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Projective manifolds with hyperplane sections being five-sheeted covers of projective space

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Abstract. Let L be a very ample line bundle on a smooth complex projective variety X of dimension ≥ 7. We classify the polarized manifolds (X, L) such that there exists a smooth member A of $|L|$ endowed with a branched covering of degree five $\pi : A \to \boldsymbol{P}^n$. The cases of $\deg \pi = 2$ and 3 are already studied by Lanteri-Palleschi-Sommese.

1. Introduction.

Let X be an $(n+1)$-dimensional smooth complex projective variety and L a very ample line bundle on X. Consider the following condition:

($*$) There exists a smooth member $A \in |L|$ such that there exists a branched covering $\pi : A \to \boldsymbol{P}^n$ of degree d.

Needless to say, the following "obvious" pairs (X, L) satisfy $(*)$: $(\boldsymbol{P}^{n+1}, \mathscr{O}_{\boldsymbol{P}^{n+1}}(d))$ and $(H_d^{n+1}, \mathscr{O}_{H_d^{n+1}}(1))$, where H_d^{n+1} is a smooth hypersurface of degree d in $\boldsymbol{P}^{n+2}$.

The study of (X, L) satisfying $(*)$ is a natural generalization of a classical problem of Castelnuovo [**C**]. The classical problem is to classify the pairs (X, L) satisfying $(*)$ when $n = 1$ and $d = 2$, and was solved by Serrano [**Se**], Sommese-Van de Ven [**S-V**], independently. When $n = 1$ and $d = 3$, Fania [**Fa**] studied the pairs (X, L). In cases $n \geq d = 2$ [**L-P-S 1**], $n > d = 3$ [**L-P-S 2**], Lanteri-Palleschi-Sommese classified the pairs.

Surprisingly, in case $n > d \in \{2, 3\}$, it turns out that the results of the classifications are simple; this relies on topological restrictions imposed X by A. In fact, in case $d = 2$, the "non-obvious" pairs never arise in the classification. In case $d = 3$, the "non-obvious" pair is only $(Y, 3\mathscr{L})$, where $(Y, \mathscr{L})$ is a Del Pezzo manifold of degree 1, i.e., a polarized manifold satisfying $-K_Y = n\mathscr{L}$ and $\mathscr{L}^{n+1} = 1$.

So, what kind of "non-obvious" pairs (X, L) arise in case $n > d > 3$?

The purpose of this paper is to give a complete classification of the pairs (X, L) that satisfy $(*)$ under the condition $n > d = 5$. Our result is as follows:

THEOREM 1.1. *Let X be a smooth projective variety with $\dim X = n + 1 > 6$. Then there exists a very ample line bundle L on X that satisfies the condition $(*)$ and $d = 5$ if and only if (X, L) is one of the following*:

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(1) $(\boldsymbol{P}^{n+1}, \mathscr{O}_{\boldsymbol{P}^{n+1}}(5))$;
(2) $(H_5^{n+1}, \mathscr{O}_{H_5^{n+1}}(1))$;
(3) $(Y, 5\mathscr{L})$;
(4) $(V_{10}, \mathscr{O}_{V_{10}}(5))$, *where* V_{10} *is a smooth weighted hypersurface of degree* 10 *in the weighted projective space* $\boldsymbol{P}(5,2,1^{n+1})$; *or*
(5) $(W_{20}, \mathscr{O}_{W_{20}}(5))$, *where* W_{20} *is a smooth weighted hypersurface of degree* 20 *in* $\boldsymbol{P}(5,4,1^{n+1})$.

No less than three "non-obvious" pairs (3)–(5) show up.

Lanteri-Palleschi-Sommese [**L-P-S 1**], [**L-P-S 2**] in cases $n > d \in \{2,3\}$ use the classification theory of polarized varieties via sectional genus.

The difficulty in our study is that a polarized manifold $(X, \mathscr{H})$ with $\Delta(X, \mathscr{H}) = d(X, \mathscr{H}) = 1$ and sectional genus ≥ 3 arises; the classification problem of polarized manifolds with these invariants is yet to be solved completely (cf. [**Fu 3**, (6.18)]).

Our study involves a new strategy although the starting point of the proof is inspired by the ideas of Lanteri-Palleschi-Sommese. The keys of the proof are as follows:

(i) To show the very ampleness of $\mathscr{O}_{W_{20}}(5)$. (Proposition 3.3)
(ii) To characterize $(X, \mathscr{H})$ with $\Delta(X, \mathscr{H}) = \mathscr{H}^{n+1} = 1$ and large sectional genus that satisfies the assumption of Theorem 1.1. (Theorem 6.2)

For (i), after finding a basis of $H^0(\mathscr{O}_{W_{20}}(5))$, we check that the freeness, the separation of points and the separation of tangent vectors for $|\mathscr{O}_{W_{20}}(5)|$.
For (ii), our strategy is to find the generators of the graded ring of $(X, \mathscr{H})$

$$R(X, \mathscr{H}) := \bigoplus_{l=0}^{\infty} H^0(X, l\mathscr{H}),$$

and the relations among them. Using the ladder method, we reduce this to describing the structure of $R(X_1, \mathscr{H}_{X_1})$ in terms of generators and relations, where X_1 is a smooth curve section of X that is an intersection of n-general members of $|\mathscr{H}|$. By the Riemann-Roch theorem and some ring-theoretic arguments, we can describe the structure of $R(X, \mathscr{H})$ successfully.

The paper is organized as follows. In Section 2, we give some notation, definitions and general facts. In Section 3, we prove (i), consequently the 'if' part in Theorem 1.1 is proved. From Section 4, we concentrate on proving the 'only if' part. In Section 4, we prove a basic result on $h^0(A, \pi^*\mathscr{O}_{\boldsymbol{P}^n}(1))$. Section 5 is devoted to the cases (1) and (2) of Theorem 1.1. Section 6 is devoted to the proof of (ii) (Theorem 6.2), as a consequence we see that the polarized manifolds (3)–(5) actually show up.

After submitting the paper, the author was informed about a paper of Lanteri [**Lan**] by the referee. Lanteri has given a classification of the pairs (X, L) in question [**Lan**, Theorem 3.5]. However, his classification result contains one doubtful case: in fact, his result says that the cases (1)–(4) in our main theorem (Theorem 1.1) arise. But he gave only a numerical characterization and invariants for the case (5). In contrast, this paper determines the structure of a unique polarized manifold appearing in that case, completely.

2. Notation and background.

In this paper, we work over the complex number field $\boldsymbol{C}$. We use the standard notation from algebraic geometry as in [**H**]. The words "Cartier divisors", "line bundles" and "invertible sheaves" are used interchangeably, and "vector bundles" and "locally free sheaves", too. The tensor products of line bundles are denoted additively, while we use multiplicative notation for intersection products in Chow rings.

A *branched covering of degree* d means a finite surjective morphism of degree d. A *manifold* means a smooth variety. A line bundle on a variety is said to be *spanned* if it is generated by global sections.

A *polarized variety* means a pair $(V, \mathscr{L})$ where V is a projective variety and $\mathscr{L}$ is an ample line bundle on V. Set $m = \dim V$.

A member of $|\mathscr{L}|$ is called a *rung* of $(V, \mathscr{L})$ if it is an irreducible and reduced subscheme of V. A rung D of $(V, \mathscr{L})$ is said to be *regular* if the restriction map $H^0(V, \mathscr{L}) \to H^0(D, \mathscr{L}_D)$ is surjective. A sequence $V = V_m \supset V_{m-1} \supset \cdots \supset V_1$ of subvarieties of V is called a *ladder* of $(V, \mathscr{L})$ if each V_j is a rung of $(V_{j+1}, \mathscr{L}_{j+1})$ for $j \geq 1$, where $\mathscr{L}_j$ is the restriction of $\mathscr{L}$ to V_j.

The Δ-*genus* of $(V, \mathscr{L})$ is defined as $\Delta(V, \mathscr{L}) = m + d(V, \mathscr{L}) - h^0(V, \mathscr{L})$, where $d(V, \mathscr{L}) := \mathscr{L}^m$ is the *degree* of $(V, \mathscr{L})$. For a manifold V, the *sectional genus* of $(V, \mathscr{L})$, denoted by $g(V, \mathscr{L})$, is defined by the formula

$$2g(V, \mathscr{L}) - 2 = (K_V + (m-1)\mathscr{L})\mathscr{L}^{m-1}.$$

A polarized variety $(V, \mathscr{L})$ is called a *scroll over* a smooth curve C if it is of the form $(\boldsymbol{P}(\mathscr{E}), H(\mathscr{E}))$ for some locally free sheaf $\mathscr{E}$ on C, where $H(\mathscr{E})$ denotes the tautological line bundle of $\boldsymbol{P}(\mathscr{E})$.

For an integer $r \geq 1$, a line bundle $\mathscr{L}$ on V is said to be r-*generated* if the graded ring $R(V, \mathscr{L}) = \bigoplus_{i=0}^{\infty} H^0(V, i\mathscr{L})$ is generated by the global sections of $\mathscr{L}, \ldots, r\mathscr{L}$. In particular $\mathscr{L}$ is said to be *simply generated* if it is one-generated.

The following is used in the study of polarized manifolds with small Δ-genera:

PROPOSITION 2.1 (Fujita). *Let $(M, \mathscr{L})$ be an m-dimensional polarized manifold having a ladder. Assume that $g := g(M, \mathscr{L}) \geq \Delta(M, \mathscr{L}) =: \Delta$ and $\mathscr{L}^m \geq 2\Delta + 1$. Then $\mathscr{L}$ is simply generated, $g = \Delta$ and $H^q(M, t\mathscr{L}) = 0$ for any integers t, q with $0 < q < m$.*

For the proof, we refer to [**Fu 3**, Chapter I (3.5)].

The following lemma is trivial but useful in studying the structure of graded rings:

LEMMA 2.2. *Let $(V, \mathscr{L})$ be a polarized variety, D a rung of $(V, \mathscr{L})$ defined by $\delta \in H^0(V, \mathscr{L})$, and $\rho_t : H^0(V, t\mathscr{L}) \to H^0(D, t\mathscr{L}_D)$ the restriction map. Then* $\mathrm{Ker}(\rho_t) = \delta H^0(V, (t-1)\mathscr{L})$.

A *weighted projective space* $\boldsymbol{P}(e_0, \ldots, e_N)$ is defined to be $\mathrm{Proj}(\boldsymbol{C}[s_0, \ldots, s_N])$, where $\mathrm{wt}(s_0, \ldots, s_N) = (e_0, \ldots, e_N) \in \boldsymbol{N}^{\oplus(N+1)}$. A projective variety W is called a *weighted complete intersection* of type $(a_1, \ldots, a_c)$ in $\boldsymbol{P}(e_0, \ldots, e_N)$ (w.c.i., for short) if the following two conditions are satisfied:

(1) $W \cong \mathrm{Proj}(\boldsymbol{C}[s_0, \ldots, s_N]/(F_1, \ldots, F_c))$, where $(F_1, \ldots, F_c)$ is a regular sequence and each F_i is a homogeneous polynomial of degree $a_i > 0$;
(2) $V_+(F_1, \ldots, F_c) \cap (\bigcup_{1<k}(s_j = 0 \mid k \nmid e_j)) = \varnothing$ in $\boldsymbol{P}(e_0, \ldots, e_N)$.

We put $S(e_0, \ldots, e_N) := \bigcup_{1<k}(s_j = 0 \mid k \nmid e_j)$.

PROPOSITION 2.3 (Mori). *Let D be an effective ample divisor of an m-dimensional projective manifold M. Assume D is a w.c.i. of type $(a_1, \ldots, a_c)$ in $\boldsymbol{P}(e_0, \ldots, e_N)$. Then the following hold.*

(1) *If $m \geq 4$, M is a w.c.i. of type $(a_1, \ldots, a_c)$ in $\boldsymbol{P}(e_0, \ldots, e_N, a)$ for some integer $a > 0$.*
(2) *If $m = 3$ and there exists a positive integer a such that $\mathcal{O}_M(D) \otimes \mathcal{O}_D \cong \mathcal{O}_D(a)$, then M satisfies the same conclusion of* (1) *for such $a > 0$.*

For the proof, see [**M**, Corollary 3.8 and Proposition 3.10].

3. Some special examples: the 'if' part of the Theorem.

In this section we consider the three special classes (3)–(5) of polarized manifolds appearing in Theorem 1.1. These classes are constructed from polarized manifolds $(M, \mathscr{L})$ with $\Delta(M, \mathscr{L}) = d(M, \mathscr{L}) = 1$.

We begin with the following fact:

FACT 3.1. *Let $(M, \mathscr{L})$ be an m-dimensional polarized manifold with $\Delta(M, \mathscr{L}) = \mathscr{L}^m = 1$, and let $H_1, \ldots, H_{m-1}$ be general members of $|\mathscr{L}|$. For each integer $1 \leq k \leq m-1$, we put $X_k := \bigcap_{k \leq i \leq m-1} H_i$. Then the following hold.*

(1) *The base locus* $\mathrm{Bs}|\mathscr{L}|$ *consists of a single point.*
(2) *The linear system $|b^*\mathscr{L} - E|$ defines a flat surjective morphism $f : \tilde{M} \to \boldsymbol{P}^{m-1}$, where $b : \tilde{M} \to M$ is the blowing up at* $\mathrm{Bs}|\mathscr{L}|$ *and E is the exceptional divisor lying over* $\mathrm{Bs}|\mathscr{L}|$. *The set E is a section of f, and every fiber of f is an integral curve of arithmetic genus $g(M, \mathscr{L}) \geq 1$.*
(3) *X_k is a k-dimensional submanifold of M, and $X_1 \subset \cdots \subset X_{m-1} \subset M$ is a regular ladder of $(M, \mathscr{L})$.*

For the proof, we refer to [**Fu 2**, Section 13].

PROPOSITION 3.2. *Let $(M, \mathscr{L})$ be as in Fact* 3.1, *and let $d \geq 2$ be an integer such that $L := d\mathscr{L}$ is spanned. Then there exists a smooth member A of $|L|$ with a finite surjective morphism of degree d,*

$$\pi : A \longrightarrow \boldsymbol{P}^{m-1}.$$

PROOF. From Fact 3.1 (2), we obtain the flat surjective morphism $f : \tilde{M} \to \boldsymbol{P}^{m-1}$. Now, since L is spanned, there exists a smooth member A of $|L|$ not passing through $\mathrm{Bs}|\mathscr{L}|$. Since $H^i(M, (1-d)\mathscr{L}) = 0$ for $i = 0, 1$ by the Kodaira vanishing theorem, we see that $h^0(A, \mathscr{L}_A) = m$, especially $|\mathscr{L}_A| = |\mathscr{L}|_A$. Therefore, combining these and $\mathscr{L}_A^{m-1} = d$, we see that $|\mathscr{L}_A|$ gives a branched covering of degree d from A to $\boldsymbol{P}^{m-1}$. □

EXAMPLE 1. Let $(X, L) = (Y, 5\mathscr{L})$, where $(Y, \mathscr{L})$ is an $(n+1)$-dimensional Del Pezzo manifold of degree 1, i.e., $-K_Y = n\mathscr{L}$ with $\mathscr{L}^{n+1} = 1$. We see $\Delta(Y, \mathscr{L}) = 1$. The very ampleness of $5\mathscr{L}$ follows from the facts that $2\mathscr{L}$ is spanned [**Fu 2**, Section 14] and that $3\mathscr{L}$ is very ample [**L-P-S 2**, (1.2)]. Hence, by Proposition 3.2, there exists a smooth five-sheeted cover of $\boldsymbol{P}^n$ that is a member of $|5\mathscr{L}|$.

EXAMPLE 2. Let $(X, L) = (V_{10}, \mathscr{O}_{10}(5))$, where V_{10} is an $(n+1)$-dimensional smooth weighted hypersurface of degree 10 in $\boldsymbol{P}(5, 2, 1^{n+1})$. We see that $\Delta(V_{10}, \mathscr{O}_{V_{10}}(1)) = \mathscr{O}_{V_{10}}(1)^{n+1} = 1$. Moreover, it follows from $g(V_{10}, \mathscr{O}_{V_{10}}(1)) = 2$ that $(V_{10}, \mathscr{O}_{V_{10}}(1))$ is a sectionally hyperelliptic polarized manifold of type $(-)$ [**Fu 2**, Sections 15 and 16]. Therefore $\mathscr{O}_{V_{10}}(5)$ is very ample due to [**Laf**, Theorem 3.3]. Consequently we obtain a smooth five-sheeted cover of $\boldsymbol{P}^n$ in $|\mathscr{O}_{V_{10}}(5)|$.

EXAMPLE 3. Let $(X, L) = (W_{20}, \mathscr{O}_{W_{20}}(5))$, where W_{20} is an $(n+1)$-dimensional smooth weighted hypersurface of degree 20 in $\boldsymbol{P}(5, 4, 1^{n+1})$. Since we have $\Delta(W_{20}, \mathscr{O}_{W_{20}}(1)) = \mathscr{O}_{W_{20}}(1)^{n+1} = 1$, we get a five-sheeted cover of $\boldsymbol{P}^n$ in $|\mathscr{O}_{W_{20}}(5)|$ from the following

PROPOSITION 3.3. *The line bundle $\mathscr{O}_{W_{20}}(5)$ is very ample.*

PROOF. We prove the conclusion with the following steps:

(a) $\mathrm{Bs}|\mathscr{O}_{W_{20}}(5)| = \varnothing$;
(b) The morphism φ associated with $|\mathscr{O}_{W_{20}}(5)|$ is injective;
(c) The linear system $|\mathscr{O}_{W_{20}}(5)|$ separates the tangent vectors.

By combining 5-generatedness of $\mathscr{O}_{W_{20}}(1)$ and [**Laf**, Theorem 2.2], the rational map φ is an embedding outside the single point $p := \mathrm{Bs}|\mathscr{O}_{W_{20}}(1)|$.

Let $x, y, z_0, \dots, z_n$ generate the graded ring $R(W_{20}, \mathscr{O}_{W_{20}}(1))$, where $\deg(x, y, z_0, \dots, z_n) = (5, 4, 1, \dots, 1)$.

(a) We see that $H^0(\mathscr{O}_{W_{20}}(5))$ is generated by the sections

$$x, yz_0, \dots, yz_n, z_{j_1} \cdots z_{j_5}, \text{ with } 0 \le j_1 \le \cdots \le j_5 \le n.$$

Therefore it follows that

$$\mathrm{Bs}|\mathscr{O}_{W_{20}}(5)| = (x = 0) \cap \left(\bigcap_{0 \le i \le n} (z_i = 0) \right),$$

which is empty since W_{20} does not meet the locus $S(5, 4, 1^{n+1})$.

(b) Suppose that $\varphi(p) = \varphi(q)$ for some $q \in W_{20}$. Then we see that $z_i(q) = 0$ for any $0 \le i \le n$, which implies $q \in \mathrm{Bs}|\mathscr{O}_{W_{20}}(1)|$. Therefore $p = q$.

(c) Let τ be a non-zero tangent vector in $T_p(W_{20})$. We need to show that there exists a section $\sigma \in H^0(\mathscr{O}_{W_{20}}(5))$ satisfying the following conditions:

$$\sigma(p) = 0 \text{ and } d\sigma(\tau) \neq 0.$$

We claim that $\sigma_i := yz_i$ satisfies the above conditions for some $0 \leq i \leq n$. The former condition is satisfied for all σ_i since $z_i(p) = 0$. We prove that the latter holds. Suppose that there exists non-zero $\tau \in T_p(W_{20})$ such that $d\sigma_i(\tau) = 0$ for all i. Since $d\sigma_i(\tau) = y(p)dz_i(\tau)$ and $y(p) \neq 0$, we see that $dz_i(\tau) = 0$ for all i. Hence it follows that

$$\tau \in T_p(\Gamma), \text{ where } \Gamma := \bigcap_{1 \leq i \leq n} (z_i = 0).$$

From $dz_0(\tau) = 0$, we have $\Gamma \cdot \mathscr{O}_{W_{20}}(1) \geq 2$, which contradicts $\mathscr{O}_{W_{20}}(1)^{n+1} = 1$. This concludes the proof. □

4. The 'only if' part.

We are now going to classify the polarized manifolds in question.

Suppose that (X, L) satsifies $(*)$ and $n > d = 5$. Let $\pi : A \to \boldsymbol{P}^n$ denote the finite morphism of degree 5. Then a Barth-type theorem of Lazarsfeld [**Laz**, Theorem 1] implies that $H^2(A, \boldsymbol{Z}) \cong H^2(\boldsymbol{P}^n, \boldsymbol{Z}) \cong \boldsymbol{Z}$ and $H^1(A, \mathscr{O}_A) = 0$. Therefore $\mathrm{Pic}(A) \cong \boldsymbol{Z}$, generated by $\pi^*\mathscr{O}_{\boldsymbol{P}^n}(1)$. The Lefschetz hyperplane section theorem implies $\mathrm{Pic}(X) \cong \boldsymbol{Z}$. We denote by $\mathscr{H}$ the ample generator of $\mathrm{Pic}(X)$; we have $\mathscr{H}_A = \pi^*\mathscr{O}_{\boldsymbol{P}^n}(1)$. Combining the ampleness of $\mathscr{H}_A$ and the fact that Δ-genus is non-negative for every polarized manifold [**Fu 3**, Chapter I (4.2)], we see

$$n + 1 \leq h^0(A, \mathscr{H}_A) \leq n + 5.$$

In fact, we have the following

Proposition 4.1. *$h^0(A, \mathscr{H}_A) = n + 1$ or $n + 2$.*

Proof. At first, suppose $h^0(A, \mathscr{H}_A) = n+5$. Then we have $\Delta(A, \mathscr{H}_A) = 0$. Therefore, by [**Fu 3**, Chapter I (5.10)], $(A, \mathscr{H}_A)$ is either (i) $(\boldsymbol{P}^n, \mathscr{O}_{\boldsymbol{P}^n}(1))$, (ii) $(\boldsymbol{Q}^n, \mathscr{O}_{\boldsymbol{Q}^n}(1))$ or (iii) a scroll over $\boldsymbol{P}^1$. Cases (i), (ii) cannot occur by $\mathscr{H}_A^n = 5$. Case (iii) also cannot occur because of $\mathrm{Pic}(A) \cong \boldsymbol{Z}$.

Secondly, suppose $h^0(A, \mathscr{H}_A) = n + 4$. Then we obtain $\Delta(A, \mathscr{H}_A) = 1$. By Proposition 2.1, we have $g(A, \mathscr{H}_A) = 1$. Therefore it follows from [**Fu 3**, (12.3)] that $(A, \mathscr{H}_A)$ is either a Del Pezzo manifold or a scroll over an elliptic curve. The latter case is ruled out because of $\mathrm{Pic}(A) \cong \boldsymbol{Z}$. The former case is also ruled out by the following reason: if $(A, \mathscr{H}_A)$ is a Del Pezzo manifold of degree 5, then we see that A is the Grassmann variety parametrizing lines in $\boldsymbol{P}^4$, $\mathrm{Gr}(5, 2)$, by combining the result of [**Fu 3**, (8.11)] and our assumption $n > 5$. But $\mathrm{Gr}(5, 2)$ cannot be ample divisors on X by virtue of [**Fu 1**, (5.2)].

Lastly, we suppose $h^0(A, \mathscr{H}_A) = n + 3$. By Proposition 2.1, we see that $g(A, \mathscr{H}_A) = \Delta(A, \mathscr{H}_A) = 2$ and that $\mathscr{H}_A$ is simply generated, hence very ample. According to [**I**], we have $\dim A \leq 4$, which contradicts our assumption. □

From now on, we will discuss the case $h^0(A, \mathscr{H}_A) = n + 2$ in Section 5 and the case $h^0(A, \mathscr{H}_A) = n + 1$ in Section 6.

5. Case where $h^0(A, \mathscr{H}_A) = n + 2$.

In this section we treat the case $h^0(A, \mathscr{H}_A) = n + 2$. The aim of this section is to prove the following

PROPOSITION 5.1. *If $h^0(A, \mathscr{H}_A) = n+2$, then (X, L) is either $(\boldsymbol{P}^{n+1}, \mathscr{O}_{\boldsymbol{P}^{n+1}}(5))$ or $(H_5^{n+1}, \mathscr{O}_{H_5^{n+1}}(1))$.*

The following lemma is a special case of [**L-P-S 1**, (1.3)]:

LEMMA 5.2 (Lanteri-Palleschi-Sommese). *If $h^0(A, \mathscr{H}_A) = n+2$, then the morphism $q : A \to \boldsymbol{P}^{n+1}$ associated to $|\mathscr{H}_A|$ is birational and its image $q(A)$ is a hypersurface (possibly singular) of degree 5 in $\boldsymbol{P}^{n+1}$.*

REMARK 5.3. By virtue of the Bertini theorem, we obtain a smooth k-dimensional rung A_k of $(A_{k+1}, \mathscr{H}_{A_{k+1}})$ inductively, with $A_n := A$. Put $C := A_1$. Then one can easily obtain an inequality

$$g(C, \mathscr{H}_C) \geq \Delta(C, \mathscr{H}_C). \tag{$\star$}$$

LEMMA 5.4. *The ladder $C \subset A_2 \subset \cdots \subset A_n$ is regular.*

PROOF. It suffices to prove $H^1(A_k, \mathscr{O}_{A_k}) = 0$ for all $k \geq 2$. By the Lefschetz hyperplane section theorem [**Fu 3**, (7.1.4)], we have $H^1(A_k, \mathscr{O}_{A_k}) \cong H^1(A_{k-1}, \mathscr{O}_{A_{k-1}})$ for all $k \geq 3$. Combining these and $H^1(A, \mathscr{O}_A) = 0$, we obtain the assertion. □

By Lemma 5.2, the smooth curve C is the normalization of $q(C)$, which is a plane quintic curve of arithmetic genus 6. Since $h^0(A_{k+1}, \mathscr{H}_{A_{k+1}}) = k+3$ for all k by virtue of Lemma 5.4, we have $\Delta(C, \mathscr{H}_C) = 3$.

LEMMA 5.5. *The line bundle $\mathscr{H}_C$ is simply generated.*

PROOF. We prove that $g(C, \mathscr{H}_C) = 6$ as follows. We have inequalities

$$3 \leq g(C, \mathscr{H}_C) \leq 6.$$

Indeed, the right inequality is obvious and the left is obtained by combining ($\star$) and $\Delta(C, \mathscr{H}_C) = 3$. We have $K_A = r\mathscr{H}_A$ for some integer r due to $\mathrm{Pic}(A) \cong \boldsymbol{Z}$. By the sectional genus formula

$$2g(A, \mathscr{H}_A) - 2 = \big(K_A + (n-1)\mathscr{H}_A\big)\mathscr{H}_A^{n-1} = 5(r + n - 1),$$

we see that $g(A, \mathscr{H}_A) - 1$ is divisible by 5. Combining this and the above inequlities, we obtain $g(C, \mathscr{H}_C) = 6$.

It follows from $g(C, \mathscr{H}_C) = 6 = p_a(q(C))$ that $\mathscr{H}_C$ is very ample, i.e., $C \cong q(C)$. Moreover $q(C)$ is a smooth plane curve. Therefore $\mathscr{H}_C$ is simply generated. □

PROOF OF PROPOSITION 5.1. By combining Lemma 5.4, 5.5 and [**Fu 3**, Chapter

I (2.5)], we see that $\mathscr{H}_A$ is very ample. Thus

$$(A, \mathscr{H}_A) \cong \big(H_5^n, \mathscr{O}_{H_5^n}(1)\big).$$

We can write $L = l\mathscr{H}$ with some integer $l \geq 1$. It follows from $5 = \mathscr{H}_A^n = l\mathscr{H}^{n+1}$ that $(l, \mathscr{H}^{n+1})$ is either $(1,5)$ or $(5,1)$.

The case $(l, \mathscr{H}^{n+1}) = (1,5)$. The ladder $C \subset \cdots \subset A \subset X$ is regular, hence $\Delta(X, L) = 3$. Therefore, from $h^0(X, L) = n+3$, it follows $(X, L) \cong (H_5^{n+1}, \mathscr{O}_{H_5^{n+1}}(1))$.

The case $(l, \mathscr{H}^{n+1}) = (5,1)$. Since $H^i(X, -4\mathscr{H}) = 0$ for $i = 0, 1$ due to the Kodaira vanishing theorem, we see that $h^0(X, \mathscr{H}) = n+2$, hence we have $\Delta(X, \mathscr{H}) = 0$. Since $\mathscr{H}^{n+1} = 1$, we obtain $(X, L) \cong (\boldsymbol{P}^{n+1}, \mathscr{O}_{\boldsymbol{P}^{n+1}}(5))$. □

6. Case where $h^0(A, \mathscr{H}_A) = n+1$.

In this section, we deal with the case $h^0(A, \mathscr{H}_A) = n+1$. The heart of this section is to prove Theorem 6.2.

Lemma 6.1. *If $h^0(A, \mathscr{H}_A) = n+1$, then we have $L = 5\mathscr{H}$, $\mathscr{H}^{n+1} = 1$ and $\Delta(X, \mathscr{H}) = 1$.*

Proof. We see that $L = l\mathscr{H}$ for $l \neq 1$ as follows. Suppose $l = 1$. Then $|\mathscr{H}_A|$ gives an embedding of A into $\boldsymbol{P}^n$, which contradicts $\deg \pi = 5$. From this, we see $l \neq 1$.

Therefore we have $(l, \mathscr{H}^{n+1}) = (5,1)$. Furthermore, from the Kodaira vanishing theorem, it follows $h^0(X, \mathscr{H}) = h^0(A, \mathscr{H}_A) = n+1$. Hence we obtain $\Delta(X, \mathscr{H}) = 1$. □

Let $H_1, \dots, H_n$ be general members of $|\mathscr{H}|$, and put $X_k := \bigcap_{k \leq i \leq n} H_i$ for all $1 \leq k \leq n$. Recalling Fact 3.1 (3), we see that X_k is a k-dimensional manifold. We put $p := \mathrm{Bs}|\mathscr{H}|$.

We now consider the morphism associated to $|L|$

$$\varphi_L : X \longrightarrow \boldsymbol{P}(|L|),$$

which is an embedding of X, and $\varphi_L(X_1)$ is a smooth curve of degree 5. Then we obtain $g(X, \mathscr{H}) = g(\varphi_L(X_1)) = 0, 1, 2$ or 6 (see [**H**, p. 354]).

The case $g(X, \mathscr{H}) = 0$. From [**Fu 3**, (12.1)], we see $\Delta(X, \mathscr{H}) = 0$, which is absurd.

The case $g(X, \mathscr{H}) = 1$. By virtue of a result of Fujita [**Fu 3**, (6.5)], we see that $(X, \mathscr{H})$ is a Del Pezzo manifold of degree 1, hence we are in the case of (3) in Theorem 1.1.

The case $g(X, \mathscr{H}) = 2$. From [**Fu 2**, Section 15 and Appendix 1] and $n \geq 6$, $(X, \mathscr{H})$ is a sectionally hyperelliptic polarized manifold of type $(-)$, which is also classified by Fujita. We are in the case (4).

THE CASE $g(X, \mathscr{H}) = 6$. Then we see that X_1 is isomorphic to a smooth plane quintic curve. What we are going to prove is the following

THEOREM 6.2. *If $h^0(A, \mathscr{H}_A) = n+1$ and $g(X, \mathscr{H}) = 6$, then*

$$(X, \mathscr{H}) \cong \bigl(W_{20}, \mathscr{O}_{W_{20}}(1)\bigr).$$

We will use the ladder method to prove this, where the key is to describe the structure of $R(X_2, \mathscr{H}_{X_2})$ explicitly. In fact, in order to get the conclusion, we need the description of the structure of $R(X_1, \mathscr{H}_{X_1})$ and the surjectivity of the restriction map

$$\rho : R\bigl(X_2, \mathscr{H}_{X_2}\bigr) \longrightarrow R\bigl(X_1, \mathscr{H}_{X_1}\bigr).$$

We first describe the structure of $R(X_1, \mathscr{H}_{X_1})$:

PROPOSITION 6.3. *Under the assumption of Theorem 6.2, there exists an isomorphism*

$$R\bigl(X_1, \mathscr{H}_{X_1}\bigr) \cong \boldsymbol{C}[x, y, z]/(F_{20}),$$

where $\mathrm{wt}(x, y, z) = (5, 4, 1)$ *and F_{20} is an irreducible weighted homogeneous polynomial of degree* 20.

PROOF. Using the Riemann-Roch theorem for X_1, we find the generators of $R(X_1, \mathscr{H}_{X_1})$ and the relations among them. We proceed in three steps.

STEP 1. We show that the dimension of $H^0(l\mathscr{H}_{X_1})$ for $l \geq 1$ is as follows:

l	$h^0(l\mathscr{H}_{X_1})$	l	$h^0(l\mathscr{H}_{X_1})$
1	1	6	3
2	1	7	3
3	1	8	4
4	2	9	5
5	3	10	6

and $h^0(l\mathscr{H}_{X_1}) = l - 5$ for all $l \geq 11$. Indeed, by the Riemann-Roch theorem, we obtain

$$h^0\bigl(l\mathscr{H}_{X_1}\bigr) = h^0\bigl((10 - l)\mathscr{H}_{X_1}\bigr) + l - 5,$$

which implies the latter assertion. We prove the former. Note that $h^0(5\mathscr{H}_{X_1}) = 3$ since $|L_{X_1}|$ gives an embedding of X_1 into $\boldsymbol{P}^2$. By Fact 3.1 (3), we see $h^0(\mathscr{H}_{X_k}) = k$ in particular $h^0(\mathscr{H}_{X_1}) = 1$, thus $h^0(9\mathscr{H}_{X_1}) = 5$. From the well-known fact that a smooth plane quintic curve has neither g^1_2 nor g^1_3, we have $h^0(2\mathscr{H}_{X_1}) = h^0(3\mathscr{H}_{X_1}) = 1$, thus $h^0(8\mathscr{H}_{X_2}) = 4, h^0(7\mathscr{H}_{X_1}) = 3$. Then we see $h^0(6\mathscr{H}_{X_1}) = 3$ and $h^0(4\mathscr{H}_{X_1}) = 2$. Therefore the former assertion is proved.

Let z be a basis of $H^0(\mathscr{H}_{X_1})$. Choose $y \in H^0(4\mathscr{H}_{X_1})$ such that $H^0(4\mathscr{H}_{X_1}) = \langle y, z^4 \rangle$.

Moreover, choose $x \in H^0(5\mathscr{H}_{X_1})$ such that $H^0(5\mathscr{H}_{X_1}) = \langle x, yz, z^5 \rangle$.

Step 2. We claim that the graded ring $R(X_1, \mathscr{H}_{X_1})$ is generated by x, y, z. Indeed, it suffices to prove that there exist some monomials in x, y, z which form a basis of $H^0(l\mathscr{H}_{X_1})$ for each l. Note that

$$h^0\big(l\mathscr{H}_{X_1}\big) - h^0\big((l-1)\mathscr{H}_{X_1}\big) = \delta \in \{0, 1\}.$$

The cases $6 \leq l \leq 11$. We may assume $\delta = 1$: otherwise, we have $H^0(l\mathscr{H}_{X_1}) = zH^0((l-1)\mathscr{H}_{X_1})$. Therefore we only consider the cases $l = 8, 9, 10$. Each monomial in x, y contained in $H^0(l\mathscr{H}_{X_1})$ has a pole of order exactly l at p. Comparing their orders of poles, we see from Step 1 that the following monomials are linearly independent for each $8 \leq l \leq 10$, hence form a basis for $H^0(l\mathscr{H}_{X_1})$:

l	monomials in $H^0(l\mathscr{H}_{X_1})$
8	y^2, xz^3, yz^4, z^8
9	$xy, y^2z, xz^4, yz^5, z^9$
10	$x^2, xyz, y^2z^2, xz^5, yz^6, z^{10}$.

Therefore the assertion holds in these cases.

The cases $l \geq 12$. We see $\delta = 1$ from Step 1. We prove the assertion by induction. When $l = 12$, it is easy to see that the following monomials are linearly independent as before, hence form a basis of $H^0(12\mathscr{H}_{X_1})$:

$$y^3, x^2z^2, xyz^3, y^2z^4, xz^7, yz^8, z^{12}.$$

Suppose $l > 12$ and that the assertion holds for $l - 1$. It is easily shown that

> for two coprime positive integers a, b and an integer l with $l \geq (a-1)(b-1)$, the equation $ai + bj = l$ has at least one solution (i, j) of non-negative integers.

Set $(a, b) = (5, 4)$. Then, since $l > 12$, there exists at least one section written as $x^i y^j$ $(i, j \geq 0)$ in $H^0(l\mathscr{H}_{X_1})$, not contained in $zH^0((l-1)\mathscr{H}_{X_1})$. Hence $H^0(l\mathscr{H}_{X_1}) = \boldsymbol{C}x^i y^j \bigoplus zH^0((l-1)\mathscr{H}_{X_1})$. From the assumption of induction, the assertion holds. This proves our claim.

By Step 2, there exists a surjective homomorphism of graded rings

$$\Phi : \boldsymbol{C}[x, y, z] \longrightarrow R\big(X_1, \mathscr{H}_{X_1}\big).$$

Step 3. We show that there exists an irreducible homogeneous polynomial F_{20} of degree 20 in $\boldsymbol{C}[x, y, z]$ such that $\operatorname{Ker}(\Phi) = (F_{20})$. Indeed, there exist no relations of degree $l < 20$ because the equation $5i + 4j = l$ has at most one solution (i, j) of non-negative integers. For $l = 20$, there are exactly 16 monomials in x, y, z contained in $H^0(20\mathscr{H}_{X_1})$. On the other hand, $h^0(20\mathscr{H}_{X_1}) = 15$. Hence there exists one relation F_{20}

of degree 20, which is written as

$$F_{20} = x^4 + y^5 + z\psi_{19}(x, y, z)$$

after we replace x and y by suitable scalar multiples, where ψ_{19} is a homogeneous polynomial in x, y, z of degree 19. The irreducibility of F_{20} is proved as follows. One can easily show that $x^4 + y^5$ is irreducible. Write $F_{20}(x, y, z) = P_1(x, y, z)P_2(x, y, z)$ with some $P_1, P_2 \in \boldsymbol{C}[x, y, z]$. Then we may assume $P_1(x, y, 0) = 1$ without loss of generality. Hence $P_1(x, y, z) = 1 + z\xi_1$ and $P_2 = x^4 + y^5 + z\xi_2$, where ξ_1, ξ_2 are polynomials in x, y, z. We obtain that

$$\psi_{19}(x, y, z) = \xi_1\bigl(x^4 + y^5 + z\xi_2\bigr) + \xi_2.$$

It follows that $\xi_1 = 0$. Indeed, otherwise, the highest term of the right-hand side has degree ≥ 20, which is absurd. Therefore F_{20} is irreducible. Furthermore, combining this and the fact that $\mathrm{ht}(\mathrm{Ker}(\Phi)) \leq \dim \boldsymbol{C}[x, y, z] - \dim R(X_1, \mathscr{H}_{X_1}) = 1$, we see $\mathrm{Ker}(\Phi) = (F_{20})$. $\square$

Next we will show the surjectivity of the restriction map ρ. Let $\mathbf{s} = \{s_0, \ldots, s_N\}$ be a minimal set of generators of $R(X_2, \mathscr{H}_{X_2})$. Then there exists an isomorphism

$$R\bigl(X_2, \mathscr{H}_{X_2}\bigr) \cong \boldsymbol{C}[s_0, \ldots, s_N]/(F_1, \ldots, F_h),$$

where $F_1, \ldots, F_h$ are homogeneous polynomials in $\boldsymbol{C}[s_0, \ldots, s_N]$. Put $I_{\mathbf{s}} := (F_1, \ldots, F_h)$.

It follows from Fact 3.1 (3) that the vector space $H^0(\mathscr{H}_{X_2})$ is of dimension 2, hence has a basis $\{s, t\}$ such that $\rho(s) = z$ and $(t)_0 = X_1$. We may assume that $\mathbf{s}$ contains these two elements.

Lemma 6.4. *The sequence t, s contained in $\mathfrak{m} := R(X_2, \mathscr{H}_{X_2})_+$ is regular.*

Proof. Let m be a homogeneous element of degree a in $R(X_2, \mathscr{H}_{X_2})$ such that $tm = 0$. We see that $R(X_2, \mathscr{H}_{X_2})_+$ has no zero-divisors since $X_2 \cong \mathrm{Proj}(R(X_2, \mathscr{H}_{X_2}))$ is integral. Hence, if $a > 0$, then we obtain $m = 0$. If $a = 0$, then the minimality of $\mathbf{s}$ implies that $I_{\mathbf{s}}$ has no generators of degree one. Thus we have $m = 0$. Therefore t is $R(X_2, \mathscr{H}_{X_2})$-regular. By the same argument, we see that s is $R(X_2, \mathscr{H}_{X_2})/(t)$-regular since $X_1 \cong \mathrm{Proj}(R(X_2, \mathscr{H}_{X_2})/(t))$ is integral. As a consequence, the assertion follows. $\square$

In order to prove Proposition 6.6, we need some information about generators of $I_{\mathbf{s}}$. Let

$$\rho_l : H^0\bigl(l\mathscr{H}_{X_2}\bigr) \twoheadrightarrow H^0\bigl(l\mathscr{H}_{X_2}\bigr)/\langle t\rangle \hookrightarrow H^0\bigl(l\mathscr{H}_{X_1}\bigr)$$

denote the restriction map. Here we show the following lemma:

Lemma 6.5. *The ideal $I_{\mathbf{s}}$ has no generators in degrees ≤ 5.*

Proof. We first prove that

$$\mathrm{Im}(\rho_5) = H^0\big(5\mathscr{H}_{X_1}\big). \tag{$\dagger$}$$

It follows that $\mathrm{rank}(\rho_5) \geq 3$. Indeed, the morphism $\varphi_L|_{X_1} : X_1 \to \boldsymbol{P}(\mathrm{Im}(\rho_5))$ is an embedding of a curve of genus 6. Consequently (†) holds by virtue of Step 1 in the proof of Proposition 6.3.

Subsequently, we find a basis of $H^0(l\mathscr{H}_{X_2})$ for $1 \leq l \leq 5$ by using Lemma 2.2.

For $l = 1$, there exist no relations in $H^0(\mathscr{H}_{X_2})$ because of the minimality of $\mathbf{s}$.

For $l = 2$, there exist no relations. In fact, it follows $H^0(2\mathscr{H}_{X_2}) = \langle s^2, st, t^2\rangle$. Indeed, let $\eta \in H^0(2\mathscr{H}_{X_2})$. We can write $\rho_2(\eta) = cz^2$ with some $c \in \boldsymbol{C}$. Then, from Lemma 2.2, it follows that η is a linear combination of s^2, st, t^2. These three monomials are linearly independent because each order of pole along X_1 differs from that of the others.

For $l = 3$, there are no relations: we see that $H^0(3\mathscr{H}_{X_2}) = \langle s^3, s^2t, st^2, t^3\rangle$ by the same argument as in the case $l = 2$.

For $l = 4$. Note that $1 \leq \mathrm{rank}(\rho_4) \leq h^0(4\mathscr{H}_{X_1}) = 2$. We first suppose $\mathrm{rank}(\rho_4) = 1$. Then $H^0(4\mathscr{H}_{X_2}) = \langle s^4, s^3t, s^2t^2, st^3, t^4\rangle$ holds, which implies that there exist no relations. By (†), there exist sections $u, v \in H^0(5\mathscr{H}_{X_2})$ such that $\rho_5(u) = x, \rho_5(v) = yz$. Since it follows from Lemma 2.2 that

$$H^0\big(5\mathscr{H}_{X_2}\big) = \langle u, v, s^5, s^4t, s^3t^2, s^2t^3, st^4, t^5\rangle,$$

there exist no relations in $H^0(5\mathscr{H}_{X_2})$.

Next we suppose $\mathrm{rank}(\rho_4) = 2$. Let w denote a section such that $\rho_4(w) = y$. Then we see

$$\begin{aligned} H^0\big(4\mathscr{H}_{X_2}\big) &= \langle w, s^4, s^3t, s^2t^2, st^3, t^4\rangle, \\ H^0\big(5\mathscr{H}_{X_2}\big) &= \langle u, sw, tw, s^5, s^4t, s^3t^2, s^2t^3, st^4, t^5\rangle, \end{aligned}$$

where u is a section such that $\rho_5(u) = x$. Therefore there exist no relations. □

Proposition 6.6. *The restriction map*

$$\rho : R\big(X_2, \mathscr{H}_{X_2}\big) \longrightarrow R\big(X_1, \mathscr{H}_{X_1}\big)$$

is surjective.

Proof. It suffices to prove that $H^1(l\mathscr{H}_{X_2}) = 0$ for every $l \geq 0$, which is equivalent to showing that $R(X_2, \mathscr{H}_{X_2})$ is a Cohen-Macaulay ring (see [**W**, (2.4)]).

We find a regular sequence of length 3 contained in $\mathfrak{m}$. The sequence t, s is regular by Lemma 6.4. Let $u \in H^0(5\mathscr{H}_{X_2})$ denote a section such that $\rho_5(u) = x$. We assert that u is $R(X_2, \mathscr{H}_{X_2})/(t, s)$-regular. Indeed, $\mathrm{Proj}(R(X_2, \mathscr{H}_{X_2})/(t, s))$ is an integral scheme p because of $\mathscr{H}_{X_2}^2 = 1$. Thus we see that $(R(X_2, \mathscr{H}_{X_2})/(t, s))_+$ has no zero-divisors. Let m be a homogeneous element of degree a in $R(X_2, \mathscr{H}_{X_2})/(t, s)$ such that $um = 0$. If $a > 0$, then we have $m = 0$ obviously. If $a = 0$, then we have $m = 0$ by Lemma 6.5. Therefore t, s, u form a regular sequence. □

At last, we can prove Theorem 6.2 as follows.

PROOF OF THEOREM 6.2. Combining Proposition 6.3 and 6.6, we see that X_2 is a weighted hypersurface of degree 20 in $\boldsymbol{P}(5,4,1^2)$. Furthermore, the assertion follows from Proposition 2.3. $\square$

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