J. Math. Soc. Japan
Vol. 56, No. 3, 2004

The simplest quartic fields with ideal class groups of exponents less than or equal to 2

By Stéphane R. LOUBOUTIN

(Received Nov. 5, 2001)
(Revised Feb. 10, 2003)

Abstract. The simplest quartic fields are the real cyclic quartic number fields defined by the irreducible quartic polynomials $x^4 - mx^3 - 6x^2 + mx + 1$, where m runs over the positive rational integers such that the odd part of $m^2 + 16$ is squarefree. We give an explicit lower bound for their class numbers which is much better than the previous known ones obtained by A. Lazarus. Then, using it, we determine the simplest quartic fields with ideal class groups of exponents ≤ 2.

1. Introduction.

For any positive rational integer m such that the odd part of $m^2 + 16$ is square-free, the quartic polynomial $x^4 - mx^3 - 6x^2 + mx + 1$ defines a real cyclic quartic number field K_m (see section 3). These fields K_m are called the *simplest quartic fields*. By using Stark's effective versions of the Brauer-Siegel theorem (see [**Sta**]), A. Lazarus obtained lower bounds for the class numbers h_{K_m} of these simplest quartic fields K_m and determined all these K_m's with $h_{K_m} \leq 2$ for m even. However, in the case that m is odd he could only prove that $m \leq 10^{14}$ if $h_{K_m} = 1$, which is of no practical use for the determination of all these K_m's with $h_{K_m} = 1$. First, we will obtain in Theorem 9 a much better lower bound for the *relative class numbers* $h_{K_m}^* := h_{K_m}/h_{k_m}$ of the simplest quartic fields K_m than his, where h_{k_m} is the class number of the quadratic subfield k_m of K_m. By our lower bound, we obtain $m \leq 381$ if $h_{K_m} = 1$ and $m \leq 649$ if $h_{K_m} \leq 2$, and therefore we can easily complete the determination of all these K_m's with $h_{K_m} \leq 2$. Next, we will explain why our lower bound for $h_{K_m}^*$ proves that there are only finitely many simplest quartic fields K_m whose ideal class groups have exponents ≤ 2 and we will determine all such K_m (see [**Lou1**] for the solution to the same problem for the imaginary cyclic quartic fields):

THEOREM 1. *There exist exactly* 22 *simplest quartic fields* K_m *whose ideal class groups are of exponents* ≤ 2. *This is the case if and only if* $m \in \{1, 2, 4, 5, 6, 8, 9, 10, 11, 15, 24\}$, *in which cases* $h_{K_m} = 1$, $m \in \{7, 12, 13, 16, 20\}$, *in which cases* $h_{K_m} = 2$, *or* $m \in \{17, 19, 23, 27, 39, 45\}$, *in which cases* $h_{K_m} = 4$.

For preliminaries, in the next section 2, we will prove some facts on general real cyclic quartic fields. We will obtain a lower bound for the products of their relative class numbers and relative regulators (see Theorem 4), and we will give a necessary condition for their ideal class groups to have exponents ≤ 2 (see Lemma 5). In section

---

2000 *Mathematics Subject Classification*. Primary: 11R16, 11R29, 11R42, 11Y40.
Key Words and Phrases. quartic field, simplest quartic field, class number, class group, zeta function.

3, we then apply these facts to the simplest quartic fields to prove Theorem 1. Our methods would work also for other families of real cyclic quartic fields, like the one treated in [**Wa**].

To conclude this introduction, we would like to thank the referee for her/his careful reading of the preliminary versions of this paper.

2. Real cyclic quartic fields.

Let K be a real cyclic quartic field and k be its real quadratic subfield. Let d_K, f_K, Cl_K, h_K, U_K, Reg_K and $\mathrm{Res}_{s=1}(\zeta_K)$ (resp. d_k, f_k, Cl_k, h_k, U_k, $\mathrm{Reg}_k = \log\varepsilon_k$ and $\mathrm{Res}_{s=1}(\zeta_k)$) be the discriminant, conductor, ideal class group, class number, unit group, regulator and residue at $s=1$ of the Dedekind zeta function ζ_K of K (resp. k), where $\varepsilon_k > 1$ is the fundamental unit of k. Let σ be a generator of the Galois group of K. Finally, let $U_K^* = \{\varepsilon \in U_K; N_{K/k}(\varepsilon) \in \{\pm 1\}\}$ denote the so-called group of *relative units* of K. If $\pm 1 \neq \varepsilon \in U_K^*$, then $\varepsilon^\sigma \in U_K^*$ and

$$\mathrm{Reg}(\varepsilon_k, \varepsilon, \varepsilon^\sigma) = 2\,\mathrm{Reg}_k\,\mathrm{Reg}_\varepsilon^*,$$

where

$$\mathrm{Reg}_\varepsilon^* := \log^2|\varepsilon| + \log^2|\varepsilon^\sigma| > 0.$$

It is known that there exists some so-called *generating relative unit* $\varepsilon_* \in U_K^*$ such that $\{-1, \varepsilon_*, \varepsilon_*^\sigma\}$ generate U_K^* (see [**Gras**] and [**Has**, Satz 22]), and we set

$$\mathrm{Reg}_K^* := \mathrm{Reg}_{\varepsilon^*}^* = \log^2|\varepsilon_*| + \log^2|\varepsilon_*^\sigma| > 0.$$

By the following Lemma, this does not depend on the choice of the generating relative unit ε_*:

LEMMA 2 (See [**Has**, Satz 16]). *It holds that* $\mathrm{Reg}(\varepsilon_k, \varepsilon_*, \varepsilon_*^\sigma) = 2\,\mathrm{Reg}_k\,\mathrm{Reg}_K^* = Q_K\,\mathrm{Reg}_K$ *for some* $Q_K \in \{1, 2\}$ *such that* $Q_K = 2$ *if and only if* $\langle -1, N_{K/k}(U_K)\rangle = U_k$.

PROOF. Noting that $N_{K/k}(\eta) = \eta^2$ for $\eta \in U_k$, we obtain that the kernel of

$$U_K \xrightarrow{N_{K/k}} U_k \longrightarrow U_k/\langle -1, U_k^2\rangle$$

is equal to $U_k U_K^*$. Hence, the index $Q_K := (U_K : U_k U_K^*)$ divides 2. □

Since $f_k > 1$ divides f_K and $d_K = f_k f_K^2$ (by the conductor-discriminant formula), we cannot have $d_K = d_k^2$ $(= f_k^2)$. Hence, K/k is ramified and h_k divides h_K (by class field theory). Hence, $h_K^* := h_K/h_k$ is a positive integer that divides h_K, which we call the *relative class number* of K. According to the analytic class number formula, we have

$$h_K^* = \frac{Q_K f_K}{8\,\mathrm{Reg}_K^*}|L(1, \chi_K)|^2, \tag{1}$$

where χ_K is any one of the two conjugate quartic Dirichlet characters associated with K (note that $d_K/d_k = f_K^2$). Our first aim is to obtain an explicit lower bound for $h_K^*\,\mathrm{Reg}_K^*$ (see Theorem 4 below). Then, using an upper bound for $\mathrm{Reg}_{K_m}^*$ (see Proposition 8 below), we will obtain a lower bound for $h_{K_m}^*$ (see Theorem 9 below).

LEMMA 3. *Set* $\kappa = 2 + \gamma - \log(4\pi) = 0.046\cdots$, *where* $\gamma = 0.577\cdots$ *denotes Euler's constant.*

1. *Let* K *be a totally real quartic number field of discriminant* $d_K \geq 6 \cdot 10^{12}$. *Then,* $\zeta_K(1 - (2/\log d_K)) \leq 0$ *implies*

$$\mathrm{Res}_{s=1}(\zeta_K) \geq \frac{2}{e \log d_K}, \tag{2}$$

and $1 - (2/\log d_K) \leq \beta < 1$ *and* $\zeta_K(\beta) = 0$ *imply*

$$\mathrm{Res}_{s=1}(\zeta_K) \geq \frac{1-\beta}{4e}. \tag{3}$$

2. (*See* [**Lou3**, Corollaire 5A(a)]). *Let* k *be a real quadratic number field. Then,*

$$\mathrm{Res}_{s=1}(\zeta_k) \leq \frac{1}{2}(\log d_k + \kappa). \tag{4}$$

Moreover (*see* [**Lou3**, Corollaire 7B]), $1/2 \leq \beta < 1$ *and* $\zeta_k(\beta) = 0$ *imply*

$$\mathrm{Res}_{s=1}(\zeta_k) \leq \frac{1-\beta}{8} \log^2 d_k. \tag{5}$$

PROOF. We need only to prove (2) and (3). According to [**Lou5**, proof of Lemma 3], $1/2 \leq \beta < 1$ and $\zeta_K(\beta) \leq 0$ imply

$$\mathrm{Res}_{s=1}(\zeta_K) \geq (1-\beta) d_K^{(\beta-1)/2} (1 + \lambda_4(1-\beta)) \left(1 - \frac{8 d_K^{(1-\beta)/8}}{d_K^{1/8}}\right), \tag{6}$$

where $\lambda_4 = 2(\gamma + \log(4\pi)) - 1 = 5.216\cdots$.

To obtain (2), we choose $\beta = 1 - (2/\log d_K)$ in (6) and note that

$$g(x) := \left(1 + \frac{2\lambda_4}{\log x}\right)\left(1 - \frac{8e^{1/4}}{x^{1/8}}\right)$$

satisfies $g(x) \geq 1$ for $x \geq 6 \cdot 10^{12}$.

To obtain (3), we use (6) to obtain

$$\mathrm{Res}_{s=1}(\zeta_N) \geq \frac{1-\beta}{e} h(d_N),$$

where $h(x) := 1 - 8e^{1/4} x^{-1/8}$ satisfies $h(x) \geq 1/4$ for $x \geq 2 \cdot 10^9$. □

THEOREM 4. *Let* K *be a real cyclic quartic field and let* k *be its real quadratic subfield. If* $d_K \geq 6 \cdot 10^{12}$, *then we have*

$$h_K^* \mathrm{Reg}_K^* \geq \frac{Q_K \sqrt{d_K/d_k}}{2e(\log d_K)(\log d_k + \kappa)} \geq \frac{f_K}{6e(\log f_K + \kappa/2)^2}, \tag{7}$$

where κ *is as in* Lemma 3.

PROOF. First, assume that $\zeta_k(1 - (2/\log d_K)) \leq 0$. Since $\zeta_K(s)/\zeta_k(s) = |L(s, \chi_K)|^2$ for s real, we have $\zeta_K(1 - (2/\log d_K)) \leq 0$ and using (2) and (4) we obtain

$$|L(1,\chi_K)|^2 = \frac{\mathrm{Res}_{s=1}(\zeta_K)}{\mathrm{Res}_{s=1}(\zeta_k)} \geq \frac{4}{e(\log d_K)(\log d_k + \kappa)}. \tag{8}$$

Second, assume that $\zeta_k(1-(2/\log d_K)) > 0$. Then, there exists β in the range $1-(2/\log d_K) \leq \beta < 1$ such that $\zeta_k(\beta) = 0$. Therefore, $\zeta_K(\beta) = 0 \leq 0$ and using (3), (5) and $d_K \geq d_k^2$, we obtain

$$|L(1,\chi_K)|^2 = \frac{\mathrm{Res}_{s=1}(\zeta_K)}{\mathrm{Res}_{s=1}(\zeta_k)} \geq \frac{2}{e\log^2 d_k} \geq \frac{4}{e(\log d_K)(\log d_k)}. \tag{9}$$

Since the right hand side of (9) is greater than the right hand side of (8), we conclude that (8) is always valid and, using (1), we obtain the desired result. □

The following Lemma will be used in Proposition 11 to prove that there are only finitely many simplest quartic fields (to be defined below) with ideal class groups of exponent ≤ 2.

LEMMA 5. *Let K be a real cyclic quartic field. Let k denote its real quadratic subfield. If the exponent* $\exp(\mathrm{Cl}_K)$ *of the ideal class group* Cl_K *of K is ≤ 2, then* $\exp(\mathrm{Cl}_k) \leq 2$, $h_K^* \leq 2^{T+t-2}$ *and $t \leq 2$, where T (resp. t) is the number of prime ideals of k ramified in K/k (resp. in $k/\boldsymbol{Q}$).*

PROOF. Let $N : \mathrm{Cl}_K \to \mathrm{Cl}_k$ and let $j : \mathrm{Cl}_k \to \mathrm{Cl}_K$ denote the norm and canonical map, and let $\mathrm{Cl}_{K/k}^{amb}$ denote the subgroup of the ambiguous classes of K (the ideal classes $\mathscr{C} \in \mathrm{Cl}_K$ which satisfy $\mathscr{C}^\sigma = \mathscr{C}$). Recall that $\#\mathrm{Cl}_{K/k}^{amb}$ divides $2^{T-1}h_k$ (see [**Lang**, Chapter 13, Lemma 4.1, page 307]). Since at least one finite place of k is ramified in K/k (the rational primes which are ramified in $k/\boldsymbol{Q}$ are totally ramified in $K/\boldsymbol{Q}$), the norm map N is onto, which proves the first assertion, and $\#\ker N = h_K/h_k = h_K^*$. If $\exp(\mathrm{Cl}_K) \leq 2$, then $\exp(\mathrm{Cl}_k) \leq 2$, which implies that h_k divides 2^{t-1}, $\ker j \circ N = \mathrm{Cl}_{K/k}^{amb}$ (for $j \circ N(\mathscr{C}) = \mathscr{C}\mathscr{C}^\sigma = \mathscr{C}^{\sigma-1}$) and $h_K^* = \#\ker N$ divides $\#\ker j \circ N = \#\mathrm{Cl}_{K/k}^{amb}$, hence divides $2^{T-1}h_k$, hence divides 2^{T+t-2}.

Finally, let G_K^+ denote the maximal real subfield of the genus field G_K of K. Then, G_K^+/K is an unramified abelian extension and the 4-rank of the ideal class group of K is greater than or equal to the 4-rank of the Galois group $\mathrm{Gal}(G_K^+/K)$, by class field theory. Since the 4-rank of $\mathrm{Gal}(G_K/K)$ is equal to $t-1$ and since the degree of the extension G_K/G_K^+ is equal to 1 if G_K is real and to 2 is G_K is imaginary, the 4-rank of $\mathrm{Gal}(G_K^+/K)$ is $\geq t-2$, and the proof is complete. □

3. Simplest quartic fields.

For any rational integer m, we consider the quartic polynomial

$$P_m(x) = x^4 - mx^3 - 6x^2 + mx + 1$$

of discriminant $d_m = 4\Delta_m^3$, where

$$\Delta_m := m^2 + 16.$$

Since $P_m(-x) = P_{-m}(x)$, we may and we will assume that $m \geq 0$. The reader will easily check that $P_m(x)$ has no rational root (for $P_m(\pm 1) = -4 \neq 0$), and that $P_m(x)$ is

$\boldsymbol{Q}$-irreducible, except for $m \in \{0,3\}$ (in which cases we have $P_0(x) = x^4 - 6x^2 + 1 = (x^2 - 2x - 1)(x^2 + 2x - 1)$ and $P_3(x) = x^4 - 3x^3 - 6x^2 + 3x + 1 = (x^2 - 4x - 1)(x^2 + x - 1)$). Hence, from now on, we assume that $m \geq 1$ and $m \neq 3$. Since $(1-x)^4 P_m((1+x)/(1-x)) = -4P_m(x)$, if θ is any complex root of $P_m(x)$ then $\sigma(\theta) := (\theta - 1)/(\theta + 1)$, $\sigma^2(\theta) = -1/\theta$ and $\sigma^3(\theta) = -(\theta + 1)/(\theta - 1)$ are the other complex roots of $P_m(x)$. Since $P_m(\pm 1) = -4 < 0$ and $P_m(0) = 1 > 0$, all the roots of $P_m(x)$ are real and if we denote by α_m the largest one, then we have

$$\alpha_m > 1 > \sigma(\alpha_m) > 0 > \sigma^2(\alpha_m) > -1 > \sigma^3(\alpha_m).$$

Hence, $P_m(x)$ defines a real cyclic quartic number field $K_m := \boldsymbol{Q}(\alpha_m)$ and σ gives a generator of the Galois group $\mathrm{Gal}(K_m/\boldsymbol{Q})$. Set $\beta_m = \alpha_m - \alpha_m^{-1} > 0$. Then $\beta_m^2 - m\beta_m - 4 = 0$ (use $\alpha_m^{-2} P_m(\alpha_m) = \alpha_m^2 - m\alpha_m - 6 + m\alpha_m^{-1} + \alpha_m^{-2} = 0$) and $\beta_m = (m + \sqrt{\Delta_m})/2$. In particular, $k_m = \boldsymbol{Q}(\sqrt{\Delta_m})$ is the quadratic subfield of the real cyclic quartic field $K_m = \boldsymbol{Q}(\alpha_m)$ and

$$N_{K_m/k_m}(\alpha_m) = -1. \tag{10}$$

Since $\alpha_m > 1$ and $\alpha_m^2 - \beta_m \alpha_m - 1 = 0$, we obtain

$$\alpha_m = \frac{1}{2}((m + \sqrt{\Delta_m})/2 + \sqrt{(\Delta_m + m\sqrt{\Delta_m})/2}\,). \tag{11}$$

In the same way, $\sigma(\beta_m) = (m - \sqrt{\Delta_m})/2$ and

$$\sigma(\alpha_m) = \frac{1}{2}((m - \sqrt{\Delta_m})/2 + \sqrt{(\Delta_m - m\sqrt{\Delta_m})/2}\,). \tag{12}$$

Note also that

$$K_m = \boldsymbol{Q}(\alpha_m) = \boldsymbol{Q}(\sqrt{(\Delta_m + m\sqrt{\Delta_m})/2}\,).$$

We will say that K_m is a *simplest quartic field* if $m \geq 1$ is such that the odd part of Δ_m is square-free, which implies $m \neq 3$. We have:

Proposition 6. *Assume that $m \geq 1$ and that the odd part of $\Delta_m = m^2 + 16$ is square-free. Let f_{K_m} and f_{k_m} denote the conductors of the simplest quartic field K_m and of its real quadratic subfield k_m. Then,*

$$(f_{K_m}, f_{k_m}) = \begin{cases} (\Delta_m, \Delta_m) & \text{if } m \equiv 1 \pmod 2 \\ (\Delta_m, \Delta_m/4) & \text{if } m \equiv 2 \pmod 4 \\ (\Delta_m/2, \Delta_m/4) & \text{if } m \equiv 4 \pmod 8 \\ (\Delta_m/2, \Delta_m/16) & \text{if } m \equiv 0 \pmod 8. \end{cases}$$

In particular, different values of m define different simplest quartic fields, f_{K_m} is odd if and only if m is odd, and f_{k_m} is even if and only if $m \equiv 4 \pmod 8$.

Proof. Let us content ourselves with a simple proof of the first case (see [**Gras**, Proposition 8] for a proof of the remaining cases). Since $k_m = \boldsymbol{Q}(\sqrt{\Delta_m})$ of discriminant $\Delta_m \equiv 1 \pmod 4$ is the quadratic subfield of the cyclic quartic field K_m, we obtain that Δ_m^3 divides d_{K_m} (use the conductor-discriminant formula). Since $(\alpha_m^3+1)/2$ is a root of $X^4-((m^3+15m+4)/2)X^3+((3m^3-12m^2+45m-192)/4)X^2-((m^3-12m^2+15m-196)/4)X-((3m^2+49)/4)\in \boldsymbol{Z}[X]$, it is an algebraic integer of K_m and d_{K_m} divides $d(1,\alpha_m,\alpha_m^2,(\alpha_m^3+1)/2)=d(1,\alpha_m,\alpha_m^2,\alpha_m^3)/4=d_m/4=\Delta_m^3$. □

PROPOSITION 7. *Set* $c=\prod_{p\equiv 1 \pmod 4}(1-2p^{-2})=0.894\cdots$. *Then,* $\#\{1\le m\le x;$ *the odd part of* Δ_m *is square-free*$\}$ *is asymptotic to* cx, *and* $\#\{1\le m\le x;$ m *is odd and* Δ_m *is square-free*$\}$ *is asymptotic to* $(1/2)cx$.

3.1. Lower bounds for class numbers.

PROPOSITION 8. *Assume that* $m\ge 1$ *and that the odd part of* $\Delta_m=m^2+16$ *is square-free. Then,* α_m *is a generating relative unit and*

$$\mathrm{Reg}^*_{K_m}=\log^2\alpha_m+\log^2\sigma(\alpha_m)\le\frac{1}{4}\log^2\Delta_m. \tag{13}$$

PROOF. For the proof of the first assertion, see [**Gras**, Proposition 8], or adapt the method in the proof of [**Wa**, Section 2]. To prove (13), we note that it holds true for $m=1,2$ (use (11) and (12)). Thus, we assume that $m\ge 3$, which implies $P_m(4)=-60m+161<0$ and $\alpha_m>4$. Now, $\alpha_m^{1-\sigma}=(2+\sqrt{\Delta_m}+\sqrt{(2+\sqrt{\Delta_m})^2-4})/2$, by (11) and (12). Hence, $\alpha_m^{1-\sigma}<2+\sqrt{\Delta_m}$, $1<\alpha_m^{1-\sigma}\sqrt{\Delta_m}<(2+\sqrt{\Delta_m})\sqrt{\Delta_m}<(1+\sqrt{\Delta_m})^2$ and $1<\alpha_m^{1-\sigma}/\sqrt{\Delta_m}<1+(2/\sqrt{\Delta_m})$. Using $(1/\sigma(\alpha_m))=1+(2/(\alpha_m-1))$, we obtain

$$\begin{aligned}&\log^2\alpha_m+\log^2\sigma(\alpha_m)-\frac{1}{4}\log^2\Delta_m\\&\quad=(\log(\alpha_m^{1-\sigma}\sqrt{\Delta_m}))\left(\log\left(\frac{\alpha_m^{1-\sigma}}{\sqrt{\Delta_m}}\right)\right)-2(\log\alpha_m)\left(\log\frac{1}{\sigma(\alpha_m)}\right)\\&\quad<2(F(1+\sqrt{\Delta_m})-F(\alpha_m)),\quad\text{where } F(x):=(\log x)\left(\log\left(1+\frac{2}{x-1}\right)\right),\quad x>1.\end{aligned}$$

Now, $x(x^2-1)F'(x)=(x^2-1)\log(1+2/(x-1))-2x\log x<G(x):=2(x+1-x\log x)$ where $G'(x)=-2\log x<0$ for $x>1$. Hence, $G(x)\le G(4)<0$ and $F'(x)<0$ for $x>4$, and $F(1+\sqrt{\Delta_m})-F(\alpha_m)<0$, for $\alpha_m<\sqrt{\Delta_m}$, by (11). □

THEOREM 9. *Assume that* $m\ge 1$ *and that the odd part of* $\Delta_m=m^2+16$ *is square-free. Let* f_{K_m} *denote the conductor of the simplest quartic field* K_m. *Then,*

$$h^*_{K_m}\ge\frac{2f_{K_m}}{3e(\log f_{K_m}+0.35)^4}. \tag{14}$$

In particular, $h^*_{K_m}>1$ *for* $f_{K_m}\ge 73000$ (*hence for* $m\ge 382$) *and* $h^*_{K_m}>2$ *for* $f_{K_m}\ge 210000$ (*hence for* $m\ge 649$).

PROOF. The right hand side of (14) being less than one for $f_{K_m}<7\cdot 10^4$, we may assume that $f_{K_m}\ge 7\cdot 10^4$, which implies $d_{K_m}=f_{k_m}f_{K_m}^2>f_{K_m}^3/8\ge 4\cdot 10^{13}$ (for $f_{k_m}\ge f_{K_m}/8$, by Proposition 6). Hence, using (7), (13) and the bounds $\Delta_m\le 2f_{K_m}$ (by Proposition 6) and $(\log 2+\kappa/2)/2\le 0.35$, we obtain (14). □

3.2. Computation of the unit index Q_{K_m}.

Proposition 10. *Assume that $m \geq 1$ and that the odd part of $\Delta_m = m^2 + 16$ is square-free. Let $1 < \varepsilon_{k_m} = (x_m + y_m\sqrt{\Delta_m})/2 \in \boldsymbol{Q}(\sqrt{\Delta_m})$ denote the fundamental unit of the real quadratic subfield $k_m = \boldsymbol{Q}(\sqrt{\Delta_m})$ of the simplest quartic field K_m.*

1. *If $N_{k_m/\boldsymbol{Q}}(\varepsilon_{k_m}) = 1$, then $Q_{K_m} = 1$.*
2. *If $N_{k_m/\boldsymbol{Q}}(\varepsilon_{k_m}) = -1$, then $Q_{K_m} = 2$ if and only if at least one of the two rational integers $4x_m + \Delta_m y_m \pm 2m$ is a perfect square.*
3. *If $m \geq 2$ is even, then, $Q_{K_m} = 1$ if $m \neq 4$, and $Q_{K_m} = 2$ if $m = 4$.*

Proof. To begin with, we note that $Q_{K_m} = 2$ if and only if $\pm\varepsilon_{k_m}\alpha_m^{1-\sigma}$ is a square in K_m (see [**Gras**, Proposition 1]), hence if and only if $\eta_m := \varepsilon_{k_m}\alpha_m^{1-\sigma}$ is a square in K_m (for $\varepsilon_{k_m} > 1$ and $\alpha_m > 1 > \sigma(\alpha_m) > 0$). Assume that $N_{k_m/\boldsymbol{Q}}(\varepsilon_{k_m}) = +1$. Then, $\eta_m^{1+\sigma} = N_{k_m/\boldsymbol{Q}}(\varepsilon_{k_m})\alpha_m^{1-\sigma^2} = \alpha_m^2/N_{K_m/k_m}(\alpha_m) = -\alpha_m^2$ (use (10)) is not a square in K_m. Hence, η_m is not a square in K_m, and $Q_{K_m} = 1$, which proves the first assertion. Let us now prove the second assertion. To begin with, using $\alpha_m^4 + 1 = \alpha_m(m\alpha_m^2 + 6\alpha_m - m)$, we obtain:

$$\begin{aligned}
\alpha_m^{1-\sigma} + \alpha_m^{-(1-\sigma)} &= \frac{\alpha_m(\alpha_m+1)}{\alpha_m - 1} + \frac{\alpha_m - 1}{\alpha_m(\alpha_m+1)} \\
&= \frac{\alpha_m^4 + 2\alpha_m^3 + 2\alpha_m^2 - 2\alpha_m + 1}{\alpha_m(\alpha_m^2 - 1)} \\
&= \frac{(m+2)\alpha_m^2 + 8\alpha_m - (m+2)}{\alpha_m^2 - 1} \\
&= m + 2 + (8/\beta_m) = 2 + \sqrt{\Delta_m}.
\end{aligned}$$

Now, $N_{K_m/k_m}(\eta_m) = \varepsilon_{k_m}^2$ (use (10)) is a square in k_m. Hence, by [**Lou4**, Proposition 3.1], η_m is a square in K_m if and only if $\mathrm{Tr}_{K_m/k_m}(\eta_m) + 2\sqrt{N_{K_m/k_m}(\eta_m)} = \varepsilon_{k_m}(\alpha_m^{1-\sigma} + \alpha_m^{\sigma^2-\sigma^3}) + 2\varepsilon_{k_m} = \varepsilon_{k_m}(\alpha_m^{1-\sigma} + \alpha_m^{-(1-\sigma)} + 2) = \varepsilon_{k_m}(4 + \sqrt{\Delta_m})$ is a square in k_m (for $\mathrm{Tr}_{K_m/k_m}(\eta_m) - 2\sqrt{N_{K_m/k_m}(\eta_m)} = \varepsilon_{k_m}\sqrt{\Delta_m}$ cannnot be a square in k_m, since $N_{k_m/\boldsymbol{Q}}(\varepsilon_{k_m}\sqrt{\Delta_m}) = \Delta_m$ is not a square in $\boldsymbol{Q}$). Finally, by [**Lou4**, Corollary 3.3], $\varepsilon_{k_m}(4 + \sqrt{\Delta_m})$ (of absolute norm m^2 a square in $\boldsymbol{Q}$) is a square in k_m if and only if $T_m + 2m$ or $T_m - 2m$ is a square in $\boldsymbol{Q}$, where $T_m = 4x_m + \Delta_m y_m$ is the trace of $\varepsilon_{k_m}(4 + \sqrt{\Delta_m})$. Let us finally prove the last assertion. Assume that $m \neq 4, 8$ is even. In that case, $\varepsilon_{k_m} = ((m/2) + (\sqrt{\Delta_m}/2))/2$ is of norm -1 and $Q_{K_m} = 1$, for neither $4x_m + \Delta_m y_m + 2m = (m+4)^2/2$ nor $4x_m + \Delta_m y_m - 2m = (m^2+16)/2$ is a perfect square (if $\Delta_m/2 = (m^2+16)/2$ is a perfect square then $\Delta_m = m^2 + 16$ must be a perfect 2-power, which implies $m = 0$ or $m = 4$). Now, if $m = 8$, then $\varepsilon_{k_m} = (1 + \sqrt{5})/2 = (1 + (\sqrt{\Delta_m}/4))/2$ is of norm -1 and $Q_{K_m} = 1$, for neither $4x_m + \Delta_m y_m + 2m = 40$ nor $4x_m + \Delta_m y_m - 2m = 8$ is a perfect square. Finally, if $m = 4$ then $\varepsilon_{k_m} = 1 + \sqrt{2} = (2 + (\sqrt{\Delta_m}/2))/2$ is of norm -1 and $Q_{K_m} = 2$, for $4x_m + \Delta_m y_m - 2m = 16$ is a perfect square. □

3.3. Computation of class numbers.

Let χ_{K_m} be any one of the two conjugate primitive quartic Dirichlet characters modulo f_{K_m} associated with a simplest quartic field K_m of conductor f_{K_m}. According to

(1), Proposition 8 and to the explicit formula for $L(1,\chi)$ for even primitive Dirichlet characters, we have

$$h_{K_m}^* = \frac{Q_{K_m}}{2(\log^2 \alpha_m + \log^2 \sigma(\alpha_m))}\left|\sum_{1\le l\le f_{K_m}/2}\chi_{K_m}(l)\log\sin(l\pi/\Delta_m)\right|^2$$

(where Q_{K_m} is computed by using Proposition 10), which provides us with a simple technique for computing efficiently h_{K_m} for m not too large. Let us now explain how one can efficiently determine such a χ_{K_m} (see also [**Lou5**]). To begin with, we note that if $\Delta = \prod_{i=1}^t p_i$ is the product of $t\ge 1$ pairwise distinct odd primes $p_i \equiv 1 \pmod 4$ then we can enumerate all the 2^t primitive quartic characters $\psi_{n,\Delta}$, $0\le n\le 2^t-1$ whose components modulo each p_i are primitive quartic characters. Indeed, for a given prime $p\equiv 1 \pmod 4$, set $g_p = \min\{g\ge 1; g^{(p-1)/2}\equiv -1 \pmod p\}$, $G_p = g_p^{(p-1)/4} \bmod p$ and let ϕ_p be the quartic character mod p defined by

$$\phi_p(x) = \zeta_4^{n(x)}, \quad \text{where } n(x) = \min\{n\ge 0; x^{(p-1)/4}\equiv G_p^n \pmod p\}\in\{0,1,2,3\}$$

(for $\gcd(x,p)=1$). To each $n\in\{0,1,\dots,2^t-1\}$ of 2-adic expansion $n=\sum_{i=1}^t a_i 2^{i-1}$, $a_i\in\{0,1\}$, we associate the primitive mod Δ quartic character

$$\psi_{n,\Delta} = \prod_{i=1}^t \phi_{p_i}^{(-1)^{a_i}}.$$

1. First, assume that m is odd. Then, $f_{K_m} = \Delta_m = \prod_{i=1}^t p_i$ is a product of $t\ge 1$ pairwise distinct odd primes $p_i\equiv 1 \pmod 4$ and there exists a unique odd $n = n_m \in \{0,1,\dots,2^{t-1}-1\}$ such that the primitive quartic character ψ_{n_m,Δ_m} is one of the two conjugate primitive quartic characters χ_{K_m} associated with K_m. The following algorithm provides us with an efficient technique for determining this unique $n=n_m$:

1. $E := \{0,1,\dots,2^{t-1}-1\}$, $p:=3$.
2. $n_{\min} := \min(E)$, $n_{\max} := \max(E)$.
3. If $n_{\min} = n_{\max}$ then go to step 6.
4. While p divides Δ_m, or $P_m(x)$ has no root in $\boldsymbol{Z}/p\boldsymbol{Z}$, do $p :=$ next prime.

(Now, since $P_m(x)$ has at least one root in $\boldsymbol{Z}/p\boldsymbol{Z}$ and since p does not divide the discriminant $d_m = 4\Delta_m^3$ of $P_m(x)$, it holds that p splits in K_m and $\chi_{K_m}(p)=+1$.)

5. Exclude all n with $\psi_{n,\Delta_m}(p)\ne 1$ from E. Then, go to step 2.
6. Return$(n_{\min})$.

2. Now, assume that m is even. Let χ_4^- denote the only primitive quadratic Dirichlet character modulo 4 (hence, $\chi_4^-(-1)=-1$), χ_8^+ be the only primitive even quadratic Dirichlet character modulo 8 and χ_{16}^+ be any one of the two conjugate primitive even quartic Dirichlet characters modulo 16.

1. If $m\equiv 2 \pmod 4$, then $f_{K_m} = 4\Delta_m'$, where $\Delta_m' = (m/2)^2+4 = \prod_{i=1}^t p_i \equiv 5 \pmod 8$ is a product of $t\ge 1$ pairwise distinct odd primes $p_i\equiv 1 \pmod 4$, and there exists a unique $n=n_m\in\{0,1,\dots,2^{t-1}-1\}$ such that the primitive quartic character $\chi_4^-\psi_{n_m,\Delta_m'}$ is one of the two conjugate primitive quartic characters χ_{K_m} associated with K_m.
2. If $m\equiv 4 \pmod 8$, then $f_{K_m} = 16\Delta_m'$, where $\Delta_m' = ((m/4)^2+1)/2 = \prod_{i=1}^t p_i \equiv 1 \pmod 4$ is a product of $t\ge 1$ pairwise distinct odd primes $p_i\equiv 1 \pmod 4$.

(a) If $m \equiv \pm 4 \pmod{32}$, then $\Delta_m' \equiv 1 \pmod 8$ and there exists a unique $n = n_m \in \{0, 1, \dots, 2^t - 1\}$ such that the primitive quartic character $\chi_{16}^+ \psi_{n_m, \Delta_m'}$ is one of the two conjugate primitive quartic characters χ_{K_m} associated with K_m.

(b) If $m \equiv \pm 12 \pmod{32}$, then $\Delta_m' \equiv 5 \pmod 8$ and there exists a unique $n = n_m \in \{0, 1, \dots, 2^t - 1\}$ such that the primitive quartic character $\chi_4^- \chi_{16}^+ \psi_{n_m, \Delta_m'}$ is one of the two conjugate primitive quartic characters χ_{K_m} associated with K_m.

3. If $m \equiv 8 \pmod{16}$, then $f_{K_m} = 8\Delta_m'$, where $\Delta_m' = 4(m/8)^2 + 1 = \prod_{i=1}^t p_i \equiv 5 \pmod 8$ is a product of $t \geq 1$ pairwise distinct odd primes $p_i \equiv 1 \pmod 4$, and there exists a unique $n = n_m \in \{0, 1, \dots, 2^{t-1} - 1\}$ such that the primitive quartic character $\chi_4^- \chi_8^+ \psi_{n_m, \Delta_m'}$ is one of the two conjugate primitive quartic characters χ_{K_m} associated with K_m.
4. Finally, if $m \equiv 0 \pmod{16}$, then $f_{K_m} = 8\Delta_m'$, where $\Delta_m' = 16(m/16)^2 + 1 = \prod_{i=1}^t p_i \equiv 1 \pmod 8$ is a product of $t \geq 1$ pairwise distinct odd primes $p_i \equiv 1 \pmod 4$, and there exists a unique $n = n_m \in \{0, 1, \dots, 2^{t-1} - 1\}$ such that the primitive quartic character $\chi_8^+ \psi_{n_m, \Delta_m'}$ is one of the two conjugate primitive quartic characters χ_{K_m} associated with K_m.

3.4. Bounds for the relative class numbers and conductors of the simplest quartic fields with ideal class groups of exponents ≤ 2.

Proposition 11. *Assume that $m \geq 1$, that the odd part of $\Delta_m = m^2 + 16$ is square-free and that the exponent of the ideal class group of the simplest quartic field K_m is ≤ 2.*

1. *If m is odd, then either* (i) *$\Delta_m = p \equiv 1 \pmod 8$ is prime or* (ii) *$\Delta_m = p_1 p_2$ is the product of two distinct odd primes $p_1 \equiv p_2 \equiv 5 \pmod 8$. Moreover, $h_{K_m}^* \leq 4$ and $m \leq 750$.*
2. *If $m \geq 2$ is even, then, $h_{K_m}^* \leq 16$ and $m \leq 2800$.*

Proof. By Lemma 5, at most two primes are ramified in k_m. Moreover, if $m \geq 1$ is odd and $d_{k_m} = \Delta_m = p_1 p_2$ is a product of two primes, then $p_1 \equiv p_2 \equiv 1 \pmod 4$ and $p_1 \equiv p_2 \pmod 8$, for $\Delta_m = m^2 + 16 \equiv 1 \pmod 8$. If we had $p_1 \equiv p_2 \equiv 1 \pmod 8$, then the genus field G_{K_m} of K_m would be real, G_{K_m}/K_m would be an unramified cyclic extension and the 4-rank of the ideal class group of K would be ≥ 1, a contradiction. Hence, we are in case (i) or (ii). Now, with the notation of Lemma 5, we have $t = T = 1$ in case (i) and $t = T = 2$ in case (ii). Hence $T + t - 2 \leq 2$ and $h_{K_m}^* \leq 4$ in both cases. Finally, using (14) we obtain $f_{K_m} \leq 560000$, which implies $m \leq 748$ (for $m^2 + 16 = \Delta_m = f_{K_m}$, by Proposition 6). Let us now prove the second assertion. With the notation of Lemma 5, we must have $t \leq 2$. Since $T \leq 2 + t$ (since a prime ideal of k_m which is ramified in K_m/k_m but unramified in $k_m/\boldsymbol{Q}$ must lie above 2), we obtain $t + T - 2 \leq 2t \leq 4$ and $h_K^* \leq 2^4 = 16$. Finally, using (14) we obtain $f_{K_m} \leq 3800000$, which implies $m \leq 2756$ (for $m^2 + 16 = \Delta_m \leq 2f_{K_m}$, by Proposition 6). $\square$

3.5. The exponent 2 class group problem for the simplest quartic fields: proof of Theorem 1.

To begin with, we note that if the exponent of the ideal class group of the simplest quartic field K_m is ≤ 2, then h_{K_m} is a perfect 2-power. First, let us deal with the case that $m \geq 1$ is odd. According to computations based on Section 3.3, only 18 out of the simplest quartic fields K_m, m odd and $1 \leq m \leq 750$, have class numbers of the form $h_{K_m} = 2^{e_m}$, $e_m \geq 0$: $m \in \{1, 5, 7, 9, 11, 13, 15, 17, 19, 23, 27, 33, 39, 45, 69, 87, 255, 549\}$. Moreover, only 11 out of these 18 values are such that $h_{K_m} > 2$, namely $m \in \{17, 19, 23, 27, 33, 39, 45, 69, 87, 255, 549\}$. Since Δ_m is a product of three distint primes for $m \in \{33, 87\}$ whereas Δ_m is a product of two distinct primes $p_1 \equiv p_2 \equiv 1 \pmod 8$ for $m \in \{69, 255, 549\}$, by Proposition 11, it remains to compute the structure of the ideal class groups of the 6 quartic fields K_m, $m \in \{17, 19, 23, 27, 39, 45\}$. We obtain the following Table 1, which completes the proof of Theorem 1 in the case that m is odd. Now, as explained in the introduction, the case that m is even is simpler to deal with:

TABLE 1.

m	17	19	23	27	39	45
Δ_m	$5 \cdot 61$	$13 \cdot 29$	$5 \cdot 109$	$5 \cdot 149$	$29 \cdot 53$	$13 \cdot 157$
h_{k_m}	2	2	2	2	2	2
Q_{K_m}	1	1	1	1	1	1
$h_{K_m}^*$	2	2	2	2	2	2
h_{K_m}	4	4	4	4	4	4
Cl_{K_m}	[2, 2]	[2, 2]	[2, 2]	[2, 2]	[2, 2]	[2, 2]

LEMMA 12 (*See also* [**LMW**]). *Assume that $m \geq 2$ is even and that the odd part of $\Delta_m = m^2 + 16$ is square-free. If the exponent of the ideal class group of K_m is ≤ 2 then $h_{k_m} = 2^{t_{k_m} - 1}$, where t_{k_m} denotes the number of distinct prime divisors of the discriminant of k_m. Moreover, the only such $m \leq 1000$ are $m \in \{2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 24, 26, 32, 34, 38, 40, 44, 46, 50, 52, 56, 62, 68, 76, 82, 86, 88, 92, 98, 104, 106, 118, 124, 136, 148, 184, 188, 202, 232, 254, 292, 358, 392, 488, 568, 968\}$ (46 values).*

PROOF. The exponent of the ideal class group of k_m must be ≤ 2 (by Lemma 5), and since all the odd prime divisors p of d_{k_m} satisfy $p \equiv 1 \pmod 4$, the 2-rank of the ideal class group of k_m is equal to $t_{k_m} - 1$. □

Now, according to computations based on Section 3.3, only 11 out of these 46 values of m are such that the class numbers of the simplest quartic fields K_m are of the form $h_{K_m} = 2^{e_m}$, $e_m \geq 0$, namely $m \in \{2, 4, 6, 8, 10, 12, 16, 18, 20, 24, 32\}$. Moreover, only 2 out of these 11 values are such that $h_{K_m} > 2$, namely $m \in \{18, 32\}$. Finally, using the Pari software for algebraic number fields to compute the structure of the ideal class groups of these 2 quartic fields, we obtain that neither $\mathrm{Cl}_{K_{18}} = [4]$ nor $\mathrm{Cl}_{K_{32}} = [4, 2]$ is elementary, which completes the proof of Theorem 1 in the case that m is even.

All our computations were carried out on a personal microcomputer by using Pr. Y. Kida's UBASIC language (for class number computations) and Pari GP (for the determination of the structures of the ideal class groups of K_{17}, K_{19}, K_{23}, K_{27}, K_{39}, K_{17}, K_{45}, K_{18}, and K_{32}).

References

[Gras] M. N. Gras, Table numérique du nombre de classes et des unités des extensions cycliques réelles de degré 4 de $\boldsymbol{Q}$, Publ. Math. Fac. Sci. Besançon, fasc **2**, 1977/1978.

[Has] H. Hasse, Arithmetische Bestimmung von Grundeinheit und Klassenzahl in zyklischen kubischen und biquadratischen Zahlkörpern, Mathematische Abhandlungen, Band 3, Walter de Gruyter, Berlin, 1975, 285–379, Originally published 1950, MR **57#** 5648.

[Lang] S. Lang, Cyclotomic Fields I and II, Combined second edition, Grad. Texts in Math., **121**, Springer-Verlag, New York, 1990, MR **91c**:11001.

[Laz1] A. J. Lazarus, Class numbers of simplest quartic fields, In: Number theory (Banff, AB, 1988), Walter de Gruyter, Berlin, 1990, 313–323, MR **92d**:11119.

[Laz2] A. J. Lazarus, On the class number and unit index of simplest quartic fields, Nagoya Math. J., **121** (1991), 1–13, MR **92a**:11129.

[LMW] S. Louboutin, R. A. Mollin and H. C. Williams, Class groups of exponent two in real quadratic fields, Advances in Number theory (Kingston, ON, 1991), Oxford Sci. Publ., Oxford Univ. Press, New York, 1993, 499–513, MR **96j**:11153.

[Lou1] S. Louboutin, Determination of all nonquadratic imaginary cyclic number fields of 2-power degrees with ideal class groups of exponents ≤ 2, Math. Comp., **64** (1995), no. 229, 323–340, MR **95c**:11124.

[Lou2] S. Louboutin, Class number problems for cubic number fields, Nagoya Math. J., **138** (1995), 199–208, MR **96f**:11145.

[Lou3] S. Louboutin, Majorations explicites du résidu au point 1 des fonctions zêta des corps de nombres, J. Math. Soc. Japan, **50** (1998), 57–69, MR **99a**:11131.

[Lou4] S. Louboutin, Hasse unit indices of dihedral octic CM-fields, Math. Nachr., **215** (2000), 107–113, MR **2001f**:11186.

[Lou5] S. Louboutin, The exponent three class group problem for some real cyclic cubic number fields, Proc. Amer. Math. Soc., **130** (2002), 353–361, MR **200h**:11106.

[Sta] H. M. Stark, Some effective cases of the Brauer-Siegel Theorem, Invent. Math., **23** (1974), 135–152, MR **49#** 7218.

[Wa] L. C. Washington, A family of cyclic quartic fields arising from modular curves, Math. Comp., **57** (1991), no. 196, 763–775, MR **92a**:11120.

Stéphane R. Louboutin
Institut de Mathématiques de Luminy, UMR 6206
163, avenue de Luminy, Case 907
13288 Marseille Cedex 9
France
E-mail: loubouti@iml.univ-mrs.fr