J. Math. Soc. Japan
Vol. 58, No. 4 (2006) pp. 1079–1117

Spaces of initial conditions of Garnier system and its degenerate systems in two variables

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(Received Jan. 13, 2005)
(Revised Nov. 4, 2005)

Abstract. We construct spaces of initial conditions of Garnier system and its degenerate systems in two variables and describe them as symplectic manifolds. These systems are expressed as polynomial Hamiltonian systems on all affine charts.

0. Introduction.

In this paper, we construct spaces of initial conditions of Garnier system $\mathscr{H}_J$, $J = 11111$ and its degenerate systems $\mathscr{H}_J$, $J = 1112, 113, 122, 14, 23, 5$ in two variables [**2**], which are completely integrable Hamiltonian systems of degree 2 of the form

$$dq_k = \sum_{i=1,2} \frac{\partial H_{Ji}}{\partial p_k} ds_i, \quad dp_k = -\sum_{i=1,2} \frac{\partial H_{Ji}}{\partial q_k} ds_i, \quad k = 1, 2.$$

The Hamiltonians for all J are certain polynomials of q_1, q_2, p_1, p_2 whose coefficients are rational functions of $s = (s_1, s_2)$ holomorphic in a domain $B_J \subset \boldsymbol{C}^2$. (The explicit forms of the Hamiltonians are given in Section 1.) We remark that the label J is a partition of 5. As is explained in [**2**], these systems are obtained as monodromy preserving deformation equations of second order linear ordinary differential equations with regular or irregular singular points and apparent singular points. Let us assign 1 to a regular singular point and $r + 1$ to an irregular singular point of Poincaré rank r. Then we can express, by a sequence of positive integers, the numbers of regular singular points and of irregular singular points with the data of Poincaré ranks. For example, $\mathscr{H}_{11111}$ (or $\mathscr{H}_{1112}$) is a monodromy preserving deformation system of a linear differenatial equation with five regular singular points (or with three regular singular points and an irregular singular point of Poincaré rank 1).

Each $\mathscr{H}_J$ defines a nonsingular foliation of the trivial fiber space $\boldsymbol{C}^4 \times B_J \ni (q, p, s)$, $q = (q_1, q_2)$, $p = (p_1, p_2)$, $s = (s_1, s_2)$, because $\partial H_{Ji}/\partial p_k$, $\partial H_{Ji}/\partial q_k$, $i, k = 1, 2$ are holomorphic on $\boldsymbol{C}^4 \times B_J$. However, since the differential system is nonlinear, the leaves or the solution surfaces may not be prolonged along some curves in B_J, in other words, they may have movable singularities. Therefore it is preferable to obtain a fiber space over B_J containing the $\boldsymbol{C}^4 \times B_J$ as a fiber subspace in which every solution surface can be prolonged along any curve in B_J. If there exists such a fiber space, then the space and their fibers will be called the *defining manifold* and the *spaces of initial conditions*

2000 *Mathematics Subject Classification.* Primary 33E17; Secondary 34M55.
Key Words and Phrases. Garnier systems, degenerate Garnier systems, spaces of initial conditions.

respectively. The space of initial conditions is the space which parametrizes all the solutions. The most typical and well known such spaces are those for Painlevé systems [**6**], [**7**], [**8**] and (non-degenerate) Garnier systems in n variables [**3**]. For these systems, the spaces of initial conditions have been constructed on the basis of the so-called *Painlevé property*, namely *movable singularities are at most poles.* We remark here that, for these systems, we can construct certain fiber spaces parametrizing all the meromorphic solutions without using the Painlevé property, and the Painlevé property guarantees that the spaces are the spaces of initial conditions, namely they parametrize all the solutions.

In this paper, we construct the spaces E_J over B_J every fiber $E_J(s)$ $(s \in B_J)$ of which parametrizes all the *meromorphic* solutions near the point s for $\mathscr{H}_J$ $J = 11111, 1112, 113, 122, 14, 23, 5$ and describe them so that every $E_J(s)$ is a symplectic manifold. It is known that the system $\mathscr{H}_{11111}$ has Painlevé property (see [**1**]) but the present author does not know if the other systems for $J = 1112, 113, 122, 14, 23, 5$ have the property. However, since it is strongly expected that the other systems have also the Painlevé property, the author called the fibers $E_J(s)$, $(s \in B_J)$ the spaces of initial conditions in the title of this paper and in the top of this introduction.

As was cited above, the spaces of initial conditions for (non-degenerate) Garnier system in n variables were constructed by H. Kimura ([**3**]), and symplectic structures in the spaces were introduced by K. Kobayashi ([**5**]). H. Kimura has also constructed the spaces $E_{1112}(s)$, $(s \in B_{1112})$ ([**4**]) without introducing symplectic structures.

We explain briefly how to obtain the fiber spaces E_J over B_J. For every J, we first compactify the fiber $\boldsymbol{C}^4 \times s \ni (q,p) \times s$ suitably. As such a compact manifold we choose four dimensional Hirzebruch manifold $\overline{\Sigma}_{\nu_J}$ which is a $\boldsymbol{P}^2$-bundle over $\boldsymbol{P}^2$. The manifold $\overline{\Sigma}_{\nu_J} \times s$ is covered by nine affine charts. Then we write the system $\mathscr{H}_J$ in the coordinates of all charts. We see that on certain three charts the differential systems are polynomial Hamiltonian systems, however on the other charts they are not Hamiltonian systems and have pole singularities on a divisor $D_J \times s$, $s \in B_J$. We next determine the so-called accessible singular points of the solutions meromorphic in the original coordinate system (q,p) on $D_J \times s$. An accessible singular point is a point through which many solution surfaces may pass. We see that the set of accessible singular points is a disjoint union of $|J|$ connected components $A_k(s)$ each of which is isomorphic to $\boldsymbol{P}^1$, where $|J|$ is the length m of $J = n_1 n_2 \dots n_m$ $(n_1 + n_2 + \cdots + n_m = 5)$ (i.e. $|J|$ is the number of singular points of the corresponding linear diffrential equation). We assign to each component $A_k(s)$ an element n_k of $J = n_1 n_2 \dots n_m$. We then make quadratic transformation $Q_{A_k(s)}$ along each $A_k(s)$. We see that the transformed differential system has yet pole singularities on the exceptional divisor $D_k^{(1)}(s) = Q_{A_k(s)}(A_k(s))$. Therefore we have to determine the accessible singular points and make quadratic transformation again. After repeating such quadratic transformations several times and auxiliary transformations, we can arrive at a holomorphic system, namely we can obtain coordinate systems which separate infinitely many solution surfaces of the original system $\mathscr{H}_J$ passing through any point on $A_k(s)$.

Let $\overline{E}_J(s)$ be the compact manifold obtained from $\overline{\Sigma}_{\nu_J} \times s$ by the composition of all the quadratic transformations and auxiliary transformations. Then we obtain $E_J(s)$ by removing the inaccessible singular points almost all of which are the points on the so-called vertical leaves. The fiber space $E_J = \bigsqcup_{s \in B_J} E_J(s)$ is what we want to obtain. The space is covered by $2|J|+3$ charts each of which is isomorphic to $\boldsymbol{C}^4 \times B_J$. We notice that the original polynomial Hamiltonian system is also expressed in every chart as a *polynomial* Hamiltonian system. The number $2|J|+3$ of affine charts of $E_J(s)$ is understood as follows: The $E_J(s)$ is a disjoint union of $\boldsymbol{C}^4$ and $(|J|+1)$ $\boldsymbol{C}^2$-bundles over $\boldsymbol{P}^1$ and each $\boldsymbol{C}^2$-bundle over $\boldsymbol{P}^1$ is covered by 2 affine charts $\boldsymbol{C}^4$ and therefore $E_J(s)$ is covered by $2(|J|+1)+1$ affine charts $\boldsymbol{C}^4$. If the construction of the spaces of initial conditions might be possible for general Garnier system in n variables and its degenerate systems, the corresponding space $E_J(s)$ might be covered by $n(|J|+1)+1$ affine charts $\boldsymbol{C}^{2n}$ because $E_J(s)$ is expected to be a disjoint union of $\boldsymbol{C}^{2n}$ and $(|J|+1)$ $\boldsymbol{C}^n$-bundles over $\boldsymbol{P}^{n-1}$.

We state our results more precisely. The number of quadratic transformations along $A_k(s)$ is $2n_k$ where n_k is the positive integer assigned as above. The first n_k quadratic transformations are simultaneous replacement of every point on some curves by $\boldsymbol{P}^2$ and the second n_k transformations are simultaneous replacement of every point on some surfaces by $\boldsymbol{P}^1$. In the case where $n_k \geqslant 2$, we have to insert some simple change of variables after the n_k-th transformation and make certain change of variables after the last transformation by investigating carefully the 2-form $dq_1 \wedge dp_1 + dq_2 \wedge dp_2$ in order to obtain good *symplectic coordinate systems* $(q^*, p^*) = (q_1^*, q_2^*, p_1^*, p_2^*)$, where we say that a coordinate system $(q^*, p^*) = (q_1^*, q_2^*, p_1^*, p_2^*)$ is symplectic if it satisfies

$$dq_1 \wedge dp_1 + dq_2 \wedge dp_2 = dq_1^* \wedge dp_1^* + dq_2^* \wedge dp_2^*.$$

We notice that the Hamiltonians H_i, $i=1,2$ in the coordinate system (q,p,s) are changed to $H_i(*)$, $i=1,2$ in (q^*, p^*, s) determined by

$$\sum_{i=1,2} dq_i \wedge dp_i + \sum_{i=1,2} dH_i \wedge ds_i = \sum_{i=1,2} dq_i^* \wedge dp_i^* + \sum_{i=1,2} dH_i(*) \wedge ds_i.$$

The image of $A_k(s)$ in $E_J(s)$ by a sequence of quadratic transformations is a $\boldsymbol{C}^2$-bundle over $\boldsymbol{P}^1$.

Lastly we notice that there exist Bäcklund transformations which act on some parameters as permutations in the case where J has several same elements. Since the transformations also act as permutations of the corresponding components $A_k(s)$, a coordinate system for a component $A_{k'}(s)$ derives coordinate systems for the other components $A_k(s)$. However the coordinate systems thus obtained are not so good, which means that the relation between the coordinate system and the original one is not of simple form, therefore we do not make use of the Bäcklund transformations in this paper.

This paper is organized as follows. In Section 1, we give the explicit forms of the Hamiltonians of the systems $\mathscr{H}_J$ due to H. Kimura ([**2**]). In Section 2, we state our results in seven theorems, which are proved in the next section. In Section 3, we firstly explain how to compactify the original phase space $\boldsymbol{C}^4 \times s$ and we secondly determine the accessible singluar points. After these preliminary studies, we proceed to make a sequence of quadratic transformations for each connected component $A_k(s)$ of the accessible singular points. Since the calculations for all components cost too many pages, we only study in this paper the case of $A_2(s)$ for $\mathscr{H}_{11111}$ and the case of $A_1(s)$ for $\mathscr{H}_{1112}$. The former is the case of two times quadratic transformations and the latter is that of four times quadratic transformations. The readers who are interested in the other cases can consult the author's doctor thesis [**9**]. In the last part of this paper, we put seven figures indicating the processes of quadratic transformations for all $\mathscr{H}_J$.

1. Hamiltonians of the all systems.

We list the explicit forms of the Hamiltonians H_{J1} and H_{J2} (abbreviated as H_1 and H_2 respectively) of the systems $\mathscr{H}_J$ ($J = 11111, 1112, 113, 122, 14, 23, 5$) due to H. Kimura ([**2**]).

$\mathscr{H}_{11111}$:

$$\begin{aligned}
s_1(s_1-1)H_1 = &\left\{q_1(q_1-1)(q_1-s_1) - \frac{s_1(s_1-1)}{s_1-s_2}q_1q_2\right\}p_1^2 \\
&+ 2q_1q_2\left(q_1 - \frac{s_1(s_2-1)}{s_2-s_1}\right)p_1p_2 + q_1q_2\left(q_2 - \frac{s_2(s_1-1)}{s_1-s_2}\right)p_2^2 \\
&- \left\{(\alpha_0-1)q_1(q_1-1) + \alpha_1 q_1(q_1-s_1) + \alpha_2(q_1-1)(q_1-s_1)\right. \\
&\qquad \left. + \alpha_3 q_1\left(q_1 - \frac{s_1(s_2-1)}{s_2-s_1}\right) - \alpha_2\frac{s_1(s_1-1)}{s_1-s_2}q_2\right\}p_1 \\
&+ \left\{(\alpha_\infty+2\nu)q_1q_2 + \alpha_2\frac{s_1(s_2-1)}{s_2-s_1}q_2 + \alpha_3\frac{s_2(s_1-1)}{s_1-s_2}q_1\right\}p_2 \\
&+ \nu(\nu+\alpha_\infty)q_1, \\
s_2(s_2-1)H_2 = &\, q_1q_2\left(q_1 - \frac{s_1(s_2-1)}{s_2-s_1}\right)p_1^2 + 2q_1q_2\left(q_2 - \frac{s_2(s_1-1)}{s_1-s_2}\right)p_1p_2 \\
&+ \left\{q_2(q_2-1)(q_2-s_2) - \frac{s_2(s_2-1)}{s_2-s_1}q_1q_2\right\}p_2^2 \\
&+ \left\{(\alpha_\infty+2\nu)q_1q_2 + \alpha_2\frac{s_1(s_2-1)}{s_2-s_1}q_2 + \alpha_3\frac{s_2(s_1-1)}{s_1-s_2}q_1\right\}p_1 \\
&- \left\{(\alpha_0-1)q_2(q_2-1) + \alpha_1 q_2(q_2-s_2) + \alpha_3(q_2-1)(q_2-s_2)\right. \\
&\qquad \left. + \alpha_2 q_2\left(q_2 - \frac{s_2(s_1-1)}{s_1-s_2}\right) - \alpha_3\frac{s_2(s_2-1)}{s_2-s_1}q_1\right\}p_2 \\
&+ \nu(\nu+\alpha_\infty)q_2, \qquad\qquad \left(\nu = -\frac{1}{2}(\alpha_0+\alpha_1+\alpha_2+\alpha_3-1+\alpha_\infty)\right).
\end{aligned}$$

$\mathscr{H}_{1112}$:

$$\begin{aligned}
s_1^2 H_1 = &\, q_1^2(q_1-s_1)p_1^2 + 2q_1^2q_2p_1p_2 + q_1q_2(q_2-s_2)p_2^2 \\
&- \left\{(\alpha_0+\alpha_2-1)q_1^2 + \alpha_1 q_1(q_1-s_1) + \eta(q_1-s_1) + \eta s_1 q_2\right\}p_1 \\
&- \left\{(\alpha_0+\alpha_1-1)q_1q_2 + \alpha_2 q_1(q_2-s_2) - \eta(s_2-1)q_2\right\}p_2 + \nu(\nu+\alpha_\infty)q_1,
\end{aligned}$$

$$
\begin{aligned}
s_2(s_2-1)H_2 =& q_1^2q_2p_1^2 + 2q_1q_2(q_2-s_2)p_1p_2 \\
&+ \left\{q_2(q_2-1)(q_2-s_2) + \frac{s_2(s_2-1)}{s_1}q_1q_2\right\}p_2^2 \\
&- \{(\alpha_0+\alpha_1-1)q_1q_2 + \alpha_2q_1(q_2-s_2) - \eta(s_2-1)q_2\}p_1 \\
&- \left\{(\alpha_0-1)q_2(q_2-1) + \alpha_1q_2(q_2-s_2) + \alpha_2(q_2-1)(q_2-s_2)\right. \\
&\quad \left. + \frac{s_2(s_2-1)}{s_1}(\alpha_2q_1+\eta q_2)\right\}p_2 + \nu(\nu+\alpha_\infty)q_2, \\
&\qquad \left(\nu = -\frac{1}{2}(\alpha_0+\alpha_1+\alpha_2-1+\alpha_\infty)\right).
\end{aligned}
$$

$\mathscr{H}_{113}$:

$$
\begin{aligned}
H_1 =& q_1^3p_1^2 + 2q_1^2\left(q_2+\frac{1}{s_2}\right)p_1p_2 + q_1\left\{q_2\left(q_2+\frac{1}{s_2}\right) - \left(\frac{s_1}{s_2^2}+\frac{1}{2}\right)q_1\right\}p_2^2 \\
&- \left\{(\alpha_0+\alpha_1-1)q_1^2 + \eta\left(q_1+\frac{q_2}{s_2}\right)\right\}p_1 \\
&- \left\{(\alpha_0+\alpha_1-1)q_1q_2 + \frac{\alpha_1}{s_2}q_1 - \eta\left(\frac{s_1}{s_2^2}-\frac{1}{2}\right)q_2 + \frac{\eta}{s_2}\right\}p_2 + \nu(\nu+\alpha_\infty)q_1, \\
H_2 =& q_1^2\left(q_2+\frac{1}{s_2}\right)p_1^2 + 2q_1\left\{q_2\left(q_2+\frac{1}{s_2}\right) - \left(\frac{s_1}{s_2^2}+\frac{1}{2}\right)q_1\right\}p_1p_2 \\
&+ \left\{q_2^2\left(q_2+\frac{1}{s_2}\right) + \left(\frac{s_1^2}{s_2^3}-\frac{s_2}{4}\right)q_1^2 - \left(\frac{s_1}{s_2^2}+\frac{3}{2}\right)q_1q_2 - \frac{q_1}{s_2}\right\}p_2^2 \\
&- \left\{(\alpha_0+\alpha_1-1)q_1q_2 + \frac{\alpha_1}{s_2}q_1 - \eta\left(\frac{s_1}{s_2^2}-\frac{1}{2}\right)q_2 + \frac{\eta}{s_2}\right\}p_1 \\
&- \left[(\alpha_0+\alpha_1-1)q_2^2 - \left\{\alpha_0-1+\alpha_1\left(\frac{s_1}{s_2^2}+\frac{1}{2}\right)\right\}q_1\right. \\
&\quad \left. + \left\{\eta\left(\frac{s_1^2}{s_2^3}-\frac{s_2}{4}\right) + \frac{\alpha_1}{s_2}\right\}q_2 - \eta\left(\frac{s_1}{s_2^2}+\frac{1}{2}\right)\right]p_2 + \nu(\nu+\alpha_\infty)q_2, \\
&\qquad \left(\nu = -\frac{1}{2}(\alpha_0+\alpha_1-1+\alpha_\infty)\right).
\end{aligned}
$$

$\mathscr{H}_{122}$:

$$
\begin{aligned}
s_1^2H_1 =& q_1^2(q_1-s_1)p_1^2 + 2q_1^2q_2p_1p_2 + q_1q_2^2p_2^2 \\
&- \{(\alpha_0-1)q_1^2 + \alpha_1q_1(q_1-s_1) + \eta_1(q_1-s_1) + \eta_1s_1q_2\}p_1 \\
&- \{(\alpha_0+\alpha_1-1)q_1q_2 + \eta_0s_2q_1 + \eta_1q_2\}p_2 + \nu(\nu+\alpha_\infty)q_1,
\end{aligned}
$$

$$
\begin{aligned}
-s_2H_2 =& q_1^2q_2p_1^2+2q_1q_2^2p_1p_2+q_2^2(q_2-1)p_2^2\\
&-\{(\alpha_0+\alpha_1-1)q_1q_2+\eta_0s_2q_1+\eta_1q_2\}p_1\\
&-\left\{(\alpha_0-1)q_2(q_2-1)+\alpha_1q_2^2+\frac{\eta_0s_2}{s_1}q_1+\eta_0s_2(q_2-1)\right\}p_2+\nu(\nu+\alpha_\infty)q_2,\\
&\left(\nu=-\frac{1}{2}(\alpha_0+\alpha_1-1+\alpha_\infty)\right).
\end{aligned}
$$

$\mathscr{H}_{14}$:

$$
\begin{aligned}
H_1 =& p_1^2-2s_2p_1p_2-\left(q_2+s_2q_1+s_1-\frac{1}{2}s_2^2\right)p_2^2-\{q_1(q_1+s_2)-q_2\}p_1,\\
&-\left\{q_1q_2+\left(s_1-\frac{1}{2}s_2^2\right)q_1+s_2q_2+1-\alpha_0\right\}p_2-\nu q_1,\\
H_2 =& -s_2p_1^2-2\left(q_2+s_2q_1+s_1-\frac{1}{2}s_2^2\right)p_1p_2\\
&-\left\{s_2q_1^2+q_1q_2+\left(s_1-\frac{1}{2}s_2^2\right)q_1-s_2q_2-s_2\left(s_1-\frac{1}{2}s_2^2\right)\right\}p_2^2,\\
&-\left\{q_1q_2+\left(s_1-\frac{1}{2}s_2^2\right)q_1+s_2q_2-\alpha_0+1\right\}p_1\\
&-\left[q_2^2-\left\{\alpha_0-1+s_2\left(s_1-\frac{1}{2}s_2^2\right)\right\}q_1+\left(s_1-\frac{1}{2}s_2^2\right)q_2\right]p_2-\nu q_2, \quad (\nu=-\alpha_\infty).
\end{aligned}
$$

$\mathscr{H}_{23}$:

$$
\begin{aligned}
H_1 =& (q_1-s_1)p_1^2+2q_2p_1p_2-\frac{1}{2}\{q_1(q_1-s_1)-q_2+2(\alpha_0-1)\}p_1\\
&-\frac{1}{2}(q_1q_2-2\eta s_2)p_2-\frac{1}{2}\nu q_1,\\
-s_2H_2 =& q_2p_1^2-q_2^2p_2^2-\frac{1}{2}(q_1q_2-2\eta s_2)p_1\\
&-\frac{1}{2}\{q_2^2-2\eta s_2(q_1-s_1)-2(\alpha_0-1)q_2\}p_2-\frac{1}{2}\nu q_2, \qquad (\nu=-\alpha_\infty).
\end{aligned}
$$

$\mathscr{H}_5$:

$$
\begin{aligned}
H_1 =& (q_2^2-q_1-s_1)p_1^2+2q_2p_1p_2+p_2^2+2(q_1^2-s_1^2+s_2q_2)p_1\\
&+2(q_1q_2+s_1q_2+s_2)p_2+2\nu q_1,\\
H_2 =& q_2p_1^2+2p_1p_2+2(q_1q_2+s_1q_2+s_2)p_1+2(q_2^2-q_1+s_1)p_2+2\nu q_2,\\
&\left(\nu=\alpha+\frac{1}{2}\right).
\end{aligned}
$$

Here q_1, q_2, p_1, p_2 and s_1, s_2 are complex variables and $\alpha_0, \alpha_1, \ldots$ are complex constants. We notice that Hamiltonians $H_1 = H_{J1}$ and $H_2 = H_{J2}$ are polynomials of $q = (q_1, q_2)$ and $p = (p_1, p_2)$ whose coefficients are rational functions of $s = (s_1, s_2)$ holomorphic in

B_J where

$$B_{11111} = \boldsymbol{C}^2 \setminus \{s_1 s_2 (s_1 - 1)(s_2 - 1)(s_1 - s_2) = 0\}, \quad B_{1112} = \boldsymbol{C}^2 \setminus \{s_1 s_2 (s_2 - 1) = 0\},$$
$$B_{113} = \boldsymbol{C}^2 \setminus \{s_2 = 0\}, \quad B_{122} = \boldsymbol{C}^2 \setminus \{s_1 = 0\}, \quad B_{14} = B_{23} = B_5 = \boldsymbol{C}^2.$$

2. Main results.

We give our main results, namely, the descriptions of the fiber spaces E_J for all systems $\mathscr{H}_J$, $J = 11111, 1112, 113, 122, 14, 23, 5$. For every J, E_J is covered by $2|J|+3$ charts $V^* \times B_J \ni (q_1^*, q_2^*, p_1^*, p_2^*, s_1, s_2)$ each of which is isomorphic to $\boldsymbol{C}^4 \times B_J$. Note that $V^0 \times B_J$ is the original space in which the original Hamiltonians $H_{Ji}(q,p,s)$, $i = 1, 2$ are defined and then we write the coordinate system of $V^0 \times B_J$ as $(q_1, q_2, p_1, p_2, s_1, s_2)$ omitting the label 0. In the following theorems, we use the notation

$$V(x_i = 0) = \{(x_1, \dots, x_n) \in \boldsymbol{C}^n \mid x_i = 0\}$$

for an affine space $V = \boldsymbol{C}^n \ni (x_1, \dots, x_n)$.

Theorem 1. *The space E_{11111} for the system $\mathscr{H}_{11111}$ is obtained by glueing thirteen copies*

$$V^* \times B_{11111} \ni \big(q_1^*, q_2^*, p_1^*, p_2^*, s_1, s_2\big), \quad * = 0, 1, 2, 01, 02, 11, 12, 21, 22, 31, 32, \infty 1, \infty 2$$

of $\boldsymbol{C}^4 \times B_{11111}$ via the following symplectic transformations

$$q_1 = \frac{1}{q_1^1}, \quad q_2 = \frac{q_2^1}{q_1^1}, \quad p_1 = -q_1^1\big(\nu + q_1^1 p_1^1 + q_2^1 p_2^1\big), \quad p_2 = q_1^1 p_2^1,$$
$$q_1 = \frac{q_1^2}{q_2^2}, \quad q_2 = \frac{1}{q_2^2}, \quad p_1 = q_2^2 p_1^2, \quad p_2 = -q_2^2\big(\nu + q_1^2 p_1^2 + q_2^2 p_2^2\big),$$
$$q_1 = q_1^{01}, \quad q_2 = p_2^{01}\big(\alpha_0 - q_2^{01} p_2^{01}\big) - \frac{s_2 q_1^{01}}{s_1} + s_2, \quad p_1 = \frac{s_2}{s_1 p_2^{01}} + p_1^{01}, \quad p_2 = \frac{1}{p_2^{01}},$$
$$q_1^1 = q_1^{02}, \quad q_2^1 = p_2^{02}\big(\alpha_0 - q_2^{02} p_2^{02}\big) + s_2 q_1^{02} - \frac{s_2}{s_1}, \quad p_1^1 = -\frac{s_2}{p_2^{02}} + p_1^{02}, \quad p_2^1 = \frac{1}{p_2^{02}},$$
$$q_1 = q_1^{11}, \quad q_2 = p_2^{11}\big(\alpha_1 - q_2^{11} p_2^{11}\big) - q_1^{11} + 1, \quad p_1 = \frac{1}{p_2^{11}} + p_1^{11}, \quad p_2 = \frac{1}{p_2^{11}},$$
$$q_1^1 = q_1^{12}, \quad q_2^1 = p_2^{12}\big(\alpha_1 - q_2^{12} p_2^{12}\big) + q_1^{12} - 1, \quad p_1^1 = -\frac{1}{p_2^{12}} + p_1^{12}, \quad p_2^1 = \frac{1}{p_2^{12}},$$
$$q_1 = p_1^{21}\big(\alpha_2 - q_1^{21} p_1^{21}\big), \quad q_2 = q_2^{21}, \quad p_1 = \frac{1}{p_1^{21}}, \quad p_2 = p_2^{21},$$
$$q_1^2 = p_1^{22}\big(\alpha_2 - q_1^{22} p_1^{22}\big), \quad q_2^2 = q_2^{22}, \quad p_1^2 = \frac{1}{p_1^{22}}, \quad p_2^2 = p_2^{22},$$

$$q_1 = q_1^{31}, \quad q_2 = p_2^{31}\big(\alpha_3 - q_2^{31}p_2^{31}\big), \quad p_1 = p_1^{31}, \quad p_2 = \frac{1}{p_2^{31}},$$

$$q_1^1 = q_1^{32}, \quad q_2^1 = p_2^{32}\big(\alpha_3 - q_2^{32}p_2^{32}\big), \quad p_1^1 = p_1^{32}, \quad p_2^1 = \frac{1}{p_2^{32}},$$

$$q_1^1 = p_1^{\infty 1}\big(\alpha_\infty - q_1^{\infty 1}p_1^{\infty 1}\big), \quad q_2^1 = q_2^{\infty 1}, \quad p_1^1 = \frac{1}{p_1^{\infty 1}}, \quad p_2^1 = p_2^{\infty 1},$$

$$q_1^2 = q_1^{\infty 2}, \quad q_2^2 = p_2^{\infty 2}\big(\alpha_\infty - q_2^{\infty 2}p_2^{\infty 2}\big), \quad p_1^2 = p_1^{\infty 2}, \quad p_2^2 = \frac{1}{p_2^{\infty 2}},$$

where

$$B_{11111} = \boldsymbol{C}^2 \setminus \big\{s_1(s_1-1)s_2(s_2-1) = 0\big\}, \quad \nu = -\frac{1}{2}(\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 - 1 + \alpha_\infty).$$

Each fiber $E_{11111}(s)$ is a disjoint union of $V^0 \simeq \boldsymbol{C}^4$ and

$$V^1\big(q_1^1 = 0\big) \cup V^2\big(q_2^2 = 0\big), \quad V^{01}\big(p_2^{01} = 0\big) \cup V^{02}\big(p_2^{02} = 0\big),$$
$$V^{11}\big(p_2^{11} = 0\big) \cup V^{12}\big(p_2^{12} = 0\big), \quad V^{21}\big(p_1^{21} = 0\big) \cup V^{22}\big(p_1^{22} = 0\big),$$
$$V^{31}\big(p_2^{31} = 0\big) \cup V^{32}\big(p_2^{32} = 0\big), \quad V^{\infty 1}\big(p_1^{\infty 1} = 0\big) \cup V^{\infty 2}\big(p_2^{\infty 2} = 0\big)$$

each of which except V^0 is a $\boldsymbol{C}^2$-bundle over $\boldsymbol{P}^1$. Since any $\boldsymbol{C}^2$-bundle over $\boldsymbol{P}^1$ is a disjoint union of $\boldsymbol{C}^3$ and $\boldsymbol{C}^2$, we have

$$E_{11111}(s) = \boldsymbol{C}^4 \sqcup 6(\boldsymbol{C}^3 \sqcup \boldsymbol{C}^2).$$

The Hamiltonians $H_i(*) = H_i(*; q^*, p^*, s)$, $i = 1, 2$ in every chart $V^* \times B_{11111}$ are polynomials of $q^* = (q_1^*, q_2^*)$ and $p^* = (p_1^*, p_2^*)$ whose coefficients are rational functions of $s = (s_1, s_2)$ holomorphic in B_{11111}.

Theorem 2. *The space E_{1112} for the system $\mathscr{H}_{1112}$ is obtained by glueing eleven copies*

$$V^* \times B_{1112} \ni \big(q_1^*, q_2^*, p_1^*, p_2^*, s_1, s_2\big), \quad * = 0, 1, 2, 01, 02, 11, 12, 21, 22, \infty 1, \infty 2$$

of $\boldsymbol{C}^4 \times B_{1112}$ via the following symplectic transformations

$$q_1 = \frac{1}{q_1^1}, \quad q_2 = \frac{q_2^1}{q_1^1}, \quad p_1 = -q_1^1\big(\nu + q_1^1p_1^1 + q_2^1p_2^1\big), \quad p_2 = q_1^1p_2^1,$$

$$q_1 = \frac{q_1^2}{q_2^2}, \quad q_2 = \frac{1}{q_2^2}, \quad p_1 = q_2^2p_1^2, \quad p_2 = -q_2^2\big(\nu + q_1^2p_1^2 + q_2^2p_2^2\big),$$

$$q_1 = q_1^{01}, \quad q_2 = p_2^{01}\big(\alpha_0 - q_2^{01}p_2^{01}\big) - \frac{s_2q_1^{01}}{s_1} + s_2, \quad p_1 = \frac{s_2}{s_1p_2^{01}} + p_1^{01}, \quad p_2 = \frac{1}{p_2^{01}},$$

$$q_1^1 = q_1^{02}, \quad q_2^1 = p_2^{02}\big(\alpha_0 - q_2^{02}p_2^{02}\big) + s_2 q_1^{02} - \frac{s_2}{s_1}, \quad p_1^1 = -\frac{s_2}{p_2^{02}} + p_1^{02}, \quad p_2^1 = \frac{1}{p_2^{02}},$$

$$q_1 = q_1^{11}, \quad q_2 = q_2^{11}, \quad p_1 = -\frac{\eta(q_2^{11}-1)}{(q_1^{11})^2} + \frac{\alpha_1}{q_1^{11}} + p_1^{11}, \quad p_2 = \frac{\eta}{q_1^{11}} + p_2^{11},$$

$$q_1^2 = q_1^{12}, \quad q_2^2 = q_2^{12}, \quad p_1^2 = \frac{\eta(q_2^{12}-1)}{(q_1^{12})^2} + \frac{\alpha_1}{q_1^{12}} + p_1^{12}, \quad p_2^2 = -\frac{\eta}{q_1^{12}} + p_2^{12},$$

$$q_1 = q_1^{21}, \quad q_2 = p_2^{21}\big(\alpha_2 - q_2^{21}p_2^{21}\big), \quad p_1 = p_1^{21}, \quad p_2 = \frac{1}{p_2^{21}},$$

$$q_1^1 = q_1^{22}, \quad q_2^1 = p_2^{22}\big(\alpha_2 - q_2^{22}p_2^{22}\big), \quad p_1^1 = p_1^{22}, \quad p_2^1 = \frac{1}{p_2^{22}},$$

$$q_1^1 = p_1^{\infty 1}\big(\alpha_\infty - q_1^{\infty 1}p_1^{\infty 1}\big), \quad q_2^1 = q_2^{\infty 1}, \quad p_1^1 = \frac{1}{p_1^{\infty 1}}, \quad p_2^1 = p_2^{\infty 1},$$

$$q_1^2 = q_1^{\infty 2}, \quad q_2^2 = p_2^{\infty 2}\big(\alpha_\infty - q_2^{\infty 2}p_2^{\infty 2}\big), \quad p_1^2 = p_1^{\infty 2}, \quad p_2^2 = \frac{1}{p_2^{\infty 2}},$$

where

$$B_{1112} = \boldsymbol{C}^2 \setminus \big\{s_1 s_2(s_2-1) = 0\big\}, \quad \nu = -\frac{1}{2}(\alpha_0 + \alpha_1 + \alpha_2 - 1 + \alpha_\infty).$$

Each fiber $E_{1112}(s)$ is a disjoint union of $V^0 \simeq \boldsymbol{C}^4$ and

$$V^1\big(q_1^1 = 0\big) \cup V^2(q_2^2 = 0), \quad V^{01}\big(p_2^{01} = 0\big) \cup V^{02}(p_2^{02} = 0),$$
$$V^{11}\big(q_1^{11} = 0\big) \cup V^{12}\big(q_1^{12} = 0\big), \quad V^{21}\big(p_2^{21} = 0\big) \cup V^{22}\big(p_2^{22} = 0\big),$$
$$V^{\infty 1}\big(p_1^{\infty 1} = 0\big) \cup V^{\infty 2}\big(p_2^{\infty 2} = 0\big)$$

each of which except V^0 is a $\boldsymbol{C}^2$-bundle over $\boldsymbol{P}^1$. Since any $\boldsymbol{C}^2$-bundle over $\boldsymbol{P}^1$ is a disjoint union of $\boldsymbol{C}^3$ and $\boldsymbol{C}^2$, we have

$$E_{1112}(s) = \boldsymbol{C}^4 \sqcup 5(\boldsymbol{C}^3 \sqcup \boldsymbol{C}^2).$$

The Hamiltonians $H_i(*) = H_i(*; q^*, p^*, s)$, $i = 1, 2$ in every chart $V^* \times B_{1112}$ are polynomials of $q^* = (q_1^*, q_2^*)$ and $p^* = (p_1^*, p_2^*)$ whose coefficients are rational functions of $s = (s_1, s_2)$ holomorphic in B_{1112}.

THEOREM 3. *The space E_{113} for the system $\mathscr{H}_{113}$ is obtained by glueing nine copies*

$$V^* \times B_{113} \ni \big(q_1^*, q_2^*, p_1^*, p_2^*, s_1, s_2\big), \quad * = 0, 1, 2, 01, 02, 11, 12, \infty 1, \infty 2$$

of $\boldsymbol{C}^4 \times B_{113}$ via the following symplectic transformations

$$q_1 = \frac{1}{q_1^1}, \quad q_2 = \frac{q_2^1}{q_1^1}, \quad p_1 = -q_1^1\big(\nu + q_1^1 p_1^1 + q_2^1 p_2^1\big), \quad p_2 = q_1^1 p_2^1,$$

$$q_1 = \frac{q_1^2}{q_2^2}, \quad q_2 = \frac{1}{q_2^2}, \quad p_1 = q_2^2 p_1^2, \quad p_2 = -q_2^2\big(\nu + q_1^2 p_1^2 + q_2^2 p_2^2\big),$$

$$q_1 = q_1^{01}, \quad q_2 = p_2^{01}\big(\alpha_0 - q_2^{01} p_2^{01}\big) - \left(\frac{2s_1 + s_2^2}{2s_2}\right) q_1^{01} - \frac{1}{s_2},$$
$$p_1 = \frac{2s_1 + s_2^2}{2s_2 p_2^{01}} + p_1^{01}, \quad p_2 = \frac{1}{p_2^{01}},$$

$$q_1^1 = q_1^{02}, \quad q_2^1 = p_2^{02}\big(\alpha_0 - p_2^{02} q_2^{02}\big) - \frac{q_1^{02}}{s_2} - \frac{2s_1 + s_2^2}{2s_2}, \quad p_1^1 = \frac{1}{s_2 p_2^{02}} + p_1^{02}, \quad p_2^1 = \frac{1}{p_2^{02}},$$

$$q_1 = q_1^{11}, \quad q_2 = q_2^{11}, \quad p_1 = -\frac{\eta (q_2^{11})^2}{(q_1^{11})^3} + \frac{\eta}{(q_1^{11})^2} + \frac{\alpha_1}{q_1^{11}} + p_1^{11}, \quad p_2 = \frac{\eta q_2^{11}}{(q_1^{11})^2} + p_2^{11},$$

$$q_1^2 = q_1^{12}, \quad q_2^2 = q_2^{12}, \quad p_1^2 = -\frac{\eta}{(q_1^{12})^3} + \frac{\eta q_2^{12}}{(q_1^{12})^2} + \frac{\alpha_1}{q_1^{12}} + p_1^{12}, \quad p_2^2 = -\frac{\eta}{q_1^{12}} + p_2^{12},$$

$$q_1^1 = p_1^{\infty 1}\big(\alpha_\infty - q_1^{\infty 1} p_1^{\infty 1}\big), \quad q_2^1 = q_2^{\infty 1}, \quad p_1^1 = \frac{1}{p_1^{\infty 1}}, \quad p_2^1 = p_2^{\infty 1},$$

$$q_1^2 = q_1^{\infty 2}, \quad q_2^2 = p_2^{\infty 2}\big(\alpha_\infty - q_2^{\infty 2} p_2^{\infty 2}\big), \quad p_1^2 = p_1^{\infty 2}, \quad p_2^2 = \frac{1}{p_2^{\infty 2}},$$

where

$$B_{113} = \boldsymbol{C}^2 \setminus \{s_2 = 0\}, \quad \nu = -\frac{1}{2}\big(\alpha_0 + \alpha_1 - 1 + \alpha_\infty\big).$$

Each fiber $E_{113}(s)$ is a disjoint union of $V^0 \simeq \boldsymbol{C}^4$ and

$$V^1\big(q_1^1 = 0\big) \cup V^2\big(q_2^2 = 0\big), \quad V^{01}\big(p_2^{01} = 0\big) \cup V^{02}\big(p_2^{02} = 0\big),$$
$$V^{11}\big(q_1^{11} = 0\big) \cup V^{12}\big(q_1^{12} = 0\big), \quad V^{\infty 1}\big(p_1^{\infty 1} = 0\big) \cup V^{\infty 2}\big(p_2^{\infty 2} = 0\big)$$

each of which except V^0 is a $\boldsymbol{C}^2$-bundle over $\boldsymbol{P}^1$. Since any $\boldsymbol{C}^2$-bundle over $\boldsymbol{P}^1$ is a disjoint union of $\boldsymbol{C}^3$ and $\boldsymbol{C}^2$, we have

$$E_{113}(s) = \boldsymbol{C}^4 \sqcup 4(\boldsymbol{C}^3 \sqcup \boldsymbol{C}^2).$$

The Hamiltonians $H_i(*) = H_i(*; q^*, p^*, s)$, $i = 1, 2$ in every chart $V^* \times B_{113}$ are polynomials of $q^* = (q_1^*, q_2^*)$ and $p^* = (p_1^*, p_2^*)$ whose coefficients are rational functions of $s = (s_1, s_2)$ holomorphic in B_{113}.

Theorem 4. *The space E_{122} for the system $\mathscr{H}_{122}$ is obtained by glueing nine copies*

$$V^* \times B_{122} \ni \big(q_1^*, q_2^*, p_1^*, p_2^*, s_1, s_2\big), \quad * = 0, 1, 2, 01, 02, 11, 12, \infty 1, \infty 2$$

of $\boldsymbol{C}^4 \times B_{122}$ via the following symplectic transformations

$$q_1 = \frac{1}{q_1^1}, \quad q_2 = \frac{q_2^1}{q_1^1}, \quad p_1 = -q_1^1\big(\nu + q_1^1 p_1^1 + q_2^1 p_2^1\big), \quad p_2 = q_1^1 p_2^1,$$

$$q_1 = \frac{q_1^2}{q_2^2}, \quad q_2 = \frac{1}{q_2^2}, \quad p_1 = q_2^2 p_1^2, \quad p_2 = -q_2^2\big(\nu + q_1^2 p_1^2 + q_2^2 p_2^2\big),$$

$$q_1 = q_1^{01}, \quad q_2 = q_2^{01}, \quad p_1 = \frac{\eta_0 s_2}{s_1 q_2^{01}} + p_1^{01}, \quad p_2 = -\frac{\eta_0 s_2 (q_1^{01} - s_1)}{s_1 (q_2^{01})^2} + \frac{\alpha_0}{q_2^{01}} + p_2^{01},$$

$$q_1^1 = q_1^{02}, \quad q_2^1 = q_2^{02}, \quad p_1^1 = -\frac{\eta_0 s_2}{q_2^{02}} + p_1^{02}, \quad p_2^1 = \frac{\eta_0 s_2 (s_1 q_1^{02} - 1)}{s_1 (q_2^{02})^2} + \frac{\alpha_0}{q_2^{02}} + p_2^{02},$$

$$q_1 = q_1^{11}, \quad q_2 = q_2^{11}, \quad p_1 = -\frac{\eta_1 (q_2^{11} - 1)}{(q_1^{11})^2} + \frac{\alpha_1}{q_1^{11}} + p_1^{11}, \quad p_2 = \frac{\eta_1}{q_1^{11}} + p_2^{11},$$

$$q_1^2 = q_1^{12}, \quad q_2^2 = q_2^{12}, \quad p_1^2 = \frac{\eta_1 (q_2^{12} - 1)}{(q_1^{12})^2} + \frac{\alpha_1}{q_1^{12}} + p_1^{12}, \quad p_2^2 = -\frac{\eta_1}{q_1^{12}} + p_2^{12},$$

$$q_1^1 = p_1^{\infty 1}\big(\alpha_\infty - q_1^{\infty 1} p_1^{\infty 1}\big), \quad q_2^1 = q_2^{\infty 1}, \quad p_1^1 = \frac{1}{p_1^{\infty 1}}, \quad p_2^1 = p_2^{\infty 1},$$

$$q_1^2 = q_1^{\infty 2}, \quad q_2^2 = p_2^{\infty 2}\big(\alpha_\infty - q_2^{\infty 2} p_2^{\infty 2}\big), \quad p_1^2 = p_1^{\infty 2}, \quad p_2^2 = \frac{1}{p_2^{\infty 2}},$$

where

$$B_{122} = \boldsymbol{C}^2 \setminus \{s_1 = 0\}, \quad \nu = -\frac{1}{2}(\alpha_0 + \alpha_1 - 1 + \alpha_\infty).$$

Each fiber $E_{122}(s)$ is a disjoint union of $V^0 \simeq \boldsymbol{C}^4$ and

$$V^1\big(q_1^1 = 0\big) \cup V^2\big(q_2^2 = 0\big), \quad V^{01}\big(q_2^{01} = 0\big) \cup V^{02}\big(q_2^{02} = 0\big),$$
$$V^{11}\big(q_1^{11} = 0\big) \cup V^{12}\big(q_1^{12} = 0\big), \quad V^{\infty 1}\big(p_1^{\infty 1} = 0\big) \cup V^{\infty 2}\big(p_2^{\infty 2} = 0\big)$$

each of which except V^0 is a $\boldsymbol{C}^2$-bundle over $\boldsymbol{P}^1$. Since any $\boldsymbol{C}^2$-bundle over $\boldsymbol{P}^1$ is a disjoint union of $\boldsymbol{C}^3$ and $\boldsymbol{C}^2$, we have

$$E_{122}(s) = \boldsymbol{C}^4 \sqcup 4(\boldsymbol{C}^3 \sqcup \boldsymbol{C}^2).$$

The Hamiltonians $H_i(*) = H_i(*; q^*, p^*, s)$, $i = 1, 2$ in every chart $V^* \times B_{122}$ are polynomials of $q^* = (q_1^*, q_2^*)$ and $p^* = (p_1^*, p_2^*)$ whose coefficients are rational functions of $s = (s_1, s_2)$ holomorphic in B_{122}.

THEOREM 5. *The space E_{14} for the system $\mathscr{H}_{14}$ is obtained by glueing seven copies*

$$V^* \times B_{14} \ni \big(q_1^*, q_2^*, p_1^*, p_2^*, s_1, s_2\big), \quad * = 0, 1, 2, 01, 02, \infty 1, \infty 2$$

of $\boldsymbol{C}^4 \times B_{14}$ via the following symplectic transformations

$$q_1 = \frac{1}{q_1^1}, \quad q_2 = \frac{q_2^1}{q_1^1}, \quad p_1 = -q_1^1\bigl(\nu + q_1^1 p_1^1 + q_2^1 p_2^1\bigr), \quad p_2 = q_1^1 p_2^1,$$

$$q_1 = \frac{q_1^2}{q_2^2}, \quad q_2 = \frac{1}{q_2^2}, \quad p_1 = q_2^2 p_1^2, \quad p_2 = -q_2^2\bigl(\nu + q_1^2 p_1^2 + q_2^2 p_2^2\bigr),$$

$$q_1 = q_1^{01}, \quad q_2 = p_2^{01}\bigl(\alpha_0 - q_2^{01} p_2^{01}\bigr) - s_2 q_1^{01} - \frac{2s_1 + s_2^2}{2}, \quad p_1 = \frac{s_2}{p_2^{01}} + p_1^{01}, \quad p_2 = \frac{1}{p_2^{01}},$$

$$q_1^1 = q_1^{02}, \quad q_2^1 = p_2^{02}\bigl(\alpha_0 - q_2^{02} p_2^{02}\bigr) - \left(\frac{2s_1 + s_2^2}{2}\right) q_1^{02} - s_2,$$
$$p_1^1 = \frac{2s_1 + s_2^2}{2p_2^{02}} + p_1^{02}, \quad p_2^1 = \frac{1}{p_2^{02}},$$

$$q_1^1 = q_1^{\infty 1}, \quad q_2^1 = q_2^{\infty 1},$$
$$p_1^1 = -\frac{1}{(q_1^{\infty 1})^4} + \frac{2q_2^{\infty 1}}{(q_1^{\infty 1})^3} + \frac{1 - \alpha_0 + 2\alpha_\infty}{q_1^{\infty 1}} + p_1^{\infty 1}, \quad p_2^1 = -\frac{1}{(q_1^{\infty 1})^2} + p_2^{\infty 1},$$

$$q_1^2 = q_1^{\infty 2}, \quad q_2^2 = q_2^{\infty 2},$$
$$p_1^2 = \frac{(q_1^{\infty 2})^2}{(q_2^{\infty 2})^3} - \frac{1}{(q_2^{\infty 2})^2} + p_1^{\infty 2}, \quad p_2^2 = -\frac{(q_1^{\infty 2})^3}{(q_2^{\infty 2})^4} + \frac{2q_1^{\infty 2}}{(q_2^{\infty 2})^3} + \frac{1 - \alpha_0 + 2\alpha_\infty}{q_2^{\infty 2}} + p_2^{\infty 2},$$

where

$$B_{14} = \boldsymbol{C}^2, \quad \nu = -\alpha_\infty.$$

Each fiber $E_{14}(s)$ is a disjoint union of $V^0 \simeq \boldsymbol{C}^4$ and

$$V^1\bigl(q_1^1 = 0\bigr) \cup V^2\bigl(q_2^2 = 0\bigr), \quad V^{01}\bigl(p_2^{01} = 0\bigr) \cup V^{02}\bigl(p_2^{02} = 0\bigr),$$
$$V^{\infty 1}\bigl(q_1^{\infty 1} = 0\bigr) \cup V^{\infty 2}\bigl(q_2^{\infty 2} = 0\bigr)$$

each of which except V^0 is a $\boldsymbol{C}^2$-bundle over $\boldsymbol{P}^1$. Since any $\boldsymbol{C}^2$-bundle over $\boldsymbol{P}^1$ is a disjoint union of $\boldsymbol{C}^3$ and $\boldsymbol{C}^2$, we have

$$E_{14}(s) = \boldsymbol{C}^4 \sqcup 3(\boldsymbol{C}^3 \sqcup \boldsymbol{C}^2).$$

The Hamiltonians $H_i(*) = H_i(*; q^*, p^*, s)$, $i = 1, 2$ in every chart $V^* \times B_{14}$ are polynomials of $q^* = (q_1^*, q_2^*)$ and $p^* = (p_1^*, p_2^*)$ whose coefficients are rational functions of $s = (s_1, s_2)$ holomorphic in B_{14}.

Theorem 6. *The space E_{23} for the system $\mathscr{H}_{23}$ is obtained by glueing seven copies*

$$V^* \times B_{23} \ni \bigl(q_1^*, q_2^*, p_1^*, p_2^*, s_1, s_2\bigr), \quad * = 0, 1, 2, 01, 02, \infty 1, \infty 2$$

of $\boldsymbol{C}^4 \times B_{23}$ *via the following symplectic transformations*

$$q_1 = \frac{1}{q_1^1}, \quad q_2 = \frac{q_2^1}{q_1^1}, \quad p_1 = -q_1^1\big(\nu + q_1^1 p_1^1 + q_2^1 p_2^1\big), \quad p_2 = q_1^1 p_2^1,$$

$$q_1 = \frac{q_1^2}{q_2^2}, \quad q_2 = \frac{1}{q_2^2}, \quad p_1 = q_2^2 p_1^2, \quad p_2 = -q_2^2\big(\nu + q_1^2 p_1^2 + q_2^2 p_2^2\big),$$

$$q_1 = q_1^{01}, \quad q_2 = q_2^{01}, \quad p_1 = -\frac{\eta s_2}{q_2^{01}} + p_1^{01}, \quad p_2 = \frac{\eta s_2(q_1^{01} - s_1)}{(q_2^{01})^2} + \frac{\alpha_0}{q_2^{01}} + p_2^{01},$$

$$q_1^1 = q_1^{02}, \quad q_2^1 = q_2^{02}, \quad p_1^1 = \frac{\eta s_1 s_2}{q_2^{02}} + p_1^{02}, \quad p_2^1 = -\frac{\eta s_2(s_1 q_1^{02} - 1)}{(q_2^{02})^2} + \frac{\alpha_0}{q_2^{02}} + p_2^{02},$$

$$q_1^1 = q_1^{\infty 1}, \quad q_2^1 = q_2^{\infty 1},$$
$$p_1^1 = -\frac{1}{2(q_1^{\infty 1})^3} + \frac{q_2^{\infty 1}}{2(q_1^{\infty 1})^2} + \frac{1 - \alpha_0 + 2\alpha_\infty}{q_1^{\infty 1}} + p_1^{\infty 1}, \quad p_2^1 = -\frac{1}{2q_1^{\infty 1}} + p_2^{\infty 1},$$

$$q_1^2 = q_1^{\infty 2}, \quad q_2^2 = q_2^{\infty 2},$$
$$p_1^2 = \frac{q_1^{\infty 2}}{2(q_2^{\infty 2})^2} + p_1^{\infty 2}, \quad p_2^2 = -\frac{(q_1^{\infty 2})^2}{2(q_2^{\infty 2})^3} + \frac{1}{2(q_2^{\infty 2})^2} + \frac{1 - \alpha_0 + 2\alpha_\infty}{q_2^{\infty 2}} + p_2^{\infty 2},$$

where

$$B_{23} = \boldsymbol{C}^2, \quad \nu = -\alpha_\infty.$$

Each fiber $E_{23}(s)$ *is a disjoint union of* $V^0 \simeq \boldsymbol{C}^4$ *and*

$$V^1\big(q_1^1 = 0\big) \cup V^2\big(q_2^2 = 0\big), \quad V^{01}\big(q_2^{01} = 0\big) \cup V^{02}\big(q_2^{02} = 0\big),$$
$$V^{\infty 1}\big(q_1^{\infty 1} = 0\big) \cup V^{\infty 2}\big(q_2^{\infty 2} = 0\big)$$

each of which except V^0 *is a* $\boldsymbol{C}^2$*-bundle over* $\boldsymbol{P}^1$*. Since any* $\boldsymbol{C}^2$*-bundle over* $\boldsymbol{P}^1$ *is a disjoint union of* $\boldsymbol{C}^3$ *and* $\boldsymbol{C}^2$*, we have*

$$E_{23}(s) = \boldsymbol{C}^4 \sqcup 3(\boldsymbol{C}^3 \sqcup \boldsymbol{C}^2).$$

The Hamiltonians $H_i(*) = H_i(*; q^*, p^*, s)$, $i = 1, 2$ *in every chart* $V^* \times B_{23}$ *are polynomials of* $q^* = (q_1^*, q_2^*)$ *and* $p^* = (p_1^*, p_2^*)$ *whose coefficients are rational functions of* $s = (s_1, s_2)$ *holomorphic in* B_{23}.

THEOREM 7. *The space* E_5 *for the system* $\mathscr{H}_5$ *is obtained by glueing five copies*

$$V^* \times B_5 \ni \big(q_1^*, q_2^*, p_1^*, p_2^*, s_1, s_2\big), \quad * = 0, 1, 2, \infty 1, \infty 2$$

of $\boldsymbol{C}^4 \times B_5$ *via the following symplectic transformations*

$$q_1 = \frac{1}{q_1^1}, \quad q_2 = \frac{q_2^1}{q_1^1}, \quad p_1 = -q_1^1\big(\nu + q_1^1 p_1^1 + q_2^1 p_2^1\big), \quad p_2 = q_1^1 p_2^1,$$

$$q_1 = \frac{q_1^2}{q_2^2}, \quad q_2 = \frac{1}{q_2^2}, \quad p_1 = q_2^2 p_1^2, \quad p_2 = -q_2^2\big(\nu + q_1^2 p_1^2 + q_2^2 p_2^2\big),$$

$$q_1^1 = q_1^{\infty 1}, \quad q_2^1 = q_2^{\infty 1},$$

$$p_1^1 = -\frac{2(q_2^{\infty 1})^4}{(q_1^{\infty 1})^5} + \frac{6(q_2^{\infty 1})^2}{(q_1^{\infty 1})^4} - \frac{2}{(q_1^{\infty 1})^3} + \frac{2(s_1 + s_2 q_2^{\infty 1})}{(q_1^{\infty 1})^2} - \frac{2\alpha}{q_1^{\infty 1}} + p_1^{\infty 1},$$

$$p_2^1 = \frac{2(q_2^{\infty 1})^3}{(q_1^{\infty 1})^4} - \frac{4 q_2^{\infty 1}}{(q_1^{\infty 1})^3} - \frac{2 s_2}{q_1^{\infty 1}} + p_2^{\infty 1},$$

$$q_1^2 = q_1^{\infty 2}, \quad q_2^2 = q_2^{\infty 2},$$

$$p_1^2 = -\frac{2}{(q_2^{\infty 2})^3} + \frac{2 q_1^{\infty 2}}{(q_2^{\infty 2})^2} - \frac{2 s_1}{q_2^{\infty 2}} + p_1^{\infty 2},$$

$$p_2^2 = -\frac{2}{(q_2^{\infty 2})^5} + \frac{6 q_1^{\infty 2}}{(q_2^{\infty 2})^4} - \frac{2(q_1^{\infty 2})^2}{(q_2^{\infty 2})^3} + \frac{2(s_1 q_1^{\infty 2} + s_2)}{(q_2^{\infty 2})^2} - \frac{2\alpha}{q_2^{\infty 2}} + p_2^{\infty 2},$$

where

$$B_5 = \boldsymbol{C}^2, \quad \nu = \alpha + \frac{1}{2}.$$

Each fiber $E_5(s)$ is a disjoint union of $V^0 \simeq \boldsymbol{C}^4$ and

$$V^1\big(q_1^1 = 0\big) \cup V^2\big(q_2^2 = 0\big), \quad V^{\infty 1}\big(q_1^{\infty 1} = 0\big) \cup V^{\infty 2}\big(q_2^{\infty 2} = 0\big)$$

each of which except V^0 is a $\boldsymbol{C}^2$-bundle over $\boldsymbol{P}^1$. Since any $\boldsymbol{C}^2$-bundle over $\boldsymbol{P}^1$ is a disjoint union of $\boldsymbol{C}^3$ and $\boldsymbol{C}^2$, we have

$$E_5(s) = \boldsymbol{C}^4 \sqcup 2(\boldsymbol{C}^3 \sqcup \boldsymbol{C}^2).$$

The Hamiltonians $H_i(*) = H_i(*; q^*, p^*, s)$, $i = 1, 2$ in every chart $V^* \times B_5$ are polynomials of $q^* = (q_1^*, q_2^*)$ and $p^* = (p_1^*, p_2^*)$ whose coefficients are rational functions of $s = (s_1, s_2)$ holomorphic in B_5.

3. Proof of theorems.

3.1. Compactification of the original phase spaces.

We first explain the four dimensional Hirzebruch manifold $\overline{\Sigma}_\nu$ which we choose as a compactification of the original phase space $\boldsymbol{C}^4 \times s$, ν being a complex constant depending on the system $\mathscr{H}_J$. The manifold is a $\boldsymbol{P}^2$-bundle over $\boldsymbol{P}^2$.

Let $\xi := (\xi_0, \xi_1, \xi_2)$ be the homogeneous coordinates of $\boldsymbol{P}^2$, $U_i := \{(\xi_0/\xi_i, \xi_1/\xi_i, \xi_2/\xi_i) \mid \xi_i \neq 0\} \simeq \boldsymbol{C}^2$ be the i-th affine chart. Set $W_i := U_i \times \boldsymbol{P}^2 (i = 0, 1, 2)$ and let $\eta_i := {}^t(\eta_{i0}, \eta_{i1}, \eta_{i2})$ be the homogeneous coordinates of the second component $\boldsymbol{P}^2$

of W_i. Then we define $\overline{\Sigma}_\nu$ to be the quotient space of $\bigsqcup_{0\le i\le 2} W_i$ by the relations

$$\eta_i = g_{i0}\cdot\eta_0,$$

$$g_{10} = \begin{pmatrix} \xi_0^2 & 0 & 0 \\ -\nu\xi_0\xi_1 & -\xi_1^2 & -\xi_1\xi_2 \\ 0 & 0 & \xi_0\xi_1 \end{pmatrix}, \quad g_{20} = \begin{pmatrix} \xi_0^2 & 0 & 0 \\ 0 & \xi_0\xi_2 & 0 \\ -\nu\xi_0\xi_2 & -\xi_1\xi_2 & -\xi_2^2 \end{pmatrix}$$

up to multiplication of nonzero constant. Set

$$W_{ij} := \{(\xi_0/\xi_i, \xi_1/\xi_i, \xi_2/\xi_i, \eta_{i0}/\eta_{ij}, \eta_{i1}/\eta_{ij}, \eta_{i2}/\eta_{ij}) \mid \xi_i, \eta_{ij} \neq 0\} \simeq \boldsymbol{C}^4, \quad 0 \leqslant i,\ j \leqslant 2,$$

then we see that $\{W_{ij}\}_{0\le i,j\le 2}$ form an atlas consisting of affine charts of the manifold $\overline{\Sigma}_\nu$ and

$$W_i = \bigcup_{0\leqslant j\leqslant 2} W_{ij}.$$

We notice that $\overline{\Sigma}_\nu$ is isomorphic to $T^*\boldsymbol{P}^2 \sqcup (\boldsymbol{P}^2\times\boldsymbol{P}^1)$ if $\nu=0$ and to $\boldsymbol{P}^2\times\boldsymbol{P}^2$ if $\nu\neq 0$.

Let us extend the original system $\mathscr{H}_J$ defined on $\boldsymbol{C}^4\times B_J \ni (q_1,q_2,p_1,p_2,s_1,s_2)$ to that on $\overline{\Sigma}_\nu\times B_J$ assuming that $(q,p)=(q_1,q_2,p_1,p_2)$ is the coordinate system of W_{00} namely

$$q_1 = \frac{\xi_1}{\xi_0}, \quad q_2 = \frac{\xi_2}{\xi_0}, \quad p_1 = \frac{\eta_{01}}{\eta_{00}}, \quad p_2 = \frac{\eta_{02}}{\eta_{00}}.$$

Denote by $\mathscr{H}_J^{(0)}$ the extended system on $\overline{\Sigma}_\nu\times B_J$. Notice that the patching matrices g_{i0}, $i=1,2$ are given by

$$g_{10} = \begin{pmatrix} 1 & 0 & 0 \\ -\nu q_1 & -q_1^2 & -q_1q_2 \\ 0 & 0 & q_1 \end{pmatrix}, \quad g_{20} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & q_2 & 0 \\ -\nu q_2 & -q_1q_2 & -q_2^2 \end{pmatrix}$$

up to nonzero constant multiplication.

We see that the transformations from the original chart W_{00} to the charts W_{i0}, $i=1,2$ are symplectic. In fact, by setting

$$q_1^1 = \frac{\xi_0}{\xi_1}, \quad q_2^1 = \frac{\xi_2}{\xi_1}, \quad p_1^1 = \frac{\eta_{11}}{\eta_{10}}, \quad p_2^1 = \frac{\eta_{12}}{\eta_{10}},$$
$$q_1^2 = \frac{\xi_1}{\xi_2}, \quad q_2^2 = \frac{\xi_0}{\xi_2}, \quad p_1^2 = \frac{\eta_{21}}{\eta_{20}}, \quad p_2^2 = \frac{\eta_{22}}{\eta_{20}},$$

we have

$$q_1 = \frac{1}{q_1^1}, \quad q_2 = \frac{q_2^1}{q_1^1}, \quad p_1 = -q_1^1(\nu + q_1^1 p_1^1 + q_2^1 p_2^1), \quad p_2 = q_1^1 p_2^1,$$
$$q_1 = \frac{q_1^2}{q_2^2}, \quad q_2 = \frac{1}{q_2^2}, \quad p_1 = q_2^2 p_1^2, \quad p_2 = -q_2^2(\nu + q_1^2 p_1^2 + q_2^2 p_2^2),$$

which yield

$$dp_1 \wedge dq_1 + dp_2 \wedge dq_2 = dp_1^1 \wedge dq_1^1 + dp_2^1 \wedge dq_2^1 = dp_1^2 \wedge dq_1^2 + dp_2^2 \wedge dq_2^2.$$

Therefore the original Hamiltonian system is written also as a Hamiltonian system on each $W_{i0} \ni (q_1^i, q_2^i, p_1^i, p_2^i)$, $i = 1, 2$. Moreover, we can verify that the Hamiltonians become polynomials of the dependent variables $(q^i, p^i) = (q_1^i, q_2^i, p_1^i, p_2^i)$ whose coefficients are rational functions of $s = (s_1, s_2)$ holomorphic in B_J if the values of $\nu = \nu_J$ are chosen as

$$\nu_{11111} = -\frac{1}{2}(\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 - 1 + \alpha_\infty), \quad \nu_{1112} = -\frac{1}{2}(\alpha_0 + \alpha_1 + \alpha_2 - 1 + \alpha_\infty),$$
$$\nu_{113} = \nu_{122} = -\frac{1}{2}(\alpha_0 + \alpha_1 - 1 + \alpha_\infty), \quad \nu_{14} = \nu_{23} = -\alpha_\infty, \quad \nu_5 = \alpha + \frac{1}{2}.$$

Hereafter, we fix the values of ν_J as above.

We see however that the differential systems on $W_{ij} \times B_J$, $j \neq 0$ are not Hamiltonian systems and have pole singularities on $W_{ij} \backslash W_{i0}$.

Setting

$$W^0 = \bigcup_{i=0}^{2} W_{i0},$$

we state these facts as

Proposition 3.1. *For every J, the extended differential system $\mathscr{H}_J^{(0)}$ on $\overline{\Sigma}_{\nu_J} \times B_J$ is a polynomial Hamiltonian system on $W^0 \times B_J$ with coefficients holomorphic in $B_J \ni s = (s_1, s_2)$ but it has pole singularities on $D_J \times B_J$ where*

$$D_J = \overline{\Sigma}_{\nu_J} \backslash W^0.$$

3.2. Accessible singularities on $D_J \times B_J$.

Let us next determine the set of accessible singular points of the system $\mathscr{H}_J^{(0)}$ on $D_J \times B_J$. By definition, an accessible singular point is a point through which (potentially infinitely many) solution surfaces of the system $\mathscr{H}_J^{(0)}$ in $W^0 \times B_J$ pass holomorphically.

For example, let us investigate the form of the system $\mathscr{H}_{11111}^{(0)}$ on $W_{01} \times B_{11111} (\subset W_0 \times B_{11111})$. By setting $\xi_0 = \eta_{01} = 1$, we take $(\xi_1, \xi_2, \eta_{00}, \eta_{02})$ as the coordinates of W_{01}. In terms of them, the system is written as

$$e(s)\eta_{00}d\xi_1 = \sum_{i=1,2} P_i(\xi_1,\xi_2,\eta_{00},\eta_{02},s)ds_i,$$

$$e(s)\eta_{00}d\xi_2 = \sum_{i=1,2} Q_i(\xi_1,\xi_2,\eta_{00},\eta_{02},s)ds_i,$$

$$e(s)\eta_{00}d\eta_{00} = \sum_{i=1,2} X_i(\xi_1,\xi_2,\eta_{00},\eta_{02},s)ds_i,$$

$$e(s)\eta_{00}d\eta_{02} = \sum_{i=1,2} Y_i(\xi_1,\xi_2,\eta_{00},\eta_{02},s)ds_i,$$

where $e(s) = s_1s_2(s_1-1)(s_2-1)(s_1-s_2)$ and $P_i, Q_i, X_i, Y_i \in \boldsymbol{C}[\xi_1,\xi_2,\eta_{00},\eta_{02},s_1,s_2]$ (polynomial ring) are given by

$$\begin{aligned}
P_1 =& O(\eta_{00}) + 2s_2(s_2-1)\xi_1\big[\{(s_1-s_2)\xi_1 + s_1(s_2-1)\}\xi_2\eta_{02} \\
&\qquad + (\xi_1-1)(\xi_1-s_1)(s_1-s_2) - s_1(s_1-1)\xi_2\big], \\
P_2 =& O(\eta_{00}) + 2s_1(s_1-1)\xi_1\xi_2\big[\{(s_1-s_2)\xi_2 - s_2(s_1-1)\}\eta_{02} + (s_1-s_2)\xi_1 + s_1(s_2-1)\big], \\
Q_1 =& O(\eta_{00}) + 2s_2(s_2-1)\xi_1\xi_2\big[\{(s_1-s_2)\xi_2 - s_2(s_1-1)\}\eta_{02} + (s_1-s_2)\xi_1 + s_1(s_2-1)\big], \\
Q_2 =& O(\eta_{00}) + 2s_1(s_1-1)\xi_2\big[\{(s_2-s_1)\xi_2 + s_2(s_1-1)\}\xi_1 \\
&\qquad + \{(\xi_2-1)(\xi_2-s_2)(s_2-s_1) - s_2(s_2-1)\xi_1\}\eta_{02}\big], \\
X_1 =& X_2 = O(\eta_{00}), \\
Y_1 =& O(\eta_{00}) + s_2(s_2-1)\big[\{(s_1-s_2)\xi_2 - s_2(s_1-1)\}\xi_2\eta_{02}^3 \\
&\qquad + \{2(s_1-s_2)\xi_1\xi_2 + s_2(s_1-1)\xi_1 + 2s_1(s_2-1)\xi_2\}\eta_{02}^2 \\
&\qquad + \{(s_1-s_2)\xi_1^2 - 2(s_1^2-s_2)\xi_1 - s_1(s_1-1)\xi_2 + s_1(s_1-s_2)\}\eta_{02} \\
&\qquad + s_1(s_1-1)\xi_1\big], \\
Y_2 =& O(\eta_{00}) + s_1(s_1-1)\big[s_2(s_2-1)\xi_2\eta_{02}^3 \\
&\qquad - \big\{(s_1-s_2)\xi_2^2 + s_2(s_2-1)\xi_1 - 2(s_1-s_2^2)\xi_2 + s_2(s_1-s_2)\big\}\eta_{02}^2 \\
&\qquad - \{2(s_1-s_2)\xi_1\xi_2 - 2s_2(s_1-1)\xi_1 - s_1(s_2-1)\xi_2\}\eta_{02} \\
&\qquad - (s_1-s_2)\xi_1^2 - s_1(s_2-1)\xi_1\big],
\end{aligned}$$

$O(\eta_{00})$ denoting a polynomial of $\xi_1, \xi_2, \eta_{00}, \eta_{02}, s_1, s_2$ with a factor η_{00}. Therefore, accessible singular points are those satisfying the equations

$$\eta_{00} = 0, \quad P_i = Q_i = Y_i = 0, \quad i = 1, 2.$$

We see that the equations have the following three solutions

$$\eta_{00}=0, \quad s_1s_2-s_2\xi_1-s_1\xi_2=0, \quad s_1-s_2\eta_{02}=0;$$
$$\eta_{00}=0, \quad 1-\xi_1-\xi_2=0, \quad 1-\eta_{02}=0;$$
$$\eta_{00}=0, \quad \xi_1=0, \quad \eta_{02}=0,$$

which are denoted by $A_0(s)\cap W_{01}$, $A_1(s)\cap W_{01}$ and $A_2(s)\cap W_{01}$ respectively. Observing the system $\mathscr{H}^{(0)}_{11111}$ on all $W_{ij}\times B_{11111}$, $j\neq 0$, we can determine the accessible singular points of $\mathscr{H}^{(0)}_{11111}$.

By the same way as above, we can obtain

PROPOSITION 3.2. *The set of accessible singular points of the system $\mathscr{H}^{(0)}_J$ for each $s=(s_1,s_2)\in B_J$ is a disjoint union of $|J|$ connected components $A_k(s)\simeq \boldsymbol{P}^1$ given by*

$\mathscr{H}^{(0)}_{11111}$:

$$\begin{aligned} A_0(s)=&\big\{(\xi,\eta_0,s)\in W_0\times B \mid s_1s_2\xi_0-s_2\xi_1-s_1\xi_2=0,\eta_{00}=0,s_1\eta_{01}-s_2\eta_{02}=0\big\}\\ &\cup\big\{(\xi,\eta_1,s)\in W_1\times B \mid s_1s_2\xi_0-s_2\xi_1-s_1\xi_2=0,\eta_{10}=0,\eta_{11}+s_2\eta_{12}=0\big\}\\ &\cup\big\{(\xi,\eta_2,s)\in W_2\times B \mid s_1s_2\xi_0-s_2\xi_1-s_1\xi_2=0,\eta_{20}=0,s_1\eta_{21}+\eta_{22}=0\big\}, \end{aligned}$$

$$\begin{aligned} A_1(s)=&\big\{(\xi,\eta_0,s)\in W_0\times B \mid \xi_0-\xi_1-\xi_2=0,\eta_{00}=0,\eta_{01}-\eta_{02}=0\big\}\\ &\cup\big\{(\xi,\eta_1,s)\in W_1\times B \mid \xi_0-\xi_1-\xi_2=0,\eta_{10}=0,\eta_{11}+\eta_{12}=0\big\}\\ &\cup\big\{(\xi,\eta_2,s)\in W_2\times B \mid \xi_0-\xi_1-\xi_2=0,\eta_{20}=0,\eta_{21}+\eta_{22}=0\big\}, \end{aligned}$$

$$\begin{aligned} A_2(s)=&\big\{(\xi,\eta_0,s)\in W_0\times B \mid \xi_1=\eta_{00}=\eta_{02}=0\big\}\\ &\cup\big\{(\xi,\eta_2,s)\in W_2\times B \mid \xi_1=\eta_{20}=\eta_{22}=0\big\}, \end{aligned}$$

$$\begin{aligned} A_3(s)=&\big\{(\xi,\eta_0,s)\in W_0\times B \mid \xi_2=\eta_{00}=\eta_{01}=0\big\}\\ &\cup\big\{(\xi,\eta_1,s)\in W_1\times B \mid \xi_2=\eta_{10}=\eta_{11}=0\big\}, \end{aligned}$$

$$\begin{aligned} A_\infty(s)=&\big\{(\xi,\eta_1,s)\in W_1\times B \mid \xi_0=\eta_{10}=\eta_{12}=0\big\}\\ &\cup\big\{(\xi,\eta_2,s)\in W_2\times B \mid \xi_0=\eta_{20}=\eta_{21}=0\big\}, \end{aligned}$$

$\mathscr{H}^{(0)}_{1112}$:

$$\begin{aligned} A_0(s)=&\big\{(\xi,\eta_0,s)\in W_0\times B \mid s_1s_2\xi_0-s_2\xi_1-s_1\xi_2=0,\eta_{00}=0,s_1\eta_{01}-s_2\eta_{02}=0\big\}\\ &\cup\big\{(\xi,\eta_1,s)\in W_1\times B \mid s_1s_2\xi_0-s_2\xi_1-s_1\xi_2=0,\eta_{10}=0,\eta_{11}+s_2\eta_{12}=0\big\}\\ &\cup\big\{(\xi,\eta_2,s)\in W_2\times B \mid s_1s_2\xi_0-s_2\xi_1-s_1\xi_2=0,\eta_{20}=0,s_1\eta_{21}+\eta_{22}=0\big\}, \end{aligned}$$

$$\begin{aligned} A_1(s)=&\big\{(\xi,\eta_0,s)\in W_0\times B \mid \xi_1=\eta_{00}=\eta_{02}=0\big\}\\ &\cup\big\{(\xi,\eta_2,s)\in W_2\times B \mid \xi_1=\eta_{20}=\eta_{22}=0\big\}, \end{aligned}$$

$$\begin{aligned} A_2(s)=&\big\{(\xi,\eta_0,s)\in W_0\times B \mid \xi_2=\eta_{00}=\eta_{01}=0\big\}\\ &\cup\big\{(\xi,\eta_1,s)\in W_1\times B \mid \xi_2=\eta_{10}=\eta_{11}=0\big\}, \end{aligned}$$

$$A_\infty(s) = \big\{(\xi,\eta_1,s)\in W_1\times B \mid \xi_0=\eta_{10}=\eta_{12}=0\big\}$$
$$\cup\big\{(\xi,\eta_2,s)\in W_2\times B \mid \xi_0=\eta_{20}=\eta_{21}=0\big\},$$

$\mathscr{H}^{(0)}_{113}$:
$$A_0(s) = \big\{(\xi,\eta_0,s)\in W_0\times B \mid 2\xi_0+(2s_1+s_2^2)\xi_1+2s_2\xi_2=0,$$
$$\eta_{00}=0, 2s_2\eta_{01}-(2s_1+s_2^2)\eta_{02}=0\big\}$$
$$\cup\big\{(\xi,\eta_1,s)\in W_1\times B \mid 2\xi_0+(2s_1+s_2^2)\xi_1+2s_2\xi_2=0,$$
$$\eta_{10}=0, s_2\eta_{11}-\eta_{12}=0\big\}$$
$$\cup\big\{(\xi,\eta_2,s)\in W_2\times B \mid 2\xi_0+(2s_1+s_2^2)\xi_1+2s_2\xi_2=0,$$
$$\eta_{20}=0, 2\eta_{21}-(2s_1+s_2^2)\eta_{22}=0\big\},$$
$$A_1(s) = \big\{(\xi,\eta_0,s)\in W_0\times B \mid \xi_1=\eta_{00}=\eta_{02}=0\big\}$$
$$\cup\big\{(\xi,\eta_2,s)\in W_2\times B \mid \xi_1=\eta_{20}=\eta_{22}=0\big\},$$
$$A_\infty(s) = \big\{(\xi,\eta_1,s)\in W_1\times B \mid \xi_0=\eta_{10}=\eta_{12}=0\big\}$$
$$\cup\big\{(\xi,\eta_2,s)\in W_2\times B \mid \xi_0=\eta_{20}=\eta_{21}=0\big\},$$

$\mathscr{H}^{(0)}_{122}$:
$$A_0(s) = \big\{(\xi,\eta_0,s)\in W_0\times B \mid \xi_2=\eta_{00}=\eta_{01}=0\big\}$$
$$\cup\big\{(\xi,\eta_1,s)\in W_1\times B \mid \xi_2=\eta_{10}=\eta_{11}=0\big\},$$
$$A_1(s) = \big\{(\xi,\eta_0,s)\in W_0\times B \mid \xi_1=\eta_{00}=\eta_{02}=0\big\}$$
$$\cup\big\{(\xi,\eta_2,s)\in W_2\times B \mid \xi_1=\eta_{20}=\eta_{22}=0\big\},$$
$$A_\infty(s) = \big\{(\xi,\eta_1,s)\in W_1\times B \mid \xi_0=\eta_{10}=\eta_{12}=0\big\}$$
$$\cup\big\{(\xi,\eta_2,s)\in W_2\times B \mid \xi_0=\eta_{20}=\eta_{21}=0\big\},$$

$\mathscr{H}^{(0)}_{14}$:
$$A_0(s) = \big\{(\xi,\eta_0,s)\in W_0\times B \mid (s_1+s_2^2/2)\xi_0+s_2\xi_1+\xi_2=0, \eta_{00}=0, \eta_{01}-s_2\eta_{02}=0\big\}$$
$$\cup\big\{(\xi,\eta_1,s)\in W_1\times B \mid (s_1+s_2^2/2)\xi_0+s_2\xi_1+\xi_2=0,$$
$$\eta_{10}=0, \eta_{11}-(s_1+s_2^2/2)\eta_{12}=0\big\}$$
$$\cup\big\{(\xi,\eta_2,s)\in W_2\times B \mid (s_1+s_2^2/2)\xi_0+s_2\xi_1+\xi_2=0,$$
$$\eta_{20}=0, (s_1+s_2^2/2)\eta_{21}-s_2\eta_{22}=0\big\},$$
$$A_\infty(s) = \big\{(\xi,\eta_1,s)\in W_1\times B \mid \xi_0=\eta_{10}=\eta_{12}=0\big\}$$
$$\cup\big\{(\xi,\eta_2,s)\in W_2\times B \mid \xi_0=\eta_{20}=\eta_{21}=0\big\},$$

$\mathscr{H}^{(0)}_{23}$:
$$A_0(s) = \big\{(\xi,\eta_0,s)\in W_0\times B \mid \xi_2=\eta_{00}=\eta_{01}=0\big\}$$
$$\cup\big\{(\xi,\eta_1,s)\in W_1\times B \mid \xi_2=\eta_{10}=\eta_{11}=0\big\},$$

$$A_\infty(s) = \{(\xi, \eta_1, s) \in W_1 \times B \mid \xi_0 = \eta_{10} = \eta_{12} = 0\}$$
$$\cup \{(\xi, \eta_2, s) \in W_2 \times B \mid \xi_0 = \eta_{20} = \eta_{21} = 0\},$$

$\mathscr{H}_5^{(0)}$:
$$A_\infty(s) = \{(\xi, \eta_1, s) \in W_1 \times B \mid \xi_0 = \eta_{10} = \eta_{12} = 0\}$$
$$\cup \{(\xi, \eta_2, s) \in W_2 \times B \mid \xi_0 = \eta_{20} = \eta_{21} = 0\}.$$

Remark 1. We notice that, although some $A_k(s)$ are expressed by three cordinate systems W_0, W_1 and W_2, they can be expressed by any two of them. We used the three systems in order to hold a symmetry. In the procedures of quadratic transformations, we use the coordinate systems W_0 and W_1.

Remark 2. We assign positive integers n_k to the above $A_k(s)$ as follows: $n_k = 1$ for all k in case of $J = 11111$; $n_k = 1$ for $k \neq 1$ and $n_1 = 2$ in case of $J = 1112$; $n_k = 1$ for $k \neq 1$ and $n_1 = 3$ in case of $J = 113$; $n_\infty = 1$ and $n_0 = n_1 = 2$ in case of $J = 122$; $n_1 = 1$ and $n_\infty = 4$ in case of $J = 14$; $n_0 = 2$ and $n_\infty = 3$ in case of $J = 23$; $n_\infty = 5$ in case of $J = 5$. These integers indicate the numbers of quadratic transformations.

3.3. Coordinate systems for $n_k = 1$.

In the following two subsections, we show how to make quadratic transformations along the components $A_k(s)$ of the accessible singular points, namely how to obtain coordinate systems which separate completely the solution surfaces passing through $A_k(s)$. As was stated in Introduction, the number of quadratic transformations along $A_k(s)$ is $2n_k$ where n_k is an element of the partition J given in Remark 2 after Proposition 3.2. In this subsection, we explain the case of $n_k = 1$ while the case of $n_k \geqslant 2$ is studied in the next subsection.

The quadratic transformation along a set A is denoted by Q_A. We remark the superscript (n) of a letter indicates that it is concerned with an n-th quadratic transformation. Notice that we need two affine coordinate systems for each $A_k(s)$, because $A_k(s)$ is isomorphic to $\boldsymbol{P}^1$.

In this subsection, we only show the quadratic transformations along $A_2(s)$, namely along $A_2(s) \cap W_0$ and $A_2(s) \cap W_2$ of $\mathscr{H}_{11111}$ as an example of the case of $n_k = 1$. The corrdinate systems for other $A_k(s)$ with $n_k = 1$ are obtained in the same way as in the example in this subsection.

3.3.1. Coordinate system for $A_2(s) \cap W_0$ of $\mathscr{H}_{11111}$.

The first quadratic transformation along $A_2(s) \cap W_0$. Note that $A_2(s) \cap W_0 \subset W_{01}$ and

$$A_2(s) \cap W_0 = \{(\xi_1, \xi_2, \eta_{00}, \eta_{02}) \in W_{01} \simeq \boldsymbol{C}^4 \mid \xi_2 \in \boldsymbol{C}, \xi_1 = \eta_{00} = \eta_{02} = 0\}.$$

We replace every point $(\xi_1, \xi_2, \eta_{00}, \eta_{02}) = (0, \xi_2, 0, 0)$ with $\xi_2 \in \boldsymbol{C}$ by $\boldsymbol{P}^2$ simultaneously. Let $(\xi_2, x_{20}^{(1)}, y_{20}^{(1)}, z_{20}^{(1)}) \in \boldsymbol{C}^4$, $(\xi_2, x_{21}^{(1)}, y_{21}^{(1)}, z_{21}^{(1)}) \in \boldsymbol{C}^4$ and $(\xi_2, x_{22}^{(1)}, y_{22}^{(1)}, z_{22}^{(1)}) \in \boldsymbol{C}^4$ be coordinate systems of $V_{21}^{(1)}(s) = Q_{A_2(s) \cap W_0}(W_{01} \times s)$ defined by

$$\xi_1 = x_{20}^{(1)}, \qquad \eta_{00} = x_{20}^{(1)} y_{20}^{(1)}, \quad \eta_{02} = x_{20}^{(1)} z_{20}^{(1)},$$
$$\xi_1 = x_{21}^{(1)} y_{21}^{(1)}, \quad \eta_{00} = y_{21}^{(1)}, \qquad \eta_{02} = y_{21}^{(1)} z_{21}^{(1)},$$
$$\xi_1 = x_{22}^{(1)} z_{22}^{(1)}, \quad \eta_{00} = y_{22}^{(1)} z_{22}^{(1)}, \quad \eta_{02} = z_{22}^{(1)},$$

then exceptinal divisor $D_{21}^{(1)}(s) = Q_{A_2(s)\cap W_0}(A_2(s)\cap W_0)$ is given by

$$\Big\{\big(\xi_2, x_{20}^{(1)}, y_{20}^{(1)}, z_{20}^{(1)}\big) \,\big|\, x_{20}^{(1)} = 0\Big\} \cup \Big\{\big(\xi_2, x_{21}^{(1)}, y_{21}^{(1)}, z_{21}^{(1)}\big) \,\big|\, y_{21}^{(1)} = 0\Big\}$$
$$\cup \Big\{\big(\xi_2, x_{22}^{(1)}, y_{22}^{(1)}, z_{22}^{(1)}\big) \,\big|\, z_{22}^{(1)} = 0\Big\}.$$

Let us write our system in the three coordinate systems near the exceptional divisor. In the first coordinate system, it is written as

$$\begin{aligned}
e(s)x_{20}^{(1)} d\xi_2 &= \frac{O\big(x_{20}^{(1)}\big)}{y_{20}^{(1)}} ds_1 + \frac{O\big(x_{20}^{(1)}\big)}{y_{20}^{(1)}} ds_2,\\
e(s)x_{20}^{(1)} dx_{20}^{(1)} &= \frac{O\big(x_{20}^{(1)}\big)}{y_{20}^{(1)}} ds_1 + \frac{O\big(x_{20}^{(1)}\big)}{y_{20}^{(1)}} ds_2,\\
e(s)x_{20}^{(1)} dy_{20}^{(1)} &= \Big\{O\big(x_{20}^{(1)}\big) + \big(\alpha_2 y_{20}^{(1)} - 1\big) P_{21}(\xi_2, s)\Big\} ds_1\\
&\quad + \Big\{O\big(x_{20}^{(1)}\big) + \big(\alpha_2 y_{20}^{(1)} - 1\big) P_{22}(\xi_2, s)\Big\} ds_2,\\
e(s)x_{20}^{(1)} dz_{20}^{(1)} &= \frac{O\big(x_{20}^{(1)}\big)}{y_{20}^{(1)}} ds_1 + \frac{O\big(x_{20}^{(1)}\big)}{y_{20}^{(1)}} ds_2
\end{aligned}$$

in a neighborhood of $D_{21}^{(1)}(s) = \big\{x_{20}^{(1)} = 0\big\}$, in the second coordinate system, it is written as

$$\begin{aligned}
e(s)y_{21}^{(1)} d\xi_2 &= O\big(y_{21}^{(1)}\big) ds_1 + O\big(y_{21}^{(1)}\big) ds_2,\\
e(s)y_{21}^{(1)} dx_{21}^{(1)} &= \Big\{O\big(y_{21}^{(1)}\big) + \big(\alpha_2 - x_{21}^{(1)}\big) P_{23}(\xi_2, s)\Big\} ds_1\\
&\quad + \Big\{O\big(y_{21}^{(1)}\big) + \big(\alpha_2 - x_{21}^{(1)}\big) P_{24}(\xi_2, s)\Big\} ds_2,\\
e(s)y_{21}^{(1)} dy_{21}^{(1)} &= O\big(y_{21}^{(1)}\big) ds_1 + O\big(y_{21}^{(1)}\big) ds_2,\\
e(s)y_{21}^{(1)} dz_{21}^{(1)} &= O\big(y_{21}^{(1)}\big) ds_1 + O\big(y_{21}^{(1)}\big) ds_2
\end{aligned}$$

in a neighborhood of $D_{21}^{(1)}(s) = \big\{y_{21}^{(1)} = 0\big\}$, and in the third coordinate system it is written as

$$\begin{aligned}
e(s)z_{22}^{(1)}d\xi_2 &= \frac{O\big(z_{22}^{(1)}\big)}{y_{22}^{(1)}}ds_1 + \frac{O\big(z_{22}^{(1)}\big)}{y_{22}^{(1)}}ds_2,\\
e(s)z_{22}^{(1)}dx_{22}^{(1)} &= \left\{O\big(z_{22}^{(1)}\big) + \left(\alpha_2 - \frac{x_{22}^{(1)}}{y_{22}^{(1)}}\right)C_{21}(s)\right\}ds_1\\
&\quad + \left\{O\big(z_{22}^{(1)}\big) + \left(\alpha_2 - \frac{x_{22}^{(1)}}{y_{22}^{(1)}}\right)C_{22}(s)\right\}ds_2,\\
e(s)z_{22}^{(1)}dy_{22}^{(1)} &= \left\{O\big(z_{22}^{(1)}\big) + \big(\alpha_2 y_{22}^{(1)} - x_{22}^{(1)}\big)P_{25}\big(\xi_2, x_{22}^{(1)}, s\big)\right\}ds_1\\
&\quad + \left\{O\big(z_{22}^{(1)}\big) + \big(\alpha_2 y_{22}^{(1)} - x_{22}^{(1)}\big)P_{26}\big(\xi_2, x_{22}^{(1)}, s\big)\right\}ds_2,\\
e(s)dz_{22}^{(1)} &= \left(O\big(z_{22}^{(1)}\big) + \frac{P_{27}\big(\xi_2, x_{22}^{(1)}, s\big)}{y_{22}^{(1)}}\right)ds_1 + \left(O\big(z_{22}^{(1)}\big) + \frac{P_{28}\big(\xi_2, x_{22}^{(1)}, s\big)}{y_{22}^{(1)}}\right)ds_2
\end{aligned}$$

in a neighborhood of $D_{21}^{(1)}(s) = \big\{z_{22}^{(1)} = 0\big\}$. Here $P_{2m}(*)$ denote a polynomial of $*$ and $C_{2m}(s)$ denotes a polynomial of $s = (s_1, s_2)$. Note that there is no points satisfying $P_{27}\big(\xi_2, x_{22}^{(1)}, s\big) = 0$ and $P_{28}\big(\xi_2, x_{22}^{(1)}, s\big) = 0$, and $C_{2m}(s) \neq 0$ for $s \in B_{11111}$. In the following, $P_{km}(*)$ and $C_{km}(*)$ always denotes polynomials of some variables $*$ and $C_{km}(s) \neq 0$ for $s \in B_J$. Investigating carefully these systems in a neighborhood of $D_{21}^{(1)}(s)$ in the same way as in the deriving Proposition 3.1, we can verity that the set of accessible singular points $A_{21}^{(1)}(s)$ is given by

$$A_{21}^{(1)}(s) = \left\{\big(\xi_2, x_{21}^{(1)}, y_{21}^{(1)}, z_{21}^{(1)}\big) = \big(\xi_2, \alpha_2, 0, z_{21}^{(1)}\big)\right\} \subset D_{21}^{(1)}(s).$$

The second quadratic transformation along $A_{21}^{(1)}(s)$. Let $\big(\xi_2, z_{21}^{(1)}, x_{20}^{(2)}, y_{20}^{(2)}\big) \in \boldsymbol{C}^4$ and $\big(\xi_2, z_{21}^{(1)}, x_{21}^{(2)}, y_{21}^{(2)}\big) \in \boldsymbol{C}^4$ be coordinate systems of $V_{21}^{(2)}(s) = Q_{A_{21}^{(1)}(s)}(V_{21}^{(1)}(s))$ defined by

$$\begin{aligned}
x_{21}^{(1)} &= \alpha_2 + x_{20}^{(2)}, & y_{21}^{(1)} &= x_{20}^{(2)}y_{20}^{(2)},\\
x_{21}^{(1)} &= \alpha_2 + x_{21}^{(2)}y_{21}^{(2)}, & y_{21}^{(1)} &= y_{21}^{(2)},
\end{aligned}$$

then the exceptional divisor $D_{21}^{(2)}(s) = Q_{A_{21}^{(1)}(s)}\big(A_{21}^{(1)}(s)\big)$ is given by

$$\left\{\big(\xi_2, z_{21}^{(1)}, x_{20}^{(2)}, y_{20}^{(2)}\big) \mid x_{20}^{(2)} = 0\right\} \cup \left\{\big(\xi_2, z_{21}^{(1)}, x_{21}^{(2)}, y_{22}^{(2)}\big) \mid y_{21}^{(2)} = 0\right\}.$$

We can verify that our system is written in the second coordinate system as

$$
\begin{aligned}
e(s)d\xi_2 &= \sum_{i=1,2} P_{2i}\big(\xi_2, z_{21}^{(1)}, x_{21}^{(2)}, y_{21}^{(2)}, s\big)ds_i,\\
e(s)dz_{21}^{(1)} &= \sum_{i=1,2} Q_{2i}\big(\xi_2, z_{21}^{(1)}, x_{21}^{(2)}, y_{21}^{(2)}, s\big)ds_i,\\
e(s)dx_{21}^{(2)} &= \sum_{i=1,2} X_{2i}\big(\xi_2, z_{21}^{(1)}, x_{21}^{(2)}, y_{21}^{(2)}, s\big)ds_i,\\
e(s)dy_{21}^{(2)} &= \sum_{i=1,2} Y_{2i}\big(\xi_2, z_{21}^{(1)}, x_{21}^{(2)}, y_{21}^{(2)}, s\big)ds_i.
\end{aligned}
$$

Here $P_{2i}, Q_{2i}, X_{2i}, Y_{2i}$, $i = 1, 2$ are certain polynomials of $\xi_2, z_{21}^{(1)}, x_{21}^{(2)}, y_{21}^{(2)}$ and s. This means that the differential system has no singular points in $\big(\xi_2, z_{21}^{(1)}, x_{21}^{(2)}, y_{21}^{(2)}, s\big)$-space $\boldsymbol{C}^4 \times B_{11111}$. On the other hand, we can verify that the points $\big(\xi_2, z_{21}^{(1)}, x_{20}^{(2)}, y_{20}^{(2)}, s\big) = \big(\xi_2, z_{21}^{(1)}, 0, 0\big)$ are inaccessible singular points, because our system is written as

$$
\begin{aligned}
e(s)d\xi_2 &= \sum_{i=1,2} P'_{2i}\big(\xi_2, z_{21}^{(1)}, x_{20}^{(2)}, y_{20}^{(2)}, s\big)ds_i,\\
e(s)dz_{21}^{(1)} &= \sum_{i=1,2} \left(Q'_{2i}\big(\xi_2, z_{21}^{(1)}, x_{20}^{(2)}, y_{20}^{(2)}, s\big) + \frac{C_{2i}(s)}{y_{20}^{(2)}}\right)ds_i,\\
e(s)dx_{20}^{(2)} &= \sum_{i=1,2} \left(X'_{2i}\big(\xi_2, z_{21}^{(1)}, x_{20}^{(2)}, y_{20}^{(2)}, s\big) + \frac{P_{2i}(\xi_2)}{y_{20}^{(2)}}\right)ds_i,\\
e(s)dy_{20}^{(2)} &= \sum_{i=1,2} Y'_{2i}\big(\xi_2, z_{21}^{(1)}, x_{20}^{(2)}, y_{20}^{(2)}, s\big)ds_i
\end{aligned}
$$

in a neighborhood of $\big(\xi_2, z_{21}^{(1)}, x_{20}^{(2)}, y_{20}^{(2)}\big) = \big(\xi_2, z_{21}^{(1)}, 0, 0\big)$ and $C_{2i}(s) \neq 0$, $i = 1, 2$, where $P'_{2i}, Q'_{2i}, X'_{2i}, Y'_{2i}$, $i = 1, 2$ are certain polynomials of $\xi_2, z_{21}^{(1)}, x_{20}^{(2)}, y_{20}^{(2)}$ and s.

Thus we have obtained a coordinate system $\big(\xi_2, z_{21}^{(1)}, x_{21}^{(2)}, y_{21}^{(2)}\big) \in \boldsymbol{C}^4$ which separates the solution surfaces passing through $A_2(s) \cap W_0 = A_2(s) \cap W_{01}$. If we set

$$
q_1^{21} = x_{21}^{(1)}, \quad q_2^{21} = -y_{21}^{(2)}, \quad p_1^{21} = w_{21}^{(1)}, \quad p_2^{21} = z_{21}^{(2)}
$$

then we have

$$
dq_1 \wedge dp_1 + dq_2 \wedge dp_1 = dq_1^{21} \wedge dp_1^{21} + dq_2^{21} \wedge dp_2^{21}.
$$

We should notice that the transformation from (q, p) to (q^{21}, p^{21}) is symplectic and then our system $\mathscr{H}_{11111}$ is also written as an Hamiltonian system in the variables (q^{21}, p^{21}). In our teminology, (q^{21}, p^{21}) is a symplectic coordinate system. We can verify that the Hamiltonians in (q^{21}, p^{21}) are polynomials of the variables whose coefficients are rational functions of $s = (s_1, s_2)$ holomorphic in B_{11111}.

3.3.2. Coordinate system for $A_2(s) \cap W_2$ of $\mathscr{H}_{11111}$.

The first quadratic transformation along $A_2(s) \cap W_2$. Note that $A_2(s) \cap W_2 \subset W_{21}$ and

$$A_2(s) \cap W_2 = \{(\xi_0, \xi_1, \eta_{20}, \eta_{22}) \in W_{21} \simeq \boldsymbol{C}^4 \mid \xi_0 \in \boldsymbol{C}, \xi_1 = \eta_{20} = \eta_{22} = 0\}.$$

We replace every point $(\xi_0, \xi_1, \eta_{20}, \eta_{22}) = (\xi_0, 0, 0, 0)$ with $\xi_0 \in \boldsymbol{C}$ by $\boldsymbol{P}^2$ simultaneously. Let $(\xi_0, X_{20}^{(1)}, Y_{20}^{(1)}, Z_{20}^{(1)}) \in \boldsymbol{C}^4$, $(\xi_0, X_{21}^{(1)}, Y_{21}^{(1)}, Z_{21}^{(1)}) \in \boldsymbol{C}^4$ and $(\xi_0, X_{22}^{(1)}, Y_{22}^{(1)}, Z_{22}^{(1)}) \in \boldsymbol{C}^4$ be coordinate systems of $V_{22}^{(1)}(s) = Q_{A_2(s)\cap W_2}(W_{21} \times s)$ defined by

$$\begin{aligned}
\xi_1 &= X_{20}^{(1)}, & \eta_{20} &= X_{20}^{(1)} Y_{20}^{(1)}, & \eta_{22} &= X_{20}^{(1)} Z_{20}^{(1)},\\
\xi_1 &= X_{21}^{(1)} Y_{21}^{(1)}, & \eta_{20} &= Y_{21}^{(1)}, & \eta_{22} &= Y_{21}^{(1)} Z_{21}^{(1)},\\
\xi_1 &= X_{22}^{(1)} Z_{22}^{(1)}, & \eta_{20} &= Y_{22}^{(1)} Z_{22}^{(1)}, & \eta_{22} &= Z_{22}^{(1)},
\end{aligned}$$

then the exceptional divisor $D_{22}^{(1)}(s) = Q_{A_2(s)\cap W_2}(A_2(s) \cap W_2)$ is given by

$$\{X_{20}^{(1)} = 0\} \cup \{Y_{21}^{(1)} = 0\} \cup \{Z_{22}^{(1)} = 0\}$$

and the set of accessible singular points $A_{22}^{(1)}(s)$ is given by

$$A_{22}^{(1)}(s) = \Big\{ \big(\xi_1, X_{21}^{(1)}, Y_{21}^{(1)}, Z_{21}^{(1)}\big) = \big(\xi_1, \alpha_2, 0, Z_{21}^{(1)}\big) \Big\} \subset D_{22}^{(1)}(s).$$

The second quadratic transformation along $A_{22}^{(1)}(s)$. We next replace the points $(\xi_1, X_{21}^{(1)}, Y_{21}^{(1)}, Z_{21}^{(1)}) = (\xi_1, \alpha_2, 0, Z_{21}^{(1)})$ with $(\xi_1, Z_{21}^{(1)}) \in \boldsymbol{C}^2$ by $\boldsymbol{P}^1$ simultaneously. Let $(\xi_1, Z_{21}^{(1)}, X_{20}^{(2)}, Y_{20}^{(2)}) \in \boldsymbol{C}^4$ and $(\xi_1, Z_{21}^{(1)}, X_{21}^{(2)}, Y_{21}^{(2)}) \in \boldsymbol{C}^4$ be coordinate systems of $V_{22}^{(2)}(s) = Q_{A_{22}^{(1)}(s)}(V_{22}^{(1)}(s))$ defined by

$$\begin{aligned}
X_{21}^{(1)} &= \alpha_2 + X_{20}^{(2)}, & Y_{21}^{(1)} &= X_{20}^{(2)} Y_{20}^{(2)},\\
X_{21}^{(1)} &= \alpha_2 + X_{21}^{(2)} Y_{21}^{(2)}, & Y_{21}^{(1)} &= Y_{21}^{(2)},
\end{aligned}$$

then the exceptional divisor $D_{22}^{(2)}(s) = Q_{A_{22}^{(1)}(s)}(A_{22}^{(1)}(s))$ is given by

$$\{X_{20}^{(2)} = 0\} \cup \{Y_{21}^{(2)} = 0\}.$$

We see that the differential system has no singular points in the $(\xi_0, Z_{21}^{(1)}, X_{21}^{(2)}, Y_{21}^{(2)}, s)$-space $\boldsymbol{C}^4 \times B$ and moreover the points $(\xi_0, Z_{21}^{(1)}, X_{20}^{(2)}, Y_{20}^{(2)}) = (\xi_0, Z_{21}^{(1)}, 0, 0)$ are inaccessible.

Thus we have obtained a coordinate system $(\xi_1, Z_{21}^{(1)}, X_{21}^{(2)}, Y_{21}^{(2)}) \in \boldsymbol{C}^4$ which separates the solution surfaces passing through $A_2(s) \cap W_2 = A_2(s) \cap W_{21}$. By setting

$$q_1^{22} = -X_{21}^{(2)}, \quad q_2^{22} = \xi_0, \quad p_1^{22} = Y_{21}^{(2)}, \quad p_2^{22} = Z_{21}^{(1)},$$

we obtain a symplectic coordinate system (q^{22}, p^{22}) for $A_2(s) \cap W_2$. We can also verify that the Hamiltonians with respect to the coordinates are polynomials of these variables.

3.3.3. Coordinate systems for $A_2(s)$ of $\mathscr{H}_{11111}$.

Here we summarize the results obtained in **3.3.1** and **3.3.2** for the later convenience.

The set $A_2(s) = (A_2(s) \cap W_0) \cup (A_2(s) \cap W_2) \subset D_{11111} \times s$ is a component of the accessible singular points. Let $Q_{A_2(s)}$ be the quadratic transformation defined by $Q_{A_2(s) \cap W_0}$ and $Q_{A_2(s) \cap W_2}$. Then $D_2^{(1)}(s) := D_{21}^{(1)}(s) \cup D_{22}^{(1)}(s)$ is the exceptional divisor defined by $Q_{A_2(s)}(A_2(s))$. The set $A_2^{(1)}(s) := A_{21}^{(1)}(s) \cup A_{22}^{(1)}(s)$ is that of accessible singular points in $D_2^{(1)}(s)$. Let $Q_{A_2^{(1)}(s)}$ be the quadratic transformation defined by $Q_{A_{21}^{(1)}(s)}$ and $Q_{A_{22}^{(1)}(s)}$, then $D_2^{(2)}(s) := D_{21}^{(2)}(s) \cup D_{22}^{(2)}(s)$ is the exceptional divisor defined by $Q_{A_2^{(1)}(s)}\big(A_2^{(1)}(s)\big)$. The set $D_2^{(2)}(s) \setminus D_2^{(1)}(s)$ is a parameter space which separates the solution surfaces passing through $A_2(s)$, where $D_2^{(1)}(s)$ also denotes its proper image by $Q_{A_2^{(1)}(s)}$. The neighborhood of the parameter space $D_2^{(2)}(s) \setminus D_2^{(1)}(s)$ is covered by two affine charts $\boldsymbol{C}^4$ whose coordinate systems are $\big(q_1^{21}, q_2^{21}, p_1^{21}, p_2^{21}\big)$ and $\big(q_1^{22}, q_2^{22}, p_1^{22}, p_2^{22}\big)$. We note

$$D_2^{(2)}(s) \setminus D_2^{(1)}(s) = \big\{p_1^{21} = 0\big\} \cup \big\{p_1^{22} = 0\big\}.$$

We remark that $D_{11111} \times s$ and $D_2^{(1)}(s)$ denoting their proper images by quadratic transformations are inaccessible singular points which may contain the points of the so-called vertical leaves.

3.4. Coordinate systems for $n_k \geqslant 2$.

In this subsection, we explain the quadratic transformations along $A_k(s)$ in the case of $n_k \geqslant 2$. Being different from the case of $n_k = 1$, we insert a simple change of variables after the n_k-th quadratic transformation and make a suitable change of variables after the last quadratic transformation, in order to obtain a good symplectic coordinate system. The last procedure is very important because it not only produces symplectic coordinates but also resolves a kind of singularity of the differential system.

Here, we only show the case of $A_1(s)$ of $\mathscr{H}_{1112}$ with $n_1 = 2$ as an example of the case of $n_k \geqslant 2$. In this case, note that $A_1(s) = (A_1(s) \cap W_0) \cup (A_1(s) \cap W_2)$. We can study the other cases in the same way as this example.

3.4.1. Coordinate system for $A_1(s) \cap W_0$ of $\mathscr{H}_{1112}$.

The first quadratic transformation along $A_1(s) \cap W_0$. Note that $A_1(s) \cap W_0 \subset W_{01}$ and

$$A_1(s) \cap W_0 = \big\{(\xi_1, \xi_2, \eta_{00}, \eta_{02}) \in W_{01} \simeq \boldsymbol{C}^4 \mid \xi_2 \in \boldsymbol{C}, \xi_1 = \eta_{00} = \eta_{02} = 0\big\}.$$

We replace every point $(\xi_1, \xi_2, \eta_{00}, \eta_{02}) = (0, \xi_2, 0, 0)$ with $\xi_2 \in \boldsymbol{C}$ by $\boldsymbol{P}^2$ simultaneously. Let $\big(\xi_2, x_{10}^{(1)}, y_{10}^{(1)}, z_{10}^{(1)}\big) \in \boldsymbol{C}^4$, $\big(\xi_2, x_{11}^{(1)}, y_{11}^{(1)}, z_{11}^{(1)}\big) \in \boldsymbol{C}^4$ and $\big(\xi_2, x_{12}^{(1)}, y_{12}^{(1)}, z_{12}^{(1)}\big) \in \boldsymbol{C}^4$ be coordinate systems of $V_{11}^{(1)}(s) = Q_{A_1(s)\cap W_0}(W_{01} \times s)$ defined by

$$\begin{aligned} \xi_1 &= x_{10}^{(1)}, & \eta_{00} &= x_{10}^{(1)} y_{10}^{(1)}, & \eta_{02} &= x_{10}^{(1)} z_{10}^{(1)}, \\ \xi_1 &= x_{11}^{(1)} y_{11}^{(1)}, & \eta_{00} &= y_{11}^{(1)}, & \eta_{02} &= y_{11}^{(1)} z_{11}^{(1)}, \\ \xi_1 &= x_{12}^{(1)} z_{12}^{(1)}, & \eta_{00} &= y_{12}^{(1)} z_{12}^{(1)}, & \eta_{02} &= z_{12}^{(1)}, \end{aligned}$$

then the exceptional divisor $D_{11}^{(1)}(s) = Q_{A_1(s)\cap W_0}(A_1(s) \cap W_0)$ is give by

$$\big\{x_{10}^{(1)} = 0\big\} \cup \big\{y_{11}^{(1)} = 0\big\} \cup \big\{z_{12}^{(1)} = 0\big\}.$$

Let us write our system in the three coordinate systems near the exceptional divisor. In the first coordinate system, it is written as

$$\begin{aligned} e(s) x_{10}^{(1)} d\xi_2 &= O\big(x_{10}^{(1)}\big) ds_1 + \frac{O\big(x_{10}^{(1)}\big)}{y_{10}^{(1)}} ds_2, \\ e(s) x_{10}^{(1)} dx_{10}^{(1)} &= O\big(x_{10}^{(1)}\big) ds_1 + O\big(x_{10}^{(1)}\big) ds_2, \\ e(s) x_{10}^{(1)} dy_{10}^{(1)} &= \Big\{O\big(x_{10}^{(1)}\big) + y_{10}^{(1)} P_{12}(\xi_2, s)\Big\} ds_1 + \Big\{O\big(x_{10}^{(1)}\big) + y_{10}^{(1)} P_{13}(\xi_2, s)\Big\} ds_2, \\ e(s) x_{10}^{(1)} dz_{10}^{(1)} &= \Big[O\big(x_{10}^{(1)}\big) + C_{11}(s)\big\{(\xi_2 - 1) z_{10}^{(1)} + 1\big\}\Big] ds_1 \\ &\qquad + \left(\frac{O\big(x_{10}^{(1)}\big)}{y_{10}^{(1)}} + P_{14}\big(\xi_2, z_{10}^{(1)}, s\big)\right) ds_2 \end{aligned}$$

in a neighborhood of $D_{11}^{(1)}(s) = \big\{x_{10}^{(1)} = 0\big\}$. Here $e(s) = s_1^2 s_2 (s_2 - 1)$. In the second coordinate system, it is written as

$$\begin{aligned} e(s) y_{11}^{(1)} d\xi_2 &= O\big(y_{11}^{(1)}\big) ds_1 + O\big(y_{11}^{(1)}\big) ds_2, \\ e(s) y_{11}^{(1)} dx_{11}^{(1)} &= \Big\{O\big(y_{11}^{(1)}\big) + P_{15}(\xi_2, s)\Big\} ds_1 + \Big\{O\big(y_{11}^{(1)}\big) + P_{16}(\xi_2, s)\Big\} ds_2, \\ e(s) y_{11}^{(1)} dy_{11}^{(1)} &= O\big(y_{11}^{(1)}\big) ds_1 + O\big(y_{11}^{(1)}\big) ds_2, \\ e(s) y_{11}^{(1)} dz_{11}^{(1)} &= \Big\{O\big(y_{11}^{(1)}\big) + C_{12}(s)\Big\} ds_1 + \Big\{O\big(y_{11}^{(1)}\big) + C_{13}(s)\Big\} ds_2 \end{aligned}$$

in a neighborhood of $D_{11}^{(1)}(s) = \big\{y_{11}^{(1)} = 0\big\}$, and in the third coordinate system it is written as

$$
\begin{aligned}
e(s)z_{12}^{(1)}d\xi_2 &= O\bigl(z_{12}^{(1)}\bigr)ds_1 + \frac{O\bigl(z_{12}^{(1)}\bigr)}{y_{12}^{(1)}}ds_2,\\
e(s)z_{12}^{(1)}dx_{12}^{(1)} &= \Bigl\{O\bigl(z_{12}^{(1)}\bigr) + C_{14}(s)\bigl(x_{12}^{(1)}+\xi_2-1\bigr)\Bigr\}ds_1\\
&\qquad + \left(\frac{O\bigl(z_{12}^{(1)}\bigr)}{y_{12}^{(1)}} + P_{17}\bigl(\xi_2, x_{12}^{(1)}, s\bigr)\right)ds_2ds_2,\\
e(s)z_{12}^{(1)}dy_{12}^{(1)} &= \Bigl\{O\bigl(z_{12}^{(1)}\bigr) + y_{12}^{(1)}C_{15}(s)\Bigr\}ds_1 + \Bigl\{O\bigl(z_{12}^{(1)}\bigr) + y_{12}^{(1)}C_{16}(s)\Bigr\}ds_2,\\
e(s)z_{12}^{(1)}dz_{12}^{(1)} &= O\bigl(z_{12}^{(1)}\bigr)ds_1 + O\bigl(z_{12}^{(1)}\bigr)ds_2
\end{aligned}
$$

in a neighborhood of $D_{11}^{(1)}(s) = \bigl\{z_{12}^{(1)} = 0\bigr\}$. We can verity that the set of accessible singular points $A_{11}^{(1)}(s)$ is given by

$$
A_{11}^{(1)}(s) = \Bigl\{\bigl(\xi_2, x_{10}^{(1)}, y_{10}^{(1)}, z_{10}^{(1)}\bigr) = (\xi_2, 0, 0, -1/(\xi_2-1))\Bigr\} \subset D_{11}^{(1)}(s).
$$

The second quadratic transformation along $A_{11}^{(1)}(s)$. We next replace the points $\bigl(\xi_2, x_{10}^{(1)}, y_{10}^{(1)}, z_{10}^{(1)}\bigr) = (\xi_2, 0, 0, -1/(\xi_2-1))$ with $\xi_2 \in \boldsymbol{C} \setminus \{\xi_2 = 1\}$ by $\boldsymbol{P}^2$ simultaneously. For a while, we assume that $\xi_2 \neq 1$, namely we exclude the point $\xi_2 = 1$. Let $\bigl(\xi_2, x_{10}^{(2)}, y_{10}^{(2)}, z_{10}^{(2)}\bigr) \in \boldsymbol{C}^4$, $\bigl(\xi_2, x_{11}^{(2)}, y_{11}^{(2)}, z_{11}^{(2)}\bigr) \in \boldsymbol{C}^4$ and $\bigl(\xi_2, x_{12}^{(2)}, y_{12}^{(2)}, z_{12}^{(2)}\bigr) \in \boldsymbol{C}^4$ be coordinate systems of $V_{11}^{(2)}(s) = Q_{A_{11}^{(1)}(s)}\bigl(V_{11}^{(1)}(s)\bigr)$ defined by

$$
\begin{aligned}
x_{10}^{(1)} &= x_{10}^{(2)}, & y_{10}^{(1)} &= x_{10}^{(2)}y_{10}^{(2)}, & z_{10}^{(1)} &= -1/(\xi_2-1) + x_{10}^{(2)}z_{10}^{(2)},\\
x_{10}^{(1)} &= x_{11}^{(2)}y_{11}^{(2)}, & y_{10}^{(1)} &= y_{11}^{(2)}, & z_{10}^{(1)} &= -1/(\xi_2-1) + y_{11}^{(2)}z_{11}^{(2)},\\
x_{10}^{(1)} &= x_{12}^{(2)}z_{12}^{(2)}, & y_{10}^{(1)} &= y_{12}^{(2)}z_{12}^{(2)}, & z_{10}^{(1)} &= -1/(\xi_2-1) + z_{12}^{(2)},
\end{aligned}
$$

then the exceptional divisor $D_{11}^{(2)}(s) = Q_{A_{11}^{(1)}(s)}(A_{11}^{(1)}(s))$ is given by

$$
\bigl\{x_{10}^{(2)} = 0\bigr\} \cup \bigl\{y_{11}^{(2)} = 0\bigr\} \cup \bigl\{z_{12}^{(2)} = 0\bigr\}.
$$

Let us write our system in the three coordinate systems near the exceptional divisor. In the first coordinate, it is written as

$$
\begin{aligned}
e(s)x_{10}^{(2)}(\xi_2-1)d\xi_2 &= \frac{O\bigl(x_{10}^{(2)}\bigr)}{y_{10}^{(2)}}ds_1 + \frac{O\bigl(x_{10}^{(2)}\bigr)}{y_{10}^{(2)}}ds_2,\\
e(s)x_{10}^{(2)}(\xi_2-1)dx_{10}^{(2)} &= \frac{O\bigl(x_{10}^{(2)}\bigr)}{y_{10}^{(2)}}ds_1 + \frac{O\bigl(x_{10}^{(2)}\bigr)}{y_{10}^{(2)}}ds_2,
\end{aligned}
$$

$$e(s)x_{10}^{(2)}(\xi_2-1)^2dy_{10}^{(2)} = \Big[O\big(x_{10}^{(2)}\big) + \big\{\eta(\xi_2-1)y_{10}^{(2)}+1\big\}P_{11}(\xi_2,s)\Big]ds_1$$
$$+\Big[O\big(x_{10}^{(2)}\big) + \big\{\eta(\xi_2-1)y_{10}^{(2)}+1\big\}P_{12}(\xi_2,s)\Big]ds_2,$$
$$e(s)\big(x_{10}^{(2)}\big)^2(\xi_2-1)^3dz_{10}^{(2)} = \left(O\big(x_{10}^{(2)}\big) + \frac{\big\{\eta(\xi_2-1)y_{10}^{(2)}+1\big\}P_{13}\big(\xi_2,z_{10}^{(2)},s\big)}{y_{10}^{(2)}}\right)ds_1$$
$$+\left(O\big(x_{10}^{(2)}\big) + \frac{\big\{\eta(\xi_2-1)y_{10}^{(2)}+1\big\}P_{14}\big(\xi_2,z_{10}^{(2)},s\big)}{y_{10}^{(2)}}\right)ds_2$$

in a neighborhood of $D_{11}^{(2)}(s) = \big\{x_{10}^{(2)}=0\big\}$, in the second coordinate system, it is written as

$$e(s)y_{11}^{(2)}(\xi_2-1)d\xi_2 = O\big(y_{11}^{(2)}\big)ds_1 + O\big(y_{11}^{(2)}\big)ds_2,$$
$$e(s)y_{11}^{(2)}(\xi_2-1)^2dx_{11}^{(2)} = \Big[O\big(y_{11}^{(2)}\big) + \big\{\eta(\xi_2-1)+x_{11}^{(2)}\big\}P_{15}\big(\xi_2,x_{11}^{(2)},s\big)\Big]ds_1$$
$$+\Big[O\big(y_{11}^{(2)}\big) + \big\{\eta(\xi_2-1)+x_{11}^{(2)}\big\}P_{16}\big(\xi_2,x_{11}^{(2)},s\big)\Big]ds_2,$$
$$e(s)y_{11}^{(2)}(\xi_2-1)^2dy_{11}^{(2)} = \frac{O\big(y_{11}^{(2)}\big)}{x_{11}^{(2)}}ds_1 + \frac{O\big(y_{11}^{(2)}\big)}{x_{11}^{(2)}}ds_2,$$
$$e(s)\big(y_{11}^{(2)}\big)^2(\xi_2-1)^3dz_{11}^{(2)} = O\big(y_{11}^{(2)}\big)ds_1 + \frac{O\big(y_{11}^{(2)}\big)}{x_{11}^{(2)}}ds_2$$

in a neighborhood of $D_{11}^{(2)}(s) = \big\{y_{11}^{(2)}=0\big\}$, and in the third coordinate system it is written as

$$e(s)z_{12}^{(2)}(\xi_2-1)d\xi_2 = \frac{O\big(z_{12}^{(2)}\big)}{y_{12}^{(2)}}ds_1 + \frac{O\big(z_{12}^{(2)}\big)}{y_{12}^{(2)}}ds_2,$$
$$e(s)\big(z_{12}^{(2)}\big)^2(\xi_2-1)^3dx_{12}^{(2)} = \left(O\big(z_{12}^{(2)}\big) + \frac{P_{17}\big(\xi_2,x_{12}^{(2)},s\big)}{y_{12}^{(2)}}\right)ds_1$$
$$+\left(O\big(z_{12}^{(2)}\big) + \frac{P_{18}\big(\xi_2,x_{12}^{(2)},s\big)}{y_{12}^{(2)}}\right)ds_2,$$
$$e(s)\big(z_{12}^{(1)}\big)^2(\xi_2-1)^3dy_{12}^{(1)} = \left(O\big(z_{12}^{(2)}\big) + \frac{P_{19}\big(\xi_2,y_{12}^{(2)},s\big)}{x_{12}^{(2)}}\right)ds_1$$
$$+\left(O\big(z_{12}^{(2)}\big) + \frac{P_{110}\big(\xi_2,y_{12}^{(2)},s\big)}{x_{12}^{(2)}}\right)ds_2,$$
$$e(s)z_{12}^{(1)}(\xi_2-1)^3dz_{12}^{(2)} = \frac{O\big(z_{12}^{(2)}\big)}{x_{12}^{(2)}y_{12}^{(2)}}ds_1 + \left(O\big(z_{12}^{(2)}\big) + \frac{P_{111}\big(\xi_2,x_{12}^{(2)},y_{12}^{(2)},s\big)}{x_{12}^{(2)}y_{12}^{(2)}}\right)ds_2$$

in a neighborhood of $D_{11}^{(2)}(s) = \{z_{12}^{(2)} = 0\}$. We can verity that the set of accessible singular points $A_{11}^{(2)}(s)$ is given by

$$A_{11}^{(2)}(s) = \Big\{\big(\xi_2, x_{10}^{(2)}, y_{10}^{(2)}, z_{10}^{(2)}\big) = \big(\xi_2, 0, -1/(\eta(\xi_2-1)), z_{10}^{(2)}\big)\Big\} \subset D_{11}^{(1)}(s).$$

The third quadratic transformation along $A_{11}^{(2)}(s)$. Here we insert a change of variables

$$\xi_2 = \xi_2, \quad x_{10}^{(2)} = x_{10}^{(2)}, \quad y_{10}^{(2)} = 1/v_{10}^{(2)}, \quad z_{10}^{(2)} = z_{10}^{(2)},$$

namely, a change of local coordinates of a neighborhood of the set $A_{11}^{(2)}(s)$. The change of variables is necessary for obtaing a symplectic coordinate system.

We next replace the points $\big(\xi_2, z_{10}^{(2)}, x_{10}^{(2)}, v_{10}^{(2)}\big) = \big(\xi_2, z_{10}^{(2)}, 0, -\eta(\xi_2-1)\big)$ with $\big(\xi_2, z_{10}^{(2)}\big) \in \boldsymbol{C}^2 \setminus \{\xi_2 = 1\}$ by $\boldsymbol{P}^1$ simultaneously. Let $\big(\xi_2, z_{10}^{(2)}, x_{10}^{(3)}, y_{10}^{(3)}\big) \in \boldsymbol{C}^4$ and $\big(\xi_2, z_{10}^{(2)}, x_{11}^{(3)}, y_{11}^{(3)}\big) \in \boldsymbol{C}^4$ be coordinate systems of $V_{11}^{(3)}(s) = Q_{A_{11}^{(2)}(s)}\big(V_{11}^{(2)}(s)\big)$ defined by

$$\begin{aligned} x_{10}^{(2)} &= x_{10}^{(3)}, & v_{10}^{(2)} &= -\eta(\xi_2-1) + x_{10}^{(3)} y_{10}^{(3)}, \\ x_{10}^{(2)} &= x_{11}^{(3)} y_{11}^{(3)}, & v_{10}^{(2)} &= -\eta(\xi_2-1) + y_{11}^{(3)}, \end{aligned}$$

then the exceptional divisor $D_{11}^{(3)}(s) = Q_{A_{11}^{(2)}(s)}(A_{11}^{(2)}(s))$ is give by

$$\{x_{10}^{(3)} = 0\} \cup \{y_{11}^{(3)} = 0\}.$$

Let us write our system in the two coordinate systems near the exceptional divisor. In the first coordinate, it is written as

$$\begin{aligned} e(s)x_{10}^{(3)}(\xi_2-1)d\xi_2 &= O\big(x_{10}^{(3)}\big)ds_1 + O\big(x_{10}^{(3)}\big)ds_2, \\ e(s)x_{10}^{(3)}(\xi_2-1)^3 dz_{10}^{(2)} &= \left(\frac{O\big(x_{10}^{(3)}\big)}{-\eta(\xi_2-1) + x_{10}^{(3)} y_{10}^{(3)}}\right)ds_1 \\ &\quad + \left(\frac{O\big(x_{10}^{(3)}\big) + \big(\alpha_1 - y_{10}^{(3)}\big)P_{11}(\xi_2, s)}{-\eta(\xi_2-1) + x_{10}^{(3)} y_{10}^{(3)}}\right)ds_2, \\ e(s)x_{10}^{(3)}(\xi_2-1)dx_{10}^{(3)} &= O\big(x_{10}^{(3)}\big)ds_1 + O\big(x_{10}^{(3)}\big)ds_2, \\ e(s)x_{10}^{(3)}(\xi_2-1)^2 dy_{10}^{(3)} &= \Big\{O\big(x_{10}^{(3)}\big) + \big(\alpha_1 - y_{10}^{(3)}\big)P_{12}(\xi_2, s)\Big\}ds_1 \\ &\quad + \Big\{O\big(x_{10}^{(3)}\big) + \big(\alpha_1 - y_{10}^{(3)}\big)P_{13}(\xi_2, s)\Big\}ds_2 \end{aligned}$$

in a neighborhood of $D_{11}^{(3)}(s) = \{x_{10}^{(3)} = 0\}$, in the second coordinate system, it is written as

$$
\begin{aligned}
e(s)y_{11}^{(3)}(\xi_2-1)d\xi_2 &= O\big(y_{11}^{(3)}\big)ds_1 + O\big(y_{11}^{(3)}\big)ds_2,\\
e(s)y_{11}^{(3)}(\xi_2-1)^2dz_{10}^{(2)} &= \left(\frac{O\big(y_{11}^{(3)}\big)}{\big\{-\eta(\xi_2-1)+y_{11}^{(3)}\big\}x_{11}^{(3)}}\right)ds_1\\
&\quad + \left(\frac{O\big(y_{11}^{(3)}\big)+\big(\alpha_1 x_{11}^{(3)}-1\big)P_{14}(\xi_2,s)}{\big\{-\eta(\xi_2-1)+y_{11}^{(3)}\big\}\big(x_{11}^{(3)}\big)^2}\right)ds_2,\\
e(s)y_{11}^{(3)}(\xi_2-1)^2dx_{11}^{(3)} &= \Big\{O\big(y_{11}^{(3)}\big)+\big(\alpha_1 x_{11}^{(3)}-1\big)P_{15}(\xi_2,s)\Big\}ds_1\\
&\quad + \Big\{O\big(y_{11}^{(3)}\big)+\big(\alpha_1 x_{11}^{(3)}-1\big)P_{16}(\xi_2,s)\Big\}ds_2,\\
e(s)\big(y_{11}^{(3)}\big)^2(\xi_2-1)^2dy_{11}^{(3)} &= \frac{O\big(y_{11}^{(3)}\big)}{x_{11}^{(3)}}ds_1 + \frac{O\big(y_{11}^{(3)}\big)}{x_{11}^{(3)}}ds_2
\end{aligned}
$$

in a neighborhood of $D_{11}^{(3)}(s) = \big\{y_{11}^{(3)} = 0\big\}$. We can verity that the set of accessible singular points $A_{11}^{(3)}(s)$ is given by

$$
A_{11}^{(3)}(s) = \Big\{\big(\xi_2, z_{10}^{(2)}, x_{10}^{(3)}, y_{10}^{(3)}\big) = \big(\xi_2, z_{10}^{(2)}, 0, \alpha_1\big)\Big\} \subset D_{11}^{(3)}(s).
$$

The fourth quadratic transformation along $A_{11}^{(3)}(s)$. We next replace the points $\big(\xi_2, z_{10}^{(2)}, x_{10}^{(3)}, y_{10}^{(3)}\big) = \big(\xi_2, z_{10}^{(2)}, 0, \alpha_1\big)$ with $\big(\xi_2, z_{10}^{(2)}\big) \in \boldsymbol{C}^2 \setminus \{\xi_2 = 1\}$ by $\boldsymbol{P}^1$ simultaneously. Let $\big(\xi_2, z_{10}^{(2)}, x_{10}^{(4)}, y_{10}^{(4)}\big) \in \boldsymbol{C}^4$ and $\big(\xi_2, z_{10}^{(2)}, x_{11}^{(4)}, y_{11}^{(4)}\big) \in \boldsymbol{C}^4$ be coordinate systems of $V_{11}^{(4)}(s) = Q_{A_{11}^{(3)}(s)}\big(V_{11}^{(3)}(s)\big)$ defined by

$$
\begin{aligned}
x_{10}^{(3)} &= x_{10}^{(4)}, & y_{10}^{(3)} &= \alpha_1 + x_{10}^{(4)}y_{10}^{(4)},\\
x_{10}^{(3)} &= x_{11}^{(4)}y_{11}^{(4)}, & y_{10}^{(3)} &= \alpha_1 + y_{11}^{(4)},
\end{aligned}
$$

then the exceptional divisor $D_{11}^{(4)}(s) = Q_{A_{11}^{(3)}(s)}\big(A_{11}^{(3)}(s)\big)$ is given by

$$
\big\{x_{10}^{(4)} = 0\big\} \cup \big\{y_{11}^{(4)} = 0\big\}.
$$

We can verify that the differential system is written as

$$
\begin{aligned}
e(s)(\xi_2-1)d\xi_2 &= \sum_{i=1,2} P'_{1i}\big(\xi_2, z_{10}^{(2)}, x_{10}^{(4)}, y_{10}^{(4)}, s\big)ds_i,\\
e(s)(\xi_2-1)^3dz_{10}^{(2)} &= \sum_{i=1,2} \frac{Q'_{1i}\big(\xi_2, z_{10}^{(2)}, x_{10}^{(4)}, y_{10}^{(4)}, s\big)}{O\big(x_{10}^{(4)}\big)-\eta(\xi_2-1)}ds_i,
\end{aligned}
$$

$$e(s)(\xi_2-1)dx_{10}^{(4)} = \sum_{i=1,2} X'_{1i}\big(\xi_2, z_{10}^{(2)}, x_{10}^{(4)}, y_{10}^{(4)}, s\big)ds_i,$$

$$e(s)(\xi_2-1)^2dy_{10}^{(4)} = \sum_{i=1,2} Y'_{1i}\big(\xi_2, z_{10}^{(2)}, x_{10}^{(4)}, y_{10}^{(4)}, s\big)ds_i$$

in the coordinates $\big(\xi_2, z_{10}^{(2)}, x_{10}^{(4)}, y_{10}^{(4)}\big)$ and s where $P'_{1i}, Q'_{1i}, X'_{1i}, Y'_{1i}$, $i=1,2$ are certain polynomials of $\xi_2, z_{10}^{(2)}, x_{10}^{(4)}, y_{10}^{(4)}$ and s. Therefore the differential system in the cordinates $\big(\xi_2, z_{10}^{(2)}, x_{10}^{(4)}, y_{10}^{(4)}\big)$ is holomorphic in a neighborhood of $\big\{x_{10}^{(4)}=0\big\}$ except for $\xi_2=1$. On the other hand, we can verify that

$$e(s)(\xi_2-1)d\xi_2 = \sum_{i=1,2} P''_{1i}\big(\xi_2, z_{10}^{(2)}, x_{11}^{(4)}, y_{11}^{(4)}, s\big)ds_i,$$

$$e(s)(\xi_2-1)^3dz_{10}^{(2)} = \sum_{i=1,2} \frac{Q''_{1i}\big(\xi_2, z_{10}^{(2)}, x_{11}^{(4)}, y_{11}^{(4)}, s\big)}{O\big(y_{11}^{(4)}\big)-\eta(\xi_2-1)}ds_i,$$

$$e(s)(\xi_2-1)^2dx_{11}^{(4)} = \sum_{i=1,2} X''_{1i}\big(\xi_2, z_{10}^{(2)}, x_{11}^{(4)}, y_{11}^{(4)}, s\big)ds_i,$$

$$e(s)(\xi_2-1)^2dy_{11}^{(4)} = \left(Y''_{11}\big(\xi_2, z_{10}^{(2)}, x_{11}^{(4)}, y_{11}^{(4)}, s\big) + \frac{C_{11}(s)(\xi_2-1)^2}{x_{11}^{(4)}}\right)ds_1, + \left(Y''_{12}\big(\xi_2, z_{10}^{(2)}, x_{11}^{(4)}, y_{11}^{(4)}, s\big) + \frac{(\xi_2-1)^2P_{11}(\xi_2)}{x_{11}^{(4)}}\right)ds_2$$

in a neighborhood of $\big(\xi_2, z_{10}^{(2)}, x_{11}^{(4)}, y_{11}^{(4)}\big) = \big(\xi_2, z_{10}^{(2)}, 0, 0\big)$ with $\xi_2 \neq 1$, $P''_{1i}, Q''_{1i}, X''_{1i}, Y''_{1i}$, $i=1,2$ are certain polynomials of $\xi_2, z_{10}^{(2)}, x_{11}^{(4)}, y_{11}^{(4)}$ and s, which shows that the points $\big(\xi_2, z_{10}^{(2)}, x_{11}^{(4)}, y_{11}^{(4)}\big) = \big(\xi_2, z_{10}^{(2)}, 0, 0\big)$ with $\xi_2 \neq 1$ are inaccessible singular points.

Thus we have obtained a coordinate system $\big(\xi_2, z_{10}^{(2)}, x_{10}^{(4)}, y_{10}^{(4)}\big) \in \boldsymbol{C}^4$ which separates the solution surfaces passing through $A_1(s) \cap W_0 = A_1(s) \cap W_{01}$ with $\xi_2 \neq 1$. It is related to the original coordinate system (q_1, q_2, p_1, p_2) by

$$q_1 = x_{10}^{(4)}, \quad q_2 = \xi_2, \quad p_1 = -\frac{\eta(\xi_2-1)}{\big(x_{10}^{(4)}\big)^2} + \frac{\alpha_1}{x_{10}^{(4)}} + y_{10}^{(4)},$$

$$p_2 = \frac{\eta}{x_{10}^{(4)}} - \eta(\xi_2-1)z_{10}^{(2)} - \frac{\alpha_1}{\xi_2-1} + \left(\alpha_1 z_{10}^{(2)} - \frac{y_{10}^{(4)}}{\xi_2-1}\right)x_{10}^{(4)} + z_{10}^{(2)}y_{10}^{(4)}\big(x_{10}^{(4)}\big)^2.$$

Auxiliary transformation. We notice that the coordinate system $\big(\xi_2, z_{10}^{(2)}, x_{10}^{(4)}, y_{10}^{(4)}\big)$ is not symplectic and the form of the differential system is very complicated. Therefore we proceed to finding another good coordinate system. For this purpose, let us calculate the 2-form $dq_1 \wedge dp_1 + dq_2 \wedge dp_2$ in this coordinate system:

$$
\begin{aligned}
&dq_1 \wedge dp_1 + dq_2 \wedge dp_2 \\
&\quad = dx_{10}^{(4)} \wedge dy_{10}^{(4)} + \Big\{ -\eta(\xi_2 - 1) + x_{10}^{(4)} \big(\alpha_1 + x_{10}^{(4)} y_{10}^{(4)}\big) \Big\} d\xi_2 \wedge dz_{10}^{(2)} \\
&\qquad + \left(\alpha_1 z_{10}^{(2)} + 2 z_{10}^{(2)} x_{10}^{(4)} y_{10}^{(4)} - \frac{y_{10}^{(4)}}{\xi_2 - 1} \right) d\xi_2 \wedge dx_{10}^{(4)} \\
&\qquad + \left(z_{10}^{(2)} \big(x_{10}^{(4)}\big)^2 - \frac{x_{10}^{(4)}}{\xi_2 - 1} \right) d\xi_2 \wedge dy_{10}^{(4)} \\
&\quad = dx_{10}^{(4)} \wedge dy_{10}^{(4)} + d\xi_2 \wedge d\left\{ \Big\{ -\eta(\xi_2 - 1) + x_{10}^{(4)} \big(\alpha_1 + x_{10}^{(4)} y_{10}^{(4)}\big) \Big\} z_{10}^{(2)} - \frac{x_{10}^{(4)} y_{10}^{(4)}}{\xi_2 - 1} \right\} \\
&\quad = dx_{10}^{(4)} \wedge dy_{10}^{(4)} + d\xi_2 \wedge d\left\{ \Big\{ -\eta(\xi_2 - 1) + x_{10}^{(4)} \big(\alpha_1 + x_{10}^{(4)} y_{10}^{(4)}\big) \Big\} z_{10}^{(2)} \right. \\
&\qquad\qquad\qquad\qquad\qquad \left. - \frac{x_{10}^{(4)} y_{10}^{(4)}}{\xi_2 - 1} - \frac{\alpha_1}{\xi_2 - 1} \right\}.
\end{aligned}
$$

Therefore, setting

$$
w_{10}^{(2)} = \Big\{ -\eta(\xi_2 - 1) + x_{10}^{(4)} \big(\alpha_1 + x_{10}^{(4)} y_{10}^{(4)}\big) \Big\} z_{10}^{(2)} - \frac{x_{10}^{(4)} y_{10}^{(4)}}{\xi_2 - 1} - \frac{\alpha_1}{\xi_2 - 1},
$$

we have symplectic coordinates $\big(\xi_2, w_{10}^{(2)}, x_{10}^{(4)}, y_{10}^{(4)}\big)$. Futhermore, in this coordinate system, we can verify that our system is written as

$$
\begin{aligned}
e(s) d\xi_2 &= \sum_{i=1,2} P_{1i}\big(\xi_2, w_{10}^{(2)}, x_{10}^{(4)}, y_{10}^{(4)}, s\big) ds_i, \\
e(s) dw_{10}^{(2)} &= \sum_{i=1,2} Q_{1i}\big(\xi_2, w_{10}^{(2)}, x_{10}^{(4)}, y_{10}^{(4)}, s\big) ds_i, \\
e(s) dx_{10}^{(4)} &= \sum_{i=1,2} X_{1i}\big(\xi_2, w_{10}^{(2)}, x_{10}^{(4)}, y_{10}^{(4)}, s\big) ds_i, \\
e(s) dy_{10}^{(4)} &= \sum_{i=1,2} Y_{1i}\big(\xi_2, w_{10}^{(2)}, x_{10}^{(4)}, y_{10}^{(4)}, s\big) ds_i,
\end{aligned}
$$

where $P_{1i}, Q_{1i}, X_{1i}, Y_{1i}$, $i = 1, 2$ are certain polynomials of $\xi_2, w_{10}^{(2)}, x_{10}^{(4)}, y_{10}^{(4)}$ and s. This means that the differential system has no singular points in $\big(\xi_2, w_{10}^{(2)}, x_{10}^{(4)}, y_{10}^{(4)}, s\big)$-space $\boldsymbol{C}^4 \times B_{1112}$. We remark that the system has no singularity on $\xi_2 = 1$. We write this affine symplectic coordinate system as $\big(q_1^{11}, q_2^{11}, p_1^{11}, p_2^{11}\big)$, namely

$$
q_1^{11} = x_{10}^{(4)}, \quad q_2^{11} = \xi_2, \quad p_1^{11} = y_{10}^{(4)}, \quad p_2^{11} = w_{10}^{(2)}.
$$

3.4.2. Coordinate system for $A_1(s) \cap W_2$ of $\mathscr{H}_{1112}$.

The first quadratic transformation along $A_1(s) \cap W_2$. Note that $A_1(s) \cap W_2 \subset W_{21}$ and

$$A_1(s) \cap W_2 = \bigl\{(\xi_0, \xi_1, \eta_{20}, \eta_{22}) \in W_{21} \simeq \boldsymbol{C}^4 \mid \xi_1 \in \boldsymbol{C}, \xi_0 = \eta_{20} = \eta_{22} = 0\bigr\}.$$

We replace every point $(\xi_0, \xi_1, \eta_{20}, \eta_{22}) = (\xi_0, 0, 0, 0)$ with $\xi_0 \in \boldsymbol{C}$ by $\boldsymbol{P}^2$ simultaneously. Let $\bigl(\xi_0, X_{10}^{(1)}, Y_{10}^{(1)}, Z_{10}^{(1)}\bigr) \in \boldsymbol{C}^4$, $\bigl(\xi_0, X_{11}^{(1)}, Y_{11}^{(1)}, Z_{11}^{(1)}\bigr) \in \boldsymbol{C}^4$ and $\bigl(\xi_0, X_{12}^{(1)}, Y_{12}^{(1)}, Z_{12}^{(1)}\bigr) \in \boldsymbol{C}^4$ be coordinate systems of $V_{12}^{(1)}(s) = Q_{A_1(s) \cap W_2}(W_{21} \times s)$ defined by

$$\begin{aligned}
\xi_1 &= X_{10}^{(1)}, & \eta_{20} &= X_{10}^{(1)} Y_{10}^{(1)}, & \eta_{22} &= X_{10}^{(1)} Z_{10}^{(1)}, \\
\xi_1 &= X_{11}^{(1)} Y_{11}^{(1)}, & \eta_{20} &= Y_{11}^{(1)}, & \eta_{22} &= Y_{11}^{(1)} Z_{11}^{(1)}, \\
\xi_1 &= X_{12}^{(1)} Z_{12}^{(1)}, & \eta_{20} &= Y_{12}^{(1)} Z_{12}^{(1)}, & \eta_{22} &= Z_{12}^{(1)},
\end{aligned}$$

then the exceptional divisor $D_{12}^{(1)}(s) = Q_{A_1(s) \cap W_2}(A_1(s) \cap W_2)$ is given by

$$\bigl\{X_{10}^{(1)} = 0\bigr\} \cup \bigl\{Y_{11}^{(1)} = 0\bigr\} \cup \bigl\{Z_{12}^{(1)} = 0\bigr\}$$

and the set of accessible singular points $A_{12}^{(1)}(s)$ is given by

$$A_{12}^{(1)}(s) = \Bigl\{\bigl(\xi_0, X_{10}^{(1)}, Y_{10}^{(1)}, Z_{10}^{(1)}\bigr) = (\xi_0, 0, 0, 1/(\xi_0 - 1)\Bigr\} \subset D_{12}^{(1)}(s).$$

The second quadratic transformation along $A_{12}^{(1)}(s)$. We next replace the points $\bigl(\xi_0, X_{10}^{(1)}, Y_{10}^{(1)}, Z_{10}^{(1)}\bigr) = (\xi_0, 0, 0, 1/(\xi_0 - 1))$ with $\xi_0 \in \boldsymbol{C} \setminus \{\xi_0 = 1\}$ by $\boldsymbol{P}^2$ simultaneously. Note that we assume $\xi_0 \neq 1$ for a while. Let $\bigl(X_{10}^{(2)}, Y_{10}^{(2)}, Z_{10}^{(2)}\bigr) \in \boldsymbol{C}^3$, $\bigl(X_{11}^{(2)}, Y_{11}^{(2)}, Z_{11}^{(2)}\bigr) \in \boldsymbol{C}^3$ and $\bigl(X_{12}^{(2)}, Y_{12}^{(2)}, Z_{12}^{(2)}\bigr) \in \boldsymbol{C}^3$ be coordinate systems of $V_{12}^{(2)}(s) = Q_{A_{12}^{(1)}(s)}\bigl(V_{12}^{(1)}(s)\bigr)$ defined by

$$\begin{aligned}
X_{10}^{(1)} &= X_{10}^{(2)}, & Y_{10}^{(1)} &= X_{10}^{(2)} Y_{10}^{(2)}, & Z_{10}^{(1)} &= 1/(\xi_0 - 1) + X_{10}^{(2)} Z_{10}^{(2)}, \\
X_{10}^{(1)} &= X_{11}^{(2)} Y_{11}^{(2)}, & Y_{10}^{(1)} &= Y_{11}^{(2)}, & Z_{10}^{(1)} &= 1/(\xi_0 - 1) + Y_{11}^{(2)} Z_{11}^{(2)}, \\
X_{10}^{(1)} &= X_{12}^{(2)} Z_{12}^{(2)}, & Y_{10}^{(1)} &= Y_{12}^{(2)} Z_{12}^{(2)}, & Z_{10}^{(1)} &= 1/(\xi_0 - 1) + Z_{12}^{(2)},
\end{aligned}$$

then the exceptional divisor $D_{12}^{(2)}(s) = Q_{A_{12}^{(1)}(s)}\bigl(A_{12}^{(1)}(s)\bigr)$ is given by

$$\bigl\{X_{10}^{(2)} = 0\bigr\} \cup \bigl\{Y_{11}^{(2)} = 0\bigr\} \cup \bigl\{Z_{12}^{(2)} = 0\bigr\}$$

and the set of accessible singular points $A_{12}^{(2)}(s)$ is given by

$$A_{12}^{(2)}(s)=\left\{\left(\xi_0,X_{10}^{(2)},Y_{10}^{(2)},Z_{10}^{(2)}\right)=\left(\xi_0,0,1/(\eta(\xi_0-1)),Z_{10}^{(2)}\right)\right\}\subset D_{12}^{(2)}(s).$$

The third quadratic transformation along $\boldsymbol{A_{12}^{(2)}(s)}$. Here we insert a change of variables

$$\xi_0=\xi_0,\quad X_{10}^{(2)}=X_{10}^{(2)},\quad Y_{10}^{(2)}=1/V_{10}^{(2)},\quad Z_{10}^{(1)}=Z_{10}^{(1)}$$

namely, a change of local coordinates of a neighborhood of the set $A_{12}^{(2)}(s)$. We next replace the points $\left(\xi_0,Z_{10}^{(2)},X_{10}^{(2)},V_{10}^{(2)}\right)=\left(\xi_0,Z_{10}^{(2)},0,\eta(\xi_0-1)\right)$ with $\left(\xi_0,Z_{10}^{(2)}\right)\in \boldsymbol{C}^2\setminus\{\xi_0=1\}$ by $\boldsymbol{P}^1$ simultaneously. Let $\left(\xi_0,Z_{10}^{(2)},X_{10}^{(3)},Y_{10}^{(3)}\right)\in\boldsymbol{C}^4$ and $\left(\xi_0,Z_{10}^{(2)},X_{11}^{(3)},Y_{11}^{(3)}\right)\in\boldsymbol{C}^4$ be coordinate systems of $V_{12}^{(3)}(s)=Q_{A_{12}^{(2)}(s)}\left(V_{12}^{(2)}(s)\right)$ defined by

$$\begin{aligned}X_{10}^{(2)}&=X_{10}^{(3)}, & V_{10}^{(2)}&=\eta(\xi_0-1)+X_{10}^{(3)}Y_{10}^{(3)},\\ X_{10}^{(2)}&=X_{11}^{(3)}Y_{11}^{(3)}, & V_{10}^{(2)}&=\eta(\xi_0-1)+Y_{11}^{(3)},\end{aligned}$$

then the exceptional divisor $D_{12}^{(3)}(s)=Q_{A_{12}^{(2)}(s)}\left(A_{12}^{(2)}(s)\right)$ is given by

$$\left\{X_{10}^{(3)}=0\right\}\cup\left\{Y_{11}^{(3)}=0\right\}$$

and the set of accessible singular points $A_{12}^{(3)}(s)$ is given by

$$A_{12}^{(3)}(s)=\left\{\left(\xi_0,Z_{10}^{(2)},X_{10}^{(3)},Y_{10}^{(3)}\right)=\left(\xi_0,Z_{10}^{(2)},0,\alpha_1\right)\right\}\subset D_{12}^{(3)}(s).$$

The fourth quadratic transformation along $\boldsymbol{A_{12}^{(3)}(s)}$. We next replace the points $\left(\xi_0,Z_{10}^{(2)},X_{10}^{(3)},Y_{10}^{(3)}\right)=\left(\xi_0,Z_{10}^{(2)},0,\alpha_1\right)$ with $\left(\xi_0,Z_{10}^{(2)}\right)\in\boldsymbol{C}^2\setminus\{\xi_0=1\}$ by $\boldsymbol{P}^1$ simultaneously. Let $\left(\xi_2,Z_{10}^{(2)},X_{10}^{(4)},Y_{10}^{(4)}\right)\in\boldsymbol{C}^4$ and $\left(\xi_2,Z_{10}^{(2)},X_{11}^{(4)},Y_{11}^{(4)}\right)\in\boldsymbol{C}^4$ be coordinate systems of $V_{12}^{(4)}(s)=Q_{A_{12}^{(3)}(s)}\left(V_{12}^{(3)}(s)\right)$ defined by

$$\begin{aligned}X_{10}^{(3)}&=X_{10}^{(4)}, & Y_{10}^{(3)}&=\alpha_1+X_{10}^{(4)}Y_{10}^{(4)},\\ X_{10}^{(3)}&=X_{11}^{(4)}Y_{11}^{(4)}, & Y_{10}^{(3)}&=\alpha_1+Y_{11}^{(4)},\end{aligned}$$

then the exceptional divisor $D_{12}^{(4)}(s)=Q_{A_{12}^{(3)}(s)}\left(A_{12}^{(3)}(s)\right)$ is given by

$$\left\{X_{10}^{(4)}=0\right\}\cup\left\{Y_{11}^{(4)}=0\right\}.$$

We see that, in the $\left(\xi_0,Z_{10}^{(2)},X_{10}^{(4)},Y_{10}^{(4)},s\right)$-space $\boldsymbol{C}^4\times B$, the differential system is holomorphic in a neighborhood of $\left\{X_{10}^{(4)}=0\right\}$ except for $\xi_0=1$, moreover, the points $\left(\xi_0,Z_{10}^{(2)},X_{11}^{(4)},Y_{11}^{(4)}\right)=\left(\xi_0,Z_{10}^{(2)},0,0\right)$ with $\xi_0\neq 1$ are inaccessible.

Thus we have obtained a coordinate system $\left(\xi_0, Z_{10}^{(2)}, X_{10}^{(4)}, Y_{10}^{(4)}\right) \in \boldsymbol{C}^4$ which separates the solution surfaces passing through $A_1(s) \cap W_2 = A_1(s) \cap W_{21}$ with $\xi_0 \neq 1$.

Auxiliary transformation. Setting

$$W_{10}^{(2)} = \left\{\eta(\xi_0 - 1) + X_{10}^{(4)}\left(\alpha_1 + X_{10}^{(4)} Y_{10}^{(4)}\right)\right\} Z_{10}^{(2)} - \frac{X_{10}^{(4)} Y_{10}^{(4)}}{\xi_0 - 1} - \frac{\alpha_1}{\xi_0 - 1},$$

we have symplectic coordinates $\left(\xi_0, W_{10}^{(2)}, X_{10}^{(4)}, Y_{10}^{(4)}\right)$ and so we write them as

$$q_1^{12} = X_{10}^{(4)}, \quad q_2^{12} = \xi_0, \quad p_1^{12} = Y_{10}^{(4)}, \quad p_2^{12} = W_{10}^{(2)}.$$

We notice that the differential system in this coordinate system has no singularity on $q_2^{12} = \xi_0 = 1$.

3.4.3. Coordinate systems for $A_1(s)$ of $\mathscr{H}_{1112}$.

Now we summarize the results obtained in **3.4.1** and **3.4.2**. The set $A_1(s) = (A_1(s) \cap W_0) \cup (A_1(s) \cap W_2) \subset D_{1112} \times s$ is a component of accessible singular points. Let $Q_{A_1(s)}$ be the quadratic transformation determined by $Q_{A_1(s) \cap W_0}$ and $Q_{A_1(s) \cap W_2}$. Then the set $D_1^{(1)}(s) := D_{11}^{(1)}(s) \cup D_{12}^{(1)}(s)$ is the exceptional divisor $Q_{A_1(s)}(A_1(s))$ and $A_1^{(1)}(s) := A_{11}^{(1)}(s) \cup A_{11}^{(1)}(s)$ is the set of accessible singular points. In the same way, for $n = 2, 3, 4$, let $Q_{A_1^{(n-1)}(s)}$ be the quadratic transformation determined by $Q_{A_{11}^{(n-1)}(s)}$ and $Q_{A_{12}^{(n-1)}(s)}$, and set $D_1^{(n)}(s) := D_{11}^{(n)}(s) \cup D_{12}^{(n)}(s)$, $A_1^{(n)}(s) := A_{11}^{(n)}(s) \cup A_{12}^{(n)}(s)$. Then $D_1^{(n)}(s) = Q_{A_1^{(n-1)}(s)}(A_1^{(n-1)}(s))$ and $A_1^{(n)}(s) \subset D_1^{(n)}(s)$, $n = 2, 3$ are the sets of accessible singular points.

The set $D_1^{(4)}(s) \setminus D_1^{(3)}(s)$ is a parameter space which separates the solution surfaces passing through $A_1(s)$, where $D_1^{(3)}(s)$ also denotes the proper image of itself by $Q_{A_1^{(3)}(s)}$. The neighborhood of $D_1^{(4)}(s) \setminus D_1^{(3)}(s)$ is covered by two affine charts $\boldsymbol{C}^4$ whose coordinate systems are $(q_1^{11}, q_2^{11}, p_1^{11}, p_2^{11})$ and $(q_1^{12}, q_2^{12}, p_1^{12}, p_2^{12})$. Note that

$$D_1^{(4)}(s) \setminus D_1^{(3)}(s) = \left\{q_1^{11} = 0\right\} \cup \left\{q_1^{12} = 0\right\}.$$

We remark that $D_{1112} \times s$ and $D_1^{(n)}(s)$, $n = 1, 2$ denoting their proper images by quadratic transformations are also inaccessible singular points which may include the points of the so-called vertical leaves.

3.5. Construction of spaces of initial conditions.

Here we explain how to construct spaces of initial conditions $E_J(s)$ for every J.

Let $A_k(s) \subset D_J \times s$, $s \in B_j$ be the components of the accessible singular points of the extended differential system $\mathscr{H}_J^{(0)}$ on $\overline{\Sigma}_{\nu_J} \times s$. For each $A_k(s)$, we have made a sequence of quadratic transformations and defined the divisors $D_k^{(n)}(s)$ $(n = 1, 2, \ldots, 2n_k)$.

Let Φ_s be the composition of all quadratic transformations and let

$$\overline{E}_J(s) = \Phi_s\left(\overline{\Sigma}_{\nu_J} \times s\right).$$

Then we obtain the spaces of initial condisions $E_J(s)$ as

$$E_J(s) = \overline{E}_J(s) \setminus \left((D_J \times s) \cup \bigcup_k \left(\bigcup_{n=1}^{2n_k-1} D_k^{(n)}(s) \right) \right),$$

which is covered by $2|J|+3$ affine charts: $W_{i0}(\simeq \boldsymbol{C}^4) \ni (q_1^i, q_2^i, p_1^i, p_2^i)$, $i = 0,1,2$ and $V^{k1}(\simeq \boldsymbol{C}^4) \ni (q_1^{k1}, q_2^{k1}, p_1^{k1}, p_2^{k1})$, $V^{k2}(\simeq \boldsymbol{C}^4) \ni (q_1^{k2}, q_2^{k2}, p_1^{k2}, p_2^{k2})$ for $k = 0, \ldots, \infty$. Here W_{00} is the original chart whose coordinate system is (q_1, q_2, p_1, p_2).

By following the above calculations, we can obtain the pacting relations given in our theorems. The other assertions are immediate consequences of these relations.

The following figures give some informations on the processes of quadratic transformations.

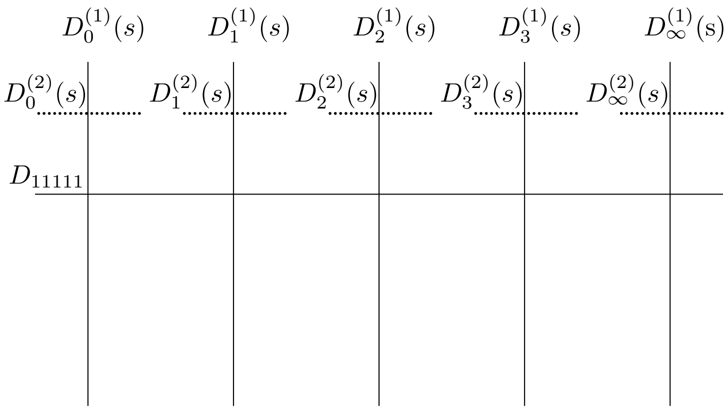

Figure 1. $J = 11111$.

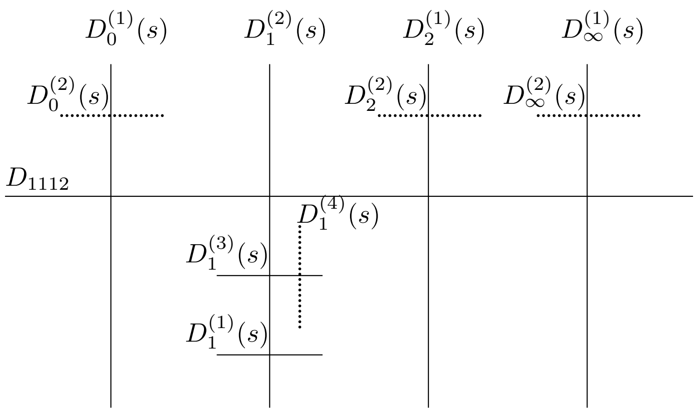

Figure 2. $J = 1112$.

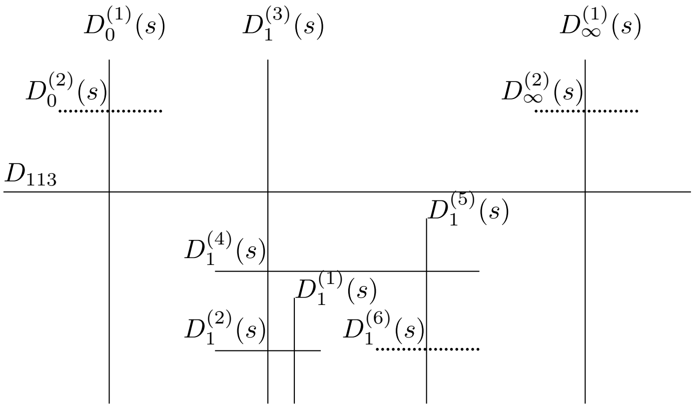

Figure 3. $J = 113$.

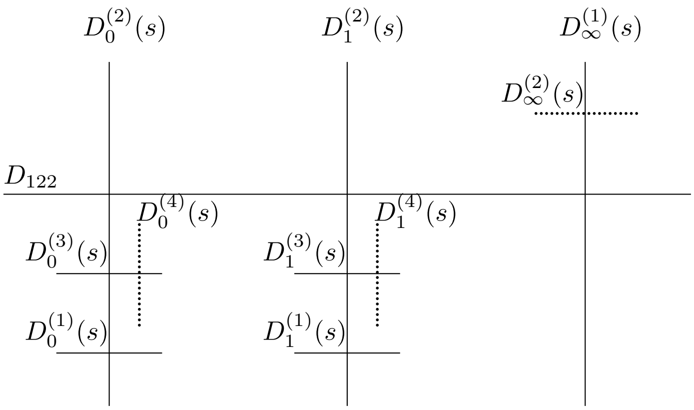

Figure 4. $J = 122$.

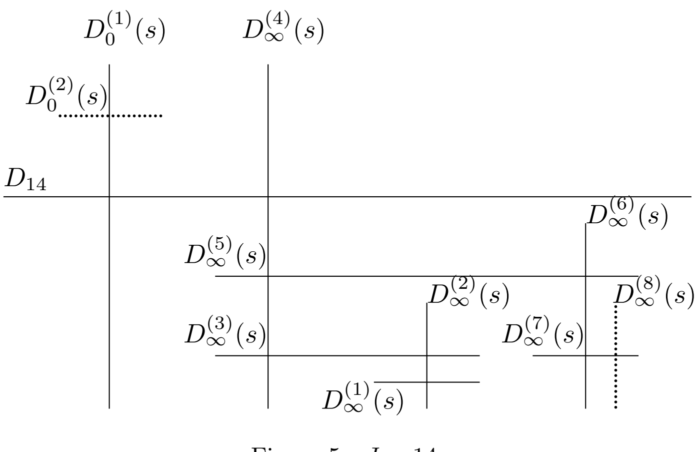

Figure 5. $J = 14$.

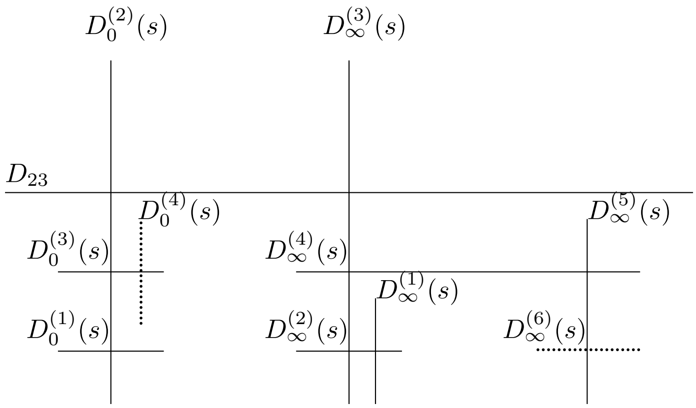

Figure 6. $J = 23$.

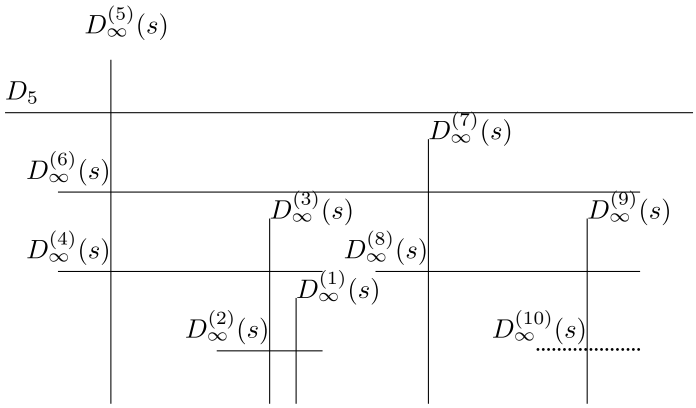

Figure 7. $J = 5$.

Acknowledgment. The author would like to express his sincere gratitude to Professor Kyoichi Takano and Professor Hironobu Kimura who gave helpful advice and encouragement.

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