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Weierstrass-type representation for harmonic maps into general symmetric spaces via loop groups

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Abstract. In [**19**] a method was presented, which constructs via loop group splittings all harmonic maps into a compact symmetric space. The present paper generalizes this method to all spaces G/K, where G is an arbitrary Lie group (semisimple or not) and K is the fixpoint group of some involution of G. The method is illustrated by a number of examples.

1. Introduction.

In [**19**] a Weierstrass-type representation was introduced for harmonic maps φ from Riemannian surfaces M to compact symmetric spaces $N = G/K$, where K is a closed subgroup of G. The paper associates first with a harmonic map $\varphi : M \to N$ a frame $F : M \to G$, such that the canonical projection $\pi : G \to N$ forms with φ and F a commutative diagram. Then, after choosing conformal coordinates on M one decomposes the Maurer Cartan form $F^{-1}\mathrm{d}F = \alpha' + \alpha''$, where α' is of type (1,0) and α'' of type (0,1).

Similar to [**4**] and [**41**], [**19**] introduces a parameter λ from $S^1 \subset \boldsymbol{C}$. Then it turns out that the integrability for α_λ is equivalent with the harmonicity of the map φ. This permits to integrate the Maurer Cartan equation $F^{-1}\mathrm{d}F = \alpha_\lambda$, where F now depends on λ ("extended frame"). The most crucial feature of [**19**] then is that one can split F everywhere into $F = H \cdot F_+$ where H is holomorphic in z ("holomorphic extended frame") and F_+ is holomorphic in λ in the open unit disk. Moreover, the Maurer Cartan form $\eta = H^{-1}dH$ of H is a holomorphic differential one-form. It is obvious that any η of this form trivially satisfies the integrability condition for the differential equation $\eta = H^{-1}dH$. We have thus replaced the Maurer-Cartan form of the extended frame F with its non-linear integrability condition with η, which satisfies its integrability condition trivially. It is crucial to observe, that from η one can reconstruct the (associated family of the) original harmonic map. Namely: starting, conversely, from an arbitary holomorphic one form η one can obtain a harmonic map $\varphi : M \to N$ as follows: first, one integrates the ordinary differential equation $\mathrm{d}H = H \cdot \eta$, $H(z = 0) = I$. Then one carries out an Iwasawa splitting $H = F \cdot V_+$ and sees that F is an extended frame for a harmonic map $\varphi = F \bmod K$.

1.1.

Let $\boldsymbol{g}$ be a finite-dimensional real Lie algebra, satisfying the following conditions:

- There exists an involutive automorphism of $\boldsymbol{g}$, $\sigma \in \mathrm{End}(\boldsymbol{g}) \setminus \{\mathrm{Id}\}$, $\sigma^2 = \mathrm{Id}$;
- $\mathrm{Ker}(\sigma - \mathrm{Id}) \cap Z(\boldsymbol{g}) = \{0\}$, where $Z(\boldsymbol{g}) = \{\xi \in \boldsymbol{g} \mid [\xi, \eta] = 0 \text{ for all } \eta \in \boldsymbol{g}\}$ is the center of $\boldsymbol{g}$.

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Then $\boldsymbol{g}$ admits the splitting

$$\boldsymbol{g} = \boldsymbol{k} \oplus \boldsymbol{p}, \tag{1.1.1}$$

where $\boldsymbol{k}$ and $\boldsymbol{p}$ are the eigenspaces of σ

$$\boldsymbol{k} = \mathrm{Ker}(\sigma - \mathrm{Id}), \qquad \boldsymbol{p} = \mathrm{Ker}(\sigma + \mathrm{Id}). \tag{1.1.2}$$

In the following we consider two connected Lie groups $K \subset G$, K closed in G, such that

$$\mathrm{Lie}(K) = \boldsymbol{k} \subset \mathrm{Lie}(G) = \boldsymbol{g}.$$

We assume as in [**3**] and [**4**] that G admits a faithful linear continuous representation. Actually, we assume that there exists some complex matrix group $G^{\boldsymbol{C}}$, the universal complexification of G in the sense of [**30**, Section 17.5], which has G as a real form. The vector space, on which $G^{\boldsymbol{C}}$ acts linearly will be denoted by V.

1.2.

In this paper we consider harmonic maps from non-compact simply connected, Riemann surfaces $\boldsymbol{D}$ to pseudo-Riemannian general symmetric spaces G/K. First we need to define the notion of general symmetric space (see, e.g., [**35**]):

We consider a connected matrix group G, σ an automorphism of order two and denote

$$\mathrm{Fix}\,\sigma = \{g \in G \mid \sigma(g) = g\}$$

and by $(\mathrm{Fix}\,\sigma)_0$ the connected component of $\mathrm{Fix}\,\sigma$ containing the identity I. Let K be a subgroup of G such that $(\mathrm{Fix}\,\sigma)_0 \subset K \subset \mathrm{Fix}\,\sigma$, then G/K is called a general symmetric space. Starting in Section 1.3 we will assume in addition that we have a non-degenerate symmetric bilinear form invariant by $\boldsymbol{g}$.

Then we have ([**3**]) the semidirect product $G \equiv H.M$, where H is a connected, reductive subgroup and M is a connected, simply connected solvable subgroup of G. For our purposes it will be important that the groups under consideration are invariant under σ. If we would only want a Levi factor to be invariant, then it would suffice to refer to [**39**] and for uniqueness questions also [**40**]. However, we also want a reductive subgroup to be invariant under σ. Therefore we prove:

THEOREM. *There exists a choice of the subgroups H and M as above, such that $\sigma H = H$ and $\sigma M = M$.*

PROOF. We write $G \equiv S.R$, where S is semisimple and R is the radical of G. Since $\mathrm{Lie}(R)$ is a maximal solvable ideal, it is easy to see $\sigma\mathrm{Lie}(R) = \mathrm{Lie}(R)$. Thus we obtain $\sigma R = R$. By [**30**, Theorem 18.4.3], and our assumptions we know $R = LM$, where L is maximal compact in R and abelian and M is simply connected. Analyzing the proof of loc.cit. it is easy to observe that M is preserved by σ. We will show next that one can, if necessary, replace L by some conjugate maximal compact subgroup $\tilde{L} = hLh^{-1}$, such that $\tilde{L}$ is invariant under σ. For the proof we will use mostly Lie algebras. To simplify notation we abbreviate $\boldsymbol{s} = \mathrm{Lie}(S)$ and $\boldsymbol{m} = \mathrm{Lie}(M)$ etc. and recall the descending central series: $\boldsymbol{r}^{(0)} = \boldsymbol{r}$ and $\boldsymbol{r}^{(j+1)} = [\boldsymbol{r}^{(j)}, \boldsymbol{r}^{(j)}]$. Moreover, since σ is semisimple and leaves every $\boldsymbol{r}^{(j)}$ invariant, there are σ-invariant subspaces W_j, $j = 0, 1, \ldots, n$ such that $W_j + W_{j+1} + \ldots + W_n = \boldsymbol{r}^{(j)}$ and the sum is direct. First we note that

$$\sigma L = mLm^{-1} \quad \text{for some } m \in M, \tag{1.2.1}$$

since maximal compact subgroups are conjugate to each other. Therefore for every l in L we have some $l' \in L$ such that $\sigma l = ml'm^{-1}$. Applying σ again we obtain $l = [\sigma(m)m]l''[\sigma(m)m]^{-1}$. Multiplying from the left by $(l'')^{-1}$ and observing that M is normal in R, we conclude that $l = l''$, and that $\sigma(m)m = c$ commutes with every $l \in L$:

$$\sigma(m)m = c \in C, \tag{1.2.2}$$

where

$$C = \{r \in M; rl = lr \ \text{ for all } \ l \in L\}. \tag{1.2.3}$$

Since the exponential map $\exp : \boldsymbol{m} \to M$ is a diffeomorphism, we can write $m = \exp(\hat{m})$ etc. Decomposing $\hat{m}$ relative to the subspaces W_j the idea of the proof is to replace iteratively L by some conjugate subgroup $\hat{L}$ such that the "m" associated with $\hat{L}$ and σ has fewer and fewer components. Altogether, we will finally obtain some $h \in M$ such that $\tilde{L} = hLh^{-1}$ is invariant under σ, i.e. is so that for every $l \in L$ there exists some $l' \in L$ such that $\sigma(hlh^{-1}) = hl'h^{-1}$ holds.

Applying (1.2.1) we obtain $\sigma(h) \cdot ml''m^{-1} \cdot \sigma(h^{-1}) = hl'h^{-1}$.

Therefore $[h^{-1}\sigma(h)m]l''[h^{-1}\sigma(h)m]^{-1} = l'$. As above one derives from this $l' = l''$ and we obtain

$$(\sigma(h))^{-1}hd = m, \tag{1.2.4}$$

for some $d \in C$. Taking logarithms and comparing the coefficients in W_0 we obtain from (1.2.2) and (1.2.4) the equations

$$\sigma(\hat{m}_0) + \hat{m}_0 = \hat{c}_0 \tag{1.2.5}$$

and

$$-\sigma(\hat{h}_0) + \hat{h}_0 + \hat{d}_0 = \hat{m}_0. \tag{1.2.6}$$

Next we note that every subspace invariant under σ decomposes into its eigenspaces relative to 1 and -1. With the obvious notation we thus obtain from (1.2.5) and (1.2.6) the equations

$$\hat{c}_0 = 2\hat{m}_0{}^{(+)} \tag{1.2.7}$$

and

$$2\hat{h}_0{}^{(-)} + \hat{d}_0{}^{(-)} = \hat{m}_0{}^{(-)}, \tag{1.2.8}$$

and

$$\hat{d}_0{}^{(+)} = \hat{m}_0{}^{(+)} = \frac{1}{2}\hat{c}_0{}^{(+)} = \hat{c}_0. \tag{1.2.9}$$

Note that m in equation (1.2.1) is only unique up to multiplication with elements of C. Thus, in view of (1.2.7) we would like to replace m with $m' = m/\sqrt{c}$. To make this precise, we write $c = \exp(\hat{c})$ and note $\mathrm{Ad}(l)\hat{c} = \hat{c}$. Thus "$\sqrt{c}$"$= \exp(\hat{c}/2) = d$ is still in C, and the W_0 component of the logarithm of $m' = md^{-1}$ is in the $-1-$eigenspace of σ. This means, that we can assume w.r.g. $\hat{m}_0 = \hat{m}_0^{(-)}$. Now we set $s = \exp(\hat{m}_0^{(-)}/2)$ and $L^\sharp = sLs^{-1}$. Then $\sigma(s) = s^{-1}$ and for

$l^\sharp \in L^\sharp$ we obtain $\sigma(l^\sharp) = \sigma(sls^{-1}) = s^{-1}ml'm^{-1}s = m^\sharp(sl's^{-1})m^{\sharp^{-1}}$, where $m^\sharp = s^{-1}ms^{-1}$. We note the crucial observation that the logarithm of $m^\sharp$ does not have any W_0 component anymore. Repeating the procedure for σ, $L^\sharp$ and $m^\sharp$ we obtain some new maximal compact subgroup and some new "m", the logarithm of which does not contain any W_1-component nor any W_0-component. Repeating this procedure finally yields a maximal compact subgroup $\tilde{L}$, which is invariant under σ. It now remains to prove that one can find not only some semisimple subgroup of G, which is invariant under σ, but one, which also commutes with $A = \hat{L}$. But this follows precisely from the first part of the proof of [**30**, Theorem 18.4.2]. $\square$

1.3.

Let $F : \boldsymbol{D} \to G$ be a smooth map, $\boldsymbol{D} = D^1$ or $\boldsymbol{C}$; $D^1 = \{z \in \boldsymbol{C} \mid |\, z \,| < 1\}$.

In view of (1.1.1), the Maurer-Cartan form associated to F, $\alpha = F^{-1}\mathrm{d}F \in \Lambda^1(\boldsymbol{D}, \boldsymbol{g})$, decomposes canonically

$$\alpha = \alpha_{\boldsymbol{k}} + \alpha_{\boldsymbol{p}} \in \Lambda^1(\boldsymbol{D}, \boldsymbol{k}) \oplus \Lambda^1(\boldsymbol{D}, \boldsymbol{p}). \tag{1.3.1}$$

Denoting by $z, \bar{z}$ the complex coordinates in $\boldsymbol{D} \subset \boldsymbol{R}^2 \equiv \boldsymbol{C}$, we have $T\boldsymbol{D} = T^{(1,0)}\boldsymbol{D} \oplus T^{(0,1)}\boldsymbol{D}$, and $\mathrm{d} = \partial + \bar{\partial}$. Then $\alpha_{\boldsymbol{p}}$ splits accordingly,

$$\alpha_{\boldsymbol{p}} = \alpha'_{\boldsymbol{p}} + \alpha''_{\boldsymbol{p}} \in \Lambda^{(1,0)}(\boldsymbol{D}, \boldsymbol{p}^{\boldsymbol{C}}) \oplus \Lambda^{(0,1)}(\boldsymbol{D}, \boldsymbol{p}^{\boldsymbol{C}}). \tag{1.3.2}$$

We consider the symmetric bracket on $\Lambda^1(\boldsymbol{D}, \boldsymbol{g}^{\boldsymbol{C}})$

$$[\alpha \wedge \beta] = ([\alpha', \beta''] - [\alpha'', \beta'])\mathrm{d}z \wedge \mathrm{d}\bar{z},$$

for $\alpha = \alpha'\mathrm{d}z + \alpha''\mathrm{d}\bar{z}$, $\beta = \beta'\mathrm{d}z + \beta''\mathrm{d}\bar{z} \in \Lambda^1(\boldsymbol{D}, \boldsymbol{g}^{\boldsymbol{C}}) = \Lambda^{(1,0)}(\boldsymbol{D}, \boldsymbol{g}^{\boldsymbol{C}}) \oplus \Lambda^{(0,1)}(\boldsymbol{D}, \boldsymbol{g}^{\boldsymbol{C}})$, $\boldsymbol{g}^{\boldsymbol{C}} \subset gl(V)$.

Composing F with the projection π of the principal K-bundle $(G, \pi, N = G/K)$, one obtains the mapping φ which closes the diagram

$$\begin{tikzcd} & G \arrow[d, "\pi"] \\ \boldsymbol{D} \arrow[ur, "F"] \arrow[r, "\varphi"'] & N = G/K. \end{tikzcd}$$

We can characterize the harmonicity of the map φ (the quality of being minimizer of the energy map (1.5.3); [**22**], [**23**], [**42**], [**19**]), in terms of the Maurer-Cartan form α associated to its lift F, as follows

THEOREM. *Assume G/K carries a non-degenerate symmetric bilinear form invariant by G. Then the following statements are equivalent*:

a) *The map φ associated with F is harmonic.*

b) *The form α satisfies the integrability and harmonicity equations*:

$$\begin{cases} \mathrm{d}\alpha_{\boldsymbol{k}} + \dfrac{1}{2}[\alpha_{\boldsymbol{k}} \wedge \alpha_{\boldsymbol{k}}] = -[\alpha'_{\boldsymbol{p}} \wedge \alpha''_{\boldsymbol{p}}] \\ \bar{\partial}\alpha'_{\boldsymbol{p}} + [\alpha_{\boldsymbol{k}} \wedge \alpha'_{\boldsymbol{p}}] = 0. \end{cases} \tag{1.3.3}$$

c) *The forms $\alpha_{\boldsymbol{k}} \in \Lambda^1(\boldsymbol{D}, \boldsymbol{k})$ and $\lambda^{-1}\alpha'_{\boldsymbol{p}} + \lambda\alpha''_{\boldsymbol{p}} \in \Lambda^1(\boldsymbol{D}, \boldsymbol{p}) \cap (\Lambda^{(1,0)}(\boldsymbol{D}, \boldsymbol{p}^{\boldsymbol{C}}) \oplus \Lambda^{(0,1)}(\boldsymbol{D}, \boldsymbol{p}^{\boldsymbol{C}}))$ satisfy the equations* (1.3.3), *for all $\lambda \in S^1$.*

d) *The “loopified form”*

$$\alpha_\lambda = \lambda^{-1}\alpha'_{\boldsymbol{p}} + \alpha_{\boldsymbol{k}} + \lambda\alpha''_{\boldsymbol{p}} \in \Lambda^1(\boldsymbol{D},\boldsymbol{g}) \tag{1.3.4}$$

is integrable for all $\lambda \in S^1$, i.e., it satisfies for every $\lambda \in S^1$ the integrability condition:

$$\mathrm{d}\alpha_\lambda + \frac{1}{2}[\alpha_\lambda \wedge \alpha_\lambda] = 0. \tag{1.3.5}$$

PROOF. b) $\Leftrightarrow$ c). Straightforward, replacing in (1.3.3) $\alpha'_{\boldsymbol{p}}$ with $\lambda^{-1}\alpha'_{\boldsymbol{p}}$ and $\alpha''_{\boldsymbol{p}}$ with $\lambda\alpha''_{\boldsymbol{p}}$, where $\lambda \in S^1$.

c) $\Leftrightarrow$ d) The relation (1.3.5) rewrites as

$$\begin{aligned}\lambda^{-1}\{\bar\partial\alpha'_{\boldsymbol{p}} + [\alpha'_{\boldsymbol{p}} \wedge \alpha_{\boldsymbol{k}}]\} + \lambda\{\partial\alpha''_{\boldsymbol{p}} + [\alpha''_{\boldsymbol{p}} \wedge \alpha_{\boldsymbol{k}}]\}\\ +\{\mathrm{d}\alpha_{\boldsymbol{k}} + [\alpha'_{\boldsymbol{p}} \wedge \alpha''_{\boldsymbol{p}}] + \frac{1}{2}[\alpha_{\boldsymbol{k}} \wedge \alpha_{\boldsymbol{k}}]\} = 0,\end{aligned} \tag{1.3.6}$$

for all $\lambda \in S^1$. Then the three braces vanish separately, provided the last equation of (1.3.3), its conjugate and the first equation of (1.3.3) vanish, whence the whole system (1.3.3) vanishes.

On the other hand, inserting λ^{-1}, λ and 1 respectively into the equations (1.3.3) and adding, we obtain (1.3.6), whence (1.3.5).

a) $\Leftrightarrow$ b). Since σ is an involution, one obtains by a direct computation that its eigenspaces satisfy the relations

$$[\boldsymbol{k},\boldsymbol{k}] \subset \boldsymbol{k},\ \ [\boldsymbol{k},\boldsymbol{p}] \subset \boldsymbol{p},\ \ [\boldsymbol{p},\boldsymbol{p}] \subset \boldsymbol{k}. \tag{1.3.7}$$

Then the integrability condition for a 1-form α,

$$\mathrm{d}\alpha + \frac{1}{2}[\alpha \wedge \alpha] = 0 \tag{1.3.8}$$

rewrites, using (1.3.7) and (1.3.1),

$$\begin{cases}\mathrm{d}\alpha_{\boldsymbol{k}} + \frac{1}{2}\{[\alpha_{\boldsymbol{k}} \wedge \alpha_{\boldsymbol{k}}] + [\alpha_{\boldsymbol{p}} \wedge \alpha_{\boldsymbol{p}}]\} = 0\\ \mathrm{d}\alpha_{\boldsymbol{p}} + [\alpha_{\boldsymbol{k}} \wedge \alpha_{\boldsymbol{p}}] = 0.\end{cases} \tag{1.3.9}$$

Thus, if F and ϕ are as in the discussion preceding the theorem, then $\alpha = F^{-1}dF$ satisfies (1.3.8), whence (1.3.9). Conversely, since $\boldsymbol{D}$ is simply connected and open, any α satisfying (1.3.9), i.e. satisfying (1.3.8), integrates to an F via $\alpha = F^{-1}dF$.

The harmonicity of $\varphi = F \bmod K$ is then equivalent to α satisfying the following equation (1.3.10), which we prove independently in Proposition 1.4 below:

$$\bar\partial\alpha'_{\boldsymbol{p}} + [\alpha_{\boldsymbol{k}} \wedge \alpha'_{\boldsymbol{p}}] = 0. \tag{1.3.10}$$

Assuming this, it is easy to observe that the system (1.3.3) is equivalent to the system (1.3.8), (1.3.10).

Indeed, using (1.3.2), the first equation in (1.3.9) becomes exactly the first one in (1.3.3). Also, using $\partial\alpha'_{\boldsymbol{p}} = \bar\partial\alpha''_{\boldsymbol{p}} = 0$, the second equation in (1.3.9) rewrites

$$(\bar\partial\alpha'_{\boldsymbol{p}} + [\alpha_{\boldsymbol{k}} \wedge \alpha'_{\boldsymbol{p}}]) + (\partial\alpha''_{\boldsymbol{p}} + [\alpha_{\boldsymbol{k}} \wedge \alpha''_{\boldsymbol{p}}]) = 0,$$

where the two parentheses are the complex conjugate of each other. Hence, this equation is just a consequence of (1.3.10), which is exactly the second equation of (1.3.3). □

1.4.

The basic result, used in the proof of Theorem 1.3, which characterizes the harmonicity of the mapping φ is provided by the following

PROPOSITION. *Assume G/K carries a non-degenerate symmetric bilinear form invariant by G. The map $\varphi : \boldsymbol{D} \to G/K$ induced by $F : \boldsymbol{D} \to G$ is harmonic if and only if the associated Maurer-Cartan form $\alpha = F^{-1}\mathrm{d}F$ satisfies* (1.3.10).

REMARK. This result is restricted to harmonic maps from 2-dimensional domains to general symmetric spaces. We sketch its proof briefly throughout sections 1.4–1.7, following a procedure similar to [**12**], [**7**].

SKETCH OF PROOF. We describe first the (right) Maurer-Cartan form of the homogeneous space $N = G/K$ and characterize its tangent bundle.

a) The tangent space TN is characterized by the property that there exist a canonical isomorphism of K-bundles

$$\psi_0 : G \times_K (\boldsymbol{g}/\boldsymbol{k}) \to T(G/K), \tag{1.4.1}$$

given by

$$\psi_0(g, \hat{\xi}) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} [\exp\, t\, (\mathrm{Ad}\, g\, \xi)].\hat{g}, \tag{1.4.2}$$

where $(g, \hat{\xi}) \in G \times_K \boldsymbol{g}/\boldsymbol{k}$, and $\hat{g} \equiv g \bmod K$.

b) In the case when $[\boldsymbol{k}, \boldsymbol{p}] \subset \boldsymbol{p}$, which is always satisfied under our assumptions (1.3.7), we have the isomorphism of K-bundles

$$\psi_1 : G \times_K (\boldsymbol{g}/\boldsymbol{k}) \to [\boldsymbol{p}] \equiv G \times_K \boldsymbol{p},$$
$$\text{given by} \quad \psi_1(g, \hat{\xi}) = (g, \pi_{\boldsymbol{p}}(\xi)),$$

where $\hat{\xi} \equiv \xi \bmod \boldsymbol{k}$ and $\pi_{\boldsymbol{p}} : \boldsymbol{g} \to \boldsymbol{p}$ is the projection associated to the decomposition (1.1.1). We consider the map

$$\beta_0 = \psi_1 \circ \psi_0^{-1} : TN \to [\boldsymbol{p}], \qquad \beta_0(X_{\hat{g}}) = (g, \pi_{\boldsymbol{p}}(\xi)),$$

where $X_{\hat{g}} \in T_{\hat{g}}N$, and where $\xi \in \boldsymbol{g}$ is determined by the relation

$$X_{\hat{g}} = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} [\exp\, t\, (\mathrm{Ad} g\, \xi)].\, \hat{g} \in T_{\hat{g}}N. \tag{1.4.3}$$

Then it is easy to verify that β_0 is an isomorphism of K-bundles, closing the diagram

$$\begin{tikzcd} TN \arrow[rr, "\beta_0"] \arrow[dr, "\psi_0^{-1}"'] & & {[\boldsymbol{p}] = G \times_K \boldsymbol{p}} \\ & G \times_K \boldsymbol{g}/\boldsymbol{k} \arrow[ur, "\psi_1"'] & \end{tikzcd}$$

c) The bundle $[\boldsymbol{p}] = G \times_K \boldsymbol{p}$ is a subbundle of $[\boldsymbol{g}] \equiv G \times_K \boldsymbol{g}$, via $[\boldsymbol{p}] \overset{i}{\hookrightarrow} [\boldsymbol{g}]$, which is induced by the natural injection.

d) $[\boldsymbol{g}]$ can be canonically identified with the trivial bundle $N \times \boldsymbol{g}$, via the diffeomorphism

$$\psi_2 : [\boldsymbol{g}] \to N \times \boldsymbol{g}, \qquad \psi_2(g,\xi) = (\hat{g}, \mathrm{Ad}g\ \xi) \ \text{ for all } \ (g,\xi) \in [\boldsymbol{g}].$$

e) As a consequence, TN can be identified with $[\boldsymbol{p}] \subset [\boldsymbol{g}] \overset{\psi_2}{\cong} N \times \boldsymbol{g}$, whence there exists the canonical injective mapping

$$\tilde{\beta} : TN \to N \times \boldsymbol{g}, \qquad \tilde{\beta} = \psi_2 \circ i \circ \beta_0,$$

$$\tilde{\beta}(X_{\hat{g}}) = (g, \mathrm{Ad}\, g \pi_{\boldsymbol{p}}(\xi)) \ \text{ for all } \ X_{\hat{g}} \in T_{\hat{g}}N,$$

where ξ is determined by the relation (1.4.3).

f) The right Maurer-Cartan form of the homogeneous space $N = G/K$ is the mapping [**12**, p. 6]

$$\beta : TN \to \boldsymbol{g},\ \beta = pr_2 \circ \tilde{\beta}, \qquad \beta(X_{\hat{g}}) = \mathrm{Ad}g\ \pi_{\boldsymbol{p}}\xi \ \text{ for all } \ X_{\hat{g}} \in T_{\hat{g}}N$$

and represents a vector-valued 1-form $\beta \in \Lambda^1(N, \boldsymbol{g})$, which is equivariant with respect to the action of G on TN (the left translation) and the action of G on $\boldsymbol{g}$ (the adjoint action), i.e., $g^*\beta = \mathrm{Ad}g \circ \beta$ for all $g \in G$. One can verify that its pullback to G satisfies $(\pi^*\beta)_g = \mathrm{Ad}g\theta_1$, where $\theta_1 = \pi_{\boldsymbol{p}} \circ \theta$, with $\theta \in \Lambda^1(G, \boldsymbol{g})$ being the (left) Maurer-Cartan 1-form of G,

$$\theta(\zeta_g) = g_*^{-1}(\zeta_g), \quad \text{for all } \ \zeta_g \in T_gG.$$

g) The map $\beta' = \beta \circ \partial\varphi = \varphi^*\beta\ |_{T^{(1,0)}D}$ satisfies the relation

$$\beta' = \mathrm{Ad}F \circ \alpha'_{\boldsymbol{p}}. \tag{1.4.4}$$

1.5.

Note that the generalization of the notion of harmonicity of a map $\varphi : \boldsymbol{D} \to N$ to a pseudo-Riemannian manifold N can be chosen in several ways. At one hand one can consider the variational problem [**4**, (2.1.2)] under compactness conditions. One obtains [**4**, (2.1.7), (2.2.1)]. Assuming that the metric on N is induced from a G-invariant form, then the equation obtained from the variational problem is equivalent to the condition $\tau(\varphi) = 0$, where τ is the tension of the map φ

$$\tau(\varphi) = \mathrm{Tr}_\gamma(\bigtriangledown \mathrm{d}\varphi), \tag{1.5.1}$$

where γ is the pseudo-metric of N, and $\bigtriangledown$ is the connection on the bundle of forms $\Lambda^1(\boldsymbol{D}, \varphi^{-1}TN)$, induced from the (trivial) Levi-Civita connection of the Riemannian flat manifold $\boldsymbol{D}$ and the Levi-Civita connection $\overset{N}{\bigtriangledown}$ of N.

As the manifold $\boldsymbol{D}$ is of real dimension 2, the tension $\tau(\varphi)$ is a multiple of the induced covariant derivative $(\varphi^{-1}\overset{N}{\bigtriangledown})_{(\partial/\partial\bar{z})}\varphi_*(\partial/\partial z)$, and its vanishing represents the holomorphicity condition for the section $\varphi_*(\partial/\partial z)$ of the induced bundle $(\varphi^{-1}TN,\ pr_1,\ \boldsymbol{D})$, ([**34**], [**12**], [**7**]).

Hence the harmonicity of the map φ can be rewritten in terms of $\overset{N}{\nabla}$

$$\overset{N}{\nabla}_{\varphi_*(\partial/\partial\bar{z})}\varphi_*\left(\frac{\partial}{\partial z}\right)=0, \tag{1.5.2}$$

or briefly $\overset{N}{\nabla}{}''\partial\varphi=0$.

REMARK. In general, if $\overset{N}{\nabla}$ is not the Levi-Civita, but an arbitrarily given affine connection, then (1.5.1) characterizes *affine harmonic maps* ([**28**, Definition 2.1, p. 407]) and (1.5.2) characterizes $\overset{N}{\nabla}$-harmonic maps. In our case, where the range of the harmonic map is a general symmetric space with two-dimensional contractive domain, the two definitions mentioned just above are equivalent. Moreover, for the Levi Civita connection $\overset{N}{\nabla}$, the two definitions (1.5.1) and (1.5.2) both provide the classical harmonic maps $\varphi:\boldsymbol{D}\to N$, i.e. the extremals of the energy functional

$$E(\varphi)=\int_{N_*}|d\varphi|^2 \mathrm{d}\,\mathrm{vol}_g, \tag{1.5.3}$$

where $N_*\subset N$ is a compact subdomain of N. Moreover, $\tau(\varphi)=0$ is exactly the Euler-Lagrange equation of the energy functional $E(\varphi)$. Since on general symmetric spaces $\overset{\text{can}}{\nabla}=\overset{N}{\nabla}$ (see 1.6-f below), the $\overset{\text{can}}{\nabla}$-harmonic maps coincide with the (classical) harmonic maps, the minimizers of (1.5.3) ([**28**, Proposition 2.3, p. 408]).

1.6.

On the general symmetric space $N=G/K$, the connection $\overset{N}{\nabla}$ has a specific form, which permits the reformulation of (1.5.2) in terms of the Maurer-Cartan form α associated to F.

The following steps lead to an explicit expression for $\overset{N}{\nabla}$:

a) The left translation on G provides by left shifts of $\boldsymbol{p}$ a horizontal distribution, which is right K-invariant and hence provides a connection on the K-principal bundle $(G,\pi,N=G/K)$, which induces on the associated bundle $(TN,\pi,N)\equiv([\boldsymbol{p}],\tilde{\pi},N)$ a G-invariant canonical connection $\overset{\text{can}}{\nabla}$ of the general symmetric space N.

K. Nomizu ([**36**]) has shown that any $\mathrm{Ad}(K)$-invariant bilinear form $\gamma:\boldsymbol{m}\times\boldsymbol{m}\to\boldsymbol{m}$ induces via

$$\gamma(\eta,\xi)=(\nabla_{\bar{\xi}}\bar{\eta})_{\hat{e}},\quad \text{for all }\ \xi,\eta\in\boldsymbol{m}, \tag{1.6.1}$$

where $\bar{\tau}=d/dt\,|_{t=0}\,(\exp\, t\,[\mathrm{Ad}\,g\,\tau]).\hat{g}$, for all $\tau\in\boldsymbol{p}$, a linear connection ∇ on $([\boldsymbol{p}],\pi,N)\cong(TN,\pi,N)$. In general, the torsion of such a connection is ([**28**, p. 405])

$$T(\xi,\eta)\equiv\nabla_\xi\eta-\nabla_\eta\xi-[\xi,\eta]=\gamma(\xi,\eta)-\gamma(\eta,\xi)-\pi_{\boldsymbol{p}}([\xi,\eta]). \tag{1.6.2}$$

In particular, for $\gamma\equiv 0$ in (1.6.1) one obtains exactly $\overset{\text{can}}{\nabla}$ ([**28**, Proposition 1.4, p. 404]), called also *the canonical affine connection of the second kind of* N. The torsion of $\overset{\text{can}}{\nabla}$ is given by

$$\overset{c}{T}_{\hat{e}}(\xi,\eta)=-\pi_{\boldsymbol{p}}([\xi,\eta]),\quad \text{for all }\ \xi,\eta\in[\boldsymbol{p}]_{\hat{e}}\equiv\boldsymbol{p}\subset\boldsymbol{g}.$$

Note that $T \equiv 0 \Leftrightarrow [\boldsymbol{p},\boldsymbol{p}] \subset \boldsymbol{k}$.

b) The connection associated via (1.6.1) to the $\mathrm{Ad}(K)$-invariant bilinear form

$$\gamma(\xi,\eta) = \frac{1}{2}\pi_{\boldsymbol{p}}([\xi,\eta]), \quad \text{for all } \ \xi,\eta \in \boldsymbol{p}$$

is called *the canonical affine connection of the first kind of* N. Using (1.6.2), it is easy to see that this connection is torsionless. Moreover, by **[12]** it coincides with the connection $\overset{N}{\nabla}$.

c) If $\overset{\text{can}}{\nabla}$ is torsion-free, it coincides with $\overset{N}{\nabla}$ if and only if

$$\gamma_{\hat{e}}(\xi,\pi_{\boldsymbol{p}}[\eta,\mu]) = \gamma_{\hat{e}}(\pi_{\boldsymbol{p}}[\xi,\eta],\mu) \ \text{ for all } \ \xi,\eta,\mu \in [\boldsymbol{p}]_{\hat{e}} \equiv \boldsymbol{p} \subset \boldsymbol{g}.$$

This relation is obviously fulfilled if $[\boldsymbol{p},\boldsymbol{p}] \subset \boldsymbol{k}$.

d) The two connections $\overset{\text{can}}{\nabla}$ and $\overset{N}{\nabla}$ are described by the relations

$$\beta(\overset{\text{can}}{\nabla}_X Y) = X_\beta(Y) - [\beta(X),\beta(Y)], \tag{1.6.3}$$

$$\beta(\overset{N}{\nabla}_X Y) = X_\beta(Y) - [\beta(X),\beta(Y)] + \frac{1}{2}\tilde{\pi}([\beta(X),\beta(Y)]), \tag{1.6.4}$$

for all $X,Y \in \Gamma(TN)$, where $\tilde{\pi}$ is the projection $\tilde{\pi} : N \times \boldsymbol{g} \to N \times_{\boldsymbol{k}} \boldsymbol{p}$.

e) The canonical connection $\overset{\text{can}}{\nabla}$ coincides with the connection $\overset{\text{d}}{\nabla} = \pi_{[\boldsymbol{p}]} \circ \tilde{\nabla}$ if and only if $[\boldsymbol{p},\boldsymbol{p}] \subset \boldsymbol{k}$, where $\tilde{\nabla} \equiv \mathrm{d}$ is the trivial connection ($\equiv$ flat differentiation).

f) As a consequence, in view of part c), if the relations (1.3.7) are satisfied, then $\overset{N}{\nabla} \equiv \overset{\text{d}}{\nabla}$. Hence, in this case, the Levi-Civita connection $\overset{N}{\nabla}$ on N is given by the relation $\beta_0 \circ \overset{N}{\nabla} = \pi_{[\boldsymbol{p}]} \circ \tilde{\nabla} \circ \beta$, which is expressed explicitly by the equation

$$\beta_0(\overset{N}{\nabla}_X(Y)) = \pi_{[\boldsymbol{p}]}(\tilde{\nabla}_X(\beta(Y))) \ \text{ for all } \ X,Y \in TN, \tag{1.6.5}$$

where $\pi_{[\boldsymbol{p}]}$ is the canonical projection along the fibers of $[\boldsymbol{k}] = G \times_K \boldsymbol{k}$, induced on $N \times \boldsymbol{g}$ via

$$\pi_{[\boldsymbol{p}]} : N \times \boldsymbol{g} \to [\boldsymbol{p}],\ \pi_{[\boldsymbol{p}]}(g,\xi) = (g,\xi_{\boldsymbol{p}}) \ \text{ for all } \ (g,\xi) \in N \times \boldsymbol{g},\ \xi_{\boldsymbol{p}} = \pi_{\boldsymbol{p}}\, \mathrm{Ad}g(\xi).$$

1.7.

Combining the previous results with (1.3.7), we insert $X = \partial/\partial\bar{z}$, and $Y = \partial/\partial z$ into (1.6.5), and obtain the harmonicity condition (1.5.2) rewritten in the form:

$$\overset{N}{\nabla}_{\varphi_*(\partial/\partial\bar{z})}\varphi_*\left(\frac{\partial}{\partial z}\right) = 0 \ \Leftrightarrow \ \pi_{[\boldsymbol{p}]}\left(\tilde{\nabla}_{(\partial/\partial\bar{z})}\left(\beta\varphi_*\left(\frac{\partial}{\partial z}\right)\right)\right) = 0.$$

Setting $\beta' = \beta \circ \partial\varphi = \beta\varphi_*(\partial/\partial z)$, this is equivalent with

$$\pi_{[\boldsymbol{p}]}(\bar{\partial}\beta') = 0.$$

Then, using (1.4.4), the condition above becomes succesively

$$\pi_{[\boldsymbol{p}]}(\bar{\partial}\mathrm{Ad}F\alpha'_{\boldsymbol{p}}) = 0 \ \Leftrightarrow \ \pi_{[\boldsymbol{p}]}(\bar{\partial}(F\alpha'_{\boldsymbol{p}}F^{-1})) = 0 \Leftrightarrow$$

$$\begin{aligned}
&\pi_{[\boldsymbol{p}]}(F\bar{\partial}\alpha'_{\boldsymbol{p}}F^{-1}+\bar{\partial}F\wedge\alpha'_{\boldsymbol{p}}F^{-1}-F\alpha'_{\boldsymbol{p}}\wedge F^{-1}.F^{-1}\bar{\partial}F.F^{-1})=0\Leftrightarrow\\
&\pi_{[\boldsymbol{p}]}\{F(\bar{\partial}\alpha'_{\boldsymbol{p}}+[F^{-1}\bar{\partial}F\wedge\alpha'_{\boldsymbol{p}}])F^{-1}\}=0\Leftrightarrow\\
&\pi_{[\boldsymbol{p}]}\circ\mathrm{Ad}F(\bar{\partial}\alpha'_{\boldsymbol{p}}+[F^{-1}\bar{\partial}F\wedge\alpha'_{\boldsymbol{p}}])=0\Leftrightarrow\\
&\pi_{\boldsymbol{p}}\mathrm{Ad}F^{-1}\mathrm{Ad}F(\bar{\partial}\alpha'_{\boldsymbol{p}}+[(\alpha''_{\boldsymbol{p}}+\alpha''_{\boldsymbol{k}})\wedge\alpha'_{\boldsymbol{p}}])=0\Leftrightarrow\\
&\bar{\partial}\alpha'_{\boldsymbol{p}}+[\alpha_{\boldsymbol{k}}\wedge\alpha'_{\boldsymbol{p}}]=0. \qquad (1.7.1)
\end{aligned}$$

Thus we obtain exactly (1.3.10), whence the conclusion in Proposition 1.4.

REMARK. In (1.7.1) we used besides the splittings

$$\alpha_{\boldsymbol{k}}=\alpha'_{\boldsymbol{k}}+\alpha''_{\boldsymbol{k}}\in\Lambda^1(\boldsymbol{D},\boldsymbol{k})=\Lambda^{(1,0)}(\boldsymbol{D},\boldsymbol{k})\oplus\Lambda^{(0,1)}(\boldsymbol{D},\boldsymbol{k}),$$

$$F^{-1}\bar{\partial}F=\alpha''_{\boldsymbol{k}}+\alpha''_{\boldsymbol{p}}\in\Lambda^{(0,1)}(\boldsymbol{D},\boldsymbol{g})=\Lambda^{(0,1)}(\boldsymbol{D},\boldsymbol{k})\oplus\Lambda^{(0,1)}(\boldsymbol{D},\boldsymbol{p}),$$

also the relation $[\alpha'_{\boldsymbol{k}}\wedge\alpha'_{\boldsymbol{p}}]=0$ and $[\alpha''_{\boldsymbol{p}}\wedge\alpha'_{\boldsymbol{p}}]\in\Lambda^2(\boldsymbol{D},\mathrm{Ker}(\pi_{\boldsymbol{p}}))$, which is a consequence of (1.3.7). The result above was obtained in [**28**, Proposition 3.1, p. 409] and leads to

$$\overset{N}{\bigtriangledown}{}''\partial\varphi=0\Leftrightarrow\overset{\mathrm{can}}{\bigtriangledown}{}''\partial\varphi=0\Leftrightarrow\tilde{\bigtriangledown}''\partial\varphi=0\Leftrightarrow\bar{\partial}\alpha'_{\boldsymbol{p}}+[\alpha_{\boldsymbol{k}}\wedge\alpha'_{\boldsymbol{p}}]=0.$$

2. Loop groups.

In the last chapter, we have seen that introducing a parameter $\lambda\in S^1$ reduces the number of equations to one and thus changes the discussion of harmonic maps to the investigation of PDE's with parameter. In this chapter we will first discuss loop groups and then apply the results to harmonic maps.

2.1.

As before we consider a connected real Lie group G. We assume that G is faithfully represented by matrices in $\mathscr{M}(n,\boldsymbol{R})$. On $\mathscr{M}(n,\boldsymbol{C})$ we consider the norm

$$|A|=\max_j\left(\sum_{i=1}^n|A_{ij}|\right). \qquad (2.1.1)$$

Then

$$|AB|\le|A|\cdot|B| \text{ and } |I|=1. \qquad (2.1.2)$$

Next we consider the Wiener algebra

$$\mathscr{A}=\left\{f:S^1\to\boldsymbol{C}\;\middle|\;f=\sum_{k\in\boldsymbol{Z}}f_k\lambda^k,\ \sum_{k\in\boldsymbol{Z}}|f_k|<\infty\right\}. \qquad (2.1.3)$$

This is a Banach algebra and so is

$$\mathscr{M}(n,\mathscr{A})=\left\{A:S^1\to\mathscr{M}(n,\boldsymbol{C})\;\middle|\;A=\sum_{k\in\boldsymbol{Z}}A_k\lambda^k,\ \sum_{k\in\boldsymbol{Z}}|A_k|<\infty\right\}. \qquad (2.1.4)$$

Similarly

$$\Lambda \boldsymbol{g}^{\boldsymbol{C}} = \{\mathscr{A} : S^1 \to \boldsymbol{g}^{\boldsymbol{C}} \subset \mathscr{M}(n,\boldsymbol{C}) \mid A_{ij} \in \mathscr{A}\} \tag{2.1.5}$$

is a Banach Lie algebra with closed subalgebras

$$\Lambda^+ \boldsymbol{g}^{\boldsymbol{C}} = \left\{ A \in \Lambda \boldsymbol{g}^{\boldsymbol{C}} \;\middle|\; A = \sum_{k\in\boldsymbol{Z}} A_k \lambda^k,\ A_k = 0 \ \text{ if } k < 0 \right\} \tag{2.1.6}$$

$$\Lambda^- \boldsymbol{g}^{\boldsymbol{C}} = \left\{ A \in \Lambda \boldsymbol{g}^{\boldsymbol{C}} \;\middle|\; A = \sum_{k\in\boldsymbol{Z}} A_k \lambda^k,\ A_k = 0 \ \text{ if } k > 0 \right\}. \tag{2.1.7}$$

Clearly,

$$\Lambda^+ \boldsymbol{g}^{\boldsymbol{C}} + \Lambda^- \boldsymbol{g}^{\boldsymbol{C}} = \Lambda \boldsymbol{g}^{\boldsymbol{C}} \tag{2.1.8}$$

$$\Lambda^+ \boldsymbol{g}^{\boldsymbol{C}} \cap \Lambda^- \boldsymbol{g}^{\boldsymbol{C}} = \boldsymbol{g}^{\boldsymbol{C}}. \tag{2.1.9}$$

On the group level one can proceed similarly.

For this we note, that according to [**30**, Chapter VII], one can always define for G a “universal complexification” $G^{\boldsymbol{C}}$. In this paper, we will assume that $G^{\boldsymbol{C}}$ has a faithful linear representation, extending the faithful representation of G.

Next, by a classical result for the Wiener algebra we have

$$A \in \mathscr{M}(n,\mathscr{A}) \text{ is invertible} \Leftrightarrow \det A \neq 0 \text{ or all } \lambda \in S^1. \tag{2.1.10}$$

We can thus define

$$GL(n,\mathscr{A}) = \{A \in \mathscr{M}(n,\mathscr{A}) \mid \det A \neq 0 \ \text{ for all } \ \lambda \in S^1\}. \tag{2.1.11}$$

Then we have

$$GL(n,\mathscr{A}) \text{ is a Banach Lie group with Lie algebra } \mathscr{M}(n,\mathscr{A}). \tag{2.1.12}$$

Analogously to the Lie algebra level we set

$$\Lambda G^{\boldsymbol{C}} = \{A \in GL(n,\mathscr{A}) \mid A(\lambda) \in G^{\boldsymbol{C}} \ \text{ for all } \ \lambda \in S^1\}, \tag{2.1.13}$$

$$\Lambda^+ G^{\boldsymbol{C}} = \left\{ A \in GL(n,\boldsymbol{C}) \;\middle|\; A = \sum_{k\in\boldsymbol{Z}} \lambda_k A^k,\ A_k = 0 \ \text{ if } k < 0 \right\} \tag{2.1.14}$$

$$\Lambda^- G^{\boldsymbol{C}} = \left\{ A \in GL(n,\boldsymbol{C}) \;\middle|\; A = \sum_{k\in\boldsymbol{Z}} \lambda_k A^k,\ A_k = 0 \ \text{ if } k > 0 \right\}. \tag{2.1.15}$$

Then $\Lambda G^{\boldsymbol{C}}$ is a Banach Lie subgroup of $GL(n,\mathscr{A})$ with Lie algebra $\Lambda \boldsymbol{g}^{\boldsymbol{C}}$. Similarly, $\Lambda^+ G^{\boldsymbol{C}}$ and $\Lambda^- G^{\boldsymbol{C}}$ are Banach Lie subgroups of $\Lambda G^{\boldsymbol{C}}$ with Lie algebras $\Lambda^+ \boldsymbol{g}^{\boldsymbol{C}}$ and $\Lambda^- \boldsymbol{g}^{\boldsymbol{C}}$ respectively.

In view of (1.3.4), we can expect that our discussion will require restrictions with regard to the distribution of powers of λ in the coefficients of the matrices occurring in our investigation. It seems that this restriction is exactly incorporated by using twisted algebras. For this we extend the automorphism σ of G to $\Lambda G^{\boldsymbol{C}}$:

$$(\sigma A)(\lambda) = \sigma(A(-\lambda)). \tag{2.1.16}$$

Then σ is an automorphism of $\Lambda G^{\boldsymbol{C}}$ and we set

$$\Lambda G_{\sigma}^{\boldsymbol{C}} = \{A \in \Lambda G^{\boldsymbol{C}} \mid \sigma A = A\}. \tag{2.1.17}$$

Then $\Lambda G_{\sigma}^{\boldsymbol{C}}$ is a Banach Lie group. On the Lie algebra level we obtain

$$\Lambda \boldsymbol{g}_{\sigma}^{\boldsymbol{C}} = \{A \in \Lambda \boldsymbol{g}^{\boldsymbol{C}} \mid d\sigma A = A\}. \tag{2.1.18}$$

Expanding here $A = \sum_{k\in\boldsymbol{Z}} A_k \lambda^k$ we see

$$A \in \Lambda \boldsymbol{g}_{\sigma}^{\boldsymbol{C}} \quad \Leftrightarrow \quad d\sigma A_{2k} = A_{2k} \text{ and } d\sigma A_{2k+1} = -A_{2k+1}. \tag{2.1.19}$$

Thus, setting $\boldsymbol{k} = \{A \in \boldsymbol{g}; d\sigma A = A\}$ and $\boldsymbol{p} = \{A \in \boldsymbol{g}; d\sigma A = -A\}$, we see that for every $A \in \Lambda \boldsymbol{g}_{\sigma}^{\boldsymbol{C}}$ we have $A_{2k} \in \boldsymbol{k}$ and $A_{2k+1} \in \boldsymbol{p}$.

Analogously one defines the Banach Lie groups $\Lambda^{\pm} G_{\sigma}^{\boldsymbol{C}}$ with associated Lie algebras $\Lambda^{\pm} \boldsymbol{g}_{\sigma}^{\boldsymbol{C}}$.

NOTE. An extended framing $F : \boldsymbol{D} \times S^1 \to G$ is analytic in $\lambda \in \boldsymbol{C}^*$ and therefore $F(z, \overline{z}, \cdot) \in \Lambda G_{\sigma}^{\boldsymbol{C}}$. Since F even takes values in G for all $z, \lambda \in S^1$, it is natural to consider also the Banach Lie group

$$\Lambda G_{\sigma} = \{A \in \Lambda G_{\sigma}^{\boldsymbol{C}} \mid A(\lambda) \in G \text{ for all } \lambda \in S^1\}; \tag{2.1.20}$$

the corresponding Lie algebra will be denoted by $\Lambda \boldsymbol{g}_{\sigma}$.

For our purposes two results are of importance

THEOREM (Birkhoff decomposition). *Every $A \in \Lambda G_{\sigma}^{\boldsymbol{C}}$ can be written in the form*

$$A = A_- W A_+,$$

where $A_- \in \Lambda^- G_{\sigma}^{\boldsymbol{C}}$, $A_+ \in \Lambda^+ G_{\sigma}^{\boldsymbol{C}}$ and W is a homomorphism of S^1 into a maximal toral subgroup of $G^{\boldsymbol{C}}$.

A proof of this result can be found in [**3**, Section 4.5]. The second important result is

THEOREM (Iwasawa decomposition, [**3**, Theorem 6.5, p. 604]). *Every $A \in \Lambda G_{\sigma}^{\boldsymbol{C}}$ can be written in the form*

$$A = L \cdot W \cdot B_+,$$

where $L \in \Lambda G_{\sigma}$ and $B_+ \in \Lambda^+ G_{\sigma}^{\boldsymbol{C}}$ and W is as in [**3**].

2.2. Factorization of generalized loop groups.

In many instances it turns out that the choice of $\lambda \in S^1$ is too narrow to facilitate discussion of geometric objects sufficiently well. One therefore uses sometimes generalized loop groups and the corresponding generalized Birkhoff and Iwasawa decompositions. The necessity for this was already recognized in [**21**] and later generalized in [**10**]. We present here briefly the main features of this generalization.

For G and K as above we have the (classical, finite-dimensional) Iwasawa decompositions

$$K^{\boldsymbol{C}} = K \cdot B, \qquad G^{\boldsymbol{C}} = G \cdot \tilde{B},$$

where B is a Borel subgroup of $K^{\boldsymbol{C}}$ and $\tilde{B}$ is a Borel subgroup of $G^{\boldsymbol{C}}$.

We fix $\varepsilon \in (0,1)$ and set

$$C_\varepsilon = \{z \in \boldsymbol{C} |\ |z| = \varepsilon\} \subset \boldsymbol{C} \cup \{\infty\} = \bar{\boldsymbol{C}},$$
$$C^\varepsilon = C_\varepsilon \cup C_{1/\varepsilon},$$
$$E = \left\{\lambda \in \bar{\boldsymbol{C}} |\ |\lambda| \in \left(\varepsilon, \frac{1}{\varepsilon}\right)\right\},\ I = \bar{\boldsymbol{C}} \backslash \bar{E}.$$

Then it is natural to consider the following loop groups and loop algebras (see, e.g., [**10**])

$$\begin{cases} \Lambda^\varepsilon G = \{g : C^\varepsilon \to G^{\boldsymbol{C}} |\ \overline{g(\lambda)} = g(1/\bar{\lambda}), \lambda \in C^\varepsilon\} \\ \Lambda^\varepsilon_E G = \{g \in \Lambda^\varepsilon G |\ g \text{ extends holomorphically to } E\} \\ \Lambda^\varepsilon_{I,\tilde{B}} G = \{g \in \Lambda^\varepsilon G |\ g \text{ extends holomorphically to } I \text{ and } g(0) \in \tilde{B}\} \end{cases}$$

$$\begin{cases} \Lambda^\varepsilon G_\sigma = \{g \in \Lambda^\varepsilon G |\ \sigma g(\lambda) = g(-\lambda)\} \\ \Lambda^\varepsilon_E G_\sigma = \{g \in \Lambda^\varepsilon G_\sigma |\ g \text{ extends holomorphically to } E\} \\ \Lambda^\varepsilon_{I,\tilde{B}} G_\sigma = \{g \in \Lambda^\varepsilon G_\sigma |\ g \text{ extends holomorphically to } I \text{ and } g(0) \in \tilde{B}\} \end{cases}$$

$$\begin{cases} \mathrm{Lie}(\Lambda^\varepsilon G) = \Lambda^\varepsilon \boldsymbol{g} = \{\xi : C^\varepsilon \to \boldsymbol{g}^{\boldsymbol{C}} |\ \overline{\xi(\lambda)} = \xi(1/\lambda),\ \text{ for all }\ \lambda \in C^\varepsilon\} \\ \mathrm{Lie}(\Lambda^\varepsilon G_\sigma) = \Lambda^\varepsilon \boldsymbol{g}_\sigma = \{\xi \in \Lambda^\varepsilon \boldsymbol{g} |\ \sigma\xi(\lambda) = \xi(-\lambda)\} \\ \mathrm{Lie}(\Lambda^\varepsilon_E G_\sigma) = \Lambda^\varepsilon_E \boldsymbol{g}_\sigma = \{\xi \in \Lambda^\varepsilon \boldsymbol{g}_\sigma |\ \xi \text{ extends holomorphically to } E\} \\ \mathrm{Lie}(\Lambda^\varepsilon_{I,\tilde{B}} G_\sigma) = \Lambda^\varepsilon_{I,\tilde{B}} \boldsymbol{g}_\sigma = \{\xi \in \Lambda^\varepsilon \boldsymbol{g}_\sigma |\ \xi \text{ extends holomorphically to } I \text{ and } \xi(0) \in \tilde{\boldsymbol{b}}\}, \end{cases}$$

where $\tilde{\boldsymbol{b}} = \mathrm{Lie}(\tilde{B})$. With these notations we have the Iwasawa loop group decomposition ([**5**], [**10**]) $\Lambda^\varepsilon G \sim \Lambda^\varepsilon_E G \cdot \Lambda^\varepsilon_{I,\tilde{B}} G$ and [**28**, Corollary 5.4, p. 415])

$$\Lambda^\varepsilon G_\sigma \sim \Lambda^\varepsilon_E G_\sigma \cdot \Lambda^\varepsilon_{I,\tilde{B}} G_\sigma, \tag{2.2.1}$$

where the intersection of the right factors is $\{e\}$. Moreover, one can show that in (2.2.1) one can replace the right side, up to a real analytic diffeomorphism, by the product of the corresponding groups. The symbol "$\sim$" expresses the fact that the right side is not necessarily equal to the left side, but does contain at least an open neighbourhood of the identity element. A characterization of when the right side is open and dense would be of great interest.

2.3. Actions of loop groups.

From (2.2.1) we obtain the following group actions [**5**], [**13**], [**21**], [**28**]:

a) The group $\Lambda^\varepsilon_{I,\tilde{B}} G_\sigma$ acts on $\Lambda^\varepsilon_E G_\sigma$ via

$$g \sharp h = (g \cdot h)_E,$$

where $g \in \Lambda^{\varepsilon}_{I,\tilde{B}} G_\sigma$ and $h \in \Lambda^{\varepsilon}_{E} G_\sigma$ and the right side is the first factor of $g \cdot h$ in its decomposition (2.2.1). Note that this group action is not globally defined, since in (2.2.1) there is (in general) no equality.

b) If $\tilde{F}$ is an extended framing then $g \in \Lambda^{\varepsilon}_{I,\tilde{B}} G_\sigma$ acts on $\tilde{F}$ via

$$(g\sharp\tilde{F})(p) = g\sharp(\tilde{F}(p)), \text{ where } p \in \boldsymbol{D}. \tag{2.3.1}$$

It is easy to see ([**28**, Proposition 6.1, p. 417]) that $g\sharp\tilde{F}$ is again an extended framing. However, due to the non-global nature of the group action used, the framing $g\sharp\tilde{F}$ may have singularities at points p, where $\tilde{F}$ does not have any singularity.

c) The action (2.3.1) induces canonically an action of $\Lambda^{\varepsilon}_{I,\tilde{B}} G_\sigma$ on $\Lambda^{\varepsilon}_{E} G_\sigma / K$ ([**28**]). This extends via (2.3.1) to an action of $\Lambda^{\varepsilon}_{I,\tilde{B}} G_\sigma$ on all S^1-families of harmonic maps.

3. Generalized Weierstrass representation of general harmonic maps.

3.1.

Following the procedure of [**19**] we want to construct a “holomorphic potential” and a “normalized potential” (called originally “meromorphic potential” in [**19**]) for each harmonic map. First we note that the proof of [**19**, Appendix] generalizes immediately to our setting and yields

THEOREM. *Let $F = F(z, \bar{z}, \lambda)$ be the extended frame of some harmonic map, $z \in \boldsymbol{D}$, $\lambda \in S^1$. Then there exists some $V_+ : \boldsymbol{D} \to \Lambda^+ G^{\boldsymbol{C}}_\sigma$ such that*

$$C(z, \lambda) = F(z, \bar{z}, \lambda) V_+(z, \bar{z}, \lambda) \tag{3.1.1}$$

is holomorphic on $\boldsymbol{D} \setminus T_1$, where T_1 is the discrete subset of $\boldsymbol{D}$, and the generalized Iwasawa decomposition needs some middle term.

For the rest of this paper we *normalize the extended frames* of harmonic maps so that they satisfy $F(z_0, \bar{z}_0, \lambda) = I$ at some fixed base point $z_0 \in \boldsymbol{D}$.

DEFINITION. Any map C as in (3.1.1) will be called a *holomorphic extended frame*. Moreover, the Maurer-Cartan form of any holomorphic extended frame

$$\eta = C^{-1} dC \tag{3.1.2}$$

will be called a *holomorphic potential* for the harmonic map φ (or for the extended frame F). As usual one verifies

LEMMA. *Every holomorphic potential η is of the form*

$$\eta = \lambda^{-1}\eta_{-1} + \lambda^0 \eta_0 + \lambda^1 \eta_1 + \dots \tag{3.1.3}$$

where η_j is a holomorphic $(1,0)$-form on $\boldsymbol{D}$.

We note that the integrability condition is trivially satisfied for any holomorphic $(1,0)$-form on $\boldsymbol{D}$. Thus, we have

PROPOSITION. *Assume η is a holomorphic $(1,0)$-form on $\boldsymbol{D}$ with values in $\Lambda \boldsymbol{g}^{\boldsymbol{C}}_\sigma$ which is of the form* (3.1.3). *Then the equation*

$$C^{-1}dC = \eta\,,\; C(z_0,\lambda) = I \tag{3.1.4}$$

is globally solvable on $\boldsymbol{D}$*. Moreover, decomposing* C *via Iwasawa splitting*

$$C = FV_+\,; \qquad F \in \Lambda G_\sigma,\; V_+ \in \Lambda^+ G_\sigma^{\boldsymbol{C}}, \tag{3.1.5}$$

we obtain an extended frame of some harmonic map $\varphi : \boldsymbol{D}\backslash S \to G/K$*, where*

$$S = \{z \in \boldsymbol{D} \mid C(z,\,\cdot\,) \notin \Lambda G_\sigma \cdot \Lambda^+ G_\sigma^{\boldsymbol{C}}\}.$$

REMARK. Since $C(z_0,\lambda) = I$ for all $\lambda \in S^1$, we know that $\boldsymbol{D}\backslash S$ does contain an open neighbourhood of z_0.

3.2.

As in [**19**] the results of Section 3.1 are used to construct normalized potentials. First we note

LEMMA. *We retain the notations and the assumptions of Theorem* 3.1. *If* $C = C_-C_+$ *with* $C_- \in \Lambda^- G_\sigma^{\boldsymbol{C}}$, $C_+ \in \Lambda^+ G_\sigma^{\boldsymbol{C}}$ *and* $C_- = I + \mathscr{O}(\lambda^{-1})$ *on some open subset* $U \subset \boldsymbol{D}\backslash S$*, then* C_- *is holomorphic on* U *and*

$$C_-^{-1} dC_- = \lambda^{-1}\xi_{-1}\mathrm{dz} \tag{3.2.1}$$

for some holomorphic map ξ_{-1} *on* U*.*

PROOF. By our assumptions C is a holomorphic map from U to the open set $\Lambda^- G_\sigma^{\boldsymbol{C}} \cdot \Lambda^+ G_\sigma^{\boldsymbol{C}}$. Since the splitting map

$$\Lambda^- G_\sigma^{\boldsymbol{C}} \cdot \Lambda^+ G_\sigma^{\boldsymbol{C}} \to \Lambda^- G_\sigma^{\boldsymbol{C}} \times \Lambda^+ G_\sigma^{\boldsymbol{C}}$$

is also complex analytic, the maps C_- and C_+ are holomorphic. Therefore $C_-^{-1}\mathrm{d}C_-$ is a holomorphic 1-form. The usual argument shows that it is of the form $\lambda^{-1}\xi_{-1}\mathrm{dz}$. The main question is for which $z \in \boldsymbol{D}\backslash S$ one can split $C = C_-C_+$ analytically, and how the (additional) singular set $\hat{S}$ looks like; let $U = (\boldsymbol{D}\backslash S)\backslash\hat{S}$. To address this issue we follow the argument of [**19**]. □

THEOREM. *We retain the notations and the assumptions of Theorem* 3.1. *Then for every holomorphic extended frame* $C : \boldsymbol{D}\backslash S \to \Lambda G_\sigma$ *there is a holomorphic function* $\tau_{\boldsymbol{D}\backslash S} : \boldsymbol{D}\backslash S \to \boldsymbol{C}$ *such* C *can be split* $C = C_-C_+$ *analytically exactly on the open set on which* $\tau_{\boldsymbol{D}\backslash S} \neq 0$*. Thus* $\hat{S}$ *is discrete in* $\boldsymbol{D}\backslash S$*.*

Moreover, considered as functions on $\boldsymbol{D}\backslash S$*,* C_- *and* C_+ *are meromorphic. In particular, the normalized potential* $\xi = C_-^{-1}dC_-$ *is a meromorphic differential* $(1,0)$*- form on* $\boldsymbol{D}\backslash S$*.*

PROOF. Following the analogous argument of [**19**] we consider a representation $\breve{\pi}$ of $\Lambda GL(n,\mathscr{A})$ in the group of automorphisms of an infinite dimensional Grassmannian like manifold Gr. Considering the dual determinant bundle and a holomorphic (highest weight) section τ we set $\tau_{\mathscr{M}}(z) = \tau(H(z)p_0)$, where p_0 denotes the canonical base point of Gr relative to τ. Then H splits analytically exactly at all points z, where $\tau_{\mathscr{M}}(z) \neq 0$. Pulling back the line bundle $\det^*$ on Gr to $\mathscr{M}$ via $\mathscr{M} \to Gr$, $z \to H(z)\dot{p}_0$, we obtain a holomorphic line bundle L^* on $\mathscr{M}$ with holomorphic section τ_L induced from $\tau_{\mathscr{M}}$ as defined above. Since $\mathscr{M}$ is Stein, L^* is trivial, and $\tau_{\mathscr{M}}$ can be considered as a complex valued function on $\mathscr{M}$. □

DEFINITION. The differential $(1,0)$-form

$$\xi = C_-^{-1} dC_- \tag{3.2.2}$$

is called *the normalized potential* for the harmonic map φ (or the *extended frame associated to* φ).

REMARK. We reiterate that every normalized potential ξ is of the form $\xi = \lambda^{-1}\xi_{-1}\mathrm{dz}$ where ξ_{-1} is meromorphic on some open subset $U = (\boldsymbol{D}\backslash S)\backslash\hat{S}$.

3.3.

The main feature of the procedure developed in [**19**] is that every harmonic map can be constructed from some holomorphic or normalized potential. We recall that the integrability condition is trivially satisfied for the potentials under consideration.

THEOREM A. *Let $\eta = \lambda^{-1}\eta_{-1} + \lambda^0\eta_0 + \lambda^1\eta_1 + \dots$ be a holomorphic $(1,0)$-form with values in $p^{\boldsymbol{C}}$ defined on a simply connected subset $\mathscr{L} \subset \boldsymbol{C}$. Let C be the solution on $\mathscr{L}$ to the ODE $dC = C\eta$, $C(z_0,\lambda) = I$ for all $\lambda \in S^1$. Then C is the extended holomorphic frame of some harmonic map $\varphi : \mathscr{L}\backslash T \to G/K$. More precisely, splitting $C = FV_+$ on some open subset $\mathscr{L}\backslash T$, we obtain the extended frame F of the harmonic map $\varphi : \mathscr{L}\backslash T \to G/K$, given by $\varphi = F \bmod K$.*

PROOF. It is easy to see that the Maurer-Cartan form of F has the form (1.3.4). The rest is straightforward. □

THEOREM B. *Let $\xi = \lambda^{-1}\xi_{-1}\mathrm{dz}$ be a meromorphic $(1,0)$-form on $\hat{L} \subset \boldsymbol{C}$ and assume that there exists a globally meromorphic solution C to the ODE $dC = C\xi$, $C(z_0,\lambda) = I$ for all $\lambda \in S^1$. Then C is the extended holomorphic frame of some harmonic map $\varphi : \hat{L}\backslash\hat{T} \to G/K$. More precisely, splitting $C = FV_+$ on some open subset $\hat{L}\backslash\hat{T}$, we obtain the extended frame F of the harmonic map $\varphi : \hat{L}\backslash\hat{T} \to G/K$, given by $\varphi = F \bmod K$.*

PROOF. First we remove the points from $\hat{L}$ where C has a pole. Then proceed as in the proof of Theorem A. □

Finally, we would like to address the question to what extent the harmonic maps are uniquely determined by the associated analytical potentials. To avoid lengthy technical assumptions we state only a local result:

THEOREM C. *Let $\varphi : \boldsymbol{D} \to G/K$ be harmonic and fix a base point $z_0 \in \boldsymbol{D}$. Then the normalized potential ξ associated with φ is holomorphic in a neighborhood of z_0 and it is uniquely determined by φ. Moreover, given a normalized potential ξ which is holomorphic near z_0, then Theorem B constructs a unique associated family of harmonic maps defined in some neighborhood of z_0.*

PROOF. Similar to [**19**]. □

REMARK. Theorem C above shows that harmonic maps and normalized potentials are essentially in a 1-1 relation.

4. Finite type harmonic maps.

4.1.

Among the harmonic maps investigated in the literature those *of finite type* play a particularly prominent role. We follow here primarily the approach of [**11**], [**10**].

For $d \in 2\boldsymbol{N}+1$ we set

$$\Lambda_d = \left\{ \sum_{n=-d}^{d} \xi_n \lambda^n \in \Lambda \boldsymbol{g}_\sigma \;\middle|\; \xi_d \neq 0 \right\}. \tag{4.1.1}$$

Note, if $\xi \in \Lambda_d$, then $\xi_{d-1} \in \boldsymbol{k}^{\boldsymbol{C}}$, since d is odd. Decomposing $\boldsymbol{k}^{\boldsymbol{C}}$ in the form $\boldsymbol{k}^{\boldsymbol{C}} = \boldsymbol{n} + \boldsymbol{h} + \overline{\boldsymbol{n}}$ with $\boldsymbol{b} = \boldsymbol{h} + \boldsymbol{n}$ being a Borel subalgebra, we can project any $\tau \in \boldsymbol{k}^{\boldsymbol{C}}$ onto $\boldsymbol{b}$ (see [**11**, (2.5)])

$$r(\tau) = \tau_{\boldsymbol{n}} + \frac{1}{2}\tau_{\boldsymbol{h}}. \tag{4.1.2}$$

Using this notation we obtain

THEOREM. *For each $d \in 2\boldsymbol{N}+1$ and $\xi_* \in \Lambda_d$, there exists an open ball U of $O \in \boldsymbol{R}^2$ where*

$$\frac{\partial \xi}{\partial z} = [\xi, \lambda^{-1}\xi_d + r(\xi_{d-1})],\; \xi(z_0) = \xi_* \tag{4.1.3}$$

is integrable, for $\xi = \sum_{n=-d}^{d} \xi_n \lambda^n$. Moreover, in this case, the $\Lambda \boldsymbol{g}_\sigma$-valued 1-form given by

$$\alpha = (\lambda^{-1}\xi_d + r(\xi_{d-1}))\mathrm{d}z + (\lambda \xi_{-d} + \overline{r(\xi_{d-1})})\mathrm{d}\overline{z} \tag{4.1.4}$$

satisfies the Maurer-Cartan equations (1.3.8)*. In addition, the extended frame F defined by*

$$F^{-1}dF = \alpha,\; F(z_0, \lambda) = I$$

on U induces the harmonic map $\varphi = F \bmod K$ on U.

PROOF. The proof can be taken almost verbatim from [**11**, Chapter 2, Theorem 2.5]. However, instead of the Killing form we take the non-degenerate invariant bilinear form κ which we have assumed to exist. At this point we also need to restrict to open balls U around O, since the argument for the completeness given in [**11**] doesn't apply in our case. □

DEFINITION. Harmonic maps obtained by this construction outlined in the last Theorem will be called *of finite type*.

We will see in the next section that harmonic maps of finite type have particularly simple holomorphic potentials.

4.2.

Consider the potential

$$\eta = \lambda^{d-1}\xi_* \mathrm{d}z, \tag{4.2.1}$$

where $\xi_* \in \Lambda_d$.

Then the holomorphic extended frame C defined by $dC = C\eta$, $C(z_0, \lambda) = I$, is of the form

$$C(z, \lambda) = \exp(z\lambda^{d-1}\xi_*). \tag{4.2.2}$$

Consider the Iwasawa splitting (locally around z_0)

$$C = \hat{F}\hat{V}_+. \tag{4.2.3}$$

For the purposes of this section we will use the freedom in choosing the coefficient V_0 in V_+ at λ^0 and require $V_0 \in B$, where $K^{\mathbf{C}} = KB$ is the (classical) Iwasawa decomposition of $K^{\mathbf{C}}$. At any rate, by the general theory $\hat{F}$ defines an associate family of harmonic maps.

DEFINITION. Maps obtained from potentials of the form (4.2.1) are called *of Symes finite type*.

The main result of this section is

THEOREM. a) *Every map of finite type is of Symes finite type.*
b) *Every map of Symes finite type is of finite type.*

PROOF. a) Let $\phi : D \to G/K$ be of finite type and ξ_* as in the last Theorem. Let F denote the extended framing of ϕ. On the other hand, by the arguments at the beginning of this section, and starting from $\eta = \lambda^{d-1}\xi_* dz$, using (4.2.2) and (4.2.3) we obtain some frame $\hat{F}$. Then

$$\hat{\xi} = \hat{F}^{-1}\xi_*\hat{F} \in \Lambda_d, \tag{4.2.4}$$

which follows from $\hat{\xi} = \hat{F}^{-1}\xi_*\hat{F} = \hat{V}_+\xi_*\hat{V}_+^{-1}$ and the fact that ξ_* is in Λ_d.

It is straightforward to verify

$$\partial_z\hat{\xi} = [\hat{\xi}, \hat{F}^{-1}\partial_z\hat{F}], \quad \hat{\xi}(z_0,\lambda) = \xi_*. \tag{4.2.5}$$

Next we use the fact that under our assumptions we have

$$\eta_* dz = \lambda^{d-1}\xi_* dz = C^{-1}dC = dCC^{-1}. \tag{4.2.6}$$

Inserting the unique decomposition (4.2.3) into the right side yields

$$\eta_* dz = d\hat{F}\hat{F}^{-1} + \hat{F}d\hat{V}_+\hat{V}_+^{-1}\hat{F}^{-1}, \tag{4.2.7}$$

from which we derive

$$\hat{\eta} dz = \lambda^{d-1}\hat{\xi} dz = \hat{F}^{-1}d\hat{F} - d\hat{V}_+\hat{V}_+^{-1}. \tag{4.2.8}$$

Therefore, $\alpha = \hat{F}^{-1}d\hat{F}$ is the projection of $\hat{\eta} dz$ along $\Lambda^+ g_\sigma{}^{\mathbf{C}}$. In particular, we obtain $\lambda^{-1}\hat{\eta}_{-1}dz = \alpha'{}_p = \lambda^{-1}\hat{\xi}_{-d}$ and $\hat{\eta}_0 dz = \hat{\xi}_{-d+1}dz = \alpha_k - (d\hat{V}_+\hat{V}_+^{-1})_0$, whence $\alpha_k = (\hat{\xi}_{-d+1}dz)_k$, the projection of $\hat{\xi}_{-d+1}dz$ onto $\boldsymbol{k}$ along $\mathrm{Lie}B$. But it is straightforward to verify (see e.g. [**20**, Section 2.3]) that

$$(\hat{\xi}_{-d+1}dz)_k = r(\hat{\xi}_{-d+1})dz + \overline{r(\hat{\xi}_{-d+1})}d\bar{z}. \tag{4.2.9}$$

Hence

$$\hat{F}^{-1}d\hat{F} = (\lambda^{-1}\hat{\xi}_{-d} + r(\hat{\xi}_{-d+1}))\mathrm{d}z + (\lambda\hat{\xi}_d + \overline{r(\xi_{-d+1})})\mathrm{d}\overline{z} \tag{4.2.10}$$

and we have $\hat{\xi}(0) = \xi_*$. The uniqueness of the solution with values in Λ_d to the differential equation (4.1.3) now shows $\xi = \hat{\xi}$ and $F^{-1}dF = \hat{F}^{-1}d\hat{F}$. Thus $F = A\hat{F}$ with some matrix A

independent of z. Evaluating at the base point z_0 yields $A = I$. This shows that the harmonic map (of Symes finite type) derived from $\eta = \lambda^{d-1}\xi_*$ coincides with the given associated harmonic map of finite type derived in 4.1.

b) This was actually part of the argument in the proof of a). □

5. The Lie group case.

5.1.

As an application of the theory presented in the paper we would like to consider harmonic maps into Lie groups, which were discussed in a somewhat different setting via loop groups in [**41**], [**4**].

In the context of this paper we consider a real Lie group G as a symmetric space

$$G \sim (G \times G)/\Delta, \tag{5.1.1}$$

where

$$\Delta = \{(g,g) \,|\, g \in G\}. \tag{5.1.2}$$

The canonical projection π is given by

$$\pi : G \times G \to G, \ (g,h) \to g^{-1}h. \tag{5.1.3}$$

Thus our approach requires to consider the loop group with values in $(G \times G)^{\mathbf{C}} = G^{\mathbf{C}} \times G^{\mathbf{C}}$. Similar to [**21**] we set

$$\mathscr{H} = G \times G \tag{5.1.4}$$

and

$$\Lambda \mathscr{H}^{\mathbf{C}} = \Lambda G^{\mathbf{C}} \times \Lambda G^{\mathbf{C}}. \tag{5.1.5}$$

Analogously we set

$$\Lambda \mathscr{H} = \Lambda G \times \Lambda G. \tag{5.1.6}$$

For our approach we also need some group $\Lambda^+ \mathscr{H}^{\mathbf{C}}$. It is natural to consider pairs $(g(\lambda), h(\lambda))$ of functions $\lambda \in S^1$, which have holomorphic extensions to the interior of the unit disk

$$\Lambda^+ \mathscr{H}^{\mathbf{C}} = \Lambda^+ G^{\mathbf{C}} \times \Lambda^+ G^{\mathbf{C}}, \tag{5.1.7}$$

and analogously

$$\Lambda^- \mathscr{H}^{\mathbf{C}} = \Lambda^- G^{\mathbf{C}} \times \Lambda^- G^{\mathbf{C}}. \tag{5.1.8}$$

Finally, we mention that the involution $\sigma(f,g) = (g,f)$ of $\Lambda \mathscr{H}$ has the fixpoint group Δ and thus defines the symmetric space G. Therefore, by our theory, we need to twist $\Lambda \mathscr{H}^{\mathbf{C}}$ by σ and obtain the twisted groups $\Lambda \mathscr{H}^{\mathbf{C}}_\sigma$, $\Lambda^{\pm} \mathscr{H}^{\mathbf{C}}_\sigma$ and $\Lambda \mathscr{H}_\sigma$.

Note that for $(f,g) \in \Lambda \mathscr{H}^{\mathbf{C}}$ the twisting condition is

$$f(\lambda) = g(-\lambda). \tag{5.1.9}$$

While this makes the second component in our group basically superfluous, we will nevertheless continue to use $\Lambda \mathscr{H}^{\mathbf{C}}_\sigma$, since we want to illustrate the general theory with this example.

5.2.

The rest of this example follows closely [**14**, Section 9]. We consider a harmonic map $\varphi : \boldsymbol{D} \to G$ and lift it to an extended framing

$$\tilde{F} : \boldsymbol{D} \to G \times G,\ \tilde{F} = (e, \varphi). \tag{5.2.1}$$

For the Maurer-Cartan form we obtain

$$\tilde{\alpha} = \tilde{F}^{-1} d\tilde{F} = (0, \varphi^{-1} d\varphi). \tag{5.2.2}$$

Abbreviating $\alpha = \varphi^{-1} d\varphi$ we need to decompose $\tilde{\alpha} = (0, \alpha)$ in the form $\tilde{\alpha} = \tilde{\alpha}_{\boldsymbol{k}} + \tilde{\alpha}_{\boldsymbol{p}}$, where $\boldsymbol{k}$, $\boldsymbol{p} \subset \boldsymbol{h} = \boldsymbol{g} \times \boldsymbol{g}$ are defined by σ:

$$\boldsymbol{k} = \mathrm{Lie}(\Delta) = \{(A, A) \mid A \in \boldsymbol{g}\}, \tag{5.2.3}$$

$$\boldsymbol{p} = \{(A, -A) \mid A \in \boldsymbol{g}\}. \tag{5.2.4}$$

Hence

$$\tilde{\alpha}_{\boldsymbol{k}} = \frac{1}{2}(\alpha, \alpha),\ \tilde{\alpha}_{\boldsymbol{p}} = \frac{1}{2}(-\alpha, \alpha). \tag{5.2.5}$$

Next we need to introduce the loop parameter λ. Decomposing $\alpha'_{\boldsymbol{p}} + \alpha''_{\boldsymbol{p}} = \alpha_{\boldsymbol{p}}$ into the $(1,0)$-part $\alpha'_{\boldsymbol{p}}$ and the $(0,1)$-part $\alpha''_{\boldsymbol{p}}$ we define

$$\tilde{\alpha}_\lambda = \lambda^{-1} \tilde{\alpha}'_{\boldsymbol{p}} + \tilde{\alpha}_{\boldsymbol{k}} + \lambda \tilde{\alpha}''_{\boldsymbol{p}}, \tag{5.2.6}$$

which yields by a straightforward computation

$$\tilde{\alpha}_\lambda = \left(\frac{1}{2}(1-\lambda^{-1})\alpha'_{\boldsymbol{p}} + \frac{1}{2}(1-\lambda)\alpha''_{\boldsymbol{p}}\,,\ \frac{1}{2}(1+\lambda^{-1})\alpha'_{\boldsymbol{p}} + \frac{1}{2}(1+\lambda)\alpha''_{\boldsymbol{p}} \right). \tag{5.2.7}$$

At this point it is useful to recall ([**4**, Proposition 4.2])

THEOREM. *A smooth map $\varphi : \boldsymbol{D} \to G$ is harmonic if and only if the* 1*-form*

$$\alpha_\lambda = \frac{1}{2}(1+\lambda^{-1})\alpha'_{\boldsymbol{p}} + \frac{1}{2}(1+\lambda)\alpha''_{\boldsymbol{p}} \tag{5.2.8}$$

is integrable for all $\lambda \in S^1$.

REMARK. 1. For $G = U(n)$ this is a classical result (see e.g. [**41**]).

2. The Theorem also holds, of course, if one replaces λ by $-\lambda$. Thus (5.2.8) is equivalent with $\tilde{\alpha}_\lambda = (\alpha_{-\lambda}, \alpha_\lambda)$ being integrable.

Since α_λ is integrable, we can solve

$$F^{-1} dF = \alpha_\lambda. \tag{5.2.9}$$

For $G = U(n)$ this is Uhlenbeck's "extended framing". In our setting, an extended framing associated with $\tilde{\alpha}_\lambda$ is given by

$$\mathscr{F}(z, \bar{z}, \lambda) = (F(z, \bar{z}, -\lambda), F(z, \bar{z}, \lambda)). \tag{5.2.10}$$

REMARK. Note that we usually normalize framings in a way, such that at some base point $z_* \in \boldsymbol{C}$ we have $\varphi(z_*) = I$ and $F(z_*, \overline{z}_*, \lambda) = I$, for all $\lambda \in S^1$. Then also $F(z_*, \overline{z}_*, -\lambda) = I$, for all $\lambda \in S^1$. But a glance at (5.2.7) shows that $F(z, \overline{z}, -1) =$const., whence $F(z, \overline{z}, -1) = I$. Since $\varphi(z)$ and $F(z, \overline{z}, 1)$ satisfy the same differential equation with same initial condition, $\varphi(z) = F(z, \overline{z}, 1)$ follows. Incidentally we have shown that the framing F is "based at $\lambda = -1$", i.e., $F(z, \overline{z}, -1) = I$. Since our extended framing $\mathscr{F}$ is not based at any $\lambda \in S^1$, we do not use "based loop groups", opposite to [**41**] or [**4**].

As a consequence, in the discussion above, the associated family φ_λ of harmonic maps containing $\varphi_{\pm 1}$ is given by

$$\varphi_\lambda(z, \overline{z}) = \mathscr{F}(z, \overline{z}, \lambda) \bmod \Delta = F(z, \overline{z}, -\lambda)^{-1} F(z, \overline{z}, \lambda). \tag{5.2.11}$$

We would like to point out that Uhlenbeck's extended framing $F(z, \overline{z}, \lambda)$ yields a harmonic map into G only for $\lambda = \pm 1$, while our setting produces naturally an S^1-family of harmonic maps into G.

5.3.

In Section 3.1 we have introduced holomorphic potentials and normalized potentials. The general theory states that the extended framing $\mathscr{F} \in \Lambda \mathscr{H}_\sigma$ can be multiplied by some $\mathscr{V}_+ \in \Lambda^+ \mathscr{H}_\sigma^{\boldsymbol{C}}$ such that

$$\mathscr{C}(z, \lambda) = \mathscr{F}(z, \overline{z}, \lambda) \mathscr{V}_+(z, \overline{z}, \lambda) \tag{5.3.1}$$

is "holomorphic" in z. For the present discussion we thus obtain

$$\mathscr{C}_1(z, \lambda) = \mathscr{F}(z, \overline{z}, \lambda) \mathscr{V}_+(z, \overline{z}, \lambda), \tag{5.3.2}$$

where $\mathscr{C}_1$ denotes the first component of $\mathscr{C}$. The second component is determined by (5.1.8). Hence, all holomorphic potentials are of the form

$$\tilde{\eta} = (\eta(z, \lambda) dz, \eta(z, -\lambda) dz), \tag{5.3.3}$$

where

$$\eta = \lambda^{-1} \eta_{-1} + \lambda^0 \eta_0 + \lambda \eta_1 + \dots \tag{5.3.4}$$

where all the matrix functions η_k are holomorphic in z. Analogously, normalized potentials are of the form

$$\tilde{\xi} = (\xi, -\xi), \tag{5.3.5}$$

$$\xi = \lambda^{-1} \xi_{-1}, \ \ \xi_{-1} \text{ meromorphic.} \tag{5.3.6}$$

5.4.

For the discussion of maps of finite type, we consider "Λ_d", i.e., those elements in $\mathrm{Lie}(\Lambda \mathscr{H}_\sigma)$, which only involve finitely many powers of λ. Then these elements need to be shifted by λ^{d-1}. In our setting we thus obtain

THEOREM. *Harmonic maps $\varphi : \boldsymbol{D} \to G$ of finite type are exactly those maps which are* *obtained from potentials of the type*

$$\tilde{\eta} = (\lambda^{d-1}\mu(\lambda)dz, (-\lambda)^{d-1}\mu(-\lambda)dz) = \lambda^{d-1}(\mu(\lambda), \mu(-\lambda))dz, \tag{5.4.1}$$

where $\mu \in \Lambda_d$, *i.e.*, $\mu = \sum_{-d \le j \le d} \mu_j \lambda^j$, *d odd*, $\mu_d \neq 0$.

5.5.

Finally, we would like to mention some explicit examples. Comparing [**4**, 4.2.1] to (5.2.8) we see that our framing $\mathscr{F}$ is of the form (5.2.10), where F is the framing considered in [**4**]. In view of (5.3.3) and (5.3.5) it thus suffices to consider one component. Therefore the potentials used in this section can be read off directly from the potentials used in [**4**]. In particular

1. For the nilpotent group

$$G = \left\{ \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \middle| \, a \in \boldsymbol{R} \right\}, \tag{5.5.1}$$

one gets the potential $\xi = (A(\lambda), A(-\lambda))$, where $A(\lambda) = ((1-\lambda^{-1})/2) \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} dz$ and a is a meromorphic function.

2. For the Heisenberg group of upper triangular unipotent matrices

$$G = \left\{ \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} \middle| \, a, b, c \in \boldsymbol{R} \right\}, \tag{5.5.2}$$

the potential is $\xi = (A(\lambda), A(-\lambda))$, where $A(\lambda) = ((1-\lambda^{-1})/2) \begin{pmatrix} 0 & \partial_z a & \partial_z c - a\partial_z b \\ 0 & 0 & \partial_z b \\ 0 & 0 & 0 \end{pmatrix} dz$ and $a, b, c : \boldsymbol{D} \to \boldsymbol{R}$ are meromorphic functions.

3. For the special linear Lie group

$$G = SL(2, \boldsymbol{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \, a, b, c, d \in \boldsymbol{R}, \; ad - bc = 1 \right\}, \tag{5.5.3}$$

the potential is $\tilde{\xi} = ((1-\lambda^{-1})/2, (1+\lambda^{-1})/2)\xi dz$, for ξ given in [**4**, (5.3.8)].

In all three cases one obtains the normalized potential in the sense of Section 3.2 by gauging away the coefficient at λ^0.

6. More examples.

6.1. CMC surfaces.

The immersions of constant mean curvature (CMC) surfaces can be characterized [**38**] by the harmonicity of the Gauss map $\varphi : \boldsymbol{D} \to G/K = SO(3)/SO(2) \equiv SU(2)/SU(1)$. Here $G^{\boldsymbol{C}} = SL(2, \boldsymbol{C})$ and $K = SU(1)$ is the subgroup of diagonal elements in $G = SU(2)$. The *normalized potentials* have the form [**15**, (3.2.1); Theorem 3.13]

$$\xi = \lambda^{-1} \begin{pmatrix} 0 & f \\ E/f & 0 \end{pmatrix} dz, \tag{6.1.1}$$

where f has only poles of even order ([**15**, Corollary 2.3]) and does not vanish identically. The solution to $g_-^{-1}dg_- = \xi$ can be decomposed in the form (Iwasawa splitting)

$$g_-(z,\lambda) = F(z,\lambda)g_+^{-1}(z,\lambda)$$

and we can assume that $F \in \Lambda_\sigma SU(2)$ and $g_+ \in \Lambda_\sigma^+ SL(2,\boldsymbol{C})$ are smooth in z. Then the Sym-Bobenko formula ([**15**, (1.1.4)], [**16**, (2.2.11)]) provides the associated family of immersions, which consists of CMC surfaces without branch points, $\varphi : S^1 \times \boldsymbol{D} \to su(2) \equiv so(3) \equiv \boldsymbol{R}^3$.

The *holomorphic potentials* of the form ([**19**, Section 4])

$$\eta = \lambda^{-1}\begin{pmatrix} 0 & f \\ g & 0 \end{pmatrix} dz, \tag{6.1.2}$$

with f and g holomorphic, provide CMC surfaces with umbilic points at the zeros of g and branch points at the zeros of f. E.g., we have:

a) for $f = 1, g = 0$, the punctured sphere $S^2 \setminus \{\text{one point}\}$;

b) for $f = 1, g = 1$, the right circular cylinder;

c) for $f = 1, g = cz^m$ $(c \in \boldsymbol{C}^*, m \geq 0)$, B. Smyth's CMC surface with an umbilic of order m at the origin $z = 0$ [**17**, Proposition 4.1]. This is nondegenerate only for $m = 0$ and $|c| \neq 1$ ([**17**, Proposition 4.4]).

d) for $f = 1, g = (z - z_1) \cdot \ldots \cdot (z - z_n)$, B. Smyth's CMC surface with n umbilic points at z_k, $k = 1, \ldots, n$.

Similarly, one can obtain surfaces with branch points; e.g, the potential

$$\xi = \lambda^{-1}\begin{pmatrix} 0 & z - z_0 \\ 1 & 0 \end{pmatrix} dz$$

yields a CMC surface with one branch point at $z = z_0$.

6.2. Willmore surfaces.

The DPW method was applied for a description of the Willmore surfaces in terms of potentials [**26**] by F. Helein. The Willmore surfaces S are the minimizers of the Willmore functional

$$W(S) = \int_S H^2 d\sigma = \frac{1}{4}\int_S (k_1 - k_2)2 d\sigma + 4\pi(1 - g),$$

where the variation is made within the set $\mathscr{S}$ of surfaces S immersed in $\boldsymbol{R}^3$, which are oriented and without boundary. Moreover, H denotes the mean curvature, k_1 and k_2 the principal curvatures, g the genus and $d\sigma$ the area element of $S \in \mathscr{S}$.

Willmore surfaces are characterized by the equation $\Delta H + 2H(H^2 - K) = 0$. The only CMC Willmore surfaces are S^2 and the minimal surfaces.

In the DPW approach one associates to a conformal Willmore immersion a frame in $\boldsymbol{R}^{4,1}$ which encodes the tangent plane of the surface and the conformal Gauss map ([**26**], [**1**]). Outside the umbilic set, this frame $F : U \subset \boldsymbol{D} \to G = SO(4,1)$ incorporates the conformal transform of the surface. The immersion is Willmore if and only if the loopified Maurer-Cartan form (1.3.4) of F is integrable; DPW works for (1) the noncompact subgroup $K = SO(3,1) \subset SO(4,1)$, and (2) for $K = SO(3) \times SO(1,1)$. Thus S is Willmore if (1) the induced map $\hat{F} : U \to G/K$ is harmonic, or respectively (2) if $\hat{F}$ is "roughly harmonic", i.e. it provides a harmonic map by a K-right gauge shift of $\hat{F}$. The second case has the peculiarity that $\alpha'_{\boldsymbol{p}}$ is not necessarily holomorphic; still, this alternative is constructive and the meromorphic potentials can be explicitly described [**26**, Section 4.2, Theorem 9, p. 38].

6.3. Minimal surfaces.

These, regarded as special cases of CMC surfaces, are characterized as well by the holomorphy of the Gauss map. The associated meromorphic potentials have the form [**18**, Thorem 3.1, p. 5]

$$\xi = \lambda^{-1} \begin{pmatrix} 0 & 0 \\ g & 0 \end{pmatrix} dz, \tag{6.3.1}$$

a particular form of (6.1.2). Then the classical Weierstrass representation produces directly from g and the coefficients of the extended frame F obtained from $C = FV_+$ the minimal surface [**18**, Section 4].

6.4. The tangent group case.

Given a connected real Lie group G with an involution σ, let $K = (\text{Fix } \sigma)_0 \subset G$. We set $\tilde{G} = TG$, $\tilde{K} = TK$ and consider the homogeneous space $\tilde{G}/\tilde{K} = TG/TK \sim T(G/K)$ [**6**]. We note that for $\tilde{\sigma} = (\sigma, \sigma_{*,e})$, we have $\tilde{K} = (\text{Fix } \sigma)_0 \subset \tilde{g}$. Then $\tilde{G} = TG \sim G \ltimes \boldsymbol{g}$, where $\boldsymbol{g} = \text{Lie}(G)$. Let $\boldsymbol{k} = \text{Lie}(K)$ and set $\tilde{\boldsymbol{k}} = \{(A,a) \mid A, a \in \boldsymbol{k}\} \sim \boldsymbol{k} \times \boldsymbol{k}$. Let $\boldsymbol{g} = \boldsymbol{k} \oplus \boldsymbol{p}$ be the Cartan decomposition relative to σ and $\tilde{\boldsymbol{p}} = \{(A,a) \mid A, a \in \boldsymbol{k}\} \sim \boldsymbol{p} \times \boldsymbol{p}$. Then $\tilde{\boldsymbol{g}} = \tilde{\boldsymbol{k}} \oplus \tilde{\boldsymbol{p}}$ is the induced decomposition on the tangent group level. Consider the group operation on $\tilde{G}$ given by

$$(g,X) \circ (h,Y) = (gh, X + \text{Ad}_g Y), \text{ where } (g,X),(h,Y) \in G \ltimes \boldsymbol{g} \sim TG. \tag{6.4.1}$$

Then the Lie bracket on $TG \sim \boldsymbol{g} \ltimes \boldsymbol{g}$ reads

$$[(a,A),(b,B)] \equiv \left[\begin{pmatrix} A & a \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} B & b \\ 0 & 0 \end{pmatrix} \right] = \begin{pmatrix} [A,B] & [A,b] - [B,a] \\ 0 & 0 \end{pmatrix}. \tag{6.4.2}$$

For a harmonic function $\varphi : \boldsymbol{D} \to \tilde{G}/\tilde{K}$, the corresponding lift $F : \boldsymbol{D} \to \tilde{G}$ induces a normalized potential of the form $\xi = \lambda^{-1}\xi_{-1}dz$, with $\xi_{-1} : \boldsymbol{D} \to \tilde{\boldsymbol{p}}^{\boldsymbol{C}} = \boldsymbol{p}^{\boldsymbol{C}} \ltimes \boldsymbol{p}^{\boldsymbol{C}}$.

Conversely, let any $\xi = \lambda^{-1}\xi_{-1}dz$, with $\xi_{-1} : \boldsymbol{D} \to \tilde{\boldsymbol{p}}^{\boldsymbol{C}} = \boldsymbol{p}^{\boldsymbol{C}} \ltimes \boldsymbol{p}^{\boldsymbol{C}}$ be given. We can assume that for this potential the differential equation $dC = C\xi$ has a meromorphic solution $C : \boldsymbol{D} \to \tilde{G}^{\boldsymbol{C}}$ where ξ is of the form $\xi = \lambda^{-1}(\xi^{(1)}, \xi^{(2)})dz$ and $C = (g,X)$, whence

$$C^{-1}dC = (g,X)^{-1}d(g,X) = (g^{-1}dg, (-\text{Ad}_g X + \text{Ad}_{g^{-1}} X')dz).$$

For the solution $C = (g,X)$ to this differential equation we perform an Iwasawa splitting

$$(g,X) = (g_r, X_r) \circ (g_+, X_+) \in \Lambda\tilde{G} \cdot \Lambda^+ \tilde{G}^{\boldsymbol{C}},$$

which is equivalent to

$$\begin{cases} g = g_r g_+ \\ X = X_r + \text{Ad}_{g_r} X_+. \end{cases}$$

Note that the first equation is the usual Iwasawa splitting equation for $g \in \Lambda G^{\boldsymbol{C}}$ and yields g_r and g_+. Rewriting the second equation we obtain

$$(\mathrm{Ad}_{g_r})^{-1}X = (\mathrm{Ad}_{g_r})^{-1}X_r + X_+.$$

Since the first summand is real, we can find X_+ and X_r as follows: first, decompose $(\mathrm{Ad}_{g_r})^{-1}X = Y_r + Y_+$; second, set $X_+ = Y_+$ and $X_r = X - \mathrm{Ad}_{g_r}X_+$; then g_r, g_+, X_r, X_+ are the components of the Iwasawa splitting of (g,X). By the general theory (Proposition 3.1), $(g_r, X_r) \in \Lambda\tilde{G}$ is the extended framing of some harmonic map. In particular, the Maurer-Cartan form $\alpha_\lambda = \lambda\alpha'_{\tilde{\boldsymbol{p}}} + \alpha_{\tilde{\boldsymbol{k}}} + \lambda\alpha''_{\tilde{\boldsymbol{p}}}$ of (g_r, X_r) is integrable. Moreover, $\varphi = (g_r, X_r)$ mod $\tilde{\mathscr{K}}$ is an associate framing of a harmonic map from $\boldsymbol{D}$ to $T(G/K)$.

EXAMPLE. For $G = SU(2), \sigma = \mathrm{Ad}\ \sigma_3$ we obtain (see [**15**]): $K = SU(1) = \left\{ \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix} \middle| z \in \boldsymbol{C}^* \right\} = (\mathrm{Fix}\ \sigma)_0$. Then $TG = SU(2) \ltimes su(2), TK = SU(1) \ltimes su(1)$ and $TG/TK = T(SU(2)/SU(1)) = TS^2$. Our costruction then produces the harmonic maps $\varphi : \boldsymbol{D} \to TS^2$ [**2**].

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