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Blow-up profile of a solution for a nonlinear heat equation with small diffusion

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Abstract. This paper is concerned with positive solutions of semilinear diffusion equations $u_t = \varepsilon^2 \triangle u + u^p$ in Ω with small diffusion under the Neumann boundary condition, where $p > 1$ is a constant and Ω is a bounded domain in $\boldsymbol{R}^N$ with C^2 boundary. For the ordinary differential equation $u_t = u^p$, the solution u^0 with positive initial data $u_0 \in C(\bar{\Omega})$ has a blow-up set $S^0 = \{x \in \bar{\Omega} \mid u_0(x) = \max_{y \in \bar{\Omega}} u_0(y)\}$ and a blow-up profile

$$u_*^0(x) = \left(u_0(x)^{-(p-1)} - \left(\max_{y \in \bar{\Omega}} u_0(y) \right)^{-(p-1)} \right)^{-1/(p-1)}$$

outside the blow-up set S^0. For the diffusion equation $u_t = \varepsilon^2 \triangle u + u^p$ in Ω under the boundary condition $\partial u / \partial \nu = 0$ on $\partial \Omega$, it is shown that if a positive function $u_0 \in C^2(\bar{\Omega})$ satisfies $\partial u_0 / \partial \nu = 0$ on $\partial \Omega$, then the blow-up profile $u_*^\varepsilon(x)$ of the solution u^ε with initial data u_0 approaches $u_*^0(x)$ uniformly on compact sets of $\bar{\Omega} \backslash S^0$ as $\varepsilon \to +0$.

1. Introduction.

This paper is concerned with the singularly perturbed diffusion equation

$$(1.1) \qquad \begin{cases} u_t = \varepsilon^2 \triangle u + u^p & \text{in } \Omega \times (0, T), \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial \Omega \times (0, T), \\ u(x, 0) = u_0(x) & x \in \bar{\Omega} \end{cases}$$

with a small constant $\varepsilon > 0$, where Ω is a bounded domain in $\boldsymbol{R}^N$ with C^2 boundary, ν is the unit outward normal vector on $\partial \Omega$, $p > 1$ is a constant and $u_0 \in C^2(\bar{\Omega})$ is a positive function satisfying $\partial u_0 / \partial \nu = 0$ on $\partial \Omega$. For the solution $u(x, t)$ of (1.1), the *blow-up time* T is defined by

$$T = \sup\{\tau > 0 \mid u(x, t) \text{ is bounded in } \bar{\Omega} \times (0, \tau)\}.$$

Then, $0 < T < +\infty$ and $\overline{\lim}_{t \to T} \|u(x, t)\|_{C(\bar{\Omega})} = +\infty$ hold. The *blow-up set* of the solution $u(x, t)$ is defined as the set

$$\{x \in \bar{\Omega} \mid \text{there is a sequence } (x_n, t_n) \text{ in } \bar{\Omega} \times (0, T) \text{ such that } (x_n, t_n) \to (x, T) \text{ and } u(x_n, t_n) \to +\infty \text{ as } n \to \infty\}.$$

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This set is a nonempty closed set in $\bar{\Omega}$. From standard parabolic estimates, we can obtain the *blow-up profile*, which is a continuous function defined by

$$u_*(x) = \lim_{t \to T} u(x, t)$$

outside the blow-up set.

Mizoguchi [**6**] showed the following for the Cauchy or Cauchy-Dirichlet problem with $(N-2)p < N+2$. For any nonnegative continuous function u_0 and $\delta > 0$, if $\varepsilon > 0$ is sufficiently small, then any point x in the blow-up set of the solution for the equation $u_t = \varepsilon^2 \triangle u + u^p$ with initial data u_0 satisfies the inequality $u_0(x) \geq \max_y u_0(y) - \delta$. See [**2**] and [**7**] on the blow-up time. (We can refer to [**4**] and [**5**] for related results on other equations of parabolic type. See also the references of [**8**] for other studies on singularity formation in blow-up of $u_t = \triangle u + u^p$.)

For the ordinary differential equation $u_t = u^p$, the solution u^0 with positive initial data $u_0 \in C(\bar{\Omega})$ has a blow-up set $S^0 = \{x \in \bar{\Omega} \mid u_0(x) = \max_{y \in \bar{\Omega}} u_0(y)\}$ and a blow-up profile

$$u_*^0(x) = \left(u_0(x)^{-(p-1)} - \left(\max_{y \in \bar{\Omega}} u_0(y) \right)^{-(p-1)} \right)^{-1/(p-1)}$$

outside the blow-up set S^0. In this paper, we show that *the blow-up profile $u_*^\varepsilon(x)$ of the solution u^ε of* (1.1) *approaches $u_*^0(x)$ uniformly on compact sets of $\bar{\Omega} \backslash S^0$ as $\varepsilon \to +0$.* Precisely, our main result is the following.

THEOREM 1. *Let $u_0 \in C^2(\bar{\Omega})$ be a positive function satisfying $\partial u_0 / \partial \nu = 0$ on $\partial \Omega$, and let $\delta > 0$ be a constant. Then, there exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0]$, the blow-up set of the solution u of* (1.1) *is contained in the set $S := \{x \in \bar{\Omega} \mid u_0(x) \geq \max_{y \in \bar{\Omega}} u_0(y) - \delta\}$ and the blow-up profile $u_*(x)$ satisfies the inequality*

$$\left\| u_*(x) - \left(u_0(x)^{-(p-1)} - \left(\max_{y \in \bar{\Omega}} u_0(y) \right)^{-(p-1)} \right)^{-1/(p-1)} \right\|_{C(\bar{\Omega} \backslash S)} \leq \delta.$$

2. Preliminaries.

In this section, we prove several lemmas. First, we take a cutoff function $\rho \in C^\infty(\boldsymbol{R})$ satisfying

$$\rho(z) = -1 \ \ (z \leq 1), \quad \rho(z) = 1 \ \ (4 \leq z) \quad \text{and} \quad 0 \leq \rho'(z) \leq 3/4 \ \ (z \in \boldsymbol{R}).$$

Then, this function $\rho(z)$ satisfies the following.

LEMMA 2. *Suppose that $f \in C^2(\bar{\Omega})$ is a positive function and that $\alpha \leq \min_{x \in \bar{\Omega}} f(x)/4$ is a positive constant. Then, the positive function $g \in C^2(\bar{\Omega})$ defined by*

$$g(x) := f(x) + \alpha \rho \left(\frac{\|f\|_{C(\bar{\Omega})} - f(x)}{\alpha} \right) \tag{2.1}$$

satisfies

$$\sup_{t\in[0,(p-1)^{-1}(\|f\|_C-\alpha/2)^{-(p-1)}]} \|(g(x)^{-(p-1)}-(p-1)t)^{-1/(p-1)}\|_{C^2(\bar{\Omega})}<+\infty.$$

PROOF. We first note

$$\min_{z\in[a,b]}\left(z+\alpha\rho\left(\frac{\|f\|_C-z}{\alpha}\right)\right)=a+\alpha\rho\left(\frac{\|f\|_C-a}{\alpha}\right)$$

and

$$\max_{z\in[a,b]}\left(z+\alpha\rho\left(\frac{\|f\|_C-z}{\alpha}\right)\right)=b+\alpha\rho\left(\frac{\|f\|_C-b}{\alpha}\right).$$

Then, we see $\alpha\le g(x)\le\|f\|_C-\alpha$. Hence, we also see $g(x)^{-(p-1)}-(p-1)t\ge(\|f\|_C-\alpha)^{-(p-1)}-(\|f\|_C-\alpha/2)^{-(p-1)}>0$. □

While the following lemma is rather technical, this gives a function $\bar{v}$ such that the inequality $\bar{v}_t\ge\varepsilon^2\triangle\bar{v}+\bar{v}^p$ holds in the region where $\bar{v}(x,t)\le 2^{2/(p-1)}C(T-t)^{-1/(p-1)}$. This function $\bar{v}$ plays a key role in Proof of Theorem 6 in the next section.

In this paragraph, we intuitively and informally explain the reason why the function $\bar{v}$ plays a key role, as it is the central idea of this paper. From Proposition 7 in the next section, we would have the Type-I estimate $u(x,t)\le C(T-t)^{-1/(p-1)}$ for some constant $C>0$. Then, we define a map $h:[0,+\infty)\times[0,T)\to[-\infty,+\infty)$ by

$$h(v,t):=\begin{cases}v^p & (v\le C(T-t)^{-1/(p-1)}),\\ -\infty & (\text{otherwise})\end{cases}$$

and consider the diffusion equation

$$v_t=\varepsilon^2\triangle v+h(v,t).$$

Obviously, $u(x,t)$ is also a solution of $v_t=\varepsilon^2\triangle v+h(v,t)$. On the other hand, because the function $2^{1/(p-1)}C(T-t)^{-1/(p-1)}$ is a super-solution of $v_t=\varepsilon^2\triangle v+h(v,t)$ and the inequality $\bar{v}_t\ge\varepsilon^2\triangle\bar{v}+h(\bar{v},t)$ holds in the region where $\bar{v}(x,t)\le 2^{2/(p-1)}\cdot C(T-t)^{-1/(p-1)}$, the function

$$w(x,t):=\begin{cases}\bar{v}(x,t) & (\bar{v}(x,t)\le 2^{1/(p-1)}C(T-t)^{-1/(p-1)}),\\ 2^{1/(p-1)}C(T-t)^{-1/(p-1)} & (\text{otherwise})\end{cases}$$

is a super-solution of $v_t=\varepsilon^2\triangle v+h(v,t)$. Therefore, if $u_0(x)\le w(x,0)$ holds, we would eventually have $u(x,t)\le w(x,t)$ for $t\in[0,T)$. Now, we should note that $w(x,t)$ is not a super-solution of (1.1) because $2^{1/(p-1)}C(T-t)^{-1/(p-1)}$ is not a super-solution of (1.1). We end the intuitive and informal explanation here, and we give the strict argument in Step 3 of Proof of Theorem 6.

LEMMA 3. *Suppose that $f\in C^2(\bar{\Omega})$ is a positive function and that $\alpha\le\min_{x\in\bar{\Omega}}f(x)/4$ is a positive constant. Let $g\in C^2(\bar{\Omega})$ be defined by* (2.1). *Then, for any $C>0$, there exist D and $\beta_0>0$ such that for any positive constants $\beta\le\beta_0$, $\varepsilon\le\beta$ and $T\le(p-1)^{-1}\cdot(\|f\|_C-\alpha/2)^{-(p-1)}$, the following holds*:

Let $\bar{v}(x,t)$ be a positive function defined by

$$\bar{v}(x,t):=(g(x)^{-(p-1)}-(p-1)t)^{-1/(p-1)}+\varepsilon^{2/(p-1)}(g(x)^{-(p-1)}-\omega(t))^{-2/(p-1)}+e^{Dt}\beta^2$$

on the set $\{(x,t) \in \bar{\Omega} \times [0,T) \mid g(x)^{-(p-1)} > \omega(t)\}$, *where* $\omega \in C^1([0,T))$ *is defined by*

$$\omega(t) := (\|g\|_C - 2\alpha)^{-(p-1)} + 2D\beta^{-(p-1)}\varepsilon(T^{1/2} - (T-t)^{1/2}).$$

Then, the inequality $\bar{v}_t \geq \varepsilon^2 \triangle \bar{v} + \bar{v}^p$ *holds in the set* $\{(x,t) \in \bar{\Omega} \times [0,T) \mid g(x)^{-(p-1)} - \omega(t) \geq (1/2)C^{-(p-1)/2}(T-t)^{1/2}\varepsilon\}$.

Proof. Throughout this proof, we denote the positive constant $(p-1)^{-1} \cdot (\|f\|_C - \alpha/2)^{-(p-1)}$ by T_0 and choose $D \geq 1$ larger if necessary. Then, let $\beta_0 > 0$ be a constant defined by $e^{DT_0}\beta_0^2 = 1$.

From Lemma 2,

$$1 + ((g(x)^{-(p-1)} - (p-1)t)^{-1/(p-1)} + 1)^{p-1} \leq D^{1/8} \tag{2.2}$$

holds. Because $(a+b)^p - a^p = \int_0^1 p(a+\sigma b)^{p-1} b \, d\sigma \leq p(a+b)^{p-1}b \leq p2^{p-1}(a^{p-1}+b^{p-1})b \leq p2^{p-1}(1+a^{p-1})(b+b^p)$ holds, by using (2.2) and $e^{Dt}\beta^2 \leq e^{DT_0}\beta_0^2 = 1$, we get

$$\begin{aligned}
&\bar{v}(x,t)^p - ((g(x)^{-(p-1)} - (p-1)t)^{-1/(p-1)} + e^{Dt}\beta^2)^p \\
&\quad \leq p2^{p-1}D^{1/8}(\varepsilon^{2/(p-1)}(g(x)^{-(p-1)} - \omega(t))^{-2/(p-1)} \\
&\qquad + \varepsilon^{2p/(p-1)}(g(x)^{-(p-1)} - \omega(t))^{-2p/(p-1)})
\end{aligned}$$

and

$$\begin{aligned}
&((g(x)^{-(p-1)} - (p-1)t)^{-1/(p-1)} + e^{Dt}\beta^2)^p - (g(x)^{-(p-1)} - (p-1)t)^{-p/(p-1)} \\
&\quad \leq p2^pD^{1/8}e^{Dt}\beta^2.
\end{aligned}$$

Hence, we obtain

$$\begin{aligned}
&\bar{v}(x,t)^p - (g(x)^{-(p-1)} - (p-1)t)^{-p/(p-1)} \\
&\quad \leq D^{1/4}(\varepsilon^{2/(p-1)}(g(x)^{-(p-1)} - \omega(t))^{-2/(p-1)} \\
&\qquad + \varepsilon^{2p/(p-1)}(g(x)^{-(p-1)} - \omega(t))^{-2p/(p-1)} + e^{Dt}\beta^2).
\end{aligned} \tag{2.3}$$

Also, from Lemma 2,

$$\begin{aligned}
&\varepsilon^2 \bar{v}_{x_i x_i}(x,t) \\
&\quad \leq \left(\sup_{t \in [0,T_0]} \|(g(x)^{-(p-1)} - (p-1)t)^{-1/(p-1)}\|_{C^2(\bar{\Omega})}\right)\varepsilon^2 \\
&\qquad + \frac{2}{p-1}\|g(x)^{-(p-1)}\|_{C^2(\bar{\Omega})}\varepsilon^{2p/(p-1)}(g(x)^{-(p-1)} - \omega(t))^{-(p+1)/(p-1)} \\
&\qquad + \frac{2(p+1)}{(p-1)^2}\|g(x)^{-(p-1)}\|^2_{C^1(\bar{\Omega})}\varepsilon^{2p/(p-1)}(g(x)^{-(p-1)} - \omega(t))^{-2p/(p-1)} \\
&\quad \leq D^{1/4}(\varepsilon^2 + \varepsilon^{2p/(p-1)}(g(x)^{-(p-1)} - \omega(t))^{-(p+1)/(p-1)} \\
&\qquad + \varepsilon^{2p/(p-1)}(g(x)^{-(p-1)} - \omega(t))^{-2p/(p-1)})
\end{aligned} \tag{2.4}$$

holds. From (2.3) and (2.4), we obtain

$$(2.5)\quad \varepsilon^2 \triangle \bar{v}(x,t) + \bar{v}(x,t)^p - (g(x)^{-(p-1)} - (p-1)t)^{-p/(p-1)}$$
$$\le D^{1/4}(\varepsilon^{2/(p-1)}(g(x)^{-(p-1)} - \omega(t))^{-2/(p-1)} + \varepsilon^{2p/(p-1)}(g(x)^{-(p-1)} - \omega(t))^{-2p/(p-1)}$$
$$+ \varepsilon^{2p/(p-1)}(g(x)^{-(p-1)} - \omega(t))^{-(p+1)/(p-1)} + \varepsilon^2 + e^{Dt}\beta^2).$$

In the region where $g(x)^{-(p-1)} - \omega(t) \le \beta^{-(p-1)}\varepsilon$, because of $\varepsilon^{-1}(g(x)^{-(p-1)} - \omega(t)) \le \beta^{-(p-1)}$ and $T_0^{-1/2} \le (T-t)^{-1/2}$,

$$\varepsilon^{2/(p-1)}(g(x)^{-(p-1)} - \omega(t))^{-2/(p-1)}$$
$$\le \beta^{-(p-1)}\varepsilon^{(p+1)/(p-1)}(g(x)^{-(p-1)} - \omega(t))^{-(p+1)/(p-1)}$$
$$\le T_0^{1/2}\beta^{-(p-1)}\varepsilon^{(p+1)/(p-1)}(g(x)^{-(p-1)} - \omega(t))^{-(p+1)/(p-1)}(T-t)^{-1/2}$$

holds. In the region where $g(x)^{-(p-1)} - \omega(t) \ge \beta^{-(p-1)}\varepsilon$, because of $\varepsilon(g(x)^{-(p-1)} - \omega(t))^{-1} \le \beta^{p-1}$,

$$\varepsilon^{2/(p-1)}(g(x)^{-(p-1)} - \omega(t))^{-2/(p-1)} \le \beta^2$$

holds. Therefore, we obtain

$$(2.6)\quad \varepsilon^{2/(p-1)}(g(x)^{-(p-1)} - \omega(t))^{-2/(p-1)}$$
$$\le D^{1/4}(\beta^{-(p-1)}\varepsilon^{(p+1)/(p-1)}(g(x)^{-(p-1)} - \omega(t))^{-(p+1)/(p-1)}(T-t)^{-1/2} + e^{Dt}\beta^2).$$

Because

$$\varepsilon(g(x)^{-(p-1)} - \omega(t))^{-1} \le 2C^{(p-1)/2}(T-t)^{-1/2}$$

holds from $g(x)^{-(p-1)} - \omega(t) \ge (1/2)C^{-(p-1)/2}(T-t)^{1/2}\varepsilon$ and

$$D^{1/4}e^{((p-1)/2)T_0 D} \le D^{1/4}\beta^{-(p-1)}$$

holds from $\beta \le \beta_0$ and $e^{DT_0}\beta_0^2 = 1$, we have

$$(2.7)\quad \varepsilon^{2p/(p-1)}(g(x)^{-(p-1)} - \omega(t))^{-2p/(p-1)}$$
$$\le 2C^{(p-1)/2}\varepsilon^{(p+1)/(p-1)}(g(x)^{-(p-1)} - \omega(t))^{-(p+1)/(p-1)}(T-t)^{-1/2}$$
$$\le D^{1/4}\beta^{-(p-1)}\varepsilon^{(p+1)/(p-1)}(g(x)^{-(p-1)} - \omega(t))^{-(p+1)/(p-1)}(T-t)^{-1/2}.$$

Because we see

$$\varepsilon \le \beta \le e^{-(1/2)T_0 D} \le D^{1/4}e^{((p-1)/2)T_0 D}T_0^{-1/2} \le D^{1/4}\beta^{-(p-1)}(T-t)^{-1/2}$$

by using $\beta \le \beta_0$ and $e^{DT_0}\beta_0^2 = 1$,

$$(2.8)\quad \varepsilon^{2p/(p-1)}(g(x)^{-(p-1)} - \omega(t))^{-(p+1)/(p-1)}$$
$$\le D^{1/4}\beta^{-(p-1)}\varepsilon^{(p+1)/(p-1)}(g(x)^{-(p-1)} - \omega(t))^{-(p+1)/(p-1)}(T-t)^{-1/2}$$

holds. From $\varepsilon \leq \beta$, we also have

$$\varepsilon^2 \leq D^{1/4} e^{Dt} \beta^2. \tag{2.9}$$

We see

$$\begin{aligned} &\bar{v}_t(x,t) - (g(x)^{-(p-1)} - (p-1)t)^{-p/(p-1)} \\ &\quad = \frac{2D}{p-1} \beta^{-(p-1)} \varepsilon^{(p+1)/(p-1)} (g(x)^{-(p-1)} - \omega(t))^{-(p+1)/(p-1)} (T-t)^{-1/2} + De^{Dt}\beta^2. \end{aligned}$$

Hence, by combining the inequalities (2.5), (2.6), (2.7), (2.8) and (2.9),

$$\varepsilon^2 \triangle \bar{v}(x,t) + \bar{v}(x,t)^p \leq \bar{v}_t(x,t)$$

holds. □

The following gives a sub-solution of (1.1).

LEMMA 4. *Suppose that a positive function $f \in C^2(\bar{\Omega})$ satisfies $\partial f/\partial \nu = 0$ on $\partial\Omega$. Let $\underline{D}_f$ be a constant defined by*

$$\underline{D}_f := \left(\frac{2}{\min_{x \in \bar{\Omega}} f(x)} \right)^p \max_{x \in \bar{\Omega}} |(\triangle f)(x)|. \tag{2.10}$$

Then, for any constant $\varepsilon > 0$ and function $u_0 \in C(\bar{\Omega})$ satisfying $2\underline{D}_f \varepsilon^2 \leq 1$ and $2\|u_0 - f\|_C \leq \min_{x \in \bar{\Omega}} f(x)$, the positive function $\underline{u}(x,t)$ defined by

$$\underline{u}(x,t) := ((f(x) - \|u_0 - f\|_C)^{-(p-1)} - (p-1)(1 - \underline{D}_f \varepsilon^2)t)^{-1/(p-1)}$$

in the set

$$\bar{\Omega} \times [0, (p-1)^{-1}(1 - \underline{D}_f \varepsilon^2)^{-1}(\|f\|_C - \|u_0 - f\|_C)^{-(p-1)})$$

is a sub-solution of (1.1).

PROOF. Let $v(x,t)$ denote the function $\triangle \underline{u} + \underline{D}_f \underline{u}^p$. Then, because $\underline{u}_t = (1 - \underline{D}_f \varepsilon^2) \underline{u}^p$ and $(\underline{u}^p)_{x_i x_i} = p\underline{u}^{p-1} \underline{u}_{x_i x_i} + p(p-1)\underline{u}^{p-2} \underline{u}_{x_i}^2$ hold, we see

$$\begin{aligned} v_t &= \triangle \underline{u}_t + \underline{D}_f p \underline{u}^{p-1} \underline{u}_t = (1 - \underline{D}_f \varepsilon^2)(\triangle \underline{u}^p + p\underline{u}^{p-1} \underline{D}_f \underline{u}^p) \\ &\geq (1 - \underline{D}_f \varepsilon^2) p \underline{u}^{p-1} (\triangle \underline{u} + \underline{D}_f \underline{u}^p) = (1 - \underline{D}_f \varepsilon^2) p \underline{u}^{p-1} v. \end{aligned} \tag{2.11}$$

Also, we have

$$\begin{aligned} v(x,0) &= (\triangle f)(x) + \underline{D}_f (f(x) - \|u_0 - f\|_C)^p \\ &\geq -\max_{x \in \bar{\Omega}} |(\triangle f)(x)| + \underline{D}_f \left(\frac{\min_{x \in \bar{\Omega}} f(x)}{2} \right)^p = 0. \end{aligned} \tag{2.12}$$

Because

$$\triangle \underline{u} + \underline{D}_f \underline{u}^p = v \geq 0$$

holds by (2.11) and (2.12), we obtain

$$\varepsilon^2 \triangle \underline{u} + \underline{u}^p - \underline{u}_t = \varepsilon^2(\triangle \underline{u} + \underline{D}_f \underline{u}^p) \geq 0.$$

Hence, because $\partial \underline{u}/\partial \nu = 0$ on $\partial\Omega$ and $\underline{u}(x,0) \leq u_0(x)$ also hold, the function $\underline{u}(x,t)$ is a sub-solution of (1.1). □

The following gives a estimate of the blow-up time.

Lemma 5. *Suppose that a positive function $f \in C^2(\bar{\Omega})$ satisfies $\partial f/\partial \nu = 0$ on $\partial\Omega$. Let $\underline{D}_f$ be the constant defined by* (2.10). *Then, for any constant $\varepsilon > 0$ and function $u_0 \in C(\bar{\Omega})$ satisfying $2\underline{D}_f\varepsilon^2 \leq 1$ and $2\|u_0 - f\|_C \leq \min_{x\in\bar{\Omega}} f(x)$, the blow-up time T of the solution $u(x,t)$ of* (1.1) *satisfies*

$$\begin{aligned}(p-1)^{-1}(\|f\|_C + \|u_0 - f\|_C)^{-(p-1)} &\leq T \\ &\leq (p-1)^{-1}(1 - \underline{D}_f\varepsilon^2)^{-1}(\|f\|_C - \|u_0 - f\|_C)^{-(p-1)}.\end{aligned}$$

Proof. Because $\min_{y\in\bar{\Omega}} f(y)/2 \leq u_0(x) \leq \|f\|_C + \|u_0 - f\|_C$ holds, we have $T \geq (p-1)^{-1}(\|f\|_C + \|u_0 - f\|_C)^{-(p-1)}$. Also, from Lemma 4, $T \leq (p-1)^{-1}(1 - \underline{D}_f\varepsilon^2)^{-1} \cdot (\|f\|_C - \|u_0 - f\|_C)^{-(p-1)}$ holds. □

3. Proof of Theorem 1.

The following theorem is the main technical result in this paper. In Proofs of not only Theorem 1 but also Theorem 2 of [**8**], this theorem is made essential use of.

Theorem 6. *Let $f \in C^2(\bar{\Omega})$ be a positive function satisfying $\partial f/\partial \nu = 0$ on $\partial\Omega$, and let δ and C be positive constants. Then, there exists $\varepsilon_0 > 0$ satisfying the following*:

Suppose that a positive constant ε and a function $u_0 \in C(\bar{\Omega})$ satisfy $\varepsilon \leq \varepsilon_0$ and $\|u_0 - f\|_{C(\bar{\Omega})} \leq \varepsilon_0$. If the solution $u(x,t)$ of (1.1) *with the blow-up time T satisfies the Type-I estimate $u(x,t) \leq C(T-t)^{-1/(p-1)}$ in $\bar{\Omega} \times [0,T)$, then the blow-up set is contained in the set $S := \{x \in \bar{\Omega} \mid f(x) \geq \max_{y\in\bar{\Omega}} f(y) - \delta\}$ and the blow-up profile $u_*(x)$ satisfies the inequality*

$$\left\| u_*(x) - \left(f(x)^{-(p-1)} - \left(\max_{y\in\bar{\Omega}} f(y) \right)^{-(p-1)} \right)^{-1/(p-1)} \right\|_{C(\bar{\Omega}\setminus S)} \leq \delta.$$

Proof. [Step 1] In this step, we take positive constants $\delta', \alpha, D, \beta_0, \beta, \varepsilon_1, T_0$ and T_1 satisfying Lemma 3 and several inequalities below. Then, we fix these constants through Steps 2 and 3.

Put $\delta' = \min\{\delta, \|f\|_C\}$. By α, we denote the positive constant

$$\min\left\{ \frac{\delta}{4}\left(\left(\|f\|_C - \frac{\delta'}{2} \right)^{-(p-1)} - \|f\|_C^{-(p-1)} \right)^{p/(p-1)} \left(\min_{x\in\bar{\Omega}} f(x) \right)^p, \frac{\delta'}{16}, \frac{\min_{x\in\bar{\Omega}} f(x)}{4} \right\}.$$

We take D and $\beta_0 > 0$ such that Lemma 3 holds for f, α and C. We also take $\beta \in (0, \beta_0]$ such that the inequality

$$e^{DT_0}\beta^2 \le \frac{\delta}{4} \tag{3.1}$$

holds, where T_0 is defined by

$$T_0 = (p-1)^{-1}\left(\|f\|_C - \frac{\alpha}{2}\right)^{-(p-1)}.$$

Then, let a constant $\varepsilon_1 > 0$ be sufficiently small such that the inequalities

$$\varepsilon_1 \le \min\{\alpha, \beta\}, \tag{3.2}$$

$$\varepsilon_1 \le \min\left\{\frac{1}{(2\underline{D}_f)^{1/2}}, \frac{\min_{x\in\bar{\Omega}} f(x)}{2}\right\}, \tag{3.3}$$

$$T_1 \le T_0, \tag{3.4}$$

$$T_1 - (p-1)^{-1}\|f\|_C^{-(p-1)} \le \frac{\delta}{4}\left(\left(\|f\|_C - \frac{\delta'}{2}\right)^{-(p-1)} - \left(\|f\|_C - \frac{\delta'}{32}\right)^{-(p-1)}\right)^{p/(p-1)}, \tag{3.5}$$

$$2D\beta^{-(p-1)}\varepsilon_1 T_0^{1/2} \le (\|f\|_C - 4\alpha)^{-(p-1)} - (\|f\|_C - 3\alpha)^{-(p-1)} \tag{3.6}$$

and

$$\varepsilon_1^{2/(p-1)} \le \frac{\delta}{4}\left(\left(\|f\|_C - \frac{\delta'}{2}\right)^{-(p-1)} - \left(\|f\|_C - \frac{\delta'}{4}\right)^{-(p-1)}\right)^{2/(p-1)} \tag{3.7}$$

hold, where $\underline{D}_f$ and T_1 are defined by (2.10) and

$$T_1 = (p-1)^{-1}(1 - \underline{D}_f\varepsilon_1^2)^{-1}(\|f\|_C - \varepsilon_1)^{-(p-1)},$$

respectively.

[Step 2] In this step, we show the following.

Let $\varepsilon \in (0, \varepsilon_1]$ and $T \in (0, T_1]$ be constants, and let $g \in C^2(\bar{\Omega})$, $\omega \in C^1([0, T))$ and a positive function $\bar{v}(x, t)$ on the set $\{(x, t) \in \bar{\Omega} \times [0, T) \mid g(x)^{-(p-1)} > \omega(t)\}$ be defined as well as Lemma 3. Then,

$$\bar{v}(x, t) \le (f(x)^{-(p-1)} - \|f\|_C^{-(p-1)})^{-1/(p-1)} + \delta$$

holds for all $(x, t) \in (\bar{\Omega}\backslash S) \times [0, T)$.

From (3.4) and (3.1),

$$e^{Dt}\beta^2 \le \frac{\delta}{4} \tag{3.8}$$

holds. Also, from $x \notin S$, we have

$$f(x) \le \|f\|_C - \delta'. \tag{3.9}$$

Because $f(x) \le \|f\|_C - 16\alpha$ holds, $\rho((\|f\|_C - f(x))/\alpha) = 1$ holds. Hence, we have

$$g(x) = f(x) + \alpha. \tag{3.10}$$

From (3.9), we also have

$$g(x) \le \|f\|_C - \frac{\delta'}{2}. \tag{3.11}$$

Because $\omega(t) \le (\|f\|_C - \delta'/4)^{-(p-1)}$ holds by using (3.4) and (3.6), from (3.11) and (3.7),

$$\varepsilon^{2/(p-1)}(g(x)^{-(p-1)} - \omega(t))^{-2/(p-1)} \le \frac{\delta}{4} \tag{3.12}$$

holds. Also, because

$$\begin{aligned}
&(g(x)^{-(p-1)} - (a+b))^{-1/(p-1)} - (g(x)^{-(p-1)} - a)^{-1/(p-1)} \\
&= \int_0^1 (p-1)^{-1}(g(x)^{-(p-1)} - (a+\sigma b))^{-p/(p-1)} b\, d\sigma \\
&\le (p-1)^{-1}(g(x)^{-(p-1)} - (a+b))^{-p/(p-1)} b
\end{aligned}$$

holds, from (3.11), (3.4) and (3.5), we have

$$\begin{aligned}
&(g(x)^{-(p-1)} - (p-1)t)^{-1/(p-1)} - (g(x)^{-(p-1)} - \|f\|_C^{-(p-1)})^{-1/(p-1)} \\
&\le (g(x)^{-(p-1)} - (p-1)T_1)^{-1/(p-1)} - (g(x)^{-(p-1)} - \|f\|_C^{-(p-1)})^{-1/(p-1)} \\
&\le (g(x)^{-(p-1)} - (p-1)T_1)^{-p/(p-1)}(T_1 - (p-1)^{-1}\|f\|_C^{-(p-1)}) \\
&\le \left(\left(\|f\|_C - \frac{\delta'}{2}\right)^{-(p-1)} - \left(\|f\|_C - \frac{\delta'}{32}\right)^{-(p-1)}\right)^{-p/(p-1)} \\
&\quad \times (T_1 - (p-1)^{-1}\|f\|_C^{-(p-1)}) \\
&\le \frac{\delta}{4}.
\end{aligned} \tag{3.13}$$

By (3.10) and (3.9), we also have

$$\begin{aligned}
&(g(x)^{-(p-1)} - \|f\|_C^{-(p-1)})^{-1/(p-1)} - (f(x)^{-(p-1)} - \|f\|_C^{-(p-1)})^{-1/(p-1)} \\
&= \int_0^1 ((f(x)+\sigma\alpha)^{-(p-1)} - \|f\|_C^{-(p-1)})^{-p/(p-1)}(f(x)+\sigma\alpha)^{-p}\alpha\, d\sigma \\
&\le \left(\left(\|f\|_C - \frac{\delta'}{2}\right)^{-(p-1)} - \|f\|_C^{-(p-1)}\right)^{-p/(p-1)} \left(\min_{x\in\bar{\Omega}} f(x)\right)^{-p}\alpha \\
&\le \frac{\delta}{4}.
\end{aligned} \tag{3.14}$$

From (3.13), (3.14), (3.12) and (3.8), we obtain the conclusion of Step 2.

[Step 3] In this step, we show the following by Steps 1 and 2.

Let $\varepsilon \in (0, \varepsilon_1]$ *be a constant, and let* $u_0 \in C(\bar{\Omega})$ *satisfy* $\|u_0 - f\|_{C(\bar{\Omega})} \le \varepsilon_1$. *Suppose that the solution* $u(x,t)$ *of* (1.1) *with the blow-up time* T *satisfies* $u(x,t) \le C(T-t)^{-1/(p-1)}$ *in* $\bar{\Omega} \times [0,T)$. *Then,*

$$u(x,t) \le (f(x)^{-(p-1)} - \|f\|_C^{-(p-1)})^{-1/(p-1)} + \delta$$

holds for all $(x,t) \in (\bar{\Omega} \backslash S) \times [0,T)$.

By Lemma 5 and (3.3), we see

$$T \le T_1. \tag{3.15}$$

We take a cutoff function $\chi \in C^\infty(\boldsymbol{R})$ satisfying $\chi(z) = 1/4$ $(z \le 0)$, $\chi(z) = z$ $(1/2 \le z)$ and $0 \le \chi'(z) \le 1$. Let $g \in C^2(\bar{\Omega})$, $\omega \in C^1([0,T))$ and a positive function $\bar{v}(x,t)$ on the set $\{(x,t) \in \bar{\Omega} \times [0,T) \mid g(x)^{-(p-1)} > \omega(t)\}$ be defined as well as Lemma 3. Also, by $\bar{u}(x,t)$, we denote the positive function

$$(g(x)^{-(p-1)} - (p-1)t)^{-1/(p-1)} + C(T-t)^{-1/(p-1)} \chi\left(\frac{g(x)^{-(p-1)} - \omega(t)}{C^{-(p-1)/2}(T-t)^{1/2}\varepsilon}\right)^{-2/(p-1)} + e^{Dt}\beta^2$$

in $(x,t) \in \bar{\Omega} \times [0,T)$. Then, we show that the inequality

$$u(x,t) \le \bar{u}(x,t) \tag{3.16}$$

holds for all $(x,t) \in \bar{\Omega} \times [0,T)$.

In order to show (3.16), we first define $G \in C(\bar{\Omega} \times [0,T))$ by

$$G(x,t) = \bar{u}_t(x,t) - (\varepsilon^2 \triangle \bar{u}(x,t) + \bar{u}(x,t)^p).$$

Then, because $\rho(u^{p-1}/(C^{p-1}(T-t)^{-1})) = -1$ holds from $u(x,t) \le C(T-t)^{-1/(p-1)}$,

$$u_t = \varepsilon^2 \triangle u + u^p + \frac{1}{2}\left(\rho\left(\frac{u^{p-1}}{C^{p-1}(T-t)^{-1}}\right) + 1\right) G(x,t) \tag{3.17}$$

holds. Also, in the region where $g(x)^{-(p-1)} - \omega(t) \ge (1/2)C^{-(p-1)/2}(T-t)^{1/2}\varepsilon$, because $\bar{u} = \bar{v}$ holds and $T \le T_0$ and $\varepsilon \le \beta$ hold from (3.15), (3.4) and (3.2), by virtue of Lemma 3,

$$\bar{u}_t \ge \varepsilon^2 \triangle \bar{u} + \bar{u}^p$$

holds. In the region where $g(x)^{-(p-1)} - \omega(t) \le (1/2)C^{-(p-1)/2}(T-t)^{1/2}\varepsilon$, because $\bar{u} \ge 4^{1/(p-1)}C(T-t)^{-1/(p-1)}$ holds from $\chi((g(x)^{-(p-1)} - \omega(t))/(C^{-(p-1)/2}(T-t)^{1/2}\varepsilon)) \le 1/2$, we also have

$$\rho\left(\frac{\bar{u}^{p-1}}{C^{p-1}(T-t)^{-1}}\right) = 1.$$

Therefore, we obtain

$$-\bar{u}_t + \varepsilon^2 \triangle \bar{u} + \bar{u}^p + \frac{1}{2}\left(\rho\left(\frac{\bar{u}^{p-1}}{C^{p-1}(T-t)^{-1}}\right) + 1\right) G(x,t) \tag{3.18}$$

$$= \frac{1}{2}\left(1 - \rho\left(\frac{\bar{u}^{p-1}}{C^{p-1}(T-t)^{-1}}\right)\right)(-\bar{u}_t + \varepsilon^2 \triangle \bar{u} + \bar{u}^p) \le 0.$$

In the region where $f(x) \geq \|f\|_C - 4\alpha$, because $g(x)^{-(p-1)} \leq \omega(0)$ holds by

$$g(x) \geq (\|f\|_C - 4\alpha) + \alpha\rho\left(\frac{\|f\|_C - (\|f\|_C - 4\alpha)}{\alpha}\right) = \|g\|_C - 2\alpha,$$

we have

$$\begin{aligned}\bar{u}(x,0) &\geq CT^{-1/(p-1)}\chi\left(\frac{g(x)^{-(p-1)} - \omega(0)}{C^{-(p-1)/2}T^{1/2}\varepsilon}\right)^{-2/(p-1)} \\ &= 16^{1/(p-1)}CT^{-1/(p-1)} \geq CT^{-1/(p-1)} \geq u_0(x).\end{aligned}$$

In the region where $f(x) \leq \|f\|_C - 4\alpha$, because $\rho((\|f\|_C - f(x))/\alpha) = 1$ holds and $\|u_0 - f\|_C \leq \alpha$ holds from (3.2), we also have

$$\bar{u}(x,0) \geq g(x) = f(x) + \alpha \geq u_0(x).$$

Therefore, we obtain

$$u_0(x) \leq \bar{u}(x,0).$$

Hence, because we also see $\partial\bar{u}/\partial\nu = 0$ on $\partial\Omega$, working the comparison theorem by (3.17) and (3.18), we eventually get (3.16), i.e., the inequality

$$u(x,t) \leq \bar{u}(x,t)$$

holds for all $(x,t) \in \bar{\Omega} \times [0,T)$.

Because $\bar{u} \leq \bar{v}$ holds, we obtain the conclusion of Step 3 by Step 2, (3.15) and (3.16).

[Step 4] In this step, we prove Theorem 6.

We take a constant $\varepsilon_1 > 0$ as in Step 3. Then, let a constant $\varepsilon_0 \in (0, \varepsilon_1]$ be sufficiently small such that

$$\varepsilon_0 \leq \min\left\{\frac{1}{(2\underline{D}_f)^{1/2}}, \frac{\min_{x\in\bar{\Omega}} f(x)}{2}\right\}, \tag{3.19}$$

$$\varepsilon_0 \leq \frac{\delta}{2}((\|f\|_C - \delta)^{-(p-1)} - \|f\|_C^{-(p-1)})^{p/(p-1)}\left(\frac{\min_{x\in\bar{\Omega}} f(x)}{2}\right)^p \tag{3.20}$$

and

$$\begin{aligned}&\|f\|_C^{-(p-1)} - (1 - \underline{D}_f\varepsilon_0^2)(\|f\|_C + \varepsilon_0)^{-(p-1)} \\ &\quad \leq \frac{\delta}{2}(p-1)((\|f\|_C - \delta)^{-(p-1)} - \|f\|_C^{-(p-1)})^{p/(p-1)}\end{aligned} \tag{3.21}$$

hold, where $\underline{D}_f$ is defined by (2.10).

Because $f(x) \le \|f\|_C - \delta$ holds from $x \notin S$ and $\min_{y\in\bar\Omega} f(y)/2 \le f(x) - \varepsilon_0$ holds from (3.19), by (3.20),

$$(3.22)\qquad \begin{aligned}&(f(x)^{-(p-1)} - \|f\|_C^{-(p-1)})^{-1/(p-1)} - ((f(x)-\varepsilon_0)^{-(p-1)} - \|f\|_C^{-(p-1)})^{-1/(p-1)}\\ &= \int_0^1 ((f(x)-\sigma\varepsilon_0)^{-(p-1)} - \|f\|_C^{-(p-1)})^{-p/(p-1)}(f(x)-\sigma\varepsilon_0)^{-p}\varepsilon_0\,d\sigma\\ &\le ((\|f\|_C-\delta)^{-(p-1)} - \|f\|_C^{-(p-1)})^{-p/(p-1)}\left(\frac{\min_{x\in\bar\Omega} f(x)}{2}\right)^{-p}\varepsilon_0\\ &\le \frac{\delta}{2}\end{aligned}$$

holds. Because $f(x) - \varepsilon_0 \le \|f\|_C - \delta$ holds from $x \notin S$, by (3.21), we have

$$(3.23)\qquad \begin{aligned}&((f(x)-\varepsilon_0)^{-(p-1)} - \|f\|_C^{-(p-1)})^{-1/(p-1)}\\ &\quad - ((f(x)-\varepsilon_0)^{-(p-1)} - (1-\underline{D}_f\varepsilon_0^2)(\|f\|_C+\varepsilon_0)^{-(p-1)})^{-1/(p-1)}\\ &\le (p-1)^{-1}((\|f\|_C-\delta)^{-(p-1)} - \|f\|_C^{-(p-1)})^{-p/(p-1)}\\ &\quad \times (\|f\|_C^{-(p-1)} - (1-\underline{D}_f\varepsilon_0^2)(\|f\|_C+\varepsilon_0)^{-(p-1)})\\ &\le \frac{\delta}{2}.\end{aligned}$$

Also, because $T \ge (p-1)^{-1}(\|f\|_C+\varepsilon_0)^{-(p-1)}$ holds from Lemma 5 and (3.19), by Lemma 4 and (3.19), we see

$$u_*(x) \ge ((f(x)-\varepsilon_0)^{-(p-1)} - (1-\underline{D}_f\varepsilon_0^2)(\|f\|_C+\varepsilon_0)^{-(p-1)})^{-1/(p-1)}.$$

Hence, from (3.22) and (3.23),

$$u_*(x) \ge (f(x)^{-(p-1)} - \|f\|_C^{-(p-1)})^{-1/(p-1)} - \delta$$

holds for all $x \in \bar\Omega\backslash S$. Therefore, we obtain the conclusion of Theorem 6 by Step 3. □

According to Friedman and McLeod [**3**] and Chen [**1**], we prove that there exists a constant $C > 0$ such that if $\varepsilon > 0$ is sufficiently small, then the solution u of (1.1) satisfies the Type-I estimate $u(x,t) \le C(T-t)^{-1/(p-1)}$.

Proposition 7. *Let $u_0 \in C^2(\bar\Omega)$ be a positive function satisfying $\partial u_0/\partial\nu = 0$ on $\partial\Omega$. Then, there exist $C > 0$ and $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0]$, the solution $u(x,t)$ of* (1.1) *with the blow-up time T satisfies $u(x,t) \le C(T-t)^{-1/(p-1)}$ in $\bar\Omega \times [0,T)$.*

Proof. We define $C > 0$ by $C = ((p-1)/2)^{-1/(p-1)}$. We also define $\varepsilon_0 > 0$ by $\varepsilon_0^2 = \min_{x\in\bar\Omega} u_0(x)^p/2|\triangle u_0(x)|$. Let $v(x,t)$ denote the function $2\varepsilon^2\triangle u(x,t) + u(x,t)^p$.
Then, we have

$$(3.24)\qquad v(x,0) \ge 0.$$

Because of $v = 2u_t - u^p$,

$$\frac{\partial v}{\partial \nu} = 0 \tag{3.25}$$

holds on $\partial\Omega$. Also, because $u_t = (v + u^p)/2$ and $\triangle u^p \geq pu^{p-1}\triangle u$ hold, we have

$$\begin{aligned} v_t &= \varepsilon^2 \triangle (v + u^p) + \frac{p}{2} u^{p-1}(v + u^p) \\ &\geq \varepsilon^2 \triangle v + \frac{p}{2} u^{p-1} v + \frac{p}{2} u^{p-1}(2\varepsilon^2 \triangle u + u^p) \\ &= \varepsilon^2 \triangle v + pu^{p-1} v. \end{aligned} \tag{3.26}$$

Because $2u_t - u^p = v \geq 0$ holds from (3.24), (3.25) and (3.26), $1/2 \leq u_t/u^p$ holds. Hence, we have

$$\frac{T-t}{2} = \int_t^T \frac{ds}{2} \leq \int_{u(x,t)}^{u(x,T)} \frac{du}{u^p} \leq \frac{u(x,t)^{-(p-1)}}{p-1}.$$

Therefore, $u(x,t) \leq C(T-t)^{-1/(p-1)}$ holds. □

Now, we prove Theorem 1.

Proof of Theorem 1. We fix a constant $C > 0$ such that Proposition 7 holds for u_0. Let a constant $\varepsilon_0 > 0$ be sufficiently small. Then, by Proposition 7, for any $\varepsilon \in (0, \varepsilon_0]$, the solution u of (1.1) satisfies $u(x,t) \leq C(T-t)^{-1/(p-1)}$. Hence, by Theorem 6 with $f := u_0$, we obtain the conclusion of Theorem 1. □

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