Deficiencies of meromorphic mappings for hypersurfaces

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Abstract. In this paper we first prove that, for every hypersurface $D$ of degree $d$ in a complex projective space, there exists a holomorphic curve $f$ from the complex plane into the projective space whose deficiency for $D$ is positive and less than one. Using this result, we construct meromorphic mappings from the complex $m$-space into the complex projective space with the same properties. We also investigate the effect of resolution of singularities to defects of meromorphic mappings.

Introduction.

The aim of this paper is to construct meromorphic mappings $f$ from $\mathbb{C}^m$ into the complex projective space $\mathbb{P}_n(\mathbb{C})$ with Nevanlinna’s deficient divisors. Throughout this paper, we assume that $n \geq 2$. The defect relation for meromorphic mappings shows that the set of Nevanlinna’s deficient divisors for $f$ is very small. Furthermore, meromorphic mappings without defect are dense in the space of all meromorphic mappings $f: \mathbb{C}^m \to \mathbb{P}_n(\mathbb{C})$ with respect to a certain kind of distance (see [M]). It therefore seems that the construction of meromorphic mappings with preassigned deficiencies is very difficult. There have been several studies on the construction of holomorphic curves with deficient hyperplanes. So far, we do not know the existence of examples of meromorphic mappings with a deficient irreducible hypersurface of high degree whose deficiency is less than one. In this paper, we prove the existence of meromorphic mappings that have a preassigned positive deficiency for a given divisor $D$ in $\mathbb{P}_n(\mathbb{C})$. We now recall the defect relation for dominant meromorphic mappings $f: \mathbb{C}^m \to \mathbb{P}_n(\mathbb{C})$ due to Griffiths’ school (cf. [S2]), that is, we have the defect relation

$$\sum_{j=1}^{q} \delta_f(D_j) \leq \frac{n+1}{d},$$

where $D_1, \cdots, D_q$ are nonsingular hypersurfaces of degree $d$ in $\mathbb{P}_n(\mathbb{C})$ intersecting normally. There has been a conjecture of Griffiths ([Gr, p. 379]) stating the defect relation for meromorphic mappings $f: \mathbb{C}^m \to \mathbb{P}_n(\mathbb{C})$ is also given by the above form under an appropriate nondegeneracy condition on $f$. Moreover, there also has been a conjecture such that the estimate

$$\delta_f(D) \leq \frac{C}{d}$$

holds under a generic condition for $D$, where $C$ is a positive constant independent of $f$ and $D$ (cf. [Si, p. 289]). However, in the case where $D$ is a singular divisor, we can construct many

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examples of meromorphic mappings $f$ such that estimates for deficiencies of the above type do not hold. This follows from the following main theorem concerning Griffiths’ conjecture that is the main result in this paper:

**Main Theorem.** Let $D \in |L(H)^{\circ d}|$ be an arbitrary divisor in $P_n(C)$, where $L(H)$ is the hyperplane bundle over $P_n(C)$ and $d$ is a positive integer. Then there exists a positive constant $\lambda(D)$ depending only on $D$ with $\lambda(D) \leq d$ that has the following property: For each positive number $\alpha$ with $\alpha \leq \lambda(D)/d$, there exists a meromorphic mapping $f : C^m \to P_n(C)$ with Zariski dense image such that $\delta_f(D) = \alpha$. Furthermore, in the case of $m \geq n$, there exists a dominant meromorphic mapping $f : C^m \to P_n(C)$ with $\delta_f(D) = \alpha$.

This theorem yields that for every irreducible hypersurface $S$ in $P_n(C)$ there exists a meromorphic mapping $f$ such that the deficiency $\delta_f(S)$ for $S$ is positive and less than one. We note here that, in general, the constant $\lambda(D)$ is dependent on the degree $d$. For instance, we have $\lambda(D) = d$ for some singular divisors. We give some concrete examples in §3. These examples show that we cannot obtain a good estimate on deficiency when $D$ has singularities. Furthermore, we investigate how the existence of singularities of $D$ affects an estimate for deficiencies in §5. The result in §5 shows that if we resolve singularities, we have an estimate for $\delta_f(D)$ depending on the structure on the singularities. The results obtained in this paper are rather pathological, but they suggest that the smoothness of divisors is a delicate matter to get a good bound for deficiencies. We note that the case of holomorphic curves is essential in the proof of our main theorem. The method used in our construction is elementary and based on the theory of entire functions of one complex variable, especially, on some properties of entire functions of order zero proved by Valiron [V2]. For this reason, we first prove the above theorem for holomorphic curves in §3. By making use of the idea of the proof for holomorphic curves, we prove the general case in §4.

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§1. Preliminaries.

We first recall some known facts on Nevanlinna theory of holomorphic curves and meromorphic mappings. Let $z = (z_1, \cdots, z_m)$ be the natural coordinate system in $C^m$, and set

\[ ||z||^2 = \sum_{u=1}^{m} z_u \bar{z}_u, \]
\[ B(r) = \{ z \in C^m; ||z|| < r \}, \]
\[ S(r) = \{ z \in C^m; ||z|| = r \}, \]
\[ d^c = \frac{\sqrt{-1}}{4\pi} (\partial - \bar{\partial}), \]
\[ \nu = dd^c ||z||^2, \quad \sigma = d^c \log ||z||^2 \wedge (dd^c \log ||z||^2)^{m-1}. \]

In the case $m = 1$, we write $\Delta(r)$ for $B(r)$ and $C(r)$ for $S(r)$, respectively.

For a $(1,1)$-current $\varphi$ of order zero on $C^m$ we set

\[ n(r, \varphi) = r^{2-2m} \langle \varphi \wedge \nu^{m-1}, \chi_{B(r)} \rangle \]
and

\[ N(r, \varphi) = \int_{1}^{r} n(t, \varphi) \frac{dt}{t}, \]

where \( \chi_{B(r)} \) denotes the characteristic function of \( B(r) \).

Let \( M \) be a compact complex manifold and \( L \to M \) a line bundle over \( M \). We denote by \( \Gamma(M, L) \) the space of all holomorphic sections of \( L \to M \). Let \( |L| = P(\Gamma(M, L)) \) be the complete linear system defined by \( L \). For a divisor \( D \) on \( M \), we denote by \( L(D) \) the line bundle over \( M \) defined by \( D \). Let \( | \cdot | \) be a hermitian fiber metric in \( L \) and let \( \omega \) be its Chern form. A meromorphic mapping \( f : \mathbb{C}^m \to M \) is said to be \textit{dominant} if

\[ \dim M = \text{rank} f := \max \{ \text{rank} df(z) ; z \in \mathbb{C}^m - I(f) \}, \]

where \( I(f) \) is the indeterminacy locus of \( f \). For a meromorphic mapping \( f : \mathbb{C}^m \to M \), we define

\[ T_f(r, L) = N(r, f^* \omega) \]

and call it the characteristic function of \( f \) with respect to \( L \). Let \( L(H) \to P_n(\mathbb{C}) \) be the hyperplane bundle over \( P_n(\mathbb{C}) \) and \( \omega_0 \) the Fubini-Study form on \( P_n(\mathbb{C}) \). In the case where \( M = P_n(\mathbb{C}) \) and \( L = L(H) \), we always take \( \omega_0 \) for \( \omega \) and we simply write \( T_f(r) \) for \( T_f(r, L(H)) \). Let \( E \) be an effective divisor on \( \mathbb{C}^m \). Then we call \( N(r, E) \) the counting function of \( E \). For a meromorphic function \( f \) on \( \mathbb{C} \) and a point \( a \in P_1(\mathbb{C}) \), we write \( N(r, a, f) \) for \( N(r, f^*a) \). Let \( L \to P_n(\mathbb{C}) \) be a positive line bundle over \( P_n(\mathbb{C}) \). Then \( L = L(H)^{\otimes d} \) for some positive integer \( d \) and \( D \in |L| \) is a hypersurface of degree \( d \) in \( P_n(\mathbb{C}) \). It is clear that, if \( L = L(H)^{\otimes d} \), then

\[ T_f(r, L) = dT_f(r) + O(1). \]

Let \( f = (f_0, \cdots, f_n) \) be a reduced representation of \( f \). It is well-known that

\[ T_f(r) = \int_{S(r)} \log \left( \max_{0 \leq j \leq n} |f_j(z)| \right) \sigma(z) + O(1). \]

This representation of the characteristic function of \( f \) is essentially due to H. Cartan. For a positive increasing function \( \Lambda(r) \) defined on \( \mathbb{R}^+ \), we define the order \( \rho_{\Lambda} \) of \( \Lambda(r) \) by

\[ \rho_{\Lambda} = \limsup_{r \to +\infty} \frac{\log \Lambda(r)}{\log r}. \]

We define the order \( \rho_f \) of \( f \) by taking \( \Lambda(r) = T_f(r) \). We now have the following well-known Nevanlinna’s inequality:

**Theorem 1.1.** Let \( f : \mathbb{C}^m \to M \) be a nonconstant meromorphic mapping and \( L \to M \) a line bundle. Then

\[ N(r, f^*D) \leq T_f(r, L) + O(1) \]

for a divisor \( D \in |L| \) with \( f(\mathbb{C}) \nsubseteq \text{Supp}D \), where \( O(1) \) stands for a bounded term as \( r \to +\infty \).

Let \( f \) and \( D \) be as in Theorem 1.1. We define Nevanlinna’s deficiency \( \delta_f(D) \) by...
\[ \delta_f(D) = 1 - \limsup_{r \to +\infty} \frac{N(r, f^* D)}{T_f(r, L)}. \]

It is clear that \( 0 \leq \delta_f(D) \leq 1 \). If \( \delta_f(D) > 0 \), then \( D \) is called a deficient divisor in the sense of Nevanlinna.

We next recall properties of entire functions of one complex variable. For a holomorphic function \( f \) on \( \mathbb{C} \), we denote by \( M(r, f) \) the maximum modulus of \( f \) on the circle \( C(r) \), that is,

\[ M(r, f) = \max_{|z|=r} |f(z)|. \]

We also note that the characteristic function of an entire function \( f \) can be written as

\[ T_f(r) = \int_{C(r)} \log^+ |f(z)| \frac{d\theta}{2\pi} + O(1), \]

where \( \log^+ x = \max\{\log x, 0\} \). Let \( \Lambda_1(r) \) and \( \Lambda_2(r) \) be positive increasing functions defined on \( \mathbb{R}^+ \). We write

\[ \Lambda_1(r) = (1 + o(1)) \Lambda_2(r) \]

provided that

\[ \lim_{r \to +\infty} \frac{\Lambda_1(r)}{\Lambda_2(r)} = 1. \]

The following theorem due to Valiron plays a specially important role in this paper (see [V1, Chapter 5] and [V2, pp. 28–29]):

**Theorem 1.2 (Valiron).** Let \( f \) be a transcendental holomorphic function on \( \mathbb{C} \). Suppose that \( T_f(r) = O((\log r)^2) \) as \( r \to +\infty \). Then

\[ \lim_{r \to +\infty} \frac{\log M(r, f)}{T_f(r)} = \lim_{r \to +\infty} \frac{N(r, 0, f)}{T_f(r)} = 1. \]

Furthermore, there exists a Borel subset \( \varepsilon(r) \) of \( C(r) \) such that

\[ \log |f(z)| = (1 + o(1)) \log M(r, f) \]

for all \( z \in C(r) \setminus \varepsilon(r) \) and \( \mu(\varepsilon(r)) \to 0 \) as \( r \to +\infty \), where \( \mu \) denotes the Haar measure on \( C(r) \) normalized so that \( \mu(C(r)) = 1 \).

**Remark 1.3.** We give here some remarks on the exceptional set \( \varepsilon(r) \). There exists the exceptional set (say \( \mathcal{E} \)) for \( f \) such that \( \varepsilon(r) = C(r) \cap \mathcal{E} \). The set \( \mathcal{E} \) is a countable union of circles not containing the origin and substanding angles at the origin whose sum \( s \) is finite. Namely, the set \( \mathcal{E} \) is given by

\[ \mathcal{E} = \bigcup_{i=1}^{+\infty} C_i, \]

where \( C_i \) is a circle that has the radius \( r_i \) and the center distance \( e_i \) from the origin. Then we have
\[ s = 2 \sum_{i=1}^{+\infty} \arcsin \left( \frac{r_i}{e_i} \right). \]

Note that the zero set of \( f \) is contained in \( \mathcal{E} \). For details, see [Ha2, pp. 75–76].

\[ \sum_{n=1}^{+\infty} \frac{1}{p_j} < +\infty. \]
Now we take the Weierstrass product

\[ \varphi(z) = \prod_{j=1}^{+\infty} \left( 1 - \frac{z}{p_j} \right)^2. \]

Then it follows from the standard estimate for the Weierstrass product (cf., e.g., Hayman [Ha1, p. 27, Theorem 1.11]) that

\[ \log |\varphi(z)| \leq \int_0^r \frac{n(t,E)}{t} dt + r \int_r^{+\infty} \frac{n(t,E)}{t^2} dt \]
\[ = N(r,E) + 2\alpha \log r + O(1) \]
\[ = \alpha (\log r)^2 + o((\log r)^2) \]

for \( z \in C(r) \). Hence we get

\[ T_\varphi(z) \leq \alpha (\log r)^2 + o((\log r)^2). \]

On the other hand, by the first main theorem, we see

\[ T_\varphi(r) \geq N(r,E) + O(1) = \alpha (\log r)^2 + o((\log r)^2). \]

Therefore, we have

\[ \|f\|_L = \alpha (\log r)^2 + o((\log r)^2). \]
\[ T_\varphi(r) = \alpha(\log r)^2 + o((\log r)^2). \]

We now define a holomorphic curve \( f = (f_0, \cdots, f_n) : \mathbf{C} \to \mathbf{P}_n(\mathbf{C}) \) as follows. Let \( \alpha_0 \) and \( \alpha_1 \) be positive real numbers with \( \alpha_1 < \alpha_0 \). By Lemma 2.1, we have entire functions \( f_0 \) and \( f_1 \) such that

\[ T_{f_j}(r) = \alpha_j(\log r)^2 + o((\log r)^2) \quad (j = 0, 1). \]

Next, let \( f_2, \cdots, f_n \) be transcendental entire functions such that \( T_{f_j}(r) = o((\log r)^2) \) for \( j = 2, \cdots, n \). We define a holomorphic curve \( f : \mathbf{C} \to \mathbf{P}_n(\mathbf{C}) \) by \( f = (f_0, \cdots, f_n) \). We now prove the following lemma that is a crucial step in our construction of holomorphic curves with deficiencies:

**Lemma 2.2.** Let \( f : \mathbf{C} \to \mathbf{P}_n(\mathbf{C}) \) be as above. Then

\[ T_f(r) = T_{f_0}(r) + o((\log r)^2) \quad \text{as} \quad r \to +\infty. \]

**Proof.** Since \( T_{f_j}(r) = O((\log r)^2) \) as \( r \to +\infty \), we have

\[ T_{f_j}(r) = \int_{C(r)} \log |f_j(z)| \frac{d\theta}{2\pi} + o((\log r)^2). \]

Hence, by using Cartan’s representation of the characteristic function, we see

\[ T_{f_0}(r) + o((\log r)^2) \leq T_f(r). \]

By Theorem 1.2, we see

\[ T_f(r) = \int_{C(r)} \log \left( \max_j |f_j(z)| \right) \frac{d\theta}{2\pi} + O(1) \]

\[ \leq \int_{C(r)} \log \left( (n+1) \max_{0 \leq j \leq n} M(r, f_j) \right) \frac{d\theta}{2\pi} + O(1) \]

\[ = \max_{0 \leq j \leq n} \log M(r, f_j) + O(1) \]

\[ = \max_{0 \leq j \leq n} (1 + o(1)) T_{f_j}(r) + O(1) \]

\[ = T_{f_0}(r) + o((\log r)^2). \]

Therefore, we have our assertion. \( \square \)

**Remark 2.3.** We can construct the above holomorphic curve \( f \) with the Zariski dense image. Its proof is however delicate, and will be given in the proof of Theorem 3.2 in §3.

§3. **Construction of holomorphic curves with deficient divisor.**

In this section we prove our main result for holomorphic curves \( f : \mathbf{C} \to \mathbf{P}_n(\mathbf{C}) \). We denote by \( \zeta = (\zeta_0, \cdots, \zeta_n) \) a homogeneous coordinate system in \( \mathbf{P}_n(\mathbf{C}) \). We first consider the case of hyperplane. We have the following that is a direct conclusion of Lemma 2.2:
THEOREM 3.1. Let α be an arbitrary positive real number less than one and let H be an arbitrary hyperplane in $\mathbb{P}_n(\mathbb{C})$. Then there exists a holomorphic curve $f : \mathbb{C} \to \mathbb{P}_n(\mathbb{C})$ with the Zariski dense image such that $\delta_f(H) = \alpha$.

PROOF. Without loss of generality, we may assume that $H = \{\zeta_1 = 0\}$. We consider a holomorphic curve $f : \mathbb{C} \to \mathbb{P}_n(\mathbb{C})$ defined in §2. We can choose such a $f$ such that $f$ has the Zariski dense image (see the proof of Theorem 3.2 below). Now we take $\alpha_0$ and $\alpha_1$ such that $1 - \alpha = \alpha_1/\alpha_0$. Note that

$$T_f(r) = N(r, 0, f) + o((\log r)^2)$$

$$= N(r, f^*H) + o((\log r)^2).$$

It follows from $T_f(r) = T_{f_0}(r) + o((\log r)^2)$ that

$$\delta_f(H) = 1 - \limsup_{r \to +\infty} \frac{N(r, f^*H)}{T_f(r)}$$

$$= 1 - \lim_{r \to +\infty} \frac{\alpha_1(\log r)^2 + o((\log r)^2)}{\alpha_0(\log r)^2 + o((\log r)^2)}$$

$$= \alpha. \quad \square$$

We next deal with the case where a given divisor $D$ is a hypersurface of degree $d$ not less than two, that is, $D \in |L(H)^{\otimes d}|$ with $d \geq 2$. Let $P(\zeta) = P(\zeta_0, \ldots, \zeta_n)$ be a homogeneous polynomial of degree $d$ and define a divisor $D$ in $\mathbb{P}_n(\mathbb{C})$ by $P = 0$. Note that $D$ may be a reducible hypersurface. We now prove the following existence theorem:

THEOREM 3.2. There exists a positive constant $\lambda(D)$ with $\lambda(D) \leq d$ depending only on $D$ that satisfies the following property: For each positive real number $\alpha$ with $\alpha \leq \lambda(D)/d$, there exists a holomorphic curve $f : \mathbb{C} \to \mathbb{P}_n(\mathbb{C})$ with the Zariski dense image such that $\delta_f(D) = \alpha$.

PROOF. We first show the existence of holomorphic curves $f$ with $\delta_f(D) = \alpha$. For a given divisor $D$, we take $f_0, \cdots, f_n$ in the following way. We write $P(\zeta)$ as follows:

$$P(\zeta) = P_1(\zeta) + P_2(\zeta) = \sum_{j=0}^n c_j \zeta_j + P_2(\zeta).$$

Let $d_j$ be the highest degree in $\zeta_j$ that are contained in $P$ and set $d = \min_{0 \leq j \leq n} d_j$. We consider the following three cases.

Case I. No $c_j$ is zero.

Take entire functions $f_0$ and $f_1$ so that $T_{f_0}(r) = \alpha_0(1 + o(1))(\log r)^2$ and $f_1 = o(f_0 + 1)$, where $o$ is a nonzero constant. We also take an entire function $f_2$ such that $T_{f_2}(r) = \alpha_1(1 + o(1))(\log r)^2$, where $\alpha_1 < \alpha_0$. Furthermore, we take transcendental entire functions $f_3, \cdots, f_n$ so that $T_{f_j}(r) = o(T_{f_{j-1}}(r))$ for $j = 3, \cdots, n$. Define a holomorphic curve $f : \mathbb{C} \to \mathbb{P}_n(\mathbb{C})$ by $f = (f_0, \cdots, f_n)$. Then, as in the proof of Lemma 2.2, we easily see

$$T_f(r) = \alpha_0(\log r)^2 + o((\log r)^2).$$
Set $F = P(f)$. We now choose $\omega$ such that $F$ does not contain the term $f_0^d, \ldots, f_0^{d-\kappa+1}$, where $\kappa$ is a nonnegative integer depending only on $D$. To this end, we rewrite $P$ as follows:

$$P(\zeta) = \sum_{j=0}^{d} a_j \zeta_0^{d-j} \zeta_1^j + c_2 \zeta_2^d + \cdots + c_n \zeta_n^d + Q(\zeta),$$

where $Q$ is a polynomial in $\zeta$ which does not contain terms $\zeta_0^{d-j} \zeta_1^j$ for $j = 0, \ldots, d$. We note that

$$\sum_{j=0}^{d} a_j f_0^{d-j} f_1^j = \sum_{j=0}^{d} a_j f_0^{d-j} (\omega(f_0 + 1))^j$$

$$= a_0 f_0^d + a_1 f_0^{d-1} \omega(f_0 + 1) + a_2 f_0^{d-2} \omega^2(f_0 + 1)^2 + \cdots + a_d \omega^d(f_0 + 1)^d$$

$$= (a_0 + a_1 \omega + a_2 \omega^2 + \cdots + a_d \omega^d) f_0^d + \text{(the lower terms of } f_0).$$

We define a polynomial $L(z)$ in $z$ by

$$L(z) = \sum_{j=0}^{d} a_j z^j. \quad (3.3)$$

Let $\omega$ be a root $\omega$ of $L(z) = 0$. Then we can write the entire function $F$ as $F = F_1 + F_2$, where

$$F_1 = \sum_{j=1}^{d} c_j' x_0^{d-j} + c_2 f_2^d + \sum_{j=3}^{n} c_j f_j^d.$$

It is easy to see that

$$c_k' = \frac{\omega^k}{k!} L^{(k)}(\omega).$$

Since $a_0 \neq 0$ and $a_d \neq 0$, we see $\omega \neq 0$. Thus, if $\omega$ is a multiple root of $L(x) = 0$ with multiplicity $k$, then

$$c_1' = \cdots = c_{k-1}' = 0.$$

Now we take $\kappa$ such that $\kappa$ is the largest multiplicity of the roots of $L(x) = 0$. Note that $1 \leq \kappa \leq d - 1$. Define $\lambda(D) = \kappa$. We assume that $\alpha < \kappa/d$. Take $\alpha_0$ and $\alpha_1$ so that $\alpha = 1 - \alpha_1/\alpha_0$. Note that $(d - \kappa)\alpha_0 < d\alpha_1$. By Theorem 1.2, we get $N(r, f^*D) = (1 + o(1)) T_F(r)$. We write $F_2$ as $\sum_{|v|=d} c_v f^v$, where $v = (v_0, \cdots, v_n)$. We now consider the following two subcases.

Subcase I: The case where $F_2$ does not contain terms with $v_0 + v_1 + v_2 = d - j$ for $0 \leq j \leq \kappa$, that is, the coefficients of those terms are zero.

We will show that

$$T_F(r) = d(1 + o(1)) T_{F_2}(r). \quad (3.4)$$

By the definition of characteristic functions, we easily see
\[ T_F(r) = \int_{C(r)} \log^+ |F(z)| \frac{d\theta}{2\pi} + O(1) \]
\[ \leq \int_{C(r)} \log^+ |f_2(z)|^d \frac{d\theta}{2\pi} + o((\log r)^2) \]
\[ \leq d(1 + o(1))T_{f_2}(r). \]

Hence we get \( T_F(r) \leq d(1 + o(1))T_{f_2}(r) \). Next we show \( d(1 + o(1))T_{f_2}(r) \leq T_F(r) \). We first assume that \( \alpha < \kappa/d \). Then \( (d - \kappa)\alpha_0 < d\alpha_1 \). For any sufficiently large \( r \) and for \( z \in C(r) \setminus \varepsilon(r) \), we have

\[ |F(z)| \geq |f_2(z)|^d \left( |c_2| - \frac{\sum_{j=1}^{d} c_j f_1^{d-j}(z)}{f_2^d(z)} \right) \]
\[ \geq |f_2(z)|^d \left( |c_2| - \frac{K \exp((1 + o(1))(d - \kappa)\alpha_0(\log r)^2)}{\exp((1 + o(1))d\alpha_1(\log r)^2)} \right) + o(1) \]
\[ \geq |f_2(z)|^d (|c_2| - |K \exp((1 + o(1))(d - \kappa)\alpha_0 - d\alpha_1)(\log r)^2| + o(1)) \]
\[ \geq \frac{|c_2|}{2} (1 + o(1)) |f_2(z)|^d, \]

where \( K \) is a some positive constant. Hence we have

\[ T_F(r) = \int_{C(r)} \log^+ |F(z)| \frac{d\theta}{2\pi} + O(1) \]
\[ \geq \int_{C(r)} \log^+ |f_2(z)|^d \frac{d\theta}{2\pi} - \mu(\varepsilon(r)) \log^+ M(r,F) \]
\[ \geq d(1 + o(1))T_{f_2}(r). \]

Thus we get (3.4). We therefore obtain \( \delta_f(D) = \alpha \). Next assume that \( \alpha = \kappa/d \). We take \( f_0, \cdots, f_n \) such that

\[ T_{f_0}(r) = \alpha_0(1 + o(1))(\log r)^2 \quad \text{and} \quad T_{f_j}(r) = o((\log r)^2) \quad (j = 1, \cdots, n). \]

Then we easily see that

\[ T_f(r) = (1 + o(1))T_{f_0}(r) \quad \text{and} \quad N(r, f^*D) \leq (d - \kappa)(1 + o(1))T_{f_0}(r). \]

We write \( F = F^{(1)} + F^{(2)} \), where \( F^{(1)} = c_\kappa f_0^{d-\kappa} \) and \( F^{(2)} = F - F^{(1)} \). Then it is easy to see that, for any sufficiently large \( r \) and for \( z \in C(r) \setminus \varepsilon(r) \),

\[ \left| \frac{F^{(2)}(z)}{F^{(1)}(z)} \right| \leq \exp(-(1 + o(1))\alpha_0(\log r)^2). \]

Hence we easily have
\[
N(r, f^*D) = \int_{C(r)} \log^+ |F(z)| \frac{d\theta}{2\pi} + o((\log r)^2)
\]
\[
\geq \int_{C(r)} \log^+ |f_0(z)|^{d-\kappa} \frac{d\theta}{2\pi} + o((\log r)^2)
\]
\[
= (d - \kappa)(1 + o(1))T_{f_0}(r).
\]
Hence we have the estimate
\[
(d - \kappa)(1 + o(1))T_{f_0}(r) = N(r, f^*D).
\]
This shows that \(\delta_f(D) = \kappa/d\).

Subcase IIb: The case where \(F_2\) contains at least one term with \(v_0 + v_1 + v_2 = d - j\) for \(0 \leq j \leq \kappa\), that is, at least one coefficient of those terms is not zero.

We write \(F\) as
\[
F = \sum_{j=1}^{d} b_j f_0^j f_2^{d-j} + F_3,
\]
where \(F_3\) does not contain the terms \(f_0^{v_0+v_1} f_2^{v_2}\) with \(v_0 + v_1 + v_2 = d - j\) for \(0 \leq j \leq \kappa\). Let \(k = \max\{j; b_j \neq 0\}\). Since \((d - \kappa)\alpha_0 \leq d\alpha_1\), we see \((d - \kappa)\alpha_0 < k\alpha_0 + (d - k)\alpha_1\). Then the growth of \(f_0^k f_2^{d-k}\) is greater than those of \(f_0^{d-k}\) and \(f_2^d\). We may assume that the zero divisor of \(f_0\) (resp. \(f_2\)) is contained in \(\mathbb{R}^+\) (resp. \(\mathbb{R}^-\)). Then the exceptional sets for \(f_0\) and \(f_2\) does not intersect. Indeed, by the construction of \(f_0\) (see the proof of Lemma 2.1), we can write \(f_0\) as follows:
\[
f_0(z) = \prod_{v=1}^{+\infty} \left(1 - \frac{z}{p_v}\right)^2,
\]
where \(p_{v+1} > p_v > 0\). Let \(\zeta_1, \zeta_2 \in C(r)\). Suppose that \(\text{Re} \ z_1 > 0\) and \(\text{Re} \ z_2 < 0\). Then, for any \(r > 0\), we have
\[
\left|1 - \frac{\zeta_1}{p_v}\right| < \left|1 - \frac{\zeta_2}{p_v}\right|
\]
for each \(v\). Hence we obtain \(|f_0(\zeta_1)| \leq |f_0(\zeta_2)|\). This implies that the set
\[
\{z \in C(r); \text{Re} \ z < 0\}
\]
do not intersect the exceptional set for \(f_0\). Hence we have our assertion. Thus we see
\[
\log M(r, f_0^k f_2^{d-k}) = (k\alpha_0 + (d - k)\alpha_1 + o(1))(\log r)^2
\]
and hence
\[
N(r, f^*D) = (k\alpha_0 + (d - k)\alpha_1 + o(1))(\log r)^2.
\]
Thus we get
This implies that there exists a holomorphic curve $f$ with $\delta_f(D) = \alpha$ for any $0 < \alpha < \kappa/d$. Next we consider case $\alpha = \kappa/d$. If there exists the term $f_0^d = \kappa$ in $F_2$, we take transcendental entire functions $f_2, \ldots, f_n$ so that $T_f(r) = o(T_{f_{j-1}}(r))$ for $j = 2, \ldots, n$. Then $\delta_f(D) = \kappa/d$. If $F_2$ does not contain the term $f_0^d$, we get a holomorphic curve $f$ with $\delta_f(D) = \alpha$ by taking $(d - \kappa)\alpha_0 = (d - k)\alpha_1$.

Case II. Some of the $\zeta_j$'s are not contained in $P$ and $P_1(\zeta) \neq 0$.

This case is essentially the same as the Case I. We may assume that $c_0 = 0$ and $c_1 \neq 0$. We set $\lambda(D) = d$. We take transcendental entire functions $f_0, \ldots, f_n$ such that $T_{f_0}(r) = (\log r)^2 + o((\log r)^2)$ and $T_{f_j}(r) = o(T_{f_{j-1}}(r))$ for $j = 2, \ldots, n$. By a suitable choice of $f_1, \ldots, f_n$, we see $f(C) \cap D \neq \emptyset$. By this choice of the $f_j$'s, it is clear that $\delta_f(D) = 1$. Now assume that $\alpha < 1$. We take $f_0$ and $f_1$ as in 8. Take transcendental entire functions $f_2, \ldots, f_n$ so that $T_{f_j}(r) = o(T_{f_{j-1}}(r))$ for $j = 2, \ldots, n$. By Lemma 2.2, we have

$$T_f(r) = T_{f_0}(r) + o((\log r)^2)$$

as $r \to +\infty$. We consider a holomorphic function $F_0$ defined by

$$F_0 = c_1f_1^d + \cdots + c_nf_n^d + F_2.$$

Then, by a method similar to the Case I, we get a holomorphic curve $f$ with $\delta_f(D) = \alpha$.

Case III. The case $d < d$.

In this case, $P = P_2$. Without loss of generality, we may assume that $d_0 = d$. Now, we set $\lambda(D) = d - d_0$. We write $P_2 = P_2^{(0)} + P_2^{(1)}$, where $P_2^{(0)}$ is the sum of monomials that contain $\zeta_0$ and $P_2^{(1)}$ does not contain $\zeta_0$. Then we may assume

$$P_2^{(0)}(\zeta) = \zeta_0^{d_0}Q_0(\zeta') + \zeta_0^{d_0-1}Q_1(\zeta') + \cdots + Q_{d_0}(\zeta'),$$

where $\zeta' = (\zeta_1, \ldots, \zeta_p)$ and $Q_j(\zeta')$ are polynomials in $\zeta'$. We may assume that $Q_0(\zeta') = Q_0(\zeta_1, \ldots, \zeta_p)$ with $p \leq n$. We take transcendental entire functions $f_0, \ldots, f_n$ as follows. Let $T_{f_j}(r) = \alpha_j(1 + o(1))(\log r)^2$ with $\alpha_1 < \alpha_0$ for $j = 0, 1$. For $j = 2, \ldots, p$, we set $f_j = b_jf_1 + q_j$, where $b_j$ are constants and $q_j$ are transcendental entire functions such that $T_{q_j}(r) = o((\log r)^2)$ and $T_{q_j}(r) = o(T_{q_{j-1}}(r))$. We also take $T_{f_j}(r) = o(T_{f_{j-1}}(r))$ for $j = p + 1, \ldots, n$. Note that for any sufficiently large $r$ and for $z \in C(r) \setminus \epsilon(r)$, we see

$$\log |f_j(z)| = \alpha_1(\log r)^2 + o((\log r)^2)$$
for \( j = 2, \cdots, p \). Define a holomorphic curve \( f \) by \( f = (f_0, \cdots, f_n) \). Let \( F_2 = P_2^{(0)}(f) + P_2^{(1)}(f) \) and \( F_2^{(j)} = P_2^{(j)}(f) \) for \( j = 0, 1 \). Now choose \( b_1, \cdots, b_p \) so that the function \( f_1^{d-d_0} \) is contained in \( F_2^{(0)} \). We define \( \Lambda(I) = \{ (\lambda_0, \lambda_1) \in (\mathbb{Z}_{\geq 0}); \lambda_0 + \lambda_1 = I \} \). Set \( f_{01}^{\lambda} = f_0^{\lambda_0} f_1^{\lambda_1} \) for \( \lambda \in \Lambda(I) \). Then we have

\[
F_2^{(0)} = \sum_{k=0}^{d} \sum_{\lambda \in \Lambda(d-k); \lambda_0 \leq d_0} a_\lambda(z)f_0^{\lambda_0} R_\lambda(f_{p+1}, \cdots, f_n),
\]

where \( a_\lambda \) are small functions with respect to \( f_0 \). Note that \( a_{(d_0, d-d_0)} \) is a constant. Then, for any sufficiently large \( r \) and for \( z \in C(r) \setminus \varepsilon(r) \), we see

\[
|f_0^{d_0}(z)f_1^{d-d_0}(z)| = C \exp((d_0 \alpha_0 + (d - d_0) \alpha_1 + o(1))(\log r)^2),
\]

where \( C \) is a positive constant. Hence

\[
\log |F_2(z)| \leq (d_0 \alpha_0 + (d - d_0) \alpha_1 + o(1))(\log r)^2. \tag{3.5}
\]

On the other hand, we set

\[
G(z) = F_2(z) - a_{(d_0, d-d_0)} f_0(z)^{d_0} f_1(z)^{d-d_0}.
\]

Then, for any sufficiently large \( r \) and for \( z \in C(r) \setminus \varepsilon(r) \), we see

\[
|G(z)| \leq \exp((d_0 \alpha_0 + (d - d_0) \alpha_1 + o(1))(\log r)^2).
\]

Hence we easily have

\[
|F_2(z)| \geq |f_0^{d_0}(z)f_1^{d-d_0}(z)| \left( |a_{(d_0, d-d_0)}| - \left| \frac{G(z)}{f_0^{d_0}(z)f_1^{d-d_0}(z)} \right| \right) \geq \left| a_{(d_0, d-d_0)} \right| \frac{2}{\exp((d_0 \alpha_0 + (d - d_0) \alpha_1 + o(1))(\log r)^2)}. \tag{3.6}
\]

By (3.5) and (3.6), we get

\[
N(r, f^n D) = (d_0 \alpha_0 + (d - d_0) \alpha_1)(1 + o(1))(\log r)^2.
\]

Thus we obtain

\[
\delta_f(D) = 1 - \frac{d_0 \alpha_0 + (d - d_0) \alpha_1}{d \alpha_0} = \frac{\lambda(D)}{d} \left( 1 - \frac{\alpha_1}{\alpha_0} \right).
\]

Therefore, for each positive number \( \alpha \) less than \( \lambda(D)/d \), there exists a holomorphic curve \( f \) with \( \delta_f(D) = \alpha \). Furthermore, if we take entire functions \( f_1, \cdots, f_n \) such that \( T_{f_0}(r) = (1 + o(1))(\log r)^2 \) and \( T_{f_j}(r) = o((\log r)^2) \) for \( j = 1, \cdots, n \). Then we have a holomorphic curve \( f \) with \( \delta_f(D) = \lambda(D)/d \). We have now shown the existence of holomorphic curves \( f \) with the desired property. Next, we will show that the above holomorphic curves can be constructed such that they have the Zariski dense images.

**Proof of Zariski denseness of the image of \( f \).** We first consider the Case II. The proof is somewhat complicated. Hence we first give an idea of the proof as follows. Suppose
that there exists a homogeneous algebraic relation $R(f_0, \cdots, f_n) = 0$ among $f_j$'s. We rewrite $R(f_0, \cdots, f_n) = 0$ as follows:

$$-A_s f_0^s = A_{s-1} f_0^{s-1} + \cdots + A_0.$$ 

Let $\{u_\nu\}_{\nu=1}^{+\infty}$ be the zero set of $f_1$, where $|u_\nu| < |u_{\nu+1}|$ and $|u_\nu| \to +\infty$ as $\nu \to +\infty$. Then there exist a subsequence $\{u_{\nu_j}\}_{\nu=1}^{+\infty}$ of $\{u_\nu\}_{\nu=1}^{+\infty}$ and a sequence $\{z_{\nu_j}\}_{\nu=1}^{+\infty}$ contained in a neighborhood of $\{u_{\nu_j}\}_{\nu=1}^{+\infty}$ such that the growth of the left hand side of the above equality is extremely larger than that of the right hand side of it at $z_{\nu_j}$ as $j \to \infty$.

Now, we give the proof of the Case II. Since $T_{f_j}(r) = o(T_{f_{j-1}}(r))$ for $j = 2, \cdots, n$, it is easy to see that $f_2, \cdots, f_n$ have no nontrivial algebraic relation. Suppose that there exists nontrivial algebraic relation $R(f_0, \cdots, f_n) = 0$ among $f_j$'s, where $R(\zeta_0, \cdots, \zeta_n)$ is a homogeneous polynomial of degree $l$. If this relation does not contain one of the $f_j$ with $j = 0, 1$, then we easily see that $f$ has the Zariski dense image. Hence we consider another case, that is, the relation $R(f_0, \cdots, f_n) = 0$ contains both of $f_0$ and $f_1$. We recall that

$$T_{f_j}(r) = (1 + o(1))\alpha_j (\log r)^2 \quad (j = 0, 1)$$

with $\alpha_1 < \alpha_0$. We rewrite the above relation as follows:

$$A_s f_0^s + A_{s-1} f_0^{s-1} + \cdots + A_0 = 0,$$

where $A_j = A_j(f_1, \cdots, f_n)$ and $A_j \neq 0$. We also write

$$A_s = a_s f_1^s + a_{s-1} f_1^{s-1} + \cdots + a_0,$$

where $a_p = a_p(f_2, \cdots, f_n)$ for $p = 0, \cdots, t$ and $s + t \leq l$. We may assume that the zero divisor of $f_0$ (resp. $f_1$) is contained in $R^+$ (resp. $R^-$). Let $\epsilon_0(r)$ (resp. $\epsilon_1(r)$) be the exceptional set for $f_0$ (resp. $f_1$). By the same reason in the Subcase I, we see $\epsilon_0(r) \cap \epsilon_1(r) = \emptyset$. The zero set $\{u_\nu\}_{\nu=1}^{+\infty}$ of $f_1$ can be written as $\{-r_\nu\}_{\nu=1}^{+\infty}$, where $\{r_\nu\}_{\nu=1}^{+\infty}$ is a positive increasing sequence with $r_\nu \to +\infty$ ($\nu \to +\infty$). Let $\rho$ be a positive number with $\rho \ll 1$. Set

$$C_\nu = \{z \in \mathcal{C}; |z + r_\nu| = \rho r_\nu\} \quad \text{and} \quad D_\nu = \{z \in \mathcal{C}; |z + r_\nu| \leq \rho r_\nu\}.$$ 

Suppose that $a_p \neq 0$ for some $p$ with $0 \leq p \leq t$. We can take $f_2, \cdots, f_n$ such that

$$T_{a_p}(r) = (1 + o(1))(\log r)^{\sigma_p},$$

where $1 < \sigma_p < 2$. Then there exists a subsequence $\{r_{\nu_j}\}_{\nu=1}^{+\infty}$ of $\{r_\nu\}_{\nu=1}^{+\infty}$ such that $a_p$ has no zero in $D_{\nu_j}$. Indeed, we first note that $T_{a_p}(r) = o((\log r)^2)$. Suppose that $a_p$ has a zero in all $D_\nu$ except for finite numbers of $\nu$. By the construction of $f_1$, the zero divisor of $f_1$ is $\Sigma_{\nu=1}^{+\infty} 2(-r_\nu)$ (see the proof of Lemma 2.1). For sufficiently large $\nu$, we see

$$N(2r_\nu, 0, a_p) \geq N(2r_\nu, 0, f_1)/2$$

$$= (1 + o(1))\alpha_1 (\log 2r_\nu)^2/2$$

$$= (1 + o(1))\alpha_1 (\log r_\nu)^2/2.$$
Thus we have a contradiction. Let $\mathcal{E}_p$ be the exceptional set for $a_p$. Then for each $\eta$ with $0 < \eta < \rho$, there exists a number $R$ such that the sum of length of circles contained in $D_{v_j} \cap \mathcal{E}_p$ is sufficiently smaller than $(\eta/2)r_{v_j}$ for $r_{v_j} \geq R$. Note that a sequence $\{r_{v_j}\}_{n=1}^{\infty}$ grows rapidly by the construction. Thus there exists a positive number $\gamma_j$ with $(\eta/2)r_{v_j} < \gamma_j < \eta r_{v_j}$ such that

$$
\Gamma_j = \{z \in \mathbb{C}; |z + r_{v_j}| = \gamma_j\}
$$

does not intersect $\mathcal{E}_p$ (see Remark 1.3). Hence by Theorem 1.2, we see

$$
\log |a_p(z)| = (1 + o(1)) \log M(|z|, a_p)
$$

for $z \in \Gamma_j$. For each $z \in \Gamma_j$, there exists a positive number $\tau_z$ such that $|z| = \tau_z r_{v_j}$, where $|r_{v_j} - \gamma_j|/r_{v_j} \leq \tau_z \leq |r_{v_j} + \gamma_j|/r_{v_j}$. Hence $1 - \eta < \tau_z < 1 + \eta$. Then we see

$$
\log M(|z|, a_p) = (1 + o(1)) (\log |z|)^{\sigma_p}
$$

$$
= (1 + o(1)) (\log r_{v_j} + \log \tau_z)^{\sigma_p}
$$

$$
\geq (1 + o(1)) (\log r_{v_j})^{\sigma_p} (1 + \log(1 - \eta)/\log r_{v_j})^{\sigma_p}
$$

$$
= (1 + o(1)) (\log r_{v_j})^{\sigma_p}
$$

$$
= (1 + o(1)) \log M(r_{v_j}, a_p).
$$

Thus we get

$$
\log |a_p(z)| \geq (1 + o(1)) \log M(r_{v_j}, a_p)
$$

for $z \in \Gamma_j$. Set

$$
\Sigma(\gamma_j; r_{v_j}) = \{z \in \mathbb{C}; |z + r_{v_j}| \leq \gamma_j\}.
$$

Since $a_p$ is a nonvanishing holomorphic function in $\Sigma(\gamma_j; r_{v_j})$, we see

$$
\log |a_p(z)| \geq \min\{\log |a_p(z)|; z \in \Gamma_j\} \geq (1 + o(1)) \log M(r_{v_j}, a_p)
$$

for $z \in \Sigma(\gamma_j; r_{v_j})$. We now consider the case of $t \neq 0$. Let $\epsilon$ be a sufficiently small positive number less than $\min\{1, \alpha_t\}$. Then there exists a positive integer $N$ depending on $f_1$ and $\epsilon$ that has the following property: If $j \geq N$, then there exists a point $z \in (C(r_{v_j}) \setminus \varepsilon_1(r_{v_j})) \cap D(r_{v_j})$ such that

$$
\log |f_1(z)| > \alpha_t (1 - \epsilon)(\log r_{v_j})^2.
$$

Hence there exists $z_{v_j} \in C(r_{v_j})$ such that

$$
\log |f_1(z_{v_j})| = \beta (\log r_{v_j})^2,
$$

where $0 < \beta < \alpha_t - \epsilon$. Note that $z_{v_j} \in \varepsilon_1(r_{v_j})$ and $\varepsilon_1(r_{v_j})$ is contained in a small neighborhood of $-r_{v_j}$. Without loss of generality, we may assume that

$$
\frac{1}{2} \leq 1 + \frac{a_{t-1}}{a_t f_1} + \ldots + \frac{a_0}{a_t f_1} \leq 2
$$
in some neighborhood of $z_{V_j}$. Since $e_0(r_{V_j}) \cap e_1(r_{V_j}) = \emptyset$, we get
\[
\log |A(z_{V_j})f_0^s(z_{V_j})| = (1 + o(1))(s\alpha_0 + t\beta)(\log r_{V_j})^2.
\]
In the case of $t = 0$, we have
\[
\log |A(z_{V_j})f_0^s(z_{V_j})| = \log |A_0(z_{V_j})f_0^s(z_{V_j})| = (1 + o(1))s\alpha_0(\log r_{V_j})^2.
\]
On the other hand, in both of the cases, we see
\[
\log |A_{s-1}(z_{V_j})f_0^{s-1}(z_{V_j}) + \cdots + A_0(z_{V_j})| \leq (1 + o(1))((s - 1)\alpha_0 + l\beta)(\log r_{V_j})^2.
\]
We again rewrite the above algebraic relation as follows:
\[
-A_s f_0^s = A_{s-1} f_0^{s-1} + \cdots + A_0.
\]
Now we take $\beta$ such that $0 < \beta < \alpha_1 - \epsilon$ and $l\beta < \alpha_0$. Since
\[
s\alpha_0 + l\beta > s\alpha_0 > (s - 1)\alpha_0 + l\beta > 0,
\]
we have a contradiction by letting $j \to +\infty$. Therefore, we conclude that $f$ has the Zariski dense image in this case.

We next consider the Case I. Note that $f_0$ and $f_0 + 1$ have no nontrivial algebraic relation by homogeneous polynomials. Suppose that there exists nontrivial algebraic relation $R(f_0, (f_0 + 1)\alpha, \cdots, f_n) = 0$, where $R$ is a nonzero homogeneous polynomial. We rewrite this relation as follows:
\[
A_n(f_0 + 1)^u + A_{n-1}(f_0 + 1)^u-1 + \cdots + A_0 = 0.
\]
Since $f_0$ and $f_0 + 1$ have no nontrivial homogeneous algebraic relation, each
\[
A_j = A_j(f_0, f_2, \cdots, f_n)
\]
contains at least one of $f_2, \cdots, f_n$. Thus we can write the above relation as follows:
\[
-B_s f_0^s = B_{s-1} f_0^{s-1} + \cdots + B_0,
\]
where $B_j = B_j(f_2, \cdots, f_n)$. Now, we have a contradiction by the above method.

We finally consider the Case III. In this case, $f_j = b_j f_1 + q_j$ for $j = 2, \cdots, p$, where $b_j$ are constants and $q_j$ are transcendental entire functions such that $T_{q_j}(r) = o((\log r)^2)$ and $T_{q_{j-1}}(r) = o(T_{q_{j-1}}(r))$. We also recall that $T_{f_j}(r) = \alpha_1(1 + o(1))(\log r)^2$ for $j = 0, 1$ and $T_{f_j}(r) = o(T_{f_{j-1}}(r))$ for $j = p + 1, \cdots, n$. Since $T_{f_j}(r) = o(T_{f_{j-1}}(r))$ for $j = p + 1, \cdots, n$, it is easy to see that $f_{p+1}, \cdots, f_n$ have no homogeneous algebraic relation. Suppose that there exists nontrivial algebraic relation $R(f_0, \cdots, f_n) = 0$ by a homogeneous polynomial $R$. If the above relation does not contain one of the $f_0$ and $f_1$, it is easy to see that $f$ has the Zariski dense image. If the relation $R(f_0, \cdots, f_n) = 0$ contains both of the $f_0$ and $f_1$, we rewrite the above relation as follows:
\[
\sum_{j=0}^s A_j(q_1, \cdots, q_p, f_1, f_{p+1}, \cdots, f_n) f_0^j = 0.
\]
Since $T_{f_j}(r) = o((\log r)^2)$ for $j = p + 1, \cdots, n$ and $T_{f_j}(r) = o((\log r)^2)$, we now have a contradiction as in the Case II. Therefore, we have completed the proof.
Remark 3.7. We give here note on the constant $\lambda(D)$ in Theorem 3.2. Let $P(\zeta)$ be a homogeneous polynomial of degree $d$ such that $D = \{P = 0\}$. Let $d_j$ be the degree in $\zeta_j$ that are contained in $P$. Set $\tilde{d} = \min_{0 \leq j \leq n} d_j$. Let $L(z)$ be the polynomial in (3.3) and denote by $\kappa$ the largest multiplicity of roots of the equation $L(z) = 0$, where $1 \leq \kappa \leq d - 1$. Note that $\lambda(D)$ may be dependent on $d$ and $\tilde{d}$. We do not know whether $\lambda(D)$ is sharp or not. We now give a list of the constant $\lambda(D)$ obtained in the proof of Theorem 3.2:

(I) If $d = \tilde{d}$, then $\lambda(D) = \kappa$.

(II) If $\tilde{d} < d$, then $\lambda(D) = d - \tilde{d}$.

We give here some examples of irreducible hypersurfaces of degree $d$.

Example 3.8. We define an irreducible hypersurface $D_d$ of degree $d$ in $P_n(C)$ by

$$\zeta_1^d + \cdots + \zeta_n^d = 0.$$ 

Note that $D_d$ has just one singular point $(1,0,\ldots,0)$. In this case, $\lambda(D) = d$. Hence, for an arbitrary positive real number $\alpha$ not greater than one, there exists a holomorphic curve $f : C \to P_n(C)$ with the Zariski dense image such that $\delta_f(D_d) = \alpha$. Note that there exist nonconstant holomorphic curves from $C$ into $P_n(C)$ that omit the above hypersurface $D_d$ of arbitrary high degree. Let $\psi(z)$ and $\varphi(z)$ be arbitrary entire functions and $\mu$ a $d$-th root of $-1$. If we define a holomorphic curve $f : C \to P_n(C)$ by

$$f(z) = (\psi(z), \varphi(z), \mu \varphi(z), 1, \ldots, 1),$$

then $f$ omits $D_d$. Note that $f$ is linearly degenerate. This example is essentially due to P. Kiernan (cf. [G1, Part 7]).

Example 3.9. We next give an example of a nonsingular hypersurface. We define a nonsingular hypersurface $S_d$ in $P_n(C)$ of degree $d \geq 2$ by

$$\xi_0^{d-1} \xi_2 - \xi_1^d + \xi_1 \xi_2^{d-1} + \sum_{j=3}^{n} \xi_j^d = 0.$$ 

In this case, we have $\lambda(D) = 1$. For the above hypersurface $S_d$, there exists a nonconstant holomorphic curve $f : C \to P_n(C)$ omitting $S_d$ for all $d \geq 2$. Indeed, let $\varphi(z)$ be an arbitrary entire function. We take entire functions $\varphi_3, \ldots, \varphi_n$ such that

$$\varphi_3^d + \cdots + \varphi_n^d = 0.$$ 

Define a holomorphic curve $f : C \to P_n(C)$ by

$$f(z) = \left( \exp \left( \frac{d}{d-1} \varphi(z) \right), \exp \varphi(z), 1, \varphi_3(z), \ldots, \varphi_n(z) \right).$$

Then we see $f(C) \cap S_d = \emptyset$. Note that $f$ is algebraically degenerate. This example is essentially due to M. L. Green (see [G2, p. 321]).

Example 3.10. Let $n = 2$ and define an irreducible curve $C_d$ by

$$\xi_0 \xi_2^{d-1} - \xi_1^d = 0.$$
Note that \( C_d \) also has just one singular point \((1,0,0)\), if \( d \geq 3 \). We also note that \( C_d \) is a rational curve. For \( C_d \), we have \( \lambda(C_d) = d - 1 \) by Theorem 3.2. Hence, for an arbitrary positive number \( \alpha \leq (d - 1)/d \), there exists a holomorphic curve with the Zariski dense image \( f : \mathbb{C} \to \mathbb{P}_2(\mathbb{C}) \) such that \( \delta_f(D_d) = \alpha \). Note that a holomorphic curve \( f \) constructed in \( \S 2 \) has the above property. Indeed, let \( f \) be as in \( \S 2 \) and assume \( \alpha = 1 - \alpha_1/\alpha_0 \). In this case \( d > \alpha_0/\alpha_1 \). We consider an entire function \( F \) defined by

\[
F = f_0f_2^{d-1} - f_1^d.
\]

Then, by a method similar to the Case I in the proof of Theorem 3.2, we have the estimate:

\[
T_F(r) = N(r, f^*C_d) + o(\log r^2) = d^2(1 + o(1))T_{f_1}(r).
\]

Therefore, we get \( \delta_f(C_d) = \alpha \).

**Remark 3.11.** We note that, for each positive integer \( d \) not less than two, there exists a holomorphic curve \( f : \mathbb{C} \to \mathbb{P}_2(\mathbb{C}) \) such that \( f \) omits \( C_d \). In fact, if we define \( f \) by

\[
f(z) = (\exp z + \exp(1 - d)z^2, 1, \exp z^2),
\]

then we easily see \( f(\mathbb{C}) \cap C_d = \emptyset \) (cf. [G2, p. 319] and [S2, p. 178]). Note that \( f \) has the Zariski dense image.

We note here that there has been another method to construct holomorphic curves with deficiencies. The holomorphic curves constructed above is of order zero. On the other hand, N. Toda has pointed out that the above examples of holomorphic curves can be proved by making use of Ahlfors-Weyl’s method (see [W]). In his construction, he used exponential curves and obtained holomorphic curves of order one with deficiencies. Note that this method works in the case that can be reduced to the hyperplane case. Indeed, let \( F_d \) be the Fermat surface degree \( d \), that is,

\[
F_d : \zeta_1^d + \cdots + \zeta_m^d = 0.
\]

Then our method gives a holomorphic curve \( f \) with \( \delta_f(F_d) = \alpha \) \((0 < \alpha \leq 1/d)\), but we cannot construct a holomorphic curve with positive deficiency for \( F_d \) by Ahlfors-Weyl’s method. Hence it seems that our method has a wide range of applicability.

### §4. Construction of meromorphic mappings with deficient divisor.

In this section, we construct meromorphic mappings \( f : \mathbb{C}^m \to \mathbb{P}_n(\mathbb{C}) \) with deficiency for a preassigned divisor. In our construction, we essentially use the method of the construction of holomorphic curves in \( \S 3 \). Throughout this section, we assume \( m \geq 2 \). Let \( F \) be a holomorphic function on \( \mathbb{C}^m \). As the case of \( m = 1 \), we define

\[
M(r, F) = \max_{z \in S(r)} |F(z)|.
\]

**Lemma 4.1.** Let \( F \) be a holomorphic function on \( \mathbb{C}^m \). Then

\[
T_F(r) + O(1) \leq \log M(r, F) \leq \frac{1 - (r/R)^2}{(1 - r/R)^{2m}(T_F(R) + O(1))}
\]
Thus, for a fixed real number \( r \) and \( R \) with \( r < R \).

For a proof, see, e.g., Noguchi [N, Lemma 1]. Set \( \delta(k) = ((1 - 1/k)^{2m})/(1 - (1/k)^2) \) for a positive number \( k \). It is clear that \( \delta(k) \to 1 \) as \( k \to +\infty \). By Lemma 4.1, we see

\[
T_F(r/k) + O(1) \leq \log M(r/k, F) \leq \delta(k)^{-1} (T_F(r) + O(1))
\]

for positive numbers \( k \) and \( r \) with \( k > 1 \). For a holomorphic function \( F \) on \( \mathbb{C}^m \) and \( \gamma \in \mathbb{C}^m \) with \( |\gamma| = 1 \), we define \( F_T : \mathbb{C} \to \mathbb{C} \) by \( F_T(s) = F(s\gamma) \) for \( s \in \mathbb{C} \). Hence \( F_T \) is the restriction of \( F(z) \) to a complex line \( \ell_T \) through the origin defined by \( \ell_T : z = s\gamma = (s\gamma_1, \ldots, s\gamma_m) \). Set \( \gamma_0 := (1,0,\cdots,0) \).

Take a holomorphic function \( \varphi(z) \) on \( \mathbb{C} \) so that \( T_F(r) = \alpha(1 + o(1)) (\log r)^2 \). Let the zero divisor of \( \varphi \) is \( \sum_j v_j p_j \). We define a holomorphic function \( F(z) \) on \( \mathbb{C}^m \) by \( F(z) = \varphi(z_1) g(z) \), where \( z = (z_1, \cdots, z_m) \) and \( g \) is a nonzero polynomial in \( z \). For the reason why we take the polynomial \( g \), see Remark 4.5 below. Note that \( F(z) \) has zeros at \( H_j \), where \( H_j \) denotes the hyperplane through \( z_1 = p_j \) that is perpendicular to the \( z_1 \)-axis. Then it is easy to see that

\[
M(r,F) = (1 + o(1)) M(r,F_{\gamma}) = (1 + o(1)) M(r, \varphi).
\]

(4.2)

Note that \( N(r,0,F_T) \leq N(r,0,F_{\gamma}) + O(\log r) \) for sufficiently large \( r \). We will give an estimate for \( N(r,0,F) \) by \( T_F(r) \). It is clear that \( N(r,0,F) \leq T_F(r) + O(1) \). For sufficiently small positive real number \( \varepsilon \), set \( S_\varepsilon = \{ z \in \mathbb{C}^m : |z_1| < \varepsilon \} \). Let \( S(1) \) be the unit sphere in \( \mathbb{C}^m \) and denote by \( \sigma \) the invariant measure on \( S(1) \) normalized so that \( \sigma(S(1)) = 1 \). Set \( S_\varepsilon' = S(1) \cap S_\varepsilon \). Note that we cannot get a good estimate for \( N(r,0,F_T) \) if \( \gamma \in \mathbb{S}(\varepsilon) \). For instance, we see \( N(r,0,F_T) = O(\log r) \) for \( \gamma \in S(1) \) with \( \gamma = (0, \gamma_2, \cdots, \gamma_m) \). Let \( \gamma = (\gamma_1, \gamma_2, \cdots, \gamma_m) \in S(1) \setminus S_\varepsilon \) and denote by \( H_\gamma \) the hyperplane through \( z_1 = r (\in \mathbb{R} - 0) \) that is perpendicular to the \( z_1 \)-axis. Since \( H_\gamma \cap \ell_T = \{ P_r(r, z_2, \cdots, z_m) \} \), there exists just one \( s \in \mathbb{C}^m \) such that \( s\gamma_1 = r \) and \( s\gamma_j = z_j \) for \( j = 2, \cdots, m \). Hence we see \( s = r/\gamma_1 \). Since \( \gamma \in S(1) \setminus S_\varepsilon \), this implies that \( \| P_r \| = r/|\gamma_1| < r/\varepsilon \). Hence to each zero \( p \) of \( F_{\gamma} \) with \( |p| = r \) there corresponds the zero \( p' \) of \( F_T \) with \( |p'| = r/\gamma_1 \). Thus we see

\[
N(r,\varepsilon,0,F_T) \geq N(r,0,F_{\gamma}) + o((\log r)^2) = (1 + o(1)) \alpha(\log r)^2
\]

and hence

\[
N(r,0,F_T) \geq (1 + o(1)) \alpha(\log r)^2 \geq (1 + o(1)) (T_F(r) + 2\alpha(\log \varepsilon)(\log r))
\]

Now, we will use the following averaging formula (see [S1, p. 91]):

\[
N(r,0,F) = \int_{\gamma \in S(1)} N(r,0,F_T) \sigma(\gamma) + O(1).
\]

By the above formula we see

\[
N(r,0,F) \geq \int_{\gamma \in S(1) \setminus S_\varepsilon} N(r,0,F_T) \sigma(\gamma)
\]

\[
\geq \int_{\gamma \in S(1) \setminus S_\varepsilon} (1 + o(1)) (T_F(r) + 2\alpha(\log \varepsilon)(\log r)) \sigma(\gamma)
\]

\[
= (1 - \sigma(S_\varepsilon'))(1 + o(1)) (T_F(r) + 2\alpha(\log \varepsilon)(\log r))
\]

Thus, for a fixed \( \varepsilon \), we have the following estimate:

\[
(1 - \sigma(S_\varepsilon'))(1 + o(1)) (T_F(r) + 2\alpha(\log \varepsilon)(\log r)) \leq N(r,0,F) \leq T_F(r) + O(1).
\]

(4.3)
We now take transcendental entire functions $f_0, \cdots, f_n$ of one complex variable satisfying the following condition: For $j = 0, 1$, let
\[ T_f(r) = \alpha_j(1 + o(1))(\log r)^2, \]
where $\alpha_1 < \alpha_0$. For $j = 2, \cdots, n$, we take $f_j$ so that $T_f(r) = o(T_{f,j-1}(r))$. By making use of the above method, we regard $f_j$ as entire functions on $\mathbb{C}^n$, that is, we define $F_j(z) = f_j(z_1)g_j(z)$ for $j = 0, \cdots, n$, where $g_j(z)$ are some nonzero polynomials. Now, we assume that $F_j(z) = f_j(z_1)$ for $j = 0, 1$. We now define a meromorphic mapping $f : \mathbb{C}^n \to P_n(\mathbb{C})$ by $f = (F_0, \cdots, F_n)$. Then we have the following lemma:

**Lemma 4.4.** Let $f$ be as above. Then
\[ \delta(k) \log M(r, F_0) \leq (1 + o(1)) T_f(r) \leq \log M(r, F_0). \]

**Proof.** As in the proof of Lemma 2.2, we easily have
\[ (1 + o(1)) T_f(r) \leq \log M(r, F_0) \]
by definition of $T_f(r)$. On the other hand, by (4.3) we see
\[
T_f(r) \geq \int_{S(r)} \log |F_0(z)| \sigma(z) \\
= N(r, 0, F_0) + O(1) \\
\geq \delta(k)(1 + o(1)) \log M(r/k, F_0) \\
\geq \delta(k) \alpha_0(1 + o(1))(\log r)^2 \\
= \delta(k) \alpha_0(1 + o(1))(\log r)^2.
\]
Therefore we have the desired conclusion.

**Remark 4.5.** We can construct the above meromorphic mapping $f$ with the Zariski dense image. Indeed, the argument in the proof of Theorem 3.2 also works in this case. Furthermore, if $m \geq n$, we have a dominant meromorphic mapping $f$ by a suitable choice of $g_j$'s. For instance, we can make $f$ to be dominant by taking $g_0(z) \equiv 1$ and $g_j(z) = z_j$ for $j \geq 1$. Hence we may assume that $f$ is dominant if $m \geq n$.

We first give a generalization of Theorem 3.1 as follows.

**Theorem 4.6.** Let $\alpha$ be an arbitrary positive real number less than one and let $H$ be an arbitrary hyperplane in $P_n(\mathbb{C})$. Then there exists a meromorphic mapping $f : \mathbb{C}^n \to P_n(\mathbb{C})$ with the Zariski dense image such that $\delta_f(H) = \alpha$. If $m \geq n$, then there exists a dominant meromorphic mapping $f : \mathbb{C}^n \to P_n(\mathbb{C})$ with $\delta_f(H) = \alpha$.

**Proof.** Without loss of generality, we may assume that $H = \{\zeta_1 = 0\}$. We consider a meromorphic mapping $f : \mathbb{C}^m \to P_n(\mathbb{C})$ as in Lemma 4.4. Choose such a $f$ so that $f$ has the Zariski dense image. Now we take $\alpha_0$ and $\alpha_1$ such that $1 - \alpha = \alpha_1/\alpha_0$. We note that
(1 + o(1))T_{f_1}(r) = N(r, f^*H) and T_f(r) = (1 + o(1))T_{f_0}(r). Hence by (4.3), Lemma 4.4 and letting $r \to +\infty$, we see

$$1 - (1 - \sigma(S')) \left( \frac{\alpha_1}{\alpha_0} \right) \leq \delta_f(H) \leq 1 - \delta(k) \left( \frac{\alpha_1}{\alpha_0} \right).$$

Letting $k \to +\infty$ and $\varepsilon \to 0$, we get $\delta_f(H) = \alpha$. \hfill \Box

We next consider the case $D \in |L(H)^{\otimes d}| (d \geq 2)$. In this case, by making use of (4.2), (4.3) and Lemma 4.4, we have the following theorem by the same method in the proof of Theorem 3.2:

**Theorem 4.7.** Let $D \in |L(H)^{\otimes d}|$ be an arbitrary divisor in $P_n(C)$, where $d$ is a positive integer. Then there exists a positive constant $\lambda(D)$ depending only on $D$ with $\lambda(D) \leq d$ that has the following property: For each positive number $\alpha$ with $\alpha \leq \lambda(D)/d$, there exists a meromorphic mapping $f : C^m \to P_n(C)$ with the Zariski dense image such that $\delta_f(D) = \alpha$. Furthermore, if $m \geq n$, then there exists an dominant meromorphic mapping $f : C^m \to P_n(C)$ with $\delta_f(D) = \alpha$.

We note that the number $\lambda(D)$ is as same as in Remark 3.6. It follows from Theorem 4.7 that we can find many examples of singular divisors and meromorphic mappings $f : C^m \to P_n(C)$ for which Griffiths’ defect relation does not hold. For instance, we consider the examples of divisor as in §3. Namely, let $C_d$ be a curve as in Example 3.9 and $\alpha$ a positive real number less than $(d - 1)/d$. Then there exists a dominant meromorphic mapping $f : C^m \to P_2(C)$ such that $\delta_f(C_d) = \alpha$. In particular, there exists a dominant meromorphic mapping $f : C^m \to P_2(C)$ such that

$$\delta_f(C_d) = \frac{d - 2}{d}.$$ 

Hence we also have an example for which Griffiths’ defect relation does not hold. We note that there exists a dominant meromorphic mapping $f : C^2 \to P_2(C)$ omitting $C_d$ for each $d$ (see Shiffman [S2, p. 178]).

**Remark 4.8.** Let $P$ and $\tilde{d}$ be as in §3. Suppose that $d \geq 3$. We note that, if $\tilde{d} \leq d - 2$, then $D$ has a singular point. Indeed, we may assume that $\tilde{d} = d_0$. We write $P$ as follows:

$$P(\zeta) = \zeta_0^{d-k}Q_1(\zeta) + Q_2(\zeta),$$

where $Q_2(\zeta)$ does not contain $\zeta_0$ and $\zeta_0^{d-k}$ is the greatest common divisor in $P - Q_2$. Since $d - k \leq d - 2$, we see that $D$ has a singular point $(1, 0, \ldots, 0)$. Set $w_j = \zeta_j/\zeta_0$ for $j = 1, \ldots, n$. Define $\tilde{P}(w) = \zeta_0^{-d}P(\zeta)$, where $w = (w_1, \ldots, w_n)$. If $d - d_0 \geq n + 1$, then the polynomial $\tilde{P}(w)$ has a zero at $(0, \ldots, 0)$ with multiplicity at least $n + 1$. Hence $D$ is not normal crossings at $(1, 0, \ldots, 0)$. This fact shows that the hypothesis in Griffiths’ defect relation, that is, $D$ is at most simple normal crossings, cannot be simply dropped.

**§5. Effect of the resolution of singularities to deficiencies.**

In this section we investigate how affects the resolution of singularities of divisors to deficiencies. In §3, we considered an example of the singular curve $C_d$ defined by...
This curve has only one singular point $P(1,0,0)$, if $d \geq 3$. If $\pi : Q_p(P_2(C)) \to P_2(C)$ is a monoidal transformation with the center $P$, then this gives a resolution of singularity of $C$. Namely, let $\tilde{C}$ and $\check{C}$ be the total transform and the proper transform of $C_d$, respectively. We also denote by $E$ the exceptional curve. Then we have

$$\tilde{C} = (d-1)E + \check{C}$$

and $\check{C}$ is a nonsingular curve in $Q_p(P_2(C))$ (see Lemma 6.1 in §6). We define a meromorphic mapping $\tilde{f} : C^m \to Q_p(P_2(C))$ by $\tilde{f} = \pi^{-1} \circ f$. We will give an estimate for $\delta_{\tilde{f}}(\check{C})$ depending on the structure of the singularity. To this end, we have to calculate the Chern form of the line bundle $L(\check{C})$. The precise calculation of the Chern form and the resolution of singularity will be done in the next section and hence we freely use the results in §6. The following is our main result in this section:

**Proposition 5.1.** Let $\alpha$ and $f$ be as in Example 3.9. Then

$$\delta_{\tilde{f}}(\check{C}) = \frac{\alpha}{1 + (1-\alpha)(d-1)}.$$

In particular, the estimate

$$\frac{\alpha}{d} < \delta_{\tilde{f}}(\check{C}) < \frac{d-1}{2d-1}$$

is valid.

**Proof.** It suffices to give a proof in the case $m = 1$. Let $\Sigma_1$ be a Hirzebruch surface of rank one, that is, $\Sigma_1$ is a nonsingular subvariety of $P_2(C) \times P_1(C)$ defined by

$$\Sigma_1 = \{(\xi_0, \xi_1, \xi_2; \xi_0, \xi_1) \in P_2(C) \times P_1(C); \xi_2 \xi_0 - \xi_1 \xi_1 = 0\},$$

where $\xi = (\xi_0, \xi_1)$ is a homogeneous coordinate system of $P_1(C)$. Then it is well-known that $\Sigma_1 = Q_p(P_2(C))$. Let $p_1 : \Sigma_1 \to P_1(C)$ and $p_2 : \Sigma_1 \to P_2(C)$ be the natural projections. Let $\omega_1$ (resp. $\omega_2$) be the Fubini-Study form on $P_1(C)$ (resp. $P_2(C)$). We first calculate $N(r, \tilde{f}^* E)$. We note that

$$\mathcal{V}_0 \cap \check{C} = \{(1 : x : tx; 1 : t); x^d - (tx)^{d-1} = 0\}$$

and

$$\mathcal{V}_1 \cap \check{C} = \{(1 : \tau y; y; \tau : 1); (\tau y)^d - y^{d-1} = 0\}.$$

In $\mathcal{V}_0$, the exceptional curve $E$ is defined by $x = 0$ and $\check{C}$ is defined by $x - t^{d-1} = 0$. On the other hand, in $\mathcal{V}_1$, the exceptional curve $E$ is defined by $y = 0$ and $\check{C}$ is defined by $\tau^d y - 1 = 0$. Note that $\tau = 1/t$. By the construction of $f$, we see

$$N(r, \tilde{f}^* E) = N(r, 0, f_2) = o((\log r)^2).$$
Hence we have \( N(r, \tilde{f}^* \mathcal{C}) = N(r, f^* \mathcal{C}) + o((\log r)^2) \). This shows

\[
N(r, \tilde{f}^* \mathcal{C}) = d(1 + o(1))\alpha_1 (\log r)^2.
\]

Next we show

\[
T_f(r, \mathcal{L}(\mathcal{C})) = T_f(r) + (d - 1)T_{f_1}(r).
\]

By Lemma 6.4 in §6, we have \( c_1(\mathcal{L}(\mathcal{C})) = p_1^* \omega_1 + (d - 1)p_2^* \omega_2 \). By Jensen’s formula and the definition of \( \tilde{f} \), we see

\[
T_f(r, \mathcal{L}(\mathcal{C})) = \int_1^r \frac{dt}{t} \int_{\Delta(t)} \tilde{f}^* (p_1^* \omega_1 + (d - 1)p_2^* \omega_2)
\]

\[
= \int_1^r \frac{dt}{t} \int_{\Delta(t)} d\tau \log \left( 1 + \left| \frac{f_1}{f_0} \right| + \left| \frac{f_2}{f_0} \right| \right) + (d - 1) \int_1^r \frac{dt}{t} \int_{\Delta(t)} d\tau \log \left( 1 + \left| \frac{f_2}{f_1} \right| \right)
\]

\[
= \int_{C(r)} \log \left( 1 + \left| \frac{f_1(z)}{f_0(z)} \right| + \left| \frac{f_2(z)}{f_0(z)} \right| \right) \frac{d\theta}{2\pi} + (d - 1) \int_{C(r)} \log \left( 1 + \left| \frac{f_2(z)}{f_1(z)} \right| \right) \frac{d\theta}{2\pi}
\]

\[
= T_f(r) + (d - 1)T_{f_1}(r).
\]

This shows our assertion. Thus we get

\[
T_f(r, \mathcal{L}(\mathcal{C})) = (\alpha_0 + (d - 1)\alpha_1 + o(1))(\log r)^2.
\]

Hence we have

\[
\delta_f(\mathcal{C}) = 1 - \limsup_{r \to \infty} \frac{N(r, \mathcal{C})}{T_f(r, \mathcal{L}(\mathcal{C}))}
\]

\[
= 1 - \lim_{r \to \infty} \frac{d(1 + o(1))\alpha_1 (\log r)^2}{\alpha_0 + (d - 1)\alpha_1 + o(1))(\log r)^2}
\]

\[
= \frac{1}{1 + (1 - \alpha)(d - 1)}.
\]

Therefore we have the desired conclusion. \( \square \)

**Remark 5.2.** We give here a remark on the above estimate for \( \delta_f(\mathcal{C}) \). We first recall the defect relation for dominant meromorphic mappings into \( \Sigma_1 \). Let \( L \) be an ample line bundle over \( \Sigma_1 \) and \( K(\Sigma_1) \) the canonical bundle of \( \Sigma_1 \). Set

\[
\gamma(L) = \left[ \frac{K(\Sigma_1)^s}{L} \right] = \inf \{ \gamma \in \mathcal{Q} : \gamma c_1(L) + c_1(K(\Sigma_1)) > 0 \}.
\]

Let \( D_j \in |L| \) and assume that \( D_1 + \cdots + D_q \) has simple normal crossings. Then a defect relation for dominant meromorphic mappings \( f : \mathbb{C}^m \to \Sigma_1 \) is given by the following (see [S2, Corollary 3.3]):

\[
\sum_{j=1}^q \delta_f(D_j) \leq \gamma(L).
\]
We will calculate the value of \( \gamma(L) \). For the line bundle \( L = L(D) \), we may assume \( D = aE + bF_\infty \), where \( a \) and \( b \) are positive integers with \( a < b \). Let \( K_{\Sigma_1} \) be the canonical divisor of \( \Sigma_1 \). It is well-known that \( K_{\Sigma_1} = -2E - 3F_\infty \). Then we see

\[
\gamma D + K_{\Sigma_1} = (\gamma a - 2)E + (\gamma b - 3)F_\infty.
\]

By making use of Nakai’s criterion (cf. [H, p. 380]), we easily have

\[
\gamma(L) = \frac{1}{b-a} \quad \text{if} \quad 3a - 2b \geq 0 \quad \text{and} \quad \gamma(L) = \frac{2}{a} \quad \text{if} \quad 3a - 2b < 0.
\]

If \( L = L(\tilde{C}) \), we have \( \gamma(L(\tilde{C})) = 2 \). In the case where target spaces are complex projective spaces, an ample line bundle \( L \) is written as \( L = L(H)^{\otimes d} \) for some positive integer \( d \). Hence we have the defect relation and the conjecture mentioned in the Introduction. For a meromorphic mapping \( f : \mathcal{C}^n \to \Sigma_1 \) with the Zariski dense image, we expect that the above defect relation holds under a suitable condition on singularities of \( D \).

§6. Appendix.

In this section, we give the resolution of singularity of \( C_d \), and calculate the Chern classes of line bundles determined by the proper transform \( \tilde{C} \) of \( C_d \). For background materials, we refer to [GH] and [H]. We first give the resolution of singularity. Recall the singular curve \( C_d \) defined by \( \zeta_0 \zeta_2^{d-1} - \zeta_1^d = 0 \). This curve has only one singular point \( \text{P}(1,0,0) \) if \( d \geq 3 \). Then we have the following:

**Lemma 6.1.** Let \( \pi : q_0 : (P_2(\mathcal{C})) \to P_2(\mathcal{C}) \) be a monoidal transformation at the center \( P \). Denote by \( \tilde{C} \) and \( \hat{C} \) the total transform and the proper transform of \( C_d \), respectively. Then the total transform \( \tilde{C} \) is given by

\[
\tilde{C} = (d-1)E + \hat{C},
\]

where \( E \) is the exceptional curve of the first kind and the proper transform \( \hat{C} \) is nonsingular.

**Proof.** Let \( U_0 \) be the affine open set determined by \( \zeta_0 \neq 0 \) in \( P_2(\mathcal{C}) \). Set \( x = \zeta_1 / \zeta_0 \) and \( y = \zeta_2 / \zeta_0 \) in \( U_0 \). We define an affine curve \( C_0 \) such that \( C_0 = C_d \cap U_0 \). Then we have the defining equation of the affine curve \( C_0 \) as follows:

\[
x^d - y^{d-1} = 0.
\]

We now give a resolution of singularity of \( C_0 \) at \((x,y) = (0,0)\). Let \( \sigma : \tilde{U}_0 \to U_0 \) be the blowing up centered at \((0,0)\). Let \( \{(x_1,y_1)\} \) and \( \{(x_2,y_2)\} \) be local coordinate systems in \( \tilde{U}_0 \). By definition of a blowing up, we consider the following two cases:

Case I. The pull back of \( C_d \) by \( x = x_1, y = y_1x_1 \).

In this case, we have one of the affine part of the total transform of \( C_d \) defined by \( x_1^d - y_1^{d-1}x_1^{d-1} = 0 \) and the exceptional curve \( E_1 \) defined by \( x_1 = 0 \). We also have a nonsingular curve \( \hat{C}_0 \) defined by \( x_1 = y_1^{d-1} \). Since

\[
\dim_{\mathbb{C}} \mathcal{C}[\{x_1,y_1\}] / (x_1,x_1 - y_1^{d-1}) = d - 1,
\]
we see that \( \mathcal{C}_0 \) and \( E_1 \) intersects each other at \((0,0)\) with multiplicity \(d-1\).

Case II. The pull back of \( C_d \) by \( x = x_2 y_2, y = y_2 \).

In this case, we see that another part of the total transform defined by \( x_2^d y_2 - y_2^{d-1} = 0 \) and the exceptional curve \( E_2 \) is defined by \( y_2 = 0 \). We also have a nonsingular curve \( \mathcal{C}_0 \) defined by \( x_2 y_2 - 1 = 0 \). Note that the multiplicity of \( E_2 \) is \( d-1 \) and \( \mathcal{C}_0 \cap E_2 = \emptyset \).

Now we have the exceptional curve of the first kind \( E \) by patching up \( E_1 \) and \( E_2 \). We also have the proper transform \( \mathcal{C}_0 \) of \( C_0 \). By taking a completion of \( \mathcal{C}_0 \), we have the proper transform \( \mathcal{C} \) of \( C_d \). It is clear that \( \mathcal{C} \) is nonsingular. Therefore we have the desired conclusion. \( \square \)

Next we calculate the Chern form of the line bundle \( L(\mathcal{C}) \). Let \( \Sigma_1 \) be the Hirzebruch surface of rank one, that is,

\[
\Sigma_1 = \{(\xi_0 : \xi_1 : \xi_2 : \xi_3) \in \mathbb{P}_2(\mathbb{C}) \times \mathbb{P}_1(\mathbb{C}) : \xi_0 \xi_2 - \xi_1 \xi_3 = 0\}.
\]

Then it is well-known that \( \mathcal{Q}_p(\mathbb{P}_2(\mathbb{C})) = \Sigma_1 \). Let \( p_1 : \mathbb{P}_2(\mathbb{C}) \times \mathbb{P}_1(\mathbb{C}) \to \mathbb{P}_2(\mathbb{C}) \) and \( p_2 : \mathbb{P}_2(\mathbb{C}) \times \mathbb{P}_1(\mathbb{C}) \to \mathbb{P}_1(\mathbb{C}) \) be the natural projections. We also denote by \( p_j \) the restriction of \( p_j \) to \( \Sigma_1 \). Then it is clear that \( E = p_{1 \ast}^{-1}((1 : 0 : 0)) \). Let \( F_\infty = p_2^{-1}( (0 : 1) ) \). It is well-known that the divisor class group \( \text{Cl}(\Sigma_1) \) is generated by \( F_\infty \) and \( E \). To calculate the Chern form of \( L(\mathcal{C}) \), we first determine a local coordinate system on \( \Sigma_1 \). Set

\[
U_i = \{(\xi_0 : \xi_1 : \xi_2) \in \mathbb{P}_2(\mathbb{C}) : \xi_i \neq 0\} \quad \text{and} \quad V_j = \{(\xi_0 : \xi_1) \in \mathbb{P}_1(\mathbb{C}) : \xi_j \neq 0\}.
\]

We also set \( W_{ij} = (U_i \cap V_j) \cap \Sigma_1 \). Let \( t = \xi_1/\xi_0 \) on \( V_0 \) and \( \tau = \xi_0/\xi_1 \) on \( V_1 \). Let \( x = \xi_1/\xi_0 \) and \( y = \xi_2/\xi_0 \) on \((U_0 \times \mathbb{P}_1(\mathbb{C})) \cap \Sigma_1 \). If we set \( W_0 = W_{00} \) and \( W_1 = W_{01} \), then \((x,t)\) and \((y,\tau)\) give local coordinate systems on \( W_0 \) and \( W_1 \), respectively. Let \( u = \xi_0/\xi_1 \) and \( v = \xi_2/\xi_0 \) on \((U_1 \times \mathbb{P}_1(\mathbb{C})) \cap \Sigma_1 \). If we set \( W_2 = W_{10} \), then we have a local coordinate system \((u,v)\) on \( W_2 \). Let \( z = \xi_0/\xi_2 \) and \( w = \xi_1/\xi_2 \) on \((U_2 \times \mathbb{P}_1(\mathbb{C})) \cap \Sigma_1 \). Set \( W_3 = W_{21} \) and determine a local coordinate system \((z,\tau)\) on \( W_3 \). Hence we have a system of local coordinate neighborhoods \( \{W_0, \ldots , W_3\} \) as follows:

\[
W_0 = \{(1 : x : tx : 1 : t)\}, \quad W_1 = \{(1 : ty : y : \tau : 1)\},
\]

\[
W_2 = \{(u : 1 : t : 1 : t)\}, \quad W_3 = \{(z : \tau : 1 : \tau : 1)\}.
\]

The change of local coordinate systems is given by

\[
y = tx, \quad \tau = 1/t \quad \text{on} \quad W_0 \cap W_1, \quad u = 1/x \quad \text{on} \quad W_0 \cap W_2,
\]

\[
x = \tau/z, \quad \tau = 1/t \quad \text{on} \quad W_0 \cap W_3, \quad y = t/u, \quad t = 1/\tau \quad \text{on} \quad W_1 \cap W_2,
\]

\[
z = 1/y \quad \text{on} \quad W_1 \cap W_3, \quad z = u/t, \quad \tau = 1/t \quad \text{on} \quad W_2 \cap W_3.
\]

Note that \( W_j \cong \mathbb{C}^2 \) for all \( j \).

Next, we calculate transition functions \( \{\psi_{\alpha \beta}\} \) of the line bundle \( L(F_\infty) \). It is clear that \( W_0 \cap F_\infty = W_2 \cap F_\infty = \emptyset \). We also have

\[
W_1 \cap F_\infty = \{(1 : ty : y : \tau : 1); \quad \tau = 0\} \quad \text{and} \quad W_3 \cap F_\infty = \{(z : \tau : 1; \tau : 1); \quad \tau = 0\}.
\]
Hence we have transition functions \( \{ \psi_{\alpha \beta} \} \) as follows: \( \psi_{01} = \psi_{03} = t \) and \( \psi_{02} = 1 \). Note that we can determine other transition functions \( \psi_{12}, \psi_{13} \) and \( \psi_{23} \) from the following relations: \( \psi_{12} = \psi_{10} \psi_{02}, \psi_{13} = \psi_{10} \psi_{03} \) and \( \psi_{23} = \psi_{20} \psi_{03} \). Note that \( \mathcal{V}_2 \cap E = \mathcal{V}_3 \cap E = \emptyset \). Since

\[
\mathcal{V}_0 \cap E = \{(1 : x : tx; 1 : t); x = 0\} \quad \text{and} \quad \mathcal{V}_1 \cap E = \{(1 : \tau y : y; \tau : 1); y = 0\},
\]

transition functions \( \{ \varphi_{\alpha \beta} \} \) of \( L(E) \) are given by \( \varphi_{01} = \tau \) and \( \varphi_{02} = \varphi_{03} = x \). Note that the total transform \( \tilde{C} \) of \( C_d \) is represented by

\[
\tilde{C} = \{(\zeta_0 : \zeta_1 : \zeta_2 : \xi_0 : \xi_1) \in P_2(C) \times P_1(C); \xi_0 \zeta_2 = \xi_1 \xi_1, \xi_0 \zeta_2^{-1} = \xi_1^d\}
\]

in \( \Sigma_1 \). Then we have

\[
\mathcal{V}_0 \cap \tilde{C} = \{(1 : x : tx; 1 : t); x^d - (tx)^{d-1} = 0\},
\]

\[
\mathcal{V}_1 \cap \tilde{C} = \{(1 : \tau y : y; \tau : 1); (\tau y)^d - y^{d-1} = 0\},
\]

\[
\mathcal{V}_2 \cap \tilde{C} = \{(u : 1 : t; 1 : t); 1 - ut^{d-1} = 0\},
\]

\[
\mathcal{V}_3 \cap \tilde{C} = \{(z : \tau : 1; \tau : 1); \tau^d - z = 0\}.
\]

Hence we have transition functions \( \{ \Phi_{\alpha \beta} \} \) of \( L(\tilde{C}) \): \( \Phi_{01} = 1, \Phi_{02} = x^d \) and \( \Phi_{03} = y^d \). Note that

\[
\Phi_{01} = (\psi_{01})^d(\varphi_{01})^d, \quad \Phi_{02} = (\psi_{02})^d(\varphi_{02})^d \quad \text{and} \quad \Phi_{03} = (\psi_{03})^d(\varphi_{03})^d.
\]

Thus we see

\[
\tilde{C} = d(E + F_\omega)
\]

in \( \text{Cl}(\Sigma_1) \). Let \( \omega_2 \) and \( \omega_1 \) be the Fubini-Study forms on \( P_2(C) \) and \( P_1(C) \), respectively, that is, \( \omega_1 = dd^c \log \| \xi \|^2 \) and \( \omega_2 = dd^c \log \| \xi \|^2 \). We obtain the following local expressions of \( \omega_1 \) on \( V_0 \) and \( \omega_2 \) on \( U_0 \), respectively:

\[
\omega_1 = \frac{-1}{2\pi} \frac{dt \wedge dt}{(1 + |t|^2)^2}, \quad \omega_2 = \frac{-1}{2\pi} \frac{(dx \wedge dv + dy \wedge dv) + \sqrt{1 - 4\pi^2} \xi dx \wedge y dy}{(1 + |x|^2 + |y|^2)^2}\frac{(1 + |x|^2 + |y|^2)^2}{\xi dx \wedge y dy}.
\]

Next we take curves \( C_j \) on \( \Sigma_1 \) such that \( c_1(L(C_j)) = \omega_j \) for \( j = 1, 2 \). Then we see

\[
C_1 \cdot E = \int_E p_2^* \omega_1 = \int_{t \in C} \frac{-1}{2\pi} \frac{dt \wedge dt}{(1 + |t|^2)^2} = 1,
\]

\[
C_1 \cdot F_\omega = \int_{F_\omega} p_2^* \omega_1 = \int_{\{t = 0\}} \frac{-1}{2\pi(1 + |t|^2)^2} d\tau \wedge d\overline{\tau} = 0,
\]

\[
C_2 \cdot E = \int_E p_1^* \omega_2 = \int_{x = 0} p_1^* \omega_2 = 0,
\]

\[
C_2 \cdot F_\omega = \int_{F_\omega} p_1^* \omega_2 = \int_{y \in C} \frac{-1}{2\pi(1 + |y|^2)^2} \frac{dy \wedge d\overline{y}}{\xi dx \wedge y dy} = 1.
\]

Take curves \( C_j \) on \( \Sigma_1 \) such that \( c_1(L(C_j)) = \omega_j \) for \( j = 1, 2 \). We write \( C_j \) as follows: \( C_1 = a_1 E + b_1 F_\omega \) and \( C_2 = a_2 E + b_2 F_\omega \), where \( a_j \) and \( b_j \) are integers. Hence, by the above calculation, we have that \( C_1 = F_\omega \) and \( C_2 = E + F_\omega \). Thus we have
\( p_1^* \omega_1 = c_1(L(F_m)), \quad p_2^* \omega_2 = c_1(L(E) \otimes L(F_m)). \) (6.3)

Therefore, by Lemma 6.1, (6.2) and (6.3), we have the following:

**Lemma 6.4.** The Chern form of \( L(\bar{C}) \) is given by

\[
c_1(L(\bar{C})) = p_2^* \omega_2 + (d - 1) p_1^* \omega_1.
\]

**References**


[M] S. Mori, Defects of holomorphic curves into \( P_n(C) \) for rational moving targets and a space of meromorphic mappings, Complex Variables Theory and Appl., 43 (2000), 363–379.


