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The finite group action and the equivariant determinant of elliptic operators

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Abstract. If a closed oriented manifold admits an action of a finite group G, the equivariant determinant of a G-equivariant elliptic operator on the manifold defines a group homomorphism from G to S^1. The equivariant determinant is obtained from the fixed point data of the action by using the Atiyah-Singer index theorem, and the fact that the equivariant determinant is a group homomorphism imposes conditions on the fixed point data. In this paper, using the equivariant determinant, we introduce an obstruction to the existence of a finite group action on the manifold, which is obtained directly from the relation among the generators of the finite group.

1. Introduction.

Let M be a $2m$-dimensional closed connected oriented Riemannian manifold and G a compact Lie group. In this paper, we define an action of G as an orientation-preserving isometric effective action of G on M. It is a classical problem to know whether there exists an action of G on M which preserves some geometric structures of M, and various results have been obtained concerning this existence problem. Assume that M admits a G-action and let $D:\Gamma(E)\longrightarrow\Gamma(F)$ be a G-equivariant elliptic operator where E, F are complex G-vector bundles over M. Then the G-equivariant index $\mathrm{Ind}(D,g)$ of D evaluated at $g\in G$ is defined by the trace of the g-action on $\ker D$, $\mathrm{coker}\, D$ as follows:

$$\mathrm{Ind}(D,g)=\mathrm{Tr}\,(g|\ker D)-\mathrm{Tr}\,(g|\mathrm{coker}\, D)\in \boldsymbol{C}$$

(cf. [**3**]), and this equivariant index has been used for the existence problem above. For example, in [**3**] Corollary 6.16, it is proved that M does not admit any $\boldsymbol{Z}_2$-action with the fixed point set of the dimension$<m$ if m is even and the Euler characteristic of M is odd. It is also proved in [**2**] that M does not admit any G-action with $\dim G>0$ if M has a *Spin*-structure and the $\widehat{A}$-genus of M does not vanish.

When $m=1$ and M is a Riemann surface of genus $\sigma\geq 2$, M is represented as the quotient U/Λ of the hyperbolic plane U under the action of a surface Fuchsian group Λ of genus σ and M admits a biholomorphic action of a finite group G if and only if G is isomorphic to the quotient Γ/Λ for some Fuchsian group Γ containing Λ as a normal subgroup (cf. [**5**], [**8**]). The necessary and sufficient condition for the existence of Γ which admits an epimorphism $\Gamma\rightarrow G$ is obtained for a cyclic group G in [**8**] and for a dihedral group G in [**5**], and this condition

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gives information about the existence of G-action on M by combining with the Riemann-Hurwitz equation. However, it is in general difficult to examine whether M admits a G-action by using this method.

Now using the determinant of the action instead of the trace, we can define $\det(D,g)$ by

$$\det(D,g) = \det(g|\ker D)/\det(g|\mathrm{coker}\, D) \in S^1 \subset \boldsymbol{C}^*$$

(cf. [**15**]), which we call the equivariant determinant of D evaluated at $g \in G$. The equivariant determinant can be related to the Atiyah-Singer index as follows. Let G_0 denote the dense subset of G consisting of elements of finite order. If $g^p = 1$ $(p \geq 2)$ for $g \in G_0$, as was proved in Appendix of [**15**], we have

$$\det(D,g) = \exp\left(\frac{2\pi\sqrt{-1}}{p}\sum_{k=1}^{p-1}\frac{1}{1-\xi_p^{-k}}\{\mathrm{Ind}(D) - \mathrm{Ind}(D,g^k)\}\right) \tag{1}$$

where $\xi_p = e^{2\pi\sqrt{-1}/p}$ is the primitive p-th root of unity and

$$\mathrm{Ind}(D) = \mathrm{Ind}(D,1) = \dim\ker D - \dim\mathrm{coker}\, D \in \boldsymbol{Z}$$

is the numerical index of D (cf. [**3**]). The equality (1) is proved as follows.

Since $\sum_{k=1}^{p-1}\xi_p^{\nu k} = -1 \pmod p$ for any integer ν, we have

$$\sum_{k=1}^{p-1}\frac{1-\xi_p^{k\lambda}}{1-\xi_p^{-k}} = -\sum_{k=1}^{p-1}\sum_{\nu=1}^{\lambda}\xi_p^{k\nu} = \lambda \pmod p$$

for any natural number λ. Let A be an $N \times N$-matrix whose p-th power is the unit matrix and $\xi_p^{\lambda_j}$ $(1 \leq j \leq N)$ its eigenvalues where λ_j's are natural numbers such that $1 \leq \lambda_j \leq p$. Then it follows from the equality above that

$$\lambda_1 + \cdots + \lambda_N = \sum_{k=1}^{p-1}\frac{1}{1-\xi_p^{-k}}\sum_{j=1}^{N}\left(1-\xi_p^{\lambda_j k}\right) \pmod p,$$

and hence we have

$$\det(A) = \exp\left(\frac{2\pi\sqrt{-1}}{p}\sum_{k=1}^{p-1}\frac{1}{1-\xi_p^{-k}}\{N - \mathrm{Tr}\,(A^k)\}\right).$$

The equality (1) follows from the equality above.

We assume that G is a finite group hereafter. Then the equality (1) gives a relation between the equivariant determinant and the fixed point data of the G-action on M and we can obtain a necessary condition on the fixed point data for the existence of a G-action on M directly from the relation among the generators of G by virtue of the fact that the equivariant determinant is a group homomorphism. We apply this method to know whether a finite group can be a subgroup of the mapping class group of a given genus $\sigma \geq 2$, namely, whether a finite group can act biholomorphically on a compact Riemann surface of genus $\sigma \geq 2$, in section 3 and to examine whether a finite group can act on M with $m \geq 2$ so that the fixed point set consists only of isolated points in section 4.

2. An additive group homomorphism and the calculation formula.

Using the equivariant determinant, we define an invariant I_D as follows.

DEFINITION 2.1. For $g \in G$, $I_D(g) \in \boldsymbol{R}/\boldsymbol{Z}$ is defined by

$$I_D(g) = \frac{1}{2\pi\sqrt{-1}} \log \det(D, g) \quad (\mathrm{mod}\ \boldsymbol{Z}).$$

Then since the equalities

$$\det(D, gh) = \det(D, g) \det(D, h)$$
$$\frac{1}{2\pi\sqrt{-1}} \log \det(D, g)^N \equiv N \frac{1}{2\pi\sqrt{-1}} \log \det(D, g) \quad (\mathrm{mod}\ \boldsymbol{Z})$$

hold, $I_D : G \longrightarrow \boldsymbol{R}/\boldsymbol{Z}$ is an additive group homomorphism and we have the next theorem.

THOREM 2.2. *We have*
(a) $I_D(g) + I_D(h) - I_D(gh) = 0$ *for any* $g, h \in G$,
(b) $N\, I_D(g) = 0$ *for any natural number* N *and any* $g \in G$ *such that* $\det(D, g)^N = 1$.

Now for any $p \geq 2$ and any $1 \leq k \leq p-1$, we have

$$\frac{1}{1 - \xi_p^{-k}} = \frac{1}{2} - \frac{\sqrt{-1}}{2} \cot \frac{\pi k}{p} \tag{2}$$

and hence it follows from (1) that the equality

$$I_D(g) \equiv \frac{p-1}{2p} \mathrm{Ind}(D) - \frac{1}{p} \sum_{k=1}^{p-1} \frac{1}{1 - \xi_p^{-k}} \mathrm{Ind}(D, g^k) \quad (\mathrm{mod}\ \boldsymbol{Z}) \tag{3}$$

holds if $g^p = 1$ $(p \geq 2)$.

REMARK 2.3. Since I_D is an additive group homomorphism, we have $I_D(g^N) = N I_D(g)$ and $I_D(gh) = I_D(hg)$. In particular, $I_D(h) = 0 \Longleftrightarrow \det(D, h) = 1$ for any element h of the commutator subgroup of G.

We can calculate $\mathrm{Ind}(D)$, $\mathrm{Ind}(D, h)$ and hence $I_D(h)$ by using the Atiyah-Singer index theorem. Let h be an element of G of order p and $\boldsymbol{Z}_p$ the cyclic group generated by h. For an integer τ, $V(\tau)$ denotes the 2-dimensional real $\boldsymbol{Z}_p$-representation defined by

$$h|V(\tau) = \begin{pmatrix} \cos(2\pi\tau)/p & -\sin(2\pi\tau)/p \\ \sin(2\pi\tau)/p & \cos(2\pi\tau)/p \end{pmatrix}.$$

Assume that the fixed point set of h consists of points $q_1, q_2, \cdots, q_n$. Then there exist integers $0 < \tau_{ij} \leq p/2$ such that the tangent bundle $T_{q_i}M$ of M at q_i is isomorphic to the direct sum

$$T_{q_i}M = \oplus_{j=1}^m V(\tau_{ij}) \tag{4}$$

as a real $\boldsymbol{Z}_p$-representation for any i. Then $\operatorname{Ind}(D)$ and $\operatorname{Ind}(D,h)$ are calculated by using the Atiyah-Singer index theorem. In particular, using the Lefschetz Theorem (3.9) in [**3**] (see also [**12**] Theorem 14.3 in chapter III), we obtain the formula:

$$\operatorname{Ind}(D,h)=\sum_{i=1}^{n}\frac{\chi_i(h)}{\prod_{j=1}^{m}(1-\xi_p^{\tau_{ij}})(1-\xi_p^{-\tau_{ij}})}$$

where $\chi_i(h)$ is the character of the virtual representation $E_{q_i}-F_{q_i}$ evaluated at h.

First we have the next proposition (see (6.17) in [**3**]).

PROPOSITION 2.4. *Let D be the signature operator and assume that p is an odd prime number. Then we have*

$$\operatorname{Ind}(D)=\operatorname{Sign}(M)\ ,\ \ \operatorname{Ind}(D,h)=\sum_{i=1}^{n}\prod_{j=1}^{m}\left(-\sqrt{-1}\cot\frac{\pi\tau_{ij}}{p}\right)$$

where $\operatorname{Sign}(M)$ *is the signature of* M.

Let $Spin(2m)$ be the $Spin$-group, $Spin^c(2m)=Spin(2m)\times_{\boldsymbol{Z}_2}S^1$ the $Spin^c$-group, $\pi:Spin^c(2m)\longrightarrow SO(2m)$ the projection and $\rho:Spin^c(2m)\longrightarrow S^1$ the homomorphism defined by $\rho([s,z])=z^2$ for $s\in Spin(2m), z\in S^1$. Then a $Spin^c(2m)$-principal bundle P over M is called a $Spin^c$-structure of M if $P\times_{Spin^c(2m)}\boldsymbol{R}^{2m}$ is isomorphic to the tangent bundle TM (see [**12**] Appendix D). It is known that M has a $Spin^c$-structure if and only if the Bockstein image of the second Stiefel-Whitney class $w_2(TM)$ in $H^3(M;\boldsymbol{Z})$ vanishes. In particular, M has a $Spin^c$-structure if M has a $Spin$-structure or an almost complex structure. Assume that M has a $Spin^c$-structure and let $\eta=P\times_{Spin^c(2m)}\boldsymbol{C}$ be the associated complex line bundle over M defined by ρ. Note that if the $Spin^c$-structure comes from an almost complex structure, η is isomorphic to the complex line bundle $\wedge^m TM$.

There exist a short exact sequence

$$1\longrightarrow\boldsymbol{Z}_2\longrightarrow Spin^c(2m)\xrightarrow{\pi\times\rho}SO(2m)\times S^1\longrightarrow 1 \tag{5}$$

and the induced exact sequence

$$\begin{aligned}&H^1(M;\boldsymbol{Z}_2)\longrightarrow H^1(M;Spin^c(2m))\xrightarrow{\varphi}\\ &\qquad H^1(M;SO(2m))\oplus H^1(M;S^1)\cong H^1(M;SO(2m))\oplus H^2(M;\boldsymbol{Z})\xrightarrow{\psi}H^2(M;\boldsymbol{Z}_2)\end{aligned}$$

where $\varphi(P)$ is the direct sum of the oriented orthonormal frame bundle $Q\in H^1(M;SO(2m))$ of M and the first Chern class $c_1(\eta)\in H^2(M;\boldsymbol{Z})$ and $\psi(\varphi(P))$ is equal to the sum of the second Stiefel-Whitney class $w_2(TM)$ and the mod 2 reduction of $c_1(\eta)$ (see [**12**] (D.2), (D.4) in Appendix D). Hence the equivalence class of a $Spin^c$-structure on M is determined by $c_1(\eta)$ if $H^1(M;\boldsymbol{Z}_2)=0$ and the mod 2 reduction of the difference $c_1(\eta)-c_1(\eta')$ corresponding to two $Spin^c$-structures vanishes. In particular, if M has an almost complex structure, there exists an element $u\in H^2(M;\boldsymbol{Z})$ such that $c_1(\eta)=c_1(\wedge^m TM)+2u=c_1(TM)+2u$.

In this paper, we call an action of G on a $Spin^c$-manifold M a $Spin^c$-action if the action lifts to an action on the $Spin^c$-structure of M. Note that a $Spin^c$-action with respect to the $Spin^c$-structure which comes from the almost complex structure of an almost complex manifold does not necessarily preserve the almost complex structure.

REMARK 2.5. Since any action of G on M lifts to the differential action on the oriented orthonormal frame bundle Q, an action of G on M lifts to an action on the $Spin^c$-structure P if the action on Q lifts to the S^1-bundle P over Q. Here it follows from Corollary 1.4 in [**9**] that any action of a finite Abelian group G on Q lifts to an action on P if $H^1(Q;\boldsymbol{Z}) = 0$ and $c_1(\eta)$ is invariant under the action of G. For example if $m \geq 2$ and $H^1(M;\boldsymbol{Z}) = 0$, it follows from the Serre spectral sequence corresponding to the fibration $SO(2m) \to Q \to M$ that $H^1(Q;\boldsymbol{Z}) = 0$ because

$$E_2^{1,0} = H^1(M;H^0(SO(2m);\boldsymbol{Z})) = 0\,, \quad E_2^{0,1} = H^0(M;H^1(SO(2m);\boldsymbol{Z})) = 0\,,$$

and hence that any action of a finite Abelian group G lifts to a $Spin^c$-action if $c_1(\eta)$ is invariant under the G-action.

Assume that there exists a $Spin^c$-action of G on M. Then for any complex G-vector bundle E over M we can define the G-equivariant E-valued Dirac operator

$$D_E : \Gamma(S_+ \otimes E) \longrightarrow \Gamma(S_- \otimes E)$$

by using G-invariant metric connections of P and E where $S_\pm = P \times_{Spin^c(2m)} \Delta_\pm$ are the half spinor bundles. Here we follow the sign convention of the complex half spin representation $\Delta_\pm$ in [**6**], [**12**] so that we can identify the Dirac operator on an almost complex manifold with the Dolbeault operator (cf. Theorem 3.5.10 in [**6**]). This sign convention differs from the sign convention in [**1**], [**3**] in the constant $(-1)^m$. Then since h acts on $S_\pm|q_i = \Delta_\pm$ through an action on $P|q_i = Spin^c(2m)$ and an action on $P|q_i$ is determined by the induced actions on $T_{q_i}M$ and on $\eta|q_i$ up to ± 1 (see (5)), we have the next proposition (see [**1**] Theorem 8.35 and [**12**] Theorem 14.11 in chapter III, (D.19), Theorem D.15 in Appendix D).

PROPOSITION 2.6. *Let L be a complex G-line bundle over the $Spin^c$-manifold M and suppose that h acts on the fibers $\eta|q_i$, $L|q_i$ via multiplications by $\xi_p^{\kappa_i}$, $\xi_p^{\mu_i}$ respectively. Then we have*

$$\mathrm{Ind}(D_L) = e^{c_1(L)} e^{c_1(\eta)/2} \widehat{A}(TM)[M] \ , \quad \mathrm{Ind}(D_L, h) = \sum_{i=1}^{n} \varepsilon_i \xi_p^{\mu_i} \xi_p^{\nu_i/2} \prod_{j=1}^{m} \frac{1}{1-\xi_p^{-\tau_{ij}}}$$

where $\widehat{A}$ is the $\widehat{A}$-class, $[M]$ is the fundamental cycle of M, $\varepsilon_i = \pm 1$ and $\nu_i = \kappa_i - \sum_{j=1}^{m} \tau_{ij}$.

Note that the numbers ε_i, κ_i in the proposition above depend on the G-action on P and are not determined by the fixed point data of the G-action on M. But if the $Spin^c$-structure comes from an almost complex structure of M and the G-action preserves the almost complex structure, the G-action on the $Spin^c$-structure is obtained from the G-action on M and the next proposition follows from the Riemann-Roch theorem (4.3) and the holomorphic Lefschetz theorem (4.6) in [**3**] (see also Theorem 3.5.2, Theorem 3.5.10 in [**6**]).

PROPOSITION 2.7. *Assume that M has an almost complex structure and that the action of G preserves the almost complex structure. Let L be a complex G-line bundle over M. Suppose that h acts on the tangent space $T_{q_i}M$ via multiplication by a diagonal matrix with diagonal entries $(\xi_p^{\tau_{i1}}, \cdots, \xi_p^{\tau_{im}})$ and acts on the fiber $L|q_i$ via multiplication by $\xi_p^{\mu_i}$. Then we have*

$$\mathrm{Ind}(D_L) = e^{c_1(L)} \mathrm{Td}(TM)[M] \ , \quad \mathrm{Ind}(D_L, h) = \sum_{i=1}^{n} \xi_p^{\mu_i} \prod_{j=1}^{m} \frac{1}{1-\xi_p^{-\tau_{ij}}}$$

where D_L is the L-valued Dirac operator with respect to the natural Spin^c-structure of M and Td *is the Todd class.*

The number n of the fixed points of h is calculated by using the next proposition.

PROPOSITION 2.8. *We have*

$$n = \sum_{j=0}^{2m} (-1)^j \mathrm{Tr}\,(h|H^j(M;\boldsymbol{R}))\,.$$

PROOF. For $1 \leq i \leq n$, it follows from (4) that the eigenvalues of $1|T_{q_i}M - h|T_{q_i}M$ are $1 - \xi_p^{\tau_{i1}}, 1 - \xi_p^{-\tau_{i1}}, \cdots, 1 - \xi_p^{\tau_{im}}, 1 - \xi_p^{-\tau_{im}}$ and hence the determinant of $1|T_{q_i}M - h|T_{q_i}M$ is positive. Therefore the equality above is deduced from Theorem A in [**1**] (see also p. 455 in [**1**], Theorem 3.9.1(a) in [**6**]). □

3. Finite subgroup of the mapping class group.

Let M be a compact Riemann surface of genus $\sigma \geq 2$. In this section, an action of a finite group G on M is defined to be a biholomorphic action of G with respect to some complex structure of M. Then it is known that G is not a subgroup of the mapping class group Γ_σ if M does not admit any action of G (see [**10**]).

Assume that M admits an action of the cyclic group $\boldsymbol{Z}_p$ of order p generated by g and suppose that the quotient map $\pi : M \longrightarrow M/\boldsymbol{Z}_p$ is a branched covering with b branch points $y_1, \cdots, y_b \in M/\boldsymbol{Z}_p$ of order $(n_1, \cdots, n_b)$. For $1 \leq i \leq b$, set $r_i = p/n_i$. Then the Riemann-Hurwitz equation

$$2\sigma - 2 = p(2\overline{\sigma} - 2) + \sum_{i=1}^{b} (p - r_i)$$

holds where $\overline{\sigma}$ is the genus of $M/\boldsymbol{Z}_p$.

Let $L = \otimes^\ell TM$ be the tensor product of ℓ TM's and D_ℓ the L-valued Dirac operator on M. Then applying Theorem 2.2, we have the next theorem.

THOREM 3.1. *Assume that M admits an action of $G = \boldsymbol{Z}_p = \langle g \rangle$. Then for $1 \leq i \leq b$ there exists a natural number $1 \leq t_i < n_i$ which is prime to n_i such that*

$$\varphi_{\ell,z}(t_1, \cdots, t_b) \in \boldsymbol{Z}\,, \quad N\psi_{\ell,z}(t_1, \cdots, t_b) \in \boldsymbol{Z}\,, \quad \psi_{\ell,z}(t_1, \cdots, t_b) \equiv I_{D_\ell}(g^z) \pmod{\boldsymbol{Z}}$$

for any z $(1 \leq z < p)$ which is prime to p and for any ℓ $(0 \leq \ell < p)$ where

$$\begin{aligned}\varphi_{\ell,z}(t_1, \cdots, t_b) &= (1 - z)\frac{p-1}{2p}(1 - \sigma)(2\ell + 1) \\ &\qquad - \sum_{i=1}^{b} \frac{1}{n_i} \sum_{j=1}^{n_i - 1} \frac{1}{1 - \xi_{n_i}^{-j}} \left(\frac{\xi_{n_i}^{jzt_i\ell}}{1 - \xi_{n_i}^{-jzt_i}} - z\frac{\xi_{n_i}^{jt_i\ell}}{1 - \xi_{n_i}^{-jt_i}} \right), \\ \psi_{\ell,z}(t_1, \cdots, t_b) &= \frac{p-1}{2p}(1 - \sigma)(2\ell + 1) - \sum_{i=1}^{b} \frac{1}{n_i} \sum_{j=1}^{n_i - 1} \frac{\xi_{n_i}^{jzt_i\ell}}{(1 - \xi_{n_i}^{-j})(1 - \xi_{n_i}^{-jzt_i})}\end{aligned}$$

(ξ_{n_i} *is the primitive n_i-th root of unity*) *and N is a natural number such that* $\det(D_\ell, g)^N = 1$.

PROOF. Let $x \in H^2(M;\boldsymbol{Z})$ be the first Chern class $c_1(TM)$ of the tangent bundle TM. Then since

$$e^{c_1(L)} = 1 + \ell c_1(TM) = 1 + \ell x\,, \quad \mathrm{Td}(TM) = \frac{x}{1-e^{-x}} = 1 + \frac{1}{2}x$$

and $x[M] = c_1(TM)[M] = 2 - 2\sigma$, it follows from Proposition 2.7 that

$$\mathrm{Ind}(D_\ell) = \left(\ell + \frac{1}{2}\right) x[M] = (1-\sigma)(2\ell+1)\,.$$

Now let $\Omega(k)$ be the fixed point set of g^k $(1 \le k \le p-1)$ and q_i any point in $\pi^{-1}(y_i)$. Then we can see that $\pi^{-1}(y_i)$ consists of r_i points $q_i, g\cdot q_i, \cdots, g^{r_i-1}\cdot q_i$, which are fixed points of g^{r_i} and therefore it follows that

$$\pi^{-1}(y_i) \subset \Omega(k) \Longleftrightarrow \pi^{-1}(y_i) \cap \Omega(k) \neq \phi \Longleftrightarrow k = r_i j \;\; (j = 1, 2, \cdots, n_i - 1)\,.$$

Since g acts transitively on $\pi^{-1}(y_i)$, g^{r_i} acts on the tangent space of M at each point in $\pi^{-1}(y_i)$ via the same rotation angle and therefore we can suppose that g^{r_i} acts on the tangent space of M at each point in $\pi^{-1}(y_i)$ via multiplication by $\xi_p^{r_i t_i}$ where $1 \le t_i < n_i$ and t_i is prime to n_i. Let z be any integer with $1 \le z < p$ such that z is prime to p. Then since the order of g^z is p, $M/\langle g^z\rangle$ coincides with $M/\langle g\rangle$ and $(g^z)^{r_i}$ acts on the tangent space of M at each point in $\pi^{-1}(y_i)$ via multiplication by $\xi_p^{z r_i t_i}$, it follows from (3) and Proposition 2.7 that

$$\begin{aligned} I_{D_\ell}(g^z) &\equiv \frac{p-1}{2p}(1-\sigma)(2\ell+1) - \frac{1}{p}\sum_{i=1}^{b} r_i \sum_{j=1}^{n_i-1} \frac{\xi_p^{r_i j z t_i \ell}}{(1-\xi_p^{-r_i j})(1-\xi_p^{-r_i j z t_i})} \\ &= \frac{p-1}{2p}(1-\sigma)(2\ell+1) - \sum_{i=1}^{b} \frac{1}{n_i} \sum_{j=1}^{n_i-1} \frac{\xi_{n_i}^{j z t_i \ell}}{(1-\xi_{n_i}^{-j})(1-\xi_{n_i}^{-j z t_i})} \\ &= \psi_{\ell,z}(t_1,\cdots,t_b) \pmod{\boldsymbol{Z}}\,. \end{aligned}$$

Therefore it follows from Theorem 2.2 (a) that

$$0 = I_{D_\ell}(g^z) - z I_{D_\ell}(g) \equiv \psi_{\ell,z}(t_1,\cdots,t_b) - z\psi_{\ell,1}(t_1,\cdots,t_b) = \varphi_{\ell,z}(t_1,\cdots,t_b) \pmod{\boldsymbol{Z}}$$

and it follows from Theorem 2.2 (b) that

$$0 = zNI_{D_\ell}(g) = NI_{D_\ell}(g^z) \equiv N\psi_{\ell,z}(t_1,\cdots,t_b) \pmod{\boldsymbol{Z}}\,. \qquad \square$$

Approximate values of $\varphi_{\ell,z}(t_1,\cdots,t_b)$ and $\psi_{\ell,z}(t_1,\cdots,t_b)$ are obtained by using a computer and the approximate values are sufficient to decide whether $\varphi_{\ell,z}(t_1,\cdots,t_b)$ and $\psi_{\ell,z}(t_1,\cdots,t_b)$ are integers if the approximate values are accurate enough. Moreover the precise values of $\varphi_{\ell,z}(t_1,\cdots,t_b) \equiv I_{D_\ell}(g^z) - zI_{D_\ell}(g) \pmod{\boldsymbol{Z}}$ and $\psi_{\ell,z}(t_1,\cdots,t_b) \equiv I_{D_\ell}(g^z) \pmod{\boldsymbol{Z}}$ are obtained by using the next proposition, which is proved in Appendix.

PROPOSITION 3.2. *$12pI_{D_\ell}(g^z)$ is an integer and we have*

$$\begin{aligned} 12pI_{D_\ell}(g^z) &\equiv 6(p-1)(1-\sigma)(2\ell+1) \\ &\quad + \sum_{i=1}^{b} r_i \left\{ z t_i (n_i-1)(7n_i-11) + 6 \sum_{j=[((\ell+1)zt_i)/n_i]+1}^{[((\ell+n_i+1)zt_i)/n_i]} f_{n_i}\left(\left[\frac{jn_i-1}{zt_i}\right] - \ell - 1\right)\right\} \\ &\quad \pmod{12p} \end{aligned}$$

where $f_{n_i}(x) = x^2 - (n_i - 2)x - (n_i - 1)^2$ *and* $[y]$ *denotes the greatest integer such that* $[y] \leq y$.

EXAMPLE 3.3. Let M be a compact Riemann surface of genus σ. Then the necessary and sufficient condition on M to admit a $\boldsymbol{Z}_p$-action is given in Theorem 4 in [**8**] (see also Proposition 2.2 in [**7**]). In this example, we consider one hundred cases where $2 \leq \sigma,\ p \leq 11$. Then if

$$(\sigma, p) = (2,7),\,(2,11),\,(3,11),\,(4,11),\,(5,7),\,(7,11),\,(8,11),\,(9,11)\,, \tag{6}$$

the Riemann-Hurwitz equation is not satisfied for any $\overline{\sigma}$, b, r_i and hence M does not admit any $\boldsymbol{Z}_p$-action. Moreover using Theorem 4 in [**8**], we can see that M does not admit any action of $\boldsymbol{Z}_p$ if and only if (σ, p) is contained in (6) or

$$(\sigma, p) = (2,9),\,(3,5),\,(3,10),\,(4,7),\,(5,9)\,,(6,11),\,(11,7)\,. \tag{7}$$

In this example, using the Riemann-Hurwitz equation and Theorem 3.1, we prove that M does not admit any $\boldsymbol{Z}_p$-action for (σ, p) in (7).

Now using the Riemann-Hurwitz equation, we can see that

$$(\sigma, p) = (2,9) \implies (b, \{n_1, \cdots, n_b\}) = (3, \{3,3,9\})$$

$$(\sigma, p) = (3,5) \implies (b, \{n_1, \cdots, n_b\}) = (1, \{5\})$$

$$(\sigma, p) = (3,10) \implies (b, \{n_1, \cdots, n_b\}) = (3, \{5,5,5\}),\ (4, \{2,2,2,10\})$$

$$(\sigma, p) = (4,7) \implies (b, \{n_1, \cdots, n_b\}) = (1, \{7\})$$

$$(\sigma, p) = (5,9) \implies (b, \{n_1, \cdots, n_b\}) = (4, \{3,3,3,9\}),\ (1, \{9\})$$

$$(\sigma, p) = (6,11) \implies (b, \{n_1, \cdots, n_b\}) = (1, \{11\})$$

$$(\sigma, p) = (11,7) \implies (b, \{n_1, \cdots, n_b\}) = (1, \{7\})\,.$$

When $(\sigma, p) = (2,9)$, $(b, \{n_1, \cdots, n_b\}) = (3, \{3,3,9\})$, direct computation using Proposition 3.2 shows that

$$1 < \varphi_{1,2}(1,1,1) = \frac{16}{9} < 2\,, \quad 1 < \varphi_{1,2}(2,1,1) = \varphi_{1,2}(1,2,1) = \frac{10}{9} < 2\,,$$
$$0 < \varphi_{1,2}(2,2,1) = \frac{4}{9} < 1\,.$$

Moreover we have

$$2 < \varphi_{1,2}(1,1,2) < 3\,,\ 1 < \varphi_{1,2}(2,1,2) = \varphi_{1,2}(1,2,2) < 2\,,\ 0 < \varphi_{1,2}(2,2,2) < 1\,,$$

$$2 < \varphi_{1,2}(1,1,4) < 3\,,\ 1 < \varphi_{1,2}(2,1,4) = \varphi_{1,2}(1,2,4) < 2\,,\ 1 < \varphi_{1,2}(2,2,4) < 2\,,$$

$$1 < \varphi_{1,2}(1,1,5) < 2\,,\ 1 < \varphi_{1,2}(2,1,5) = \varphi_{1,2}(1,2,5) < 2\,,\ 0 < \varphi_{1,2}(2,2,5) < 1\,,$$

$$2 < \varphi_{1,2}(1,1,7) < 3\,,\ 1 < \varphi_{1,2}(2,1,7) = \varphi_{1,2}(1,2,7) < 2\,,\ 0 < \varphi_{1,2}(2,2,7) < 1\,,$$

$$2 < \varphi_{1,2}(1,1,8) < 3\,,\ 1 < \varphi_{1,2}(2,1,8) = \varphi_{1,2}(1,2,8) < 2\,,\ 1 < \varphi_{1,2}(2,2,8) < 2\,,$$

and therefore none of $\varphi_{1,2}(t_1,t_2,t_3)$ is an integer. Hence it follows from Theorem 3.1 that the Riemann surface of genus 2 does not admit any action of $\boldsymbol{Z}_9$.

When $(\sigma,p)=(3,5)$, $(b,\{n_1,\cdots,n_b\})=(1,\{5\})$, direct computation shows that

$$2<\varphi_{1,2}(1)\,,\ \varphi_{1,2}(2)\,,\ \varphi_{1,2}(3)\,,\ \varphi_{1,2}(4)<3\,.$$

Hence the Riemann surface of genus 3 does not admit any action of $\boldsymbol{Z}_5$. Therefore it is clear that the Riemann surface of genus 3 does not admit any action of $\boldsymbol{Z}_{10}$.

When $(\sigma,p)=(4,7)$, $(b,\{n_1,\cdots,n_b\})=(1,\{7\})$, direct computation shows that

$$3<\varphi_{1,2}(1)\,,\ \varphi_{1,2}(4)\,,\ \varphi_{1,2}(5)<4<\varphi_{1,2}(2)\,,\ \varphi_{1,2}(3)\,,\ \varphi_{1,2}(6)<5\,.$$

Hence the Riemann surface of genus 4 does not admit any action of $\boldsymbol{Z}_7$.

When $(\sigma,p)=(5,9)$, $(b,\{n_1,\cdots,n_b\})=(4,\{3,3,3,9\})$, direct computation shows that none of $\varphi_{1,2}(t_1,t_2,t_3,t_4)$ is an integer for $1\le t_1\le t_2\le t_3\le 2$, $1\le t_4\le 8$, $t_4\ne 3,6$. Moreover if $(\sigma,p)=(5,9)$, $(b,\{n_1,\cdots,n_b\})=(1,\{9\})$, direct computation also shows that none of $\varphi_{1,2}(t_1)$ is an integer for $1\le t_1\le 8, t_1\ne 3,6$. Hence the Riemann surface of genus 5 does not admit any action of $\boldsymbol{Z}_9$.

When $(\sigma,p)=(6,11)$, $(b,\{n_1,\cdots,n_b\})=(1,\{11\})$, direct computation shows that none of $\varphi_{1,2}(t_1)$ is an integer for $1\le t_1\le 10$. Hence the Riemann surface of genus 6 does not admit any action of $\boldsymbol{Z}_{11}$.

When $(\sigma,p)=(11,7)$, $(b,\{n_1,\cdots,n_b\})=(1,\{7\})$, direct computation shows that none of $\varphi_{1,2}(t_1)$ is an integer for $1\le t_1\le 6$. Hence the Riemann surface of genus 11 does not admit any action of $\boldsymbol{Z}_7$.

REMARK 3.4. It also follows from Theorem 7.1 in [**1**] that the compact Riemann surface of genus σ does not admit any action of $\boldsymbol{Z}_p$ if $(\sigma,p)=(3,5),(4,7),(6,11),(11,7)$.

EXAMPLE 3.5. Let M be a compact Riemann surface of genus σ $(2\le\sigma\le 11)$ which admits an action of $\boldsymbol{Z}_p$ $(3\le p\le 11)$. Let G be a finite non-Abelian group and we assume that the commutator subgroup of G contains an element γ which is expressed as the product of r g's and s_j h_j's $(0<r, G\ni g\ne 1, G\ni h_j,\ 1\le j\le u)$ which satisfies the condition that the greatest common divisor d of p and $r\mu$ is less than p where $p,q_1,\cdots,q_u$ are orders of $g,h_1,\cdots,h_u$ respectively and μ is the least common multiple of $q_1,\cdots,q_u$. For example, let G be the dihedral group $D(2p)$ generated by g, h whose orders are p, 2 respectively. Then we have $\gamma=g^{-1}h^{-1}gh=g^{p-2}$ and the greatest common divisor d of p and $r\mu=p-2$ is less than p. For other example, let G be the symmetric group of p letters $1,2,\cdots,p$, $G\ni\tau_1=(1,2)$, $\tau_2=(1,3),\cdots,\tau_{p-1}=(1,p)$ the transpositions and g an element of G defined by $g=\tau_1\tau_2\cdots\tau_{p-1}=(p,p-1,\cdots,2,1)$ whose order is p. Then we have $\gamma=1=g\tau_{p-1}\cdots\tau_2\tau_1$ and the greatest common divisor d of p and $r\mu=2$ is less than p.

Now we assume that M admits an action of G. Then it follows that

$$\begin{aligned}&1=\det(D_\ell,\gamma^\mu)=\det(D_\ell,g)^{r\mu}\det(D_\ell,h_1)^{s_1\mu}\cdots\det(D_\ell,h_u)^{s_u\mu}=\det(D_\ell,g)^{r\mu}\\&\Longrightarrow\ \det(D_\ell,g)^d=1\end{aligned}\tag{8}$$

because the commutator subgroup of G is contained in the kernel of the equivariant determinant (see Remark 2.3). Let $\boldsymbol{Z}_p$ be the cyclic group generated by g and suppose that M is the branched covering space of $M/\boldsymbol{Z}_p$ with b branch points $y_1, \cdots, y_b$ of order $(n_1, \cdots, n_b)$. Then it follows from Theorem 3.1 that there exists a natural number $1 \le t_i < n_i$ which is prime to n_i for $1 \le i \le b$ such that $d\psi_{\ell,z}(t_1, \cdots, t_b) \in \boldsymbol{Z}$ for any z $(1 \le z < p)$ which is prime to p and for any ℓ $(0 \le \ell < p)$.

Now it follows from the Riemann-Hurwitz equation and Theorem 4 in [**8**] that

$$(\sigma, p) = (2,5) \Longrightarrow (b, \{n_1, \cdots, n_b\}) = (3, \{5,5,5\})$$

$$(\sigma, p) = (7,5) \Longrightarrow (b, \{n_1, \cdots, n_b\}) = (3, \{5,5,5\})$$

$$(\sigma, p) = (3,9) \Longrightarrow (b, \{n_1, \cdots, n_b\}) = (3, \{3,9,9\})$$

$$(\sigma, p) = (4,9) \Longrightarrow (b, \{n_1, \cdots, n_b\}) = (3, \{9,9,9\})$$

$$(\sigma, p) = (11,9) \Longrightarrow (b, \{n_1, \cdots, n_b\}) = (5, \{3,9,9,9,9\})$$

$$(\sigma, p) = (7,10) \Longrightarrow (b, \{n_1, \cdots, n_b\}) = (4, \{2,10,10,10\}), (5, \{2,2,2,5,10\})$$

$$(\sigma, p) = (5,11) \Longrightarrow (b, \{n_1, \cdots, n_b\}) = (3, \{11,11,11\}).$$

When $(\sigma, p) = (2,5)$, $(b, \{n_1, \cdots, n_b\}) = (3, \{5,5,5\})$, we have $d = 1$ because d is a divisor of 5 and direct computation using Proposition 3.2 shows that $-2 < \psi_{1,1}(t_1,t_2,t_3) < -1$ for any $1 \le t_1 \le t_2 \le t_3 \le 4$. Hence none of $\psi_{1,1}(t_1,t_2,t_3)$ is an integer and therefore the Riemann surface of genus 2 does not admit any action of G if $p = 5$.

When $(\sigma, p) = (7,5)$, $(b, \{n_1, \cdots, n_b\}) = (3, \{5,5,5\})$, direct computation shows that $-8 < \psi_{1,1}(t_1,t_2,t_3) < -7$ for any $1 \le t_1 \le t_2 \le t_3 \le 4$. Hence the Riemann surface of genus 7 does not admit any action of G if $p = 5$.

When $(\sigma, p) = (3,9)$, $(b, \{n_1, \cdots, n_b\}) = (3, \{3,9,9\})$, direct computation shows that none of $3\psi_{1,1}(t_1,t_2,t_3)$ is an integer and therefore none of $\psi_{1,1}(t_1,t_2,t_3)$ is an integer for $1 \le t_1 \le t_2 \le t_3 \le 8$, $t_1,t_2,t_3 \ne 3, 6$. Hence the Riemann surface of genus 3 does not admit any action of G if $p = 9$.

When $(\sigma, p) = (4,9)$, $(b, \{n_1, \cdots, n_b\}) = (3, \{9,9,9\})$, direct computation shows that none of $3\psi_{1,1}(t_1,t_2,t_3)$ is an integer for $1 \le t_1 \le t_2 \le t_3 \le 8$, $t_1,t_2,t_3 \ne 3, 6$. Hence the Riemann surface of genus 4 does not admit any action of G if $p = 9$.

When $(\sigma, p) = (11,9)$, $(b, \{n_1, \cdots, n_b\}) = (5, \{3,9,9,9,9\})$, direct computation shows that none of $3\psi_{1,1}(t_1,t_2,t_3,t_4,t_5)$ is an integer for $1 \le t_1 \le 2$, $1 \le t_2 \le t_3 \le t_4 \le t_5 \le 8$, $t_2,t_3,t_4,t_5 \ne 3, 6$. Hence the Riemann surface of genus 11 does not admit any action of G if $p = 9$.

When $(\sigma, p) = (7,10)$, $(b, \{n_1, \cdots, n_b\}) = (4, \{2,10,10,10\})$, direct computation shows that none of $2\psi_{1,1}(t_1,t_2,t_3,t_4)$ nor none of $5\psi_{1,1}(t_1,t_2,t_3,t_4)$ is an integer for $t_1 = 1$, $1 \le t_2 \le t_3 \le t_4 \le 9$, $t_2,t_3,t_4 \ne 2, 4, 5, 6, 8$. When $(\sigma, p) = (7,10)$, $(b, \{n_1, \cdots, n_b\}) = (5, \{2,2,2,5,10\})$, direct computation also shows that none of $2\psi_{1,1}(t_1,t_2,t_3,t_4,t_5)$ nor none of $5\psi_{1,1}(t_1,t_2,t_3,t_4,t_5)$ is an integer for $t_1 = t_2 = t_3 = 1$, $1 \le t_4 \le 4$, $1 \le t_5 \le 9$, $t_5 \ne 2, 4, 5, 6, 8$. Hence the Riemann surface of genus 7 does not admit any action of G if $p = 10$.

When $(\sigma, p) = (5,11)$, $(b, \{n_1, \cdots, n_b\}) = (3, \{11,11,11\})$, direct computation shows that

$$\{(t_1,t_2,t_3) \mid \psi_{1,1}(t_1,t_2,t_3) \in \boldsymbol{Z}\} \cap \{(t_1,t_2,t_3) \mid \psi_{2,1}(t_1,t_2,t_3) \in \boldsymbol{Z}\} = \emptyset.$$

Hence the Riemann surface of genus 5 does not admit any action of G if $p = 11$.

It follows from the result above that the Riemann surface of genus σ does not admit any action of G if $(\sigma, p) = (2,5), (7,5), (3,9), (4,9), (11,9), (7,10), (5,11)$. Note that if $\sigma \equiv 0, 1$

(mod p), M can be embedded symmetrically into $\boldsymbol{R}^3$ with respect to the π-rotation around x-axis and $2\pi/p$-rotation around z-axis, and hence the Riemann surface of genus σ admits an action of the dihedral group $D(2p)$. Therefore the list of (σ,p) above does not contain (σ,p) such that $\sigma\equiv 0,1$ (mod p).

4. 0-pseudofree action of cyclic groups.

Let $\boldsymbol{Z}_p$ be the cyclic group of prime order p generated by g. Then an action of $\boldsymbol{Z}_p$ on M is called 0-pseudofree if it is not free and the fixed point set of any $h\in\boldsymbol{Z}_p$ $(h\neq 1)$ consists only of isolated points (cf. [**11**], [**14**]). In this paper, 0-pseudofree is simply called pseudofree. Then since the fixed point set of g^k is independent of k, the action of $\boldsymbol{Z}_p$ is pseudofree if and only if the fixed point set of g consists only of isolated points and the number n of the fixed points of g^k is independent of k. In this section, applying Theorem 2.2, we examine whether M admits a pseudofree action of $\boldsymbol{Z}_p$.

First we have the next theorem.

THOREM 4.1. *Assume that M admits a pseudofree action of $\boldsymbol{Z}_p=\langle g\rangle$ where p is an odd prime number. Let $q_1,q_2,\cdots,q_n$ be the fixed points of g and suppose that the tangent space $T_{q_i}M$ $(1\le i\le n)$ splits into the direct sum*

$$T_{q_i}M=\oplus_{j=1}^m V(\tau_{ij})\quad\left(0<\tau_{ij}<\frac{p}{2}\right)$$

as a real $\boldsymbol{Z}_p$-representation as in (4). *Then we have*

$$\sum_{k=1}^{(p-1)/2}\sum_{i=1}^{n}\prod_{j=1}^{2s}\cot\frac{\pi k\tau_{ij}}{p}\equiv 0\pmod{\boldsymbol{Z}}\ \text{ if }\ m=2s,$$

$$\sum_{k=1}^{(p-1)/2}\cot\frac{\pi k}{p}\sum_{i=1}^{n}\prod_{j=1}^{2s-1}\cot\frac{\pi k\tau_{ij}}{p}\equiv 0\pmod{\boldsymbol{Z}}\ \text{ if }\ m=2s-1.$$

PROOF. Let D be the signature operator. Since $pI_D(g)=0$, it follows from (2), (3) and Proposition 2.4 that

$$\begin{aligned}\boldsymbol{N}\ni&\sum_{k=1}^{p-1}\frac{1}{1-\xi_p^{-k}}\mathrm{Ind}(D,g^k)\\&=\sum_{k=1}^{(p-1)/2}2\mathrm{Re}\left\{\left(\frac{1}{2}-\frac{\sqrt{-1}}{2}\cot\frac{\pi k}{p}\right)\sum_{i=1}^{n}\prod_{j=1}^{m}\left(-\sqrt{-1}\cot\frac{\pi k\tau_{ij}}{p}\right)\right\}\\&=\begin{cases}(-1)^s\displaystyle\sum_{k=1}^{(p-1)/2}\sum_{i=1}^{n}\prod_{j=1}^{2s}\cot\frac{\pi k\tau_{ij}}{p} & (m=2s)\\(-1)^s\displaystyle\sum_{k=1}^{(p-1)/2}\cot\frac{\pi k}{p}\sum_{i=1}^{n}\prod_{j=1}^{2s-1}\cot\frac{\pi k\tau_{ij}}{p} & (m=2s-1).\end{cases}\end{aligned}$$

The theorem is deduced from the equality above. $\square$

COROLLARY 4.2. *Assume that M admits a pseudofree action of* $\boldsymbol{Z}_3$ *and let n be the number of the fixed points. Then n is even or* $n \geq 3^{[(m+1)/2]}$.

PROOF. Since $\cot(\pi/3) = 1/\sqrt{3}$ and $\cot(2\pi/3) = -1/\sqrt{3}$, it follows from Theorem 4.1 that

$$\sum_{i=1}^{n} \left(\pm \frac{1}{3^s} \right) \equiv 0 \pmod{\boldsymbol{Z}} \quad \left(s = \left[\frac{m+1}{2} \right] \right).$$

The result of the corollary immediately follows from the equality above. □

REMARK 4.3. Assume that M admits a pseudofree action of $\boldsymbol{Z}_3 = \langle g \rangle$ and let D be the signature operator. Then as is known in (6.7), (6.9) in [**3**], $\mathrm{Ind}(D,g)$ is expressed as follows:

$$\mathrm{Ind}(D,g) = \begin{cases} \mathrm{Tr}\,(g|\rho^+) - \mathrm{Tr}\,(g|\rho^-) & (\text{if } m \text{ is even}) \\ \mathrm{Tr}\,(g|\rho) - \mathrm{Tr}\,(g|\rho^*) & (\text{if } m \text{ is odd}) \end{cases}$$

where $\rho^{\pm}$ are real $\boldsymbol{Z}_3$-representations and ρ a complex $\boldsymbol{Z}_3$-representation. It follows from the equalities above that $\mathrm{Ind}(D,g) \in \boldsymbol{Z}$ if m is even and that $\mathrm{Ind}(D,g) \in \sqrt{-3}\boldsymbol{Z}$ if m is odd. The result in Corollary 4.2 is also deduce from this fact and Proposition 2.4.

For the $Spin^c$-action of cyclic groups, we have the following theorems.

THOREM 4.4. *Assume that M has a* $Spin^c$*-structure and admits a pseudofree* $Spin^c$*-action of* $\boldsymbol{Z}_2$*. If there exists a complex* $\boldsymbol{Z}_2$*-line bundle L over M such that the index* $\mathrm{Ind}(D_L)$ *of the L-valued Dirac operator* D_L *is an odd number, then we have* $n \geq 2^m$.

PROOF. It follows from Theorem 2.2 (b), (3) and Proposition 2.6 that

$$0 = 2I_{D_L}(g) \equiv \frac{1}{2} \left(\mathrm{Ind}(D_L) - \frac{1}{2^m} \sum_{i=1}^{n} \sqrt{-1}^{\lambda_i} \right) \pmod{\boldsymbol{Z}}$$

for some integer λ_i. The right-hand side of the equality above is not an integer if $\mathrm{Ind}(D_L)$ is odd and $n < 2^m$. This completes the proof. □

REMARK 4.5. In the theorem above, the index $\mathrm{Ind}(D_L)$ is equal to the index $\mathrm{Ind}(D)$ of the non-twisted Dirac operator D if L is the trivial complex line bundle with the trivial $\boldsymbol{Z}_2$-action.

The next theorem is also useful for the $Spin^c$-action of $\boldsymbol{Z}_3$, $\boldsymbol{Z}_5$.

THOREM 4.6. *Assume that M has a* $Spin^c$*-structure and admits a pseudofree* $Spin^c$*-action of* $\boldsymbol{Z}_p$ *where p is an odd prime number and that the action lifts to an action on a complex line bundle L over M. Let* δ *be the distance from* $((p-1)/2)\mathrm{Ind}(D_L)$ *to* $p\boldsymbol{Z}$ *defined by* $\delta = \min_{s \in \boldsymbol{Z}} |((p-1)/2)\mathrm{Ind}(D_L) - ps|$ *where* D_L *is the L-valued Dirac operator. Then we have*

$$n \geq \frac{\delta}{3(p-1)} \left(2 \sin \frac{\pi}{p} \right)^{m+1}.$$

Moreover if $\det(D_L, g) = 1$*, we have*

$$n \geq \frac{\delta}{p-1} \left(2 \sin \frac{\pi}{p} \right)^{m+1}.$$

PROOF. Set

$$K_1 = \sum_{k=1}^{p-1} \frac{1}{1-\xi_p^{-k}} \left\{ \mathrm{Ind}(D_L, g^{2k}) - 2\mathrm{Ind}(D_L, g^k) \right\}, \quad K_2 = \sum_{k=1}^{p-1} \frac{1}{1-\xi_p^{-k}} \mathrm{Ind}(D_L, g^k).$$

Then since $|1-\xi_p^t| \geq |1-\xi_p|$ for any integer t which is not a multiple of p, it follows from Proposition 2.6 that

$$\begin{aligned} |K_1| &\leq \sum_{k=1}^{p-1} \sum_{i=1}^{n} \frac{1}{|1-\xi_p^{-k}|} \left\{ \frac{1}{\prod_{j=1}^m |1-\xi_p^{-2k\tau_{ij}}|} + 2\frac{1}{\prod_{j=1}^m |1-\xi_p^{-k\tau_{ij}}|} \right\} \\ &\leq \frac{3n(p-1)}{|1-\xi_p|^{m+1}} = \frac{3n(p-1)}{(2\sin(\pi/p))^{m+1}}. \end{aligned}$$

Moreover it follows from Theorem 2.2 (a) and (3) that

$$\begin{aligned} &2I_{D_L}(g) - I_{D_L}(g^2) = 0 \Longleftrightarrow \frac{p-1}{2p}\mathrm{Ind}(D_L) + \frac{1}{p}K_1 \equiv 0 \pmod{\boldsymbol{Z}} \\ &\Longleftrightarrow \frac{p-1}{2}\mathrm{Ind}(D_L) + K_1 \equiv 0 \pmod{p}. \end{aligned}$$

Hence we have $|K_1| \geq \delta$ and therefore it follows that

$$\frac{3n(p-1)}{(2\sin(\pi/p))^{m+1}} \geq \delta \Longleftrightarrow n \geq \frac{\delta}{3(p-1)} \left(2\sin\frac{\pi}{p}\right)^{m+1}.$$

If $\det(D_L, g) = 1 \Longleftrightarrow I_{D_L}(g) = 0$, it follows from (3) that

$$\frac{p-1}{2p}\mathrm{Ind}(D_L) - \frac{1}{p}K_2 \equiv 0 \pmod{\boldsymbol{Z}} \Longleftrightarrow \frac{p-1}{2}\mathrm{Ind}(D_L) - K_2 \equiv 0 \pmod{p},$$

which implies that $|K_2| \geq \delta$. Hence it follows from the same argument as above that

$$\delta \leq |K_2| \leq \frac{n(p-1)}{(2\sin(\pi/p))^{m+1}} \Longrightarrow n \geq \frac{\delta}{p-1} \left(2\sin\frac{\pi}{p}\right)^{m+1}. \qquad \square$$

Under the notation in the theorem above, we obtain the next corollary immediately from Proposition 2.8.

COROLLARY 4.7. *If*

$$\sum_{j=0}^{2m} \dim H^j(M; \boldsymbol{R}) < \frac{\delta}{3(p-1)} \left(2\sin\frac{\pi}{p}\right)^{m+1},$$

then M does not admit any Spin^c-action of $\boldsymbol{Z}_p$. Moreover if $\det(D_L, g) = 1$ *and*

$$\sum_{j=0}^{2m} \dim H^j(M; \boldsymbol{R}) < \frac{\delta}{p-1} \left(2\sin\frac{\pi}{p}\right)^{m+1},$$

then M does not admit any Spin^c-action of $\boldsymbol{Z}_p$.

EXAMPLE 4.8. Let p be a prime number, Σ_{pk} the compact Riemann surfaces of genus pk and S^2 the 2-dimensional sphere. Let $T = S^2 \times \cdots \times S^2$ be the $m-1$-times product of S^2 and $M_{pk} = \Sigma_{pk} \times T$ a $2m$-dimensional almost complex manifold with

$$c_1(TM_{pk}) = (2-2pk)y + \sum_{j=1}^{m-1} 2z_j \in H^2(M_{pk}; \boldsymbol{Z}) \cong H^2(\Sigma_{pk}; \boldsymbol{Z}) \oplus \oplus_{j=1}^{m-1} H^2(S^2; \boldsymbol{Z})$$

where y is the positive generator of $H^2(\Sigma_{pk}; \boldsymbol{Z}) \cong \boldsymbol{Z}$ and $z_1, \cdots, z_{m-1}$ are the positive generators of $H^2(S^2; \boldsymbol{Z}) \cong \boldsymbol{Z}$. Hence M_{pk} has a $Spin^c$-structure with

$$c_1(\eta) = (2s+2-2pk)y + \sum_{j=1}^{m-1} (2t_j+2)z_j \in H^2(M_{pk}; \boldsymbol{Z})$$

for some integers s, t_j. If the $Spin^c$-structure of M_{pk} comes from appropriate almost complex structures of Σ_{pk}, S^2, the integers s, t_j's are equal to 0 and both of Σ_{pk} and S^2 admit pseudofree $Spin^c$-actions of $\boldsymbol{Z}_p$ with 2 fixed points, and therefore the diagonal $Spin^c$-action of $\boldsymbol{Z}_p$ on M_{pk} is pseudofree and has 2^m fixed points.

Now since the total Chern class $c(TM_{pk})$ is equal to $(1+(2-2pk)y)\prod_{j=1}^{m-1}(1+2z_j)$, it follows from Proposition 2.6 that

$$\begin{aligned} \mathrm{Ind}(D) &= e^{sy+\sum_{j=1}^{m-1} t_j z_j} \frac{(2-2pk)y}{1-e^{-(2-2pk)y}} \prod_{j=1}^{m-1} \frac{2z_j}{1-e^{-2z_j}} [M_{pk}] = (s+1-pk) \prod_{j=1}^{m-1} (t_j+1) \\ &\equiv (s+1) \prod_{j=1}^{m-1} (t_j+1) \pmod{p}. \end{aligned}$$

Hence it follows from Theorem 4.4 that any pseudofree $Spin^c$-action of $\boldsymbol{Z}_2$ on M_{2k} has n fixed points with $n \geq 2^m$ if none of s, t_j's is $-1 \pmod 2$. In particular, if the $Spin^c$-structure of M_{2k} comes from the almost complex structures of Σ_{2k}, S^2, then any pseudofree $Spin^c$-action of $\boldsymbol{Z}_2$ on M_{2k} has more than or equal to 2^m fixed points. If none of s, t_j's is $-1 \pmod 3$, we have $\delta = 1$ and hence it follows from Theorem 4.6 that any pseudofree $Spin^c$-action of $\boldsymbol{Z}_3$ on M_{3k} has n fixed points with $n \geq (2\sin(\pi/3))^{m+1}/6$. If none of s, t_j's is $-1 \pmod 5$, we have $\delta = 1$ or 2 and hence it also follows from Theorem 4.6 that any pseudofree $Spin^c$-action of $\boldsymbol{Z}_5$ on M_{5k} has n fixed points with $n \geq (2\sin(\pi/5))^{m+1}/12$.

Moreover it follows from Corollary 4.2 that M_{3k} does not admit any pseudofree action of $\boldsymbol{Z}_3$ with $1, 3, 5, \cdots, 3^{[(m+1)/2]} - 2$ fixed points.

EXAMPLE 4.9. Let $M = \boldsymbol{CP}^2 \times \boldsymbol{CP}^k$ $(k \geq 3)$ be the product of complex projective spaces and assume that M admits a pseudofree action of the cyclic group $\boldsymbol{Z}_p = \langle g \rangle$ of odd prime order p. Then we have

$$H^2(M; \boldsymbol{Z}) = H^2(\boldsymbol{CP}^2; \boldsymbol{Z}) \oplus H^2(\boldsymbol{CP}^k; \boldsymbol{Z}) = \{\lambda x + \mu y \,|\, \lambda, \mu \in \boldsymbol{Z}\} = \boldsymbol{Z} \oplus \boldsymbol{Z}$$

where $x \in H^2(\boldsymbol{CP}^2; \boldsymbol{Z}) \cong \boldsymbol{Z}$ and $y \in H^2(\boldsymbol{CP}^k; \boldsymbol{Z}) \cong \boldsymbol{Z}$ are the positive generators and

$$g^* : H^2(M; \boldsymbol{Z}) = \boldsymbol{Z} \oplus \boldsymbol{Z} \longrightarrow H^2(M; \boldsymbol{Z}) = \boldsymbol{Z} \oplus \boldsymbol{Z}$$

is represented by a 2×2 integral matrix $A = (a_{ij})$ whose p-th power is equal to the unit matrix E. Since the $\boldsymbol{Z}_p$-action preserves the volume element x^2y^k, it follows that $g^*(x^2y^k) = (g^*x)^2(g^*y)^k =$

$x^2y^k \in H^{2(2+k)}(M;\boldsymbol{Z})$ and hence that the x^2y^k-coefficient of $(a_{11}x+a_{21}y)^2(a_{12}x+a_{22}y)^k$ is equal to 1. Let ξ_p^u, ξ_p^v be the eigenvalues of A. Then since $\det(A)=1$, it follows that $v=-u$ and hence that

$$\boldsymbol{Z} \ni \mathrm{Tr}\,(A) = 2\mathrm{Re}\,(\xi_p^u) = 2\cos\frac{2\pi u}{p}\,.$$

Therefore $\mathrm{Tr}\,(A)$ is equal to -1 or 2 if $p=3$ and is equal to 2 if $p\geq 5$.

If $p=3$ and $\mathrm{Tr}\,(A)=-1$, it follows from the Hamilton-Cayley's theorem that $A^2+A+E=0$, which is equivalent to the equalities $a_{11}^2+a_{11}+1+a_{12}a_{21}=0$, $a_{11}+a_{22}=-1$. Therefore A is expressed as

$$\begin{pmatrix} s & t \\ -((s^2+s+1)/t) & -(s+1) \end{pmatrix} \quad (s,t\in\boldsymbol{Z})\,.$$

Then the x^2y^k-coefficient of $(a_{11}x+a_{21}y)^2(a_{12}x+a_{22}y)^k$ is equal to

$$\begin{aligned} f(s) &= \sum_{j=0}^{2}\binom{2}{j}\binom{k}{k-j}s^{2-j}\left(-\frac{s^2+s+1}{t}\right)^j t^j(-(s+1))^{k-j} \\ &= (-1)^k(s+1)^{k-2}\left\{s^2(s+1)^2+2ks(s+1)(s^2+s+1)+\frac{k(k-1)}{2}(s^2+s+1)^2\right\} \\ &= (-1)^k(s+1)^{k-2}\left\{s^2(s+1)^2+\frac{k}{2}(s^2+s+1)\left((k+3)\left(s+\frac{1}{2}\right)^2+\frac{3k-7}{4}\right)\right\}. \end{aligned}$$

Here we have $f(s)=0$ if $s=-1$ and

$$|f(s)| \geq \frac{k}{2}\left((k+3)\left(\pm\frac{1}{2}\right)^2+\frac{3k-7}{4}\right) = \frac{k(k-1)}{2} \geq 3$$

if $s\neq -1$, and therefore $f(s)\neq 1$ for any s. Hence we have $\mathrm{Tr}\,(A)=2$ for any odd prime p.

Then using the Hamilton-Cayley's theorem, we can show that $A^p = pA-(p-1)E \Longleftrightarrow A = E$ by induction and hence that $g^*x=x$, $g^*y=y$. Therefore g acts trivially on

$$\bigoplus_{r=0}^{2+k} H^{2r}(M;\boldsymbol{R}) \cong \bigoplus_{s=0}^{2}\bigoplus_{t=0}^{k}\left(H^{2s}(\boldsymbol{CP}^2;\boldsymbol{R})\otimes H^{2t}(\boldsymbol{CP}^k;\boldsymbol{R})\right) \cong \boldsymbol{R}^{3(k+1)}$$

and hence it follows from Proposition 2.8 that g has $3(k+1)$ fixed points. For example, if $k<p$, the fixed point set of the $\boldsymbol{Z}_p$-action on $\boldsymbol{CP}^j$ ($j=2$ or k) defined by

$$g\cdot[z_0:z_1:z_2:\cdots:z_j] \longrightarrow [z_0:\xi_p z_1:\xi_p^2 z_2:\cdots:\xi_p^j z_j] \tag{9}$$

consists of $j+1$ points. Hence the diagonal action of $\boldsymbol{Z}_p$ on M is pseudofree and has $3(k+1)$ fixed points.

Now we give a *Spin*^{c}-structure of M which comes from the almost complex structure with $c_1(\eta)=c_1(TM)=3x+(k+1)y$. Then since $c_1(\eta)=3x+(k+1)y$ is invariant under the action of $\boldsymbol{Z}_p$ and $H^1(M;\boldsymbol{Z})=0$, the action of $\boldsymbol{Z}_p$ lifts to an action on the *Spin*^{c}-structure as we see in

Remark 2.5. Let D be the non-twisted Dirac operator on M. Then it follows from Proposition 2.6 that

$$\begin{aligned}
\mathrm{Ind}(D) &= e^{(3x+(k+1)y)/2}\widehat{A}(M)[M] \\
&= x^2y^k\text{-coefficient of } \left(\frac{x}{1-e^{-x}}\right)^3 \left(\frac{y}{1-e^{-y}}\right)^{k+1} \\
&= \left(\frac{1}{2\pi\sqrt{-1}} \oint_{C_1(z)} \frac{e^{2z}}{(e^z-1)^3} e^z\,dz\right) \left(\frac{1}{2\pi\sqrt{-1}} \oint_{C_2(w)} \frac{e^{kw}}{(e^w-1)^{k+1}} e^w\,dw\right)
\end{aligned}$$

(where $C_1(z), C_2(w)$ are sufficiently small counterclockwise loops around the origin)

$$= \left(\frac{1}{2\pi\sqrt{-1}} \oint_{C_3(u)} \frac{(u+1)^2}{u^3}\,du\right) \left(\frac{1}{2\pi\sqrt{-1}} \oint_{C_4(v)} \frac{(v+1)^k}{v^{k+1}}\,dv\right)$$

(via the substitution $u = e^z - 1$, $v = e^w - 1$)

$$= u^2v^k\text{-coefficient of } (u+1)^2(v+1)^k = 1\,.$$

Hence we have $\delta = 1$ for $p = 3$ and $\delta = 2$ for $p = 5$ in Theorem 4.6, and it follows that

$$\begin{aligned}
3(3-1)\cdot 3(k+1) &\geq \left(2\sin\frac{\pi}{3}\right)^{2+k+1} \quad (p=3)\,, \\
3(5-1)\cdot 3(k+1) &\geq 2\left(2\sin\frac{\pi}{5}\right)^{2+k+1} \quad (p=5)\,,
\end{aligned}$$

which implies that $k \leq 5$ if $p = 3$ and that $k \leq 37$ if $p = 5$.

Moreover since $3(k+1) < 3^{[(3+k+1)/2]}$ for any $k \geq 3$, it follows from Corollary 4.2 that M does not admit any pseudofree action of $\boldsymbol{Z}_3$ if k is even. Hence M does not admit any pseudofree action of $\boldsymbol{Z}_3$ if $k = 4$ or $k \geq 6$ and any pseudofree action of $\boldsymbol{Z}_5$ if $k \geq 38$.

Let G be the finite non-Abelian group defined in Example 3.5. Then if $p = 3$, 5, the greatest common divisor d is equal to 1 and hence we have $\det(D, g) = 1$. Therefore if M admits a *Spin*^{c}-action of G, it follows also from Theorem 4.6 (or Corollary 4.7) that

$$\begin{aligned}
(3-1)\cdot 3(k+1) &\geq \left(2\sin\frac{\pi}{3}\right)^{2+k+1} \quad (p=3)\,, \\
(5-1)\cdot 3(k+1) &\geq 2\left(2\sin\frac{\pi}{5}\right)^{2+k+1} \quad (p=5)\,.
\end{aligned}$$

The inequalities above imply that M does not admit any pseudofree *Spin*^{c}-action of G if $p = 3$ and that $k \leq 29$ if $p = 5$ and M admits a pseudofree *Spin*^{c}-action of G.

APPENDIX. Here we give the proof of Proposition 3.2. Let a be any complex number such that $a^n = 1$ and $a \neq 1$. Then for $|t| < 1$, we have

$$\begin{aligned}
\frac{1}{(1-at)^2} &= \sum_{i=0}^{\infty}(i+1)a^it^i = \sum_{j=0}^{\infty}\sum_{s=0}^{n-1}(nj+s+1)a^st^{nj+s} \\
&= \sum_{j=0}^{\infty} t^{nj} \sum_{s=0}^{n-1}(s+1)a^st^s + n\sum_{j=0}^{\infty} jt^{nj} \sum_{s=0}^{n-1} a^st^s
\end{aligned}$$

$$= \frac{\sum_{s=0}^{n-1}(s+1)a^s t^s}{1-t^n} + \frac{nt^n \sum_{s=0}^{n-1} a^s t^s}{(1-t^n)^2} = \frac{\sum_{s=0}^{n-1}\{(n-s-1)t^n+s+1\}\,a^s t^s}{(1-t^n)^2}.$$

Set $g(t) = \sum_{s=0}^{n-1}\{(n-s-1)t^n+s+1\}\,a^s t^s$. Then we have

$$g'(t) = \sum_{s=0}^{n-1}\left\{(n^2-n-s^2-s)a^s t^{n+s-1} + (s^2+s)a^s t^{s-1}\right\}$$

$$g''(t) = \sum_{s=0}^{n-1}\left[\left\{-s^3-ns^2+(n-1)^2 s+n(n-1)^2\right\}a^s t^{n+s-2} + (s^3-s)a^s t^{s-2}\right],$$

and hence it follows that

$$g(1) = n\sum_{s=0}^{n-1} a^s = 0\,, \quad g'(1) = (n^2-n)\sum_{s=0}^{n-1} a^s = 0\,.$$

Therefore we have

$$\frac{1}{(1-a)^2} = \lim_{t\to 1-0}\frac{1}{(1-at)^2} = \lim_{t\to 1-0}\frac{g''(t)}{\{(1-t^n)^2\}''}$$
$$= \frac{\sum_{s=0}^{n-1}\left\{-ns^2+(n^2-2n)s+n(n-1)^2\right\}a^s}{2n^2} = -\sum_{s=0}^{n-1}\frac{f_n(s)}{2n}a^s$$

where $f_n(s) = s^2-(n-2)s-(n-1)^2$. Hence, if km is not a multiple of n, it follows that

$$\frac{\xi_n^{km\ell}}{(1-\xi_n^{-k})(1-\xi_n^{-km})} = \frac{\xi_n^{k(m\ell+m+1)}}{(1-\xi_n^{k})(1-\xi_n^{km})} = \xi_n^{k(m\ell+m+1)}\frac{1-\xi_n^{km}}{1-\xi_n^{k}}\frac{1}{(1-\xi_n^{km})^2}$$
$$= -\xi_n^{k(m\ell+m+1)}\sum_{\nu=0}^{m-1}\xi_n^{k\nu}\sum_{s=0}^{n-1}\frac{f_n(s)}{2n}\xi_n^{kms} = -\sum_{s=0}^{n-1}\frac{f_n(s)}{2n}\sum_{\nu=0}^{m-1}\xi_n^{k((\ell+s+1)m+1+\nu)}.$$

Thus we have

$$-\sum_{k=1}^{n-1}\frac{\xi_n^{km\ell}}{(1-\xi_n^{-k})(1-\xi_n^{-km})} = \sum_{s=0}^{n-1}\frac{f_n(s)}{2n}\sum_{\nu=0}^{m-1}\sum_{k=1}^{n-1}\xi_n^{k((\ell+s+1)m+1+\nu)}$$
$$= \sum_{s=0}^{n-1}\frac{f_n(s)}{2n}\sum_{\nu=0}^{m-1}\left(-1+\sum_{k=0}^{n-1}\xi_n^{k((\ell+s+1)m+1+\nu)}\right)$$
$$= -\frac{m}{2n}\sum_{s=0}^{n-1}f_n(s) + \frac{1}{2n}\sum_{s=0}^{n-1}f_n(s)\sum_{\nu=0}^{m-1}\sum_{k=0}^{n-1}\xi_n^{k((\ell+s+1)m+1+\nu)}$$
$$= -\frac{m}{2n}\sum_{s=0}^{n-1}f_n(s) + \frac{1}{2}\sum_{s=0}^{n-1}\vartheta_{(n,\ell,m)}(s)\,f_n(s)$$

where

$$\vartheta_{(n,\ell,m)}(s) = \#\{\nu\in\mathbf{Z}\,|\,0\le\nu\le m-1\,,\ (\ell+s+1)m+1+\nu = jn \text{ for some integer } j\}$$
$$= \#\left\{j\in\mathbf{Z}\,|\,\left[\frac{(\ell+s+1)m}{n}\right]+1\le j\le\left[\frac{(\ell+s+2)m}{n}\right]\right\}$$

because $\sum_{k=0}^{n-1}\xi_n^{k((\ell+s+1)m+1+\nu)}$ is equal to n if $(\ell+s+1)m+1+\nu$ is a multiple of n and is equal to 0 if $(\ell+s+1)m+1+\nu$ is not a multiple of n.

Here we have

$$\sum_{s=0}^{n-1} f_n(s) = \sum_{s=0}^{n-1} s^2 - (n-2)\sum_{s=0}^{n-1} s - (n-1)^2 \sum_{s=0}^{n-1} 1 = -\frac{1}{6}n(n-1)(7n-11)$$

and

$$\sum_{s=0}^{n-1} \vartheta_{(n,\ell,m)}(s) f_n(s) = \sum_{j=[((\ell+1)m)/n]+1}^{[((\ell+n+1)m)/n]} f_n\left(\left[\frac{jn-1}{m}\right]-\ell-1\right)$$

because the set of (s, j) such that

$$0 \le s \le n-1\,,\ \left[\frac{(\ell+s+1)m}{n}\right]+1 \le j \le \left[\frac{(\ell+s+2)m}{n}\right]$$

coincides with the set of (s, j) such that

$$\left[\frac{(\ell+1)m}{n}\right]+1 \le j \le \left[\frac{(\ell+n+1)m}{n}\right]\ ,\ s = \left[\frac{jn-1}{m}\right]-\ell-1\,.$$

Hence we have

$$-\sum_{k=1}^{n-1} \frac{\xi_n^{km\ell}}{(1-\xi_n^{-k})(1-\xi_n^{-km})} = \frac{m}{12}(n-1)(7n-11) + \frac{1}{2}\sum_{j=[((\ell+1)m)/n]+1}^{[((\ell+n+1)m)/n]} f_n\left(\left[\frac{jn-1}{m}\right]-\ell-1\right)$$

and therefore it follows from (3) that

$$\begin{aligned} I_{D_\ell}(g^z) &\equiv \frac{p-1}{2p}(1-\sigma)(2\ell+1) - \frac{1}{p}\sum_{i=1}^{b} r_i \sum_{j=1}^{n_i-1} \frac{\xi_{n_i}^{jzt_i\ell}}{(1-\xi_{n_i}^{-j})(1-\xi_{n_i}^{-jzt_i})} \\ &= \frac{p-1}{2p}(1-\sigma)(2\ell+1) \\ &\quad + \frac{1}{p}\sum_{i=1}^{b} r_i \left\{ \frac{zt_i}{12}(n_i-1)(7n_i-11) + \frac{1}{2}\sum_{j=[((\ell+1)zt_i)/n_i)]+1}^{[((\ell+n_i+1)zt_i)/n_i]} f_{n_i}\left(\left[\frac{jn_i-1}{zt_i}\right]-\ell-1\right)\right\}. \\ &\quad (\mathrm{mod}\ \boldsymbol{Z})\,. \end{aligned}$$

The equality of Proposition 3.2 follows immediately from the equality above.

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