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Deficiencies of meromorphic mappings for hypersurfaces

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Abstract. In this paper we first prove that, for every hypersurface D of degree d in a complex projective space, there exists a holomorphic curve f from the complex plane into the projective space whose deficiency for D is positive and less than one. Using this result, we construct meromorphic mappings from the complex m-space into the complex projective space with the same properties. We also investigate the effect of resolution of singularities to defects of meromorphic mappings.

Introduction.

The aim of this paper is to construct meromorphic mappings f from $\boldsymbol{C}^m$ into the complex projective space $\boldsymbol{P}_n(\boldsymbol{C})$ with Nevanlinna's deficient divisors. Throughout this paper, we assume that $n \geq 2$. The defect relation for meromorphic mappings shows that the set of Nevanlinna's deficient divisors for f is very small. Furthermore, meromorphic mappings without defect are dense in the space of all meromorphic mappings $f : \boldsymbol{C}^m \to \boldsymbol{P}_n(\boldsymbol{C})$ with respect to a certain kind of distance (see [**M**]). It therefore seems that the construction of meromorphic mappings with preassigned deficiencies is very difficult. There have been several studies on the construction of holomorphic curves with deficient hyperplanes. So far, we do not know the existence of examples of meromorphic mappings with a deficient irreducible hypersurface of high degree whose deficiency is less than one. In this paper, we prove the existence of meromorphic mappings that have a preassigned positive deficiency for a given divisor D in $\boldsymbol{P}_n(\boldsymbol{C})$. We now recall the defect relation for dominant meromorphic mappings $f : \boldsymbol{C}^m \to \boldsymbol{P}_n(\boldsymbol{C})$ due to Griffiths' school (cf. [**S2**]), that is, we have the defect relation

$$\sum_{j=1}^{q} \delta_f(D_j) \leq \frac{n+1}{d},$$

where $D_1, \cdots, D_q$ are nonsingular hypersurfaces of degree d in $\boldsymbol{P}_n(\boldsymbol{C})$ intersecting normally. There has been a conjecture of Griffiths ([**Gr**, p. 379]) stating the defect relation for meromorphic mappings $f : \boldsymbol{C}^m \to \boldsymbol{P}_n(\boldsymbol{C})$ is also given by the above form under an appropriate nondegeneracy condition on f. Moreover, there also has been a conjecture such that the estimate

$$\delta_f(D) \leq \frac{C}{d}$$

holds under a generic condition for D, where C is a positive constant independent of f and D (cf. [**Si**, p. 289]). However, in the case where D is a singular divisor, we can construct many

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examples of meromorphic mappings f such that estimates for deficiencies of the above type do not hold. This follows from the following main theorem concerning Griffiths' conjecture that is the main result in this paper:

MAIN THEOREM. *Let $D \in |L(H)^{\otimes d}|$ be an arbitrary divisor in $\boldsymbol{P}_n(\boldsymbol{C})$, where $L(H)$ is the hyperplane bundle over $\boldsymbol{P}_n(\boldsymbol{C})$ and d is a positive integer. Then there exists a positive constant $\lambda(D)$ depending only on D with $\lambda(D) \le d$ that has the following property: For each positive number α with $\alpha \le \lambda(D)/d$, there exists a meromorphic mapping $f : \boldsymbol{C}^m \to \boldsymbol{P}_n(\boldsymbol{C})$ with Zariski dense image such that $\delta_f(D) = \alpha$. Furthermore, in the case of $m \ge n$, there exists a dominant meromorphic mapping $f : \boldsymbol{C}^m \to \boldsymbol{P}_n(\boldsymbol{C})$ with $\delta_f(D) = \alpha$.*

This theorem yields that for every irreducible hypersurface S in $\boldsymbol{P}_n(\boldsymbol{C})$ there exists a meromorphic mapping f such that the deficiency $\delta_f(S)$ for S is positive and less than one. We note here that, in general, the constant $\lambda(D)$ is dependent on the degree d. For instance, we have $\lambda(D) = d$ for some singular divisors. We give some concrete examples in §3. These examples show that we cannot obtain a good estimate on deficiency when D has singularities. Furthermore, we investigate how the existence of singularities of D affects an estimate for deficiencies in §5. The result in §5 shows that if we resolve singularities, we have an estimate for $\delta_f(D)$ depending on the structure on the singularities. The results obtained in this paper are rather pathological, but they suggest that the smoothness of divisors is a delicate matter to get a good bound for deficiencies. We note that the case of holomorphic curves is essential in the proof of our main theorem. The method used in our construction is elementary and based on the theory of entire functions of one complex variable, especially, on some properties of entire functions of order zero proved by Valiron [**V2**]. For this reason, we first prove the above theorem for holomorphic curves in §3. By making use of the idea of the proof for holomorphic curves, we prove the general case in §4.

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§1. Preliminaries.

We first recall some known facts on Nevanlinna theory of holomorphic curves and meromorphic mappings. Let $z = (z_1, \cdots, z_m)$ be the natural coordinate system in $\boldsymbol{C}^m$, and set

$$\|z\|^2 = \sum_{\upsilon=1}^{m} z_\upsilon \bar{z}_\upsilon, \qquad B(r) = \{z \in \boldsymbol{C}^m;\ \|z\| < r\},$$

$$S(r) = \{z \in \boldsymbol{C}^m;\ \|z\| = r\}, \qquad d^c = \frac{\sqrt{-1}}{4\pi}(\bar{\partial} - \partial),$$

$$\upsilon = dd^c\|z\|^2, \qquad \sigma = d^c \log\|z\|^2 \wedge (dd^c \log\|z\|^2)^{m-1}.$$

In the case $m = 1$, we write $\Delta(r)$ for $B(r)$ and $C(r)$ for $S(r)$, respectively.

For a (1,1)-current φ of order zero on $\boldsymbol{C}^m$ we set

$$n(r, \varphi) = r^{2-2m}\langle \varphi \wedge \upsilon^{m-1},\ \chi_{B(r)} \rangle$$

and

$$N(r,\varphi) = \int_1^r n(t,\varphi)\frac{dt}{t},$$

where $\chi_{B(r)}$ denotes the characteristic function of $B(r)$.

Let M be a compact complex manifold and $L \to M$ a line bundle over M. We denote by $\Gamma(M,L)$ the space of all holomorphic sections of $L \to M$. Let $|L| = \boldsymbol{P}(\Gamma(M,L))$ be the complete linear system defined by L. For a divisor D on M, we denote by $L(D)$ the line bundle over M defined by D. Let $|\cdot|$ be a hermitian fiber metric in L and let ω be its Chern form. A meromorphic mapping $f : \boldsymbol{C}^m \to M$ is said to be *dominant* if

$$\dim M = \operatorname{rank} f := \max\{\operatorname{rank} d f(z); z \in \boldsymbol{C}^m - I(f)\},$$

where $I(f)$ is the indeterminacy locus of f. For a meromorphic mapping $f : \boldsymbol{C}^m \to M$, we define

$$T_f(r,L) = N(r, f^*\omega)$$

and call it the characteristic function of f with respect to L. Let $L(H) \to \boldsymbol{P}_n(\boldsymbol{C})$ be the hyperplane bundle over $\boldsymbol{P}_n(\boldsymbol{C})$ and ω_0 the Fubini-Study form on $\boldsymbol{P}_n(\boldsymbol{C})$. In the case where $M = \boldsymbol{P}_n(\boldsymbol{C})$ and $L = L(H)$, we always take ω_0 for ω and we simply write $T_f(r)$ for $T_f(r,L(H))$. Let E be an effective divisor on $\boldsymbol{C}^m$. Then we call $N(r,E)$ the counting function of E. For a meromorphic function f on $\boldsymbol{C}$ and a point $a \in \boldsymbol{P}_1(\boldsymbol{C})$, we write $N(r, a, f)$ for $N(r, f^*a)$. Let $L \to \boldsymbol{P}_n(\boldsymbol{C})$ be a positive line bundle over $\boldsymbol{P}_n(\boldsymbol{C})$. Then $L = L(H)^{\otimes d}$ for some positive integer d and $D \in |L|$ is a hypersurface of degree d in $\boldsymbol{P}_n(\boldsymbol{C})$. It is clear that, if $L = L(H)^{\otimes d}$, then

$$T_f(r, L) = d\, T_f(r) + O(1).$$

Let $f = (f_0, \cdots, f_n)$ be a reduced representation of f. It is well-known that

$$T_f(r) = \int_{S(r)} \log\left(\max_{0 \le j \le n} |f_j(z)|\right)\sigma(z) + O(1).$$

This representation of the characteristic function of f is essentially due to H. Cartan. For a positive increasing function $\Lambda(r)$ defined on $\boldsymbol{R}^+$, we define the order ρ_Λ of $\Lambda(r)$ by

$$\rho_\Lambda = \limsup_{r\to+\infty} \frac{\log \Lambda(r)}{\log r}.$$

We define the order ρ_f of f by taking $\Lambda(r) = T_f(r)$. We now have the following well-known Nevanlinna's inequality:

THEOREM 1.1. *Let $f : \boldsymbol{C}^m \to M$ be a nonconstant meromorphic mapping and $L \to M$ a line bundle. Then*

$$N(r, f^*D) \le T_f(r,L) + O(1)$$

for a divisor $D \in |L|$ with $f(\boldsymbol{C}) \not\subseteq \operatorname{Supp} D$, where $O(1)$ stands for a bounded term as $r \to +\infty$.

Let f and D be as in Theorem 1.1. We define Nevanlinna's deficiency $\delta_f(D)$ by

$$\delta_f(D) = 1 - \limsup_{r \to +\infty} \frac{N(r, f^*D)}{T_f(r, L)}.$$

It is clear that $0 \leq \delta_f(D) \leq 1$. If $\delta_f(D) > 0$, then D is called a *deficient divisor in the sense of Nevanlinna*.

We next recall properties of entire functions of one complex variable. For a holomorphic function f on $\boldsymbol{C}$, we denote by $M(r, f)$ the maximum modulus of f on the circle $C(r)$, that is,

$$M(r, f) = \max_{|z|=r} |f(z)|.$$

We also note that the characteristic function of an entire function f can be written as

$$T_f(r) = \int_{C(r)} \log^+ |f(z)| \frac{d\theta}{2\pi} + O(1),$$

where $\log^+ x = \max\{\log x, 0\}$. Let $\Lambda_1(r)$ and $\Lambda_2(r)$ be positive increasing functions defined on $\mathbf{R}^+$. We write

$$\Lambda_1(r) = (1 + o(1))\Lambda_2(r)$$

provided that

$$\lim_{r \to +\infty} \frac{\Lambda_1(r)}{\Lambda_2(r)} = 1.$$

The following theorem due to Valiron plays a specially important role in this paper (see [**V1**, Chapter 5] and [**V2**, pp. 28–29]):

THEOREM 1.2 (Valiron). *Let f be a transcendental holomorphic function on $\boldsymbol{C}$. Suppose that $T_f(r) = O((\log r)^2)$ as $r \to +\infty$. Then*

$$\lim_{r \to +\infty} \frac{\log M(r, f)}{T_f(r)} = \lim_{r \to +\infty} \frac{N(r, 0, f)}{T_f(r)} = 1.$$

Furthermore, there exists a Borel subset $\varepsilon(r)$ of $C(r)$ such that

$$\log |f(z)| = (1 + o(1)) \log M(r, f)$$

for all $z \in C(r) \setminus \varepsilon(r)$ and $\mu(\varepsilon(r)) \to 0$ as $r \to +\infty$, where μ denotes the Haar measure on $C(r)$ normalized so that $\mu(C(r)) = 1$.

REMARK 1.3. We give here some remarks on the exceptional set $\varepsilon(r)$. There exists the exceptional set (say $\mathscr{E}$) for f such that $\varepsilon(r) = C(r) \cap \mathscr{E}$. The set $\mathscr{E}$ is a countable union of circles not containing the origin and substanding angles at the origin whose sum s is finite. Namely, the set $\mathscr{E}$ is given by

$$\mathscr{E} = \bigcup_{i=1}^{+\infty} C_i,$$

where C_i is a circle that has the radius r_i and the center distance e_i from the origin. Then we have

$$s = 2\sum_{i=1}^{+\infty} \arcsin\left(\frac{r_i}{e_i}\right).$$

Note that the zero set of f is contained in $\mathscr{E}$. For details, see [**Ha2**, pp. 75–76].

§2. Two lemmas.

In this section, we prove two lemmas needed later. We first show the existence of entire functions of order zero with an approximating growth of preassigned characteristic functions. We now have the following lemma:

LEMMA 2.1. *Let α be an arbitrary positive real number. Then there exists a transcendental entire function φ on $\boldsymbol{C}$ such that*

$$T_\varphi(r) = \alpha(\log r)^2 + o((\log r)^2)$$

as $r \to +\infty$.

PROOF. Take $p_j = \exp(j/\alpha)$ $(j = 1, 2, \cdots)$ and define an effective divisor E on $\boldsymbol{C}$ by $E = \sum_{j=1}^{\infty} 2p_j$. If $p_j < t$, then $j < \alpha \log t$. Hence we have

$$n(t,E) = 2\alpha \log t + c(t),$$

where $|c(t)| < 2$. Thus we see

$$N(r,E) = \alpha(\log r)^2 + O(\log r).$$

Note that $\sum_{j=1}^{+\infty} 1/p_j < +\infty$. Now we take the Weierstrass product

$$\varphi(z) = \prod_{j=1}^{+\infty}\left(1 - \frac{z}{p_j}\right)^2.$$

Then it follows from the standard estimate for the Weierstrass product (cf., e.g., Hayman [**Ha1**, p. 27, Theorem 1.11]) that

$$\begin{aligned}
\log|\varphi(z)| &\le \int_0^r \frac{n(t,E)}{t}dt + r\int_r^\infty \frac{n(t,E)}{t^2}dt \\
&= N(r,E) + 2\alpha \log r + O(1) \\
&= \alpha(\log r)^2 + o((\log r)^2)
\end{aligned}$$

for $z \in C(r)$. Hence we get

$$T_\varphi(z) \le \alpha(\log r)^2 + o((\log r)^2).$$

On the other hand, by the first main theorem, we see

$$T_\varphi(r) \ge N(r,E) + O(1) = \alpha(\log r)^2 + o((\log r)^2).$$

Therefore, we have

$$T_\varphi(r) = \alpha(\log r)^2 + o((\log r)^2). \qquad \square$$

We now define a holomorphic curve $f = (f_0, \cdots, f_n) : \boldsymbol{C} \to \boldsymbol{P}_n(\boldsymbol{C})$ as follows. Let α_0 and α_1 be positive real numbers with $\alpha_1 < \alpha_0$. By Lemma 2.1, we have entire functions f_0 and f_1 such that

$$T_{f_j}(r) = \alpha_j(\log r)^2 + o((\log r)^2) \quad (j = 0, 1).$$

Next, let $f_2, \cdots, f_n$ be transcendental entire functions such that $T_{f_j}(r) = o((\log r)^2)$ for $j = 2, \cdots, n$. We define a holomorphic curve $f : \boldsymbol{C} \to \boldsymbol{P}_n(\boldsymbol{C})$ by $f = (f_0, \cdots, f_n)$. We now prove the following lemma that is a crucial step in our construction of holomorphic curves with deficiencies:

LEMMA 2.2. *Let $f : \boldsymbol{C} \to \boldsymbol{P}_n(\boldsymbol{C})$ be as above. Then*

$$T_f(r) = T_{f_0}(r) + o((\log r)^2) \quad \text{as } r \to +\infty.$$

PROOF. Since $T_{f_j}(r) = O((\log r)^2)$ as $r \to +\infty$, we have

$$T_{f_j}(r) = \int_{C(r)} \log |f_j(z)| \frac{d\theta}{2\pi} + o((\log r)^2).$$

Hence, by using Cartan's representation of the characteristic function, we see

$$T_{f_0}(r) + o((\log r)^2) \leq T_f(r).$$

By Theorem 1.2, we see

$$\begin{aligned} T_f(r) &= \int_{C(r)} \log \left(\max_j |f_j(z)| \right) \frac{d\theta}{2\pi} + O(1) \\ &\leq \int_{C(r)} \log \left((n+1) \max_{0 \leq j \leq n} M(r, f_j) \right) \frac{d\theta}{2\pi} + O(1) \\ &= \max_{0 \leq j \leq n} \log M(r, f_j) + O(1) \\ &= \max_{0 \leq j \leq n} (1 + o(1)) T_{f_j}(r) + O(1) \\ &= T_{f_0}(r) + o((\log r)^2). \end{aligned}$$

Therefore, we have our assertion. $\square$

REMARK 2.3. We can construct the above holomorphic curve f with the Zariski dense image. Its proof is however delicate, and will be given in the proof of Theorem 3.2 in §3.

§3. Construction of holomorphic curves with deficient divisor.

In this section we prove our main result for holomorphic curves $f : \boldsymbol{C} \to \boldsymbol{P}_n(\boldsymbol{C})$. We denote by $\zeta = (\zeta_0, \cdots, \zeta_n)$ a homogeneous coordinate system in $\boldsymbol{P}_n(\boldsymbol{C})$. We first consider the case of hyperplane. We have the following that is a direct conclusion of Lemma 2.2:

THEOREM 3.1. *Let α be an arbitrary positive real number less than one and let H be an arbitrary hyperplane in $\boldsymbol{P}_n(\boldsymbol{C})$. Then there exists a holomorphic curve $f: \boldsymbol{C} \to \boldsymbol{P}_n(\boldsymbol{C})$ with the Zariski dense image such that $\delta_f(H) = \alpha$.*

PROOF. Without loss of generality, we may assume that $H = \{\zeta_1 = 0\}$. We consider a holomorphic curve $f: \boldsymbol{C} \to \boldsymbol{P}_n(\boldsymbol{C})$ defined in §2. We can choose such a f such that f has the Zariski dense image (see the proof of Theorem 3.2 below). Now we take α_0 and α_1 such that $1-\alpha = \alpha_1/\alpha_0$. Note that

$$\begin{aligned} T_{f_1}(r) &= N(r, 0, f_1) + o((\log r)^2) \\ &= N(r, f^*H) + o((\log r)^2). \end{aligned}$$

It follows from $T_f(r) = T_{f_0}(r) + o((\log r)^2)$ that

$$\begin{aligned} \delta_f(H) &= 1 - \limsup_{r\to+\infty} \frac{N(r, f^*H)}{T_f(r)} \\ &= 1 - \lim_{r\to+\infty} \frac{\alpha_1(\log r)^2 + o((\log r)^2)}{\alpha_0(\log r)^2 + o((\log r)^2)} \\ &= \alpha. \end{aligned}$$

□

We next deal with the case where a given divisor D is a hypersurface of degree d not less than two, that is, $D \in |L(H)^{\otimes d}|$ with $d \geq 2$. Let $P(\zeta) = P(\zeta_0, \cdots, \zeta_n)$ be a homogeneous polynomial of degree d and define a divisor D in $\boldsymbol{P}_n(\boldsymbol{C})$ by $P = 0$. Note that D may be a reducible hypersurface. We now prove the following existence theorem:

THEOREM 3.2. *There exists a positive constant $\lambda(D)$ with $\lambda(D) \leq d$ depending only on D that satisfies the following property: For each positive real number α with $\alpha \leq \lambda(D)/d$, there exists a holomorphic curve $f: \boldsymbol{C} \to \boldsymbol{P}_n(\boldsymbol{C})$ with the Zariski dense image such that $\delta_f(D) = \alpha$.*

PROOF. We first show the existence of holomorphic curves f with $\delta_f(D) = \alpha$. For a given divisor D, we take $f_0, \cdots, f_n$ in the following way. We write $P(\zeta)$ as follows:

$$P(\zeta) = P_1(\zeta) + P_2(\zeta) = \sum_{j=0}^{n} c_j \zeta_j^d + P_2(\zeta).$$

Let d_j be the highest degree in ζ_j that are contained in P and set $\tilde{d} = \min_{0\leq j\leq n} d_j$. We consider the following three cases.

Case I. No c_j is zero.

Take entire functions f_0 and f_1 so that $T_{f_0}(r) = \alpha_0(1+o(1))(\log r)^2$ and $f_1 = \omega(f_0+1)$, where ω is a nonzero constant. We also take an entire function f_2 such that $T_{f_2}(r) = \alpha_1(1+o(1))(\log r)^2$, where $\alpha_1 < \alpha_0$. Furthermore, we take transcendental entire functions $f_3, \cdots, f_n$ so that $T_{f_j}(r) = o(T_{f_{j-1}}(r))$ for $j = 3, \cdots, n$. Define a holomorphic curve $f: \boldsymbol{C} \to \boldsymbol{P}_n(\boldsymbol{C})$ by $f = (f_0, \cdots, f_n)$. Then, as in the proof of Lemma 2.2, we easily see

$$T_f(r) = \alpha_0(\log r)^2 + o((\log r)^2).$$

Set $F = P(f)$. We now choose ω such that F does not contain the term $f_0^d, \cdots, f_0^{d-\kappa+1}$, where κ is a nonnegative integer depending only on D. To this end, we rewrite P as follows:

$$P(\zeta) = \sum_{j=0}^{d} a_j \zeta_0^{d-j} \zeta_1^j + c_2 \zeta_2^d + \cdots + c_n \zeta_n^d + Q(\zeta),$$

where Q is a polynomial in ζ which does not contain terms $\zeta_0^{d-j}\zeta_1^j$ for $j = 0, \cdots, d$. We note that

$$\begin{aligned}\sum_{j=0}^{d} a_j f_0^{d-j} f_1^j &= \sum_{j=0}^{d} a_j f_0^{d-j} (\omega(f_0+1))^j \\ &= a_0 f_0^d + a_1 f_0^{d-1}\omega(f_0+1) + a_2 f_0^{d-2}\omega^2(f_0+1)^2 + \cdots + a_d \omega^d (f_0+1)^d \\ &= (a_0 + a_1\omega + a_2\omega^2 + \cdots + a_d\omega^d) f_0^d + (\text{the lower terms of } f_0).\end{aligned}$$

We define a polynomial $L(z)$ in z by

$$L(z) = \sum_{j=0}^{d} a_j z^j. \tag{3.3}$$

Let ω be a root ω of $L(z) = 0$. Then we can write the entire function F as $F = F_1 + F_2$, where

$$F_1 = \sum_{j=1}^{d} c'_j f_0^{d-j} + c_2 f_2^d + \sum_{j=3}^{n} c_j f_j^d.$$

It is easy to see that

$$c'_k = \frac{\omega^k}{k!} L^{(k)}(\omega).$$

Since $a_0 \neq 0$ and $a_d \neq 0$, we see $\omega \neq 0$. Thus, if ω is a multiple root of $L(x) = 0$ with multiplicity k, then

$$c'_1 = \cdots = c'_{k-1} = 0.$$

Now we take κ such that κ is the largest multiplicity of the roots of $L(x) = 0$. Note that $1 \leq \kappa \leq d-1$. Define $\lambda(D) = \kappa$. We assume that $\alpha < \kappa/d$. Take α_0 and α_1 so that $\alpha = 1 - \alpha_1/\alpha_0$. Note that $(d-\kappa)\alpha_0 < d\alpha_1$. By Theorem 1.2, we get $N(r, f^*D) = (1+o(1))T_F(r)$. We write F_2 as $\sum_{|\nu|=d} c_\nu f^\nu$, where $\nu = (\nu_0, \cdots, \nu_n)$. We now consider the following two subcases.

Subcase $\mathrm{I_a}$: The case where F_2 does not contain terms with $\nu_0 + \nu_1 + \nu_2 = d - j$ for $0 \leq j \leq \kappa$, that is, the coefficients of those terms are zero.

We will show that

$$T_F(r) = d(1+o(1))T_{f_2}(r). \tag{3.4}$$

By the definition of characteristic functions, we easily see

$$
\begin{aligned}
T_F(r) &= \int_{C(r)} \log^+ |F(z)| \frac{d\theta}{2\pi} + O(1) \\
&\le \int_{C(r)} \log^+ |f_2(z)|^d \frac{d\theta}{2\pi} + o((\log r)^2) \\
&\le d(1+o(1))T_{f_2}(r).
\end{aligned}
$$

Hence we get $T_F(r) \le d(1+o(1))T_{f_2}(r)$. Next we show $d(1+o(1))T_{f_2}(r) \le T_F(r)$. We first assume that $\alpha < \kappa/d$. Then $(d-\kappa)\alpha_0 < d\alpha_1$. For any sufficiently large r and for $z \in C(r) \setminus \varepsilon(r)$, we have

$$
\begin{aligned}
|F(z)| &\ge |f_2(z)|^d \left(|c_2| - \left| \frac{\sum_{j=\kappa}^d c'_j f_1^{d-j}(z)}{f_2^d(z)} \right| - \left| \sum_{j=3}^n c_j \frac{f_j^d(z)}{f_2^d(z)} \right| - \left| \frac{F_2(z)}{f_2^d(z)} \right| \right) \\
&\ge |f_2(z)|^d \left(|c_2| - \left| \frac{K \exp((1+o(1))(d-\kappa)\alpha_0 (\log r)^2)}{\exp((1+o(1))d\alpha_1 (\log r)^2)} \right| + o(1) \right) \\
&\ge |f_2(z)|^d (|c_2| - |K \exp((1+o(1))((d-\kappa)\alpha_0 - d\alpha_1)(\log r)^2)| + o(1)) \\
&\ge \frac{|c_2|}{2}(1+o(1))|f_2(z)|^d,
\end{aligned}
$$

where K is a some positive constant. Hence we have

$$
\begin{aligned}
T_F(r) &= \int_{C(r)} \log^+ |F(z)| \frac{d\theta}{2\pi} + O(1) \\
&\ge \int_{C(r)} \log^+ |f_2(z)|^d \frac{d\theta}{2\pi} - \mu(\varepsilon(r)) \log^+ M(r,F) \\
&\ge d(1+o(1))T_{f_2}(r).
\end{aligned}
$$

Thus we get (3.4). We therefore obtain $\delta_f(D) = \alpha$. Next assume that $\alpha = \kappa/d$. We take $f_0, \cdots, f_n$ such that

$$
T_{f_0}(r) = \alpha_0(1+o(1))(\log r)^2 \quad \text{and} \quad T_{f_j}(r) = o((\log r)^2) \quad (j = 1, \cdots, n).
$$

Then we easily see that

$$
T_f(r) = (1+o(1))T_{f_0}(r) \quad \text{and} \quad N(r, f^*D) \le (d-\kappa)(1+o(1))T_{f_0}(r).
$$

We write $F = F^{(1)} + F^{(2)}$, where $F^{(1)} = c'_\kappa f_0^{d-\kappa}$ and $F^{(2)} = F - F^{(1)}$. Then it is easy to see that, for any sufficiently large r and for $z \in C(r) \setminus \varepsilon(r)$,

$$
\left| \frac{F^{(2)}(z)}{F^{(1)}(z)} \right| \le \exp(-(1+o(1))\alpha_0 (\log r)^2).
$$

Hence we easily have

$$
\begin{aligned}
N(r, f^*D) &= \int_{C(r)} \log^+ |F(z)| \frac{d\theta}{2\pi} + o((\log r)^2) \\
&\geq \int_{C(r)} \log^+ |f_0(z)|^{d-\kappa} \frac{d\theta}{2\pi} + o((\log r)^2) \\
&= (d-\kappa)(1+o(1))T_{f_0}(r).
\end{aligned}
$$

Hence we have the estimate

$$(d-\kappa)(1+o(1))T_{f_0}(r) = N(r, f^*D).$$

This shows that $\delta_f(D) = \kappa/d$.

Subcase $\mathrm{I_b}$: The case where F_2 contains at least one term with $\nu_0 + \nu_1 + \nu_2 = d - j$ for $0 \leq j \leq \kappa$, that is, at least one coefficient of those terms is not zero.

We write F as

$$F = \sum_{j=1}^{d} b_j f_0^j f_2^{d-j} + F_3,$$

where F_3 does not contain the terms $f_0^{\nu_0+\nu_1} f_2^{\nu_2}$ with $\nu_0 + \nu_1 + \nu_2 = d - j$ for $0 \leq j \leq \kappa$. Let $k = \max\{j; b_j \neq 0\}$. Since $(d-\kappa)\alpha_0 \leq d\alpha_1$, we see $(d-\kappa)\alpha_0 < k\alpha_0 + (d-k)\alpha_1$. Then the growth of $f_0^k f_2^{d-k}$ is greater than those of $f_0^{d-\kappa}$ and f_2^d. We may assume that the zero divisor of f_0 (resp. f_2) is contained in $\boldsymbol{R}^+$ (resp. $\boldsymbol{R}^-$). Then the exceptional sets for f_0 and f_2 does not intersect. Indeed, by the construction of f_0 (see the proof of Lemma 2.1), we can write f_0 as follows:

$$f_0(z) = \prod_{\nu=1}^{+\infty} \left(1 - \frac{z}{p_\nu}\right)^2,$$

where $p_{\nu+1} > p_\nu > 0$. Let $\zeta_1, \zeta_2 \in C(r)$. Suppose that Re $\zeta_1 > 0$ and Re $\zeta_2 < 0$. Then, for any $r > 0$, we have

$$\left|1 - \frac{\zeta_1}{p_\nu}\right| < \left|1 - \frac{\zeta_2}{p_\nu}\right|$$

for each ν. Hence we obtain $|f_0(\zeta_1)| \leq |f_0(\zeta_2)|$. This implies that the set

$$\{z \in C(r);\ \mathrm{Re}\, z < 0\}$$

does not intersect the exceptional set for f_0. Hence we have our assertion. Thus we see

$$\log M(r, f_0^k f_2^{d-k}) = (k\alpha_0 + (d-k)\alpha_1 + o(1))(\log r)^2$$

and hence

$$N(r, f^*D) = (k\alpha_0 + (d-k)\alpha_1 + o(1))(\log r)^2.$$

Thus we get

$$\begin{aligned}\delta_f(D) &= 1 - \frac{k\alpha_0 + (d-k)\alpha_1}{d\alpha_0} \\ &= \frac{(d-k)(\alpha_0 - \alpha_1)}{d\alpha_0} \\ &< \frac{(d-k)\alpha_0 - (d-k-\kappa)\alpha_0}{d\alpha_0} \\ &= \frac{\kappa}{d}.\end{aligned}$$

This implies that there exists a holomorphic curve f with $\delta_f(D) = \alpha$ for any $0 < \alpha < \kappa/d$. Next we consider case $\alpha = \kappa/d$. If there exists the term $f_0^{d-\kappa}$ in F_2, we take transcendental entire functions $f_2, \cdots, f_n$ so that $T_{f_j}(r) = o(T_{f_{j-1}}(r))$ for $j = 2, \cdots, n$. Then $\delta_f(D) = \kappa/d$. If F_2 does not contain the term $f_0^{d-\kappa}$, we get a holomorphic curve f with $\delta_f(D) = \alpha$ by taking $(d-k-\kappa)\alpha_0 = (d-k)\alpha_1$.

Case II. Some of the ζ_j's are not contained in P and $P_1(\zeta) \not\equiv 0$.

This case is essentially the same as the Case I. We may assume that $c_0 = 0$ and $c_1 \neq 0$. We set $\lambda(D) = d$. We take transcendental entire functions $f_0, \cdots, f_n$ such that $T_{f_0}(r) = (\log r)^2 + o((\log r)^2)$ and $T_{f_j}(r) = o(T_{f_{j-1}}(r))$ for $j = 1, \cdots, n$. By a suitable choice of $f_1, \cdots, f_n$, we see $f(\boldsymbol{C}) \cap D \neq \emptyset$. By this choice of the f_j's, it is clear that $\delta_f(D) = 1$. Now assume that $\alpha < 1$. We take f_0 and f_1 as in §2. Take transcendental entire functions $f_2, \cdots, f_n$ so that $T_{f_j}(r) = o(T_{f_{j-1}}(r))$ for $j = 2, \cdots, n$. By Lemma 2.2, we have

$$T_f(r) = T_{f_0}(r) + o((\log r)^2)$$

as $r \to +\infty$. We consider a holomorphic function F_0 defined by

$$F_0 = c_1 f_1^d + \cdots + c_n f_n^d + F_2.$$

Then, by a method similar to the Case I, we get a holomorphic curve f with $\delta_f(D) = \alpha$.

Case III. The case $\tilde{d} < d$.

In this case, $P = P_2$. Without loss of generality, we may assume that $d_0 = \tilde{d}$. Now, we set $\lambda(D) = d - d_0$. We write $P_2 = P_2^{(0)} + P_2^{(1)}$, where $P_2^{(0)}$ is the sum of monomials that contain ζ_0 and $P_2^{(1)}$ does not contain ζ_0. Then we may assume

$$P_2^{(0)}(\zeta) = \zeta_0^{d_0} Q_0(\zeta') + \zeta_0^{d_0-1} Q_1(\zeta') + \cdots + Q_{d_0}(\zeta'),$$

where $\zeta' = (\zeta_1, \cdots, \zeta_n)$ and $Q_j(\zeta')$ are polynomials in ζ'. We may assume that $Q_0(\zeta') = Q_0(\zeta_1, \cdots, \zeta_p)$ with $p \leq n$. We take transcendental entire functions $f_0, \cdots, f_n$ as follows. Let $T_{f_j}(r) = \alpha_j(1 + o(1))(\log r)^2$ with $\alpha_1 < \alpha_0$ for $j = 0, 1$. For $j = 2, \cdots, p$, we set $f_j = b_j f_1 + q_j$, where b_j are constants and q_j are transcendental entire functions such that $T_{q_2}(r) = o((\log r)^2)$ and $T_{q_j}(r) = o(T_{q_{j-1}}(r))$. We also take $T_{f_j}(r) = o(T_{f_{j-1}}(r))$ for $j = p+1, \cdots, n$. Note that for any sufficiently large r and for $z \in C(r) \setminus \varepsilon(r)$, we see

$$\log |f_j(z)| = \alpha_1 (\log r)^2 + o((\log r)^2)$$

for $j=2,\cdots,p$. Define a holomorphic curve f by $f=(f_0,\cdots,f_n)$. Let $F_2=P_2^{(0)}(f)+P_2^{(1)}(f)$ and $F_2^{(j)}=P_2^{(j)}(f)$ for $j=0,1$. Now choose $b_1,\cdots,b_p$ so that the function $f_1^{d-d_0}$ is contained in $F_2^{(0)}$. We define $\Lambda(l)=\{(\lambda_0,\lambda_1)\in(\mathbf{Z}_{\geq 0})^2;\lambda_0+\lambda_1=l\}$. Set $f_{01}^{\lambda}=f_0^{\lambda_0}f_1^{\lambda_1}$ for $\lambda\in\Lambda(l)$. Then we have

$$F_2^{(0)}=\sum_{k=0}^{d}\sum_{\lambda\in\Lambda(d-k);\lambda_0\leq d_0}a_\lambda(z)f_{01}^{\lambda}R_\lambda(f_{p+1},\cdots,f_n),$$

where a_λ are small functions with respect to f_0. Note that $a_{(d_0,d-d_0)}$ is a constant. Then, for any sufficiently large r and for $z\in C(r)\setminus\varepsilon(r)$, we see

$$|f_0^{d_0}(z)f_1^{d-d_0}(z)|=C\exp((d_0\alpha_0+(d-d_0)\alpha_1+o(1))(\log r)^2),$$

where C is a positive constant. Hence

$$\log|F_2(z)|\leq(d_0\alpha_0+(d-d_0)\alpha_1+o(1))(\log r)^2. \tag{3.5}$$

On the other hand, we set

$$G(z)=F_2(z)-a_{(d_0,d-d_0)}f_0(z)^{d_0}f_1(z)^{d-d_0}.$$

Then, for any sufficiently large r and for $z\in C(r)\setminus\varepsilon(r)$, we see

$$|G(z)|\leq\exp((d_0\alpha_0+(d-d_0)\alpha_1+o(1))(\log r)^2).$$

Hence we easily have

$$\begin{aligned}|F_2(z)|&\geq|f_0^{d_0}(z)f_1^{d-d_0}(z)|\left(|a_{(d_0,d-d_0)}|-\left|\frac{G(z)}{f_0^{d_0}(z)f_1^{d-d_0}(z)}\right|\right)\\&\geq\frac{|a_{(d_0,d-d_0)}|}{2}\exp((d_0\alpha_0+(d-d_0)\alpha_1+o(1))(\log r)^2).\end{aligned} \tag{3.6}$$

By (3.5) and (3.6), we get

$$N(r,f^*D)=(d_0\alpha_0+(d-d_0)\alpha_1)(1+o(1))(\log r)^2.$$

Thus we obtain

$$\delta_f(D)=1-\frac{d_0\alpha_0+(d-d_0)\alpha_1}{d\alpha_0}=\frac{\lambda(D)}{d}\left(1-\frac{\alpha_1}{\alpha_0}\right).$$

Therefore, for each positive number α less than $\lambda(D)/d$, there exists a holomorphic curve f with $\delta_f(D)=\alpha$. Furthermore, if we take entire functions $f_1,\cdots,f_n$ such that $T_{f_0}(r)=(1+o(1))(\log r)^2$ and $T_{f_j}(r)=o((\log r)^2)$ for $j=1,\cdots,n$. Then we have a holomorphic curve f with $\delta_f(D)=\lambda(D)/d$. We have now shown the existence of holomorphic curves f with the desired property. Next, we will show that the above holomorphic curves can be constructed such that they have the Zariski dense images.

PROOF OF ZARISKI DENSENESS OF THE IMAGE OF f. We first consider the Case II. The proof is somewhat complicated. Hence we first give an idea of the proof as follows. Suppose that there exists a homogeneous algebraic relation $R(f_0,\cdots,f_n)=0$ among f_j's. We rewrite $R(f_0,\cdots,f_n)=0$ as follows:

$$-A_s f_0^s = A_{s-1} f_0^{s-1} + \cdots + A_0.$$

Let $\{u_\nu\}_{\nu=1}^{+\infty}$ be the zero set of f_1, where $|u_\nu| < |u_{\nu+1}|$ and $|u_\nu| \to +\infty$ as $\nu \to +\infty$. Then there exist a subsequence $\{u_{\nu_j}\}_{j=1}^{+\infty}$ of $\{u_\nu\}_{\nu=1}^{+\infty}$ and a sequence $\{z_{\nu_j}\}_{j=1}^{+\infty}$ contained in a neighborhood of $\{u_{\nu_j}\}_{j=1}^{+\infty}$ such that the growth of the left hand side of the above equality is extremely larger than that of the right hand side of it at z_{ν_j} as $j \to \infty$.

Now, we give the proof of the Case II. Since $T_{f_j}(r) = o(T_{f_{j-1}}(r))$ for $j=2,\cdots,n$, it is easy to see that $f_2,\cdots,f_n$ have no homogeneous algebraic relation. Suppose that there exists nontrivial algebraic relation $R(f_0,\cdots,f_n)=0$ among f_j's, where $R(\zeta_0,\cdots,\zeta_n)$ is a homogeneous polynomial of degree l. If this relation does not contain one of the f_j with $j=0,1$, then we easily see that f has the Zariski dense image. Hence we consider another case, that is, the relation $R(f_0,\cdots,f_n)=0$ contains both of f_0 and f_1. We recall that

$$T_{f_j}(r) = (1+o(1))\alpha_j(\log r)^2 \quad (j=0,1)$$

with $\alpha_1 < \alpha_0$. We rewrite the above relation as follows:

$$A_s f_0^s + A_{s-1} f_0^{s-1} + \cdots + A_0 = 0,$$

where $A_j = A_j(f_1,\cdots,f_n)$ and $A_s \not\equiv 0$. We also write

$$A_s = a_t f_1^t + a_{t-1} f_1^{t-1} + \cdots + a_0,$$

where $a_p = a_p(f_2,\cdots,f_n)$ for $p=0,\cdots,t$ and $s+t \le l$. We may assume that the zero divisor of f_0 (resp. f_1) is contained in $\boldsymbol{R}^+$ (resp. $\boldsymbol{R}^-$). Let $\varepsilon_0(r)$ (resp. $\varepsilon_1(r)$) be the exceptional set for f_0 (resp. f_1). By the same reason in the Subcase I$_\mathrm{b}$, we see $\varepsilon_0(r)\cap\varepsilon_1(r)=\emptyset$. The zero set $\{u_\nu\}_{\nu=1}^{+\infty}$ of f_1 can be written as $\{-r_\nu\}_{\nu=1}^{+\infty}$, where $\{r_\nu\}_{\nu=1}^{+\infty}$ is a positive increasing sequence with $r_\nu \to +\infty$ $(\nu \to +\infty)$. Let ρ be a positive number with $\rho \ll 1$. Set

$$C_\nu = \{z \in \boldsymbol{C};\ |z+r_\nu| = \rho r_\nu\} \quad \text{and} \quad D_\nu = \{z \in \boldsymbol{C};\ |z+r_\nu| \le \rho r_\nu\}.$$

Suppose that $a_p \not\equiv 0$ for some p with $0 \le p \le t$. We can take $f_2,\cdots,f_n$ such that

$$T_{a_p}(r) = (1+o(1))(\log r)^{\sigma_p},$$

where $1 < \sigma_p < 2$. Then there exists a subsequence $\{r_{\nu_j}\}_{j=1}^{+\infty}$ of $\{r_\nu\}_{\nu=1}^{+\infty}$ such that a_p has no zero in D_{ν_j}. Indeed, we first note that $T_{a_p}(r) = o((\log r)^2)$. Suppose that a_p has a zero in all D_ν except for finite numbers of ν. By the construction of f_1, the zero divisor of f_1 is $\sum_{\nu=1}^{+\infty} 2(-r_\nu)$ (see the proof of Lemma 2.1). For sufficiently large ν, we see

$$\begin{aligned} N(2r_\nu,0,a_p) &\ge N(2r_\nu,0,f_1)/2 \\ &= (1+o(1))\alpha_1(\log 2r_\nu)^2/2 \\ &= (1+o(1))\alpha_1(\log r_\nu)^2/2. \end{aligned}$$

Thus we have a contradiction. Let $\mathscr{E}_p$ be the exceptional set for a_p. Then for each η with $0<\eta<\rho$, there exists a number R such that the sum of length of circles contained in $D_{\nu_j}\cap\mathscr{E}_p$ is sufficiently smaller than $(\eta/2)r_{\nu_j}$ for $r_{\nu_j}\geq R$. Note that a sequence $\{r_\nu\}_{\nu=1}^{+\infty}$ grows rapidly by the construction. Thus there exists a positive number γ_j with $(\eta/2)r_{\nu_j}<\gamma_j<\eta r_{\nu_j}$ such that

$$\Gamma_j=\{z\in\boldsymbol{C};\ |z+r_{\nu_j}|=\gamma_j\}$$

does not intersect $\mathscr{E}_p$ (see Remark 1.3). Hence by Theorem 1.2, we see

$$\log|a_p(z)|=(1+o(1))\log M(|z|,a_p)$$

for $z\in\Gamma_j$. For each $z\in\Gamma_j$, there exists a positive number τ_z such that $|z|=\tau_z r_{\nu_j}$, where $|r_{\nu_j}-\gamma_j|/r_{\nu_j}\leq\tau_z\leq|r_{\nu_j}+\gamma_j|/r_{\nu_j}$. Hence $1-\eta<\tau_z<1+\eta$. Then we see

$$\begin{aligned}
\log M(|z|,a_p)&=(1+o(1))(\log|z|)^{\sigma_p}\\
&=(1+o(1))(\log r_{\nu_j}+\log\tau_z)^{\sigma_p}\\
&\geq(1+o(1))(\log r_{\nu_j})^{\sigma_p}(1+\log(1-\eta)/\log r_{\nu_j})^{\sigma_p}\\
&=(1+o(1))(\log r_{\nu_j})^{\sigma_p}\\
&=(1+o(1))\log M(r_{\nu_j},a_p).
\end{aligned}$$

Thus we get

$$\log|a_p(z)|\geq(1+o(1))\log M(r_{\nu_j},a_p)$$

for $z\in\Gamma_j$. Set

$$\Sigma(\gamma_j;r_{\nu_j})=\{z\in\boldsymbol{C};\ |z+r_{\nu_j}|\leq\gamma_j\}.$$

Since a_p is a nonvanishing holomorphic function in $\Sigma(\gamma_j;r_{\nu_j})$, we see

$$\log|a_p(z)|\geq\min\{\log|a_p(z)|;z\in\Gamma_j\}\geq(1+o(1))\log M(r_{\nu_j},a_p)$$

for $z\in\Sigma(\gamma_j;r_{\nu_j})$. We now consider the case of $t\neq 0$. Let ε be a sufficiently small positive number less than $\min\{1,\alpha_1\}$. Then there exists a positive integer N depending on f_1 and ε that has the following property: If $j\geq N$, then there exists a point $z\in(C(r_{\nu_j})\setminus\varepsilon_1(r_{\nu_j}))\cap D(r_{\nu_j})$ such that

$$\log|f_1(z)|>\alpha_1(1-\varepsilon)(\log r_{\nu_j})^2.$$

Hence there exists $z_{\nu_j}\in C(r_{\nu_j})$ such that

$$\log|f_1(z_{\nu_j})|=\beta(\log r_{\nu_j})^2,$$

where $0<\beta<\alpha_1-\varepsilon$. Note that $z_{\nu_j}\in\varepsilon_1(r_{\nu_j})$ and $\varepsilon_1(r_{\nu_j})$ is contained in a small neighborhood of $-r_{\nu_j}$. Without loss of generality, we may assume that

$$\frac{1}{2}\leq\left|1+\frac{a_{t-1}}{a_t f_1}+\cdots+\frac{a_0}{a_t f_1^t}\right|\leq 2$$

in some neighborhood of z_{ν_j}. Since $\varepsilon_0(r_{\nu_j}) \cap \varepsilon_1(r_{\nu_j}) = \emptyset$, we get

$$\log |A_s(z_{\nu_j}) f_0^s(z_{\nu_j})| = (1+o(1))(s\alpha_0 + t\beta)(\log r_{\nu_j})^2.$$

In the case of $t = 0$, we have

$$\log |A_s(z_{\nu_j}) f_0^s(z_{\nu_j})| = \log |a_0(z_{\nu_j}) f_0^s(z_{\nu_j})| = (1+o(1))s\alpha_0(\log r_{\nu_j})^2.$$

On the other hand, in the both of two cases, we see

$$\log |A_{s-1}(z_{\nu_j}) f_0^{s-1}(z_{\nu_j}) + \cdots + A_0(z_{\nu_j})| \leq (1+o(1))((s-1)\alpha_0 + l\beta)(\log r_{\nu_j})^2.$$

We again rewrite the above algebraic relation as follows:

$$-A_s f_0^s = A_{s-1} f_0^{s-1} + \cdots + A_0.$$

Now we take β such that $0 < \beta < \alpha_1 - \varepsilon$ and $l\beta < \alpha_0$. Since

$$s\alpha_0 + t\beta > s\alpha_0 > (s-1)\alpha_0 + l\beta > 0,$$

we have a contradiction by letting $j \to +\infty$. Therefore, we conclude that f has the Zariski dense image in this case.

We next consider the Case I. Note that f_0 and $f_0 + 1$ have no nontrivial algebraic relation by homogeneous polynomials. Suppose that there exists nontrivial algebraic relation $R(f_0, (f_0+1)\omega, \cdots, f_n) = 0$, where R is a nonzero homogeneous polynomial. We rewrite this relation as follows:

$$A_u(f_0+1)^u + A_{u-1}(f_0+1)^{u-1} + \cdots + A_0 = 0.$$

Since f_0 and $f_0 + 1$ have no nontrivial homogeneous algebraic relation, each

$$A_j = A_j(f_0, f_2, \cdots, f_n)$$

contains at least one of $f_2, \cdots, f_n$. Thus we can write the above relation as follows:

$$-B_s f_0^s = B_{t-1} f_0^{s-1} + \cdots + B_0,$$

where $B_j = B_j(f_2, \cdots, f_n)$. Now, we have a contradiction by the above method.

We finally consider the Case III. In this case, $f_j = b_j f_1 + q_j$ for $j = 2, \cdots, p$, where b_j are constants and q_j are transcendental entire functions such that $T_{q_2}(r) = o((\log r)^2)$ and $T_{q_j}(r) = o(T_{q_{j-1}}(r))$. We also recall that $T_{f_j}(r) = \alpha_1(1+o(1))(\log r)^2$ for $j = 0, 1$ and $T_{f_j}(r) = o(T_{f_{j-1}}(r))$ for $j = p+1, \cdots, n$. Since $T_{f_j}(r) = o(T_{f_{j-1}}(r))$ for $j = p+1, \cdots, n$, it is easy to see that $f_{p+1}, \cdots, f_n$ have no homogeneous algebraic relation. Suppose that there exists nontrivial algebraic relation $R(f_0, \cdots, f_n) = 0$ by a homogeneous polynomial R. If the above relation does not contain one of the f_0 and f_1, it is easy to see that f has the Zariski dense image. If the relation $R(f_0, \cdots, f_n) = 0$ contains both of the f_0 and f_1, we rewrite the above relation as follows:

$$\sum_{j=0}^{s} A_j(q_1, \cdots, q_p, f_1, f_{p+1}, \cdots, f_n) f_0^j = 0.$$

Since $T_{f_j}(r) = o((\log r)^2)$ for $j = p+1, \cdots, n$ and $T_{q_j}(r) = o((\log r)^2)$, we now have a contradiction as in the Case II. Therefore, we have completed the proof. $\square$

REMARK 3.7. We give here note on the constant $\lambda(D)$ in Theorem 3.2. Let $P(\zeta)$ be a homogeneous polynomial of degree d such that $D = \{P = 0\}$. Let d_j be the degree in ζ_j that are contained in P. Set $\tilde{d} = \min_{0\le j\le n} d_j$. Let $L(z)$ be the polynomial in (3.3) and denote by κ the largest multiplicity of roots of the equation $L(z) = 0$, where $1 \le \kappa \le d-1$. Note that $\lambda(D)$ may be dependent on d and $\tilde{d}$. We do not know whether $\lambda(D)$ is sharp or not. We now give a list of the constant $\lambda(D)$ obtained in the proof of Theorem 3.2:

(I) If $d = \tilde{d}$, then $\lambda(D) = \kappa$.
(II) If $\tilde{d} < d$, then $\lambda(D) = d - \tilde{d}$.

We give here some examples of irreducible hypersurfaces of degree d.

EXAMPLE 3.8. We define an irreducible hypersurface D_d of degree d in $\boldsymbol{P}_n(\boldsymbol{C})$ by

$$\zeta_1^d + \cdots + \zeta_n^d = 0.$$

Note that D_d has just one singular point $(1,0,\cdots,0)$. In this case, $\lambda(D) = d$. Hence, for an arbitrary positive real number α not greater than one, there exists a holomorphic curve $f : \boldsymbol{C} \to \boldsymbol{P}_n(\boldsymbol{C})$ with the Zariski dense image such that $\delta_f(D_d) = \alpha$. Note that there exist nonconstant holomorphic curves from $\boldsymbol{C}$ into $\boldsymbol{P}_n(\boldsymbol{C})$ that omit the above hypersurface D_d of arbitrary high degree. Let $\psi(z)$ and $\varphi(z)$ be arbitrary entire functions and μ a d-th root of -1. If we define a holomorphic curve $f : \boldsymbol{C} \to \boldsymbol{P}_n(\boldsymbol{C})$ by

$$f(z) = (\psi(z), \varphi(z), \mu\varphi(z), 1, \cdots, 1),$$

then f omits D_d. Note that f is linearly degenerate. This example is essentially due to P. Kiernan (cf. [**G1**, Part 7]).

EXAMPLE 3.9. We next give an example of a nonsingular hypersurface. We define a nonsingular hypersurface S_d in $\boldsymbol{P}_n(\boldsymbol{C})$ of degree $d \ge 2$ by

$$\zeta_0^{d-1}\zeta_2 - \zeta_1^d + \zeta_1\zeta_2^{d-1} + \sum_{j=3}^{n} \zeta_j^d = 0.$$

In this case, we have $\lambda(D) = 1$. For the above hypersurface S_d, there exists a nonconstant holomorphic curve $f : \boldsymbol{C} \to \boldsymbol{P}_n(\boldsymbol{C})$ omitting S_d for all $d \ge 2$. Indeed, let $\varphi(z)$ be an arbitrary entire function. We take entire functions $\varphi_3, \cdots, \varphi_n$ such that

$$\varphi_3^d + \cdots + \varphi_n^d = 0.$$

Define a holomorphic curve $f : \boldsymbol{C} \to \boldsymbol{P}_n(\boldsymbol{C})$ by

$$f(z) = \left(\exp\left(\frac{d}{d-1}\varphi(z)\right), \exp\varphi(z), 1, \varphi_3(z), \cdots, \varphi_n(z)\right).$$

Then we see $f(\boldsymbol{C}) \cap S_d = \emptyset$. Note that f is algebraically degenerate. This example is essentially due to M. L. Green (see [**G2**, p. 321]).

EXAMPLE 3.10. Let $n = 2$ and define an irreducible curve C_d by

$$\zeta_0\zeta_2^{d-1} - \zeta_1^d = 0.$$

Note that C_d also has just one singular point $(1,0,0)$, if $d \geq 3$. We also note that C_d is a rational curve. For C_d, we have $\lambda(C_d) = d-1$ by Theorem 3.2. Hence, for an arbitrary positive number $\alpha \leq (d-1)/d$, there exists a holomorphic curve with the Zariski dense image $f : \boldsymbol{C} \to \boldsymbol{P}_2(\boldsymbol{C})$ such that $\delta_f(D_d) = \alpha$. Note that a holomorphic curve f constructed in §2 has the above property. Indeed, let f be as in §2 and assume $\alpha = 1 - \alpha_1/\alpha_0$. In this case $d > \alpha_0/\alpha_1$. We consider an entire function F defined by

$$F = f_0 f_2^{d-1} - f_1^d.$$

Then, by a method similar to the Case I in the proof of Theorem 3.2, we have the estimate:

$$T_F(r) = N(r, f^*C_d) + o((\log r)^2) = d(1+o(1))\, T_{f_1}(r).$$

Therefore, we get $\delta_f(C_d) = \alpha$.

REMARK 3.11. We note that, for each positive integer d not less than two, there exists a holomorphic curve $f : \boldsymbol{C} \to \boldsymbol{P}_2(\boldsymbol{C})$ such that f omits C_d. In fact, if we define f by

$$f(z) = (\exp z + \exp(1-d)z^2, 1, \exp z^2),$$

then we easily see $f(\boldsymbol{C}) \cap C_d = \emptyset$ (cf. [**G2**, p. 319] and [**S2**, p. 178]). Note that f has the Zariski dense image.

We note here that there has been another method to construct holomorphic curves with deficiencies. The holomorphic curves constructed above is of order zero. On the other hand, N. Toda has pointed out that the above examples of holomorphic curves can be proved by making use of Ahlfors-Weyl's method (see [**W**]). In his construction, he used exponential curves and obtained holomorphic curves of order one with deficiencies. Note that this method works in the case that can be reduced to the hyperplane case. Indeed, let F_d be the Fermat surface degree d, that is,

$$F_d : \zeta_0^d + \cdots + \zeta_n^d = 0.$$

Then our method gives a holomorphic curve f with $\delta_f(F_d) = \alpha$ $(0 < \alpha \leq 1/d)$, but we cannot construct a holomorphic curve with positive deficiency for F_d by Ahlfors-Weyl's method. Hence it seems that our method has a wide range of applicability.

§4. Construction of meromorphic mappings with deficient divisor.

In this section, we construct meromorphic mappings $f : \boldsymbol{C}^m \to \boldsymbol{P}_n(\boldsymbol{C})$ with deficiency for a preassigned divisor. In our construction, we essentially use the method of the construction of holomorphic curves in §3. Throughout this section, we assume $m \geq 2$. Let F be a holomorphic function on $\boldsymbol{C}^m$. As the case of $m = 1$, we define

$$M(r,F) = \max_{z \in S(r)} |F(z)|.$$

LEMMA 4.1. *Let F be a holomorphic function on $\boldsymbol{C}^m$. Then*

$$T_F(r) + O(1) \leq \log M(r,F) \leq \frac{1-(r/R)^2}{(1-r/R)^{2m}} (T_F(R) + O(1))$$

for arbitrary positive numbers r and R with $r < R$.

For a proof, see, e.g., Noguchi [**N**, Lemma 1]. Set $\delta(k) = ((1-1/k)^{2m})/(1-(1/k)^2)$ for a positive number k. It is clear that $\delta(k) \to 1$ as $k \to +\infty$. By Lemma 4.1, we see

$$T_F(r/k) + O(1) \le \log M(r/k, F) \le \delta(k)^{-1}(T_F(r) + O(1))$$

for positive numbers k and r with $k > 1$. For a holomorphic function F on $\boldsymbol{C}^m$ and $\gamma \in \boldsymbol{C}^m$ with $||\gamma|| = 1$, we define $F_\gamma : \boldsymbol{C} \to \boldsymbol{C}$ by $F_\gamma(s) = F(s\gamma)$ for $s \in \boldsymbol{C}$. Hence F_γ is the restriction of $F(z)$ to a complex line ℓ_γ through the origin defined by $\ell_\gamma : z = s\gamma = (s\gamma_1, \cdots, s\gamma_m)$. Set $\gamma_0 := (1, 0, \cdots, 0)$. Take a holomorphic function $\varphi(z)$ on $\boldsymbol{C}$ so that $T_\varphi(r) = \alpha(1+o(1))(\log r)^2$. Let the zero divisor of φ is $\sum_j \nu_j p_j$. We define a holomorphic function $F(z)$ on $\boldsymbol{C}^m$ by $F(z) = \varphi(z_1)g(z)$, where $z = (z_1, \cdots, z_m)$ and g is a nonzero polynomial in z. For the reason why we take the polynomial g, see Remark 4.5 below. Note that $F(z)$ has zeros at H_j, where H_j denotes the hyperplane through $z_1 = p_j$ that is perpendicular to the z_1-axis. Then it is easy to see that

$$M(r, F) = (1+o(1))M(r, F_{\gamma_0}) = (1+o(1))M(r, \varphi). \tag{4.2}$$

Note that $N(r, 0, F_\gamma) \le N(r, 0, F_{\gamma_0}) + O(\log r)$ for sufficiently large r. We will give an estimate for $N(r, 0, F)$ by $T_F(r)$. It is clear that $N(r, 0, F) \le T_F(r) + O(1)$. For sufficiently small positive real number ε, set $S_\varepsilon = \{z \in \boldsymbol{C}^m; |z_1| < \varepsilon\}$. Let $S(1)$ be the unit sphere in $\boldsymbol{C}^m$ and denote by σ the invariant measure on $S(1)$ normalized so that $\sigma(S(1)) = 1$. Set $S'_\varepsilon = S(1) \cap S_\varepsilon$. Note that we cannot get a good estimate for $N(r, 0, F_\gamma)$ if $\gamma \in S'(\varepsilon)$. For instance, we see $N(r, 0, F_\gamma) = O(\log r)$ for $\gamma \in S(1)$ with $\gamma = (0, \gamma_2, \cdots, \gamma_m)$. Let $\gamma = (\gamma_1, \gamma_2, \cdots, \gamma_m) \in S(1) \setminus S_\varepsilon$ and denote by H_r the hyperplane through $z_1 = r$ $(r \in \boldsymbol{R}_{>0})$ that is perpendicular to the z_1-axis. Since $H_r \cap \ell_\gamma = \{\mathrm{P}_r(r, z_2, \cdots, z_m)\}$, there exists just one $s \in \boldsymbol{C}^*$ such that $s\gamma_1 = r$ and $s\gamma_j = z_j$ for $j = 2, \cdots, m$. Hence we see $s = r/\gamma_1$. Since $\gamma \in S(1) \setminus S_\varepsilon$, this implies that $\|\mathrm{P}_r\| = r/|\gamma_1| < r/\varepsilon$. Hence to each zero p of F_{γ_0} with $|p| = r$ there corresponds the zero p' of F_γ with $|p'| = r/\gamma_1$. Thus we see

$$N(r/\varepsilon, 0, F_\gamma) \ge N(r, 0, F_{\gamma_0}) + o((\log r)^2) = (1+o(1))\alpha(\log r)^2$$

and hence

$$N(r, 0, F_\gamma) \ge (1+o(1))\alpha(\log \varepsilon r)^2 \ge (1+o(1))(T_{F_{\gamma_0}}(r) + 2\alpha(\log \varepsilon)(\log r)).$$

Now, we will use the following averaging formula (see [**S1**, p. 91]):

$$N(r, 0, F) = \int_{\gamma \in S(1)} N(r, 0, F_\gamma)\sigma(\gamma) + O(1).$$

By the above formula we see

$$\begin{aligned} N(r, 0, F) &\ge \int_{\gamma \in S(1) \setminus S_\varepsilon} N(r, 0, F_\gamma)\sigma(\gamma) \\ &\ge \int_{\gamma \in S(1) \setminus S_\varepsilon} (1+o(1))(T_F(r) + 2\alpha(\log \varepsilon)(\log r))\sigma(\gamma) \\ &= (1-\sigma(S'_\varepsilon))(1+o(1))(T_F(r) + 2\alpha(\log \varepsilon)(\log r)). \end{aligned}$$

Thus, for a fixed ε, we have the following estimate:

$$(1-\sigma(S'_\varepsilon))(1+o(1))(T_F(r) + 2\alpha(\log \varepsilon)(\log r)) \le N(r, 0, F) \le T_F(r) + O(1). \tag{4.3}$$

We now take transcendental entire functions $f_0, \cdots, f_n$ of one complex variable satisfying the following condition: For $j = 0, 1$, let

$$T_{f_j}(r) = \alpha_j(1+o(1))((\log r)^2),$$

where $\alpha_1 < \alpha_0$. For $j = 2, \cdots, n$, we take f_j so that $T_{f_j}(r) = o(T_{f_{j-1}}(r))$. By making use of the above method, we regard f_j as entire functions on $\boldsymbol{C}^m$, that is, we define $F_j(z) = f_j(z_1)g_j(z)$ for $j = 0, \cdots, n$, where $g_j(z)$ are some nonzero polynomials. Now, we assume that $F_j(z) = f_j(z_1)$ for $j = 0, 1$. We now define a meromorphic mapping $f : \boldsymbol{C}^m \to \boldsymbol{P}_n(\boldsymbol{C})$ by $f = (F_0, \cdots, F_n)$. Then we have the following lemma:

LEMMA 4.4. *Let f be as above. Then*

$$\delta(k) \log M(r, F_0) \leq (1+o(1))T_f(r) \leq \log M(r, F_0).$$

PROOF. As in the proof of Lemma 2.2, we easily have

$$(1+o(1))T_f(r) \leq \log M(r, F_0)$$

by definition of $T_f(r)$. On the other hand, by (4.3) we see

$$\begin{aligned} T_f(r) &\geq \int_{S(r)} \log |F_0(z)| \sigma(z) \\ &= N(r, 0, F_0) + O(1) \\ &\geq \delta(k)(1+o(1)) \log M(r/k, F_0) \\ &\geq \delta(k)\alpha_0(1+o(1))((\log(r/k))^2) \\ &= \delta(k)\alpha_0(1+o(1))((\log r)^2). \end{aligned}$$

Therefore we have the desired conclusion. □

REMARK 4.5. We can construct the above meromorphic mapping f with the Zariski dense image. Indeed, the argument in the proof of Theorem 3.2 also works in this case. Furthermore, if $m \geq n$, we have a dominant meromorphic mapping f by a suitable choice of g_j's. For instance, we can make f to be dominant by taking $g_0(z) \equiv 1$ and $g_j(z) = z_j$ for $j \geq 1$. Hence we may assume that f is dominant if $m \geq n$.

We first give a generalization of Theorem 3.1 as follows.

THEOREM 4.6. *Let α be an arbitrary positive real number less than one and let H be an arbitrary hyperplane in $\boldsymbol{P}_n(\boldsymbol{C})$. Then there exists a meromorphic mapping $f : \boldsymbol{C}^m \to \boldsymbol{P}_n(\boldsymbol{C})$ with the Zariski dense image such that $\delta_f(H) = \alpha$. If $m \geq n$, then there exists a dominant meromorphic mapping $f : \boldsymbol{C}^m \to \boldsymbol{P}_n(\boldsymbol{C})$ with $\delta_f(H) = \alpha$.*

PROOF. Without loss of generality, we may assume that $H = \{\zeta_1 = 0\}$. We consider a meromorphic mapping $f : \boldsymbol{C}^m \to \boldsymbol{P}_n(\boldsymbol{C})$ as in Lemma 4.4. Choose such a f so that f has the Zariski dense image. Now we take α_0 and α_1 such that $1 - \alpha = \alpha_1/\alpha_0$. We note that

$(1+o(1))T_{f_1}(r) = N(r, f^*H)$ and $T_f(r) = (1+o(1))T_{f_0}(r)$. Hence by (4.3), Lemma 4.4 and letting $r \to +\infty$, we see

$$1-(1-\sigma(S'_\varepsilon))\left(\frac{\alpha_1}{\alpha_0}\right) \le \delta_f(H) \le 1-\delta(k)\left(\frac{\alpha_1}{\alpha_0}\right).$$

Letting $k \to +\infty$ and $\varepsilon \to 0$, we get $\delta_f(H) = \alpha$. □

We next consider the case $D \in |L(H)^{\otimes d}|$ $(d \ge 2)$. In this case, by making use of (4.2), (4.3) and Lemma 4.4, we have the following theorem by the same method in the proof of Theorem 3.2:

THEOREM 4.7. *Let $D \in |L(H)^{\otimes d}|$ be an arbitrary divisor in $\boldsymbol{P}_n(\boldsymbol{C})$, where d is a positive integer. Then there exists a positive constant $\lambda(D)$ depending only on D with $\lambda(D) \le d$ that has the following property: For each positive number α with $\alpha \le \lambda(D)/d$, there exists a meromorphic mapping $f : \boldsymbol{C}^m \to \boldsymbol{P}_n(\boldsymbol{C})$ with the Zariski dense image such that $\delta_f(D) = \alpha$. Furthermore, if $m \ge n$, then there exists a dominant meromorphic mapping $f : \boldsymbol{C}^m \to \boldsymbol{P}_n(\boldsymbol{C})$ with $\delta_f(D) = \alpha$.*

We note that the number $\lambda(D)$ is as same as in Remark 3.6. It follows from Theorem 4.7 that we can find many examples of singular divisors and meromorphic mappings $f : \boldsymbol{C}^m \to \boldsymbol{P}_n(\boldsymbol{C})$ for which Griffiths' defect relation does not hold. For instance, we consider the examples of divisor as in §3. Namely, let C_d be a curve as in Example 3.9 and α a positive real number less than $(d-1)/d$. Then there exists a dominant meromorphic mapping $f : \boldsymbol{C}^m \to \boldsymbol{P}_2(\boldsymbol{C})$ such that $\delta_f(C_d) = \alpha$. In particular, there exists a dominant meromorphic mapping $f : \boldsymbol{C}^m \to \boldsymbol{P}_2(\boldsymbol{C})$ such that

$$\delta_f(C_d) = \frac{d-2}{d}.$$

Hence we also have an example for which Griffiths' defect relation does not hold. We note that there exists a dominant meromorphic mapping $f : \boldsymbol{C}^2 \to \boldsymbol{P}_2(\boldsymbol{C})$ omitting C_d for each d (see Shiffman [**S2**, p. 178]).

REMARK 4.8. Let P and $\tilde{d}$ be as in §3. Suppose that $d \ge 3$. We note that, if $\tilde{d} \le d-2$, then D has a singular point. Indeed, we may assume that $\tilde{d} = d_0$. We write P as follows:

$$P(\zeta) = \zeta_0^{d-k} Q_1(\zeta) + Q_2(\zeta),$$

where $Q_2(\zeta)$ does not contain ζ_0 and ζ_0^{d-k} is the greatest common divisor in $P - Q_2$. Since $d-k \le d-2$, we see that D has a singular point $(1,0,\cdots,0)$. Set $w_j = \zeta_j/\zeta_0$ for $j = 1,\cdots,n$. Define $\tilde{P}(w) = \zeta_0^{-d}P(\zeta)$, where $w = (w_1,\cdots,w_n)$. If $d - d_0 \ge n+1$, then the polynomial $\tilde{P}(w)$ has a zero at $(0,\cdots,0)$ with multiplicity at least $n+1$. Hence D is not normal crossings at $(1,0,\cdots,0)$. This fact shows that the hypothesis in Griffiths' defect relation, that is, D is at most simple normal crossings, cannot be simply dropped.

§5. Effect of the resolution of singularities to deficiencies.

In this section we investigate how affects the resolution of singularities of divisors to deficiencies. In §3, we considered an example of the singular curve C_d defined by

$$C_d : \zeta_0 \zeta_2^{d-1} - \zeta_1^d = 0.$$

This curve has only one singular point $\mathrm{P}(1,0,0)$, if $d \geq 3$. If $\pi : Q_{\mathrm{P}}(\boldsymbol{P}_2(\boldsymbol{C})) \to \boldsymbol{P}_2(\boldsymbol{C})$ is a monoidal transformation with the center P, then this gives a resolution of singularity of C. Namely, let $\tilde{C}$ and $\overline{C}$ be the total transform and the proper transform of C_d, respectively. We also denote by E the exceptional curve. Then we have

$$\tilde{C} = (d-1)E + \overline{C}$$

and $\overline{C}$ is a nonsingular curve in $Q_{\mathrm{P}}(\boldsymbol{P}_2(\boldsymbol{C}))$ (see Lemma 6.1 in §6). We define a meromorphic mapping $\tilde{f} : \boldsymbol{C}^m \to Q_{\mathrm{P}}(\boldsymbol{P}_2(\boldsymbol{C}))$ by $\tilde{f} = \pi^{-1} \circ f$. We will give an estimate for $\delta_{\tilde{f}}(\overline{C})$ depending on the structure of the singularity. To this end, we have to calculate the Chern form of the line bundle $L(\tilde{C})$. The precise calculation of the Chern form and the resolution of singularity will be done in the next section and hence we freely use the results in §6. The following is our main result in this section:

PROPOSITION 5.1. *Let α and f be as in Example* 3.9. *Then*

$$\delta_{\tilde{f}}(\overline{C}) = \frac{\alpha}{1 + (1-\alpha)(d-1)}.$$

In particular, the estimate

$$\frac{\alpha}{d} < \delta_{\tilde{f}}(\overline{C}) < \frac{d-1}{2d-1}$$

is valid.

PROOF. It suffices to give a proof in the case $m = 1$. Let Σ_1 be a Hirzebruch surface of rank one, that is, Σ_1 is a nonsingular subvariety of $\boldsymbol{P}_2(\boldsymbol{C}) \times \boldsymbol{P}_1(\boldsymbol{C})$ defined by

$$\Sigma_1 = \{(\zeta_0, \zeta_1, \zeta_2; \xi_0, \xi_1) \in \boldsymbol{P}_2(\boldsymbol{C}) \times \boldsymbol{P}_1(\boldsymbol{C});\ \zeta_2 \xi_0 - \zeta_1 \xi_1 = 0\},$$

where $\xi = (\xi_0, \xi_1)$ is a homogeneous coordinate system of $\boldsymbol{P}_1(\boldsymbol{C})$. Then it is well-known that $\Sigma_1 = Q_{\mathrm{P}}(\boldsymbol{P}_2(\boldsymbol{C}))$. Let $p_1 : \Sigma_1 \to \boldsymbol{P}_1(\boldsymbol{C})$ and $p_2 : \Sigma_1 \to \boldsymbol{P}_2(\boldsymbol{C})$ be the natural projections. Let ω_1 (resp. ω_2) be the Fubini-Study form on $\boldsymbol{P}_1(\boldsymbol{C})$ (resp. $\boldsymbol{P}_2(\boldsymbol{C})$). We first calculate $N(r, \tilde{f}^*E)$. We note that

$$\mathscr{U}_0 \cap \tilde{C} = \{(1 : x : tx; 1 : t);\ x^d - (tx)^{d-1} = 0\}$$

and

$$\mathscr{U}_1 \cap \tilde{C} = \{(1 : \tau y : y; \tau : 1);\ (\tau y)^d - y^{d-1} = 0\}.$$

In $\mathscr{U}_0$, the exceptional curve E is defined by $x = 0$ and $\overline{C}$ is defined by $x - t^{d-1} = 0$. On the other hand, in $\mathscr{U}_1$, the exceptional curve E is defined by $y = 0$ and $\overline{C}$ is defined by $\tau^d y - 1 = 0$. Note that $\tau = 1/t$. By the construction of f, we see

$$\begin{aligned} N(r, \tilde{f}^*E) &= N(r, 0, f_2) \\ &= o((\log r)^2). \end{aligned}$$

Hence we have $N(r, \tilde{f}^*\bar{C}) = N(r, f^*C) + o((\log r)^2)$. This shows

$$N(r, \tilde{f}^*\bar{C}) = d(1+o(1))\alpha_1(\log r)^2.$$

Next we show

$$T_{\tilde{f}}(r, L(\bar{C})) = T_f(r) + (d-1)T_{f_1}(r).$$

By Lemma 6.4 in §6, we have $c_1(L(\bar{C})) = p_1^*\omega_1 + (d-1)p_2^*\omega_2$. By Jensen's formula and the definition of $\tilde{f}$, we see

$$\begin{aligned}
T_{\tilde{f}}(r, L(\bar{C})) &= \int_1^r \frac{dt}{t}\int_{\Delta(t)} \tilde{f}^*(p_1^*\omega_1 + (d-1)p_2^*\omega_2) \\
&= \int_1^r \frac{dt}{t}\int_{\Delta(t)} dd^c \log\left(1 + \left|\frac{f_1}{f_0}\right| + \left|\frac{f_2}{f_0}\right|\right) + (d-1)\int_1^r \frac{dt}{t}\int_{\Delta(t)} dd^c \log\left(1 + \left|\frac{f_2}{f_1}\right|\right) \\
&= \int_{C(r)} \log\left(1 + \left|\frac{f_1(z)}{f_0(z)}\right| + \left|\frac{f_2(z)}{f_0(z)}\right|\right)\frac{d\theta}{2\pi} + (d-1)\int_{C(r)} \log\left(1 + \left|\frac{f_2(z)}{f_1(z)}\right|\right)\frac{d\theta}{2\pi} \\
&= T_f(r) + (d-1)T_{f_1}(r).
\end{aligned}$$

This shows our assertion. Thus we get

$$T_{\tilde{f}}(r, L(\bar{C})) = (\alpha_0 + (d-1)\alpha_1 + o(1))(\log r)^2.$$

Hence we have

$$\begin{aligned}
\delta_{\tilde{f}}(\bar{C}) &= 1 - \limsup_{r\to+\infty} \frac{N(r, \bar{C})}{T_{\tilde{f}}(r, L(C))} \\
&= 1 - \lim_{r\to+\infty} \frac{d(1+o(1))\alpha_1(\log r)^2}{(\alpha_0 + (d-1)\alpha_1 + o(1))(\log r)^2} \\
&= \frac{\alpha}{1 + (1-\alpha)(d-1)}.
\end{aligned}$$

Therefore we have the desired conclusion. □

REMARK 5.2. We give here a remark on the above estimate for $\delta_{\tilde{f}}(\bar{C})$. We first recall the defect relation for dominant meromorphic mappings into Σ_1. Let L be an ample line bundle over Σ_1 and $K(\Sigma_1)$ the canonical bundle of Σ_1. Set

$$\gamma(L) = \left[\frac{K(\Sigma_1)^*}{L}\right] = \inf\{\gamma \in \boldsymbol{Q};\ \gamma c_1(L) + c_1(K(\Sigma_1)) > 0\}.$$

Let $D_j \in |L|$ and assume that $D_1 + \cdots + D_q$ has simple normal crossings. Then a defect relation for dominant meromorphic mappings $f : \boldsymbol{C}^m \to \Sigma_1$ is given by the following (see [**S2**, Corollary 3.3]):

$$\sum_{j=1}^q \delta_f(D_j) \le \gamma(L).$$

We will calculate the value of $\gamma(L)$. For the line bundle L, there exists an ample divisor D such that $L = L(D)$. We may assume $D = aE + bF_\infty$, where a and b are positive integers with $a < b$. Let K_{Σ_1} be the canonical divisor of Σ_1. It is well-known that $K_{\Sigma_1} = -2E - 3F_\infty$. Then we see

$$\gamma D + K_{\Sigma_1} = (\gamma a - 2)E + (\gamma b - 3)F_\infty.$$

By making use of Nakai's criterion (cf. [**H**, p. 380]), we easily have

$$\gamma(L) = \frac{1}{b-a} \quad \text{if} \quad 3a - 2b \geq 0 \qquad \text{and} \qquad \gamma(L) = \frac{2}{a} \quad \text{if} \quad 3a - 2b < 0.$$

If $L = L(\overline{C})$, we have $\gamma(L(\overline{C})) = 2$. In the case where target spaces are complex projective spaces, an ample line bundle L is written as $L = L(H)^{\otimes d}$ for some positive integer d. Hence we have the defect relation and the conjecture mentioned in the Introduction. For a meromorphic mapping $f : \boldsymbol{C}^m \to \Sigma_1$ with the Zariski dense image, we expect that the above defect relation holds under a suitable condition on singularities of D.

§6. Appendix.

In this section, we give the resolution of singularity of C_d, and calculate the Chern classes of line bundles determined by the proper transform $\overline{C}$ of C_d. For background materials, we refer to [**GH**] and [**H**]. We first give the resolution of singularity. Recall the singular curve C_d defined by $\zeta_0\zeta_2^{d-1} - \zeta_1^d = 0$. This curve has only one singular point $\mathrm{P}(1,0,0)$ if $d \geq 3$. Then we have the following:

LEMMA 6.1. *Let $\pi : Q_{\mathrm{P}}(\boldsymbol{P}_2(\boldsymbol{C})) \to \boldsymbol{P}_2(\boldsymbol{C})$ be a monoidal transformation at the center* P. *Denote by $\tilde{C}$ and $\overline{C}$ the total transform and the proper transform of C_d, respectively. Then the total transform $\tilde{C}$ is given by*

$$\tilde{C} = (d-1)E + \overline{C},$$

where E is the exceptional curve of the first kind and the proper transform $\overline{C}$ is nonsingular.

PROOF. Let U_0 be the affine open set determined by $\zeta_0 \neq 0$ in $\boldsymbol{P}_2(\boldsymbol{C})$. Set $x = \zeta_1/\zeta_0$ and $y = \zeta_2/\zeta_0$ in U_0. We define an affine curve C_0 such that $C_0 = C_d \cap U_0$. Then we have the defining equation of the affine curve C_0 as follows:

$$x^d - y^{d-1} = 0.$$

We now give a resolution of singularity of C_0 at $(x,y) = (0,0)$. Let $\varpi : \tilde{U}_0 \to U_0$ be the blowing up centered at $(0,0)$. Let $\{(x_1,y_1)\}$ and $\{(x_2,y_2)\}$ be local coordinate systems in $\tilde{U}_0$. By definition of a blowing up, we consider the following two cases:

Case I. The pull back of C_d by $x = x_1$, $y = y_1x_1$.

In this case, we have one of the affine part of the total transform of C_d defined by $x_1^d - y_1^{d-1}x_1^{d-1} = 0$ and the exceptional curve E_1 defined by $x_1 = 0$. We also have a nonsingular curve $\overline{C}_0$ defined by $x_1 = y_1^{d-1}$. Since

$$\dim_{\boldsymbol{C}} \boldsymbol{C}[[x_1,y_1]]/(x_1, x_1 - y_1^{d-1}) = d-1,$$

we see that $\bar{C_0}$ and E_1 intersects each other at $(0,0)$ with multiplicity $d-1$.

Case II. The pull back of C_d by $x = x_2y_2$, $y = y_2$.

In this case, we see that another part of the total transform defined by $x_2^d y_2^d - \underline{y_2}^{d-1} = 0$ and the exceptional curve E_2 is defined by $y_2 = 0$. We also have a nonsingular curve C_0' defined by $x_2^d y_2 - 1 = 0$. Note that the multiplicity of E_2 is $d-1$ and $C_0' \cap E_2 = \emptyset$.

Now we have the exceptional curve of the first kind E by patching up E_1 and E_2. We also have the proper transform $\bar{C_0}$ of C_0. By taking a completion of $\bar{C_0}$, we have the proper transform $\bar{C}$ of C_d. It is clear that $\bar{C}$ is nonsingular. Therefore we have the desired conclusion. □

Next we calculate the Chern form of the line bundle $L(\bar{C})$. Let Σ_1 be the Hirzebruch surface of rank one, that is,

$$\Sigma_1 = \{(\zeta_0 : \zeta_1 : \zeta_2; \xi_0 : \xi_1) \in \boldsymbol{P}_2(\boldsymbol{C}) \times \boldsymbol{P}_1(\boldsymbol{C});\ \xi_0\zeta_2 - \xi_1\zeta_1 = 0\}.$$

Then it is well-known that $Q_{\mathrm{P}}(\boldsymbol{P}_2(\boldsymbol{C})) = \Sigma_1$. Let $p_1 : \boldsymbol{P}_2(\boldsymbol{C}) \times \boldsymbol{P}_1(\boldsymbol{C}) \to \boldsymbol{P}_2(\boldsymbol{C})$ and $p_2 : \boldsymbol{P}_2(\boldsymbol{C}) \times \boldsymbol{P}_1(\boldsymbol{C}) \to \boldsymbol{P}_1(\boldsymbol{C})$ be the natural projections. We also denote by p_j the restriction of p_j to Σ_1. Then it is clear that $E = {p_1}^{-1}((1:0:0))$. Let $F_\infty = {p_2}^{-1}((0:1))$. It is well-known that the divisor class group $\mathrm{Cl}(\Sigma_1)$ is generated by F_∞ and E. To calculate the Chern form of $L(\bar{C})$, we first determine a local coordinate system on Σ_1. Set

$$U_i = \{(\zeta_0 : \zeta_1 : \zeta_2) \in \boldsymbol{P}_2(\boldsymbol{C});\ \zeta_i \neq 0\} \quad \text{and} \quad V_j = \{(\xi_0 : \xi_1) \in \boldsymbol{P}_1(\boldsymbol{C});\ \xi_j \neq 0\}.$$

We also set $W_{ij} = (U_i \times V_j) \cap \Sigma_1$. Let $t = \xi_1/\xi_0$ on V_0 and $\tau = \xi_0/\xi_1$ on V_1. Let $x = \zeta_1/\zeta_0$ and $y = \zeta_2/\zeta_0$ on $(U_0 \times \boldsymbol{P}_1(\boldsymbol{C})) \cap \Sigma_1$. If we set $\mathscr{U}_0 = W_{00}$ and $\mathscr{U}_1 = W_{01}$, then (x,t) and (y,τ) give local coordinate systems on $\mathscr{U}_0$ and $\mathscr{U}_1$, respectively. Let $u = \zeta_0/\zeta_1$ and $v = \zeta_2/\zeta_1$ on $(U_1 \times \boldsymbol{P}_1(\boldsymbol{C})) \cap \Sigma_1$. If we set $\mathscr{U}_2 = W_{10}$, then we have a local coordinate system (u,t) on $\mathscr{U}_2$. Let $z = \zeta_0/\zeta_2$ and $w = \zeta_1/\zeta_2$ on $(U_2 \times \boldsymbol{P}_1(\boldsymbol{C})) \cap \Sigma_1$. Set $\mathscr{U}_3 = W_{21}$ and determine a local coordinate system (z,τ) on $\mathscr{U}_3$. Hence we have a system of local coordinate neighborhoods $\{\mathscr{U}_0, \cdots, \mathscr{U}_3\}$ as follows:

$$\mathscr{U}_0 = \{(1 : x : tx; 1 : t)\}, \quad \mathscr{U}_1 = \{(1 : \tau y : y; \tau : 1)\},$$
$$\mathscr{U}_2 = \{(u : 1 : t; 1 : t)\}, \quad \mathscr{U}_3 = \{(z : \tau : 1; \tau : 1)\}.$$

The change of local coordinate systems is given by

$$\begin{aligned}
&y = tx,\ \tau = 1/t \quad \text{on} \quad \mathscr{U}_0 \cap \mathscr{U}_1, && u = 1/x \quad \text{on} \quad \mathscr{U}_0 \cap \mathscr{U}_2,\\
&x = \tau/z,\ \tau = 1/t \quad \text{on} \quad \mathscr{U}_0 \cap \mathscr{U}_3, && y = t/u,\ t = 1/\tau \quad \text{on} \quad \mathscr{U}_1 \cap \mathscr{U}_2,\\
&z = 1/y \quad \text{on} \quad \mathscr{U}_1 \cap \mathscr{U}_3, && z = u/t,\ \tau = 1/t \quad \text{on} \quad \mathscr{U}_2 \cap \mathscr{U}_3.
\end{aligned}$$

Note that $\mathscr{U}_j \cong \boldsymbol{C}^2$ for all j.

Next, we calculate transition functions $\{\psi_{\alpha\beta}\}$ of the line bundle $L(F_\infty)$. It is clear that $\mathscr{U}_0 \cap F_\infty = \mathscr{U}_2 \cap F_\infty = \emptyset$. We also have

$$\mathscr{U}_1 \cap F_\infty = \{(1 : \tau y : y; \tau : 1);\ \tau = 0\} \quad \text{and} \quad \mathscr{U}_3 \cap F_\infty = \{(z : \tau : 1; \tau : 1);\ \tau = 0\}.$$

Hence we have transition functions $\{\psi_{\alpha\beta}\}$ as follows: $\psi_{01} = \psi_{03} = t$ and $\psi_{02} = 1$. Note that we can determine other transition functions ψ_{12}, ψ_{13} and ψ_{23} from the following relations: $\psi_{12} = \psi_{10}\psi_{02}$, $\psi_{13} = \psi_{10}\psi_{03}$ and $\psi_{23} = \psi_{20}\psi_{03}$. Note that $\mathscr{U}_2 \cap E = \mathscr{U}_3 \cap E = \emptyset$. Since

$$\mathscr{U}_0 \cap E = \{(1 : x : tx; 1 : t); x = 0\} \quad \text{and} \quad \mathscr{U}_1 \cap E = \{(1 : \tau y : y; \tau : 1); y = 0\},$$

transition functions $\{\varphi_{\alpha\beta}\}$ of $L(E)$ are given by $\varphi_{01} = \tau$ and $\varphi_{02} = \varphi_{03} = x$. Note that the total transform $\tilde{C}$ of C_d is represented by

$$\tilde{C} = \{(\zeta_0 : \zeta_1 : \zeta_2; \xi_0 : \xi_1) \in \boldsymbol{P}_2(\boldsymbol{C}) \times \boldsymbol{P}_1(\boldsymbol{C}); \xi_0\zeta_2 = \xi_1\zeta_1,\ \zeta_0\zeta_2^{d-1} = \zeta_1^d\}$$

in Σ_1. Then we have

$$\mathscr{U}_0 \cap \tilde{C} = \{(1 : x : tx; 1 : t);\ x^d - (tx)^{d-1} = 0\},$$

$$\mathscr{U}_1 \cap \tilde{C} = \{(1 : \tau y : y; \tau : 1);\ (\tau y)^d - y^{d-1} = 0\},$$

$$\mathscr{U}_2 \cap \tilde{C} = \{(u : 1 : t; 1 : t);\ 1 - ut^{d-1} = 0\},$$

$$\mathscr{U}_3 \cap \tilde{C} = \{(z : \tau : 1; \tau : 1);\ \tau^d - z = 0\}.$$

Hence we have transition functions $\{\phi_{\alpha\beta}\}$ of $L(\tilde{C})$: $\phi_{01} = 1$, $\phi_{02} = x^d$ and $\phi_{03} = y^d$. Note that $\phi_{01} = (\psi_{01})^d(\varphi_{01})^d$, $\phi_{02} = (\psi_{02})^d(\varphi_{02})^d$ and $\phi_{03} = (\psi_{03})^d(\varphi_{03})^d$. Thus we see

$$\tilde{C} = d(E + F_\infty) \tag{6.2}$$

in $\mathrm{Cl}(\Sigma_1)$. Let ω_2 and ω_1 be the Fubini-Study forms on $\boldsymbol{P}_2(\boldsymbol{C})$ and $\boldsymbol{P}_1(\boldsymbol{C})$, respectively, that is, $\omega_1 = dd^c \log\|\xi\|^2$ and $\omega_2 = dd^c \log\|\zeta\|^2$. We obtain the following local expressions of ω_1 on V_0 and ω_2 on U_0, respectively:

$$\omega_1 = \frac{\sqrt{-1}}{2\pi}\frac{dt \wedge \overline{dt}}{(1+|t|^2)^2}, \quad \omega_2 = \frac{\sqrt{-1}}{2\pi}\left(\frac{dx \wedge \overline{dx} + dy \wedge \overline{dy}}{1+|x|^2+|y|^2} - \frac{(\bar{x}dx + \bar{y}dy) \wedge (x\overline{dx} + y\overline{dy})}{(1+|x|^2+|y|^2)^2}\right).$$

Next we take curves C_j on Σ_1 such that $c_1(L(C_j)) = \omega_j$ for $j = 1, 2$. Then we see

$$C_1 \cdot E = \int_E p_2^*\omega_1 = \int_{t \in \boldsymbol{C}} \frac{\sqrt{-1}}{2\pi}\frac{dt \wedge \overline{dt}}{(1+|t|^2)^2} = 1,$$

$$C_1 \cdot F_\infty = \int_{F_\infty} p_2^*\omega_1 = \int_{\{\tau=0\}} \frac{\sqrt{-1}d\tau \wedge \overline{d\tau}}{2\pi(1+|\tau|^2)^2} = 0,$$

$$C_2 \cdot E = \int_E p_1^*\omega_2 = \int_{\{x=0\}} p_1^*\omega_2 = 0,$$

$$C_2 \cdot F_\infty = \int_{F_\infty} p_1^*\omega_2 = \int_{y \in \boldsymbol{C}} \frac{\sqrt{-1}dy \wedge \overline{dy}}{2\pi(1+|y|^2)^2} = 1.$$

Take curves C_j on Σ_1 such that $c_1(L(C_j)) = \omega_j$ for $j = 1, 2$. We write C_j as follows: $C_1 = a_1E + b_1F_\infty$ and $C_2 = a_2E + b_2F_\infty$, where a_j and b_j are integers. Hence, by the above calculation, we have that $C_1 = F_\infty$ and $C_2 = E + F_\infty$. Thus we have

$$p_1^*\omega_1 = c_1(L(F_\infty)), \qquad p_2^*\omega_2 = c_1(L(E) \otimes L(F_\infty)). \tag{6.3}$$

Therefore, by Lemma 6.1, (6.2) and (6.3), we have the following:

LEMMA 6.4. *The Chern form of $L(\bar{C})$ is given by*

$$c_1(L(\bar{C})) = p_2^*\omega_2 + (d-1)p_1^*\omega_1.$$

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