,\xi)$$

$$= (N'+1)\sum_{|\tau|=N'+1}\int_0^1 \tau!^{-1}(1-\theta)^{N'}(2\pi)^{-l}\iint e^{-iy\eta}$$

$$\times\{a_{N'}^{(\tau)}(x,\xi+\eta)\bar{A}_{(\tau)}(x+\theta y,\xi)-\bar{A}^{(\tau)}(x,\xi+\eta)a_{N'(\tau)}(x+\theta y,\xi)\}\,d\theta dy d\eta$$

$$= \bar{r}_{N'1}(x,\xi)-\bar{r}_{N'2}(x,\xi).$$

Now, as to (2.25), we remember putting $a_{N'}=\sum_{|\gamma|\le N'}\gamma!^{-1}\alpha_{Nn}^{(p+\gamma)}\beta_{N(q+\gamma)}$ and $\bar{A}=A(1-\tilde{\alpha}_{Nn})$, and so can see that the first term in (2.25) is vanishing, since

$$\operatorname{supp}\alpha_{Nn}\cap\operatorname{supp}(1-\tilde{\alpha}_{Nn})=\emptyset.$$

We put

$$\bar{r}_{N'1}(x,\xi)=(N'+1)\sum_{|\tau|=N'+1}\int_0^1\tau!^{-1}(1-\theta)^{N'}(2\pi)^{-l}\iint e^{-iy\eta}g_1(x,\xi;y,\eta)\,d\theta dy d\eta,$$

where

$$g_1(x,\xi;y,\eta)=\langle\eta\rangle^{-2l}\langle D_y\rangle^{2l}\langle y\rangle^{-2l}\langle D_\eta\rangle^{2l}a_{N'}^{(\tau)}(x,\xi+\eta)\bar{A}_{(\tau)}(x+\theta y,\xi).$$

Then we can write by putting $\langle D\rangle^{2l}=\sum_{|\gamma|\le l}C_{l\gamma}D^{2\gamma}$ and for multi-indices $m,k\in\boldsymbol{N}^l$

$$|g_1{}^{(\alpha)}_{(\beta)}(x,\xi;y,\eta)|$$

$$=\left|\langle\eta\rangle^{-2l}\left(\sum_{|m|\le l}C_{lm}D_y^{2m}\right)\langle y\rangle^{-2l}\left(\sum_{|k|\le l}C_{lk}D_\eta^{2k}\right)\sum_{\alpha'\le\alpha}C_{\alpha\alpha'}\sum_{\beta'\le\beta}C_{\beta\beta'}\right.$$

$$\left.\times\, a_{N'}{}^{(\tau+\alpha')}_{(\beta')}(x,\xi+\eta)\bar{A}^{(\alpha-\alpha')}_{(\tau+\beta-\beta')}(x+\theta y,\xi)\right|$$

$$= \left| \langle \eta \rangle^{-2l} \sum_{|m| \le l} C_{lm} \sum_{|k| \le l} C_{lk} \sum_{\alpha' \le \alpha} C_{\alpha\alpha'} \sum_{\beta' \le \beta} C_{\beta\beta'} \sum_{j=0}^{2m} C_{mj} \right.$$

$$\left. \times D_y^{2m-j} \langle y \rangle^{-2l} D_\eta^{2k} a_{N'}{}^{(\tau+\alpha')}_{(\beta')}(x, \xi + \eta) D_y^j \bar{A}^{(\alpha-\alpha')}_{(\tau+\beta-\beta')}(x + \theta y, \xi) \right|.$$

From the estimates (2.11), (2.10) and (2.14), we have

$$|g_1{}^{(\alpha)}_{(\beta)}(x, \xi; y, \eta)|$$

$$\le C \langle y \rangle^{-2l} \langle \eta \rangle^{-2l} K^{2(1+N)} C^{(N'+1)+|\alpha+\beta|+2l} n^{2-(N'+1)-|p|-|\alpha|} (N'+1)! |p+q|! |\alpha+\beta|!.$$

Moreover, putting

$$\bar{r}_{N'2}(x, \xi) = (N'+1) \sum_{|\tau|=N'+1} \int_0^1 \tau!^{-1} (1-\theta)^{N'} (2\pi)^{-l} \iint e^{-iy\eta} g_2(x, \xi; y, \eta)\, d\theta dy d\eta,$$

where

$$g_2(x, \xi; y, \eta) = \langle \eta \rangle^{-2l} \langle D_y \rangle^{2l} \bar{A}^{(\tau)}(x, \xi + \eta) a_{N'(\tau)}(x + \theta y, \xi),$$

we can estimate

$$|g_2{}^{(\alpha)}_{(\beta)}(x, \xi; y, \eta)|$$

$$= \left| \langle \eta \rangle^{-2l} \sum_{|m| \le l} C_{lm} \sum_{\alpha' \le \alpha} C_{\alpha\alpha'} \sum_{\beta' \le \beta} C_{\beta\beta'} \sum_{\tau' \le \tau} C_{\tau\tau'} \right.$$

$$\left. \times A^{(\tau'+\alpha')}_{(\beta')}(x, \xi + \eta) \partial_\xi^{\tau-\tau'} (1 - \tilde{\alpha}_{Nn}(\xi + \eta)) D_y^{2m} a_{N'}{}^{(\alpha-\alpha')}_{(\tau+\beta-\beta')}(x + \theta y, \xi) \right|$$

$$\le C \langle \eta \rangle^{2-2l} K^{3(1+N)} n^{2-(N'+1)-|p|-|\alpha|} (N'+1)! |p+q|! |\alpha+\beta|!.$$

Hence, we can see, by Lemma 2.2,

$$\|\bar{r}_{N'} v(t, \cdot)\| \le C (N'+1)! |p+q|! n^{2-(N'+1)-p} \|v(t, \cdot)\|.$$

From (2.24), it follows

$$\|[\alpha_{Nn}^{(p)}(D_x) \beta_{N(q)}(x), \bar{A}(x, D_x)] v\| \le c_0 e^{-\varepsilon_0 n} \|v\|. \tag{2.26}$$

Thus we get from (2.22) and (2.26)

$$F_{pq} = \tilde{f}_{N'pq} v + \tilde{r}_{N'pq} v + [\alpha_{Nn}^{(p)}(D_x) \beta_{N(q)}(x), \bar{A}(x, D_x)] v. \tag{2.27}$$

Step 4. Energy estimate.

We can write from (2.13)

$$\frac{d}{dt}(\|v_{p,q}\|^2) = 2\,\mathrm{Re}(A v_{p,q}, v_{p,q}) + 2\,\mathrm{Re}(\tilde{f}_{N'pq} v, v_{p,q}) + 2\,\mathrm{Re}(\tilde{r}_{N'pq} v, v_{p,q})$$

$$+ 2\,\mathrm{Re}([\alpha_{Nn}^{(p)}(D_x) \beta_{N(q)}(x), \bar{A}(x, D_x)] v, v_{p,q}), \tag{2.28}$$

in view of (2.27). In order to estimate the right hand side of (2.28), each term can be estimated in Lemma 2.3, Lemma 2.4 or Lemma 2.5 separately.

Lemma 2.3. *There is a positive constant* C_0 *satisfying*

$$2\,\mathrm{Re}(A(x,D_x)v_{p,q}(t,\cdot),v_{p,q}(t,\cdot)) \geq C_0 n^2\|v_{p,q}(t,\cdot)\|^2. \tag{2.29}$$

Proof. We can see from (2.14)

$$\begin{aligned}\mathrm{Re}(a_\Lambda(x,D_x)v_{p,q},v_{p,q}) &= (\{a_\Lambda(x,D_x)-a_\Lambda(x_0,D_x)\}v_{p,q},v_{p,q})\\ &\quad+(\{a_\Lambda(x_0,D_x)-a_\Lambda(x_0,n\xi_0)\}v_{p,q},v_{p,q})\\ &\quad+(a_\Lambda(x_0,n\xi_0)v_{p,q},v_{p,q}).\end{aligned}\tag{2.30}$$

Note that the first term in (2.30) gives, for some operator $\bar{a}_\Lambda(x,D_x)$,

$$(\{a_\Lambda(x,D_x)-a_\Lambda(x_0,D_x)\}v_{p,q},v_{p,q}) = ((x-x_0)\bar{A}_0(x,D_x)v_{p,q},v_{p,q}), \tag{2.31}$$

and, since $|x-x_0|\leq r_0$ on $x\in\operatorname{supp}\beta_{N(q)}$, there is a positive constant C_1 satisfying

$$(2.31) \leq C_1 r_0 n^2\|v_{p,q}\|^2. \tag{2.32}$$

Concerning the second term in (2.30), for some function $\bar{a}_\Lambda$, we have

$$(a_\Lambda(x_0,\xi)-a_\Lambda(x_0,n\xi_0))v_{p,q} = (\xi-n\xi_0)\bar{a}_\Lambda(x_0,n\xi_0+\theta(\xi-n\xi_0))v_{p,q}. \tag{2.33}$$

Noting $|\xi-n\xi_0|\leq r_0 n$ on $\operatorname{supp}\alpha_{Nn}^{(p)}$, we get, for some positive constant C_2,

$$(2.33) \leq C_2 r_0 n^2\|v_{p,q}\|^2. \tag{2.34}$$

Concerning the last term in (2.30), in veiw of

$$\mathrm{Re}\,a_\Lambda(x_0,n\xi_0) = \mathrm{Re}\,a(x_0-i\rho\omega,\xi_0)n^2,$$

we have, for some constant $c>0$,

$$\mathrm{Re}\,a_\Lambda(x_0,n\xi_0) \geq cn^2$$

because of the assumption of (2.9). Hence it can be estimated from below that

$$\mathrm{Re}(a_\Lambda(x_0,n\xi_0)v_{p,q},v_{p,q}) \geq cn^2\|v_{p,q}\|^2. \tag{2.35}$$

Collecting (2.32), (2.34) and (2.35) in (2.30), and choosing r_0 small enough such that

$$C_1 r_0 + C_2 r_0^2 < c,$$

we get, for some constant $c_a>0$,

$$\mathrm{Re}(a_\Lambda(x,D_x)v_{p,q},v_{p,q}) \geq c_a n^2\|v_{p,q}\|^2. \tag{2.36}$$

Moreover, from (2.14),

$$\mathrm{Re}(A(x,D_x)v_{p,q},v_{p,q}) \geq c_a n^2\|v_{p,q}\|^2 - c_b n\|v_{p,q}\|^2. \tag{2.37}$$

Hence (2.29) holds. □

In the following lemma, we give the estimate of the second term in (2.28).

Lemma 2.4. *There is a positive constant C_1 satisfying*

$$\|\tilde{f}_{N'pq}v(t,\cdot)\| \le C_1 \sum_{1\le|\mu+\nu|\le N'} C^{|\mu+\nu|}n^{2-|\nu|}\|v_{p+\mu,q+\nu}(t,\cdot)\|. \tag{2.38}$$

Proof. To begin with, let us remember $\tilde{f}_{N'pq}(x,\xi)$ in (2.22).

$$\begin{aligned}
&\tilde{f}_{N'pq}(x,\xi)\\
&= (2\pi)^{-2l}\sum_{1\le|\mu+\nu|\le N'}\mu!^{-1}\nu!^{-1}(-1)^{|\nu|}\iiiint \eta^{\nu}e^{-iy\eta}y'^{\mu}e^{-iy'\eta'}\\
&\quad\times \alpha_{Nn}^{(p)}(\xi+\eta+\eta')\beta_{N(q)}(x+y)\partial_x^{\mu}\partial_{\xi}^{\nu}\tilde{A}(x,\xi+\eta)\,dyd\eta dy'd\eta'\\
&= (2\pi)^{-2l}\sum_{1\le|\mu+\nu|\le N'}\mu!^{-1}\nu!^{-1}(-1)^{|\nu|}\iiiint (-i)^{\nu}\partial_y^{\nu}(e^{-iy\eta})(-1)^{|\nu|}(-i)^{\mu}\partial_{\eta'}^{\mu}(e^{-iy'\eta'})(-1)^{|\mu|}\\
&\quad\times \alpha_{Nn}^{(p)}(\xi+\eta+\eta')\beta_{N(q)}(x+y)\partial_x^{\mu}\partial_{\xi}^{\nu}\tilde{A}(x,\xi+\eta)\,dyd\eta dy'd\eta'\\
&= (2\pi)^{-2l}\sum_{1\le|\mu+\nu|\le N'}\mu!^{-1}\nu!^{-1}(-1)^{|\nu|}\iiiint D_y^{\nu}(e^{-iy\eta})(-1)^{|\nu|}\partial_{\eta'}^{\mu}(e^{-iy'\eta'})(-1)^{|\mu|}\\
&\quad\times \alpha_{Nn}^{(p)}(\xi+\eta+\eta')\beta_{N(q)}(x+y)D_x^{\mu}\partial_{\xi}^{\nu}\tilde{A}(x,\xi+\eta)\,dyd\eta dy'd\eta'\\
&= (2\pi)^{-2l}\sum_{1\le|\mu+\nu|\le N'}\mu!^{-1}\nu!^{-1}(-1)^{|\nu|}\iiiint e^{-iy\eta-iy'\eta'}\\
&\quad\times \alpha_{Nn}^{(p+\mu)}(\xi+\eta+\eta')\beta_{N(q+\nu)}(x+y)\tilde{A}_{(\mu)}^{(\nu)}(x,\xi+\eta)\,dyd\eta dy'd\eta'.
\end{aligned} \tag{2.39}$$

Now, in veiw of the fact

$$(2\pi)^{-l}\iint e^{-iy'\eta'}h(\eta')\,dy'd\eta' = \int e^{-iy'0}\hat{h}(y')\,dy' = h(0),$$

(2.39) gives

$$\begin{aligned}
&\tilde{f}_{N'pq}(x,\xi)\\
&\quad= (2\pi)^{-l}\sum_{1\le|\mu+\nu|\le N'}\mu!^{-1}\nu!^{-1}(-1)^{|\nu|}\iint e^{-iy\eta}\\
&\qquad\times \tilde{A}_{(\mu)}^{(\nu)}(x,\xi+\eta)\alpha_{Nn}^{(p+\mu)}(\xi+\eta)\beta_{N(q+\nu)}(x+y)\,dyd\eta.
\end{aligned} \tag{2.40}$$

Thus, we can see that (2.40) represents the symbol of the operator product, that is

$$\tilde{f}_{N'pq}(x,D_x) = \sum_{1\le|\mu+\nu|\le N'}\mu!^{-1}\nu!^{-1}(-1)^{|\nu|}\tilde{A}_{(\mu)}^{(\nu)}(x,D_x)\alpha_{Nn}^{(p+\mu)}(D_x)\beta_{N(q+\nu)}(x). \tag{2.41}$$

Now, from (2.14)

$$\tilde{A}(x,D_x) = a_{\Lambda}(x,D_x)\tilde{\alpha}_{Nn}(D_x) + b_{\Lambda}(x,D_x)\tilde{\alpha}_{Nn}(D_x). \tag{2.42}$$

Applying $(A\tilde{\alpha}_{Nn})^{(\nu)}_{(\mu)}(x, D_x)$ to Lemma 2.2, we get

$$\|(a_A\tilde{\alpha}_{Nn})^{(\nu)}_{(\mu)}(x, D_x)v_{p+\mu, q+\nu}\| \leq C_a c_a^{|\mu+\nu|}|\mu+\nu|! n^{2-|\nu|}\|v_{p+\mu, q+\nu}\|, \tag{2.43}$$

and

$$\|(b_A\tilde{\alpha}_{Nn})^{(\nu)}_{(\mu)}(x, D_x)v_{p+\mu, q+\nu}\| \leq C_b c_b^{|\mu+\nu|}|\mu+\nu|! n^{1-|\nu|}\|v_{p+\mu, q+\nu}\|. \tag{2.44}$$

From (2.41), we can write

$$\tilde{f}_{N'pq}(x, D_x)v = \sum_{1\leq|\mu+\nu|\leq N'} \mu!^{-1}\nu!^{-1}(-1)^{|\nu|}\tilde{A}^{(\nu)}_{(\mu)}(x, D_x)v_{p+\mu, q+\nu}.$$

Thus, from (2.43) and (2.44), we get

$$\begin{aligned}
&\sum_{1\leq|\mu+\nu|\leq N'} \mu!^{-1}\nu!^{-1}\|(a_A\tilde{\alpha}_{Nn} + b_A\tilde{\alpha}_{Nn})^{(\nu)}_{(\mu)}(x, D_x)v_{p+\mu, q+\nu}\| \\
&\quad\leq C\sum_{1\leq|\mu+\nu|\leq N'} \mu!^{-1}\nu!^{-1}C^{|\mu+\nu|}|\mu+\nu|! n^{2-|\nu|}\|v_{p+\mu, q+\nu}\| \\
&\quad\leq C\sum_{1\leq|\mu+\nu|\leq N'} C^{|\mu+\nu|}\frac{|\mu+\nu|!}{|\mu|!|\nu|!}n^{2-|\nu|}\|v_{p+\mu, q+\nu}\|.
\end{aligned}$$

(2.42) implies (2.38). □

Finally, the third term on the right hand side in (2.28) can be estimated as follows.

LEMMA 2.5. *There is a positive constant* C_2 *satisfying*

$$\|\tilde{r}_{N'pq}v(t,\cdot)\| \leq C_2 C^{N'+1}(N'+1)!|p+q|! n^{2-(N'+1)-|p|}\|v(t,\cdot)\|. \tag{2.45}$$

PROOF. From $\tilde{r}_{N'pq}(x,\xi)$ in (2.22),

$$\begin{aligned}
&\tilde{r}_{N'pq}(x,\xi) \\
&\quad= (N'+1)\int_0^1 (1-\theta)^{N'}(2\pi)^{-2l}\sum_{|\mu+\nu|=N'+1}\mu!^{-1}\nu!^{-1}(-1)^{|\nu|} \\
&\qquad\times\iiiint e^{-iy\eta-iy'\eta'}\alpha^{(p+\mu)}_{Nn}(\xi+\eta+\eta')\beta_{N(q+\nu)}(x+y) \\
&\qquad\times\tilde{A}^{(\nu)}_{(\mu)}(x+\theta y', \xi+\eta-\theta\eta)\,d\theta dy d\eta dy' d\eta' \\
&\quad= (N'+1)\int_0^1 (1-\theta)^{N'}(2\pi)^{-2l}\sum_{|\mu+\nu|=N'+1}\mu!^{-1}\nu!^{-1}(-1)^{|\nu|} \\
&\qquad\times\iiiint e^{-iy\eta-iy'\eta'}h(x,\xi; y,\eta,y',\eta')\,d\theta dy d\eta dy' d\eta',
\end{aligned}$$

where

$$\begin{aligned}
h(x,\xi; y,\eta,y',\eta') &= \langle\eta\rangle^{-2l}\langle D_y\rangle^{2l}\langle y\rangle^{-2l}\langle D_\eta\rangle^{2l}\langle\eta'\rangle^{-2l}\langle D_{y'}\rangle^{2l}\langle y'\rangle^{-2l}\langle D_{\eta'}\rangle^{2l} \\
&\quad\times\alpha^{(p+\mu)}_{Nn}(\xi+\eta+\eta')\beta_{N(q+\nu)}(x+y)\tilde{A}^{(\nu)}_{(\mu)}(x+\theta y', \xi+\eta-\theta\eta).
\end{aligned}$$

By putting $\langle D\rangle^{2l} = \sum_{|\gamma|\le l} C_{l\gamma} D^{2\gamma}$, and for multi-indices $d, m, n, k \in \boldsymbol{N}^l$,

$$
\begin{aligned}
&|h^{(\alpha)}_{(\beta)}(x,\xi; y,\eta, y',\eta')| \\
&\quad = \left| \langle\eta\rangle^{-2l} \left(\sum_{|d|\le l} C_{ld} D_y^{2d} \right) \langle y\rangle^{-2l} \left(\sum_{|n|\le l} C_{ln} D_\eta^{2n} \right) \langle\eta'\rangle^{-2l} \left(\sum_{|m|\le l} C_{lm} D_{y'}^{2m} \right) \langle y'\rangle^{-2l} \right. \\
&\qquad \times \left(\sum_{|k|\le l} C_{lk} D_{\eta'}^{2k} \right) \sum_{\alpha'\le\alpha} C_{\alpha\alpha'} \sum_{\beta'\le\beta} C_{\beta\beta'} \alpha_{Nn}^{(p+\mu+\alpha')}(\xi+\eta+\eta') \beta_{N(q+v+\beta')}(x+y) \\
&\qquad \left. \times \tilde{A}^{(v+\alpha-\alpha')}_{(\mu+\beta-\beta')}(x+\theta y', \xi+\eta-\theta\eta) \right|, \\
&\quad = \left| \langle\eta\rangle^{-2l} \langle\eta'\rangle^{-2l} \sum_{|d|\le l} C_{ld} \sum_{|n|\le l} C_{ln} \sum_{|m|\le l} C_{lm} \sum_{|k|\le l} C_{lk} \sum_{\alpha'\le\alpha} C_{\alpha\alpha'} \sum_{\beta'\le\beta} C_{\beta\beta'} \sum_{j=0}^{2m} C_{mj} \right. \\
&\qquad \times \sum_{i=0}^{2n} C_{mi} \sum_{h=0}^{2d} C_{dh} D_y^{2d-h} \langle y\rangle^{-2l} D_{y'}^{2m-j} \langle y'\rangle^{-2l} D_\eta^{2n-i} D_{\eta'}^{2k} \alpha_{Nn}^{(p+\mu+\alpha')}(\xi+\eta+\eta') \\
&\qquad \left. \times D_y^h \beta_{N(q+v+\beta')}(x+y) D_\eta^i D_{y'}^j \tilde{A}^{(v+\alpha-\alpha')}_{(\mu+\beta-\beta')}(x+\theta y', \xi+\eta-\theta\eta) \right|. \qquad (2.46)
\end{aligned}
$$

In (2.46), by the estimates (2.11), (2.10) and (2.14),

$$
\begin{aligned}
&|h^{(\alpha)}_{(\beta)}(x,\xi; y,\eta, y',\eta')| \\
&\quad \le C\langle y\rangle^{-2l}\langle\eta\rangle^{-2l}\langle y'\rangle^{-2l}\langle\eta'\rangle^{-2l} K^{2(1+N)} |p+\mu+\alpha'+4l|!|q+v+\beta'+2l|! n^{-|p+\mu+\alpha'|} \\
&\qquad \times C^{|\mu+v+\alpha-\alpha'+\beta-\beta'+4l|} |\mu+v+\alpha-\alpha'+\beta-\beta'+4l| n^{2-|v+\alpha-\alpha'|} \\
&\quad \le C_K \langle y\rangle^{-2l}\langle\eta\rangle^{-2l}\langle y'\rangle^{-2l}\langle\eta'\rangle^{-2l} C^{N'+1}(N'+1)!|\alpha+\beta|!|p+q|! n^{2-(N'+1)-|p|-|\alpha|}.
\end{aligned}
$$

Hence, (2.45) holds. □

Collecting Lemma 2.3, Lemma 2.4, Lemma 2.5 and (2.26) in (2.28), we conclude that

$$
\begin{aligned}
\frac{d}{dt}\|v_{p,q}\| &\ge C_0 n^2 \|v_{p,q}(t,\cdot)\| - C_1 \sum_{1\le|\mu+v|\le N'} C^{|\mu+v|} n^{2-|v|} \|v_{p+\mu,q+v}(t,\cdot)\| \\
&\quad - C_2 C^{N'+1}(N'+1)!|p+q|! n^{2-(N'+1)-|p|} \|v(t,\cdot)\| - c_0 e^{-\varepsilon_0 n}\|v(t,\cdot)\|. \qquad (2.47)
\end{aligned}
$$

Multiplying (2.47) by $M^{|p+q|} n^{-|q|}$ and summing up them for $0 \le |p+q| \le N-1$, we have the following estimate.

Lemma 2.6. *There are positive constants c', c'' and δ satisfying*

$$
\frac{d}{dt} S_N \ge c' n^2 S_N - n^2 c'' e^{-\delta n}\|v\|, \qquad (2.48)
$$

where

$$S_N = \sum_{0 \le |p+q| \le N-1} M^{|p+q|} n^{-|q|} \|v_{p,q}\|.$$

Proof. As to the second term in (2.47), it follows that

$$\sum_{0 \le |p+q| \le N-1} M^{|p+q|} n^{-|q|} C_1 \sum_{1 \le |\mu+\nu| \le N'} C^{|\mu+\nu|} n^{2-|\nu|} \|v_{p+\mu, q+\nu}\|$$

$$\le C_1 \sum_{0 \le |p+q| \le N-1} \sum_{0 \le |\mu+\nu| \le N'} CM^{-1} M^{|p+\mu+q+\nu|} n^{-|q|} n^{2-|q+\nu|} \|v_{p+\mu, q+\nu}\| \tag{2.49}$$

in veiw of $(CM^{-1})^{|\mu+\nu|} \le CM^{-1}$ if $M > C$. Moreover, by taking $N' = N - |p+q|$,

$$(2.49) \le C_1 CM^{-1} n^2 \sum_{0 \le |p+q+\mu+\nu| \le N} M^{|p+\mu+q+\nu|} n^{-|q+\nu|} \|v_{p+\mu, q+\nu}\|$$

$$\le C_1 CM^{-1} n^2 \left\{ S_N + \sum_{|p+q|=N} M^{|p+q|} n^{-|q|} \|v_{p,q}\| \right\}. \tag{2.50}$$

Here, we choose $M = 1/c'$ such that $C_1 CM^{-1} \le C_0$. The second term in (2.50) is estimated as follows. From (2.10) and (2.11), we have

$$\begin{aligned} \sum_{|p+q|=N} M^{|p+q|} n^{-|q|} \|v_{p,q}\| &\le \sum_{|p+q|=N} M^{|p+q|} n^{-|p+q|} K^{2(1+N)} |p+q|! \|v\| \\ &\le CM^N K^{2(1+N)} N! n^{-N} \|v\| \\ &\le C' K^2 e^{-(eMK^2)^{-1} n} \|v\| \\ &\le C'' e^{-c_1 n} \|v\|. \end{aligned} \tag{2.51}$$

Then we put $N = n/eMK^2$, and $e^{(-eMK^2)^{-1} n} = e^{-c_1 n}$ since letting $K = 1/c''$. The third term in (2.47) is estimatd by

$$\sum_{|p+q|=N} M^{|p+q|} n^{-|q|} C_2 C^{N'+1} (N'+1)! |p+q|! n^{2-(N'+1)-|p|} \|v\|$$

$$\begin{aligned} &\le C_2 C^{N'+1} (N'+1)! n^2 \sum_{|p+q|=N} M^{|p+q|} n^{-|p+q|} |p+q|! \|v\| \\ &\le C_2 C^{N'+1} n^2 M^N (N'+1)! N! n^{-N} \|v\| \\ &\le C_2 n^2 e^{-c_2 n} \|v\|. \end{aligned} \tag{2.52}$$

Hence, putting $\delta = \min\{\varepsilon_0, c_1, c_2\}$ where ε_0, c_1 and c_2 are represented in (2.47), (2.51) and (2.52) respectively, we can prove (2.48) from (2.50), (2.51) and (2.52). □

From (2.48) in Lemma 2.6, we have

$$S_N(t) \ge e^{c' n^2 t} S_N(0) - n^2 c'' e^{-\delta n} \int_0^t e^{c' n^2 (t-s)} \|v(s)\| \, ds. \tag{2.53}$$

Step 5. Initial data.

Here, we consider the function $\hat{\phi}(\xi)$ be a function whose support is located in small neighbourhood of ξ_0,

$$\phi(\eta) \geq 0, \quad \mathrm{supp}[\phi] \subset \left\{\xi; |\xi - \xi_0| < \frac{1}{2} r_0\right\} \quad \text{and} \quad \int |\phi(\xi)|^2 \, d\xi = 1.$$

Let us note that $\alpha_N(\xi) = 1$ on the support of $\hat{\phi}$. Then we put the initial data as follows.

$$\hat{v}_0(\xi) = \hat{\phi}(\xi - n\xi_0), \tag{2.54}$$

namely, $v_0(x) = e^{in\xi_0 x}\phi(x)$. As to $S_N(0)$ in (2.53), since

$$\|\alpha_{Nn}(D_x)\beta_N(x)v_0\| = \|\alpha_{Nn}(D_x)v_0\| \geq C$$

we can see

$$S_N(0) = \sum_{0 \leq |p+q| \leq N-1} M^{|p+q|} n^{-|q|} \|\alpha_{Nn}^{(p)}(D_x)\beta_{N(q)}(x)v_0\| \geq \tilde{C}. \tag{2.55}$$

And, we have from (1.6) in Definition 1.1

$$\begin{aligned} \|v(t,\cdot)\| &\leq \|e^{\rho\langle D_x\rangle}u(t,\cdot)\| \\ &\leq C\|e^{(\rho+\mu(t))\langle D_x\rangle}u_0\| \\ &\leq C\|e^{-\rho\omega\cdot\xi+\rho\langle\xi\rangle+\mu(t)\langle\xi\rangle}\hat{v}_0\| \\ &\leq cne^{\rho n+\mu(t)n}. \end{aligned} \tag{2.56}$$

Thus, since $\mu(t)$ is a monotone increasing function continuously defined in $[0, T]$ with $\mu(0) = 0$, we can see

$$\int_0^t e^{c'n^2(t-s)}\|v(s)\| \, ds \leq cne^{c'n^2t+\rho n} \int_0^t e^{\mu(s)n - c'sn^2} \, ds \leq C' e^{\rho n+\mu(t)n} e^{c'n^2 t}. \tag{2.57}$$

Consequently, from (2.57) and (2.55), (2.53) can be estimated by

$$\begin{aligned} S_N(t) &\geq e^{c'n^2t} S_N(0) - C'c''n^2 e^{-(\delta-\rho)n+\mu(t)n} e^{c'n^2t} \\ &\geq Ce^{c'n^2t}, \end{aligned} \tag{2.58}$$

where $\mu(t)$ should be taken as satisfying $\mu(t) < \delta - \rho$ for any ρ possessing $\rho < \delta$. On the other hand, $S_N(t)$ can be estimated by, from (2.56),

$$\begin{aligned} S_N(t) &= \sum_{0 \leq |p+q| \leq N-1} M^{|p+q|} n^{-|q|} \|\alpha_{Nn}^{(p)}(D_x)\beta_{N(q)}(x)v\| \\ &\leq C(M)N\|v\| \\ &\leq C(M)n^2 e^{(\rho+\mu(t))n}. \end{aligned} \tag{2.59}$$

Hence, two inequalities (2.58) and (2.59) are not compatible by defining an initial data as (2.54). Thus we completed the proof of Theorem 1.2.

ACKNOWLEDGMENT. The author is greatly indebted to Professor Kunihiko Kajitani for supervising the work. Also his thanks to all members of Professor Kajitani's seminar for valuable discussions and comments.

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