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Non-commutative valuation rings of $K(X;\sigma,\delta)$ over a division ring K

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Abstract. Let K be a division ring with a σ-derivation δ, where σ is an endomorphism of K and $K(X;\sigma,\delta)$ be the quotient division ring of the Ore extension $K[X;\sigma,\delta]$ over K in an indeterminate X. First, we describe non-commutative valuation rings of $K(X;\sigma,\delta)$ which contain $K[X;\sigma,\delta]$. Suppose that (σ,δ) is compatible with V, where V is a total valuation ring of K, then $R^{(1)} = V[X;\sigma,\delta]_{J(V)[X;\sigma,\delta]}$, the localization of $V[X;\sigma,\delta]$ at $J(V)[X;\sigma,\delta]$, is a total valuation ring of $K(X;\sigma,\delta)$. Applying the description above, then, second, we describe non-commutative valuation rings B of $K(X;\sigma,\delta)$ such that $B \cap K = V$, $X \in B$ and $B \subseteq R^{(1)}$, which is the aim of this paper. In the end of each section we give several examples to display some of the various phenomena.

0. Introduction.

Let K be a division ring, σ be an endomorphism of K and δ be a σ-derivation, i.e., an additive map $\delta : K \to K$ such that $\delta(ab) = \sigma(a)\delta(b) + \delta(a)b$ for all $a, b \in K$. As usual, $K[X;\sigma,\delta] = \{f(X) \mid f(X) = a_n X^n + \cdots + a_0, a_i \in K\}$ is the Ore extension over K with $Xa = \sigma(a)X + \delta(a)$ for any $a \in K$, where X is an indeterminate. It is a very interesting problem to describe all non-commutative valuation rings of $K(X;\sigma,\delta)$, the left quotient division ring of $K[X;\sigma,\delta]$. However, the problem seems rather difficult as can be seen in [**BT**], [**BS**], [**KMP**], [**XKM$_1$**], [**XKM$_2$**]. Let V be a total valuation ring of K and (σ,δ) be compatible with V. Then, in [**BT**], they proved that $R^{(1)} = V[X;\sigma,\delta]_{J(V)[X;\sigma,\delta]}$, the localization of $V[X;\sigma,\delta]$ at $J(V)[X;\sigma,\delta]$, is a total valuation ring of $K(X;\sigma,\delta)$ such that $R^{(1)} \cap K = V$, $X \in R^{(1)}$ and we have obtained more detailed results on $R^{(1)}$, based on some properties of (σ,δ) (see [**XKM$_2$**]).

The aim of this paper is to describe non-commutative valuation rings B of $K(X;\sigma,\delta)$ such that $R^{(1)} \supseteq B$, $B \cap K = V$ and $X \in B$.

The paper is organized as follows:

In Section 1, we shall study non-commutative valuation rings of $K(X;\sigma,\delta)$ containing $R = K[X;\sigma,\delta]$, adopting the methods and results in [**C**], [**LL**] and [**LM**]. If δ is not a quasi-algebraic σ-derivation, then there are no proper non-commutative valuation rings of $K(X;\sigma,\delta)$ containing R (Corollary 1.2). In the case where δ is a quasi-algebraic σ-derivation and $\sigma \in \mathrm{Aut}(K)$, the group of all automorphisms of K, we shall describe all non-commutative valuation rings of $K(X;\sigma,\delta)$ containing R (Theorem 1.7). If $\sigma \notin \mathrm{Aut}(K)$ and δ is a quasi-algebraic σ-derivation, then there is a monic invariant polynomial $p(X)$ of minimal non-zero degree such that $K[X;\sigma,\delta]p(X)$ is a maximal

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ideal. Set $\hat{K} = \bigcup_{i=1}^{\infty} p(X)^{-i} K p(X)^{i}$, a division ring containing K. We can extend σ and δ to $\hat{\sigma}$ and $\hat{\delta}$ with $\hat{\sigma} \in \mathrm{Aut}(\hat{K})$ and obtain $\hat{R} = \bigcup_{i=1}^{\infty} p(X)^{-i} R p(X)^{i} = \hat{K}[X; \hat{\sigma}, \hat{\delta}]$ with a maximal ideal $\hat{P} = \hat{R}p(X)$. If $\hat{K}$ is left algebraic over K and $Rp(X)$ is completely prime, then $\hat{R}_{\hat{P}}$ is the only proper total valuation ring of $K(X; \sigma, \delta)$ such that $\hat{R}_{\hat{P}} \supseteq R$ (Proposition 1.9). If $\hat{K}$ is not left algebraic over K, then the circumstances become very complicated so that we only treat the case where K is a field (Proposition 1.11).

Suppose that V is a total valuation ring of K and (σ, δ) is compatible with V. Then σ and δ naturally induce $\bar{\sigma}$ and $\bar{\delta}$ of $\bar{V} = V/J(V)$, a division ring and we can construct the Ore extension $\bar{V}[X; \bar{\sigma}, \bar{\delta}]$ over $\bar{V}$. By applying the results in Section 1 to $\bar{V}[X; \bar{\sigma}, \bar{\delta}]$, in Section 2, we shall study non-commutative valuation rings B of $K(X; \sigma, \delta)$ satisfying the following conditions: $B \cap K = V$, $X \in B$, $B \subseteq R^{(1)}$. If $\bar{\delta}$ is not a quasi-algebraic $\bar{\sigma}$-derivation, then $R^{(1)}$ is the only non-commutative valuation ring of $K(X; \sigma, \delta)$ satisfying the conditions. If $\bar{\sigma} \in \mathrm{Aut}(\bar{V})$, then we can describe all non-commutative valuation rings of $K(X; \sigma, \delta)$ satisfying the conditions. However, in the case where $\bar{\sigma} \notin \mathrm{Aut}(\bar{V})$ and $\bar{\delta}$ is a quasi-algebraic $\bar{\sigma}$-derivation, we can only describe non-commutative valuation rings of $K(X; \sigma, \delta)$ satisfying the additional conditions in Propositions 1.9 and 1.11 respectively.

In the end of each section, we shall give several examples designed to display some of the various phenomena.

1. The non-commutative valuation rings of $K(X; \sigma, \delta)$ containing $K[X; \sigma, \delta]$.

Throughout this paper, K is a division ring, σ is an endomorphism of K and δ is a σ-derivation.

The aim of this section is to describe non-commutative valuation rings of $K(X; \sigma, \delta)$ containing $K[X; \sigma, \delta]$.

There are three types of non-commutative valuation rings as follows: Let Q be a simple Artinian ring and let R be an order in Q, i.e., R is a prime Goldie ring. We say that R is a *Dubrovin valuation ring* of Q if R is semi-hereditary and R is local, i.e., $R/J(R)$ is a simple Artinian ring, where $J(R)$ is the Jacobson radical of R. Assume that Q is a division ring. A subring R of Q is said to be a *total* valuation ring of Q, if for any non-zero $q \in Q$, either $q \in R$ or $q^{-1} \in R$. Furthermore, a total valuation ring R of Q is said to be *invariant* if $q^{-1}Rq = R$ for all non-zero $q \in Q$. It is easy to see that a total valuation ring R is a Dubrovin valuation ring and the converse is not necessarily true. We refer to [**MMU**] for some properties of non-commutative valuation rings.

First, we shall study ideals in $K[X; \sigma, \delta]$ where properties of (σ, δ) are of critical importance. Following [**LL**], a σ-derivation δ is said to be *quasi-algebraic* if there exist $a_n = 1, a_{n-1}, \dots, a_0 \in K$, $n > 0$ such that $\sum_{i=1}^{n} a_i \delta^i = D_{a_0, \sigma^n}$, where $D_{a_0, \sigma^n}(a) = a_0 a - \sigma^n(a) a_0$ for all $a \in K$. In [**LL**], they conjectured that $K[X; \sigma, \delta]$ is not simple if and only if δ is a quasi-algebraic σ-derivation. In the case where $\sigma \in \mathrm{Aut}(K)$, Lemonnier gave a necessary and sufficient condition for $K[X; \sigma, \delta]$ to be not simple ([**L**]). In [**LLLM**], they gave an affirmative answer to the conjecture by using some results in [**LL**].

Proposition 1.1 ([**LLLM**, (3.6)]). *Let K be a division ring, σ be an endomorphism of K and δ be a σ-derivation. Then $K[X; \sigma, \delta]$ is not simple if and only if δ is a quasi-algebraic σ-derivation.*

A non-commutative valuation ring B of $K(X;\sigma,\delta)$ is said to be *proper* if $B \neq K(X;\sigma,\delta)$. Throughout this section, set $R = K[X;\sigma,\delta]$, which is a left principal ideal domain.

Corollary 1.2. *If δ is not a quasi-algebraic σ-derivation, then there are no proper non-commutative valuation rings of $K(X;\sigma,\delta)$ containing R.*

Proof. Let B be a proper non-commutative valuation ring of $K(X;\sigma,\delta)$ containing R. Then $J(B) \neq 0$ and so $J(B) \cap R \neq 0$. Hence δ is a quasi-algebraic σ-derivation by Proposition 1.1, a contradiction. □

From Corollary 1.2, we may assume that δ is a quasi-algebraic σ-derivation. The *inner order* of σ, denoted by $\circ(\sigma)$, is defined by the smallest positive integer n such that $\sigma^n = I_a$, the inner automorphism induced by a, where $a \in K$; if no such natural number n exists, then $\circ(\sigma)$ is ∞. Note that $\circ(\sigma) = n < \infty$, then $\sigma \in \mathrm{Aut}(K)$. A monic polynomial $p(X)$ with $\deg p(X) = n$ is said to be *invariant* if for any $a \in K$, $p(X)a = \sigma^n(a)p(X)$, $p(X)X = (X + c)p(X)$ for some $c \in K$. Note that $K[X;\sigma,\delta]p(X)$ is an ideal of $K[X;\sigma,\delta]$ if and only if $p(X)$ is invariant.

In the remainder of this section, except for Propositons 1.4 and 1.5, we always assume that δ is a quasi-algebraic σ-derivation and $p(X)$ is a monic invariant polynomial of minimal non-zero degree with $\deg p(X) = n$.

If $\circ(\sigma) = m < \infty$, then in [**C**], Cauchon proved that $Z(R) = Z(K)_{\sigma,\delta}[\lambda p(X)^l]$ for some non-zero $\lambda \in K$ and some natural number l, where $Z(S)$ stands for the center of S for any ring S and $Z(K)_{\sigma,\delta} = \{a \in Z(K) \mid \sigma(a) = a \text{ and } \delta(a) = 0\}$ (also see [**LL**, (2.8)]). We note that any non-zero prime ideal of R is maximal.

With this notation, we will describe all maximal ideals of R in the following lemma which is essentially due to Cauchon.

Lemma 1.3. *Suppose that δ is a quasi-algebraic σ-derivation.*

(1) *If $\circ(\sigma) = \infty$, then $P = K[X;\sigma,\delta]p(X)$ is the only maximal ideal of R.*

(2) *If $\circ(\sigma) < \infty$, then any maximal ideal of R is one of the following: $P = K[X;\sigma,\delta]p(X)$, $M = Rw(Y)$, where $Y = \lambda p(X)^l$ and $w(Y)$ runs over all irreducible polynomials of $Z(K)_{\sigma,\delta}[Y]$ different from Y.*

Proof. It is clear that $P = Rp(X)$ is a maximal ideal of R in all cases.

(1) First, we note that if $\circ(\sigma) = \infty$, then $Z(R) \subseteq K$. To prove this, on the contrary, assume that there exists a polynomial $c(X) = c_nX^n + c_{n-1}X^{n-1} + \cdots + c_0 \in Z(R)$ with $n \geq 1$, $c_n \neq 0$, then it is easy to see that σ^n is inner induced by c_n^{-1}, a contradiction. Hence P is the unique maximal ideal of R by [**C**, (6.2.13)].

(2) Let M be any maximal ideal of R with $M = Rq(X)$ for some monic invariant polynomial $q(X)$ in R and $M \neq P$. Then $q(X) = aw(Y)p(X)^m$ (see [**LL**, (2.8)]), where $a \in K$, $w(Y) \in Z(K)_{\sigma,\delta}[Y]$ and $m \geq 0$. Since $M \neq P$, $m = 0$ and $w(Y)$ must be an irreducible polynomial different from Y, showing that $M = Rw(Y)$. Conversely, it is easy to see that $Rw(Y)$ is a maximal ideal of R, where $w(Y)$ is an irreducible polynomial of $Z(K)_{\sigma,\delta}[Y]$ different from Y. □

The following two propositions are remarkable and may be implicitly known. However we shall give proofs of them for the reader's convenience.

Proposition 1.4. *δ is inner if and only if there exists a monic invariant polynomial $p(X) = X - b$ for some $b \in K$.*

Proof. Suppose that δ is inner. Then there is an element $b \in K$ such that $\delta(a) = ba - \sigma(a)b$ for all $a \in K$. Then $Xa = \sigma(a)X + \delta(a) = \sigma(a)X + ba - \sigma(a)b$ so that $(X - b)a = \sigma(a)(X - b)$. It easily follows that $(X - b)X = (X - b + \sigma(b))(X - b)$ by using $\delta(b) = b^2 - \sigma(b)b$. This means that $X - b$ is a monic invariant polynomial.

Conversely, suppose that $X - b$ is a monic invariant polynomial. Then $(X - b)a = \sigma(a)(X - b)$ and $(X - b)a = \sigma(a)X + \delta(a) - ba$ for any $a \in K$ and so $\sigma(a)(X - b) = \sigma(a)X + \delta(a) - ba$. Hence $\delta(a) = ba - \sigma(a)b$, i.e., δ is inner. $\square$

Proposition 1.5. *Suppose that K is a field and that $\sigma \neq 1$. Then δ is inner.*

Proof. Since $\sigma \neq 1$, there exists an element $b \in K$ with $\sigma(b) \neq b$. For any $a \in K$, $\sigma(a)\delta(b) + \delta(a)b = \delta(ab) = \delta(ba) = \sigma(b)\delta(a) + \delta(b)a$ and so $(\sigma(a) - a)\delta(b) = (\sigma(b) - b)\delta(a)$. Let $c = -\delta(b)/(\sigma(b) - b) \in K$. Then $\delta(a) = ca - \sigma(a)c$ for any $a \in K$, i.e., δ is inner. $\square$

We shall study the proper non-commutative valuation rings of $K(X;\sigma,\delta)$ containing R by using Lemma 1.3 and the following lemma which is implicitly known. An order S in a simple Artinian ring Q is called *Dedekind* if every one-sided ideal of S is a progenerator (see [**MR**, (5.2.10)]). A prime ideal P of S is *left localizable* if $\mathscr{C}(P) = \{c \in S \mid c \text{ is regular } \bmod P\}$ is a left Ore set and any element in $\mathscr{C}(P)$ is regular. We denote by S_P the localization of S at P if P is left localizable. If P is left and right localizable, then we say that P is *localizable*. If S is a Dedekind order in Q, then any non-zero prime ideal is a maximal ideal of S (see [**MR**, (5.4.5)]). Hence the following lemma follows from [**D**, Theorems 3 and 4, §2].

Lemma 1.6. *Let Q be a simple Artinian ring and S be a Dedekind order in Q. Then there is a one-to-one correspondence between the set of all proper Dubrovin valuation rings of Q containing S and the set of all non-zero maximal ideals of S, which is given by $\varphi : B \to J(B) \cap S$ and $\varphi^{-1} : P \to S_P$, where B is a proper Dubrovin valuation ring of Q containing S and P is a non-zero maximal ideal of S.*

If $\sigma \in \mathrm{Aut}(K)$, then R is a principal ideal domain so that it is a Dedekind order in $K(X;\sigma,\delta)$. Thus the following theorem follows from Lemmas 1.3 and 1.6.

Theorem 1.7. *Suppose that $\sigma \in \mathrm{Aut}(K)$ and δ is a quasi-algebraic σ-derivation such that $p(X)$ is a monic invariant polynomial of minimal non-zero degree.*

(1) *If $\circ(\sigma) = \infty$, then R_P is the only proper Dubrovin valuation ring of $K(X;\sigma,\delta)$ containing R, where $R = K[X;\sigma,\delta]$ and $P = Rp(X)$.*

(2) *If $\circ(\sigma) = m < \infty$, then any proper Dubrovin valuation ring of $K(X;\sigma,\delta)$ containing R is one of the following: R_P, R_M, where $M = Rw(Y)$ is as in Lemma 1.3.*

Next we shall study the case where σ is not an automorphism. Since $P = Rp(X) \supsetneq p(X)R$, we have an ascending chain of overrings; $R \subsetneq p(X)^{-1}Rp(X) \subsetneq \cdots \subsetneq p(X)^{-m}Rp(X)^m \subsetneq \cdots$. Set $\hat{R} = \bigcup_{i=1}^{\infty} p(X)^{-i}Rp(X)^i$. Furthermore, $Kp(X) \supsetneq p(X)K$ so that we have an ascending chain of division overrings; $K \subsetneq p(X)^{-1}Kp(X) \subsetneq \cdots \subsetneq$

$p(X)^{-m}Kp(X)^m \subsetneq \cdots$. And we set $\hat{K} = \bigcup_{i=1}^{\infty} p(X)^{-i}Kp(X)^i$, a division ring containing K. Let $p(X)^{-m}ap(X)^m$ be any element in $\hat{K}$ where $a \in K$. Define $\hat{\sigma}(p(X)^{-m}ap(X)^m) = p(X)^{-m}\sigma(a)p(X)^m$. It is easy to check that $\hat{\sigma}$ is well-defined and $\hat{\sigma} \in \mathrm{Aut}(\hat{K})$. By usage of $p(X)$, we have another automorphism τ of $\hat{K}$ defined by $\tau(\alpha) = p(X)\alpha p(X)^{-1}$ for any $\alpha \in \hat{K}$. We note that we have a descending chain of subdivision rings; $K \supsetneq \tau(K) \supsetneq \cdots \supsetneq \tau^m(K) \supsetneq \cdots$ and $\sigma^n = \tau|_K$, the restriction of τ to K, where $n = \deg p(X)$.

We also note that $\hat{R} = R$ if $\sigma \in \mathrm{Aut}(K)$. Using this notation we have the following:

Proposition 1.8. *Suppose that $\sigma \notin \mathrm{Aut}(K)$ and δ is a quasi-algebraic σ-derivation such that $p(X)$ is the unique monic invariant polynomial of minimal non-zero degree.*

(1) *$\hat{R} = \hat{K}[X;\hat{\sigma},\hat{\delta}]$ for some $\hat{\delta}$, a $\hat{\sigma}$-derivation.*

(2) *$\hat{\sigma} \in \mathrm{Aut}(\hat{K})$ and $\circ(\hat{\sigma}) = \infty$.*

(3) *$\hat{P} = \hat{R}p(X)$ is the unique maximal ideal of $\hat{R}$. In particular, $Rp(X)$ is completely prime if and only if $\hat{P}$ is completely prime.*

Proof. (1) First, we define $\hat{\delta}$ as follows; let $\alpha = p(X)^{-l}ap(X)^l$ be any element of $\hat{K}$, where $a \in K$. Since $p(X)^lX = (X + b_l)p(X)^l$ for some $b_l \in K$, we have $(X + b_l)a = \sigma(a)X + \delta(a) + b_la = \sigma(a)(X + b_l) + a_l$, where $a_l = \delta(a) + b_la - \sigma(a)b_l \in K$ and

$$
\begin{aligned}
p(X)^lX\alpha &= p(X)^lXp(X)^{-l}ap(X)^l \\
&= (X + b_l)ap(X)^l \\
&= (\sigma(a)(X + b_l) + a_l)p(X)^l.
\end{aligned}
$$

So

$$
\begin{aligned}
X\alpha &= p(X)^{-l}(\sigma(a)(X + b_l) + a_l)p(X)^l \\
&= p(X)^{-l}\sigma(a)p(X)^lX + p(X)^{-l}a_lp(X)^l \\
&= \hat{\sigma}(\alpha)X + \hat{\delta}(\alpha),
\end{aligned}
$$

where $\hat{\delta}(\alpha) = p(X)^{-l}a_lp(X)^l \in \hat{K}$.

To show that $\hat{\delta}$ is a $\hat{\sigma}$-derivation, let $\alpha, \beta \in \hat{K}$. Then

$$
\begin{aligned}
X(\alpha + \beta) &= X\alpha + X\beta \\
&= \hat{\sigma}(\alpha)X + \hat{\delta}(\alpha) + \hat{\sigma}(\beta)X + \hat{\delta}(\beta) \\
&= (\hat{\sigma}(\alpha) + \hat{\sigma}(\beta))X + \hat{\delta}(\alpha) + \hat{\delta}(\beta)
\end{aligned}
$$

and $X(\alpha + \beta) = \hat{\sigma}(\alpha + \beta)X + \hat{\delta}(\alpha + \beta)$, which shows that $\hat{\delta}(\alpha + \beta) = \hat{\delta}(\alpha) + \hat{\delta}(\beta)$. Further, $X\alpha\beta = (\hat{\sigma}(\alpha)X + \hat{\delta}(\alpha))\beta = \hat{\sigma}(\alpha)\hat{\sigma}(\beta)X + \hat{\sigma}(\alpha)\hat{\delta}(\beta) + \hat{\delta}(\alpha)\beta$ and $X\alpha\beta = \hat{\sigma}(\alpha)\hat{\sigma}(\beta)X + \hat{\delta}(\alpha\beta)$. Thus $\hat{\delta}(\alpha\beta) = \hat{\sigma}(\alpha)\hat{\delta}(\beta) + \hat{\delta}(\alpha)\beta$ and so $\hat{\delta}$ is a $\hat{\sigma}$-derivation.

Next we prove that $\hat{R} = \hat{K}[X;\hat{\sigma},\hat{\delta}]$. Since $\hat{R}$ is a ring, it follows from the definition that $\hat{R} \supseteq \hat{K}[X;\hat{\sigma},\hat{\delta}]$. To prove the converse inclusion, let $\alpha = p(X)^{-l}f(X)p(X)^l \in \hat{R}$, where $f(X) = \Sigma a_iX^i$ and l is a non-negative integer. Since $p(X)^lX = (X + b_l)p(X)^l$ for some b_l in K, we have

$$\begin{aligned} p(X)^{-l}Xp(X)^{l} &= p(X)^{-l}(X+b_l-b_l)p(X)^{l} \\ &= p(X)^{-l}(X+b_l)p(X)^{l}+p(X)^{-l}(-b_l)p(X)^{l} \\ &= X+p(X)^{-l}(-b_l)p(X)^{l}, \end{aligned}$$

which belongs to $\hat{K}[X;\hat{\sigma},\hat{\delta}]$. Hence $\alpha = p(X)^{-l}f(X)p(X)^{l} = \Sigma p(X)^{-l}a_ip(X)^{l}p(X)^{-l}\cdot X^ip(X)^{l} \in \hat{K}[X;\hat{\sigma},\hat{\delta}]$.

(2) We have proved that $\hat{\sigma} \in \mathrm{Aut}(\hat{K})$. To prove that $\circ(\hat{\sigma}) = \infty$, on the contrary, assume that $\hat{\sigma}^m = I_\alpha$ for some $m \geq 1$ and $\alpha \in \hat{K}$, i.e., $\hat{\sigma}^m(\beta) = \alpha\beta\alpha^{-1}$ for all $\beta \in \hat{K}$. Write $\alpha = p(X)^{-l}ap(X)^{l}$ and $\beta = p(X)^{-l}bp(X)^{l}$ for some $l \geq 0$ and $a, b \in K$. Then $\hat{\sigma}^m(\beta) = p(X)^{-l}\sigma^m(b)p(X)^{l}$ and $\alpha\beta\alpha^{-1} = p(X)^{-l}ap(X)^{l}p(X)^{-l}bp(X)^{l}p(X)^{-l}a^{-1}p(X)^{l} = p(X)^{-l}aba^{-1}p(X)^{l}$. Hence $\sigma^m(b) = aba^{-1}$ for any $b \in K$, contradicting to the assumption $\sigma \notin \mathrm{Aut}(K)$ and so $\circ(\hat{\sigma}) = \infty$.

(3) For any $\alpha = p(X)^{-l}f(X)p(X)^{l} \in \hat{R}$, we have $p(X)\alpha = p(X)^{-(l-1)}f(X)p(X)^{l-1}\cdot p(X) \in \hat{P}$, which implies that $\hat{P}$ is an ideal of $\hat{R}$. We know from Lemma 1.3 that there is a unique maximal ideal, say, $\hat{M} = \hat{R}\beta$, so that $p(X) = \gamma\beta$ for some $\gamma \in \hat{R}$. Write $\gamma = p(X)^{-m}r(X)p(X)^{m}$ and $\beta = p(X)^{-m}g(X)p(X)^{m}$, where $m \geq 0$ and $r(X), g(X) \in R$. So $p(X) = p(X)^{-m}r(X)g(X)p(X)^{m}$ and thus $p(X) = r(X)g(X)$. Since τ is extended to an automorphism of $\hat{R}$ which is a conjugation by $p(X)$, we have $\hat{R}\beta = \tau^m(\hat{R}\beta) = \hat{R}\tau^m(\beta) = \hat{R}g(X)$ and thus $Rg(X) \subseteq \hat{R}g(X) \cap R \subseteq Rp(X)$ so that $g(X) = c(X)p(X)$ for some $c(X) \in R$. It follows that $P = Rp(X) = Rr(X)g(X) \subseteq Rg(X) = Rc(X)p(X) \subseteq Rp(X)$. Hence $\hat{P} = \hat{R}g(X)$, which shows that $\hat{P}$ is the unique maximal ideal of $\hat{R}$.

To show the last statement, suppose that $\hat{P}$ is completely prime, then P is completely prime, because $P = \hat{P} \cap R$. Conversely, suppose that P is completely prime and assume that $\alpha\beta \in \hat{P}$ with $\alpha \notin \hat{P}$, where $\alpha, \beta \in \hat{R}$. So $\alpha\beta = \gamma p(X)$ for some $\gamma \in \hat{R}$. Write $\alpha = p(X)^{-l}f(X)p(X)^{l}$, $\beta = p(X)^{-l}g(X)p(X)^{l}$ and $\gamma = p(X)^{-l}r(X)p(X)^{l}$ for some $l \geq 0$ and $f(X), g(X), r(X) \in R$. Then we have $f(X)g(X) = r(X)p(X)$ and so either $f(X) \in P$ or $g(X) \in P$. If $f(X) \in P$, then $f(X) = c(X)p(X)$ for some $c(X) \in R$. Thus $\alpha = p(X)^{-l}f(X)p(X)^{l} = p(X)^{-l}c(X)p(X)^{l}p(X) \in \hat{P}$, a contradiction. Hence $g(X) \in P$ and so $\beta \in \hat{P}$, proving that $\hat{P}$ is completely prime. □

$\hat{K}$ is said to be (left) *algebraic* over K if for any $\alpha \in \hat{K}$, there exist $c_i \in K$, not all zero, such that $\sum_{i=0}^{m} c_i\alpha^i = 0$.

In the case where $P = Rp(X)$ is completely prime and $\hat{K}$ is algebraic over K, we have the following:

Proposition 1.9. *Let δ be a quasi-algebraic σ-derivation with $\sigma \notin \mathrm{Aut}(K)$. Suppose that $\hat{K}$ is algebraic over K and $P = Rp(X)$ is a completely prime ideal, where $p(X)$ is the unique monic invariant polynomial of minimal non-zero degree. Then $\hat{R}_{\hat{P}}$ is the unique proper total valuation ring of $K(X;\sigma,\delta)$ containing R.*

Proof. We know from Theorem 1.7 and Proposition 1.8 that $\hat{R}_{\hat{P}}$ is a Dubrovin valuation ring of $K(X;\sigma,\delta)$ containing R. Since $\hat{P}$ is a completely prime ideal, we have $\hat{R}_{\hat{P}}$ is total (see the proof of [**MMU**, (8.13)]). To show the uniqueness, let B be any proper total valuation ring of $K(X;\sigma,\delta)$ containing R. Then we shall prove that

$B \supseteq \hat{R}$. It suffices to prove that $B \supseteq \hat{K}$. Let $\alpha = p(X)^{-l}ap(X)^{l} \in \hat{K}$ for some $a \in K$ and $l \geq 0$. Since B is total, we have either $\alpha \in B$ or $\alpha^{-1} \in B$. If $\alpha \in B$, then there is nothing to do. So we may assume $\alpha^{-1} \in B$. Since $\hat{K}$ is algebraic over K, there are a natural number m and elements $c_i \in K$, not all zero, such that $\sum_{i=0}^{m} c_i\alpha^{-i} = 0$. We may assume that $c_0 \neq 0$, $c_m \neq 0$. Thus $\tau^{l}(c_0) + \sum_{i=1}^{m} \tau^{l}(c_i)a^{-i} = \tau^{l}(0) = 0$ and so $1 = -\sum_{i=1}^{m} \tau^{l}(c_0^{-1}c_i)a^{-i}$. Multiplying a on the both side, we have $a = -\sum_{i=1}^{m} \tau^{l}(c_0^{-1}c_i)a^{-i+1}$. Hence we have $\alpha = \tau^{-l}(a) = -\sum_{i=1}^{m} c_0^{-1}c_i\tau^{-l}(a^{-i+1}) = -\sum_{i=1}^{m} c_0^{-1}c_i\alpha^{-i+1} \in B$. Since $B \supseteq \hat{R}$, we have $B = \hat{R}_{\hat{P}}$ by Theorem 1.7. □

Corollary 1.10. *Let δ be a quasi-algebraic σ-derivation with $\sigma \notin \mathrm{Aut}(K)$ and δ be inner. Suppose that $\hat{K}$ is algebraic over K. Then $\hat{R}_{\hat{P}}$ is the unique proper total valuation ring of $K(X;\sigma,\delta)$ containing R.*

Proof. Since δ is inner, there exists a monic invariant polynomial $p(X) = X - b$ for some $b \in K$ by Proposition 1.4. Hence $R = K[Y,\sigma]$, where $Y = p(X)$. This means that $P = Rp(X)$ is a completely prime ideal and so $\hat{R}_{\hat{P}}$ is the unique proper total valuation ring of $K(X;\sigma,\delta)$ containing R. □

If $\hat{K}$ is not algebraic over K, then it becomes very complicated, even in the case where K is a field. If K is a field, then there exists a monic invariant polynomial $p(X) = X - b$ for some $b \in K$ by Propositions 1.4 and 1.5 and so, as in Proposition 1.9, $\hat{R}_{\hat{P}}$ is a proper total valuation ring of $K(X;\sigma,\delta)$ containing R. But there are many others proper total valuation rings of $K(X;\sigma,\delta)$ containing R, as it will be seen in the following:

Proposition 1.11. *Suppose that K is a field with $\sigma \notin \mathrm{Aut}(K)$ and δ is a quasi-algebraic σ-derivation. Let $p(X) = X - b$ be an invariant polynomial and $\hat{P} = \hat{R}p(X)$. If $\hat{K}$ is not algebraic over K, then*

(1) $\mathrm{tr.deg}_K \hat{K} = \infty$ *and*

(2) *Let $\mathfrak{B} = \{t_i \in \hat{K} \mid i \in \Lambda\}$ be a transcendental basis of $\hat{K}/K$, where Λ is an index set with α as its ordinal number. Then there exist at least total valuation rings A_i and B_i $(i \in \Lambda)$ of $K(X;\sigma,\delta)$ satisfying the following*:

(i) *$\hat{R}_{\hat{P}} \supseteq A_i$ for all $i \in \Lambda$ and A_i are incomparable each other.*

(ii) *$B_1 \subsetneq B_2 \subsetneq \cdots \subsetneq \hat{R}_{\hat{P}}$.*

Proof. (1) Let $K_i = p(X)^{-i}Kp(X)^{i}$ and assume that $K = K_0 \subsetneq K_1 \subsetneq K_2 \subsetneq \cdots \subsetneq K_l \subsetneq K_{l+1}$ such that K_i is algebraic over K_{i-1} for all $0 < i \leqq l$ and K_{l+1} is transcendental over K_l. If $l > 1$, then there exists an element $t = p(X)^{-(l+1)}ap(X)^{l+1} \in K_{l+1}$ which is transcendental over K_l, where $a \in K$. Since $\tau(t) = p(X)^{-l}ap(X)^{l} \in K_l$, it is algebraic over K_{l-1}, i.e., there exist $\beta_i \in K_{l-1}$, not all zero, such that $\sum_{i=0}^{m} \beta_i\tau(t)^{i} = 0$, which implies $\sum_{i=0}^{m} p(X)^{-1}\beta_i p(X)t^{i} = 0$, i.e., t is algebraic over K_l, a contradiction. So we may assume that $t = p(X)^{-1}ap(X)$ is transcendental over K for some $a \in K$. Set $t = t_1, \ldots, t_l = p(X)^{-l}ap(X)^{l}$ for any natural number l and $\overline{K_l} = K(t_1, \ldots, t_l)$, the field generated by K and $t_1, \ldots, t_l$. Then it follows from the same method as the above that t_l is transcendental over $\overline{K_{l-1}}$ for any l by induction on l. Hence $\mathrm{tr.deg}_K \hat{K} = \infty$.

(2) Let $\mathfrak{B} = \{t_i \in \hat{K} \mid i \in \Lambda\}$ be a transcendental basis of $\hat{K}/K$, where Λ is an index set whose ordinal number is α, and let $\overline{K} = K(t_i)_{i \in \Lambda}$ be the field extension of K generated by K and $\mathfrak{B}$.

(i) For any $i \in \Lambda$, we define a valuation v_i of $\overline{K}$ as follows; $v_i(a) = 0 = v_i(t_j)$ for any $a \in K$ and $j \in \Lambda$, $j \neq i$ and $v_i(t_i) = 1$. v_i naturally determines a valuation of $\overline{K}$ (see [**G**, (18.4)]) and let $\overline{V_i}$ be the valuation ring of $\overline{K}$ corresponding to v_i. It is clear that the rings $\overline{V_i}$ are incomparable to each other. Since $\hat{K}$ is algebraic over $\overline{K}$, there is an extension V_i of $\overline{V_i}$ to $\hat{K}$ for each $i \in \Lambda$ ([**E**, (13.2)]), and the rings V_i are incomparable. Now, since $\hat{R} = \hat{K}[Y, \hat{\sigma}]$ with a maximal ideal $\hat{P} = \hat{R}Y$, where $Y = p(X)$, we have the natural homomorphism $\varphi : \hat{R}_{\hat{P}} \to \hat{K}(\cong \hat{R}_{\hat{P}}/J(\hat{R}_{\hat{P}}))$. Let $A_i = \varphi^{-1}(V_i)$ be the complete inverse image of V_i by φ which are total valuation rings of $K(X; \sigma, \delta)$ such that $\hat{R}_{\hat{P}} \supseteq A_i$ by [**XKM₁**, Proposition 1.7 (3)] and it is clear that the A_i are incomparable to each other.

(ii) Let $G = \bigoplus \boldsymbol{Z}_i$ ($i \in \Lambda$ and $\boldsymbol{Z}_i = \boldsymbol{Z}$, the ring of integers) be a totally ordered abelian group by anti-lexicographical ordering, where Λ is consider as a well-ordered set. We define a valuation w of $\overline{K}$ as follows; $w(a) = 0$ for all $a \in K$ and $w(t_i) = g_i = (\dots, 0, 1, 0, \dots) \in G$, the i-th component is one and the other components are all zeros. Let W be the valuation ring of $\overline{K}$ determined by w. As in [**XKM₁**, Example 2.5], set $F_i = \{g \in G_+ \mid \text{either } g \geq g_i \text{ or } g_i > g$ but the i-th component of g is 1 and j-th components are all zeros if $j > i$ for each $i\}$, where $G_+ = \{g \in G \mid g \geq 0\}$. Then it is easy to check that F_i are prime filters (see [**G**, p. 196]). So, by [**G**, (17.8)], $P_i = \{k \in \overline{K} \mid w(k) \in F_i\} \cup \{0\}$ are prime ideals of W. From the definitions we easily see that $F_i \subsetneq F_j$ for any $i, j \in \Lambda$ with $i > j$ so that $P_i \subsetneq P_j$. Thus we have the valuation rings W_{P_i} ($i \in \Lambda$) of $\overline{K}$ which are well ordered; $W_{P_1} \subsetneq W_{P_2} \subsetneq \cdots \subsetneq \overline{K}$. Hence, as in (i), we have a set of total valuation rings B_i ($i \in \Lambda$) of $K(X; \sigma, \delta)$ such that $B_1 \subsetneq B_2 \subsetneq \cdots \subsetneq \hat{R}_{\hat{P}}$. □

We end this section with some examples to display some of the various phenomena we have discussed. We start with the following obtained by [**LM**, (2.8)]. However we shall give an elementary proof of it for the reader's convenience.

Proposition 1.12. *Let $\sigma \in \operatorname{Aut}(K)$ with $\sigma\delta = \delta\sigma$ and $\operatorname{char} K = 0$. If δ is not inner, then it is not a quasi-algebraic σ-derivation.*

Proof. Assume that δ is a quasi-algebraic σ-derivation. Let $p(X)$ be a monic invariant polynomial of minimal non-zero degree, say, $p(X) = X^n + a_{n-1}X^{n-1} + \cdots + a_0$ ($n > 1, a_i \in K$). Since $p(X)a = \sigma^n(a)p(X)$ for all $a \in K$, we have $\sigma^n(a)a_{n-1} = n\delta(\sigma^{n-1}(a)) + a_{n-1}\sigma^{n-1}(a)$ by comparison of the coefficients of X^{n-1}. Hence $\sigma(a)a_{n-1} = n\delta(a) + a_{n-1}a$ for all $a \in K$, because $\sigma \in \operatorname{Aut}(K)$ and so $\delta(a) = (-n^{-1}a_{n-1})a - \sigma(a)(-n^{-1}a_{n-1})$, an inner derivation which is a contradiction. □

We start off with the case $\sigma = 1$ and we immediately have the following from Proposition 1.12.

Example 1.1. Let F be a field with $\operatorname{char} F = 0$, $K = F(t)$ be a rational function field over F in an indeterminate t, $\sigma = 1$ and δ be the formal differentiation with respect to t. Then δ is not quasi-algebraic.

A. Leory provided us with the following result:

Example 1.2. Let F be a field with $\operatorname{char} F = p > 0$, $K = F(t)$, where t is an indeterminate over F, $\sigma = 1$ and let δ be the formal differentiation with respect to t. Then δ is quasi-algebraic and not inner.

Proof. Since $\delta^p = 0$ and $\operatorname{char} F = p$, it follows that X^p is invariant. For any monic polynomial $p(X) = X^n + a_{n-1}X^{n-1} + \cdots + a_0$ with $n < p$, if $p(X)$ is invariant, then we have δ is inner as in Proposition 1.12, a contradiction. Hence, X^p is a monic invariant polynomial of minimal non-zero degree. Thus δ is quasi-algebraic and not inner by Proposition 1.4. □

Example 1.3 ([**Le**]). There exists a division ring with automorphism σ such that $\circ(\sigma) < \infty$ and a quasi-algebraic σ-derivation δ which is not inner. In fact, $\sigma^2 = 1$, $\delta^2 = 0$, $\sigma\delta \neq \delta\sigma$ for an example in [**Le**].

To give examples of division rings such that $\circ(\sigma) = \infty$ and δ is a quasi-algebraic σ-derivation (or, δ is not a quasi-algebraic σ-derivation), let F be a field with $\sigma \in \operatorname{Aut}(F)$ and $K = F((t,\sigma))$, the quotient ring of the skew formal power series ring $F[[t,\sigma]]$. We can naturally extend σ to an automorphism of K which is given by $\sigma(\Sigma a_n t^n) = \Sigma(\sigma(a_n)t^n)$ for any $\Sigma a_n t^n \in K$. Now we define a map δ from K to K as follows; $\delta(\Sigma a_n t^n) = \Sigma n\sigma(a_n)t^{n+1}$. Then we easily have the following properties;

(1) δ is a σ-derivation.

(2) $\sigma\delta = \delta\sigma$.

Example 1.4. Under the same notation as in the above, suppose that $\circ(\sigma) = \infty$ for $\sigma \in \operatorname{Aut}(F)$ and $\sigma(a) \neq a - 1$ for all $a \in F$.

(1) $\circ(\sigma) = \infty$ as $\sigma \in \operatorname{Aut}(K)$ and δ is not inner.

(2) If $\operatorname{char} F = 0$, then δ is not a quasi-algebraic σ-derivation.

(3) If $\operatorname{char} F = p > 0$, then δ is a quasi-algebraic σ-derivation.

Proof. (1) It is easily checked that $\circ(\sigma) = \infty$. To prove that δ is not inner, on the contrary, assume that δ is inner. Then there is an element $\beta = (\sum_{n=k}^{\infty} b_n t^n)$ $(b_n \in F, b_k \neq 0, n \in \mathbf{Z})$ such that $\delta(\alpha) = \beta\alpha - \sigma(\alpha)\beta$ for all $\alpha \in K$. So for any $a \in F$, we have $0 = \delta(a) = \beta a - \sigma(a)\beta = (\sum_{n=k}^{\infty} b_n t^n)a - \sigma(a)(\sum_{n=k}^{\infty} b_n t^n)$. By comparison of degrees in the equation, we have $b_n\sigma^n(a) = \sigma(a)b_n$ for all $n = k, k+1, \ldots.$ Hence, $\beta = b_1 t$, and $t^2 = \delta(t) = b_1 tt - tb_1 t = (b_1 - \sigma(b_1))t^2$ and thus $\sigma(b_1) = b_1 - 1$, a contradiction. Hence δ is not inner.

(2) This follows from Proposition 1.12 and (1).

(3) We easily see from the definition of δ that $\delta^p = 0$, i.e., δ is quasi-algebraic. □

Next we will give an example of a field K such that $\hat{K}$ is algebraic over K.

Example 1.5. Let $K = F(t)$ and let σ be an endomorphism of K determined by $\sigma(a) = a$ for all $a \in F$ and $\sigma(t) = t^2$. Since $p(X) = X$ is a monic invariant polynomial of minimal non-zero degree in the skew polynomial ring $K[X;\sigma]$, $\hat{K} = \bigcup_{i=1}^{\infty} X^{-i}KX^i$ and $\hat{K}$ is algebraic over K.

Proof. Since $\sigma(t) = t^2$, it easily follows that $(X^{-1}tX)^2 = X^{-1}t^2X = t$, algebraic over K. Inductively, we have $(X^{-n}tX^n)^{2^n} = X^{-(n-1)}(X^{-1}t^2X)\cdots(X^{-1}t^2X)X^{n-1} = X^{-(n-1)}t^{2^{n-1}}X^{n-1} = t$, algebraic over K. And it is easy to check that $\hat{K}$ is generated by K and $X^{-n}tX^n$. Hence $\hat{K}$ is algebraic over K. □

Finally we will give an example of a field K such that $\hat{K}$ is not algebraic over K.

Example 1.6. Let $K = F(t_0, t_1, t_2, \ldots)$, where $t_0, t_1, t_2, \ldots$ are indeterminates. Let σ be an endomorphism of K determined by $\sigma(a) = a$ for all $a \in F$, $\sigma(t_i) = t_{i+1}$ for all $i \geq 0$ and $\delta = 0$. Then $\hat{K} = \bigcup_{i=1}^{\infty} X^{-i}KX^{i}$ is not algebraic over K and $\hat{K} = F(\ldots, t_{-1}, t_0, t_1, \ldots)$, where $t_{-n} = X^{-1}t_{-n+1}X$ for any natural number n.

Proof. Let $K_{-n} = F(t_{-n}, \ldots, t_{-1}, t_0, t_1, t_2, \ldots)$, a field generated by $t_{-n}, \ldots, t_{-1}, t_0, t_1, t_2, \ldots$ over F. Since σ induces an automorphism $\hat{\sigma}$ of $\hat{K}$ which is a conjugation by X, we have $\hat{\sigma}(t_{-n}) = t_{-n+1}$ for all natural number and so $\hat{\sigma}(K_{-n}) = K_{-n+1}$. We shall prove that t_{-n} is transcendental over K_{-n+1} for any n. Assume that t_{-1} is algebraic over K_0, say, $t_{-1}^{l} + a_{l-1}t_{-1}^{l-1} + \cdots + a_0 = 0$, where $a_i \in K_0$ and $l \geq 1$. Then $t_0^{l} + \sigma(a_{l-1})t_0^{l-1} + \cdots + \sigma(a_0) = \sigma(0) = 0$, where $\sigma(a_i) \in K_1$, and so t_0 is algebraic over K_1, a contradiction. We can prove that t_{-n} is transcendental over K_{-n+1} by the same way. In particular, $\hat{K}$ is transcendental over K. The second statement is clear, because $t_{-n} = X^{-n}t_0X^{n}$ for any natural number n. □

2. Non-commutative valuation rings of $K(X;\sigma,\delta)$ contained in $R^{(1)}$.

Let K be a division ring, σ be an endomorphism of K, δ be a σ-derivation and V be a total valuation ring of K. Throughout this section, we assume that (σ,δ) is compatible with V, i.e., $\sigma(V) \subseteq V$, $\sigma(J(V)) \subseteq J(V)$, $\delta(V) \subseteq V$, $\delta(J(V)) \subseteq J(V)$. In [**BT**], they proved that $J(V)[X;\sigma,\delta]$ is left localizable and $R^{(1)} = V[X;\sigma,\delta]_{J(V)[X;\sigma,\delta]}$, the localization of $V[X;\sigma,\delta]$ at $J(V)[X;\sigma,\delta]$, is a total valuation ring of $K(X;\sigma,\delta)$ with $R^{(1)} \cap K = V$, $X \in R^{(1)}$ and we studied some properties of $R^{(1)}$ (see [$\mathbf{XKM_2}$]). In this section we shall study non-commutative valuation rings B of $K(X;\sigma,\delta)$ such that $B \cap K = V$, $B \subseteq R^{(1)}$ and $X \in B$, which are the purpose of this paper. This will be done by combining the results in Section 1 and Proposition 2.1.

A left order S in a simple Artinian ring Q is said to be left *Prüfer* if any finitely generated left S-ideal in Q is a progenerator of S-Mod, the category of left S-modules ([**MMU**, §2]). We shall start with the following general case.

Proposition 2.1. *Let S be a Dubrovin valuation ring of a simple Artinian ring Q and $\varphi : S \to \bar{S} = S/J(S)$ be the natural homomorphism. Suppose that $\mathfrak{R}$ is a left order in $\bar{S}$ and let $R = \varphi^{-1}(\mathfrak{R})$ be the complete inverse image of $\mathfrak{R}$. Then*:

(1) *R is a left order in Q.*

(2) *R is a left Prüfer order in Q if and only if $\mathfrak{R}$ is a left Prüfer order in $\bar{S}$.*

(3) *Suppose that R is left Prüfer. Let $\wp$ be a prime ideal of $\mathfrak{R}$ and $P = \varphi^{-1}(\wp)$, a prime ideal of R. Then $\wp$ is left localizable if and only if P is left localizable. Furthermore $R_P = \varphi^{-1}(\mathfrak{R}_{\wp})$.*

(4) *R is a Dubrovin valuation ring if and only if $\mathfrak{R}$ is a Dubrovin valuation ring.*

(5) *Suppose that Q is a division ring and S is a total valuation ring of Q. Then R is a total valuation ring if and only if $\mathfrak{R}$ is a total valuation ring.*

Proof. (1) First, we shall prove that for any $s \in S$, there exists a $c \in U(S) \cap R$ with $cs \in R$, where $U(S)$ is the group of units in S. If $s \in J(S)$, then there is nothing to do, because $J(S) \subseteq R$. So we may assume that $s \notin J(S)$, then $\bar{s} = \bar{c}^{-1}\bar{r}$ for some $c, r \in R$ with $\bar{c} \in \mathscr{C}_{\mathfrak{R}}(0) = \{\bar{t} \in \mathfrak{R} \mid \bar{t} \text{ is regular}\}$, equivalently, $\bar{c}\bar{S} = \bar{S}$. So by Nakayama's Lemma, $c \in U(S)$ and $c \in R$, because $\bar{c} \in \mathfrak{R}$. Let $\mathscr{C} = \{c \in R \mid c \in \mathscr{C}_S(0)\}$, which is not empty.

Second, we shall prove that, for any $q \in Q$, there is a $c \in \mathscr{C}$ with $cq \in R$. Since there is an element $d \in \mathscr{C}_S(0)$ with $dq \in S$, there exists $d_1 \in U(S) \cap R$ with $d_1 dq \in R$. For this $d_1 d$, there is a $c_1 \in U(S) \cap R$ with $c_1 d_1 d \in R$. Hence $c = c_1 d_1 d \in \mathscr{C}$ and $cq \in R$.

Now, for any $r \in R$ and $c \in \mathscr{C}$, there is a $d \in \mathscr{C}$ such that $drc^{-1} = t \in R$ and so $dr = tc$, showing that $\mathscr{C}$ is a left Ore set of R. Now it is clear that $R_{\mathscr{C}} = \{c^{-1}r \mid c \in \mathscr{C}$ and $r \in R\} = Q$.

(2) This is proved by the exactly same method as in [**M**, (3.1)].

(3) It is easy to see that $\overline{\mathscr{C}_R(P)} = \mathscr{C}_{\mathfrak{R}}(\wp)$ and $\mathscr{C}_R(P) = \{c \in R \mid \bar{c} \in \mathscr{C}_{\mathfrak{R}}(\wp)\}$.

Furthermore, by the same way as in [**MMU**, (22.6)], we have $\mathscr{C}_R(P) \subseteq \mathscr{C}_R(0)$ and $\mathscr{C}_{\mathfrak{R}}(\wp) \subseteq \mathscr{C}_{\mathfrak{R}}(0)$. Suppose that P is left localizable. Then it is easy to see that $\wp$ is left localizable. To see that $\varphi(R_P) = \mathfrak{R}_\wp$, let $c \in \mathscr{C}_R(P)$. Then $\bar{c}\bar{S} = \bar{S}$ and so $c \in U(S)$. Thus it follows that $S \supseteq R_P$ and so $\varphi(R_P) = \mathfrak{R}_\wp$ follows easily. Conversely, suppose that $\wp$ is left localizable. For any $c \in \mathscr{C}_R(P)$ and $r \in R$, there are $t \in R$ and $d \in \mathscr{C}_R(P)$ with $\bar{d}\bar{r} = \bar{t}\bar{c}$. So $dr - tc = m \in J(S)$. Since $c \in U(S)$, we have $m = nc$ for some $n \in J(S)$. Hence $dr = (t+n)c$, showing that P is left localizable. It is easy to see that $R_P = \varphi^{-1}(\mathfrak{R}_\wp)$.

(4) By [**M**, (3.1)], R is Prüfer if and only if $\mathfrak{R}$ is Prüfer. So in both directions, it suffices to prove that $J(R) = \varphi^{-1}(J(\mathfrak{R}))$. To prove this, let I be a maximal right ideal of R. Then it is enought to prove that $I \supseteq J(S)$. On the contrary, suppose that $I \supsetneqq J(S)$. Then $I + J(S) = R$ and so $IS + J(S) = S$. Hence $IS = S$ and thus $I = IR \supseteqq IJ(S) = ISJ(S) = J(S)$, a contradiction. Hence we have $R/J(R) \cong \mathfrak{R}/J(\mathfrak{R})$, i.e., R is local if and only if $\mathfrak{R}$ is local. Therefore R is a Dubrovin valuation ring if and only if $\mathfrak{R}$ is a Dubrovin valuation ring.

(5) This follows from (4) and the proof of [**MMU**, (8.13)]. □

Remark. (1) The statement (1) in Proposition 2.1 is valid if S is a left order in Q and $\bar{S} = S/J(S)$ is a simple Artinian ring.

(2) It is tempting to conclude that R is an invariant valuation ring of a division ring Q if and only if $\mathfrak{R}$ and S are invariant. However, this is not necessarily true as it will be seen in Example 2.5.

Now let $\varphi : R^{(1)} = V[X;\sigma,\delta]_{J(V[X;\sigma,\delta])} \to \overline{R^{(1)}} = R^{(1)}/J(R^{(1)}) \cong \bar{V}(X;\bar{\sigma},\bar{\delta})$ be the natural homomorphism, where $\bar{\sigma}(\bar{v}) = [\sigma(v) + J(V)]$ and $\bar{\delta}(\bar{v}) = [\delta(v) + J(V)]$ for any $\bar{v} = [v + J(V)] \in \bar{V}$. Set $R = \varphi^{-1}(\bar{V}[X;\bar{\sigma},\bar{\delta}]) = V[X;\sigma,\delta] + J(R^{(1)})$, a left Prüfer order by Proposition 2.1, because $\bar{V}[X;\bar{\sigma},\bar{\delta}]$ is a left principal ideal domain.

We shall study non-commutative valuation rings B of $K(X;\sigma,\delta)$ such that $B \cap K = V$, $R^{(1)} \supseteq B$ and $X \in B$ by applying the results of Section 1 and Proposition 2.1 to the situation above. For simplicity, we denote by $\mathscr{D}$ the set of all Dubrovin valuation rings B of $K(X;\sigma,\delta)$ such that $B \cap K = V$, $R^{(1)} \supseteq B$ and $X \in B$.

Proposition 2.2. *There is a one-to-one correspondence between $\mathscr{D}$ and the set of all Dubrovin valuation rings $\mathfrak{B}$ of $\bar{V}(X;\bar{\sigma},\bar{\delta})$ with $\mathfrak{B} \supseteq \bar{V}[X;\bar{\sigma},\bar{\delta}]$, which is given by $\varphi(B) = \mathfrak{B}$ and $\varphi^{-1}(\mathfrak{B}) = B$, where $B \in \mathscr{D}$.*

Proof. Let $B \in \mathscr{D}$ with $B \neq R^{(1)}$. Then $B \supsetneqq J(R^{(1)})$ and $\varphi(B) = B/J(R^{(1)})$ is a Dubrovin valuation ring of $\overline{R^{(1)}}$ (see [**MMU**, (6.6)]).

Conversely, let $\mathfrak{B}$ be a Dubrovin valuation ring of $\overline{V}(X;\bar{\sigma},\bar{\delta})$ containing $\overline{V}[X;\bar{\sigma},\bar{\delta}]$. Then it is easy to see that $B=\varphi^{-1}(\mathfrak{B})\in\mathscr{D}$ and $\varphi(B)=\mathfrak{B}$ by Proposition 2.1. □

Proposition 2.3. *If $\bar{\delta}$ is not a quasi-algebraic $\bar{\sigma}$-derivation, then $\mathscr{D}=\{R^{(1)}\}$.*

Proof. Let $B\in\mathscr{D}$ with $B\neq R^{(1)}$. Then $\mathfrak{B}=\varphi(B)$ is a proper Dubrovin valuation ring of $\overline{V}(X;\bar{\sigma},\bar{\delta})$ containing $\overline{V}[X;\bar{\sigma},\bar{\delta}]$ and so $J(\mathfrak{B})\cap\overline{V}[X;\bar{\sigma},\bar{\delta}]$ is a non-zero ideal of $\overline{V}[X;\bar{\sigma},\bar{\delta}]$ which is a contradiction to Proposition 1.1. □

In the remainder of this section, we assume that $\bar{\delta}$ is a quasi-algebraic $\bar{\sigma}$-derivation and let $p(X)\in V[X;\sigma,\delta]$ is a monic polynomial such that $\overline{p(X)}\in\overline{V}[X;\bar{\sigma},\bar{\delta}]$ is a monic invariant polynomial of mininal non-zero degree (the existence of such $\overline{p(X)}$ is guaranteed by Proposition 1.1). In the case $\bar{\sigma}\in\mathrm{Aut}(\overline{V})$, we shall give a complete description of $\mathscr{D}$ as follows:

Theorem 2.4. *Suppose that $\bar{\delta}$ is a quasi-algebraic $\bar{\sigma}$-derivation and $\bar{\sigma}\in\mathrm{Aut}(\overline{V})$.*

(1) *If $\circ(\bar{\sigma})=\infty$, then $\mathscr{D}=\{R^{(1)},R_P\}$, where $P=Rp(X)$.*

(2) *If $\circ(\bar{\sigma})=m<\infty$, then $\mathscr{D}=\{R^{(1)},R_P,R_M\mid P=Rp(X),M=Rw(X)$, where $w(X)\in V[X;\sigma,\delta]$ such that $\overline{w(X)}$ is an irreducible polynomial of $Z(\overline{V})_{\bar{\sigma},\bar{\delta}}[Y](Y=\bar{\lambda}\overline{p(X)}^l$ for some $\bar{\lambda}\in\overline{V}$ and $l\geq 1$ as in Lemma 1.3)$\}$. In particular, $J(R_P)=R_Pp(X)=p(X)R_P$ and $J(R_M)=R_Mw(X)=w(X)R_M$.*

Proof. Since $p(X),w(X)\in U(R^{(1)})$, we easily have $P=\varphi^{-1}(\overline{V}[X;\bar{\sigma},\bar{\delta}]\overline{p(X)})=Rp(X)=p(X)R$ and $M=\varphi^{-1}(\overline{V}[X;\bar{\sigma},\bar{\delta}]\overline{w(X)})=Rw(X)=w(X)R$, where $R=V[X;\sigma,\delta]+J(R^{(1)})$. Hence the theorem follows from Propositions 2.1, 2.2 and Theorem 1.7. □

Remark. Under the same notation and assumptions as in Theorem 2.4, $J(R^{(1)})=\bigcap_{m=1}^{\infty}R_Pp(X)^m=\bigcap_{m=1}^{\infty}R_Mw(X)^m$.

Proof. Since $J(R^{(1)})$ is a prime ideal of R_P, we have $J(R^{(1)})\subseteq\bigcap_{m=1}^{\infty}R_Pp(X)^m$. So $0=\varphi(J(R^{(1)}))\subseteq\bigcap_{m=1}^{\infty}\varphi(R_P)\overline{p(X)}^m=0$, because $\varphi(R_P)$ is a Noetherian Dubrovin valuation ring. Hence $J(R^{(1)})=\bigcap_{m=1}^{\infty}R_Pp(X)^m$ and similarly, $J(R^{(1)})=\bigcap_{m=1}^{\infty}R_Mw(X)^m$. □

The property in the Remark above will characterize R_P and R_M as follows:

Theorem 2.5. *Suppose that $\bar{\delta}$ is a quasi-algebraic $\bar{\sigma}$-derivation and $\bar{\sigma}\in\mathrm{Aut}(\overline{V})$. If B is a Dubrovin valuation ring of $K(X;\sigma,\delta)$ such that $B\cap K=V$, $X\in B$ and $J(B)=Bg(X)=g(X)B$ for some $g(X)\in V[X;\sigma,\delta]$ with $J(R^{(1)})=\bigcap_{m=1}^{\infty}Bg(X)^m$, then either $B=R_P$ or $B=R_M$, where P and M are as in Theorem 2.4.*

Proof. First note that if S is a Dubrovin valuation ring of $K(X;\sigma,\delta)$ such that $S\cap K=V$, $S\supseteq R^{(1)}$, then $S=R^{(1)}$. Assume that $S\supsetneq R^{(1)}$. Then $J(S)\subsetneq J(R^{(1)})$, and so $J(V)S=J(V)R^{(1)}S=J(R^{(1)})S=S$. Write $1=vs$ for some $v\in J(V)$ and $s\in S$. Then $s=v^{-1}\in K\cap S=V$, a contradiction. Hence $S=R^{(1)}$ follows. Now, let B be a Dubrovin valuation ring of $K(X;\sigma,\delta)$ satisfying the conditions in Theorem 2.5. Then by [**BMO**, theorem 5], $J(R^{(1)})$ is Goldie prime, i.e., a prime ideal of B such that $B/J(R^{(1)})$ is a Goldie ring. So $B_{J(R^{(1)})}$ is a Dubrovin valuation ring by [**MMU**, (14.5)]. Since any $c(X) \in V[X;\sigma,\delta]\backslash J(V)[X;\sigma,\delta]$ is a unit in $R^{(1)}$, it easily follows that $\mathscr{C}_B(J(R^{(1)})) \supseteq V[X;\sigma,\delta]\backslash J(V)[X;\sigma,\delta]$ and so $B_{J(R^{(1)})} \supseteq R^{(1)}$. Let $W = B_{J(R^{(1)})} \cap K$ and we want to show that $V = W$. $B \supseteq J(B_{J(R^{(1)})})$ implies that $J(B_{J(R^{(1)})}) = J(R^{(1)})$. Thus $J(V) = J(R^{(1)}) \cap K = J(B_{J(R^{(1)})}) \cap K = J(W)$ and so $V = W$ follows. Hence $R^{(1)} = B_{J(R^{(1)})} \supseteq B$. Thus we have either $B = R_P$ or $B = R_M$, because $B \supseteq R = V[X;\sigma,\delta] + J(R^{(1)})$. □

If $\bar{\sigma} \notin \mathrm{Aut}(\overline{V})$, then as in Section 1, let $\hat{\overline{V}} = \bigcup_{m=1}^{\infty} \overline{p(X)}^{-m}\,\overline{V}\,\overline{p(X)}^{m}$, a division ring containing $\overline{V}$ and $\hat{\overline{R}} = \bigcup_{m=1}^{\infty} \overline{p(X)}^{-m}\overline{R}\,\overline{p(X)}^{m}$, where $\overline{R} = \varphi(R) = \overline{V}[X;\bar{\sigma},\bar{\delta}]$. Then $\hat{\overline{R}} = \hat{\overline{V}}[X;\hat{\bar{\sigma}},\hat{\bar{\delta}}]$ for some $\hat{\bar{\sigma}} \in \mathrm{Aut}(\hat{\overline{V}})$, and $\hat{\bar{\delta}}$ is a $\hat{\bar{\sigma}}$-derivation by Proposition 1.8. Since $\overline{R}\,\overline{p(X)} \supsetneq \overline{p(X)}\,\overline{R}$, we have $Rp(X) = V[X;\sigma,\delta]p(X) + J(R^{(1)}) \supsetneq p(X)R$ and so we can construct $\hat{R} = \bigcup_{m=1}^{\infty} p(X)^{-m}Rp(X)^{m}$, an over ring of R. It is easy to see that $\hat{R} = \varphi^{-1}(\hat{\overline{R}})$, a Prüfer order in $K(X;\sigma,\delta)$ by Proposition 2.1. It follows from Proposition 1.8 that $\hat{\overline{P}} = \hat{\overline{R}}\,\overline{p(X)}$ is the unique maximal ideal of $\hat{\overline{R}}$ and so $\hat{R}_{\hat{P}} \in \mathscr{D}$, by Propositions 2.1 and 2.2, where $\hat{P} = \varphi^{-1}(\hat{\overline{P}})$. Hence we have:

Theorem 2.6. *Suppose that $\bar{\delta}$ is a quasi-algebraic $\bar{\sigma}$-derivation and $\bar{\sigma} \notin \mathrm{Aut}(\overline{V})$.*

(1) *$\hat{R}_{\hat{P}} \in \mathscr{D}$.*

(2) *If $\hat{\overline{V}}$ is algebraic over $\overline{V}$ and $\hat{\overline{P}}$ is completely prime, then $R^{(1)}$ and $\hat{R}_{\hat{P}}$ are only total valuation rings of $K(X;\sigma,\delta)$ in $\mathscr{D}$.*

(3) *If $\hat{\overline{V}}$ is not algebraic over $\overline{V}$ and $\overline{V}$ is a field with $\alpha = \mathrm{tr.deg}_{\overline{V}}\,\hat{\overline{V}}$, then there are $\{A_i\}, \{B_i\} \subseteq \mathscr{D}$ satisfying;*

(i) *$\hat{R}_{\hat{P}} \supsetneq A_i$ for any $i \in \Lambda$ and A_i are incomparable each other, where Λ is an index set as in Proposition* 1.11.

(ii) *$B_1 \subsetneq B_2 \subsetneq \cdots \subsetneq \hat{R}_{\hat{P}}$.*

Proof. (1) This was proved in the paragraph before Theorem 2.6.

(2) This follows from Propositions 1.9 and 2.2.

(3) This follows from Propositions 1.11 and 2.2. □

Let K_0 be a division ring and let δ be a σ-derivation with $\sigma\delta = \delta\sigma$, where σ is an endomorphism of K_0. Further, let $R = K_0[t]$ be the polynomial ring over K_0 in an indeterminate t with $at = ta$ for any $a \in K_0$, $P = tK_0[t]$, a maximal ideal of R and $V = K_0[t]_P$, the localization of R at P, is a Noetherian total valuation ring. We naturally extend σ,δ to $K = K_0(t)$ as follows; $\sigma(f(t)) = \sigma(a_0) + \sigma(a_1)t + \cdots + \sigma(a_n)t^n$ and $\delta(f(t)) = \delta(a_0) + \delta(a_1)t + \cdots + \delta(a_n)t^n$ for any $f(t) = a_0 + a_1t + \cdots + a_nt^n \in K_0[t]$.

Proposition 2.7. *Under the same notation and assumptions as the above, (σ,δ) is compatible with V and $\overline{V} \cong K_0$ naturally.*

Proof. Since $\sigma\delta = \delta\sigma$, it is easily checked that δ is a σ-derivation on K. It is also clear that $\sigma(K_0[t]) \subseteq K_0[t]$, $\sigma(tK_0[t]) \subseteq tK_0[t]$, $\delta(K_0[t]) \subseteq K_0[t]$, $\delta(tK_0[t]) \subseteq tK_0[t]$. So, to prove that (σ,δ) is compatible with V, it suffices to prove that $\delta(c(t)^{-1}) \in V$ for any $c(t) \in K_0[t]\backslash tK_0[t]$, because $J(V) = tV$. Since $0 = \delta(c(t)c(t)^{-1}) = \sigma(c(t))\delta(c(t)^{-1}) + \delta(c(t))c(t)^{-1}$, we have $\delta(c(t)^{-1}) = -\sigma(c(t)^{-1})\delta(c(t))c(t)^{-1} \in V$. The last statement is clear. □

As all examples in Section 1 satisfies $\sigma\delta = \delta\sigma$ except for Example 1.3, applying Proposition 2.7, we have:

Example 2.1. Let K_0 be a division ring which is one of the examples in Section 1, except for Example 1.3, $K = K_0(t)$ be the non-commutative rational function ring in an indeterminate t with $at = ta$ for any $a \in K_0$ and let V be as in Proposition 2.7. Then $(\bar{\sigma}, \bar{\delta})$ satisfies the same properties as that of (σ, δ) in Section 1.

Next we will give some examples of total valuation rings such that (σ, δ) is compatible with V but $(\bar{\sigma}, \bar{\delta})$ have different properties from (σ, δ). Let K_0 be a division ring with $\sigma \in \mathrm{Aut}(K_0)$, $K = K_0(t)$ and $V = K_0[t]_P$, where $P = tK_0[t]$, as before. For any $f(t) = a_0 + a_1 t + \cdots + a_n t^n \in K_0[t]$, we define $\sigma(f(t)) = \sigma(a_0) + \sigma(a_1)t^2 + \cdots + \sigma(a_n)t^{2^n}$, $\delta(f(t)) = tf(t) - \sigma(f(t))t$, an inner σ-derivation and extend σ, δ to K naturaly. It is easy to check that σ is an endomorphism of K but not automorphism and that (σ, δ) is compatible with V. Since $\overline{V} \cong K_0$ naturally, we have $\bar{\sigma} \in \mathrm{Aut}(\overline{V})$ and $\circ(\bar{\sigma}) = n < \infty$ if $\circ(\sigma) = n$ as an automorphism of K_0. To show that $\bar{\delta} = 0$, let $g^{-1}f \in V$, where $f \in K_0[t]$ and $g \in K_0[t] \backslash tK_0[t]$. Since $\delta(g^{-1}f) = \sigma(g^{-1})\delta(f) + \delta(g^{-1})f$ and $\sigma(g^{-1})\delta(f) \in J(V)$, it suffices to prove that $\delta(g^{-1}) \in J(V)$. However, $0 = \delta(gg^{-1}) = \sigma(g)\delta(g^{-1}) + \delta(g)g^{-1}$ implies $\delta(g^{-1}) = -\sigma(g)^{-1}\delta(g)g^{-1} \in J(V)$, because $\delta(g) \in J(V)$ and $\sigma(g)^{-1} \in V$. Hence $\bar{\delta} = 0$. Summarizing we have

Example 2.2. Under the same notation and assumptions as the above, we have

(1) (σ, δ) is compatible with V.

(2) $\sigma \notin \mathrm{Aut}(V)$ but $\bar{\sigma} \in \mathrm{Aut}(\overline{V})$ and $\delta \neq 0$ but $\bar{\delta} = 0$. In particular, $\circ(\bar{\sigma}) = n < \infty$ if $\circ(\sigma) = n < \infty$ as an automorphism of K_0.

Let K_0 be a division ring with $\sigma \in \mathrm{Aut}(K_0)$ and $\circ(\sigma) = \infty$ (or $\sigma \neq 1$ if K_0 is a field) and let δ be a σ-derivation of K_0 with $\sigma\delta = \delta\sigma$. Further, let $K = K_0((t, \sigma))$ be the quotient division ring of the skew formal power series ring $V = K_0[[t, \sigma]]$ which is a Noetherian total valuation ring such that $\overline{V} \cong K_0$. As in Proposition 2.7, we extend σ, δ to K as follows;

$$\sigma(\Sigma a_n t^n) = \Sigma \sigma(a_n) t^n$$

and

$$\delta(\Sigma a_n t^n) = \Sigma \delta(a_n) t^n \quad \text{for any } \Sigma a_n t^n \in K.$$

It is easily checked that $\sigma\delta = \delta\sigma$ and that (σ, δ) is compatible with V. Suppose that δ is inner as a derivation of K_0, i.e., $\delta(a) = a_0 a - \sigma(a)a_0$ for all $a \in K_0$ and some $a_0 \in K_0$. Then we will prove that δ is inner in K if and only if $\sigma(a_0) = a_0$. If $\sigma(a_0) = a_0$, then it is easily checked that δ is inner in K induced by a_0. Conversely, assume that there is an element $\alpha_0 = \Sigma b_n t^n \in K$ such that $\delta(\alpha) = \alpha_0\alpha - \sigma(\alpha)\alpha_0$ for all $\alpha \in K$. For all $a \in K$, we have $a_0 a - \sigma(a)a_0 = \delta(a) = \alpha_0 a - \sigma(a)\alpha_0 = \Sigma_n (b_n \sigma^n(a) - \sigma(a)b_n)t^n$. Thus $a_0 a - \sigma(a)a_0 = b_0 a - \sigma(a)b_0$ and so $(a_0 - b_0)a = \sigma(a)(a_0 - b_0)$. Hence $a_0 = b_0$ follows, because $\circ(\sigma) = \infty$ (or $\sigma \neq 1$ if K_0 is a field). Similarly, $0 = b_n\sigma^n(a) - \sigma(a)b_n$ implies $b_n = 0$ for any $n \neq 0$, $n \neq 1$ and $b_1\sigma(a) = \sigma(a)b_1$. So $\alpha_0 = a_0 + b_1 t$ follows. Hence, in particular, $(a_0 a - \sigma(a)a_0)t = \delta(a)t = \delta(at) = (a_0 + b_1 t)at - \sigma(a)t(a_0 + b_1 t) = (a_0 a - \sigma(a)\sigma(a_0))t + (b_1\sigma(a) - \sigma(a)\sigma(b_1))t^2$ and so $\sigma(a_0) = a_0$ follows. Hence we have:

Example 2.3. Under the same notation and assumptions as the above, we take $a_0 \in K_0$ with $\sigma(a_0) \neq a_0$. Then (σ, δ) is compatible with V and δ is not inner but $\bar{\delta}$ is inner.

We will give an example of a valuation ring V of a field K such that $\hat{K}$ is transcendental over K but $\hat{\bar{V}}$ is algebraic over $\bar{V}$.

Example 2.4. Let $K = F(t, t_0, t_1, t_2, \ldots)$ be a rational function field over a field F in indeterminates $t, t_0, t_1, t_2, \ldots$, let σ be an endomorphism of K determined by $\sigma(a) = a$ for all $a \in F$, $\sigma(t) = t^2$, $\sigma(t_i) = t_{i+1}$ for all $i \geq 0$ and $\delta = 0$. Further, let $G = \bigoplus \boldsymbol{Z}_i$ ($\boldsymbol{Z}_i = \boldsymbol{Z}, i = 0, 1, 2 \ldots$) be the totally ordered abelian group ordered by lexicographical ordering and define $v(t) = 0 = v(a)$ for all non-zero $a \in F$ and $v(t_i) = (0, \ldots, 0, 1, 0, \ldots)$, the i-th component is one and the other components are all zeros. Let V be a valuation ring of K determined by v (see [**G**, (18.4)]). We shall prove that $\bar{V} \cong K_0 = F(t)$. Let $\alpha = fg^{-1}$ be any element of K, where $f = a_0 + a_1\mathrm{t}_1 + \cdots + a_k\mathrm{t}^k, g = b_0 + b_1\mathrm{t}_1 + \cdots + b_k\mathrm{t}^k$, $a_i, b_i \in K_0$ (some a_i, b_i may be zeros if necessary) and t_i are monomials, e.g., $\mathrm{t}_i = t_0^{l_0} \cdots t_m^{l_m}$. Now assume that $v(\alpha) = 0$, equivalently, $v(f) = v(g)$. Thus, by definition of v, we have $a_0 \neq 0$ if and only if $b_0 \neq 0$. In the case where $a_0 = 0$, since $v(f) = \min\{v(\mathrm{t}_i)\}$, we may assume that $v(f) = v(\mathrm{t}_1)$. Then $f\mathrm{t}_1^{-1} = a_1 + a_2\mathrm{t}_2\mathrm{t}_1^{-1} + \cdots + a_k\mathrm{t}_k\mathrm{t}_1^{-1}$ with $v(\mathrm{t}_i\mathrm{t}_1^{-1}) > 0$ for any $i \neq 1$ and $\alpha = (f\mathrm{t}_1^{-1})(g\mathrm{t}_1^{-1})^{-1}$. Thus, in any case, we may assume that $f = f_0 + f_1$ and $g = g_0 + g_1$, where $f_0, g_0 \in K_0$, $f_0 \neq 0$, $g_0 \neq 0$ and $f_1, g_1 \in J(V)$. Since $g - g_0 = g_1 \in J(V)$ and $g_0 \in U(V)$, it follows that $gg_0^{-1} - 1 \in J(V)$ and so $g \in U(V)$. Thus in particular, $g_0^{-1} - g^{-1} \in J(V)$. Hence $\alpha - f_0g_0^{-1} = (f - f_0)g^{-1} + f_0(g^{-1} - g_0^{-1}) \in J(V)$ and so $[\alpha + J(V)] = [f_0g_0^{-1} + J(V)]$. It is clear that, for any $\alpha, \beta \in K_0$, $[\alpha + J(V)] = [\beta + J(V)]$ if and only if $\alpha = \beta$. Hence it easily follows that $\bar{V} \cong F(t)$ with $\bar{\sigma}([t + J(V)]) = [t^2 + J(V)]$ and so $\hat{\bar{V}}$ is algebraic over $\bar{V}$ by Example 1.5. Further, let $t_{-1} = X^{-1}t_0X$ as in Example 1.6. Then we have t_{-1} is transcendental over K by the exactly same way as in Example 1.6. Hence $\hat{K}$ is transcendental over K.

As we have noticed in the remark to Proposition 2.1, R is not necessarily invariant despite S and $\mathfrak{R}$ are invariant. Finally we shall give such an example.

Example 2.5. Let F be a field with $\sigma \in \mathrm{Aut}(F)$ and V be a valuation ring of F with either $\sigma(V) \not\subseteq V$ or $\sigma(V) \subsetneq V$ (see [**XKM$_1$**, Examples 2.2, 2.3, 2.5, 2.6 and 2.7]). Then $S = F[X,\sigma]_{(X)}$, the localization of $F[X,\sigma]$ at the maximal ideal $(X) = XF[X,\sigma]$, is an invariant valuation ring of $F(X,\sigma)$. Let $\varphi : S \to \bar{S} = S/J(S)$ ($\cong F$) be the natural homomorphism and let $R = \varphi^{-1}(V)$. Then R is not invariant by [**XKM$_1$**, (1.7)], though S and V are both invariant.

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