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Chart description and a new proof of the classification theorem of genus one Lefschetz fibrations

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Abstract. We introduce a graphical method, called the chart description, to describe the monodromy representation of a genus one Lefschetz fibration. Using this method, we give a new and purely combinatorial proof of the classification theorem of genus one Lefschetz fibrations.

1. Introduction.

Let M and B be compact, connected, and oriented (not necessarily closed) C^∞-manifolds of dimensions 4 and 2, respectively.

DEFINITION 1. A C^∞-map $f : M \to B$ is a *genus one Lefschetz fibration* if the following conditions are satisfied:

(a) $\partial M = f^{-1}(\partial B)$;
(b) there is a finite set of points $y_1, \dots, y_n$ $(n \geq 1)$, called the *critical values of* f, in $\mathrm{Int} B (= B - \partial B)$ such that $f | f^{-1}(B - \{y_1, \dots, y_n\}) : f^{-1}(B - \{y_1, \dots, y_n\}) \to B - \{y_1, \dots, y_n\}$ is a C^∞-fiber bundle with fiber the 2-torus T^2;
(c) for each i $(1 \leq i \leq n)$, there exists a single point $p_i \in f^{-1}(y_i)$ such that
   (1) $(df)_p : T_p(M) \to T_{f(p)}(B)$ is onto for any $p \in f^{-1}(y_i) - \{p_i\}$,
   (2) about p_i (resp. y_i), there exist local complex coordinates z_1, z_2 with $z_1(p_i) = z_2(p_i) = 0$ (resp. local complex coordinate ξ with $\xi(y_i) = 0$), so that f is locally written as $\xi = f(z_1, z_2) = z_1 z_2$ or $\overline{z_1} z_2$;
(d) for each i $(1 \leq i \leq n)$, $H_2(f^{-1}(y_i); \boldsymbol{Z}) \cong \boldsymbol{Z}$.

Throughout this paper, by a Lefschetz fibration, we always mean a genus one Lefschetz fibration.

We call a fiber $f^{-1}(y)$ a *singular fiber* if $y \in \{y_1, \dots, y_n\}$, otherwise a *general fiber*. We call M the *total space*, B the *base space*, and f the *projection*.

A singular fiber is either a smoothly immersed 2-sphere in M with a single transverse self-intersection of sign $+1$ or -1. Such a fiber is said to be *of type* I_1^+ or I_1^-, respectively (cf. [**9**], [**10**]).

DEFINITION 2. A *chiral Lefschetz fibration* is a Lefschetz fibration whose singular fibers are all of type I_1^+.

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REMARK 3. A Lefschetz fibration (cf. [**3**]) is called an achiral Lefschetz fibration in [**16**] and in [**2**] or a differentiable Lefschetz fibration in [**19**]. A chiral Lefschetz fibration is called a Lefschetz fibration in [**17**] (cf. [**15**]) or a symplectic Lefschetz fibration in [**19**].

DEFINITION 4. Lefschetz fibrations $f : M \to B$ and $f' : M' \to B'$ are *isomorphic* if there exist orientation preserving diffeomorphisms $H : M \to M'$ and $h : B \to B'$ such that $f' \circ H = h \circ f$.

Let

$$s_1 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad s_2 = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

be elements of $SL(2,\boldsymbol{Z})$. Then $SL(2,\boldsymbol{Z})$ has a group presentation

$$\langle s_1, s_2 \mid s_1 s_2 s_1 (s_2 s_1 s_2)^{-1}, (s_1 s_2)^6 \rangle.$$

(This presentation is obtained from the presentation given in Problem 1.4.24 in p. 47 of [**11**] by the substitution $s_1 = yx^{-1}, s_2 = xy^{-2}$.)

The following is the deformation theorem due to Moishezon [**17**] (which is also called "normalizing theorem of local monodromies" [**15**]).

THEOREM 5 ([**17**]). *Let $g_1, g_2, \ldots, g_n$ be elements of $SL(2,\boldsymbol{Z})$ which are conjugates of s_1 with $g_1 g_2 \cdots g_n = 1$. Then by successive application of elementary transformations, the n-tuple $(g_1, g_2, \ldots, g_n)$ can be transformed to an n-tuple $(h_1, h_2, \ldots, h_n)$ with $h_i = s_1$ for odd i and $h_i = s_2$ for even i, and n must be a multiple of* 12.

Here *elementary transformations* mean the transformations

$$\begin{aligned} &(x_1, \ldots, x_{j-1}, x_j, x_{j+1}, x_{j+2}, \ldots, x_n) \\ &\quad \mapsto \quad (x_1, \ldots, x_{j-1}, x_{j+1}, x_{j+1}^{-1} x_j x_{j+1}, x_{j+2}, \ldots, x_n), \quad \text{and} \\ &(x_1, \ldots, x_{j-1}, x_j, x_{j+1}, x_{j+2}, \ldots, x_n) \\ &\quad \mapsto \quad (x_1, \ldots, x_{j-1}, x_j x_{j+1} x_j^{-1}, x_j, x_{j+2}, \ldots, x_n) \end{aligned}$$

for $j = 1, \ldots, n-1$ (cf. [**15**], [**17**]).

This theorem implies the classification theorem of chiral Lefschetz fibrations over the 2-sphere.

THEOREM 6 ([**7**], [**17**]). *Let $f : M \to B$ and $f' : M' \to B'$ be chiral Lefschetz fibrations such that B and B' are diffeomorphic to the* 2*-sphere S^2. They are isomorphic if and only if they have the same number of critical values.*

This result was generalized by the second named author to the case where the base space B is a surface of any genus.

THEOREM 7 ([**15**]). *Let $f : M \to B$ and $f' : M' \to B'$ be chiral Lefschetz fibrations over closed base spaces. They are isomorphic if and only if $g(B) = g(B')$ and they have the same number of critical values.*

For a Lefschetz fibration $f : M \to B$, we denote by $n_+(f)$ the number of singular fibers of type I_1^+ and by $n_-(f)$ the number of singular fibers of type I_1^-. (Note that $f : M \to B$ is a chiral Lefschetz fibration if and only if $n_-(f) = 0$.) The sum $n_+(f) + n_-(f)$ is the number of the critical values of f.

THEOREM 8 (cf. [**14**], [**15**]). *Let $f : M \to B$ and $f' : M' \to B'$ be Lefschetz fibrations over closed base spaces. Suppose that $n_+(f) - n_-(f) \neq 0$. They are isomorphic if and only if $g(B) = g(B')$, $n_+(f) = n_+(f')$ and $n_-(f) = n_-(f')$.*

This theorem was proved in [**14**] for the case where B is a 2-sphere, and in [**15**] (p. 563) for the general case (see below).

It is known (cf. [**3**], [**13**]) that the Euler number $e(M)$ and the signature $\sigma(M)$ of M are related to the numbers $n_+(f)$ and $n_-(f)$ by

$$e(M) = n_+(f) + n_-(f) \quad \text{and} \quad \sigma(M) = -\frac{2}{3}(n_+(f) - n_-(f)).$$

Thus the above theorem implies that when M and M' are total spaces of Lefschetz fibrations over closed surfaces B and B' respectively, M and M' are diffeomorphic provided $g(B) = g(B')$, $e(M) = e(M')$ and $\sigma(M) = \sigma(M') \neq 0$. This is the statement given in p. 563 of [**15**].

These results were proved by considering the natural projection $SL(2, \boldsymbol{Z}) \to PSL(2, \boldsymbol{Z})$ and establishing a theorem similar to the deformation theorem (Theorem 5) in terms of $PSL(2, \boldsymbol{Z})$.

We introduce a graphical method, called the chart description method, to describe a monodromy representation of a Lefschetz fibration over a surface B. The idea can be applied to Lefschetz fibrations of any fiber genus. We concentrate on the fiber genus one case in this paper, and the higher genus case will be discussed elsewhere. Using the chart description method, we have a new and purely combinatorial proof of the above classification theorems.

This paper is organized as follows. In § 2, we recall the notion of monodromy representations of Lefschetz fibrations. In § 3, chart description of a Lefschetz fibration is defined and it is proved that any Lefschetz fibration can be described by a chart (Theorem 15). In § 4, some results on chart descriptions are given which are used later. The main theorem (Theorem 21) is stated and proved in § 5. It gives a certain kind of 'normal form' of a chart description. In § 6, we prove the classification theorem by use of Theorem 21.

2. Monodromy representation.

Let $f : M \to B$ be a Lefschetz fibration and let $S_f = \{y_1, \ldots, y_n\}$ be the set of critical values. Take a point $y_0 \in \mathrm{Int}(B) - S_f$ and fix a diffeomorphism ι from the general fiber $f^{-1}(y_0)$ to the 2-torus $T^2 = S^1 \times S^1$. We identify the (orientation preserving) mapping class group $\mathrm{MC}(f^{-1}(y_0))$ of the general fiber $f^{-1}(y_0)$ with the mapping class group $\mathrm{MC}(T^2)$ of $T^2 = S^1 \times S^1$ by the diffeomorphism ι, and the latter group is identified with $SL(2, \boldsymbol{Z})$ by sending the positive Dehn twists along $S^1 \times \{*\}$ and $\{*\} \times S^1$ to s_1 and s_2, respectively.

Since $f| f^{-1}(B - S_f) : f^{-1}(B - S_f) \to B - S_f$ is a C^∞-fiber bundle with fiber the 2-torus T^2, we have a monodromy representation

$$\rho_f : \pi_1(B - S_f, y_0) \to \mathrm{MC}(f^{-1}(y_0)) \cong \mathrm{MC}(T^2)$$

as a fiber bundle. Combining the isomorphism $\mathrm{MC}(T^2) \cong SL(2,\boldsymbol{Z})$, we have a monodromy representation

$$\rho_f : \pi_1(B - S_f, y_0) \to SL(2,\boldsymbol{Z}).$$

Such a representation depends on the choice of y_0 and the diffeomorphism ι, but it is uniquely determined up to inner automorphisms of $\mathrm{MC}(T^2)$ or of $SL(2,\boldsymbol{Z})$.

Two monodromy representations $\rho_f : \pi_1(B - S_f, y_0) \to \mathrm{MC}(T^2)$ and $\rho_{f'} : \pi_1(B' - S_{f'}, y_0{}') \to \mathrm{MC}(T^2)$ are said to be *equivalent* if there exist an element $g \in \mathrm{MC}(T^2)$ and an orientation preserving homeomorphism $h : (B, S_f, y_0) \to (B', S_{f'}, y_0{}')$ such that

$$\mathrm{conj}(g) \circ \rho_f = \rho_{f'} \circ h_\#,$$

where $\mathrm{conj}(g)$ is the inner automorphism of $\mathrm{MC}(T^2)$ by g and $h_\# : \pi_1(B - S_f, y_0) \to \pi_1(B' - S_{f'}, y_0{}')$ is the isomorphism induced by h.

THEOREM 9 ([**17**], [**16**]). *Let $f : M \to B$ and $f' : M' \to B'$ be Lefschetz fibrations such that $n_+(f) - n_-(f) \neq 0$ or B has non-empty boundary. Then they are isomorphic if and only if their monodromy representations are equivalent.*

Theorem 9 is assumed when we prove Theorem 8 in § 6. (Our idea is to use chart descriptions in order to normalize monodromy representations, and we prove that under the hypothesis of Theorem 8, f and f' have equivalent monodromy representations if $g(B) = g(B')$, $n_+(f) = n_+(f')$ and $n_-(f) = n_-(f')$.) The proof of Theorem 9 [**17**] was based on the surjectivity of ρ_f, and our argument in this paper also gives an alternative proof of the surjectivity (cf. Theorem 26).

For a while, we assume that the base space B is a closed surface.

Let D be a 2-disk in B such that $y_0 \in \partial D$ and $S_f \subset \mathrm{Int}(D)$.

Let $\alpha_1, \dots, \alpha_n$ be mutually disjoint simple paths in D except at the common starting point $y_0 \in \partial D$ such that they appear in this order around y_0 and that their terminal points are the points of S_f. We call such a system of paths, $\alpha_1, \dots, \alpha_n$, a *Hurwitz path system* or a *system of good ordered paths* (cf. [**17**]). We denote by a_i $(1 \le i \le n)$ the element of $\pi_1(D - S_f, y_0)$ represented by a loop starting at y_0, going along α_i toward the endpoint (say y_i) of α_i, turning around y_i in the positive direction and going back to y_0 along α_i. Then $\pi_1(D - S_f, y_0)$ is freely generated by the elements $a_1, \dots, a_n$. The system $a_1, \dots, a_n$ is called the *Hurwitz generator system* or the *system of good ordered generators* of $\pi_1(D - S_f, y_0)$ associated with $\alpha_1, \dots, \alpha_n$. Note that

$$a_1 \cdots a_n = [\partial D]$$

in $\pi_1(D - S_f, y_0)$ and in $\pi_1(B - S_f, y_0)$, where ∂D is a loop along the boundary ∂D of the oriented 2-disk D.

When B is a closed surface of positive genus g, let $\beta_1, \beta_2, \dots, \beta_{2g}$ be simple closed paths in B which are mutually disjoint except at the common base point y_0 such that (1) we obtain a 2-disk D' by cutting B along these paths, (2) the 2-disk D is contained in this 2-disk D', and (3) we have

$$[\partial D] = [b_1, b_2] \cdots [b_{2g-1}, b_{2g}]$$

in $\pi_1(B - S_f, y_0)$, where b_i $(1 \leq i \leq 2g)$ is the homotopy class of β_i and $[a,b]$ stands for the commutator $aba^{-1}b^{-1}$ of a and b. For example, see Figure 1, where $n = 3$ and $g = 2$.

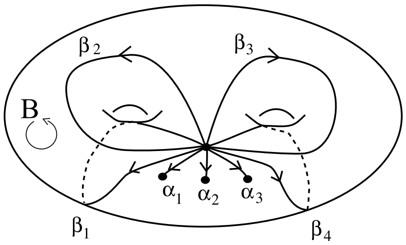

Figure 1. Hurwitz system.

A monodromy representation $\rho_f : \pi_1(B - S_f, y_0) \to SL(2, \boldsymbol{Z})$ is completely determined by the values $\rho_f(a_1), \ldots, \rho_f(a_n)$ and $\rho_f(b_1), \ldots, \rho_f(b_{2g})$.

If the terminal point of α_i is a critical value of type I_1^+ (or I_1^-, resp.), then $\rho_f(a_i)$ is a conjugate of s_1 (or of s_1^{-1}, resp.). Thus a monodromy representation $\rho_f : \pi_1(B - S_f, y_0) \to SL(2, \boldsymbol{Z})$ satisfies the following conditions:

(1) For each i $(1 \leq i \leq n)$, $\rho_f(a_i)$ is a conjugate of s_1 or of s_1^{-1}.
(2) $\rho_f(a_1) \cdots \rho_f(a_n) = [\rho_f(b_1), \rho_f(b_2)] \cdots [\rho_f(b_{2g-1}), \rho_f(b_{2g})]$.

Conversely, any homomorphism $\pi_1(B - S_f, y_0) \to SL(2, \boldsymbol{Z})$ satisfying these conditions is a monodromy representation of a Lefschetz fibration.

When we consider a Lefschetz fibration over a 2-sphere (i.e., $g = 0$), then the right-hand side of the equation in the second condition above is the identity element of $SL(2, \boldsymbol{Z})$.

3. Chart description.

DEFINITION 10. A *chart* in B is a finite graph Γ in B (possibly being empty or having *hoops* that are closed edges without vertices) whose edges are labeled with 1 or 2 and oriented so that the following conditions are satisfied:

(1) The degree of each vertex is equal to 1, 6 or 12.
(2) For a degree-6 vertex v, the six incident edges are labeled alternately with 1 and 2; and three consecutive edges are oriented inward and the other three are oriented outward (see Figure 2 where $\{i, j\} = \{1, 2\}$).
(3) For a degree-12 vertex v, the twelve incident edges are labeled alternately with 1 and 2; and all edges are oriented inward or all edges are oriented outward (see Figure 2 where $\{i, j\} = \{1, 2\}$).
(4) $\Gamma \cap \partial B = \varnothing$.

For a chart Γ, we denote by $\mathrm{Vert}(\Gamma)$ the set of all the vertices of Γ, and by S_Γ the subset of $\mathrm{Vert}(\Gamma)$ consisting of the vertices of degree 1.

An edge of a chart which is incident to a vertex v is an *incoming edge of* v (or an *outgoing edge* of v, resp.) if it is oriented toward v (or away from v, resp.).

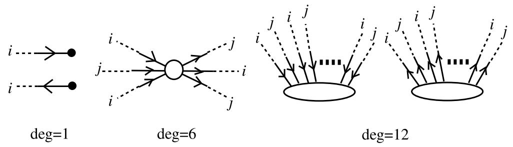

Figure 2. vertices of a chart.

A vertex of a chart (of degree 1 or 12) is *negative* (or *positive*, resp.) if all edges incident to the vertex are incoming edges (or outgoing edges, resp.). Since a degree-6 vertex has three incoming edges and three outgoing edges, we have the obvious equation:

$$\begin{aligned}&\#\{\text{positive degree-1 vertices}\} - \#\{\text{negative degree-1 vertices}\}\\ &\quad = 12\,(\#\{\text{negative degree-12 vertices}\} - \#\{\text{positive degree-12 vertices}\})\,.\end{aligned}$$

Among the six edges incident to a degree-6 vertex v of a chart, three consecutive edges are incoming edges and the other three are outgoing edges. A *middle edge of* v means the middle one of the three incoming edges or the middle one of the three outgoing edges. A *non-middle edge of* v means an edge incident to v that is not a middle edge.

Let Γ be a chart in B. A path $\eta : [0,1] \to B$ is said to be *in general position with respect to* Γ if $\eta([0,1]) \cap \Gamma$ is empty or consists of some points of $\Gamma - \mathrm{Vert}(\Gamma)$ where the path η intersects edges of Γ transversely. To each intersection of η and Γ, assign a letter s_i^{ε} where i is the label of the intersecting edge of Γ and $\varepsilon = +1$ (or $\varepsilon = -1$, resp.) if η intersects the edge from right to left (or from left to right, resp.) with respect to the orientation of the edge. (Our convention for the sign ε follows that of [**5**], which is opposite to that in Chapter 18 of [**6**].) The *intersection word* of η with respect to Γ means a word obtained by reading the letters assigned to the intersections along η. This word will be denoted by $w_\Gamma(\eta)$ (cf. [**6**]). Regarding s_1 and s_2 as the matrices given in the introduction, we assume that $w_\Gamma(\eta)$ represents an element of $SL(2,\boldsymbol{Z})$.

DEFINITION 11. Let Γ be a chart in B missing a point y_0 of B, and let S_Γ be the set of degree-1 vertices of Γ. The *monodromy representation associated with* Γ is a homomorphism

$$\rho_\Gamma : \pi_1(B - S_\Gamma, y_0) \to SL(2,\boldsymbol{Z})$$

defined as follows: For an element $x \in \pi_1(B - S_\Gamma, y_0)$, take a representative path

$$\eta : [0,1] \to B - S_\Gamma$$

so that η is in general position with respect to Γ. We define $\rho_\Gamma(x)$ to be the element of $SL(2,\boldsymbol{Z})$ represented by the intersection word $w_\Gamma(\eta)$.

LEMMA 12. *The homomorphism $\rho_\Gamma : \pi_1(B - S_\Gamma, y_0) \to SL(2,\boldsymbol{Z})$ is well-defined.*

PROOF. Let η and η' be representatives of the same element x. If a homotopy connecting η and η' misses all of the vertices of Γ, then $w_\Gamma(\eta)$ and $w_\Gamma(\eta')$ represent the same element in the free group $\langle s_1, s_2 \rangle$. If the homotopy passes through vertices of degree 6 or 12, then the words differ by an element in the normal subgroup generated by $s_i s_j s_i s_j^{-1} s_i^{-1} s_j^{-1}$ and $(s_i s_j)^6$ for $\{i,j\} = \{1,2\}$. Thus $w_\Gamma(\eta)$ and $w_\Gamma(\eta')$ represent the same element of $SL(2,\boldsymbol{Z})$. $\square$

DEFINITION 13. A *Lefschetz fibration described by a chart Γ* is a Lefschetz fibration $f : M \to B$ with $S_f = S_\Gamma$ whose monodromy representation is equal to $\rho_\Gamma : \pi_1(B - S_\Gamma, y_0) \to SL(2,\boldsymbol{Z})$, up to inner automorphisms of $SL(2,\boldsymbol{Z})$.

Since we assume that a Lefschetz fibration has at least one critical value, we also assume that a chart Γ has at least one degree-1 vertex.

REMARK 14. Let Γ be a chart in B, and let y_0 and $y_0{}'$ be points of $B - \Gamma$. Take a path η connecting these two points which is in general position with respect to Γ. Then the two monodromy representations $\rho_\Gamma : \pi_1(B - S_\Gamma, y_0) \to SL(2,\boldsymbol{Z})$ and $\rho_\Gamma{}' : \pi_1(B - S_\Gamma, y_0{}') \to SL(2,\boldsymbol{Z})$ associated with the chart Γ with base points y_0 and $y_0{}'$ respectively are related by the following commutative diagram (where $\mathrm{conj}(g)(h) = g^{-1}hg$):

$$\begin{array}{ccc} \pi_1(B - S_\Gamma, y_0) & \xrightarrow{\rho_\Gamma} & SL(2,\boldsymbol{Z}) \\ {\scriptstyle \eta_\#}\downarrow & & \downarrow{\scriptstyle \mathrm{conj}(w_\Gamma(\eta))} \\ \pi_1(B - S_\Gamma, y_0{}') & \xrightarrow{\rho_\Gamma{}'} & SL(2,\boldsymbol{Z}). \end{array}$$

THEOREM 15. *Any Lefschetz fibration $f : M \to B$ over a closed surface B can be described by a chart Γ.*

PROOF. Take a regular value $y_0 \in B - S_f$ and consider a monodromy representation $\rho_f : \pi_1(B - S_f, y_0) \to SL(2,\boldsymbol{Z})$ of the Lefschetz fibration by identifying the mapping class group $\mathrm{MC}(f^{-1}(y_0))$ with $\mathrm{MC}(T^2) \cong SL(2,\boldsymbol{Z})$. Let $\alpha_1, \dots, \alpha_n$ and $\beta_1, \dots, \beta_{2g}$ be simple paths and simple closed paths, respectively, in B as in § 2, where n is the number of critical values of f and g is the genus of the base space B, and let $a_1, \dots, a_n, b_1, \dots, b_{2g}$ be the elements of $\pi_1(B - S_f, y_0)$ associated with them (cf. § 2). We decompose a regular neighborhood of $(\cup_{i=1}^n \alpha_i) \cup (\cup_{j=1}^{2g} \beta_j)$ in B into $n+1$ disks and $n+2g$ bands as follows: Recall that, for each i $(1 \le i \le n)$, the terminal point of α_i is a critical value, say y_i. Let $N(y_i)$ be a small regular neighborhood of y_i in B for i $(0 \le i \le n)$. Let $N'(\alpha_i)$ $(1 \le i \le n)$ be a regular neighborhood of $\alpha_i \cap (B - \cup_{k=0}^n \mathrm{Int} N(y_k))$ in $B - \cup_{k=0}^n \mathrm{Int} N(y_k)$, and let $N'(\beta_j)$ $(1 \le j \le 2g)$ be a regular neighborhood of $\beta_j \cap (B - \cup_{k=0}^n \mathrm{Int} N(y_k))$ in $B - \cup_{k=0}^n \mathrm{Int} N(y_k)$. Then the union of $n+1$ disks $N(y_0), \dots, N(y_n)$ and $n+2g$ bands $N'(\alpha_1), \dots, N'(\alpha_n), N'(\beta_1), \dots, N'(\beta_{2g})$ is a regular neighborhood of $(\cup_{i=1}^n \alpha_i) \cup (\cup_{j=1}^{2g} \beta_j)$ in B.

We construct a desired chart Γ piece by piece. Define $\Gamma \cap N(y_0)$ to be empty. For each i $(1 \le i \le n)$, the monodromy $\rho_f(a_i)$ is a conjugate of s_1 or s_1^{-1}. Take a word expression of $\rho_f(a_i)$, say $w_i s_1^{\varepsilon_i} w_i^{-1}$, where w_i is a word in $\{s_1, s_1^{-1}, s_2, s_2^{-1}\}$ and $\varepsilon_i \in \{-1, 1\}$. Define $\Gamma \cap N(y_i)$ to be a radial arc in $N(y_i)$ connecting the center y_i and a point of $\partial N(y_i)$ missing $N'(\alpha_i)$ whose label is 1 and it is oriented inward (or outward, resp.) if ε_i is -1 (or 1, resp.). Define $\Gamma \cap N'(\alpha_i)$ to be a union of some proper arcs in $N'(\alpha_i)$ missing $N(y_0)$ and $N(y_i)$ such that they are labeled and oriented so that the intersection word of α_i (restricted to $N'(\alpha_i)$) is equal to the word w_i. See Figure 3, where $w_i = s_1^{-1} s_2 s_2$ and $\varepsilon_i = 1$.

For each j $(1 \le j \le 2g)$, we define $\Gamma \cap N'(\beta_j)$ to be the union of some proper arcs in $N'(\beta_j)$ missing $N(y_0)$ which are labeled and oriented such that the intersection word of β_j (restricted to

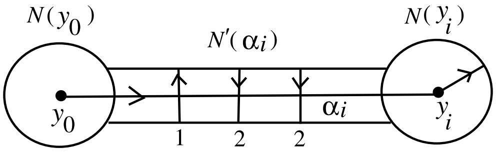

Figure 3.

$N'(\beta_j))$ is a word representing the monodromy $\rho_f(b_j)$.

We have constructed Γ on the neighborhood $N((\cup_{i=1}^n \alpha_i) \cup (\cup_{j=1}^{2g} \beta_j))$ of $(\cup_{i=1}^n \alpha_i) \cup (\cup_{j=1}^{2g} \beta_j)$. The edges of Γ (which have been constructed on $N((\cup_{i=1}^n \alpha_i) \cup (\cup_{j=1}^{2g} \beta_j)))$ intersect $\partial N((\cup_{i=1}^n \alpha_i) \cup (\cup_{j=1}^{2g} \beta_j))$ transversely. So we can consider the intersection word of the closed path $\partial N((\cup_{i=1}^n \alpha_i) \cup (\cup_{j=1}^{2g} \beta_j))$ with respect to Γ. By the construction, this word represents $\rho_f(a_1)\cdots\rho_f(a_n)([\rho_f(b_1),\rho_f(b_2)]\cdots[\rho_f(b_{2g-1}),\rho_f(b_{2g})])^{-1}$ in $SL(2,\boldsymbol{Z})$, which is the identity element of $SL(2,\boldsymbol{Z})$. Since $SL(2,\boldsymbol{Z})$ has a group presentation

$$\langle s_1, s_2 \,|\, s_1 s_2 s_1 (s_2 s_1 s_2)^{-1}, (s_1 s_2)^6 \rangle,$$

there exists a finite sequence of words in $\{s_1, s_1^{-1}, s_2, s_2^{-1}\}$ starting from the intersection word of the closed path $\partial N((\cup_{i=1}^n \alpha_i) \cup (\cup_{j=1}^{2g} \beta_j))$ with respect to Γ and terminating with the empty word such that each word is related to the previous one by one of the following transformations;

- insertion of $s_i s_i^{-1}$ or $s_i^{-1} s_i$ for $i \in \{1,2\}$,
- deletion of $s_i s_i^{-1}$ or $s_i^{-1} s_i$ for $i \in \{1,2\}$,
- insertion of $s_1 s_2 s_1 (s_2 s_1 s_2)^{-1}$ or $s_2 s_1 s_2 (s_1 s_2 s_1)^{-1}$,
- insertion of $(s_1 s_2)^6$ or $(s_1 s_2)^{-6}$.

(Note that deletion of $s_1 s_2 s_1 (s_2 s_1 s_2)^{-1}$ is obtained from insertion of $s_2 s_1 s_2 (s_1 s_2 s_1)^{-1}$ and deletion of $s_i s_i^{-1}$ and $s_i^{-1} s_i$. Deletion of $s_2 s_1 s_2 (s_1 s_2 s_1)^{-1}$, $(s_1 s_2)^6$ or $(s_1 s_2)^{-6}$ is also obtained from the transformations above.) Therefore, by the same argument as in p. 147 of [**5**] or in Chapter 18 of [**6**], we can extend the chart Γ constructed on $N((\cup_{i=1}^n \alpha_i) \cup (\cup_{j=1}^{2g} \beta_j))$ to a chart in a slightly bigger neighborhood $N'((\cup_{i=1}^n \alpha_i) \cup (\cup_{j=1}^{2g} \beta_j))$ of $(\cup_{i=1}^n \alpha_i) \cup (\cup_{j=1}^{2g} \beta_j)$ in B such that $\Gamma \cap \partial N'((\cup_{i=1}^n \alpha_i) \cup (\cup_{j=1}^{2g} \beta_j)) = \emptyset$. This is a desired chart Γ in B, since by construction, we have $\rho_\Gamma(a_i) = \rho_f(a_i)$ and $\rho_\Gamma(b_j) = \rho_f(b_j)$ for i $(1 \le i \le n)$ and j $(1 \le j \le 2g)$. □

4. Moves on charts.

In the previous section, we have seen that any Lefschetz fibration is described by a chart. Such a chart description is not unique. Here we introduce some moves on charts which do not change the isomorphism class of the Lefschetz fibration. These moves play an important role in our proof of the classification theorem.

LEMMA 16. *Let Γ and Γ' be charts in B. Suppose that there exists a* 2*-disk E in B such that Γ and Γ' are identical outside of E and that Γ and Γ' have no degree-*1 *vertices in E. Then Lefschetz fibrations described by Γ and Γ' have the same monodromy representation.*

PROOF. Take a point y_0 in $B-E$ missing Γ and Γ' and consider the monodromy representations $\rho_\Gamma : \pi_1(B-S,y_0) \to SL(2,\boldsymbol{Z})$ and $\rho_{\Gamma'} : \pi_1(B-S,y_0) \to SL(2,\boldsymbol{Z})$, where $S=\{y_1,\dots,y_n\}$ is the set of degree-1 vertices of Γ, which is equal to that of Γ'. Then $\rho_\Gamma = \rho_{\Gamma'}$. This is seen as follows: Let $\alpha_1,\dots,\alpha_n$ and $\beta_1,\dots,\beta_{2g}$ be paths and closed paths, respectively, in B as before. We may assume these paths to be disjoint from E. Then $\rho_\Gamma(a_i) = \rho_{\Gamma'}(a_i)$ and $\rho_\Gamma(b_j) = \rho_f(b_j)$ for i $(1 \le i \le n)$ and j $(1 \le j \le 2g)$. Thus $\rho_\Gamma = \rho_{\Gamma'}$. $\square$

DEFINITION 17. When two charts Γ and Γ' are in the situation of Lemma 16, we say that Γ is obtained from Γ' by a *CI-move* in E. A CI-move illustrated in Figure 4 is called a *channel change*.

Typical CI-moves are illustrated in Figure 5.

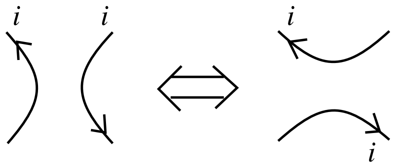

Figure 4. channel change.

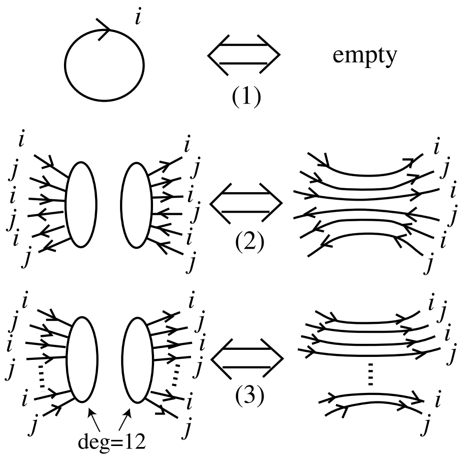

Figure 5. some CI-moves.

LEMMA 18. *Let Γ and Γ' be charts in B. Suppose that there exists a* 2*-disk E in B such that Γ and Γ' are identical outside of E and that Γ and Γ' differ by one of Figure* 6 *in E. Then Lefschetz fibrations described by Γ and Γ' have the same monodromy representation up to an equivalence.*

PROOF. By an isotopic deformation in E, we may assume that the degree-1 vertex of Γ in E and that of Γ' are located in the same position of E so that the set of degree-1 vertices of Γ is equal to that of Γ', say S. Take a point y_0 in $B-E$ missing Γ and Γ' and consider the monodromy

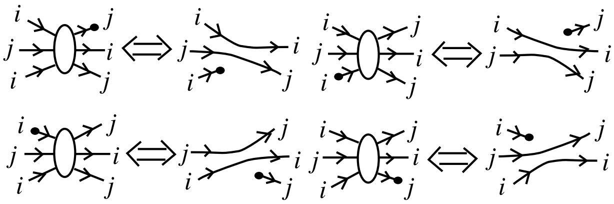

Figure 6. CII-moves.

representations $\rho_\Gamma : \pi_1(B-S, y_0) \to SL(2,\boldsymbol{Z})$ and $\rho_{\Gamma'} : \pi_1(B-S, y_0) \to SL(2,\boldsymbol{Z})$. Then we see that $\rho_\Gamma = \rho_{\Gamma'}$ by a similar argument as in the proof of the previous lemma. □

DEFINITION 19. When two charts Γ and Γ' are in the situation of Lemma 18, we say that Γ is obtained from Γ' by a *CII-move* in E.

Note that, for a CII-move, the edge connecting the degree-1 vertex and the degree-6 vertex in E is a non-middle edge of the degree-6 vertex.

By a *C-move*, we mean a CI-move, a CII-move or an isotopic deformation in B. Two charts are said to be *C-move equivalent* if they are related by a finite sequence of C-moves. By Lemmas 16 and 18, such charts describe Lefschetz fibrations with the same monodromy representation up to an equivalence. (By Theorem 21 (and the proof of Theorem 8), we see that the converse is also true under a certain condition; namely, two charts Γ and Γ' describing Lefschetz fibrations with equivalent monodromy representations are C-move equivalent provided that $n_+(\Gamma) = n_+(\Gamma')$, $n_-(\Gamma) = n_-(\Gamma')$ and $n_+(\Gamma) - n_-(\Gamma) \neq 0$, where n_+ and n_- stand for the numbers of positive and negative, respectively, degree-1 vertices of the chart.)

Note that CII-moves in this paper are called CIII-moves in [**5**] and [**6**].

Figure 7 is an example of a sequence of C-moves: The first move is a CI-move and the last two are CII-moves.

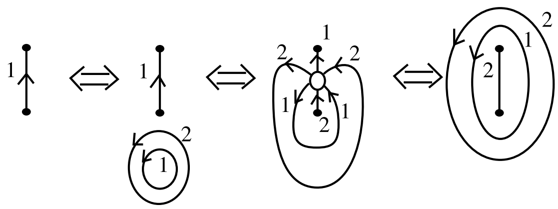

Figure 7. label change of a free edge.

5. The main theorem.

DEFINITION 20. A (d_1, d_2)*-type edge* (or a (d_1, d_2)*-edge*) of a chart Γ is an edge of Γ whose endpoints are vertices of degrees d_1 and d_2, where $d_1, d_2 \in \{1, 6, 12\}$ with $d_1 \leq d_2$.

A *free edge* of Γ is the union of a $(1,1)$-type edge and its endpoints.

An *oval nest* of Γ is the union of a free edge and some (or no) concentric hoops surrounding the free edge such that it is contained in a 2-disk, say E, in B and the remainder of Γ is outside E (cf. [**6**]).

A *nucleon* of Γ is the union of a degree-12 vertex, twelve degree-1 vertices and twelve $(1,12)$-type edges connecting these vertices.

A nucleon is *positive* (or *negative*, resp.) if the twelve degree-1 vertices are positive (or negative, resp.).

THEOREM 21. *Any chart Γ in B can be transformed, by C-moves, to a chart $\Gamma_0 \cup \Gamma_1$ in B satisfying the following conditions*:

(1) *There is a 2-disk, say E, in B such that Γ_0 is inside E and Γ_1 is outside E.*
(2) *Γ_0 consists of oval nests and nucleons.*
(3) *The nucleons of Γ_0 are all positive or all negative.*
(4) *Γ_1 does not have degree-1 vertices or degree-12 vertices.*

Furthermore, if Γ_0 has at least one nucleon, then applying C-moves, we can arrange so that all oval nests in Γ_0 are free edges with label 1 and that Γ_1 is empty.

We devote this section to proving this theorem.

LEMMA 22. *Any chart can be transformed by C-moves to a chart without $(1,6)$-type edges.*

The basic idea of this lemma and its proof is the same as that of Proposition 21 and Lemma 24 in [**5**]. The proof gives an algorithm to reduce the number of the $(1,6)$-type edges from an arbitrary chart with $(1,6)$-type edges. This process will be referred to as the *reduction process of $(1,6)$-type edges*.

PROOF. (Step 1) Suppose that there exists a $(1,6)$-type edge which is a non-middle edge of the degree-6 vertex. Apply a CII-move and eliminate the degree-6 vertex. Then one of the following occurs.

- The number of $(1,6)$-type edges decreases. So does the number of degree-6 vertices.
- The number of $(1,6)$-type edges is unchanged, but the number of degree-6 vertices decreases.

Thus the total number of $(1,6)$-type edges and degree-6 vertices decreases.

(Step 2 (1)) Suppose that there exists a $(1,6)$-type edge which is a middle edge of the degree-6 vertex. We denote the $(1,6)$-type edge by e, the degree-1 vertex by v_0 and the degree-6 vertex by v_1. We only consider the case where e is an outgoing edge of v_1, since the other case where e is an incoming edge of v_1 is treated in the same way.

Let K be the region of $B - \Gamma$ whose closure $\mathrm{Cl}(K)$ in B contains v_0, v_1 and e. Let $\overline{K}$ be a 'completion' of K; namely, $\overline{K}$ is a compact oriented surface with a map $\iota : \overline{K} \to B$ such that the restriction $\iota|\mathrm{Int}(\overline{K}) : \mathrm{Int}(\overline{K}) \to K$ is an orientation preserving homeomorphism. By a *boundary loop of* K we mean a loop in B that is the image under ι of a boundary loop of the completion of K. Here we assume that the orientation of a boundary loop is equal to the orientation induced from $\overline{K}$.

Let $f_1, f_2, \ldots, f_m$ be a sequence of edges of Γ with signs (in the exponential notation) such that the edges appear in this order when we walk along the unique boundary loop of K starting from v_0 and that the sign of an edge is positive if the orientation of the edge is the same as the orientation of the boundary loop; otherwise the sign is negative. For example, see Figure 8, where K is an open disk oriented counterclockwise in the picture. For this K, the length m is equal to 11 and the sequence $f_1, \ldots, f_m$ is

$$e_1^{-1}, e_2^{+1}, e_3^{+1}, e_4^{-1}, e_4^{+1}, e_5^{-1}, e_5^{+1}, e_6^{-1}, e_7^{+1}, e_8^{-1}, e_1^{+1}.$$

For such a sequence $f_1, \ldots, f_m$, we consider a sequence $w(f_1), \ldots, w(f_m)$ of elements of $SL(2, \boldsymbol{Z})$ defined by $w(f_k) = s_i^{\varepsilon}$ for k $(1 \le k \le m)$ where $i \in \{1, 2\}$ is the label of the edge f_k and ε is the sign of f_k. For the above example, $w(f_1), \ldots, w(f_m)$ is

$$s_1^{-1}, s_2^{+1}, s_1^{+1}, s_2^{-1}, s_2^{+1}, s_1^{-1}, s_1^{+1}, s_2^{-1}, s_1^{+1}, s_2^{-1}, s_1^{+1}.$$

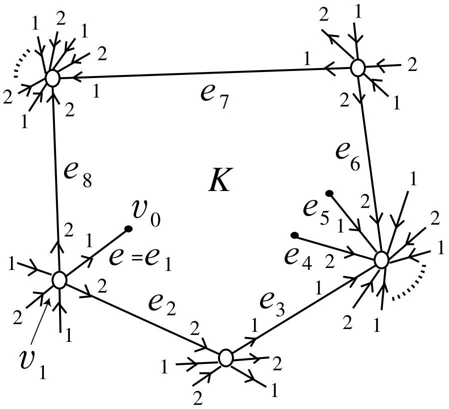

Figure 8. boundary loop of K.

Let i be the label of e and let j be the complementary label with $\{i, j\} = \{1, 2\}$. Since we are assuming that e is an outgoing edge of v_1, we have $w(f_1) = s_i^{-1}$ and $w(f_m) = s_i^{+1}$.

(Case 1) Suppose that there is a consecutive pair f_k and f_{k+1} in the sequence $f_1, \ldots, f_m$ for some k such that $w(f_k) = s_j^{+1}$ and $w(f_{k+1}) = s_i^{+1}$. In this case, the vertex between the edges f_k and f_{k+1} must be a degree-6 vertex, and f_k and f_{k+1} are non-middle edges of this vertex (see Figure 9). Move the vertex v_0 and the edge e toward the edge f_{k+1} in K by an isotopic deformation, and apply a channel change between the edges e and f_{k+1} as in Figure 9. In the result, the vertex v_0 is incident to a $(1, 6)$-type edge which is a non-middle edge of a degree-6 vertex. Go back to Step 1.

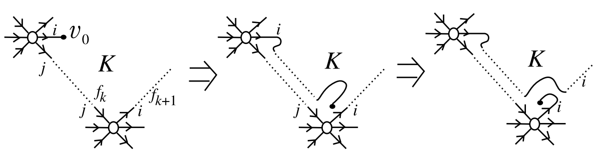

Figure 9. reduction of $(1, 6)$-type edge, 1.

(Case 2) Suppose that there is a consecutive pair f_k and f_{k+1} in the sequence $f_1, \ldots, f_m$ for some k such that $w(f_k) = s_i^{-1}$ and $w(f_{k+1}) = s_j^{-1}$. In this case, the vertex between the edges f_k and f_{k+1} must be a degree-6 vertex, and f_k and f_{k+1} are non-middle edges of this vertex (see Figure 10). Move the vertex v_0 and the edge e toward the edge f_k in K by an isotopic deformation, and apply a channel change between the edges e and f_k as in Figure 10. In the result, the vertex

v_0 is incident to a $(1,6)$-type edge which is a non-middle edge of a degree-6 vertex. Go back to Step 1.

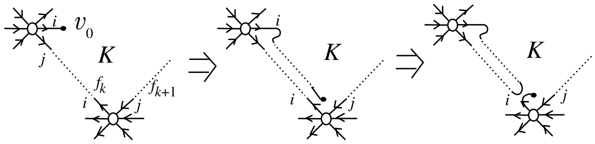

Figure 10. reduction of $(1,6)$-type edge, 2.

(Step 2 (2)) By Step 2 (1), we may assume that

(*) there is no consecutive pair f_k and f_{k+1} in the sequence $f_1,\dots,f_m$ such that $(w(f_k),w(f_{k+1})) = (s_j^{+1},s_i^{+1})$ or (s_i^{-1},s_j^{-1}).

Suppose that there is no edge in $f_1,\dots,f_m$ except f_1 and f_m incident to a degree-1 vertex; i.e., suppose that all vertices (except v_0) incident to the edges $f_1,\dots,f_m$ are vertices of degree 6 or 12. In this case, for each consecutive pair f_k and f_{k+1}, the labels of f_k and f_{k+1} are distinct. Since $w(f_1) = s_i^{-1}$ and since we assume the condition (*), the sequence $w(f_1),\dots,w(f_m)$ must be of the form

$$s_i^{-1}, s_j^{+1}, s_i^{-1}, s_j^{+1}, \dots.$$

This contradicts that $w(f_m) = s_i^{+1}$.

Therefore there must be an edge in $f_1,\dots,f_m$ except f_1 and f_m incident to a degree-1 vertex. Let f_k be the first one among such edges. Note that f_k and f_{k+1} are the same edge with opposite signs. Note also that, $w(f_{k'}) = s_i^{-1}$ for any odd integer k' with $1 \le k' \le k$, and $w(f_{k'}) = s_j^{+1}$ for any even integer k' with $2 \le k' \le k$. Therefore, according as k is even or odd, there are two cases as follows:

(1) $(w(f_{k-1}),w(f_k),w(f_{k+1})) = (s_i^{-1},s_j^{+1},s_j^{-1})$, or
(2) $(w(f_{k-1}),w(f_k),w(f_{k+1})) = (s_j^{+1},s_i^{-1},s_i^{+1})$.

Let v be the vertex incident to f_{k-1} and f_k, which is of degree 6 or 12.

(Case 1) For the case (1) with $k \ge 3$, move v_0 and the edge e toward the edge f_{k-1} in K by an isotopic deformation and apply a channel change between the edges e and f_{k-1}. Let f'_{k-1} denote the new edge incident to v obtained by this move. If v is a degree-12 vertex, then this move reduces the number of the $(1,6)$-type edges. If v is a degree-6 vertex, then f'_{k-1} or f_k is a non-middle edge of v, and these edges are $(1,6)$-type edges. Thus we can go back to Step 1.

For the case (1) with $k = 2$, note that f_2 and f_3 are the same edge (with the opposite signs) that is a $(1,6)$-type edge and is a non-middle edge of a degree-6 vertex. Go back to Step 1.

(Case 2) For the case (2), move v_0 and the edge e toward the edge f_k in K by an isotopic deformation and apply a channel change between the edges e and f_k. Then we obtain a free edge and reduce the number of the $(1,6)$-type edges.

By repetition of this procedure, we can eliminate the $(1,6)$-type edges from a chart. □

LEMMA 23. *Let $\Gamma = \Gamma_0 \cup \Gamma_1$ be a chart in B such that Γ_0 is contained in a 2-disk E in B and that Γ_1 is outside E. Up to C-moves, we can move Γ_0 into an arbitrary region of $B - \Gamma_1$ by adding Γ_0 some hoops surrounding it.*

PROOF. By an isotopic deformation and a single channel change as in Figure 11, we can move Γ_0 to the next region of $B-\Gamma_1$. This yields a single hoop surrounding Γ_0. By this process, we see the result. □

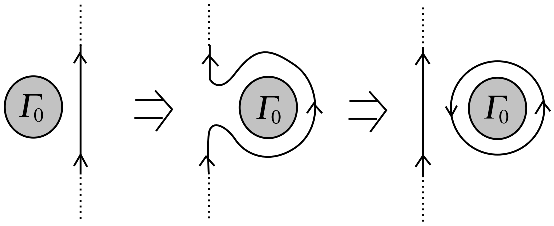

Figure 11. passing process.

LEMMA 24. (1) *We can move an oval nest of a chart into any region by adding additional hoops.*

(2) *We can move a nucleon of a chart into any region.*

PROOF. The assertion (1) is a direct consequence of Lemma 23 (cf. Lemma 19 of [**5**]). For (2), by Lemma 23, we can move a nucleon by adding some hoops. Applying a channel change between the innermost hoop surrounding the nucleon and an edge of the nucleon with the same label, we can remove the innermost hoop. Repeat this, until all hoops are removed. □

LEMMA 25. *The local replacement illustrated in Figure* 12 *is a CI-move, where the vertex is a degree*-12 *vertex and the shaded box means a subchart with four degree*-6 *vertices illustrated in Figure* 13*.*

PROOF. It is obvious by the definition of a CI-move. □

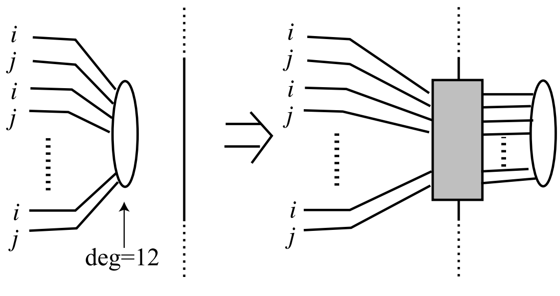

Figure 12.

PROOF OF THEOREM 21. By moves in Lemma 25 (Figure 12) and a CI-move illustrated in (3) of Figure 5, we can remove any pair of a positive degree-12 vertex and a negative one. Thus, the chart is transformed to a chart such that there exist no degree-12 vertices or such that the degree-12 vertices are all positive or all negative.

First we consider the case where there exist no degree-12 vertices in Γ. By the reduction process of $(1,6)$-edges in Lemma 22, remove all $(1,6)$-type edges from the chart. This process does not yield degree-12 vertices. Thus every degree-1 vertex is connected with another degree-1

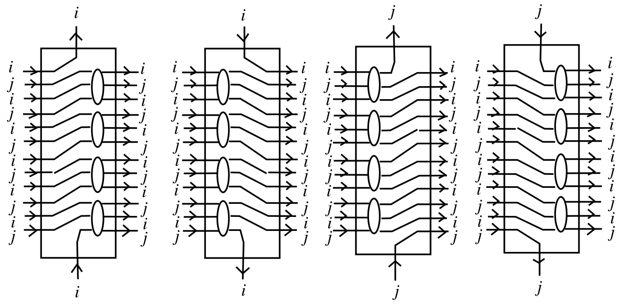

Figure 13.

vertex by a free edge. Let E be a 2-disk in B which is disjoint from the chart. By Lemma 24(1), we move all free edges into E and they become oval nests in E, that form a desired chart Γ_0. The remainder has neither degree-1 vertices nor degree-12 vertices, which is a desired chart Γ_1.

We consider the case where all degree-12 vertices are negative and there exists at least one degree-12 vertex. (The case where all degree-12 vertices are positive is treated similarly.) By the reduction process of $(1,6)$-edges in Lemma 22, remove all $(1,6)$-type edges from the chart. Since there exist no positive degree-12 vertices, all negative degree-1 vertices are connected with positive degree-1 vertices by $(1,1)$-type edges to form free edges. Let E be a 2-disk in B which is disjoint from the chart. Move the free edges into a 2-disk E as oval nests. There exist no negative degree-1 vertices outside E. By the equation

$$\begin{aligned}&\#\{\text{positive degree-1 vertices}\} - \#\{\text{negative degree-1 vertices}\}\\ &\quad = 12\,(\#\{\text{negative degree-12 vertices}\} - \#\{\text{positive degree-12 vertices}\}),\end{aligned}$$

we see that the number of (positive) degree-1 vertices outside of E is 12 times the number of (negative) degree-12 vertices. Therefore all (positive) degree-1 vertices outside E and all (negative) degree-12 vertices are connected by $(1,12)$-type edges and form nucleons. Move these nucleons into E by Lemma 24(2). The remainder has neither degree-1 vertices nor degree-12 vertices. This completes the proof of the first assertion of the theorem.

Now we prove the second assertion. Consider that $\Gamma = \Gamma_0 \cup \Gamma_1$ is as in the first assertion and suppose that Γ_0 has at least one nucleon. By C-moves illustrated in Figure 7, we may assume that all free edges in the oval nests have label 1. For an oval nest with some hoops, apply a channel change between the outermost hoop of the oval nest and an edge of a nucleon with the same label. Then the outermost hoop is removed. Repeat this, and we can remove all hoops from the oval nests. Now Γ_0 consists of free edges with label 1 and nucleons.

We prove that the chart Γ_1 lying outside E can be transformed to the empty set by induction on the number of degree-6 vertices of Γ_1.

If the number of degree-6 vertices of Γ_1 is 0, then Γ_1 has no vertices and hence it consists of hoops. Move a nucleon of Γ_0 from E toward a hoop of Γ_1. By a channel change between the hoop and an edge of the nucleon with the same label, we can remove the hoop from Γ_1. Continue this, until all hoops are removed and then Γ_1 is empty. Move the nucleon back into E.

Now we suppose that there exists at least one degree-6 vertex of Γ_1. Let v be a degree-6 vertex of Γ_1 and let e be a non-middle incoming edge of v (or a non-middle outgoing edge of v, resp.) when the nucleons are positive (or negative, resp.). By Lemma 24(2), we move a nucleon of Γ_0 from E toward the edge e and apply a channel change between the edge e and an edge of the nucleon with the same label. Then the edge e and the edge of the nucleon change into a $(1,6)$-edge and a $(6,12)$-type edge. By a CII-move, we remove the degree-6 vertex v. Then apply the reduction process of $(1,6)$-type edges in Lemma 22 to this chart in $B-E$. Since this process does not increase the number of degree-6 vertices and since there exist a single degree-12 vertex and 12 degree-1 vertices in this chart, after the reduction process of $(1,6)$-type edges, we obtain a nucleon again in $B-E$, and the remainder in $B-E$ is a chart which has fewer degree-6 vertices than Γ_1. Move the nucleon back into E. By induction hypothesis, we see that Γ_1 can be transformed to the empty set. □

6. The classification theorem.

Using Theorem 21, we show the results on classification of Lefschetz fibrations stated in § 1.

Theorems 6 and 7 are special cases of Theorem 8. We prove Theorem 8 by use of chart description.

PROOF OF THEOREM 8. The only if part is trivial. We prove the if part. Suppose that $g(B)=g(B')$, $n_+(f)=n_+(f')$, $n_-(f)=n_-(f')$ and that $n_+(f)-n_-(f)\neq 0$. Consider charts Γ in B and Γ' in B' describing the Lefschetz fibrations f and f'. By Theorem 21, Γ and Γ' are transformed to $\Gamma_0\cup\Gamma_1$ and $\Gamma'_0\cup\Gamma'_1$ as in the first assertion of Theorem 21, respectively. Since $n_+(f)-n_-(f)\neq 0$, there must be some nucleons in Γ_0. Thus, by the second assertion of Theorem 21, we may assume that Γ_0 consists of free edges with label 1 and nucleons and Γ_1 is empty. The number of free edges is $\min\{n_+(f),n_-(f)\}$. The number of degree-1 vertices appearing in the nucleons is $n_+(f)+n_-(f)-2\min\{n_+(f),n_-(f)\}$ $(=|n_+(f)-n_-(f)|)$. The number of nucleons is this number divided by 12. The nucleons in Γ_0 are all positive if $n_+(f)-n_-(f)>0$, or all negative if $n_+(f)-n_-(f)<0$. We assume that Γ is a chart of this form. This situation is the same for Γ'. Then there exits an orientation preserving diffeomorphism from B to B' which maps Γ to Γ', and by Theorem 9, we see that f and f' are isomorphic. □

The following theorem was treated in [**14**], which is a generalization of Theorem 5. Using the chart description method, we have a remarkably quick proof to this.

THEOREM 26. *Let $g_1,g_2,\dots,g_n$ be elements of $SL(2,\mathbf{Z})$ which are conjugates of s_1 or s_1^{-1} with $g_1g_2\cdots g_n=1$. Let n_+ (or n_-, resp.) be the number of g_k $(1\leq k\leq n)$ such that g_k is a conjugate of s_1 (or s_1^{-1}, resp.). So $n_++n_-=n$.*

(1) *Suppose that $n_+\neq n_-$. Then n_+-n_- is a multiple of 12, say $n_+-n_-=12\varepsilon m$ for $\varepsilon\in\{+1,-1\}$ and for a positive integer m. By successive application of elementary transformations, the n-tuple $(g_1,g_2,\dots,g_n)$ can be transformed to an n-tuple $(h_1,h_2,\dots,h_n)$ with*

$$h_k=\begin{cases} s_1^{\varepsilon} & \text{for odd } k \text{ with } 1\leq k\leq 12m,\\ s_2^{\varepsilon} & \text{for even } k \text{ with } 1\leq k\leq 12m,\\ s_1 & \text{for odd } k \text{ with } 12m<k\leq n,\\ s_1^{-1} & \text{for even } k \text{ with } 12m<k\leq n.\end{cases}$$

(2) *Suppose that $n_+ = n_-$. By successive application of elementary transformations, the n-tuple $(g_1, g_2, \dots, g_n)$ can be transformed to an n-tuple*

$$(w_1, w_1^{-1}, w_2, w_2^{-1}, \dots, w_{n_+}, w_{n_+}^{-1})$$

for some elements $w_1, \dots, w_{n_+}$ which are conjugates of s_1.

PROOF. Let B be a 2-disk and let $S = \{y_1, \dots, y_n\}$ be a set of n interior points of B. Take a point y_0 of ∂B and consider a Hurwitz path system, $\alpha_1, \dots, \alpha_n$, connecting y_0 and the points of $\{y_1, \dots, y_n\}$, and consider the associated generator system, $a_1, \dots, a_n$, of $\pi_1(B - S, y_0)$.

By the argument of the proof of Theorem 15, there exists a chart Γ in B such that the monodromy representation $\rho_\Gamma : \pi_1(B - S, y_0) \to SL(2, \mathbf{Z})$ satisfies $\rho_\Gamma(a_k) = g_k$ for k $(1 \le k \le n)$. Note that Γ has n_+ positive degree-1 vertices and n_- negative degree-1 vertices.

We note that when Γ is transformed to a chart Γ' by C-moves and isotopic deformations in B, the monodromy representations ρ_Γ and $\rho_{\Gamma'}$ are equivalent in the sense of [**6**] (p. 127) and the n-tuple $(g_1, g_2, \dots, g_n) = (\rho_\Gamma(a_1), \rho_\Gamma(a_2), \dots, \rho_\Gamma(a_n))$ is transformed to $(\rho_{\Gamma'}(a_1), \rho_{\Gamma'}(a_2), \dots, \rho_{\Gamma'}(a_n))$ by successive application of elementary transformations (cf. [**17**], p. 127 of [**6**]).

Now, transform the chart Γ to a chart $\Gamma' = \Gamma_0 \cup \Gamma_1$ as in the first assertion of Theorem 21. Applying a CI-move, we may assume that $\Gamma_1 = \emptyset$. Then $|n_+ - n_-|$ is the number of the degree-1 vertices appearing in the nucleons of Γ_0. Thus $n_+ - n_- = 12\varepsilon m$, where $\varepsilon = +1$ (or -1, resp.) if the nucleons of Γ_0 are positive (or negative, resp.) and m is the number of the nucleons.

(1) Suppose that $n_+ \neq n_-$. Then $m \neq 0$, and there exists at least one nucleon in Γ_0. By the second assertion of Theorem 21, we may assume that all oval nests of Γ_0 are free edges with label 1. By an isotopic deformation in B, we can move the chart Γ' so that the monodromy representation $\rho_{\Gamma'} : \pi_1(B - S, y_0) \to SL(2, \mathbf{Z})$ satisfies

$$\rho_{\Gamma'}(a_k) = \begin{cases} s_1^\varepsilon & \text{for odd } k \text{ with } 1 \le k \le 12m, \\ s_2^\varepsilon & \text{for even } k \text{ with } 1 \le k \le 12m, \\ s_1 & \text{for odd } k \text{ with } 12m < k \le n, \\ s_1^{-1} & \text{for even } k \text{ with } 12m < k \le n. \end{cases}$$

Therefore we have the first assertion.

(2) Suppose that $n_+ = n_-$. Then there exist no nucleons in Γ_0, and Γ_0 consists of oval nests. The number of oval nests is equal to n_+. By C-moves illustrated in Figure 7, we may assume that all free edges in the oval nests have label 1. By an isotopic deformation in B, we can move Γ' so that the monodromy representation $\rho_{\Gamma'} : \pi_1(B - S, y_0) \to SL(2, \mathbf{Z})$ satisfies that $(\rho_{\Gamma'}(a_1), \rho_{\Gamma'}(a_2), \dots, \rho_{\Gamma'}(a_{n-1}), \rho_{\Gamma'}(a_n))$ is

$$(w_1, w_1^{-1}, w_2, w_2^{-1}, \dots, w_{n_+}, w_{n_+}^{-1})$$

for some elements $w_1, \dots, w_{n_+}$ which are conjugates of s_1. $\square$

REMARK 27. Our main theorem (Theorem 21), or Theorem 26, can be used in order to describe and simplify the monodromy representations of Lefschetz fibrations f with $n_+(f) -$

$n_-(f) = 0$. In this case, we have the same number of singular fibers of type I_1^+ and of type I_1^-. We can make them in pairs with trivial surrounding monodromy consisting of a I_1^+ singular fiber and a I_1^- type singular fiber. Such a pair corresponds to a free edge in the chart description. It can be deformed to make a "twin" type singular fiber ([**14**]). Then the classification is reduced to that of torus fibrations with twin singular fibers. The classification problem of diffeomorphism types of the total spaces of such fibrations is treated in [**4**] when the base space is a sphere, or in [**20**] when the 1st Betti number of the total space is odd.

Some interesting topics are also found in [**1**], [**8**], [**12**] and [**18**].

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References

[1] R. Gompf, Sums of elliptic surfaces, J. Differential Geom., **34** (1991), 93–114.

[2] R. E. Gompf and A. I. Stipsicz, 4-manifolds and Kirby calculus, Grad. Stud. Math., **20**, Amer. Math. Soc., Providence, RI, 1999.

[3] J. Harer, Pencils of curves in 4-manifolds, Thesis, UCB, 1979.

[4] Z. Iwase, Good torus fibrations with twin singular fibers, Japan. J. Math., **10** (1984), 321–352.

[5] S. Kamada, Surfaces in R^4 of braid index three are ribbon, J. Knot Theory Ramifications, **1** (1992), 137–160.

[6] S. Kamada, Braid and knot theory in dimension four, Math. Surveys Monogr., **95**, Amer. Math. Soc., Providence, RI, 2002.

[7] A. Kas, On the deformation types of regular elliptic surfaces, In: Complex analysis and algebraic geometry, Iwanami Shoten, Tokyo, 1977, pp. 107–111.

[8] A. Kas, On the handlebody decomposition associated to a Lefschetz fibration, Pacific J. Math., **89** (1980), 89–104.

[9] K. Kodaira, On compact analytic surfaces II, Ann. of Math., **77** (1963), 563–626.

[10] K. Kodaira, On compact analytic surfaces III, Ann. of Math., **78** (1963), 1–40.

[11] W. Magnus, A. Karrass and D. Solitar, Combinatorial group theory, Second revised edition, Dover Publ. Inc., New York, 1976.

[12] R. Mandelbaum and J. R. Harper, Global monodromy of elliptic Lefschetz fibrations, In: Current trends in algebraic topology, pp. 35–41, CMS Conf. Proc., **2**, Amer. Math. Soc., Providence, RI, 1982.

[13] Y. Matsumoto, On 4-manifolds fibered by tori, II, Proc. Japan Acad., **59** (1983), 100–103.

[14] Y. Matsumoto, Torus fibrations over the 2-sphere with the simplest singular fibers, J. Math. Soc. Japan, **37** (1985), 605–636.

[15] Y. Matsumoto, Diffeomorphism types of elliptic surfaces, Topology, **25** (1986), 549–563.

[16] Y. Matsumoto, Lefschetz fibrations of genus two – A topological approach, In: Topology and Teichmüller spaces (eds. S. Kojima et al.), Proceeding of the 37-th Taniguchi Sympo., 1996, World Scientific Publ. Co., River Edge, NJ, pp. 123–148.

[17] B. Moishezon, Complex surfaces and connected sums of complex projective planes, Lecture Notes in Math., **603**, Springer, Berlin-New York, 1977.

[18] K. Sakamoto and S. Fukuhara, Classification of T^2-bundles over T^2, Tokyo J. Math., **6** (1983), 311–327.

[19] B. Siebert and G. Tian, On hyperelliptic C^∞-Lefschetz fibrations of four-manifolds, Commun. Contemp. Math., **1** (1999), 255–280.

[20] M. Toda, Diffeomorphism types of good torus fibrations with twin singular fibers, J. Math. Sci. Univ. Tokyo, **8** (2001), 365–401.

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