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Microlocal boundary value problem for regular-specializable systems

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Abstract. In the framework of microlocal analysis, a boundary value morphism is defined for solutions to the regular-specializable system of analytic linear partial differential equations. This morphism can be regarded as a microlocal counterpart of the boundary value morphism for hyperfunction solutions due to Monteiro Fernandes, and the injectivity of this morphism (that is, the Holmgren type theorem) is proved. Moreover, under a kind of hyperbolicity condition, it is proved that this morphism is surjective (that is, the solvability).

Introduction.

In microlocal analysis, it is one of the main subjects to give an appropriate formulation of the boundary value problems for hyperfunction or microfunction solutions to a system of analytic linear partial differential equations (that is, a coherent (left) $\mathscr{D}$-Module, here in this paper, we shall write *Module* or *Ring* with capital letters, instead of *sheaf of modules* or *sheaf of rings*). We shall recall the previous results:

When we impose the *non-characteristic* condition, we can obtain the following satisfactory results: Suppose that the boundary is real analytic and non-characteristic for the system. Then all the hyperfunction or microfunction solutions have boundary values as hyperfunction or microfunction solutions to the *induced system* on the boundary, and the local or microlocal uniqueness theorem (Holmgren type theorem) hold. Note that in the case of hyperfunction solutions to a differential equation, these results are given by Komatsu-Kawai [**Ko-K**] and Schapira [**Sc1**], and in the case of a system, we can prove these facts by means of the theory of *microsupports* (cf. Kashiwara-Kawai [**K-K1**]). See also Kataoka [**Kat**] for microlocal boundary value problems in the framework of the theory of *mild microfunctions*.

However, once we release the non-characteristic condition for the system, the problem is much involved; In general, we must impose some regularity condition on the solutions in order to define their boundary values as solutions to the induced system. As this condition, Oaku [**Oa1**], [**Oa2**] introduced the sheaf of *F-mild hyperfunctions* and of *F-mild microfunctions* as a microlocalization. For the *F*-mild hyperfunction or microfunction solutions to a Fuchsian system in the sense of Laurent-Monteiro Fernandes [**L-MF1**], we can obtain the local or microlocal uniqueness theorem for boundary value problem (see Oaku [**Oa1**], [**Oa2**], and cf. Oaku and Yamazaki [**O-Y**]).

On the other hand, if we assume the following condition to the Fuchsian system,

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all the hyperfunction solutions have boundary values and a local uniqueness theorem holds as in the non-characteristic case: Suppose that the system is *regular-specializable*. Then the *nearby-cycle* of the system is defined in the theory of $\mathscr{D}$-Modules. The definitions of the regular-specializable $\mathscr{D}$-Module and its nearby-cycle are initiated by Kashiwara [**Kas**], Kashiwara and Kawai [**K-K2**] and Malgrange [**Mal**] for regular-holonomic cases. Further the notion of nearby-cycle is extended to the *specializable* $\mathscr{D}$*-*Module (see Laurent [**L2**], Laurent and Malgrange [**L-Ma**] and Mebkhout [**Me**]). Note that we do not have a definition of nearby-cycle for general Fuchsian systems at this stage. After the results by Kashiwara-Oshima [**K-O**], Oshima [**Os1**] and Schapira [**Sc3**], [**Sc4**], for the hyperfunction solution sheaf to regular-specializable system Monteiro Fernandes [**MF1**] defined a boundary value morphism which takes values in hyperfunction solutions to the nearby-cycle of the system instead of the induced system. This morphism is injective (cf. [**MF2**]) and gives a generalization of the non-characteristic boundary value morphism. Moreover Laurent-Monteiro Fernandes [**L-MF2**] redefined this morphism and discussed the solvability under a kind of hyperbolicity condition (the *near-hyperbolicity*). Here we should remark that even in single equation cases, some results due to Tahara [**T**] can not be recovered by Laurent-Monteiro Fernandes [**L-MF2**]. However, since this morphism is defined only for hyperfunction solutions, a microlocal boundary value problem is not considered. Therefore in this paper, we shall microlocalize this morphism in the framework of Oaku [**Oa3**] and Oaku-Yamazaki [**O-Y**] and extend their result to our case; that is, for the regular-specializable system we shall define a injective boundary value morphism as a microlocalization of the boundary value morphism in the sense of Monteiro Fernandes [**MF1**], and prove this morphism is surjective under the near-hyperbolicity condition.

We remark that for a Fuchsian system in the sense of Tahara [**T**], Oaku [**Oa3**] defined an injective boundary value morphism under additional conditions of characteristic exponents by using a detailed study due to Tahara [**T**].

The plan of this paper is as follows: In §1, we shall introduce the notation and recall complementary results used in later sections. In §2, we shall define a general boundary morphism for a complex of sheaves under some condition. Further, we shall prove this morphism is isomorphic under the near-hyperbolicity condition in the sense of Laurent and Monteiro Fernandes [**L-MF2**] (cf. Kashiwara-Schapira [**K-S1**]). §§3 and 4 are preparations for §5; §3 is an exposition of the regular-specializable $\mathscr{D}$-Module. In §4, we recall several sheaves and in particular, a sheaf $\mathscr{C}_{N|M}$ attached to the boundary on some cotangent bundles in order to formulate our boundary value problem. We remark that roughly speaking, $\mathscr{C}_{N|M}$ is a microlocalization of the specialization of the sheaf of hyperfunctions. In §5, for any $\mathscr{C}_{N|M}$ solutions to the regular-specializable system, we shall define a boundary value morphism which takes values in microfunction solutions to the nearby-cycle of the system, and prove this morphism is injective in the zero-th cohomology (this means the microlocal uniqueness theorem). Note that the restriction of our morphism to the zero-section coincides with that in the sense of Monteiro Fernandes [**MF1**]. Finally §6 is devoted to examples.

We shall end this introduction with the following remarks: The non-characteristic, Fuchsian or regular-specializable conditions are generalized to the higher-codimensional case. If we impose non-characteristic or Fuchsian conditions, we can extend the results of the one-codimensional case mentioned above to that of the higher-codimensional case in the framework of F-mild microfunctions (see Oaku-Yamazaki [**O-Y**]). On the contrary, if we assume only the regular-specializable condition, we cannot define boundary values for any hyperfunction solution as a natural extension of the boundary values in the sense of Monteiro Fernandes [**MF1**]. Hence in this case, we need additional conditions on the system in order to obtain an appropriate formulation of the higher-codimensional boundary value problem (cf. Kashiwara-Oshima [**K-O**] and Oshima [**Os2**]).

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1. Preliminaries.

In this section, we shall fix the notation and recall known results used in later sections. General references are made to Kashiwara-Schapira [**K-S2**].

We denote by $\boldsymbol{Z}, \boldsymbol{R}$ and $\boldsymbol{C}$ the sets of all the integers, real numbers and complex numbers respectively. Moreover we set $\boldsymbol{N} := \{n \in \boldsymbol{Z}; n \geq 1\}$ and $\boldsymbol{N}_0 := \boldsymbol{N} \cup \{0\}$.

In this paper, all the manifolds are assumed to be paracompact. In general, let $\tau : E \to Z$ a vector bundle over a manifold Z. Then, set $\dot{E} := E \backslash Z$ and $\dot{\tau}$ the restriction of τ to $\dot{E}$. Let M be an $(n+1)$-dimensional real analytic manifold and N a one-codimensional closed real analytic submanifold of M. Let X and Y be complexifications of M and N respectively such that Y is a closed submanifold of X and that $Y \cap M = N$. Moreover in this paper, we assume the existence of a partial complexification of M in X; that is, there exists a $(2n+1)$-dimensional real analytic submanifold L of X containing both M and Y such that the triplet (N, M, L) is locally isomorphic to $(\boldsymbol{R}^n \times \{0\}, \boldsymbol{R}^{n+1}, \boldsymbol{C}^n \times \boldsymbol{R})$ by local coordinates $(z, \tau) = (x + \sqrt{-1}y, t + \sqrt{-1}s)$ of X around each point of N. We say such local coordinates *admissible*. By admissible coordinates we have locally the following relation:

$$\begin{array}{ccccc} N = \boldsymbol{R}^n_x \times \{0\} & \hookrightarrow & M = \boldsymbol{R}^n_x \times \boldsymbol{R}_t & & \\ \cap\big\downarrow & & i \, \cap\big\downarrow & \searrow^{i_M} & \\ Y = \boldsymbol{C}^n_z \times \{0\} & \overset{i_Y}{\hookrightarrow} & L = \boldsymbol{C}^n_z \times \boldsymbol{R}_t & \overset{i_L}{\hookrightarrow} & X = \boldsymbol{C}^n_z \times \boldsymbol{C}_\tau, \end{array}$$

and with these coordinates, we often identify $T_Y X$ and $T_Y L$ with X and L respectively. We shall mainly follow the notation in Kashiwara-Schapira [**K-S2**]; we denote by $\tilde{M}_N$ and $\tilde{L}_Y$ the normal deformations of N and Y in M and L respectively and regard $\tilde{M}_N$ as a closed submanifold of $\tilde{L}_Y$. The projection $\tau_Y : T_Y L \to Y$ induces natural mappings:

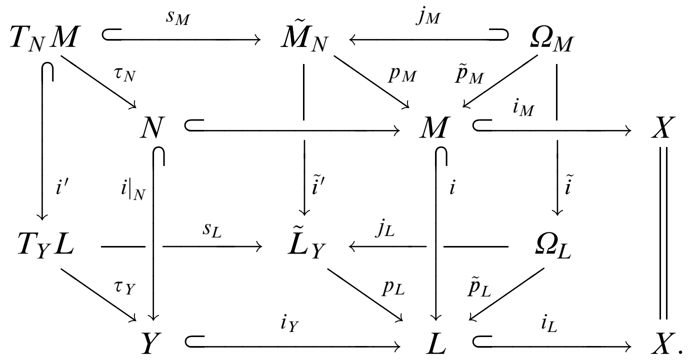
$$T^*_N Y \underset{\tau_{Y\pi}}{\longleftarrow} T_N M \underset{N}{\times} T^*_N Y \underset{\tau_{Yd}}{\overset{\sim}{\longrightarrow}} T^*_{T_N M} T_Y L,$$

and by τ_{Yd} we identify $T^*_{T_N M} T_Y L$ with $T_N M \underset{N}{\times} T^*_N Y$. Similarly by natural mappings

$$T^*_{\tilde{M}_N}\tilde{L}_Y \underset{s_{L\pi}}{\longleftarrow} T_N M \underset{\tilde{M}_N}{\times} T^*_{\tilde{M}_N}\tilde{L}_Y \underset{s_{Ld}}{\overset{\sim}{\longrightarrow}} T^*_{T_N M} T_Y L,$$

we identify $T_N M \underset{\tilde{M}_N}{\times} T^*_{\tilde{M}_N}\tilde{L}_Y$ with $T^*_{T_N M} T_Y L$.

We have the following commutative diagram:



$T_Y L \backslash T_Y Y$ has two components with respect to its fiber. We denote by $T_Y L^+$ one of them and represent (at least locally) by fixing admissible coordinates

$$T_Y L^+ = \{(z, t) \in T_Y L; t > 0\}.$$

Moreover set $T_N M^+ := T_Y L^+ \cap T_N M$. Define open embeddings f and f_N by:

$$\begin{array}{ccc} T_Y L^+ & \overset{f}{\hookrightarrow} & T_Y L \\ \uparrow\cup & \circlearrowleft & \uparrow\cup \\ T_N M^+ & \overset{f_N}{\hookrightarrow} & T_N M. \end{array}$$

Thus we regard $T_N M^+ \underset{N}{\times} T^*_N Y$ as an open set of $T^*_{T_N M} T_Y L$. Moreover f induces mappings:

$$\begin{array}{ccccc} T^*_{T_N M^+} T_Y L^+ & \overset{\sim}{\longleftarrow} & T_N M^+ \underset{T_N M}{\times} T^*_{T_N M} T_Y L & \overset{f_\pi}{\longrightarrow} & T^*_{T_N M} T_Y L \\ & & \wr\Big| & \circlearrowleft & \wr\Big| \\ & & T_N M^+ \underset{N}{\times} T^*_N Y & \overset{f_N \times 1}{\longrightarrow} & T_N M \underset{N}{\times} T^*_N Y. \end{array}$$

Hence we identify $T^*_{T_N M^+} T_Y L^+$ with $T_N M^+ \underset{N}{\times} T^*_N Y$, and f_π with $f_N \times 1$.

REMARK 1.1. To define $T_Y L^+$ (or $T_N M^+$) by means of admissible coordinates is equivalent to determining a local isomorphism $or_{Y/L} \simeq \boldsymbol{Z}_Y$ (or equivalently $or_{N/M} \simeq \boldsymbol{Z}_N$). Here $or_{Y/L}$ denotes the relative orientation sheaf.

Let $\pi_{N,M} : T^*_{\tilde{M}_N}\tilde{L}_Y \to \tilde{M}_N$ and $\pi_{N|M} : T^*_{T_N M} T_Y L \to T_N M$, be the natural projections. We denote by $\nu_*(*)$ and $\mu_*(*)$ the *specialization* and *microlocalization functors* respectively. Let F be an object of $\mathbf{D}^{\mathrm{b}}(X)$. Then, by Sato's fundamental distinguished triangle we have

$$Rj_{L*}\tilde{p}_L^{-1} i_L^! F|_{\tilde{M}_N} \otimes \omega_{M/L} \to R\Gamma_{\tilde{M}_N}(Rj_{L*}\tilde{p}_L^{-1} i_L^! F) \to R\dot{\pi}_{N,M*}\mu_{\tilde{M}_N}(Rj_{L*}\tilde{p}_L^{-1} i_L^! F) \overset{+1}{\to},$$

where $\omega_{M/L}$ denotes the *dualizing complex*. Applying the functor s_M^{-1}, we have

$$s_M^{-1}(Rj_{L*}\tilde{p}_L^{-1}i_L^!F|_{\tilde{M}_N}) = i'^{-1}s_L^{-1}Rj_{L*}\tilde{p}_L^{-1}i_L^!F = \nu_Y(i_L^!F)|_{T_NM},$$

$$\begin{aligned} s_M^{-1}R\Gamma_{\tilde{M}_N}(Rj_{L*}\tilde{p}_L^{-1}i_L^!F) &\simeq s_M^{-1}\tilde{i}'^!Rj_{L*}\tilde{p}_L^!i_L^!F \otimes \omega_{Y/L}^{\otimes -1} \simeq s_M^{-1}Rj_{M*}\tilde{i}^!\tilde{p}_L^!i_L^!F \otimes \omega_{N/M}^{\otimes -1} \\ &\simeq s_M^{-1}Rj_{M*}p_M^!i^!i_L^!F \otimes \omega_{N/M}^{\otimes -1} \simeq s_M^{-1}Rj_{M*}p_M^{-1}i_M^!F \\ &= \nu_N(i_M^!F). \end{aligned}$$

Further, since $\mu_{\tilde{M}_N}(Rj_{L*}\tilde{p}_L^{-1}i_L^!F)$ is a conic object, it is easy to see that

$$s_M^{-1}R\dot{\pi}_{N,M*}\mu_{\tilde{M}_N}(Rj_{L*}\tilde{p}_L^{-1}i_L^!F) \simeq R\dot{\pi}_{N|M*}s_{L\pi}^{-1}\mu_{\tilde{M}_N}(Rj_{L*}\tilde{p}_L^{-1}i_L^!F).$$

Hence we obtain the following distinguished triangle:

$$\nu_Y(i_L^!F)|_{T_NM} \otimes \omega_{M/L}^{\otimes -1} \to \nu_N(i_M^!F) \to R\dot{\pi}_{N|M*}s_{L\pi}^{-1}\mu_{\tilde{M}_N}(Rj_{L*}\tilde{p}_L^{-1}i_L^!F) \xrightarrow{+1}.$$

By Kashiwara-Schapira [**K-S2**, Proposition 4.3.5], we have a natural morphism

$$\begin{aligned} s_{L\pi}^{-1}\mu_{\tilde{M}_N}(Rj_{L*}\tilde{p}_L^{-1}i_L^!F) &\to \mu_{T_NM}(s_L^{-1}Rj_{L*}\tilde{p}_L^{-1}i_L^!F) \otimes \omega_{T_YL/\tilde{L}_Y} \otimes \omega_{T_NM/\tilde{M}_N}^{\otimes -1} \\ &\simeq \mu_{T_NM}(\nu_Y(i_L^!F)), \end{aligned}$$

and this morphism induces a natural morphism of distinguished triangles:

$$\begin{array}{ccccccc} \nu_Y(i_L^!F)|_{T_NM} \otimes \omega_{M/L}^{\otimes -1} & \longrightarrow & \nu_N(i_M^!F) & \longrightarrow & R\dot{\pi}_{N|M*}s_{L\pi}^{-1}\mu_{\tilde{M}_N}(Rj_{L*}\tilde{p}_L^{-1}i_L^!F) & \xrightarrow{+1} \\ \| & & \downarrow & & \downarrow & \\ \nu_Y(i_L^!F)|_{T_NM} \otimes \omega_{M/L}^{\otimes -1} & \longrightarrow & R\Gamma_{T_NM}(\nu_Y(i_L^!F)) & \longrightarrow & R\dot{\pi}_{N|M*}\mu_{T_NM}(\nu_Y(i_L^!F)) & \xrightarrow{+1} \end{array}$$

(see Proposition 4.3 (3)).

Next, we shall recall a general result. Let Z be a complex manifold, $\tau : E \to Z$ a complex vector bundle, and $\pi : E^* \to Z$ its dual bundle. Then, as in the real case (see for example Kashiwara-Schapira [**K-S2**, Section 5.5]) the action of $\boldsymbol{C}^\times := \boldsymbol{C}\setminus\{0\}$ on E induces a natural mapping $\theta_E : T^*E \to \boldsymbol{C}$. Set $S_E^{\boldsymbol{C}} := \theta_E^{-1}(0)$. Let (z,x) be local coordinates of E such that z is coordinates of Z and x is linear coordinates. Let $(z,x;\zeta,\xi)$ be associated coordinates of T^*E. Then θ_E is written explicitly as $\theta_E(z,x;\zeta,\xi) = \langle x,\xi\rangle$. Denote by $\mathbf{D}^{\mathrm{b}}_{\boldsymbol{C}^\times}(E)$ the subcategory of $\mathbf{D}^{\mathrm{b}}(E)$ consisting of $\boldsymbol{C}^\times$-conic objects, and by $\mathrm{SS}(\cdot)$ the *microsupport*. Then we have the following result which seems to be first stated in Laurent-Monteiro Fernandes [**L-MF2**, Lemma 1.1.1]:

Proposition 1.2. *The category $\mathbf{D}^{\mathrm{b}}_{\boldsymbol{C}^\times}(E)$ is the full subcategory of $\mathbf{D}^{\mathrm{b}}(E)$ consisting of objects F such that* $\mathrm{SS}(F) \subset S_E^{\boldsymbol{C}}$.

Indeed, the proof in Kashiwara-Schapira [**K-S2**, Proposition 5.4.5] still works in the complex case, and $\dot{E} \underset{\dot{E}/\boldsymbol{C}^\times}{\times} T^*(\dot{E}/\boldsymbol{C}^\times) = \dot{E} \underset{E}{\times} S_E^{\boldsymbol{C}}$. Hence by the same proof as in Kashiwara-Schapira [**K-S2**, Proposition 5.5.3] we obtain the proposition.

2. General boundary values.

In this section, we shall define our boundary value morphism.
First, by using admissible coordinates, we set (at least locally)

$$T_Y X^+ := \{(z, \tau) \in T_Y X; \operatorname{Re} \tau > 0\},$$

and consider the following commutative diagram:

$$\begin{array}{ccccc} T_Y L^+ & \overset{f}{\hookrightarrow} & T_Y L & & \\ {\scriptstyle T_Y i_L}\downarrow & & {\scriptstyle T_Y i_L}\downarrow & \searrow^{\tau_Y} & \\ T_Y X^+ & \overset{f}{\hookrightarrow} & T_Y X & \overset{\tau_X}{\twoheadrightarrow} & Y. \end{array}$$

We regard $T_Y L$ as a closed conic subset of $T_Y X$ by $T_Y i_L$. Note that both $T_Y L^+ \to T_Y L$ and $T_Y X^+ \to T_Y X$ are open embeddings. Set $\tau_X^+ := \tau_X f : T_Y X^+ \to Y$. Using admissible coordinates we define a continuous section $\sigma : Y \to \dot{T}_Y X$ by $z \mapsto (z; 1)$. Similarly we define ${}^t\sigma : Y \to \dot{T}_Y^* X$ by $z \mapsto (z; 1)$.

THEOREM 2.1. *For any $F \in \operatorname{Ob} \mathbf{D}^{\mathrm{b}}(X)$ with $\nu_Y(F) \in \operatorname{Ob} \mathbf{D}^{\mathrm{b}}_{\mathbf{C}^\times}(T_Y X)$, there exists the following natural isomorphism*:

$$f^{-1}\nu_Y(i_L^! F) \simeq f^{-1}\tau_Y^{-1}\sigma^{-1}\nu_Y(F) \otimes \omega_{L/X}.$$

PROOF. Recall that by Kashiwara-Schapira [**K-S2**, Proposition 4.2.5], we have natural morphisms:

$$\begin{array}{ccc} (T_Y i_L)^{-1}\nu_Y(F) \otimes \omega_{L/X} & \longrightarrow & \nu_Y(i_L^{-1}F) \otimes \omega_{L/X} \\ \downarrow & \circlearrowright & \downarrow \\ (T_Y i_L)^! \nu_Y(F) & \overset{\beta}{\longleftarrow} & \nu_Y(i_L^! F). \end{array}$$

Set $G := R\tau^+_{X*} f^{-1}\nu_Y(F) \in \operatorname{Ob} \mathbf{D}^{\mathrm{b}}(Y)$. Since $\nu_Y(F) \in \operatorname{Ob} \mathbf{D}^{\mathrm{b}}_{\mathbf{C}^\times}(T_Y X)$, by Kashiwara-Schapira [**K-S2**, Proposition 2.7.8], it follows that $f^{-1}\nu_Y(F) \simeq \tau_X^{+-1} G$. Hence, we see that $\sigma^{-1}\nu_Y(F) \simeq \sigma^{-1} f^{-1}\nu_Y(F) \simeq \sigma^{-1}\tau_X^{+-1} G \simeq G$. In particular, we have

$$\begin{aligned} f^{-1}(T_Y i_L)^{-1}\nu_Y(F) &\simeq (T_Y i_L)^{-1} f^{-1}\nu_Y(F) \simeq (T_Y i_L)^{-1}\tau_X^{+-1} G \simeq f^{-1}\tau_Y^{-1} G \\ &\simeq f^{-1}\tau_Y^{-1}\sigma^{-1} f^{-1}\nu_Y(F) \simeq f^{-1}\tau_Y^{-1}\sigma^{-1}\nu_Y(F). \end{aligned}$$

Moreover, we have the following chain of isomorphisms:

$$\begin{aligned} f^{-1}(T_Y i_L)^! \nu_Y(F) &\simeq f^!(T_Y i_L)^! \nu_Y(F) \simeq (T_Y i_L)^! f^! \nu_Y(F) \simeq (T_Y i_L)^! f^{-1}\nu_Y(F) \\ &\simeq (T_Y i_L)^! \tau_X^{+-1} G \simeq (T_Y i_L)^! \tau_X^{+!} G \otimes \omega_{T_Y X^+/Y}^{\otimes -1} \\ &\simeq f^! \tau_Y^! G \otimes \omega_{T_Y X^+/Y}^{\otimes -1} \simeq f^{-1}\tau_Y^{-1} G \otimes \omega_{T_Y L^+/Y} \otimes \omega_{T_Y X^+/Y}^{\otimes -1} \\ &\simeq (T_Y i_L)^{-1}\tau_X^{+-1} G \otimes \omega_{L/X} \simeq f^{-1}(T_Y i_L)^{-1}\nu_Y(F) \otimes \omega_{L/X}. \end{aligned}$$

Hence, we obtain the following commutative diagram:

$$\begin{array}{ccccc} f^{-1}\tau_Y^{-1}\sigma^{-1}\nu_Y(F)\otimes\omega_{L/X} \simeq & f^{-1}(T_Y i_L)^{-1}\nu_Y(F)\otimes\omega_{L/X} & \longrightarrow & f^{-1}\nu_Y(i_L^{-1}F)\otimes\omega_{L/X} \\ & \downarrow \wr & & \downarrow \\ & f^{-1}(T_Y i_L)^{!}\nu_Y(F) & \xleftarrow{\ \beta\ } & f^{-1}\nu_Y(i_L^{!}F), \end{array}$$

which implies that β is an epimorphism.

Next, we shall prove that β is a monomorphism. By taking admissible coordinates, we may assume that $X = \boldsymbol{C}^{n+1}$ and $L = \boldsymbol{C}^n \times \boldsymbol{R}$, hence we identify $\mathit{or}_{L/X}$ with $\boldsymbol{Z}_L$. By a distinguished triangle

$$(T_Y i_L)^{!}\nu_Y(F) \to (T_Y i_L)^{-1}\nu_Y(F) \to (T_Y i_L)^{-1} R\Gamma_{T_Y X \backslash T_Y L}(\nu_Y(F)) \xrightarrow{+1},$$

for any $p \in T_Y L^+$ and $j \in \boldsymbol{Z}$, we have the exact sequences

$$\begin{array}{ccccccc} \varinjlim_W H^j(W;F) & \longrightarrow & \varinjlim_W H^j(W\backslash L;F) & \longrightarrow & (\mathscr{H}^{j+1}\nu_Y(i_L^{!}F))_p & \longrightarrow & \varinjlim_W H^{j+1}(W;F) \\ \| & \circlearrowleft & \downarrow \rho & \circlearrowleft & \downarrow \beta & \circlearrowleft & \| \\ \varinjlim_W H^j(W;F) & \longrightarrow & H^j_{T_Y X\backslash T_Y L}(\nu_Y(F))_p & \longrightarrow & \mathscr{H}^{j+1}_{T_Y L}(\nu_Y(F))_p & \longrightarrow & \varinjlim_W H^{j+1}(W;F), \end{array}$$

where W ranges through the family of open subsets of X such that $p \notin C_Y(X\backslash W)$. In fact, by the excision we can take the same family of W to calculate the stalk of $\mathscr{H}^{j+1}\nu_Y(i_L^{!}F)$. Set $T_Y X\backslash T_Y L = \Omega^+ \sqcup \Omega^-$, where $\Omega^{\pm} := \{(z,\tau) \in T_Y X; \pm\mathrm{Im}\,\tau > 0\}$. Hence we have

$$\begin{aligned} \mathscr{H}^j_{T_Y X\backslash T_Y L}(\nu_Y(F))_p &\simeq \mathscr{H}^j_{\Omega^+}(\nu_Y(F))_p \oplus \mathscr{H}^j_{\Omega^-}(\nu_Y(F))_p \\ &\simeq \varinjlim_V H^j(V\cap\Omega^+;\nu_Y(F)) \oplus \varinjlim_V H^j(V\cap\Omega^-;\nu_Y(F)) \\ &\simeq \varinjlim_{V,U_V^+} H^j(U_V^+;F) \oplus \varinjlim_{V,U_V^-} H^j(U_V^-;F), \end{aligned}$$

where V ranges through the fundamental system of conic open neighborhoods of p in $T_Y X$, and each $U_V^{\pm}$ ranges through the family of open subsets of X such that $C_Y(X\backslash U_V^{\pm}) \cap \Omega^{\pm} \cap V = \varnothing$. We set $W^{\pm} := \{(z,\tau) \in W; \pm\mathrm{Im}\,\tau > 0\}$. Then

$$\varinjlim_W H^j(W\backslash L;F) = \varinjlim_W (H^j(W^+;F) \oplus H^j(W^-;F)).$$

Thus we can write $\rho = (\rho_+,\rho_-)$, where each $\rho_{\pm}$ is the restriction of sheaves:

$$\varinjlim_W H^j(W^{\pm};F) \to \varinjlim_{V,U_V^{\pm}} H^j(U_V^{\pm};F).$$

Suppose that $(u_+,u_-) \in \varinjlim_W (H^j(W^+;F) \oplus H^j(W^-;F))$ satisfies

$$\begin{aligned} \rho(u_+,u_-) = 0 \in {} & \varinjlim_{V,U_V^+} H^j(U_V^+;F) \oplus \varinjlim_{V,U_V^-} H^j(U_V^-;F) \\ & \simeq \varinjlim_V (H^j(V\cap\Omega^+;\nu_Y(F)) \oplus H^j(V\cap\Omega^-;\nu_Y(F))). \end{aligned}$$

Set $z_0 := \tau_Y(p) \in Y$ and $V_\varepsilon = \{(z,\tau) \in X; |z - z_0| < \varepsilon, 0 < |\tau| < \varepsilon, \operatorname{Re}\tau > -\varepsilon|\operatorname{Im}\tau|\}$ for an $\varepsilon > 0$. Then, we can find an $\varepsilon > 0$ such that $u_\pm = 0 \in H^j(V_\varepsilon; F)$ since $\mathscr{H}^j \nu_Y(F)$ is $\boldsymbol{C}^\times$-conic. Hence it follows that

$$(u_+, u_-) = 0 \in \varinjlim_W (H^j(W^+; F) \oplus H^j(W^-; F)),$$

namely, ρ is injective. Thus by Five Lemma, we can show that β is a monomorphism. Therefore, we have

$$f^{-1}\tau_Y^{-1}\sigma^{-1}\nu_Y(F) \otimes \omega_{L/X} \simeq f^{-1}(T_Y i_L)^{-1}\nu_Y(F) \otimes \omega_{L/X} \xrightarrow{\sim} f^{-1}\nu_Y(i_L^! F).$$

The proof is complete. □

Theorem 2.2. *For any* $F \in \operatorname{Ob}\mathbf{D}^{\mathrm{b}}(X)$ *with* $\nu_Y(F) \in \operatorname{Ob}\mathbf{D}^{\mathrm{b}}_{\boldsymbol{C}^\times}(T_Y X)$, *there exists the following natural isomorphism*:

$$f_\pi^{-1}\mu_{T_N M}(\nu_Y(i_L^! F)) \xrightarrow{\sim} f_\pi^{-1}\tau_{Y\pi}^{-1}\mu_N(\sigma^{-1}\nu_Y(F)) \otimes \omega_{L/X}.$$

Proof. By Theorem 2.1 and Kashiwara-Schapira [**K-S2**, Proposition 4.3.5], we obtain the following chain of isomorphisms:

$$\begin{aligned} f_\pi^{-1}\mu_{T_N M}(\nu_Y(i_L^! F)) &\simeq \mu_{T_N M^+}(f^{-1}\nu_Y(i_L^! F)) \simeq \mu_{T_N M^+}(f^{-1}\tau_Y^{-1}\sigma^{-1}\nu_Y(F)) \otimes \omega_{L/X} \\ &\simeq f_\pi^{-1}\tau_{Y\pi}^{-1}\mu_N(\sigma^{-1}\nu_Y(F)) \otimes \omega_{L/X} \otimes \omega_{T_N M^+/N} \otimes \omega_{T_Y L^+/Y}^{\otimes -1} \\ &\simeq f_\pi^{-1}\tau_{Y\pi}^{-1}\mu_N(\sigma^{-1}\nu_Y(F)) \otimes \omega_{L/X}. \end{aligned}$$

This proves the theorem. □

Definition 2.3. For any $F \in \operatorname{Ob}\mathbf{D}^{\mathrm{b}}(X)$ with $\nu_Y(F) \in \operatorname{Ob}\mathbf{D}^{\mathrm{b}}_{\boldsymbol{C}^\times}(T_Y X)$, by virtue of Theorem 2.2 we define:

$$\begin{aligned} \beta : f_\pi^{-1}s_{L\pi}^{-1}\mu_{\tilde{M}_N}(Rj_{L*}\tilde{p}_L^{-1}i_L^! F) &\to f_\pi^{-1}\mu_{T_N M}(\nu_Y(i_L^! F)) \\ &\xrightarrow{\sim} f_\pi^{-1}\tau_{Y\pi}^{-1}\mu_N(\sigma^{-1}\nu_Y(F)) \otimes \omega_{L/X}. \end{aligned}$$

Next, we shall show that β is an epimorphism under the near-hyperbolicity condition due to Laurent-Monteiro Fernandes [**L-MF2**, Definition 1.3.1]:

Definition 2.4. Let F be an object of $\mathbf{D}^{\mathrm{b}}(X)$. Then we say F is *near-hyperbolic* at $x_0 \in N$ (in *dt*-codirection) if there exist positive constants C and ε_1 such that

$$\begin{aligned} &\mathrm{SS}(F) \cap \{(z,\tau; z^*, \tau^*) \in T^*X; |z - x_0| < \varepsilon_1, |\tau| < \varepsilon_1, 0 < t\} \\ &\qquad \subset \{(z,\tau; z^*, \tau^*) \in T^*X; |t^*| \le C(|y^*|(|y| + |s|) + |x^*|)\} \end{aligned}$$

holds by admissible coordinates $(z,\tau) = (x + \sqrt{-1}y, t + \sqrt{-1}s)$ of X and associated coordinates $(z,\tau; z^*, \tau^*) = (x + \sqrt{-1}y, t + \sqrt{-1}s; x^* + \sqrt{-1}y^*, t^* + \sqrt{-1}s^*)$ of T^*X.

Theorem 2.5. *Let F be an object of* $\mathbf{D}^{\mathrm{b}}(X)$. *Assume that* $\nu_Y(F) \in \operatorname{Ob}\mathbf{D}^{\mathrm{b}}_{\boldsymbol{C}^\times}(T_Y X)$ *and F is near-hyperbolic at* $x_0 \in N$. *Then, for any* $p^* = (x_0, t_0; \sqrt{-1}\langle\xi_0, dx\rangle) \in T^*_{T_N M^+}T_Y L^+$, *the morphism β induces an isomorphism*:

$$\beta : s_{L\pi}^{-1}\mu_{\tilde{M}_N}(Rj_{L*}\tilde{p}_L^{-1}i_L^! F)_{p^*} \to \mu_N(\sigma^{-1}\nu_Y(F))_{\tau_{Y\pi}(p^*)} \otimes \omega_{L/X}.$$

Proof. By Theorem 2.2, we may show the isomorphism

$$s_{L\pi}^{-1}\mu_{\tilde{M}_N}(Rj_{L*}\tilde{p}_L^{-1}i_L^!F)_{p^*} \xrightarrow{\sim} \mu_{T_NM}(\nu_Y(i_L^!F))_{p^*}.$$

By virtue of the inverse Fourier-Sato transformation, it is enough to show that the isomorphism

$$\tilde{s}_L^{-1}\nu_{\tilde{M}_N}(Rj_{L*}\tilde{p}_L^{-1}R\Gamma_L(F))_{p_0} \xrightarrow{\sim} \nu_{T_NM}(\nu_Y(R\Gamma_L(F)))_{p_0}$$

holds at any point $p_0 = (x_0, t_0; \sqrt{-1}y_0) \in T_{T_NM^+}T_YL^+$. Here $\tilde{s}_L : T_{T_NM}T_YL \to T_{\tilde{M}_N}\tilde{L}_Y$ is a natural mapping. Since

$$\tilde{s}_L^{-1}\nu_{\tilde{M}_N}(Rj_{L*}\tilde{p}_L^{-1}R\Gamma_L(F))|_{T_NM^+} \simeq \nu_{T_NM}(\nu_Y(R\Gamma_L(F)))|_{T_NM^+} \simeq \nu_Y(R\Gamma_L(F))|_{T_NM^+},$$

we may assume that $y_0 \neq 0$. By taking suitable admissible coordinates, we may assume that $X = \boldsymbol{C}^{n+1} \supset L = \boldsymbol{C}^n \times \boldsymbol{R}$ and so on with $x_0 = 0$. We set as in Bony-Schapira [**B-S2**]

$$B(0,a) := \{(x,t) \in \boldsymbol{R}^{n+1}; |x| + |t| < a\}, \quad B'(0,a) := \{x \in \boldsymbol{R}^n; |x| < a\}.$$

Set $K_+(a,\delta) := \operatorname{Int}\gamma[B'(0,a) \cup \{(0,a\delta)\}]$. Here $\gamma[\cdot]$ means the *convex hull* and $\operatorname{Int} A$ denotes the *interior* of A. For an open convex cone $\Gamma' \subset \boldsymbol{R}^n$, we set $\Gamma'_\varepsilon := \Gamma' \cap B'(0,\varepsilon)$. Then, for any $k \in \boldsymbol{Z}$ we have

$$\mathscr{H}^k\nu_{\tilde{M}_N}(Rj_{L*}\tilde{p}_L^{-1}R\Gamma_L(F))|_{\tilde{s}_L(p_0)} = \varinjlim_{a,\delta,\Gamma'_\varepsilon} H^k(K_+(a,\delta) + \sqrt{-1}\Gamma'_\varepsilon; R\Gamma_L(F)),$$

$$\mathscr{H}^k\nu_{T_NM}(\nu_Y(R\Gamma_L(F))|_{p_0} = \varinjlim_{U(a,\delta,\Gamma'_\varepsilon)} H^k(U_+(a,\delta,\Gamma'_\varepsilon); R\Gamma_L(F)).$$

Here $\Gamma' \subset \boldsymbol{R}^n$ ranges through the family of open conic neighborhoods of y_0, $U(a,\delta,\Gamma'_\varepsilon)$ ranges through the family of open neighborhoods of $B(0,a) + \sqrt{-1}\Gamma'_\varepsilon$ in L, and we set

$$U_+(a,\delta,\Gamma'_\varepsilon) := U(a,\delta,\Gamma'_\varepsilon) \cap \{(z,t) \in L; t > 0\}.$$

Then the proof of the theorem is reduced to the following proposition. □

Proposition 2.6 [cf. [**B-S2**, Lemme 3.2]). *Let $\Gamma' \subset \boldsymbol{R}^n$ be a conic neighborhood of y_0. Then there exists a positive constant $\delta > 0$ satisfying the following: If a and ε are sufficiently small positive constants, then for any $k \in \boldsymbol{Z}$ there exist $\varepsilon', \delta' > 0$ and a conic neighborhood $\Gamma \subset \boldsymbol{R}^n$ of y_0 such that*

$$H^k(K_+(a,\delta') + \sqrt{-1}\Gamma_{\varepsilon'}; R\Gamma_L(F)) \xrightarrow{\sim} H^k(U_+(a,\delta,\Gamma'_\varepsilon); R\Gamma_L(F)).$$

Proof. The proof is very similar to that of [**B-S2**, Lemme 3.2]. We use the following lemma instead of [**B-S2**, Théorème 1.1]):

Lemma 2.7 (cf. [**B-S1**, Théorème 2.1]). *Let $\omega \subset \Omega \subset L$ be convex sets such that ω is locally compact and Ω is an open set. Let G be an object of $\mathbf{D}^{\mathrm{b}}(L)$. Set*

$$A := \{(z^*, t^*); (z,t;z^*,t^*) \in \mathrm{SS}(G) \text{ for some } (z,t) \in \Omega\}.$$

Suppose that if a hyperplane with normal vector in A crosses Ω, then this hyperplane always crosses ω. Then for any open neighborhood $\omega' \subset \Omega$ of ω, it follows that

$$R\Gamma(\Omega; G) \xrightarrow{\sim} R\Gamma(\omega'; G).$$

Proof of Lemma 2.7. Set

$$\Phi := \{V \subset \Omega; V \text{ is open}, \omega' \subset V, R\Gamma(V;G) \xrightarrow{\sim} R\Gamma(\omega';G)\}.$$

Then $\Phi \neq \emptyset$. Let $\{V_i\}_{i \in I} \subset \Phi$ be any totally ordered subset. Set $\tilde{V} := \bigcup_{i \in I} V_i$. Since L is a Lindelöf space, we can find a subsequence $\{V_j'\}_{j \in \boldsymbol{N}} \subset \{V_i\}_{i \in I}$ such that $\tilde{V} = \bigcup_{j \in \boldsymbol{N}} V_j'$ and $V_j' \subset V_k'$ if $j \leq k$. Hence $\{H^{k-1}(V_j';G)\}_{j \in \boldsymbol{N}}$ satisfies Mittag-Leffler condition for any $k \in \boldsymbol{Z}$ since $H^{k-1}(V_j';G) \simeq H^{k-1}(\omega';G)$ for any $j \in \boldsymbol{N}$. Thus we have $H^k(\tilde{V};G) \xrightarrow{\sim} H^k(\omega';G)$ (see [**K-S2**, Proposition 2.7.1]). Hence by induction on k, we see $\tilde{V} \in \Phi$. Therefore by Zorn's Lemma, there exists a maximal element $V \in \Phi$. Suppose that $V \neq \Omega$. Take $p \in \Omega \backslash V$. Then instead of Zerner's theorem, we can use the theory of microsupports to prove the existence of $W \in \Phi$ such that $p \in W$ (see the proof of [**B-S1**, Théorème 2.1] and [**K-S2**, Proposition 5.2.1, Lemma 5.2.2]). Further by the method of proof, we may assume $R\Gamma(W;G) \xrightarrow{\sim} R\Gamma(V \cap W;G)$. Thus, we have isomorphisms $R\Gamma(V;G) \simeq R\Gamma(\omega';G) \simeq R\Gamma(V;G) \simeq R\Gamma(V \cap W;G)$. Hence, by the distinguished triangle

$$R\Gamma(V \cup W;G) \to R\Gamma(V;G) \oplus R\Gamma(W;G) \to R\Gamma(V \cap W;G) \xrightarrow{+1},$$

$R\Gamma(V \cup W;G) \simeq R\Gamma(\omega';G)$ holds; that is, $V \subsetneq V \cup W \in \Phi$, which is a contradiction. □

We end the proof of Proposition 2.6 (cf. also Tahara [**T**, Lemmata 2.1.1 and 2.1.2]). Recall that $i_L : L \to X$ is the canonical embedding. By [**K-S2**, Corollary 6.4.4] we have

$$\mathrm{SS}(R\Gamma_L(F)) \subset i_L^{\#}(\mathrm{SS}(F)).$$

Thus if $(0, t_0; z_0^*, t_0^*) \in \mathrm{SS}(R\Gamma_L(F)) \cap \{(z,t;z^*,t^*) \in T^*L; |z| < \varepsilon_1, 0 < t < \varepsilon_1\}$, then by [**K-S2**, Remark 6.2.8] and the near-hyperbolicity condition, we can find a sequence $\{(z_j;\tau_j;z_j^*,\tau_j^*)\}_{j \in \boldsymbol{N}} \subset \{(z,\tau;z^*,\tau^*) \in T^*X; |t^*| \leq C(|y^*|(|y|+|s|)+|x^*|)\}$ such that $(z_j;t_j;z_j^*,t_j^*) \underset{j}{\to} (0,t_0;z_0^*,t_0^*)$ and $|s_j|\,|s_j^*| \underset{j}{\to} 0$. In particular since $|s_j| \underset{j}{\to} 0$, we see

$$\begin{aligned}&\mathrm{SS}(R\Gamma_L(F)) \cap \{(z,t;z^*,t^*) \in T^*L; |z| < \varepsilon_1, 0 < t < \varepsilon_1\} \\ &\qquad \subset \{(z,t;z^*,t^*) \in T^*L; |z| < \varepsilon_1, 0 < t < \varepsilon_1, |t^*| \leq C(|y^*|\,|y| + |x^*|)\}.\end{aligned}$$

Thus we have only to follow the argument in the proof of [**B-S2**, Lemme 3.2] to obtain

$$R\Gamma(M_{\eta,\varepsilon}; R\Gamma_L(F)) \xrightarrow{\sim} R\Gamma(U_+(a,\delta,\Gamma_\varepsilon'); R\Gamma_L(F)).$$

Here $M_{\eta,\varepsilon} := \operatorname{Int}\gamma[(B'(0,a) + \sqrt{-1}\Gamma_{\varepsilon/2}') \cup \{(0,\alpha\delta) + \sqrt{-1}\eta\}]$ for an $\eta \in \Gamma_{\varepsilon/4}'$ and an independent constant $\alpha > 0$. By the same argument as in the proof of Lemma 2.7, we have

$$R\Gamma\Big(\bigcup_{\eta \in \Gamma_{\varepsilon/4}'} M_{\eta,\varepsilon}; R\Gamma_L(F)\Big) \xrightarrow{\sim} R\Gamma\big(U_+(a,\delta,\Gamma_\varepsilon'); R\Gamma_L(F)\big).$$

We can find $\varepsilon', \delta' > 0$ and a conic neighborhood $\Gamma \subset \boldsymbol{R}^n$ of y_0 such that

$$K_+(a,\delta') + \sqrt{-1}\Gamma_{\varepsilon'} \subset \bigcup_{\eta \in \Gamma_{\varepsilon/4}'} M_{\eta,\varepsilon}.$$

The proof is complete. □

3. Regular-specializable systems.

In this section, we shall recall the basic results concerning the regular-specializable $\mathscr{D}$-Module and its nearby-cycle. Although all the contents in this section are well-known to specialists, we shall give a detailed review for the convenience of the reader. Note that a generalization to the higher-codimensional case is obtained, but we restrict ourselves to the one-codimensional case. We inherit the notation from §1. In particular, Y denotes a one-codimensional complex submanifold of X.

Let $\mathscr{D}_X$ be the Ring on X of holomorphic differential operators, and $\{\mathscr{D}_X^{(m)}\}_{m\in \boldsymbol{N}_0}$ the usual order filtration on $\mathscr{D}_X$. Let us recall the definition of the V-filtration:

Definition 3.1. Let $\mathscr{I}_Y$ be the defining Ideal of Y in $\mathscr{O}_X$ with a convention that $\mathscr{I}_Y^j = \mathscr{O}_X$ for $j \leq 0$. The *V-filtration* $\{\mathsf{F}_Y^k(\mathscr{D}_X)\}_{k\in \boldsymbol{Z}}$ (*along* Y) is a filtration on $\mathscr{D}_X|_Y$ defined by

$$\mathsf{F}_Y^k(\mathscr{D}_X) := \bigcap_{j\in \boldsymbol{Z}} \{P \in \mathscr{D}_X|_Y; P\mathscr{I}_Y^j \subset \mathscr{I}_Y^{j-k}\}.$$

It is easy to see that by admissible coordinates, this filtration is written as

$$\mathsf{F}_Y^k(\mathscr{D}_X) = \Big\{ \sum_{j-i\leq k} P_{ij}(z,\partial_z)\tau^i\partial_\tau^j \in \mathscr{D}_X|_Y \Big\}.$$

Let $\mathscr{D}_{[T_Y X]}$ be the subsheaf of $\mathscr{D}_{T_Y X}$ consisting of operators which are polynomials with respect to the fiber variables. Then the associated graded Ring with $\{\mathsf{F}_Y^k(\mathscr{D}_X)\}_{k\in\boldsymbol{Z}}$ is canonically isomorphic to $\tau_{X*}\mathscr{D}_{[T_Y X]}$, hence this graded Ring is non-commutative (for details of this filtration, we refer to Björk [**Bj**], Sabbah [**Sab**] and Schapira [**Sc2**]).

We denote by ϑ the Euler vector field on $T_Y X$. Then ϑ is characterized by $\vartheta\varphi = k\varphi$ for any $\varphi \in \mathscr{I}_Y^k/\mathscr{I}_Y^{k+1}$ and $k \in \boldsymbol{N}$, and ϑ can be represented by $\tau\partial_\tau$ by admissible coordinates.

Definition 3.2. A coherent $\mathscr{D}_X|_Y$-Module $\mathscr{M}$ is said to be *regular-specializable* (*along* Y) if there exist locally a coherent $\mathscr{O}_X$-sub-Module $\mathscr{L}$ of $\mathscr{M}$ and a non-zero polynomial $b(\alpha) \in \boldsymbol{C}[\alpha]$ such that the following conditions are satisfied:

(1) $\mathscr{L}$ generates $\mathscr{M}$ over $\mathscr{D}_X$; that is, $\mathscr{M} = \mathscr{D}_X\mathscr{L}$;

(2) $b(\vartheta)\mathscr{L} \subset (\mathsf{F}_Y^{-1}(\mathscr{D}_X) \cap \mathscr{D}_X^{(m)})\mathscr{L}$, where m is the degree $\deg b$ of $b(\alpha)$.

In what follows, we shall omit the phrase "along Y" since Y is fixed.

Remark 3.3. (1) Let $\mathscr{M}$ be a coherent $\mathscr{D}_X|_Y$-Module for which Y is non-characteristic. Then $\mathscr{M}$ is regular-specializable.

(2) By Kashiwara-Kawai [**K-K2**, Lemma 4.1.5], any regular-holonomic $f^{-1}\mathscr{D}_X$-Module is regular-specializable.

Proposition 3.4. (1) *A coherent $\mathscr{D}_X|_Y$-Module $\mathscr{M}$ is regular-specializable if and only if the following condition is satisfied*: *For any local section u of $\mathscr{M}$, there exist a non-zero polynomial $b_u(\alpha) \in \boldsymbol{C}[\alpha]$ and $Q_u \in \mathsf{F}_Y^{-1}(\mathscr{D}_X) \cap \mathscr{D}_X^{(\deg b_u)}$ such that*

$$(b_u(\vartheta) + Q_u)u = 0.$$

(2) *In an exact sequence of coherent* $\mathscr{D}_X|_Y$*-Modules*

$$0 \to \mathscr{M}' \to \mathscr{M} \to \mathscr{M}'' \to 0,$$

$\mathscr{M}$ *is regular-specializable if and only if both* $\mathscr{M}'$ *and* $\mathscr{M}''$ *are regular-specializable.*

For the proof, see Mebkhout [**Me**] or Sabbah [**Sab**].

PROPOSITION 3.5. *Let* $\mathscr{M}$ *be a coherent* $\mathscr{D}_X|_Y$*-Module. If* $\mathscr{M}$ *is regular-specializable, then* $R\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mu_Y(\mathscr{O}_X))$ *and* $R\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \nu_Y(\mathscr{O}_X))$ *are objects of* $\mathbf{D}^{\mathrm{b}}_{\boldsymbol{C}^\times}(T^*_Y X)$ *and* $\mathbf{D}^{\mathrm{b}}_{\boldsymbol{C}^\times}(T_Y X)$ *respectively.*

PROOF. Denote by $C_{T^*_Y X}(\cdot)$ the normal cone along $T^*_Y X$. Since the Hamiltonian isomorphism induces isomorphisms $T^*T_Y X \simeq T^*T^*_Y X \simeq T_{T^*_Y X}T^*X$, we identify these spaces. Then by Kashiwara-Schapira [**K-S2**, Theorem 6.4.1], for any $F \in \operatorname{Ob} \mathbf{D}^{\mathrm{b}}(X)$ we have:

$$\mathrm{SS}(\nu_Y(F)) = \mathrm{SS}(\mu_Y(F)) \subset C_{T^*_Y X}(\mathrm{SS}(F)).$$

Let (z, τ) be admissible coordinates of X and $(z, \tau; z^*, \tau^*)$ the associated coordinates of T^*X. As in §1, we use identification $T_Y X = X$ and $T^*X = T_{T^*_Y X}T^*X$ by means of (z, τ). Then under these coordinates we have (see [**K-S2**, (6.2.3)]):

$$\begin{array}{ccccc} T^*T_Y X & \overset{\sim}{\text{———}} & T^*T^*_Y X & \overset{\sim}{\text{———}} & T_{T^*_Y X}T^*X \\ \Cup & & \Cup & & \Cup \\ (z, \tau; z^*, \tau^*) & \longleftrightarrow & (z, \tau^*; z^*, -\tau) & \longleftrightarrow & (z, \tau; z^*, \tau^*) \end{array}$$

$$S^{\boldsymbol{C}}_{T^*_Y X} = \{(z, \tau^*; z^*, -\tau) \in T^*T^*_Y X; \tau\tau^* = 0\}.$$

Assume that $\mathscr{M}$ is generated by $\{u_j\}_{j=1}^J$ over $\mathscr{D}_X$. Then by virtue of Proposition 3.4, each $\mathscr{D}_X u_j$ is regular-specializable. Hence, for each j we can find a non-zero polynomial $b_j(\alpha)$ and $Q_j \in \mathscr{D}_X^{(m_j)} \cap \mathsf{F}_Y^{-1}(\mathscr{D}_X)$ such that $(b_j(\vartheta) + Q_j)u_j = 0$, where m_j denotes the degree of $b_j(\alpha)$. Set $\mathscr{L}_j := \mathscr{D}_X / \mathscr{D}_X(b_j(\vartheta) + Q_j)$. Then it follows that each $\mathscr{L}_j$ is regular-specializable and that there exists an epimorphism $\bigoplus_{j=1}^J \mathscr{L}_j \to \mathscr{M} \to 0$. Hence we have

$$\mathrm{char}(\mathscr{M}) \subset \mathrm{char}\Big(\bigoplus_{j=1}^J \mathscr{L}_j\Big) = \bigcup_{j=1}^J \mathrm{char}(\mathscr{L}_j).$$

Since the principal symbol of $b_j(\vartheta) + Q_j$ has the form of $(\tau\tau^*)^{m_j} + \tau q_j(z, \tau; z^*, \tau\tau^*)$, we have $C_{T^*_Y X}(\mathrm{char}(\mathscr{L}_j)) = \{(z, \tau; z^*, \tau^*); \tau\tau^* = 0\}$. Thus we have

$$\begin{aligned} \mathrm{SS}(R\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \nu_Y(\mathscr{O}_X))) = \mathrm{SS}(R\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mu_Y(\mathscr{O}_X))) &\subset C_{T^*_Y X}(\mathrm{char}(\mathscr{M})) \\ &\subset \bigcup_{j=1}^J C_{T^*_Y X}(\mathrm{char}(\mathscr{L}_j)) = S^{\boldsymbol{C}}_{T^*_Y X}. \end{aligned}$$

This proves the proposition by virtue of Proposition 1.2. □

We denote by $\mathscr{C}^{\boldsymbol{R}}_{Y|X} := \mu_Y(\mathscr{O}_X)[1]$ the sheaf of *real holomorphic microfunctions* on $T^*_Y X$. Then, by Proposition 3.5 and the proof in Kashiwara-Schapira [**K-S2**, Proposition 8.6.3], we obtain the following:

Corollary 3.6. *For any regular-specializable $\mathscr{D}_X|_Y$-Module $\mathscr{M}$, there exists the following distinguished triangle*:

$$R\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{O}_X)|_Y \to R\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \sigma^{-1}\nu_Y(\mathscr{O}_X)) \to R\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, {}^t\sigma^{-1}\mathscr{C}^{\boldsymbol{R}}_{Y|X}) \xrightarrow{+1}.$$

Let $\mathscr{M}$ be a coherent $\mathscr{D}_X|_Y$-Module. Recall that a V-filtration $\{\mathsf{F}^k(\mathscr{M})\}_{k\boldsymbol{Z}}$ is said to be *good* if there exist (locally) generators $\{u_j\}_{j=1}^m$ and $k_j \in \boldsymbol{Z}$ such that for any $k \in \boldsymbol{Z}$

$$\mathsf{F}^k(\mathscr{M}) = \sum_{j=1}^m \mathsf{F}_Y^{k-k_j}(\mathscr{D}_X)u_j$$

holds. The following theorem is proved by Kashiwara [**Kas**] (cf. also Björk [**Bj**]):

Theorem 3.7. *Set $G := \{\alpha \in \boldsymbol{C}; -1 \le \operatorname{Re}\alpha < 0\}$. Then, for any regular-specializable $\mathscr{D}_X$-Module $\mathscr{M}$, there exist a unique good V-filtration $\{\mathsf{F}_Y^k(\mathscr{M})\}_{k\in\boldsymbol{Z}}$ on $\mathscr{M}$ and a non-zero polynomial $b_Y(\alpha) \in \boldsymbol{C}[\alpha]$ such that $b_Y^{-1}(0) \subset G$ and for any $k \in \boldsymbol{Z}$ the following holds*:

$$b_Y(\vartheta + k)\mathsf{F}_Y^k(\mathscr{M}) \subset \mathsf{F}_Y^{k-1}(\mathscr{M}).$$

Definition 3.8. Let $\mathscr{M}$ be a regular-specializable $\mathscr{D}_X|_Y$-Module. Under the notation of Theorem 3.7, the *nearby-cycle* $\Psi_Y(\mathscr{M})$ and the *vanishing-cycle* $\Phi_Y(\mathscr{M})$ are defined by:

$$\Psi_Y(\mathscr{M}) := \mathsf{F}_Y^{-1}(\mathscr{M})/\mathsf{F}_Y^{-2}(\mathscr{M}),$$

$$\Phi_Y(\mathscr{M}) := \mathsf{F}_Y^{0}(\mathscr{M})/\mathsf{F}_Y^{-1}(\mathscr{M}).$$

Remark 3.9. Laurent [**L2**] extended the definitions of nearby and vanishing cycles to the derived category of bounded complexes with (regular-)specializable cohomologies by using the theory of second microlocalization.

Let $\iota : Y \to X$ be the natural embedding. The *inverse image* in the sense of $\mathscr{D}$-Module is defined by

$$\boldsymbol{D}\iota^*\mathscr{M} := \mathscr{O}_Y \underset{\iota^{-1}\mathscr{O}_X}{\overset{\boldsymbol{L}}{\otimes}} \iota^{-1}\mathscr{M} = \mathscr{D}_{Y\to X} \underset{\iota^{-1}\mathscr{D}_X}{\overset{\boldsymbol{L}}{\otimes}} \iota^{-1}\mathscr{M}.$$

Here $\mathscr{D}_{Y\to X} := \mathscr{O}_Y \underset{\iota^{-1}\mathscr{O}_X}{\otimes} \iota^{-1}\mathscr{D}_X$ is the *transfer bi-Module*. Then we have (cf. Laurent [**L2**], Mebkhout [**Me**] or Sabbah [**Sab**]):

Proposition 3.10. *For any regular-specializable $\mathscr{D}_X|_Y$-Module $\mathscr{M}$, $\Psi_Y(\mathscr{M})$, $\Phi_Y(\mathscr{M})$ and each cohomology of $\boldsymbol{D}\iota^*\mathscr{M}$ are coherent $\mathscr{D}_Y$-Modules. Moreover, there exists the following distinguished triangle*:

$$\Phi_Y(\mathscr{M}) \xrightarrow{\mathrm{Var}} \Psi_Y(\mathscr{M}) \longrightarrow \boldsymbol{D}\iota^*\mathscr{M} \xrightarrow{+1}.$$

Here, $\mathrm{Var} := \varphi(\vartheta)\tau$ *with* $\varphi(\zeta) := (e^{2\pi\sqrt{-1}\zeta} - 1)/\zeta$.

Let $\dot{\gamma} : \dot{T}_Y^*X \to \boldsymbol{P}_Y^*X := \dot{T}_Y^*X/\boldsymbol{C}^\times$ be the natural projection. Denote by $\mathscr{C}^{\boldsymbol{R},f}_{Y|X}$ the sheaf of *temperate real holomorphic microfunctions* on T_Y^*X (see Andronikof [**A**] for the definition). Since $\mathscr{C}^{\boldsymbol{R},f}_{Y|X}$ has the unique continuation property, Laurent [**L2**] introduced a subsheaf $\tilde{\mathscr{C}}_{Y|X}$ of $\mathscr{C}^{\boldsymbol{R},f}_{Y|X}$ as follows: If $p^* \in \dot{T}^*_Y X$, then the stalk $\tilde{\mathscr{C}}_{Y|X}|_{p^*} \subset \mathscr{C}^{\boldsymbol{R},f}_{Y|X}|_{p^*}$ is consisting of germs which have a continuation to the universal covering of $\dot{\gamma}^{-1}\dot{\gamma}(p^*)$ with finite determinations. If $p^* \in T^*_Y Y = Y$, then set $\tilde{\mathscr{C}}_{Y|X}|_{p^*} := \mathscr{C}^{\boldsymbol{R},f}_{Y|X}|_{p^*} = \mathscr{B}_{Y|X}|_{p^*}$.

REMARK 3.11. In fact, Laurent defined several sheaves in order to describe the growth condition of holomorphic microfunction solutions to a general specializable $\mathscr{D}$-Module (see [**L1**] and [**L2**]).

Denote by $\mathscr{N}_{X|Y}$ the sheaf of *Nilsson class functions* on X along Y and regard as a sheaf on Y. Then the following theorem is proved by Laurent [**L2**] (cf. also Kashiwara-Kawai [**K-K3**]):

THEOREM 3.12. (1) *There exists the following exact sequence*:

$$0 \longrightarrow \mathscr{O}_X|_Y \longrightarrow \mathscr{N}_{X|Y} \xrightarrow{\mathrm{Can}} {}^t\sigma^{-1}\tilde{\mathscr{C}}_{Y|X} \longrightarrow 0.$$

(2) *For any regular-specializable $\mathscr{D}_X|_Y$-Module $\mathscr{M}$, there exists a natural isomorphism*

$$R\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \tilde{\mathscr{C}}_{Y|X}) \xrightarrow{\sim} R\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{C}^{\boldsymbol{R}}_{Y|X}).$$

Further there exists the following isomorphism of distinguished triangles:

$$\begin{array}{ccccccc} R\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{O}_X)|_Y & \longrightarrow & R\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{N}_{X|Y}) & \xrightarrow{\mathrm{Can}} & R\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, {}^t\sigma^{-1}\tilde{\mathscr{C}}_{Y|X}) & \xrightarrow{+1} & \\ \downarrow\wr & & \downarrow\wr & & \downarrow\wr & & \\ R\mathscr{H}om_{\mathscr{D}_Y}(\boldsymbol{D}\iota^*\mathscr{M}, \mathscr{O}_Y) & \longrightarrow & R\mathscr{H}om_{\mathscr{D}_Y}(\Psi_Y(\mathscr{M}), \mathscr{O}_Y) & \xrightarrow{{}^t(\mathrm{Var})} & R\mathscr{H}om_{\mathscr{D}_Y}(\Phi_Y(\mathscr{M}), \mathscr{O}_Y) & \xrightarrow{+1} & . \end{array}$$

REMARK 3.13. (1) The isomorphism (Cauchy-Kovalevskaja type theorem)

$$R\mathscr{H}om_{\mathscr{D}_Y}(\boldsymbol{D}\iota^*\mathscr{M}, \mathscr{O}_Y) \simeq R\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{O}_X)|_Y$$

holds for Fuchsian systems in the sense of Laurent-Monteiro Fernandes [**L-MF1**].

(2) Mandai [**Man**] extended the definition of boundary values to a general Fuchsian differential equation in the complex domain.

By Corollary 3.6 and Theorem 3.12, we can obtain:

THEOREM 3.14. *Let $\mathscr{M}$ be a regular-specializable $\mathscr{D}_X|_Y$-Module. Then, a natural morphism $\mathscr{N}_{X|Y} \to \sigma^{-1}\nu_Y(\mathscr{O}_X)$ induces the following isomorphism of distinguished triangles*:

$$\begin{array}{ccccccc} R\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{O}_X)|_Y & \longrightarrow & R\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{N}_{X|Y}) & \xrightarrow{\mathrm{Can}} & R\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, {}^t\sigma^{-1}\tilde{\mathscr{C}}_{Y|X}) & \xrightarrow{+1} & \\ \| & & \downarrow\wr & & \downarrow\wr & & \\ R\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{O}_X)|_Y & \longrightarrow & R\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \sigma^{-1}\nu_Y(\mathscr{O}_X)) & \longrightarrow & R\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, {}^t\sigma^{-1}\mathscr{C}^{\boldsymbol{R}}_{Y|X}) & \xrightarrow{+1} & . \end{array}$$

In particular, there exists the following isomorphism:

$$R\mathscr{H}om_{\mathscr{D}_Y}(\Psi_Y(\mathscr{M}), \mathscr{O}_Y) \simeq R\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \sigma^{-1}\nu_Y(\mathscr{O}_X)).$$

4. Several sheaves attached to the boundary.

In this section, we recall several sheaves attached to the boundary due to Oaku [**Oa3**]. These sheaves will play essential roles for our boundary value problem. Note that in Oaku [**Oa3**] these sheaves are defined on cosphere bundles. So we shall present equivalent but slightly different definitions on cotangent bundles along the line of Oaku-Yamazaki [**O-Y**]. We refer to Oaku [**Oa3**] or Oaku-Yamazaki [**O-Y**] for the proofs. Although only the higher-codimensional case is treated in Oaku-Yamazaki [**O-Y**], the same proofs also work as in the one-codimensional case.

We inherit the notation from §2, and we denote by $\mathscr{O}_X, \mathscr{B}_M$ and $\mathscr{C}_M$ the sheaves of *holomorphic functions* on X, of *hyperfunctions* on M and of *microfunctions* on T_M^*X respectively. Further, Let $\mathscr{B}\mathscr{O}_L$ be the sheaf of *hyperfunctions with holomorphic parameters* z on L; that is,

$$\mathscr{B}\mathscr{O}_L := \mathscr{H}_L^1(\mathscr{O}_X) \otimes \mathit{or}_{L/X} \simeq i_L^!\mathscr{O}_X \otimes \mathit{or}_{L/X}[1].$$

Definition 4.1. We set:

$$\mathscr{C}_{N|M} := s_{L\pi}^{-1}\mu_{\tilde{M}_N}(Rj_{L*}\tilde{p}_L^{-1}i_L^!\mathscr{O}_X) \otimes \mathit{or}_{M/X}[n+1],$$

$$\tilde{\mathscr{C}}_{N|M} := \mu_{T_NM}(\nu_Y(i_L^!\mathscr{O}_X)) \otimes \mathit{or}_{N/L}[n+1],$$

$$\tilde{\mathscr{B}}_{N|M} := \tilde{\mathscr{C}}_{N,M}|_{T_NM}.$$

Remark 4.2. The reader may confuse the sheaf $\tilde{\mathscr{C}}_{Y|X}$ with the sheaf $\tilde{\mathscr{C}}_{N|M}$ in §3 because we used a notation similar to each other. However, these sheaves are quite different.

By virtue of the following proposition, we can regard $\mathscr{C}_{N|M}$ as a microlocalization of $\nu_N(\mathscr{B}_M)$, and $\mathscr{C}_{N|M}$ as a subsheaf of $\tilde{\mathscr{C}}_{N|M}$:

Proposition 4.3. (1) *$\mathscr{C}_{N|M}$ and $\tilde{\mathscr{C}}_{N|M}$ are concentrated in degree zero; that is, $\mathscr{C}_{N|M}$* and *$\tilde{\mathscr{C}}_{N|M}$ are regarded as sheaves on $T_{T_NM}^*T_YL$.*

(2) *A canonical morphism $s_{N|M}^*: \mathscr{C}_{N|M} \to \tilde{\mathscr{C}}_{N|M}$ is a monomorphism.*

(3) *$\mathscr{C}_{N|M}|_{T_NM} = \nu_N(\mathscr{B}_M)$ holds. Further, there exists the following commutative diagram with exact rows on T_NM:*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \nu_Y(\mathscr{B}\mathscr{O}_L)|_{T_NM} & \longrightarrow & \nu_N(\mathscr{B}_M) & \longrightarrow & \dot{\pi}_{N|M*}\mathscr{C}_{N|M} & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \nu_Y(\mathscr{B}\mathscr{O}_L)|_{T_NM} & \longrightarrow & \tilde{\mathscr{B}}_{N|M} & \longrightarrow & \dot{\pi}_{N|M*}\tilde{\mathscr{C}}_{N|M} & \longrightarrow & 0.
\end{array}$$

Note that $\nu_Y(\mathscr{B}\mathscr{O}_L)$ is concentrated in degree zero.

5. Boundary values for regular-specializable system.

We are ready to define our boundary value morphism:

Definition 5.1. Let $\mathscr{M}$ be a regular-specializable $\mathscr{D}_X|_Y$-Module. Then by Proposition 3.5, $R\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{O}_X)$ satisfies the assumption of Theorem 2.2. Thus combining Definition 2.3 with Proposition 4.3 and Theorem 3.14, we define the morphism β as:

$$\beta : f_\pi^{-1} R\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{C}_{N|M}) \to f_\pi^{-1} R\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \tilde{\mathscr{C}}_{N|M})$$
$$\xrightarrow{\sim} f_\pi^{-1} \tau_{Y\pi}^{-1} R\mathscr{H}om_{\mathscr{D}_Y}(\Psi_Y(\mathscr{M}), \mathscr{C}_N).$$

By the construction, we can obtain the following Holmgren type theorem:

Theorem 5.2. (1) *The morphism β gives a monomorphism*

$$\beta^0 : f_\pi^{-1} \mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{C}_{N|M}) \rightarrowtail f_\pi^{-1} \tau_{Y\pi}^{-1} \mathscr{H}om_{\mathscr{D}_Y}(\Psi_Y(\mathscr{M}), \mathscr{C}_N).$$

(2) *The restriction of β^0 to the zero-section $T_N M^+$ of $T^*_{T_N M^+} T_Y L^+$ coincides with the boundary value morphism due to Monteiro Fernandes* [**MF1**].

Proof. (1) follows from the fact that $s^*_{N|M} : f_\pi^{-1}\mathscr{C}_{N|M} \to f_\pi^{-1}\tilde{\mathscr{C}}_{N|M}$ is a monomorphism by Proposition 4.3.

(2) Comparing our construction with that of Laurent-Monteiro Fernandes [**L-MF2**], we easily obtain the desired result. □

Remark 5.3. By Theorem 2.1, Proposition 3.5 and Theorem 3.14, for any regular-specializable $\mathscr{D}_X|_Y$-Module $\mathscr{M}$ we have

$$f^{-1} R\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \nu_Y(\mathscr{B}\mathscr{O}_L)) \simeq f^{-1} \tau_Y^{-1} R\mathscr{H}om_{\mathscr{D}_Y}(\Psi_Y(\mathscr{M}), \mathscr{O}_Y).$$

Next we shall discuss the solvability.

Definition 5.4. Let $\mathscr{M}$ be a coherent $\mathscr{D}_X|_Y$-Module. Then we say $\mathscr{M}$ is *near-hyperbolic* at $x_0 \in N$ (in *dt*-codirection) if $R\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{O}_X)$ is near-hyperbolic in the sense of Definition 2.4. We remark that $\mathrm{SS}(R\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{O}_X)) = \mathrm{char}(\mathscr{M})$.

Remark 5.5. As is shown by Laurent-Monteiro Fernandes [**L-MF2**, Lemma 1.3.2], the near-hyperbolicity condition is weaker than the Fuchsian hyperbolicity condition due to Tahara [**T**] (cf. Bony-Schapira [**B-S2**]).

The following theorem is a direct consequence of Theorem 2.5:

Theorem 5.6. *Let $\mathscr{M}$ be a regular-specializable $\mathscr{D}_X|_Y$-Module. Assume that $\mathscr{M}$ is near-hyperbolic at $x_0 \in N$. Then, for any $p^* = (x_0, t_0; \sqrt{-1}\langle \xi_0, dx\rangle) \in T^*_{T_N M^+} T_Y L^+$,*

$$\beta : R\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{C}_{N|M})_{p^*} \to R\mathscr{H}om_{\mathscr{D}_Y}(\Psi_Y(\mathscr{M}), \mathscr{C}_N)_{\tau_{Y\pi}(p^*)}$$

is an isomorphism. In particular,

$$\beta : R\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \nu_N(\mathscr{B}_M))_{(x_0, t_0)} \to R\mathscr{H}om_{\mathscr{D}_Y}(\Psi_Y(\mathscr{M}), \mathscr{B}_N)_{x_0}$$

is an isomorphism.

6. Examples.

Example 6.1. Let $\mathscr{C}^F_{N|M}$ be the sheaf of F-mild microfunctions on $T^*_{T_N M} T_Y L$, and set $\tilde{\mathscr{C}}^A_{N|M} := \mathscr{H}^n \mu_N(\mathscr{O}_X|_Y) \otimes \mathscr{o}\mathscr{r}_{N/Y}$ (see Oaku [**Oa2**], [**Oa3**], and Oaku-Yamazaki [**O-Y**]). Let $\mathscr{M}$ be a regular-specializable $\mathscr{D}_X|_Y$-Module. Set $\mathscr{M}_Y := \mathscr{H}^0 \boldsymbol{D}\iota^* \mathscr{M} = \mathscr{O}_Y \otimes_{\iota^{-1}\mathscr{O}_X} \iota^{-1}\mathscr{M}$.

Since $\mathscr{M}$ is a Fuchsian system in the sense of Laurent-Monteiro Fernandes [**L-MF1**], by the argument in Oaku-Yamazaki [**O-Y**] we have the following commutative diagram:

$$\begin{array}{ccccc} f_\pi^{-1}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M},\mathscr{C}^F_{N|M}) & \rightarrowtail & f_\pi^{-1}\tau_{Y\pi}^{-1}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M},\tilde{\mathscr{C}}^A_{N|M}) & \xrightarrow{\sim} & f_\pi^{-1}\tau_{Y\pi}^{-1}\mathscr{H}om_{\mathscr{D}_Y}(\mathscr{M}_Y,\mathscr{C}_N) \\ \downarrow & & \downarrow & & \downarrow \\ f_\pi^{-1}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M},\mathscr{C}_{N|M}) & \rightarrowtail & f_\pi^{-1}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M},\tilde{\mathscr{C}}_{N|M}) & \xrightarrow{\sim} & f_\pi^{-1}\tau_{Y\pi}^{-1}\mathscr{H}om_{\mathscr{D}_Y}(\Psi_Y(\mathscr{M}),\mathscr{C}_N), \end{array}$$

that is, the boundary value morphism

$$\gamma^F : f_\pi^{-1}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M},\mathscr{C}^F_{N|M}) \rightarrowtail f_\pi^{-1}\tau_{Y\pi}^{-1}\mathscr{H}om_{\mathscr{D}_Y}(\mathscr{M}_Y,\mathscr{C}_N)$$

and β^0 are compatible. In particular, suppose that Y is non-characteristic for $\mathscr{M}$. Then, it is known that $\Psi_Y(\mathscr{M}) \xrightarrow{\sim} \boldsymbol{D}\iota^*\mathscr{M} \simeq \mathscr{M}_Y$ and by Oaku [**Oa3**, Propositions 2.1, 2.2] (see also Oaku-Yamazaki [**O-Y**, Proposition 5.1]) we have:

$$\tilde{\gamma}_{N|M} : R\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M},\tilde{\mathscr{C}}_{N|M}) \xrightarrow{\sim} \tau_{Y\pi}^{-1}R\mathscr{H}om_{\mathscr{D}_Y}(\mathscr{M}_Y,\mathscr{C}_N).$$

In this case we see that β^0 is equivalent to the non-characteristic boundary value morphism (see Oaku [**Oa3**]). In particular, the restriction of β^0 to the zero-section T_NM^+ is equivalent to Komatsu-Kawai [**Ko-K**] and Schapira [**Sc1**]. In addition, if $\pm dt \in T_N^*M$ is hyperbolic for $\mathscr{M}$, then the nearly-hyperbolic condition is satisfied (cf. Kashiwara-Schapira [**K-S1**]) and β is an isomorphism.

Example 6.2. Assume that $X = \boldsymbol{C}^{n+1}$ by admissible coordinates.

(1) Let $b(\alpha)$ be a non-zero polynomial with degree m, and $Q \in \mathscr{D}_X^{(m)} \cap \mathsf{F}_Y^{-1}(\mathscr{D}_X)$. Set

$$\mathscr{M} := \mathscr{D}_X/\mathscr{D}_X(b(\vartheta)+Q).$$

Then $\mathscr{M}$ is regular-specializable. Assume that

$$b(\alpha) = \prod_{j=1}^{\mu}(\alpha-\alpha_j)^{\nu_j} \quad (\alpha_i-\alpha_j \notin \boldsymbol{Z} \text{ for } 1 \le i \ne j \le \mu)$$

(note that $\sum_{j=1}^{\mu}\nu_j = m$). Then a direct calculation shows that $\Psi_Y(\mathscr{M}) \simeq \mathscr{D}_Y^m$, and β^0 is equivalent to γ in Oaku [**Oa3**, Theorem 2.4 and Remark]: Let $p^* = (x_0, t_0; \sqrt{-1}\langle\xi_0, dx\rangle)$ be a point of $T^*_{T_NM^+}T_YL^+$, and $f(x,t)$ a germ of $\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M},\mathscr{C}_{N|M})$ at p^*. Then, since $R\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M},\mathscr{N}_{X|Y}) \simeq R\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M},\sigma^{-1}\nu_Y(\mathscr{O}_X))$ by virtue of Theorem 3.14, we can see that as a germ of $\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M},\tilde{\mathscr{C}}_{N|M})$ at p^*, $f(x,t)$ has a defining function

$$F(z,\tau) = \sum_{j=1}^{\mu}\sum_{k=1}^{\nu_j} F_{jk}(z,\tau)\tau^{\alpha_j}(\log\tau)^{k-1}.$$

Here each $F_{jk}(z,\tau)$ is holomorphic on a neighborhood of $\{(z,0) \in X; |x_0 - z| < \varepsilon, \operatorname{Im} z \in \Gamma\}$ with a positive constant ε and an open convex cone Γ such that $\xi_0 \in \operatorname{Int}\Gamma^\circ$, where Γ° denotes the dual cone. Then, $\beta^0(f)$ is equivalent to $\{\mathrm{sp}_N(F_{jk}(x+\sqrt{-1}\Gamma 0, 0)); 1 \le k \le \nu_j, 1 \le j \le \mu\}$. Moreover, if the principal symbol of $b(\vartheta)+Q$ is written as $\tau^m P(z,\tau;z^*,\tau^*)$ for a hyperbolic polynomial P at dt-codirection, then the nearly-hyperbolic condition is satisfied. Note that this operator is a special case of Fuchsian hyperbolic operators due to Tahara [**T**].

(2) Take an operator $A(z, \partial_z) \in \mathscr{D}_Y^{(1)}$ at the origin and set $A^0 := 1$ and $A^{(j)} := (1/j!)A \circ A^{(j-1)} \in \mathscr{D}_Y^{(j)}$ for $j \geq 1$. Let $p^* = (0, 1; \sqrt{-1}\langle\xi, dx\rangle)$ be a point of $T^*_{T_N M^+} T_Y L^+$ and set $p_0 := (0; \sqrt{-1}\langle\xi, dx\rangle) \in T^*_N Y$. Set

$$P := (\vartheta - \alpha_1)(\vartheta - \alpha_2) - \tau A(z, \partial_z)\vartheta \in \mathscr{D}_X|_Y,$$

where $(\alpha_1, \alpha_2) \in \boldsymbol{C}^2$. Consider $\mathscr{M} := \mathscr{D}_X/\mathscr{D}_X P = \mathscr{D}_X u$, where $u := 1 \bmod P$. Then we see that $\Psi_Y(\mathscr{M}) \simeq \mathscr{D}_Y^2$ and $\Phi_Y(\mathscr{M}) \simeq \mathscr{D}_Y^2$. Let $f(x, t)$ be a germ of $\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{C}_{N|M})$ at p^*. We regard $f(x, t)$ as a germ of $\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \tilde{\mathscr{C}}_{N|M})$ at p^*. Then:

(i) If $(\alpha_1, \alpha_2) = (-1, 0)$, then

$$\Phi_Y(\mathscr{M}) = \frac{\mathsf{F}_Y^0(\mathscr{D}_X)u + \mathsf{F}_Y^1(\mathscr{D}_X)(\vartheta + 1)u}{\mathsf{F}_Y^{-1}(\mathscr{D}_X)u + \mathsf{F}_Y^0(\mathscr{D}_X)(\vartheta + 1)u} = \mathscr{D}_Y[u] + \mathscr{D}_Y[\partial_\tau(\vartheta + 1)u],$$

$$\Psi_Y(\mathscr{M}) = \frac{\mathsf{F}_Y^{-1}(\mathscr{D}_X)u + \mathsf{F}_Y^0(\mathscr{D}_X)(\vartheta + 1)u}{\mathsf{F}_Y^{-2}(\mathscr{D}_X)u + \mathsf{F}_Y^{-1}(\mathscr{D}_X)(\vartheta + 1)u} = \mathscr{D}_Y[\tau u] + \mathscr{D}_Y[(\vartheta + 1)u],$$

and $\mathrm{Var} : ([u], [\partial_\tau(\vartheta - 1)u]) \mapsto ([\tau u], 0)$. Hence $\mathscr{M}_Y \simeq \mathscr{D}_Y[(\vartheta + 1)u] \simeq \mathscr{D}_Y$. In this case $f(x, t)$ has the following defining function:

$$F(z, \tau) = U_0(z) + \frac{U_{-1}(z)}{\tau} - \sum_{j=1}^{\infty} \frac{A^{(j+1)}U_{-1}(z)}{j}\tau^j - AU_{-1}(z)\log\tau,$$

and $\beta^0(f(x, t))$ is given by $\{\mathrm{sp}_N(U_i)(x)\}_{i=-1,0}$ at p_0. If $f(x, t)$ is F-mild at p_0, then $U_{-1}(z) = 0$ and $\gamma^F(f(x, t)) = \{f(x, +0)\} = \{\mathrm{sp}_N(U_0)(x)\}$.

(ii) If $(\alpha_1, \alpha_2) = (0, 1)$, then:

$$\Phi_Y(\mathscr{M}) = \frac{\mathsf{F}_Y^1(\mathscr{D}_X)u + \mathsf{F}_Y^2(\mathscr{D}_X)\vartheta u}{\mathsf{F}_Y^0(\mathscr{D}_X)u + \mathsf{F}_Y^1(\mathscr{D}_X)\vartheta u} = \mathscr{D}_Y[\partial_\tau u] + \mathscr{D}_Y[\partial_\tau^2\vartheta u],$$

$$\Psi_Y(\mathscr{M}) = \frac{\mathsf{F}_Y^0(\mathscr{D}_X)u + \mathsf{F}_Y^1(\mathscr{D}_X)\vartheta u}{\mathsf{F}_Y^{-1}(\mathscr{D}_X)u + \mathsf{F}_Y^0(\mathscr{D}_X)\vartheta u} = \mathscr{D}_Y[u] + \mathscr{D}_Y[\partial_\tau\vartheta u],$$

and $\mathrm{Var}[\partial_\tau u] = \mathrm{Var}[\partial_\tau^2\vartheta u] = 0$. Hence $\mathscr{M}_Y \simeq \mathscr{D}_Y[u] + \mathscr{D}_Y[\partial_\tau\vartheta u] \simeq \mathscr{D}_Y^2$. In this case $f(x, t)$ has the following defining function:

$$F(z, \tau) = U_0(z) + \sum_{j=0}^{\infty} \frac{A^{(j)}U_1(z)}{j + 1}\tau^{j+1},$$

and $f(x, t)$ is always F-mild. Hence $\beta^0(f(x, t))$ at p_0 coincides with

$$\gamma^F(f(x, t)) = \{\partial_t^i f(x, +0)\}_{i=0,1} = \{\mathrm{sp}_N(U_i)(x)\}_{i=0,1}.$$

Indeed if $\tau \neq 0$, $\mathscr{M}$ is isomorphic to $\mathscr{D}_X/\mathscr{D}_X(\partial_\tau^2 - A(z; \partial_z)\partial_\tau)$ for which Y is non-characteristic.

(iii) If $(\alpha_1, \alpha_2) = (1, 1)$, then

$$\Phi_Y(\mathscr{M}) = \frac{\mathsf{F}_Y^2(\mathscr{D}_X)u}{\mathsf{F}_Y^1(\mathscr{D}_X)u} = \mathscr{D}_Y[\partial_\tau^2 u] + \mathscr{D}_Y[\partial_\tau^2(\vartheta - 1)u],$$

$$\Psi_Y(\mathscr{M}) = \frac{\mathsf{F}_Y^1(\mathscr{D}_X)u}{\mathsf{F}_Y^0(\mathscr{D}_X)u} = \mathscr{D}_Y[\partial_\tau u] + \mathscr{D}_Y[\partial_\tau(\vartheta - 1)u],$$

and $\mathrm{Var} : ([\partial_\tau^2 u], [\partial_\tau^2(\vartheta - 1)u]) \mapsto (2\pi\sqrt{-1}[\partial_\tau(\vartheta - 1)u], 0)$. Hence $\mathscr{M}_Y \simeq \mathscr{D}_Y[\partial_\tau u] \simeq \mathscr{D}_Y$. In this case $f(x,t)$ has the following defining function:

$$F(z,\tau) = \sum_{j=0}^{\infty} A^{(j)} U_0(z)\tau^{j+1} - \sum_{j=1}^{\infty}\sum_{k=1}^{j} \frac{A^{(j)} U_1(z)}{k}\tau^{j+1} + \sum_{j=0}^{\infty} A^{(j)} U_1(z)\tau^{j+1}\log\tau,$$

and $\beta^0(f(x,t))$ is given by $\{\mathrm{sp}_N(U_i)(x)\}_{i=0,1}$ at p_0. If $f(x,t)$ is F-mild at p_0, then $U_1(z) = 0$ and $\gamma^F(f(x,t)) = \{\partial_t f(x,+0)\} = \{\mathrm{sp}_N(U_0)(x)\}$.

(iv) If $(\alpha_1, \alpha_2) = (1, 2)$, then:

$$\Phi_Y(\mathscr{M}) = \frac{\mathsf{F}_Y^2(\mathscr{D}_X)u + \mathsf{F}_Y^3(\mathscr{D}_X)(\vartheta - 1)u}{\mathsf{F}_Y^1(\mathscr{D}_X)u + \mathsf{F}_Y^2(\mathscr{D}_X)(\vartheta - 1)u} = \mathscr{D}_Y[\partial_\tau^2 u] + \mathscr{D}_Y[\partial_\tau^3(\vartheta - 1)u],$$

$$\Psi_Y(\mathscr{M}) = \frac{\mathsf{F}_Y^1(\mathscr{D}_X)u + \mathsf{F}_Y^2(\mathscr{D}_X)(\vartheta - 1)u}{\mathsf{F}_Y^0(\mathscr{D}_X)u + \mathsf{F}_Y^1(\mathscr{D}_X)(\vartheta - 1)u} = \mathscr{D}_Y[\partial_\tau u] + \mathscr{D}_Y[\partial_\tau^2(\vartheta - 1)u],$$

and $\mathrm{Var} : ([\partial_\tau^2 u], [\partial_\tau^3(\vartheta - 1)u]) \mapsto (0, 2A[\partial_\tau u])$. Hence

$$\mathscr{M}_Y \simeq \frac{\mathscr{D}_Y[\partial_\tau u] + \mathscr{D}_Y[\partial_\tau^2(\vartheta - 1)u]}{\mathscr{D}_Y A[\partial_\tau u]}.$$

In this case $f(x,t)$ has the following defining function:

$$F(z,\tau) = \sum_{j=0}^{\infty} A^{(j)} U_2(z)\tau^{j+2} + U_1(z)\tau - \sum_{j=2}^{\infty}\sum_{k=1}^{j-1} \frac{jA^{(j)} U_1(z)}{k}\tau^{j+1}$$
$$+ \left(\sum_{j=0}^{\infty}(j+1)A^{(j+1)} U_1(z)\tau^j\right)\tau^2\log\tau,$$

and $\beta^0(f(x,t))$ is given by $\{\mathrm{sp}_N(U_i)(x)\}_{i=1,2}$ at p_0. $f(x,t)$ is F-mild under the condition that $AU_1(z) = 0$, and in this case $\gamma^F(f(x,t))$ at p_0 is given by

$$\gamma^F(f_3(x,t)) = \{\partial_t^i f(x,+0)\}_{i=1,2} = \{\mathrm{sp}_N(U_1)(x), 2\,\mathrm{sp}_N(U_2)(x)\}$$

with $A\partial_t f(x,+0) = A\,\mathrm{sp}_N(U_1)(x) = 0$.

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