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Functional equations of prehomogeneous zeta functions and intertwining operators

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Abstract. We establish a relation between the gamma matrices of the functional equations satisfied by zeta functions associated with prehomogeneous vector spaces and certain integrals related to the intertwining operator of degenerate principal series representations of general linear groups.

Introduction.

In the present paper, we establish a relation between the gamma matrices of the functional equations satisfied by zeta functions associated with prehomogeneous vector spaces and certain integrals related to the intertwining operator of degenerate principal series representations of GL_m.

In [**7**] and [**8**] we proved that the functional equation of the zeta functions associated with a prehomogeneous vector space can be formulated as the functional equation satisfied by $O(m)$-invariant spherical functions on a certain weakly spherical homogeneous space acted on by GL_m. The spherical functions are defined to be the Poisson transforms of distributions on the weakly spherical homogeneous space which belong to a degenerate principal series representation of GL_m and the explicit form of the functional equation satisfied by them can be determined if we can calculate the images of the distributions under the intertwining operator. This gives a new method of calculating the gamma matrices of the functional equations of prehomogeneous zeta functions.

In Section 1 we formulate our main result (Theorem 1), which expresses the gamma matrix in terms of an integral of a complex power of certain polynomial obtained as a specialization of the fundamental relative invariant of the prehomogeneous vector space under consideration. Some concrete examples are discussed in Section 2. The proof of the main result is given in Section 3.

As the examples in Section 2 show, Theorem 1 often simplifies the calculation of the gamma matrices. However the explicit evaluation of the integral $I_{ij}(\lambda)$ in Theorem 1 seems quite difficult for prehomogeneous vector spaces with complicated fundamental relative invariants. Therefore our interest in the main theorem is of theoretical nature. We shall discuss some theoretical implication of our approach in the final remark in Section 3.

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The result of the present paper can be generalized to the functional equations satisfied by the p-adic local zeta functions associated with prehomogeneous vector spaces.

1. Main result.

Let m and n be positive integers with $m > n$. Let $\mathbf{H}$ be an $\boldsymbol{R}$-subgroup of $\mathbf{GL}_m$ with no $\boldsymbol{R}$-rational characters and $\mathbf{V} = \mathbf{M}_{m,n}$ the space of m by n matrices. Then the group $\mathbf{H} \times \mathbf{GL}_n$ acts linearly on the matrix space $\mathbf{V}$ by

$$\rho(h_1, h_2)(v) = h_1 v\, {}^t h_2 \quad (h_1 \in \mathbf{H},\ h_2 \in \mathbf{GL}_n,\ v \in \mathbf{V}).$$

We assume that $(\mathbf{H} \times \mathbf{GL}_n, \rho, \mathbf{V})$ is a regular prehomogeneous vector space whose singular set $\mathbf{S}$ is an $\boldsymbol{R}$-irreducible hypersurface. Then the dual triple $(\mathbf{H} \times \mathbf{GL}_n, \rho^*, \mathbf{V}^*)$ is also a regular prehomogeneous vector space whose singular set $\mathbf{S}^*$ is an $\boldsymbol{R}$-irreducible hypersurface. Denote by $f(v)$ the fundamental relative invariant of $(\mathbf{H} \times \mathbf{GL}_n, \rho, \mathbf{V})$ over $\boldsymbol{R}$ and by $f^*(v^*)$ the fundamental relative invariant of the dual prehomogeneous vector space $(\mathbf{H} \times \mathbf{GL}_n, \rho^*, \mathbf{V}^*)$ over $\boldsymbol{R}$. Namely, $f(v)$ (resp. $f^*(v^*)$) is an $\boldsymbol{R}$-irreducible polynomial on $\mathbf{V}$ (resp. $\mathbf{V}^*$) defining $\mathbf{S}$ (resp. $\mathbf{S}^*$). Put $d = \deg f(v)$ and $d^* = \deg f^*(v^*)$. In the following, we identify $\mathbf{V}^*$ with $\mathbf{V} = \mathbf{M}_{m,n}$ by the inner product $\langle v, v^* \rangle = \mathrm{tr}({}^t v v^*)$. Then we have $\rho^*(h_1, h_2)(v^*) = {}^t h_1^{-1} v^* h_2^{-1}$.

In the following, we put

$$H = \mathbf{H}(\boldsymbol{R}), \quad GL_n = \mathbf{GL}_n(\boldsymbol{R}), \quad V = V^* = \mathbf{M}_{m,n}(\boldsymbol{R}), \quad S = \mathbf{S}(\boldsymbol{R}), \quad S^* = \mathbf{S}^*(\boldsymbol{R}).$$

Let $V - S = V_1 \cup \cdots \cup V_\nu$ and $V^* - S^* = V_1^* \cup \cdots \cup V_\nu^*$ be the decomposition of $V - S$ and $V^* - S^*$ into connected components. For i, j $(1 \le i, j \le \nu)$ and $s \in \boldsymbol{C}$ with $\mathrm{Re}\, s > 0$, we put

$$|f(v)|_i^s = \begin{cases} |f(v)|^s & (v \in V_i), \\ 0 & (v \notin V_i), \end{cases} \qquad |f^*(v^*)|_j^s = \begin{cases} |f^*(v^*)|^s & (v^* \in V_j^*), \\ 0 & (v^* \notin V_j^*). \end{cases}$$

We denote also by the same symbols $|f(v)|_i^s$ and $|f^*(v^*)|_j^s$ the tempered distributions on V and V^*, respectively, defined by the analytic continuations of the functions above. We denote by $\mathscr{S}(V)$ and $\mathscr{S}(V^*)$ the spaces of rapidly decreasing functions on the real vector spaces V and V^*, respectively. For $\phi \in \mathscr{S}(V)$ and $\phi^* \in \mathscr{S}(V^*)$, we define the local zeta functions $\Phi_i(\phi; \lambda)$, $\Phi_i^*(\phi^*; \lambda)$ $(1 \le i \le \nu)$ attached to these prehomogeneous vector spaces by

$$\Phi_i(\phi; \lambda) = \int_V |f(v)|_i^{(\lambda - mn/2)/d}\, \phi(v)\, dv,$$

$$\Phi_j^*(\phi^*; \lambda) = \int_{V^*} |f^*(v^*)|_j^{(\lambda - mn/2)/d^*}\, \phi^*(v^*)\, dv^*.$$

Denote by $\hat{\phi}$ the Fourier transform of $\phi \in \mathscr{S}(V)$:

$$\hat{\phi}(v^*) = \int_V \phi(v) \exp\left(2\pi\sqrt{-1}\langle v, v^*\rangle\right) dv.$$

Then, by the general theory ([**10**], [**6**]), the following functional equation holds:

$$\Phi_i(\phi; \lambda) = \sum_{j=1}^{\nu} \gamma_{ij}(\lambda) \Phi_j^*(\hat{\phi}; -\lambda) \quad (i = 1, \ldots, \nu), \tag{1.1}$$

where $\gamma_{ij}(\lambda)$ are meromorphic functions of λ independent of ϕ and have elementary expressions in terms of the Γ function and exponential functions.

The main result of this paper is the following theorem which gives an expression of $\gamma_{ij}(\lambda)$ as an Eulerian integral (an integral of a complex power of a polynomial).

THEOREM 1. *Put* $v_0^* = \binom{0}{E_n} \in V^*$ *and, for each* j $(1 \leq j \leq \nu)$, *take a* $g_j \in \mathbf{GL}_m(\boldsymbol{R})$ *such that* $v_j^* = {}^t g_j^{-1} v_0^* \in V_j^*$ *and* $\left|f^*(v_j^*)\right| = 1$. *Then we have*

$$\gamma_{ij}(\lambda) = \pi^{-\lambda-(m-n)n/2} I_{ij}(\lambda) \prod_{k=0}^{n-1} \frac{\Gamma\left(\frac{\lambda}{2n} + \frac{m-2k}{4}\right)}{\Gamma\left(-\frac{\lambda}{2n} - \frac{m-2(n-k)}{4}\right)},$$

where

$$I_{ij}(\lambda) = \int_{\mathbf{M}_{m-n,n}(\boldsymbol{R})} \left| f\left(g_j \begin{pmatrix} x \\ E_n \end{pmatrix}\right)\right|_i^{(\lambda-mn/2)/d} dx.$$

REMARKS.

(1) The integral $I_{ij}(\lambda)$ may be divergent for any λ (cf. Section 2 Example 1). Then we understand $I_{ij}(\lambda)$ to be the value at $\alpha = 0$ of the function $I_{ij}(\alpha, \lambda)$ defined by the analytic continuation of the integral

$$I_{ij}(\alpha, \lambda) = \int_{\mathbf{M}_{m-n,n}(\boldsymbol{R})} \left| f\left(g_j \begin{pmatrix} x \\ E_n \end{pmatrix}\right)\right|_i^{(\lambda-mn/2)/d} \det\left(E_n + {}^t x x\right)^{-\alpha} dx,$$

which is absolutely convergent when $\mathrm{Re}(\lambda) > \frac{mn}{2}$ and $\mathrm{Re}(\alpha)$ is sufficiently large.

(2) As mentioned in the introduction, the integral $I_{ij}(\lambda)$ is related to an integral defining an intertwining operator of degenerate principal series representation of GL_m. The relation will be clarified in Section 3.

2. Calculation of gamma matrices.

In this section, we give some samples of the calculation of the gamma matrices $(\gamma_{ij}(\lambda))$ based on Theorem 1.

EXAMPLE 1. The case of indefinite quadratic forms.

Let m be a positive integer greater than 1 and p, q positive integers with $p + q = m$.

Denote by $\mathbf{SO}(p,q)$ the special orthogonal group of the symmetric matrix

$$E_{p,q}=\begin{pmatrix}E_p & 0\\ 0 & -E_q\end{pmatrix}.$$

The group $\mathbf{SO}(p,q)\times\mathbf{GL}_1$ acts on the vector space $\mathbf{M}_{m,1}$ by

$$\rho(h,t)\cdot v=thv\quad (h\in\mathbf{SO}(p,q),\ t\in\mathbf{GL}_1,\ v\in\mathbf{M}_{m,1})$$

and the triple $(\mathbf{SO}(p,q)\times\mathbf{GL}_1,\rho,\mathbf{M}_{m,1})$ is a regular prehomogeneous vector space with fundamental relative invariant $f(v)={}^tvE_{p,q}v$. The dual prehomogeneous vector space $(\mathbf{SO}(p,q)\times\mathbf{GL}_1,\rho^*,\mathbf{M}_{m,1})$ is given by

$$\rho^*(h,t)v^*=t^{-1}\,{}^th^{-1}v^*\quad (h\in\mathbf{SO}(p,q),\ t\in\mathbf{GL}_1,\ v^*\in\mathbf{M}_{m,1})$$

Its fundamental relative invariant is $f^*(v^*)={}^tv^*E_{p,q}v^*$. For $i,j=1,2$, we put

$$V_i=\{v\in\mathbf{M}_{m,1}(\boldsymbol{R})=\boldsymbol{R}^m\mid \operatorname{sgn}f(v)=(-1)^{i-1}\},$$
$$V_j^*=\{v^*\in\mathbf{M}_{m,1}(\boldsymbol{R})=\boldsymbol{R}^m\mid \operatorname{sgn}f^*(v^*)=(-1)^{j-1}\}.$$

Since the degree of the fundamental relative invariants is equal to 2, we have, for $\phi,\phi^*\in\mathscr{S}(\boldsymbol{R}^m)$,

$$\Phi_i(\phi;\lambda)=\int_V\left|{}^tvE_{p,q}v\right|_i^{(2\lambda-m)/4}\phi(v)\,dv,$$
$$\Phi_j^*(\phi^*;\lambda)=\int_{V^*}\left|{}^tv^*E_{p,q}v^*\right|_j^{(2\lambda-m)/4}\phi^*(v^*)\,dv^*.$$

The explicit form of the functional equation (1.1) is well-known in this case (see, e.g., Gelfand-Shilov [**2**]):

PROPOSITION 2. *For $i=1,2$, we have*

$$\Phi_i(\hat{\phi^*};\lambda)=\sum_{j=1,2}\gamma_{ij}(\lambda)\Phi_j^*(\phi^*;-\lambda),$$

where

$$\begin{pmatrix}\gamma_{11}(\lambda) & \gamma_{12}(\lambda)\\ \gamma_{21}(\lambda) & \gamma_{22}(\lambda)\end{pmatrix}=\pi^{-(\lambda+1)}\Gamma\left(\frac{2\lambda-m+4}{4}\right)\Gamma\left(\frac{2\lambda+m}{4}\right)$$
$$\times\begin{pmatrix}\sin\pi\left(\frac{p-q-2\lambda}{4}\right) & \sin\frac{p\pi}{2}\\ \sin\frac{q\pi}{2} & \sin\pi\left(\frac{q-p-2\lambda}{4}\right)\end{pmatrix}.$$

Put $v_0^* = {}^t(0, \dots, 0, 1) \in \boldsymbol{R}^m$. Then, we can take

$$g_1 = \begin{pmatrix} 0 & & 1 \\ & \iddots & \\ 1 & & 0 \end{pmatrix}, \; g_2 = 1, \; v_1^* = {}^t(1, 0, \dots, 0) \in V_1^*, \; v_2^* = v_0^* \in V_2^*.$$

Hence

$$I_{ij}(\lambda) = \begin{cases} \displaystyle\int_{\boldsymbol{R}^{m-1}} \left|1 + (x_1^2 + \cdots + x_{p-1}^2) - (x_p^2 + \cdots + x_{m-1}^2)\right|_i^{(2\lambda - m)/4} dx & (j = 1), \\ \displaystyle\int_{\boldsymbol{R}^{m-1}} \left|(x_1^2 + \cdots + x_p^2) - (x_{p+1}^2 + \cdots + x_{m-1}^2) - 1\right|_i^{(2\lambda - m)/4} dx & (j = 2). \end{cases}$$

Then we can obtain the following proposition, from which Proposition 2 follows immediately.

Proposition 3.

$$\begin{pmatrix} I_{11}(\lambda) & I_{12}(\lambda) \\ I_{21}(\lambda) & I_{22}(\lambda) \end{pmatrix} = \pi^{(m-3)/2} \cdot \Gamma\left(\frac{2 - m - 2\lambda}{4}\right) \Gamma\left(\frac{2\lambda - m + 4}{4}\right)$$
$$\times \begin{pmatrix} \sin \pi\left(\frac{p-q-2\lambda}{4}\right) & \sin \frac{p\pi}{2} \\ \sin \frac{q\pi}{2} & \sin \pi\left(\frac{q-p-2\lambda}{4}\right) \end{pmatrix}.$$

Proof. We give a proof of the formula only for $I_{i1}(\lambda)$, since the calculation of $I_{i2}(\lambda)$ is quite similar. The integral defining $I_{i1}(\lambda)$ is divergent for every $\lambda \in \boldsymbol{C}$ and we understand

$$I_{i1}(\lambda) = I_{i1}(\lambda, \alpha)|_{\alpha = 0},$$

where

$$I_{i1}(\lambda, \alpha) = \int_{\boldsymbol{R}^{m-1}} \left|1 + (x_1^2 + \cdots + x_{p-1}^2) - (x_p^2 + \cdots + x_{m-1}^2)\right|_i^{(2\lambda - m)/4}$$
$$\times \left|1 + (x_1^2 + \cdots + x_{p-1}^2) + (x_p^2 + \cdots + x_{m-1}^2)\right|^{-\alpha} dx.$$

We put

$$|x|_+^s = \begin{cases} |x|^s & (x > 0), \\ 0 & (x \le 0), \end{cases} \qquad |x|_-^s = |-x|_+^s. \tag{2.1}$$

Then we have

$$\begin{aligned} I_{i1}(\lambda,\alpha) &= \frac{\pi^{(m-1)/2}}{\Gamma\left(\frac{p-1}{2}\right)\Gamma\left(\frac{q}{2}\right)} \int_0^\infty\int_0^\infty |1+r-t|_{\pm}^{(2\lambda-m)/4} (1+r+t)^{-\alpha} r^{(p-1)/2-1} t^{q/2-1}\, dr\, dt \\ &= \frac{\pi^{(m-1)/2}}{\Gamma\left(\frac{p-1}{2}\right)\Gamma\left(\frac{q}{2}\right)} \int_0^\infty |1+r|^{(2\lambda-p+q)/4-\alpha} r^{(p-1)/2-1}\, dr \\ &\quad \times \int_0^\infty |1-t|_{\pm}^{(2\lambda-m)/4} (1+t)^{-\alpha} t^{q/2-1}\, dt. \end{aligned}$$

Here we take the $+$ sign for $i=1$ and the $-$ sign for $i=2$. By the well-known formula for the Euler B-function, we easily have

$$\int_0^\infty |1+r|^{(2\lambda-p+q)/4-\alpha} r^{(p-1)/2-1}\, dr = \frac{\Gamma\left(\alpha+\frac{-2\lambda-m+2}{4}\right)\Gamma\left(\frac{p-1}{2}\right)}{\Gamma\left(\alpha-\frac{2\lambda-p+q}{4}\right)}.$$

Moreover, from the integral expression of the Gauss hypergeometric series

$$\int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-zt)^{-\alpha}\, dt = \frac{\Gamma(\beta)\Gamma(\gamma-\beta)}{\Gamma(\gamma)} F(\alpha,\beta,\gamma;z),$$

we have

$$\begin{aligned} &\int_0^\infty |1-t|_{\pm}^{(2\lambda-m)/4}\, t^{q/2-1}(1+t)^{-\alpha}\, dt \\ &= \begin{cases} \displaystyle\int_0^1 (1-t)^{(2\lambda-m)/4} t^{q/2-1}(1+t)^{-\alpha}\, dt & (i=1) \\ \displaystyle\int_1^\infty (t-1)^{(2\lambda-m)/4} t^{q/2-1}(1+t)^{-\alpha}\, dt & (i=2) \end{cases} \\ &= \begin{cases} \dfrac{\Gamma\left(\frac{2\lambda-m+4}{4}\right)\Gamma\left(\frac{q}{2}\right)}{\Gamma\left(\frac{2\lambda-p+q+4}{4}\right)} F\left(\alpha, \dfrac{q}{2}, \dfrac{2\lambda-p+q}{4}+1; -1\right) & (i=1) \\ \dfrac{\Gamma\left(\frac{-2\lambda+p-q}{4}\right)\Gamma\left(\frac{2\lambda-m+4}{4}\right)}{\Gamma\left(1-\frac{q}{2}\right)} F\left(\alpha, \alpha-\dfrac{2\lambda-p+q}{4}, \alpha+1-\dfrac{q}{2}; -1\right) & (i=2). \end{cases} \end{aligned}$$

Since $F(0,\beta,\gamma;z)=1$, we obtain

$$\begin{aligned} I_{i1}(\lambda) &= I_{i1}(\lambda,0) \\ &= \pi^{(m-1)/2}\cdot\frac{\Gamma\left(-\frac{2\lambda+m-2}{4}\right)}{\Gamma\left(-\frac{2\lambda-p+q}{4}\right)} \times \begin{cases} \dfrac{\Gamma\left(\frac{2\lambda-m}{4}+1\right)}{\Gamma\left(\frac{2\lambda-p+q}{4}+1\right)} & (i=1), \\ \dfrac{\Gamma\left(-\frac{2\lambda-p+q}{4}\right)\Gamma\left(\frac{2\lambda-m}{4}+1\right)}{\Gamma\left(\frac{q}{2}\right)\Gamma\left(1-\frac{q}{2}\right)} & (i=2). \end{cases} \end{aligned}$$

The proposition follows immediately from this identity and the identity $\Gamma(s)\Gamma(1-s) = \pi/\sin \pi s$. $\square$

EXAMPLE 2. The case of the discriminant of binary cubic forms.

Let us consider the prehomogeneous vector space $(\mathbf{SL}_2 \times \mathbf{GL}_1, \rho, \mathbf{M}_{4,1})$, in which the group $\mathbf{SL}_2$ acts on $\mathbf{M}_{4,1}$ via the 3rd symmetric tensor representation S^3. Let $\mathbf{H} = S^3(\mathbf{SL}_2)$ be the image of $\mathbf{SL}_2$ in $\mathbf{GL}_4$. We identify the vector space $\mathbf{M}_{4,1}$ with the space of homogeneous polynomials in the variables U, V of degree 3 by

$$v = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \longmapsto x_1U^3 + x_2U^2V + x_3UV^2 + x_4V^3.$$

Then the fundamental relative invariant $f(v)$ is given by the discriminant of the corresponding cubic form:

$$f(v) = x_2^2x_3^2 + 18x_1x_2x_3x_4 - 4x_1x_3^3 - 4x_2^3x_4 - 27x_1^2x_4^2.$$

Since we have

$${}^t\mathbf{H}^{-1} = \begin{pmatrix} 1 & & & \\ & 3 & & \\ & & 3 & \\ & & & 1 \end{pmatrix}^{-1} \mathbf{H} \begin{pmatrix} 1 & & & \\ & 3 & & \\ & & 3 & \\ & & & 1 \end{pmatrix}$$

in GL_4, the fundamental relative invariant f^* of the dual prehomogeneous vector space is given by

$$\begin{aligned} f^*\bigl({}^t(y_1, y_2, y_3, y_4)\bigr) &= f\bigl({}^t(y_1, 3y_2, 3y_3, y_4)\bigr) \\ &= 3^3\bigl(3y_2^2y_3^2 + 6y_1y_2y_3y_4 - 4y_1y_3^3 - 4y_2^3y_4 - y_1^2y_4^2\bigr). \end{aligned}$$

For $i, j = 1, 2$, we put

$$\begin{aligned} V_i &= \bigl\{v \in \mathbf{M}_{4,1}(\boldsymbol{R}) = \boldsymbol{R}^4 \mid \operatorname{sgn} f(v) = (-1)^{i-1}\bigr\}, \\ V_j^* &= \bigl\{v^* \in \mathbf{M}_{4,1}(\boldsymbol{R}) = \boldsymbol{R}^4 \mid \operatorname{sgn} f^*(v^*) = (-1)^{j-1}\bigr\}. \end{aligned}$$

Since the degree of the fundamental relative invariants is equal to 4, we have, for $\phi, \phi^* \in \mathscr{S}(\boldsymbol{R}^4)$,

$$\begin{aligned} \Phi_i(\phi; \lambda) &= \int_V |f(v)|_i^{(\lambda-2)/4}\, \phi(v)\, dv, \\ \Phi_j^*(\phi^*; \lambda) &= \int_{V^*} |f^*(v^*)|_j^{(\lambda-2)/4}\, \phi^*(v^*)\, dv^*. \end{aligned}$$

In this case the explicit form of the functional equation (1.1) was calculated by Shintani ([**11**]):

Proposition 4. *For $i = 1, 2$, we have*

$$\Phi_i(\hat{\phi^*};\lambda) = \sum_{j=1,2} \gamma_{ij}(\lambda)\Phi_j^*(\phi^*;-\lambda),$$

where

$$\begin{pmatrix} \gamma_{11}(\lambda) & \gamma_{12}(\lambda) \\ \gamma_{21}(\lambda) & \gamma_{22}(\lambda) \end{pmatrix} = \frac{3^{(3\lambda+2)/2}}{2} \cdot \pi^{-(\lambda+2)}\,\Gamma\left(\frac{\lambda}{4}+\frac{1}{2}\right)^2 \Gamma\left(\frac{\lambda}{4}+\frac{1}{3}\right)\Gamma\left(\frac{\lambda}{4}+\frac{2}{3}\right)$$
$$\times \begin{pmatrix} -\sin\left(\frac{\pi\lambda}{2}\right) & \cos\left(\frac{\pi\lambda}{4}\right) \\ 3\cos\left(\frac{\pi\lambda}{4}\right) & -\sin\left(\frac{\pi\lambda}{2}\right) \end{pmatrix}.$$

Shintani's proof is quite involved and far from trivial. As we shall see below, Theorem 1 gives a more straightforward way to the proof of Proposition 4.

Proof of Proposition 4. Put

$$v_0^* = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \quad \text{and} \quad g_j = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & (-1)^{j-1} & 0 & 1 \end{pmatrix} \begin{pmatrix} (2^2 3^3)^{-1/4} & & & \\ & 1 & & \\ & & 1 & \\ & & & (2^2 3^3)^{1/4} \end{pmatrix}.$$

Then, $v_j^* = {}^t g_j^{-1} v_0^* = (2^2 3^3)^{-1/4} \cdot {}^t(0, (-1)^j, 0, 1) \in V_j^*$ and $\left|f^*(v_j^*)\right| = 1$. Hence, we have

$$I_{ij}(\lambda) = (2^2 3^3)^{(\lambda+2)/4} \int_{\boldsymbol{R}^3} \left|f\left({}^t(x_1, x_2, x_3, 1 + (-1)^{j-1}x_2)\right)\right|_i^{(\lambda-2)/4} dx_1\,dx_2\,dx_3.$$

We put $\epsilon = (-1)^{i-1}$ and $\eta = (-1)^{j-1}$. In the following, we sometimes write $|x|_{\pm 1}^s$ for the function $|x|_{\pm}^s$ defined by (2.1). Since

$$f(v) = -27x_4^2\left\{\left(x_1 - \frac{9x_2x_3x_4 - 2x_3^3}{27x_4^2}\right)^2 - \frac{2^2\left(x_3^2 - 3x_2x_4\right)^3}{3^6 x_4^4}\right\},$$

we obtain

$$I_{ij}(\lambda) = 2^{(\lambda+2)/2}3^{3\lambda/2} \int_{\boldsymbol{R}^3} |1+\eta x_2|^{(\lambda-2)/2} \left| x_1^2 - \frac{2^2\left(x_3^2 - 3x_2(1+\eta x_2)\right)^3}{3^6(1+\eta x_2)^4}\right|_{-\epsilon}^{(\lambda-2)/4}$$
$$\times\, dx_1\,dx_2\,dx_3$$

$$= 2^{(\lambda+2)/2}3^{3\lambda/2} \sum_{\tau=\pm 1} \int_{\boldsymbol{R}^3} |1+\eta x_2|^{(\lambda-2)/2} \left(\frac{2^2 \left| x_3^2 - 3x_2(1+\eta x_2) \right|_\tau^3}{3^6 (1+\eta x_2)^4} \right)^{(\lambda-2)/4+1/2}$$

$$\times \left| x_1^2 - \tau \right|_{-\epsilon}^{(\lambda-2)/4} dx_1\, dx_2\, dx_3$$

$$= 2^{\lambda+1/2} \sum_{\tau,\mu,\nu=\pm 1} \int_{\boldsymbol{R}^2} |1+\eta x_2|_\mu^{-(\lambda+2)/2} \left(3\,|x_2|_\nu\, |1+\eta x_2|_\mu \right)^{3\lambda/4+1/2}$$

$$\times \left| x_3^2 - \mu\nu \right|_\tau^{3\lambda/4} dx_2\, dx_3 \int_{\boldsymbol{R}} \left| x_1^2 - \tau \right|_{-\epsilon}^{(\lambda-2)/4} dx_1$$

$$= 2^{\lambda+1}3^{(3\lambda+2)/4} \sum_{\tau,\mu,\nu=\pm 1} \int_{\boldsymbol{R}} |1+x_2|_{\mu\nu}^{(\lambda-2)/4} |x_2|_{\eta\nu}^{3\lambda/4+1/2} dx_2$$

$$\times \int_{\boldsymbol{R}} \left| x_3^2 - \mu \right|_\tau^{3\lambda/4} dx_3 \int_{\boldsymbol{R}} \left| x_1^2 - \tau \right|_{-\epsilon}^{(\lambda-2)/4} dx_1.$$

By the formula for the Euler B-function, the integrals can be easily calculated and we have

$$\sum_{\nu=\pm 1} \int_{\boldsymbol{R}} |1+x_2|_{\mu\nu}^{(\lambda-2)/4} |x_2|_{\eta\nu}^{3\lambda/4+1/2} dx_2 = \frac{\Gamma\left(\dfrac{\lambda+2}{4}\right)\Gamma\left(\dfrac{3\lambda+6}{4}\right)}{\Gamma(\lambda+2)} \cdot c_{\mu,\eta}^{(1)}(\lambda),$$

$$\int_{\boldsymbol{R}} \left| x_3^2 - \mu \right|_\tau^{3\lambda/4} dx_3 = \sqrt{\pi} \cdot \frac{\Gamma\left(\dfrac{3\lambda+4}{4}\right)}{\Gamma\left(\dfrac{3\lambda+6}{4}\right)} \cdot c_{\tau,\mu}^{(2)}(\lambda),$$

$$\int_{\boldsymbol{R}} \left| x_1^2 - \tau \right|_{-\epsilon}^{(\lambda-2)/4} dx_1 = \sqrt{\pi} \cdot \frac{\Gamma\left(\dfrac{\lambda+2}{4}\right)}{\Gamma\left(\dfrac{\lambda+4}{4}\right)} \cdot c_{\epsilon,\tau}^{(3)}(\lambda),$$

where

$$C^{(1)}(\lambda) = \left(c_{\mu,\eta}^{(1)}(\lambda) \right)_{\mu,\eta=\pm 1} = \begin{pmatrix} \dfrac{\sin\left(\frac{\pi\lambda}{4}\right)}{\cos\left(\frac{\pi\lambda}{2}\right)} & 1 \\ 1 & \dfrac{\sin\left(\frac{\pi\lambda}{4}\right)}{\cos\left(\frac{\pi\lambda}{2}\right)} \end{pmatrix}$$

$$C^{(2)}(\lambda) = \left(c_{\tau,\mu}^{(2)}(\lambda) \right)_{\tau,\mu=\pm 1} = \begin{pmatrix} -\dfrac{1}{\cos\left(\frac{3\pi\lambda}{4}\right)} & \dfrac{\sin\left(\frac{3\pi\lambda}{4}\right)}{\cos\left(\frac{3\pi\lambda}{4}\right)} \\ 1 & 0 \end{pmatrix}$$

$$C^{(3)}(\lambda) = \left(c_{\epsilon,\tau}^{(3)}(\lambda) \right)_{\epsilon,\tau=\pm 1} = \begin{pmatrix} 1 & 0 \\ -\dfrac{1}{\sin\left(\frac{\pi\lambda}{4}\right)} & -\dfrac{\cos\left(\frac{\pi\lambda}{4}\right)}{\sin\left(\frac{\pi\lambda}{4}\right)} \end{pmatrix}.$$

The calculation above shows that

$$\begin{pmatrix} I_{11}(\lambda) & I_{12}(\lambda) \\ I_{21}(\lambda) & I_{22}(\lambda) \end{pmatrix} = 2^{\lambda+1/2}3^{(3\lambda+2)/4} \cdot \pi \cdot \frac{\Gamma\left(\frac{\lambda+2}{4}\right)^2 \Gamma\left(\frac{3\lambda+4}{4}\right)}{\Gamma(\lambda+2)\Gamma\left(\frac{\lambda+4}{4}\right)} \cdot C^{(3)}(\lambda)C^{(2)}(\lambda)C^{(1)}(\lambda).$$

Hence we have

$$\begin{aligned} &\begin{pmatrix} I_{11}(\lambda) & I_{12}(\lambda) \\ I_{21}(\lambda) & I_{22}(\lambda) \end{pmatrix} \\ &\quad = 2^{\lambda+1/2}3^{(3\lambda+2)/4} \cdot \pi \cdot \frac{\Gamma\left(\frac{\lambda+2}{4}\right)^2 \Gamma\left(\frac{3\lambda+4}{4}\right)}{\Gamma(\lambda+2)\Gamma\left(\frac{\lambda+4}{4}\right)} \cdot \begin{pmatrix} \dfrac{\sin\left(\frac{\pi\lambda}{2}\right)}{\cos\left(\frac{\pi\lambda}{2}\right)} & -\dfrac{\cos\left(\frac{\pi\lambda}{4}\right)}{\cos\left(\frac{\pi\lambda}{2}\right)} \\ -3 \cdot \dfrac{\cos\left(\frac{\pi\lambda}{4}\right)}{\cos\left(\frac{\pi\lambda}{2}\right)} & \dfrac{\sin\left(\frac{\pi\lambda}{2}\right)}{\cos\left(\frac{\pi\lambda}{2}\right)} \end{pmatrix}. \end{aligned}$$

By Theorem 1, this implies Proposition 4. We note only that the proof requires the identities

$$\begin{aligned} \Gamma(2z) &= (2\pi)^{-1/2}\, 2^{-\frac{1}{2}+2z}\Gamma(z)\Gamma\left(z+\frac{1}{2}\right), \\ \Gamma(3z) &= (2\pi)^{-1}\, 3^{-\frac{1}{2}+3z}\Gamma(z)\Gamma\left(z+\frac{1}{3}\right)\Gamma\left(z+\frac{2}{3}\right). \end{aligned}$$

3. Proof of Theorem 1.

In [**7**] and [**8**], we proved that the functional equation (1.1) can be interpreted as a relation between $O(m)$-invariant spherical functions on the homogeneous space $X = H\backslash GL_m$. The point is to consider the family

$$\left\{(g^{-1}\mathbf{H}g \times \mathbf{GL}_n, \rho, \mathbf{M}_{m,n}) \middle| Hg \in H\backslash GL_m\right\}$$

of prehomogeneous vector spaces and to regard the associated zeta functions as functions on the parameter space $H\backslash GL_m$. Namely we consider the functions

$$\begin{aligned} H\backslash GL_m \ni Hg &\longmapsto \Phi_i(Hg, \phi; \lambda) := \int_V |f(gv)|_i^{(\lambda - mn/2)/d}\, \phi(v)\, dv, \\ H\backslash GL_m \ni Hg &\longmapsto \Phi_j^*(Hg, \phi^*; \lambda) := \int_{V^*} \left|f^*({}^t g^{-1} v^*)\right|_j^{(\lambda - mn/2)/d^*} \phi^*(v^*)\, dv^*. \end{aligned}$$

Put

$$v_0 = \begin{pmatrix} E_n \\ 0 \end{pmatrix}, \qquad v_0^* = \begin{pmatrix} 0 \\ E_n \end{pmatrix}.$$

LEMMA 5. *Assume that ϕ and ϕ^* are left $O(m)$-invariant. Then we have*

$$\Phi_i(Hg,\phi;\lambda)=Z_{mn}(\phi;\lambda)\omega_i(Hg;\lambda),$$
$$\Phi_j^*(Hg,\phi^*;\lambda)=Z_{mn}(\phi^*;\lambda)\omega_j^*(Hg;\lambda).$$

Here we put

$$Z_{mn}(\phi;\lambda)=\int_{\{v\in\mathbf{M}_{m,n}(\boldsymbol{R})\mid\operatorname{rank}v=n\}}\det({}^tvv)^{(\lambda-mn/2)/2n}\phi(v)\,dv,$$
$$\omega_i(Hg;\lambda)=\int_{O(m)}|f(gkv_0)|_i^{(\lambda-mn/2)/d}\,dk,$$
$$\omega_j^*(Hg;\lambda)=\int_{O(m)}\left|f^*({}^tg^{-1}kv_0^*)\right|_j^{(\lambda-mn/2)/d^*}\,dk,$$

where dk is the normalized Haar measure on the orthogonal group $O(m)$.

It is not difficult (and well-known) to calculate $Z_{mn}(\phi;\lambda)$ for $\phi_0(v):=\exp\bigl(-\pi\operatorname{tr}({}^tvv)\bigr)$:

$$Z_{mn}(\phi_0;\lambda)=\pi^{-(2\lambda-mn)/4}\prod_{k=0}^{n-1}\frac{\Gamma\bigl(\frac{\lambda}{2n}+\frac{m-2k}{4}\bigr)}{\Gamma\bigl(\frac{m-k}{2}\bigr)}.$$

By this identity as well as Lemma 5, the functional equation (1.1) can be rewritten as

$$\omega_i(Hg;\lambda)=|\det g|^{-n}\,\Gamma_{mn}(\lambda)\sum_{j=1}^{\nu}\gamma_{ij}(\lambda)\omega_j^*(Hg;-\lambda),\tag{3.1}$$

where

$$\Gamma_{mn}(\lambda)=\pi^{\lambda}\prod_{k=0}^{n-1}\frac{\Gamma\bigl(-\frac{\lambda}{2n}+\frac{m-2k}{4}\bigr)}{\Gamma\bigl(\frac{\lambda}{2n}+\frac{m-2k}{4}\bigr)}\tag{3.2}$$

(see [**7**, Theorem 3.11] and [**8**, Theorem 3.7]). Thus the determination of $\gamma_{ij}(\lambda)$ is equivalent to the determination of an explicit formula for the functional equation (3.1).

Let $P=P_{n,m-n}$ (resp. $P^*=P_{m-n,n}$) be the upper triangular maximal parabolic subgroup of GL_m corresponding to the partition $m=n+(m-n)$ (resp. $m=(m-n)+n$). Denote by $\mathscr{B}(GL_m/P;z_1,z_2)$ the space of hyperfunctions ψ on GL_m satisfying the condition

$$\psi(gp)=|\det p_1|^{z_1-(m-n)/2}\,|\det p_2|^{z_2+n/2}\,\psi(g)\quad\left(\forall p=\begin{pmatrix}p_1&*\\0&p_2\end{pmatrix}\in P=P_{n,m-n}\right).\tag{3.3}$$

The left regular representation of GL_m on $\mathscr{B}(GL_m/P;z_1,z_2)$ defines a representation belonging to degenerate principal series. We denote also by $\mathscr{B}(GL_m/P;z_1,z_2)^H$ the subspace of H-invariant functions.

For $z=(z_1,z_2)\in \boldsymbol{C}^2$, we put

$$\Psi_{z,i}(g)=|\det g|^{z_2+n/2}\,|f(gv_0)|_i^{n(z_1-z_2-m/2)/d}.$$

Then $\Psi_{z,i}(g)$ belongs to $\mathscr{B}(GL_m/P;z_1,z_2)^H$. Similarly, if we put

$$\Psi^*_{z,j}(g)=|\det g|^{z_1-n/2}\left|f^*({}^tg^{-1}v_0^*)\right|_j^{n(z_1-z_2-m/2)/d^*},$$

then $\Psi^*_{z,j}(g)$ belongs to $\mathscr{B}(GL_m/P^*;z_1,z_2)^H$.

Denote by $\mathscr{A}(GL_m/O(m))$ the space of real analytic functions on $GL_m/O(m)$. For any $\psi\in\mathscr{B}(GL_m/P;z_1,z_2)$, we define its Poisson transform by

$$\mathscr{P}\psi(g)=\int_{O(m)}\psi(gk)\,dk.$$

Then it is known that $\mathscr{P}\psi$ belongs to $\mathscr{A}(GL_m/O(m))$ and the image of the GL_m-equivariant mapping

$$\mathscr{P}:\mathscr{B}(GL_m/P;z_1,z_2)\to\mathscr{A}(GL_m/O(m))$$

is characterized by certain differential equations (see [**4**, Theorem 5.1]). Moreover, as is proved in [**4**] Section 5, the mapping $\mathscr{P}$ is injective for generic $z\in\boldsymbol{C}^2$.

The images of $\Psi_{z,i}$ and $\Psi^*_{z,j}$ under the Poisson transform are related to the spherical functions ω_i and ω^*_j as follows:

$$\begin{cases}\mathscr{P}\Psi_{z,i}(g)=|\det g|^{z_2+n/2}\,\omega_i(Hg;n(z_1-z_2)),\\ \mathscr{P}\Psi^*_{z,j}(g)=|\det g|^{z_1-n/2}\,\omega^*_j(Hg;n(z_1-z_2)).\end{cases}\tag{3.4}$$

Now we consider the intertwining operator

$$T_z:\mathscr{B}(GL_m/P;z_1,z_2)\longrightarrow\mathscr{B}(GL_m/P^*;z_2,z_1).$$

It is known that the operator T_z depends on z meromorphically. For a continuous function $\psi\in\mathscr{B}(GL_m/P;z_1,z_2)$, $T_z\psi$ is given by

$$T_z\psi(g)=\int_{M_{m-n,n}(\boldsymbol{R})}\psi\left(g\begin{pmatrix}X & E_{m-n}\\ E_n & 0\end{pmatrix}\right)\det\left(E_n+{}^txx\right)^{-\alpha}dx\Big|_{\alpha=0}.\tag{3.5}$$

The right hand side is understood to be the value at $\alpha=0$ of the analytic continuation of the function defined by the integral. We have the following commutative diagram:

$$\begin{array}{ccc} \mathscr{B}(GL_m/P; z_1, z_2) & \xrightarrow{\mathscr{P}} & \mathscr{A}(GL_m/O(m)) \\ {\scriptstyle T_z}\big\downarrow & & \big\downarrow{\scriptstyle \times c(z_1-z_2)} \\ \mathscr{B}(GL_m/P^*; z_2, z_1) & \xrightarrow{\mathscr{P}} & \mathscr{A}(GL_m/O(m)), \end{array} \tag{3.6}$$

where the right vertical arrow is the multiplication by

$$\begin{aligned} c(z_1 - z_2) &= \int_{M_{m-n,n}(\boldsymbol{R})} \det\left(E_n + {}^t xx\right)^{(z_1-z_2-m/2)/2} dx \\ &= \pi^{(m-n)n/2} \prod_{k=0}^{n-1} \frac{\Gamma\left(\frac{z_2-z_1}{2} - \frac{m-2(n-k)}{4}\right)}{\Gamma\left(\frac{z_2-z_1}{2} + \frac{m-2k}{4}\right)}. \end{aligned} \tag{3.7}$$

Now we rewrite the functional equation (3.1) by using (3.4). Then we have

$$\mathscr{P}\Psi_{(z_1,z_2),i}(g) = \Gamma_{mn}(\lambda) \sum_{j=1}^{\nu} \gamma_{ij}(\lambda) \mathscr{P}\Psi^*_{(z_2,z_1),j}(g),$$

where we put $\lambda = n(z_1 - z_2)$. By the commutativity of the diagram (3.6), this implies the identity

$$\mathscr{P}T_z\Psi_{(z_1,z_2),i}(g) = c(z_1 - z_2)\Gamma_{mn}(\lambda) \sum_{j=1}^{\nu} \gamma_{ij}(\lambda) \mathscr{P}\Psi^*_{(z_2,z_1),j}(g).$$

Since $\mathscr{P}$ is injective on $\mathscr{B}(GL_m/P^*; z_2, z_1)$ for generic (z_1, z_2) and the both side of the identity above are meromorphic, we obtain

$$T_z\Psi_{(z_1,z_2),i}(g) = c(z_1 - z_2)\Gamma_{mn}(\lambda) \sum_{j=1}^{\nu} \gamma_{ij}(\lambda) \Psi^*_{(z_2,z_1),j}(g). \tag{3.8}$$

Therefore, if we choose g_j as in Theorem 1, then

$$I_{ij}(\lambda) = T_z\Psi_{(z_1,z_2),i}(g_j) = c(z_1 - z_2)\Gamma_{mn}(\lambda)\gamma_{ij}(\lambda).$$

By (3.2) and (3.7), this proves Theorem 1. □

Remark.

(1) If one can prove (3.8) directly, namely, without assuming (1.1), the argument can be reversed to give an alternative proof of the functional equations of prehomogeneous zeta functions. A sufficient condition is that the subspace $\mathscr{B}(GL_m/P^*; z_2, z_1)^H$ is spanned by $\Psi^*_{(z_2,z_1),j}$ $(j = 1, \ldots, \nu)$. We note that this condition is closely related to the H-orbit structure of GL_m/P^*.

(2) In the above argument the role in the proof of functional equations played by the

Fourier transform is now played by the intertwining operator. To appreciate the meaning of this fact, let us consider the prehomogeneous vector space $(\mathbf{SL}_r \times \mathbf{SL}_r \times \mathbf{GL}_1, \rho, \mathbf{M}_r)$. Here the representation ρ is given by $\rho(h_1, h_2, t)v = th_1 v\, {}^t h_2$. Then $m = r^2$, $n = 1$, and $H = \mathbf{SL}_r \times \mathbf{SL}_r$ embedded in $\mathbf{GL}_{r^2}$. Then, the intertwining operator on the degenerate principal series with respect to the parabolic subgroup P_{1,r^2-1} plays the role of the Fourier analysis on the matrix space $\mathbf{M}_r$ in the earlier approach to functional equations. Piatetski-Shapiro and Rallis used the same parabolic subgroup P_{1,r^2-1} to construct the standard L-functions of automorphic forms on GL_r by the Rankin–Selberg method. This relation between the usual formulation in the theory of prehomogeneous vector spaces and the present one is quite similar to the relation between the Godement-Jacquet construction of the standard L-functions of GL_n ([**1**]) and the Piatetski-Shapiro–Rallis construction ([**5**, Part A, Section 3]). Thus the result above (and the result in [**7**] and [**8**]) may be viewed as an analogue of the Piatetski-Shapiro–Rallis construction for arbitrary prehomogeneous zeta functions.

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