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# A remark on Schubert cells and the duality of orbits on flag manifolds

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**Abstract.** It is known that the closure of an arbitrary $K_{\boldsymbol{C}}$-orbit on a flag manifold is expressed as a product of a closed $K_{\boldsymbol{C}}$-orbit and a Schubert cell ([**M2**], [**Sp**]). We already applied this fact to the duality of orbits on flag manifolds ([**GM**]). We refine here this result and give its new applications to the study of domains arising from the duality.

## 1. Duality of orbits on flag manifolds.

Let $G_{\boldsymbol{C}}$ be a connected complex semisimple Lie group and $G_{\boldsymbol{R}}$ a connected real form of $G_{\boldsymbol{C}}$. Let $K_{\boldsymbol{C}}$ be the complexification in $G_{\boldsymbol{C}}$ of a maximal compact subgroup $K$ of $G_{\boldsymbol{R}}$. Let $X = G_{\boldsymbol{C}}/P$ be a flag manifold of $G_{\boldsymbol{C}}$ where $P$ is an arbitrary parabolic subgroup of $G_{\boldsymbol{C}}$. Then there exists a natural one-to-one correspondence between the set of $K_{\boldsymbol{C}}$-orbits $S$ and the set of $G_{\boldsymbol{R}}$-orbits $S'$ on $X$ given by the condition:

$$S \leftrightarrow S' \Longleftrightarrow S \cap S' \text{ is non-empty and compact} \tag{A}$$

([**M3**]). In the following, we will identify orbits $S$ with $K_{\boldsymbol{C}}$-$P$ double cosets and $S'$ with $G_{\boldsymbol{R}}$-$P$ cosets.

We defined in [**GM**] a subset $C(S)$ of $G_{\boldsymbol{C}}$ by

$$C(S) = \{x \in G_{\boldsymbol{C}} \mid xS \cap S' \text{ is non-empty and compact in } X = G_{\boldsymbol{C}}/P\}$$

where $S'$ is the $G_{\boldsymbol{R}}$-orbit on $X$ given by (A).

If $S$ is closed, then $S'$ is open ([**M1**]) and so the condition

$$xS \cap S' \text{ is non-empty and compact in } G_{\boldsymbol{C}}/P$$

implies

$$xS \subset S'.$$

Hence the set $C(S)_0$ is the cycle domain (cycle space) for $S'$ ([**WW**]) where $C(S)_0$ denotes the connected component of $C(S)$ containing the identity.

On the other hand, let $S_{\mathrm{op}}$ denote the unique open $K_{\boldsymbol{C}}$-$B$ double coset in $G_{\boldsymbol{C}}$ where $B$ is a Borel subgroup of $G_{\boldsymbol{C}}$ contained in $P$. (We will keep this notation for the whole note.) Then $S'_{\mathrm{op}}$ is the unique closed $G_{\boldsymbol{R}}$-$B$ double coset in $G_{\boldsymbol{C}}$ and the condition

$$xS_{\mathrm{op}} \cap S'_{\mathrm{op}} \text{ is non-empty and compact in } G_{\boldsymbol{C}}/B$$

implies

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$$xS_{\rm op} \supset S'_{\rm op}.$$

Let $\{S_j \mid j \in J\}$ be the set of $K_{\boldsymbol{C}}$-$B$ double cosets in $G_{\boldsymbol{C}}$ of codimension one and $T_j = S_j^{\rm cl}$ denote the closure of $S_j$. The sets $T_j$ will play an important role in our constructions.

The complement of $S_{\rm op}$ in $G_{\boldsymbol{C}}$ is written as

$$\bigcup_{j\in J} T_j$$

(by Theorem 2 in Section 2). So the set $C(S_{\rm op})$ is the complement of the infinite family of complex hypersurfaces

$$gT_j^{-1} \quad (j \in J,\ g \in S'_{\rm op})$$

and hence the connected component $C(S_{\rm op})_0$ is Stein.

This domain is sometimes called the "Iwasawa domain" since it is a maximal domain where all Iwasawa decompositions can be holomorphically extended from $G_{\boldsymbol{R}}$.

In [**GM**], we defined

$$C = \bigcap C(S)$$

where we take the intersection for all $K_{\boldsymbol{C}}$-orbits $S$ on $X$ on all flag manifolds $X = G_{\boldsymbol{C}}/P$ of $G_{\boldsymbol{C}}$ and conjectured

$$C = \widetilde{D_0} Z$$

(Conjecture 1.3) where $D_0 = \widetilde{D_0}/K_{\boldsymbol{C}}$ is the domain introduced by [**AG**] (which is sometimes denoted as $\Omega_{AG}$) and $Z$ is the center of $G_{\boldsymbol{C}}$. For connected components, it means

$$C_0 = \widetilde{D_0}. \tag{B}$$

This conjecture (B) was solved recently as follows. It is proved in Proposition 8.3 of [**GM**] that

$$C_0 = C(S_{\rm op})_0.$$

In other words, $C(S)_0$ is minimal when $S = S_{\rm op}$. We believe that it is one of central facts of this theory since it gives a very strong estimate of all $C(S)$ through $C(S_{\rm op})$ only. So (B) is equivalent to the equality

$$C(S_{\rm op})_0 = \widetilde{D_0} \tag{C}$$

which was recently established by many people's contributions as follows.

The domain $C(S_{\rm op})_0$ was considered in [**BGW**] for $SU(p,q)$ (under the name "polar set") and for general cases in [**G**]. In [**G**] was conjectured (C) as well as the coincidence of $\widetilde{D_0}$ with the universal domain of all analytic extensions from the Riemann symmetric spaces.

In 1999, J. Faraut and T. Kobayashi constructed some Hermitian symmetric domains $\Xi_0$ containing $G_{\boldsymbol{R}}/K$ in the classical case and gave a proof for $\Xi_0 \subset C(S_{\rm op})_0$ in an unpublished note. Using this inclusive relation, they also showed that all the joint eigenfunctions on $G_{\boldsymbol{R}}/K$ with respect to $G_{\mathbf{R}}$-invariant differential operators on $G_{\mathbf{R}}/K$ can be holomorphically extended to the domains $\Xi_0$. It is known that $\Xi_0$ are subdomains of $\widetilde{D_0}$ and that they coincide in some cases including Hermitian cases (c.f. [**BHH**], [**KS2**]).

Later, Krötz and Stanton proved the inclusion

$$\widetilde{D_0} \subset C(S_{\mathrm{op}})_0 \tag{D}$$

for all classical cases in [**KS1**] and also applied it to holomorphic extension of solutions of invariant differential operators. Independently, [**GM**] proved the equality (C) for all classical cases and exceptional Hermitian cases. Huckleberry gave a general proof of the inclusion (D) in [**H**] using the strictly plurisubharmonicness of a funtion $\rho$ which is proved in [**BHH**]. Recently, the second author gave a general proof of (D) without complex analysis ([**M4**]).

On the other hand, Barchini proved the opposite inclusion $C(S_{\mathrm{op}})_0 \subset \widetilde{D_0}$ by a general argument in [**B**].

REMARK 1. In [**FH**], the authors deduce the equality $C_0 = \widetilde{D_0}$ from their result about $C(S)$ for closed $S$ and Proposition 8.1 in [**GM**]. As we showed above, this equality is already the consequence of Proposition 8.3 in [**GM**] and the equality (C). So it does not need the results in [**FH**].

## 2. Schubert cells in the category of $K_{\mathbf{C}}$-$B$ double cosets.

The principal idea of our considerations in [**GM**] was that $C(S)_0$ will be essentially independent of neither $S$ nor the flag manifold $X = G_{\mathbf{C}}/P$. To justify it, we need to build bridges between $C(S)$ for different $S$ and for it we need to see connections between different $K_{\mathbf{C}}$-orbits. It turns out that Schubert cells are very efficient tool for such considerations as in Section 2 and Section 8 in [**GM**]. They give a possibility to obtain an important information about general $C(S)$ from a consideration of simplest $S$. Here we refine connections between $K_{\mathbf{C}}$ -orbits and Schubert cells and give more examples of applications.

For a simple root $\alpha$ in the root system with respect to the order defined by $B$, we can define a parabolic subgroup

$$P_\alpha = B \cup B w_\alpha B$$

of $G_{\mathbf{C}}$ such that $\dim_{\mathbf{C}} P_\alpha = \dim_{\mathbf{C}} B + 1$.

LEMMA 1. *Let $S_1$ be a $K_{\mathbf{C}}$-$B$ double coset. Then we have*:

(i) *If* $\dim_{\mathbf{C}} S_1 P_\alpha = \dim_{\mathbf{C}} S_1$, *then* $S_1^{\mathrm{cl}} P_\alpha = S_1^{\mathrm{cl}}$.

(ii) *If* $\dim_{\mathbf{C}} S_1 P_\alpha = \dim_{\mathbf{C}} S_1 + 1$, *then there exists a $K_{\mathbf{C}}$-$B$ double coset $S_2$ such that* $S_1^{\mathrm{cl}} P_\alpha = S_2^{\mathrm{cl}}$.

PROOF. Though this lemma follows easily from [**M2**] Lemma 3, we will give a proof for the sake of completeness. Write $S_1 = K_{\mathbf{C}} g B$. Then we have a natural bijection

$$(g^{-1} K_{\mathbf{C}} g \cap P_\alpha) \backslash P_\alpha / B \cong K_{\mathbf{C}} \backslash K_{\mathbf{C}} g P_\alpha / B = K_{\mathbf{C}} \backslash S_1 P_\alpha / B$$

by the map $x \mapsto gx$.

(i) If $\dim_{\mathbf{C}} S_1 P_\alpha = \dim_{\mathbf{C}} S_1$, then $(g^{-1} K_{\mathbf{C}} g \cap P_\alpha) B / B$ is Zariski open in $P_\alpha / B = P^1(\mathbf{C})$ and hence it is dense. So we have

$$S_1^{\rm cl} = (K_{\boldsymbol{C}}gB)^{\rm cl} \supset S_1P_\alpha \supset S_1$$

and therefore $S_1^{\rm cl} = S_1^{\rm cl}P_\alpha$.

(ii) Suppose $\dim_{\boldsymbol{C}} S_1P_\alpha = \dim_{\boldsymbol{C}} S_1 + 1$. Then there exists a $p \in P_\alpha$ such that $(g^{-1}K_{\boldsymbol{C}}g \cap P_\alpha)pB/B$ is Zariski open in $P_\alpha/B = P^1(\boldsymbol{C})$ since the number of $K_{\boldsymbol{C}}$-$B$ double cosets in $G_{\boldsymbol{C}}$ is finite. If we write $S_2 = K_{\boldsymbol{C}}gpB$, then we have

$$(S_2)^{\rm cl} \supset S_1P_\alpha \supset S_2$$

and therefore $S_2^{\rm cl} = S_1^{\rm cl}P_\alpha$. □

THEOREM 1. *Let $S_1$ be a $K_{\boldsymbol{C}}$-$B$ double coset in $G_{\boldsymbol{C}}$ and $w$ an element of the Weyl group $W$. Then we have*:

(i) $S_1^{\rm cl}(BwB)^{\rm cl} = S_2^{\rm cl}$ *for some $K_{\boldsymbol{C}}$-$B$ double coset $S_2$.*

(ii) (*minimal expression*) *There exists a $w' \in W$ such that $w' \le w$* (*Bruhat order*), $\ell(w') = \dim_{\boldsymbol{C}} S_2 - \dim_{\boldsymbol{C}} S_1$ *and that*

$$S_1^{\rm cl}(Bw'B)^{\rm cl} = S_2^{\rm cl}.$$

*Here $\ell(w') = \dim_{\boldsymbol{C}} Bw'B - \dim_{\boldsymbol{C}} B$ is the length of $w'$.*

PROOF. (i) This follows from Lemma 1 because every Schubert cell $(BwB)^{\rm cl}$ is written as

$$(BwB)^{\rm cl} = P_{\alpha_1}\cdots P_{\alpha_\ell}$$

where $w = w_{\alpha_1}\cdots w_{\alpha_\ell}$ is a minimal expression of $w \in W$.

(ii) By Lemma 1, we can choose a subsequence $\beta_1,\ldots,\beta_q$ ($q = \dim_{\boldsymbol{C}} S_2 - \dim_{\boldsymbol{C}} S_1$) of $\alpha_1,\ldots,\alpha_\ell$ such that

$$\dim_{\boldsymbol{C}} S_1^{\rm cl}P_{\beta_1}\cdots P_{\beta_k} = \dim_{\boldsymbol{C}} S_1^{\rm cl}P_{\beta_1}\cdots P_{\beta_{k-1}} + 1$$

for $k = 1,\ldots,q$ and that

$$S_2^{\rm cl} = S_1^{\rm cl}(BwB)^{\rm cl} = S_1^{\rm cl}P_{\alpha_1}\cdots P_{\alpha_\ell} = S_1^{\rm cl}P_{\beta_1}\cdots P_{\beta_q} = S_1^{\rm cl}(Bw'B)^{\rm cl}$$

with $w' = w_{\beta_1}\cdots w_{\beta_q}$. □

REMARK 2. $S_1^{\rm cl}(BwB)^{\rm cl} = S_2^{\rm cl}$ implies $S_1^{\rm cl} \subset S_2^{\rm cl}$. But $S_1^{\rm cl} \subset S_2^{\rm cl}$ does not always imply $S_1^{\rm cl}(BwB)^{\rm cl} = S_2^{\rm cl}$ for some $w$ (c.f. [**M2**]).

DEFINITION 1. For every $K_{\boldsymbol{C}}$-$B$ double coset $S$, we can define, by Theorem 1, a subset $J(S)$ of $J$ by

$$J(S) = \{j \in J \mid S^{\rm cl}(BwB)^{\rm cl} = T_j \text{ for some } w \in W\}.$$

LEMMA 2. *Let $S$ be a non-open $K_{\boldsymbol{C}}$-$B$ double coset. Then there exists a simple root $\alpha$ such that*

$$\dim_{\boldsymbol{C}} SP_\alpha = \dim_{\boldsymbol{C}} S + 1.$$

PROOF. Write $G_{\boldsymbol{C}} = (Bw_0B)^{\rm cl} = P_{\alpha_1}\cdots P_{\alpha_m}$ with the longest element $w_0$ in $W$. If

$$\dim_{\boldsymbol{C}} SP_\alpha = \dim_{\boldsymbol{C}} S$$

for all simple roots $\alpha$, then we have, by Lemma 1,

$$G_{\boldsymbol{C}} = S^{\mathrm{cl}} G_{\boldsymbol{C}} = S^{\mathrm{cl}} P_{\alpha_1} \cdots P_{\alpha_m} = S^{\mathrm{cl}},$$

a contradiction. □

THEOREM 2. *If $\ell(w) < \mathrm{codim}_{\boldsymbol{C}} S$, then*

$$S^{\mathrm{cl}}(BwB)^{\mathrm{cl}} \subset T_j$$

*for some $j \in J(S)$.*

PROOF. Since $\mathrm{codim}_{\boldsymbol{C}} S^{\mathrm{cl}}(BwB)^{\mathrm{cl}} = d > 0$, we can choose simple roots $\alpha_1, \dots, \alpha_{d-1}$ such that

$$\mathrm{codim}_{\boldsymbol{C}} S^{\mathrm{cl}}(BwB)^{\mathrm{cl}} P_{\alpha_1} \cdots P_{\alpha_{d-1}} = 1$$

by Lemma 2. Since $(BwB)^{\mathrm{cl}} P_{\alpha_1} \cdots P_{\alpha_{d-1}} = (Bw'B)^{\mathrm{cl}}$ for some $w' \in W$, we have

$$S^{\mathrm{cl}}(BwB)^{\mathrm{cl}} \subset S^{\mathrm{cl}}(Bw'B)^{\mathrm{cl}} = T_j$$

for some $j \in J(S)$. □

## 3. Applications.

DEFINITION 2. For every subset $J'$ in $J$, we define a domain $\Omega(J')$ in $G_{\boldsymbol{C}}$ by

$$\Omega(J') = \{x \in G_{\boldsymbol{C}} \mid xT_j \cap S'_{\mathrm{op}} = \emptyset \text{ for all } j \in J'\}_0.$$

We can prove the following corollary:

COROLLARY. *Let $S$ be a closed $K_{\boldsymbol{C}}$-$P$ double coset in $G_{\boldsymbol{C}}$. Write $S = S_1^{\mathrm{cl}}$ with the dense $K_{\boldsymbol{C}}$-$B$ double coset $S_1$ in $S$. Then we have*

$$C(S)_0 = \Omega(J(S_1)).$$

REMARK 3. (i) We can see $C(S_{\mathrm{op}})_0 = \Omega(J)$. By the same argument as for $C(S_{\mathrm{op}})_0$ in Section 1, we can prove $\Omega(J')$ is Stein for every subset $J'$ in $J$. So the Steinness of $C(S)_0$ ([**W**]) becomes a corollary of this equivalence $C(S)_0 = \Omega(J(S_1))$ (c.f. [**HW**]).

(ii) It is clear that $\Omega(J') \supset \Omega(J)$ for every subset $J'$ in $J$. So we have

$$C(S)_0 \supset C(S_{\mathrm{op}})_0.$$

But this inclusion was already proved in Proposition 8.3 in [**GM**]. This is natural because the way of proof of the corollary below is essentially the same as that of Proposition 8.3 in [**GM**]. So the above corollary may be considered as its refinement.

PROOF OF COROLLARY. Let $x$ be an element on the boundary of $C(S)_0$. Then

$$xS \cap S_2'P \neq \emptyset$$

for some $G_{\boldsymbol{R}}$-$P$ double coset $S_2'P$ in the boundary of $S'$. Here we take $S_2$ as the dense $K_{\boldsymbol{C}}$-$B$ double coset contained in $S_2P$. Since $S$ is right $P$-invariant, we have

$$xS \cap S_2' \neq \emptyset \quad \text{and} \quad \dim_{\boldsymbol{C}} S_2 > \dim_{\boldsymbol{C}} S.$$

Applying Theorem 1 (ii) to the pair $(S_2^{\mathrm{cl}},\ G_{\boldsymbol{C}})$, we can take a $w \in W$ such that $\ell(w) = \operatorname{codim}_{\boldsymbol{C}} S_2$ and that

$$S_2^{\mathrm{cl}}(BwB)^{\mathrm{cl}} = G_{\boldsymbol{C}}.$$

So we have $S_2(BwB)^{\mathrm{cl}} \supset S_{\mathrm{op}}$ and hence

$$S_2' \subset S_{\mathrm{op}}'(Bw^{-1}B)^{\mathrm{cl}}.$$

Since $xS \cap S_2' \neq \emptyset$, we have

$$xS \cap S_{\mathrm{op}}'(Bw^{-1}B)^{\mathrm{cl}} \neq \emptyset.$$

Hence

$$xS(BwB)^{\mathrm{cl}} \cap S_{\mathrm{op}}' \neq \emptyset$$

which implies $xT_j \cap S_{\mathrm{op}}' \neq \emptyset$ for some $j \in J(S_1)$ by Theorem 2. Thus $x \notin \Omega(J(S_1))$.

Conversely, suppose

$$xT_j \cap S_{\mathrm{op}}' \neq \emptyset$$

for some $T_j = S(BwB)^{\mathrm{cl}} = S_1^{\mathrm{cl}}(BwB)^{\mathrm{cl}}$. Note that $j \in J(S_1)$ by Definition 1 and that we may assume $\ell(w) = \operatorname{codim}_{\boldsymbol{C}} S - 1 = \operatorname{codim}_{\boldsymbol{C}} S_1 - 1$ by Theorem 1 (ii). Then we have

$$xS \cap S_{\mathrm{op}}'(Bw^{-1}B)^{\mathrm{cl}} \neq \emptyset$$

and hence

$$xS \cap S_3' \neq \emptyset$$

for some $K_{\boldsymbol{C}}$-$B$ double coset $S_3$ such that $S_3' \subset S_{\mathrm{op}}'(Bw^{-1}B)^{\mathrm{cl}}$. Hence $S_3(BwB)^{\mathrm{cl}} \supset S_{\mathrm{op}}$ and therefore $\dim_{\boldsymbol{C}} S_3 \geq \dim_{\boldsymbol{C}} G_{\boldsymbol{C}} - \ell(w) > \dim_{\boldsymbol{C}} S$. So we have

$$S_3' \cap S' = \emptyset$$

because $S'$ is the union of $G_{\boldsymbol{R}}$-$B$ double cosets $S_4'$ satisfying $S_4 \subset S$. Hence we have

$$xS \not\subset S'$$

and therefore

$$x \notin C(S). \qquad \square$$

REMARK 4. (i) The condition $\ell(w) = \operatorname{codim}_{\boldsymbol{C}} S - 1$ does "not always" imply

$$\operatorname{codim}_{\boldsymbol{C}} S^{\mathrm{cl}}(BwB)^{\mathrm{cl}} = 1.$$

Counter examples exist already for $G_{\boldsymbol{R}} = SU(2,1)$.

(ii) The construction of the domain $\Omega(J(S_1))$ is essentially equivalent to the construction of "Schubert domain" in [**HW**]. We can see that the proof of our corollary using the results in Section 2 is extremely simple. Let us explain the connection between these two constructions introducing notations in [**HW**].

Take a Borel subgroup $B_0$ of $G_{\boldsymbol{C}}$ so that $G_{\boldsymbol{R}}B_0$ is closed in $G_{\boldsymbol{C}}$. A Borel subgroup $B$ of $G_{\boldsymbol{C}}$ is called an "Iwasawa Borel subgroup" if

$$B = g_0 B_0 g_0^{-1} \quad \text{for some } g_0 \in G_{\boldsymbol{R}}.$$

Let $Z = G_{\boldsymbol{C}}/Q$ be a flag manifold. Then we can take $Q$ so that $Q \supset B_0$. Every Schubert cell $Y$ in $Z$ for $B$ is written as

$$Y = (Bg_0wQ)^{\mathrm{cl}} = (g_0B_0wQ)^{\mathrm{cl}}$$

with some $w \in W$. Let $S$ be a closed $K_{\boldsymbol{C}}$-$Q$ double coset. (They use the symbol $C_0$ for $S$.) The "incidence variety" $H_Y$ is written as

$$H_Y = \{g \mid gS \cap Y \neq \emptyset\} = YS^{-1} = (g_0B_0wQ)^{\mathrm{cl}}S^{-1} = (S(Qw^{-1}B_0)^{\mathrm{cl}}g_0^{-1})^{-1}.$$

If $\operatorname{codim} H_Y = 1$, then

$$H_Y^{-1} = S(Qw^{-1}B_0)^{\mathrm{cl}}g_0^{-1} = T_j g_0^{-1}$$

for some $j \in J' = J(S_1)$ (where $S_1$ is the dense $K_{\boldsymbol{C}}$-$B_0$ double coset in $S$) and $g_0 \in G_{\boldsymbol{R}}$ by our notation.

They defined

$$\mathscr{Y}(S') = \{Y = (g_0B_0wQ)^{\mathrm{cl}} \mid \operatorname{codim} H_Y = 1\}.$$

(They use the symbol $D$ for $S'$. Note that the condition $Y \subset Z \setminus S'$ follows from $\operatorname{codim} H_Y = 1$ because

$$\begin{aligned} Y \cap S' = \emptyset &\Longleftrightarrow S'Y^{-1} = S'(Qw^{-1}B_0)^{\mathrm{cl}}g_0^{-1} \not\ni e \\ &\Longleftrightarrow S'(Qw^{-1}B_0)^{\mathrm{cl}} \not\ni g_0 \\ &\Longleftrightarrow S'(Qw^{-1}B_0)^{\mathrm{cl}} \cap G_{\boldsymbol{R}}B_0 = \emptyset \\ &\Longleftrightarrow S(Qw^{-1}B_0)^{\mathrm{cl}} \cap K_{\boldsymbol{C}}B_0 = \emptyset \\ &\Longleftrightarrow \operatorname{codim} S(Qw^{-1}B_0)^{\mathrm{cl}} \geq 1.) \end{aligned}$$

The Schubert domain is defined as

$$\Omega_S(S') = \left\{ G_{\boldsymbol{C}} \setminus \left( \bigcup_{Y \in \mathscr{Y}(S')} H_Y \right) \right\}_0 .$$

This definition is equivalent to our definition of $\Omega(J')$ because

$$\begin{aligned} g \notin \bigcup_{Y \in \mathscr{Y}(S')} H_Y &\iff g^{-1} \notin T_j g_0^{-1} \text{ for all } j \in J' \text{ and } g_0 \in G_{\boldsymbol{R}} \\ &\iff g^{-1} G_{\boldsymbol{R}} B_0 \cap T_j = \emptyset \text{ for all } j \in J' \\ &\iff G_{\boldsymbol{R}} B_0 \cap g T_j = \emptyset \text{ for all } j \in J'. \end{aligned}$$

REMARK 5. The problem of the description of the domain of cycles $C(S)_0$ for groups $G_{\boldsymbol{R}}$ of Hermitian type is simpler than the general case. Firstly, in this case, $D_0 = \widetilde{D_0}/K_{\boldsymbol{C}}$ has a very simple description: $D_0 \cong G_{\boldsymbol{R}}/K \times \overline{G_{\boldsymbol{R}}/K}$ (Proposition 2.2 in [**GM**]). As usual, the equality $C(S)_0 = \widetilde{D_0}$ for $S$ ($\leftrightarrow S'$) of nonholomorphic type is reduced to two inclusions. The proof of $C(S)_0 \subset \widetilde{D_0}$ in [**WZ1**] had a mistake which was corrected in [**WZ2**]. The opposite inclusion was checked in [**WZ1**] for classical Hermitian groups. In Proposition 2.4 of [**GM**], we gave a very simple proof of this inclusion for arbitrary groups of Hermitian type which is free of case-by-case considerations: the use of Schubert cells makes this fact almost trivial. The note [**WZ2**] also contains this fact with a proof referred to [**HW**] but without an appropriate reference on the preceding proof in [**GM**]. Moreover it asserts a misleading statement that the paper [**GM**] does not contain a direct proof.

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