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Borel summability of formal solutions of some first order singular partial differential equations and normal forms of vector fields

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Abstract. Let $L=\sum_{i=1}^{d} X_i(z)\partial_{z_i}$ be a holomorphic vector field degenerating at $z=0$ such that Jacobi matrix $((\partial X_i/\partial z_j)(0))$ has zero eigenvalues. Consider $Lu=F(z,u)$ and let $\tilde{u}(z)$ be a formal power series solution. We study the Borel summability of $\tilde{u}(z)$, which implies the existence of a genuine solution $u(z)$ such that $u(z)\sim\tilde{u}(z)$ as $z\to 0$ in some sectorial region. Further we treat singular equations appearing in finding normal forms of singular vector fields and study to simplify L by transformations with Borel summable functions.

0. Introduction.

Let $L=\sum_{i=1}^{d} X_i(z)\partial_{z_i}$ be a holomorphic vector field in a neighborhood of the origin in $\boldsymbol{C}^d$ and which is singular at the origin, that is, $X_i(0)=0$ for all $1\le i\le d$. Let $F(z,u)$ be a holomorphic function in a neighborhood of $(z,u)=(0,0)\in\boldsymbol{C}^{d+1}$. Let us consider $Lu=F(z,u)$, which is a singular first order semilinear partial differential equation. Let $\Sigma=\{z; X_1(z)=\cdots=X_d(z)=0\}$ and $\big((\partial X_i/\partial z_j)(z)\big)$ be the Jacobi matrix of $(X_1(z),X_2(z),\cdots,X_d(z))$. Suppose that Σ is a submanifold with codim $\Sigma=d_1$. Set $d_0=\operatorname{rank}\big((\partial X_i/\partial z_j)(0)\big)$ and let $\{\lambda_i\}_{i=1}^{d_0}$ be nonzero eigenvalues of $\big((\partial X_i/\partial z_j)(0)\big)$. Since L is singular, Cauchy Kowalevsky's Theorem is not available. The existence of holomorphic solutions of equations of this type was studied under the condition $d_0=d_1$ and Poincaré's condition on $\{\lambda_i\}_{i=1}^{d_0}$ (see [**8**], [**9**] and [**12**]). However there are formal series solutions in many other cases. In general we can not expect the convergence of these formal solutions. Gevrey type estimates of coefficients of formal solutions were obtained in [**18**]. One of our aims is to give an analytical meaning of formal solutions. In the present paper we study L with $d_1=d_0+1$. Let $\tilde{u}(z)\in\boldsymbol{C}[[z]]$ be a formal solution. For our aims firstly we simplify L by holomorphic local coordinates transformations. We show in this paper under some additional conditions that we can find a holomorphic local coordinates system $(x(z),y(z),t(z))\in\boldsymbol{C}^{d_0}\times\boldsymbol{C}^{d-d_1}\times\boldsymbol{C}$, $x(0)=y(0)=t(0)=0$, such that $\Sigma=\{x_1(z)=\cdots=x_{d_0}(z)=t(z)=0\}$ and a solution $u(x,y,t)$ which is holomorphic in

$$\{(x,y);|x|<r,\ |y|<r\}\times\{0<|t|<r_0,|\arg t-\theta|<\pi/2\gamma+\delta\} \tag{0.1}$$

for some θ and $\delta>0$, where $\gamma>0$ is a constant determined by L. Further it holds that $u(x,y,t)$ has an asymptotic expansion $u(x,y,t)\sim\sum_{n=0}^{\infty}u_n(x,y)t^n$ as $t\to 0$ in this sectorial region with remainder estimate of Gevrey type

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$$\left|u(x,y,t)-\sum_{n=0}^{N-1}u_n(x,y)t^n\right|\le AB^N\Gamma\left(\frac{N}{\gamma}+1\right)|t|^N \tag{0.2}$$

and $u(x(z),y(z),t(z))=\tilde{u}(z)$ as a formal series (Theorem 1.5).

As a special but important case, singular semilinear equations appear in finding normal forms of singular vector fields (see [**1**], [**8**]), which are more profoundly studied than general ones. It is well known that if all the eigenvalues of $\big((\partial X_i/\partial z_j)(0)\big)$ are non zero and distinct, then the formal solutions (formal transformations to normal forms) are convergent under Poincaré's condition, more generally, Siegel's one [**16**] or Bruno's one [**7**]. Furthermore we assume $d_0=d-1$ and $d_1=d$, so zero is a simple eigenvalue of $\big((\partial X_i/\partial z_j)(0)\big)$. The other aim of this paper is to simplify L, by using not only holomorphic functions in a full neighborhood of the origin but also holomorphic functions with asymptotic expansion in a sectorial region, so we find a normal form of L (Theorems 1.7 and 1.8).

There are several definitions and notions concerning formal series and functions with asymptotic series. The theory of the multi-summability of formal series has recently developed (see [**2**], [**3**]) and is important in the theory of functions with asymptotic series. Borel summability is a special case of multi-summability, that is, it is one-summability. It is shown in [**4**], [**5**] and [**6**] that formal power series solutions of ordinary differential equations are multi-summable, which also means the existence of genuine solutions in some strict sense. As for partial differential equations, the relation between formal solutions and genuine solutions were studied in [**13**] and [**14**], but the multi-summability of formal solutions was not investigated. So the sector where the asymptotic expansion is valid is not wide and there are many genuine solutions with the same asymptotic expansion. There are few results about multi-summability of formal solutions of partial differential equations. Borel summability of formal solutions of Cauchy problem of heat equation was studied in [**11**], and it is shown in [**15**] that formal solutions are multi-summable for some class of partial differential equations. We adopt in this paper the notion of Borel summability and study formal solutions of singular first order semilinear equations. We note that (0.1) and (0.2) mean that $u(x,y,t)$ is Borel summable with respect to t.

The contents of this paper is the following:

1. Notations, definitions and main results.
2. Singular first order partial differential equations on sectorial regions.
3. Coordinates transformations by holomorphic functions.
4. Normal forms of some singular vector fields by transformations with holomorphic functions on a sectorial region.
5. Borel and Laplace transforms, convolution and majorant functions.
6. Proofs of Theorems 2.3 and 2.4.
7. Existence of solutions of singular differential equations.

We study in Section 2 the existence of solutions of some first order semilinear partial differential equation on a sectorial region and give Theorems 2.3 and 2.4 which are tools to show main results. Their proofs need the theory of Borel transform and Laplace transform of holomorphic functions on sectorial regions and majorant functions. Hence they are given in section 6. In Sections 3 and 4 we start discussions by assuming Theorems 2.3 and 2.4. In Section 3 we transform L to the operator studied in Section 2 by holomorphic coordinates transformations and show one of the main results (Theorem 1.5). In Section 4 we further transform L by holomorphic functions on a sectorial region (Borel summable functions) and show the other main results (Theorems 1.7 and 1.8). We prepare in Section 5 for proving the theorems in Section 2, that is, we give briefly the properties of Borel transform, Laplace transform and majorant functions. We devote Section 6 to the proofs of Theorems 2.3 and 2.4. We use in Sections 2, 3 and 4 the results about existence of solutions of some singular semilinear ordinary or partial differential equations, so we summarize them in Section 7.

1. Notations, definitions and main results.

In this section we give notations and definitions. $\boldsymbol{N} = \{0,1,2,\cdots\}$ is the set of all nonnegative integers. For open sets V and U, $V \Subset U$ means that $\overline{V}$ is compact and $\overline{V} \subset U$. Firstly let us introduce spaces of formal series and those of holomorphic functions on sectorial regions related to formal series. We denote by $\mathscr{O}(\Omega)$ the set of all holomorphic functions on a region Ω. Let $t \in \boldsymbol{C}$. For $\theta \in \boldsymbol{R}$ and $r,\delta > 0$ set $S(\theta,\delta,r) = \{0 < |t| < r; |\arg t - \theta| < \delta\}$, $S(\theta,\delta) = S(\theta,\delta,\infty)$ and $S_{\{0\}}(\theta,\delta) = \{t \in S(\theta,\delta); 0 < |t| < \rho(\arg t)\}$, where $\rho(\cdot) > 0$ is some positive continuous function on $(\theta-\delta,\theta+\delta)$, which is called a sectorial neighborhood of $t = 0$ in $S(\theta,\delta)$.

Let $x = (x_1,x_2,\cdots,x_n) \in \boldsymbol{C}^n$ and $|x| = \max\{|x_i|; 1 \le i \le n\}$. For a multi-index $\alpha = (\alpha_1,\cdots,\alpha_n) \in \boldsymbol{N}^n$, $|\alpha| = \sum_{i=1}^n \alpha_i$ and $x^\alpha = x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_n^{\alpha_n}$. A series $\tilde{f}(x) = \sum_{\alpha\in\boldsymbol{N}^n} f_\alpha x^\alpha$, $f_\alpha \in \boldsymbol{C}$, is called a formal power series in x. The set of all such formal power series is denoted by $\boldsymbol{C}[[x]]$. The totality of all convergent power series in x, that is, all holomorphic functions in a neighborhood of $x = 0$, is denoted by $\boldsymbol{C}\{x\}$. Let $U \subset \boldsymbol{C}^n$ be an open polydisk centered at the origin and the set of all such polydisks is denoted by $\mathfrak{U}_0$. The set of all formal series power series in one variable t with coefficients in $\mathscr{O}(U)$, $\tilde{f}(x,t) = \sum_{m=0}^\infty f_m(x)t^m$ $(f_m(x) \in \mathscr{O}(U))$, is denoted by $\mathscr{O}(U)[[t]]$.

DEFINITION 1.1. Let $\tilde{f}(x,t) = \sum_{m=0}^\infty f_m(x)t^m \in \mathscr{O}(U)[[t]]$. We say that $\tilde{f}(x,t)$ has Gevrey order s in t, if for any $V \Subset U$ there are positive constants A and B such that

$$\sup_{x\in V}|f_m(x)| \le AB^m\Gamma(sm+1). \tag{1.1}$$

The set of all such formal series is denoted by $\mathscr{O}(U)[[t]]_s$.

Let us introduce spaces of holomorphic functions on sectorial regions with asymptotic expansion. For the details of this topic we refer to [**2**] and [**3**].

DEFINITION 1.2. Let $\gamma > 0$ and U be an open polydisk centered at $x = 0$. Let $f(x,t)$ be a holomorphic function on $U \times S(\theta,\delta,r)$ with asymptotic expansion $f(x,t) \sim \sum_{m=0}^\infty f_m(x)t^m$ $(f_m(x) \in \mathscr{O}(U))$ in the following sense. For any $V \Subset U$ there exist constants A and B such that for any $N \in \boldsymbol{N}$

$$\sup_{x\in V}\left| f(x,t) - \sum_{m=0}^{N-1} f_m(x)t^m \right| \le AB^N\Gamma\left(\frac{N}{\gamma}+1\right)|t|^N \tag{1.2}$$

holds in $S(\theta,\delta,r)$. The set of all such holomorphic functions is denoted by $\mathscr{A}^{\{\gamma\}}(U \times S(\theta,\delta,r))$.

Set

$$\boldsymbol{C}\{x\}[[t]]_s := \bigcup_{U\in\mathfrak{U}_0} \mathscr{O}(U)[[t]]_s, \tag{1.3}$$

$$\mathscr{O}(U)\{t\}_{\gamma,\theta} := \cup_{\delta>\pi/2\gamma}\cup_{r>0}\mathscr{A}^{\{\gamma\}}(U\times S(\theta,\delta,r)), \tag{1.4}$$

$$\boldsymbol{C}\{x\}\{t\}_{\gamma,\theta} := \bigcup_{U\in\mathfrak{U}_0} \mathscr{O}(U)\{t\}_{\gamma,\theta}. \tag{1.5}$$

For each $f(x,t)\in\boldsymbol{C}\{x\}\{t\}_{\gamma,\theta}$ the coefficients $\{f_m(x)\}_{m\in\boldsymbol{N}}$ of the asymptotic expansion are uniquely determined and we can define a homomorphism $\mathfrak{J}:\boldsymbol{C}\{x\}\{t\}_{\gamma,\theta}\Rightarrow\boldsymbol{C}\{x\}[[t]]_{1/\gamma}$ by

$$(\mathfrak{J}f)(x,t) := \sum_{m=0}^{+\infty} f_m(x)t^m \in \boldsymbol{C}\{x\}[[t]]_{1/\gamma}. \tag{1.6}$$

Since $\delta>\pi/(2\gamma)$, $\mathfrak{J}$ is not surjective but injective. Therefore, we can identify $\tilde{f}(x,t)=(\mathfrak{J}f)(x,t)\in\mathfrak{J}(\boldsymbol{C}\{x\}\{t\}_{\gamma,\theta})\subset\boldsymbol{C}\{x\}[[t]]_{1/\gamma}$ with $f(x,t)=(\mathfrak{J}^{-1}\tilde{f})(x,t)\in\boldsymbol{C}\{x\}\{t\}_{\gamma,\theta}$.

DEFINITION 1.3. Let $\tilde{f}(x,t)\in\boldsymbol{C}\{x\}[[t]]_{1/\gamma}$. If there exists $f(x,t)\in\boldsymbol{C}\{x\}\{t\}_{\gamma,\theta}$ such that $\tilde{f}=\mathfrak{J}f$, then we say that $\tilde{f}(x,t)$ is γ-Borel summable in the direction θ and $f(x,t)$ is γ-Borel sum of $\tilde{f}(x,t)$. We also say that $f(x,t)\in\boldsymbol{C}\{x\}\{t\}_{\gamma,\theta}$ is γ-Borel summable in the direction θ.

Next we give conditions on vector fields. Let $z=(z_1,\cdots,z_d)\in\boldsymbol{C}^d$, $\partial_{z_i}=\partial/(\partial z_i)$, $\partial_z=(\partial_{z_1},\cdots,\partial_{z_d})$ and $L=L(z,\partial_z)$ be a holomorphic vector field in a neighborhood W of the origin,

$$L(z,\partial_z)=\sum_{i=1}^{d}X_i(z)\partial_{z_i}. \tag{1.7}$$

L is singular at $z=0$, that is, $X_i(0)=0$ for all $1\le i\le d$. Set

$$\Sigma=\{z\in W; X_i(z)=0 \quad for\ \ i=1,2,\cdots,d\} \tag{1.8}$$

and denote the Jacobi matrix of $(X_1(z),X_2(z),\cdots,X_d(z))$ by $\big((\partial X_i/\partial z_j)(z)\big)$. We introduce conditions C.1 and C.2 on L.

C.1 Σ is a complex submanifold with codimension d_1.
C.2 The Jordan canonical form of $\big((\partial X_i/\partial z_j)(0)\big)$ is

$$\begin{pmatrix}
\lambda_1 & 0 & \cdots & \dots & \cdots & 0 & \cdots & \cdots & 0\\
\mu_1 & \lambda_2 & 0 & \dots & \cdots & 0 & \cdots & \cdots & 0\\
0 & \mu_2 & \lambda_3 & \cdots & \cdots & 0 & \cdots & \cdots & 0\\
\vdots & \ddots & \ddots & \ddots & & & & & \\
\vdots & \ddots & \ddots & \ddots & \ddots & 0 & & & \\
0 & \cdots & \cdots & 0 & \mu_{d_0-1} & \lambda_{d_0} & 0 & \cdots & 0\\
0 & \cdots & \cdots & \cdots & 0 & 0 & 0 & \cdots & 0\\
\vdots & \ddots & \ddots & \ddots & \ddots & & & & \vdots\\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 0
\end{pmatrix}.$$

Here $\lambda_i \neq 0$ for $1 \le i \le d_0$ and $\mu_i = 0$ or 1 and the convex hull of the set of d_0 points $\{\lambda_1, \cdots, \lambda_{d_0}\}$ in the complex plane does not contain the origin.

The rank of $\big((\partial X_i/\partial z_j)(0)\big)$ is d_0 and $d_0 \le d_1$. Set $d_2 = d - d_1$. Suppose that L satisfies conditions C.1 and C.2. Furthermore assume $d_0 < d_1$. We introduce other conditions. In order to do so we give some definitions and notations about $\boldsymbol{C}\{z\}$. For $\varphi_i(z) \in \boldsymbol{C}\{z\}$ $(i = 1, \cdots, p)$ we denote by $\mathscr{I}(\varphi_1, \varphi_2, \cdots, \varphi_p)$ the ideal generated by them in $\boldsymbol{C}\{z\}$. We denote by $\mathscr{I}(\Sigma)$ the ideal consisting of elements in $\boldsymbol{C}\{z\}$ vanishing on Σ. Let $\Sigma = \{z \in W;\ \zeta_i(z) = 0,\ 1 \le i \le d_1\}$, where $\zeta_i(0) = 0$ and $\{\zeta_i(z)\}_{1 \le i \le d_1}$ are functionally independent. Then $\mathscr{I}(\Sigma) = \mathscr{I}(\zeta_1, \zeta_2, \cdots, \zeta_{d_1})$ and $X_i(z) \in \mathscr{I}(\Sigma)$. It follows from C.2 that there are $\{X_{i_j}(z)\}_{j=1}^{d_0}$ which are functionally independent at $z = 0$. We may assume $\{X_j(z)\}_{j=1}^{d_0}$ are functionally independent. We can take $\psi_i(z) \in \mathscr{I}(\Sigma)$ $(1 \le i \le d_1 - d_0)$ such that $\mathscr{I}(\Sigma) = \mathscr{I}(X_1, \cdots, X_{d_0}, \psi_1, \cdots, \psi_{d_1 - d_0})$. Hence $f(z) \in \mathscr{I}(\Sigma)$ is of the form $f(z) = \sum_{j=1}^{d_0} g_j(z) X_j(z) + \sum_{k=1}^{d_1 - d_0} h_k(z)\psi_k(z)$. For $f(z) \in \mathscr{I}(\Sigma)$ the notation $f(z) \equiv O(|\psi|^p)$ mod $\mathscr{I}(X_1, \cdots, X_{d_0})$ means

$$f(z) = \sum_{j=1}^{d_0} g_j(z) X_j(z) + \sum_{\substack{\alpha \in \boldsymbol{N}^{d_1 - d_0} \\ |\alpha| = p}} h_\alpha(z)\psi(z)^\alpha,\ \ \psi(z)^\alpha = \prod_{k=1}^{d_1 - d_0} \psi_k(z)^{\alpha_k}.$$

Define

$$p(i) := \sup\{p \in \boldsymbol{N}; X_i(z) \equiv O(|\psi|^p) \quad \mathrm{mod}\ \mathscr{I}(X_1, \cdots, X_{d_0})\} \tag{1.9}$$

and set $p(i) = \infty$ for $X_i(z) \in \mathscr{I}(X_1, \cdots, X_{d_0})$. The exponent σ is defined by

$$\sigma = \min_{1 \le i \le d} p(i). \tag{1.10}$$

The exponent σ was introduced, called multiplicity and denoted by δ in **[18]**, where it is shown that it depends on neither the choice of $\{X_{i_j}(z)\}_{j=1}^{d_0}$ nor coordinates systems. If $d_1 = d_0 + 1$, by denoting $\psi_1(z)$ by $\psi(z)$, we have $\mathscr{I}(\Sigma) = \mathscr{I}(X_1, \cdots, X_{d_0}, \psi)$ and for $1 \le i \le d$

$$X_i(z) = \sum_{j=1}^{d_0} g_{i,j}(z) X_j(z) + h_i(z)\psi(z)^\sigma. \tag{1.11}$$

LEMMA 1.4. *Assume* C.1, C.2 *and* $d_1 = d_0 + 1$. *Then there are holomorphic functions* $\phi(z) \in \mathscr{I}(\Sigma)$ *and* $\rho(z)$ *in a neighborhood of* $z = 0$ *such that* $\dim\{(\mathrm{grad}\ \phi)(0), (\mathrm{grad}\ X_1)(0), \cdots, (\mathrm{grad}\ X_d)(0)\} = d_0 + 1$ *and*

$$L\phi(z) = \rho(z)\phi(z)^\sigma. \tag{1.12}$$

The proof of Lemma 1.4 is given in Section 3. Let $\varphi(z) \in \mathscr{I}(\Sigma)$ and $\varrho(z)$ be holomorphic in a neighborhood of $z = 0$ satisfying

$$\begin{cases} \dim\{(\operatorname{grad} \varphi)(0), (\operatorname{grad} X_1)(0), \cdots, (\operatorname{grad} X_d)(0)\} = d_0 + 1, \\ L\varphi(z) = \varrho(z)\varphi(z)^\sigma. \end{cases}$$

Furthermore suppose $\sigma \geq 2$. Then we shall show in Section 3 that there is a holomorphic function $k(z)$ with $k(0) \neq 0$ such that $\varrho(z)|_\Sigma = (k(z)\rho(z))|_\Sigma$ holds (Lemma 3.7). Now we introduce condition C.3 in which $\rho(z)$ is that in Lemma 1.4.

C.3 (a) $d_1 = d_0 + 1, \quad \sigma \geq 2.$
(b) $\rho(0) \neq 0.$

It follows from the above remark that $\rho(0) \neq 0$ does not depend on the choice of $\phi(z)$.

Now let us proceed to study the equation

$$Lu(z) = F(z, u(z)), \tag{1.13}$$

where $F(z,u)$ is holomorphic in a neighborhood of $(z,u) = (0,0)$ and $F(0,0) = 0$. In many cases there exist formal power series solutions of (1.13), so there is a problem. Do formal solutions have analytical interpretations? The following theorem is an answer to this problem.

THEOREM 1.5. *Assume* C.1, C.2, C.3 *and for all* $m = (m_1, \cdots, m_{d_0}) \in \boldsymbol{N}^{d_0}$

$$\sum_{i=1}^{d_0} m_i\lambda_i - \frac{\partial F}{\partial u}(0,0) \neq 0. \tag{1.14}$$

Then there exists a unique formal solution $\tilde{u}(z) \in \boldsymbol{C}[[z]]$ *of* (1.13). *Moreover there exist a holomorphic local coordinates system* $(x(z), y(z), t(z)) \in \boldsymbol{C}^{d_0} \times \boldsymbol{C}^{d-d_0-1} \times \boldsymbol{C}$ $(x(0) = y(0) = t(0) = 0)$ *in a neighborhood* Ω *of the origin such that* $\Sigma \cap \Omega = \{x_1(z) = \cdots = x_{d_0}(z) = t(z) = 0\}$ *and* $u(x,y,t) \in \boldsymbol{C}\{x,y\}\{t\}_{\sigma-1,\theta}$ *for some* θ, *which is a genuine solution of* (1.13) *such that* $\tilde{u}(z) = u(x(z), y(z), t(z))$ *holds in* $\boldsymbol{C}[[z]]$.

We show in Section 3 that under the assumptions of Theorem 1.5 L can be represented in the form

$$L = \sum_{i=1}^{d_0} (\lambda_i x_i + \mu_{i-1}x_{i-1} + A_i(x,y,t))\partial_{x_i} + \sum_{j=1}^{d_2} B_j(x,y,t)\partial_{y_j} + t^{\gamma+1}C(x,y,t)\partial_t, \tag{1.15}$$

where $\sigma = \gamma + 1$, $\lambda_i \neq 0$, $\mu_i = 1$ or 0, and the coefficients are holomorphic in a neighborhood of the origin and satisfy as $(x,y,t) \to (0,0,0)$

$$\begin{cases} A_i(0,y,t) = O(|t|^{\gamma+1}), \ A_i(x,y,t) = O\big((|x| + |y| + |t|)^2\big), \\ B_j(0,y,t) = O(|t|^{\gamma+1}), \ B_j(x,y,t) = O\big((|x| + |y| + |t|)^2\big), \\ C(0,0,0) \neq 0. \end{cases} \tag{1.16}$$

We remark that $\rho(0) \neq 0$ in C.3-(b) means $C(0,0,0) \neq 0$. It follows from C.2 and (1.14) that there is a positive constant K_0 such that for all $m = (m_1, \cdots, m_{d_0}) \in \boldsymbol{N}^{d_0}$

$$\left| \sum_{i=1}^{d_0} m_i \lambda_i - \frac{\partial F}{\partial u}(0,0) \right| \geq K_0(|m|+1). \tag{1.17}$$

The other aim of this paper is to find normal form of some singular vector fields by transformations with asymptotic developable functions. Let $L(z,\partial) = \sum_{i=1}^{d} X_i(z)\partial_{z_i}$ be a holomorphic vector field which is singular at the origin. It is assumed to satisfy the following conditions C.1' and C.2', which are more strict than C.1 and C.2.

C.1' $\Sigma = \{0\}$.
C.2' The Jordan canonical form of $\big((\partial X_i/\partial z_j)(0)\big)$ is diagonal

$$\begin{pmatrix} \lambda_1 & 0 & \cdots & \ldots & \cdots & 0 & 0 \\ 0 & \lambda_2 & 0 & \ldots & \cdots & 0 & 0 \\ 0 & 0 & \lambda_3 & \cdots & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & & & \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 & 0 \\ 0 & \cdots & \cdots & 0 & 0 & \lambda_{d-1} & 0 \\ 0 & \cdots & \cdots & \cdots & 0 & 0 & 0 \end{pmatrix}.$$

Here $\lambda_i \neq 0$ for $1 \leq i \leq d-1$ and distinct, and the convex hull of the set of $(d-1)$ points $\{\lambda_1, \cdots, \lambda_{d-1}\}$ in the complex plane does not contain the origin.

By C.1' and C.2' it holds that $\dim \Sigma = 0$, $d = d_1 = d_0 + 1$ and $\big((\partial X_i/\partial z_j)(0)\big)$ has one zero eigenvalue. For L satisfying C.1' and C.2' we have

LEMMA 1.6. *Suppose that* C.1' *and* C.2' *hold. Let* $\rho(z)$ *be that in Lemma* 1.4. *Then* $\sigma \geq 2$ *and* $\rho(0) \neq 0$ *hold.*

The proof of Lemma 1.6 is given in Section 4.

THEOREM 1.7. *Assume* C.1', C.2' *and*

$$\sum_{i=1}^{d-1} m_i \lambda_i - \lambda_k \neq 0 \tag{1.18}$$

for all $m = (m_1, \cdots, m_{d-1}) \in \boldsymbol{N}^{d-1}$ *with* $|m| \geq 2$ *and all* $1 \leq k \leq d-1$. *Then there exist a holomorphic local coordinates system* $(x(z), t(z)) \in \boldsymbol{C}^{d_0} \times \boldsymbol{C}$ $(x(0) = t(0) = 0)$ *in a neighborhood* Ω *of the origin, and functions* $\zeta_i(x,t) \in \boldsymbol{C}\{x\}\{t\}_{\sigma-1,\theta}$ $(1 \leq i \leq d-1)$ *and* $\eta(x,t) \in \boldsymbol{C}\{x\}\{t\}_{\sigma-1,\theta}$ *for some* θ *such that*

$$\begin{cases} \zeta_1(0,0) = \cdots = \zeta_{d-1}(0,0) = 0, \ \eta(x,0) = 0, \\ \left(\dfrac{\partial \zeta_i}{\partial x_j}(0,0)\right) = (\delta_{i,j}), \ \dfrac{\partial \eta}{\partial t}(0,0) \neq 0, \end{cases} \tag{1.19}$$

and by transformation $\zeta_i = \zeta_i(x,t)$ $(1 \le i \le d-1)$, $\eta = \eta(x,t)$, *L is represented in the form*

$$\sum_{i=1}^{d-1} \lambda_i(\eta)\zeta_i \frac{\partial}{\partial \zeta_i} + \eta^{\sigma} c(\eta) \frac{\partial}{\partial \eta}, \tag{1.20}$$

where $\{\lambda_i(\eta)\}_{i=1}^{d-1}$ *and* $c(\eta)$ *are polynomials in* η *with degree* $\le \sigma - 1$, $\lambda_i(0) = \lambda_i$ *and* $c(0) = 1$.

If we admit multiplications of nonvanishing functions to vector fields in the process to find normal forms of vector fields, we have

THEOREM 1.8. *Suppose that the same assumptions as those in Theorem* 1.7 *hold. Then there exist a holomorphic function* $h(z)$ *with* $h(0) \neq 0$ *and a holomorphic local coordinates system* $(x(z), t(z)) \in \boldsymbol{C}^{d_0} \times \boldsymbol{C}$ $(x(0) = t(0) = 0)$ *in a neighborhood* Ω *of the origin such that the following holds.*

Set $L_h = h(z)L$. *Then there exist* $\zeta_i(x,t) \in \boldsymbol{C}\{x\}\{t\}_{\sigma-1,\theta}$ $(1 \le i \le d-1)$ *for some* θ *such that*

$$\zeta_1(0,0) = \cdots = \zeta_{d-1}(0,0) = 0, \ \left(\frac{\partial \zeta_i}{\partial x_j}(0,0)\right) = (\delta_{i,j}), \tag{1.21}$$

and by transformation $\zeta_i = \zeta_i(x,t)$ $(1 \le i \le d-1)$, $\eta = t$, L_h *is represented in the form*

$$\sum_{i=1}^{d-1} \lambda_i(\eta)\zeta_i \frac{\partial}{\partial \zeta_i} + \eta^{\sigma} \frac{\partial}{\partial \eta}, \tag{1.22}$$

where $\{\lambda_i(\eta)\}_{i=1}^{d-1}$ *are polynomials in* η *with degree* $\le \sigma - 1$ *and* $\lambda_i(0) = \lambda_i$.

We give a simple example

$$L := L(x,t,\partial_x,\partial_t) = (\lambda x + x^2 + xt + t^2)\frac{\partial}{\partial x} + t^{\gamma+1}\frac{\partial}{\partial t}, \tag{1.23}$$

where γ is a positive integer and $\lambda > 0$. We have $\sigma = \gamma + 1$. Let us try to simplify L. Let θ be a real constant such that $0 < |\theta| < \pi/\gamma$.

First consider

$$t^{\gamma+1}\varphi'(t) = \lambda\varphi(t) + \varphi(t)^2 + t\varphi(t) + t^2. \tag{1.24}$$

We have a solution $\varphi(t) \in \boldsymbol{C}\{t\}_{\gamma,\theta}$ with $\varphi(t) \sim \sum_{n=2}^{\infty} c_n t^n$ $(c_2 = -1/\lambda)$ (Proposition 7.3 or see [**6**]). By $w = x - \varphi(t)$, $t = t$, L is transformed to

$$L(w,t,\partial_w,\partial_t) = \big((\lambda + t + 2\varphi(t))w + w^2\big)\partial_w + t^{\gamma+1}\partial_t. \tag{1.25}$$

Set $\lambda(t) = \lambda + t + 2\sum_{n=2}^{\gamma} c_n t^n$ and $A(t) = \lambda + t + 2\varphi(t) - \lambda(t)$. Then $\lambda(t)$ is a polynomial in t with degree $\le \gamma$, $A(t) \in \boldsymbol{C}\{t\}_{\gamma,\theta}$ with $A(t) \sim 2\sum_{n=\gamma+1}^{\infty} c_n t^n$ and

$$L(w,t,\partial_w,\partial_t) = \big((\lambda(t)+A(t))w+w^2\big)\partial_w + t^{\gamma+1}\partial_t.$$

Next consider

$$L(w,t,\partial_w,\partial_t)\phi(w,t) = \lambda(t)\phi(w,t). \tag{1.26}$$

The existence of a solution of (1.26) can be shown as follows. Since $A(t) = O(t^{\gamma+1})$, there is $\psi_*(t) \in \boldsymbol{C}\{t\}_{\gamma,\theta}$ with $\psi_*(0) = 0$ satisfying

$$t^{\gamma+1}\psi_*'(t) + A(t)\psi_*(t) + A(t) = 0$$

(Proposition 7.3 or see [**6**]). Set $\phi(w,t) = w + \psi_*(t)w + \psi(w,t)$. Then (1.26) becomes

$$L(w,t,\partial_w,\partial_t)\psi(w,t) = \lambda(t)\psi(w,t) - (1+\psi_*(t))w^2. \tag{1.27}$$

We can find a formal solution $\tilde{\psi}(w,t) = \sum_{n=0}^{\infty}\psi_n(w)t^n \in \mathscr{O}(U)[[t]]_{1/\gamma}$ of (1.27) for a neighborhood U of $w = 0$ such that $\psi_n(w) = O(|w|^2)$ for all n. It follows from Theorem 2.4 in Section 2 that $\tilde{\psi}(w,t)$ is γ-Borel summable. Hence the γ-Borel sum $\psi(w,t) \in \boldsymbol{C}\{w\}\{t\}_{\gamma,\theta}$ of $\tilde{\psi}(w,t)$ is a solution of (1.27). Thus $\phi(w,t) = w + \psi_*(t)w + \psi(w,t)$ is a solution of (1.26). By $\zeta(x,t) = \phi(x-\varphi(t),t)$, $\eta(x,t) = t$, L is transformed to $L = \lambda(\eta)\zeta(\partial/\partial\zeta) + \eta^{\gamma+1}(\partial/\partial\eta)$.

2. Singular first order partial differential equations on sectorial regions.

In this section we study some singular partial differential equation on a sectorial region and give the existence of solutions with asymptotic expansion, Theorems 2.3 and 2.4, which are main results in this section. Their proofs need Borel and Laplace transforms and many estimates, and are slightly long. Hence they are given in Section 6. We prove Theorem 1.5 in Section 3 by transforming L to the operator studied in this section.

Let $(x,y,t) \in \boldsymbol{C}^{d_0} \times \boldsymbol{C}^{d_2} \times \boldsymbol{C}$, $U = \{(x,y) \in \boldsymbol{C}^{d_0+d_2}; |x| < R, |y| < R\}$, γ be a positive integer and $S := S(\theta_0, \pi/2\gamma + \varepsilon_0, r) = \{0 < |t| < r; |\arg t - \theta_0| < \pi/2\gamma + \varepsilon_0\}(\varepsilon_0 > 0)$. Let $P := P(x,y,t,\partial_x,\partial_y,\partial_t)$ be a first order linear partial differential operator with holomorphic coefficients in $U \times S$,

$$P(x,y,t,\partial_x,\partial_y,\partial_t) = \sum_{i=1}^{d_0}(\lambda_i x_i + \mu_{i-1}x_{i-1} + A_i(x,y,t))\partial_{x_i} + \sum_{j=1}^{d_2}B_j(x,y,t)\partial_{y_j} + t^{\gamma+1}C(x,y,t)\partial_t, \tag{2.1}$$

where $\lambda_i \neq 0$, $\mu_i = 1$ or 0. As for the coefficients we assume $A_i(x,y,t)$, $B_j(x,y,t)$, $C(x,y,t) \in \mathscr{O}(U)\{t\}_{\gamma,\theta_0}$ and they satisfy the following conditions as $(x,y,t) \to (0,0,0)$ in S

$$\begin{cases} A_i(0,y,t) = O(|t|^{\gamma+1}),\ A_i(x,y,t) = O\big((|x|+|y|+|t|)^2\big), \\ B_j(0,y,t) = O(|t|^{\gamma+1}),\ B_j(x,y,t) = O\big((|x|+|y|+|t|)^2\big), \\ C(0,0,0) \neq 0 \end{cases} \tag{2.2}$$

(see (1.15) and (1.16)). Here $A_i(0,y,t) = O(|t|^{\gamma+1})$ means $\partial_t^j A_i(0,y,0) = 0$ for $0 \le j \le \gamma$ and $A_i(x,y,t) = O\big((|x| + |y| + |t|)^2\big)$ means $A_i(0,0,0) = \partial_{x_j} A_i(0,0,0) = \partial_{y_k} A_i(0,0,0) = \partial_t A_i(0,0,0) = 0$ $(1 \le j \le d_0,\ 1 \le k \le d_2)$, and similar notations will be often used. It also holds that the convex hull of $\{\lambda_1 . \cdots, \lambda_{d_0}\}$ in the complex plane does not contain the origin, hence, there is a constant $C > 0$ such that

$$\left|\sum_{i=1}^{d_0} m_i \lambda_i\right| \ge C|m| \quad \text{for all } m = (m_1, \cdots, m_{d_0}) \in \boldsymbol{N}^{d_0}. \tag{2.3}$$

Let $U_0 = \{u \in \boldsymbol{C}; |u| < R_0\}$ and $F(x,y,t,u) \in \mathscr{O}(U \times S \times U_0) \cap \mathscr{O}(U \times U_0)\{t\}_{\gamma,\theta_0}$ with $F(0,0,0,0) = 0$. So

$$F(x,y,t,u) = \sum_{n=0}^{\infty} F_n(x,y,t) u^n, \quad F_0(0,0,0) = 0, \tag{2.4}$$

where $F_n(x,y,t) \in \mathscr{O}(U)\{t\}_{\gamma,\theta_0}$. Now let us study a semi linear equation

$$Pu = F(x,y,t,u) \tag{2.5}$$

under the above assumptions and show the existence of a solution $u(x,y,t) \in \mathscr{O}(V)\{t\}_{\gamma,\theta_0}$ of (2.5) with $u(0,0,0) = 0$ for some open polydisk V about the origin. Since $u(x,y,t) \in \mathscr{O}(V)\{t\}_{\gamma,\theta_0} \subset \boldsymbol{C}[[x,y,t]]$, we give remarks on the existence of a formal solution $\tilde{u}(x,y,t) = \sum_{(p,q,r) \in \boldsymbol{N}^{d_0} \times \boldsymbol{N}^{d_2} \times \boldsymbol{N}} u_{p,q,r} x^p y^q t^r$ with $u_{0,0,0} = 0$. As for the existence we refer to [**18**].

PROPOSITION 2.1 ([**18**]). *Suppose that*

$$\sum_{i=1}^{d_0} m_i \lambda_i - F_1(0,0,0) \ne 0 \tag{2.6}$$

holds for all $m = (m_1, \cdots, m_{d_0}) \in \boldsymbol{N}^{d_0}$. *Then there is a unique formal solution* $\tilde{u}(x,y,t) = \sum_{(m,n) \in \boldsymbol{N}^{d_0} \times \boldsymbol{N}} u_{m,n}(y) x^m t^n$ *of* (2.5) *such that* $\{u_{m,n}(y)\}_{(m,n) \in \boldsymbol{N}^{d_0} \times \boldsymbol{N}}$ *are holomorphic in a neighborhood of* $y = 0$, $u_{0,0}(0) = 0$ *and*

$$|u_{m,n}(y)| \le AB^{|m|+n} \left(\frac{n}{\gamma}\right)! \tag{2.7}$$

holds for some constants A and B.

The estimate (2.7) is obtained in [**18**] under the condition that coefficients are holomorphic in a full neighborhood of the origin. Though the coefficients of (2.5) are in $\mathscr{O}(U)\{t\}_{\gamma,\theta}$, we can get (2.7) by a slightly modified method. Let us remember only the existence of a unique formal solution $\tilde{u}(x,y,t)$. Since P has a simpler form than that studied in [**18**], we can find a formal solution of the form $\tilde{u}(x,y,t) = \sum_{n=0}^{\infty} u_n(x,y) t^n$ such that $\{u_n(x,y)\}_{n \in \boldsymbol{N}}$ are holomorphic in a neighborhood of $(x,y) = (0,0)$. We give how to determine $u_0(x,y)$ for the later discussions. Set $A_{i,0}(x,y) = A_i(x,y,0)$, $B_{j,0}(x,y) = B_j(x,y,0)$ and

$$P_0(x,y,\partial_x,\partial_y)=\sum_{i=1}^{d_0}(\lambda_i x_i+\mu_{i-1}x_{i-1}+A_{i,0}(x,y))\partial_{x_i}+\sum_{j=1}^{d_2}B_{j,0}(x,y)\partial_{y_j},$$

which does not depend on t and ∂_t. Then $u_0:=u_0(x,y)$ satisfies

$$\big(P_0(x,y,\partial_x,\partial_y)-F_1(x,y,0)\big)u_0=F_0(x,y,0)+\sum_{k=2}^{\infty}F_k(x,y,0)u_0^k. \tag{2.8}$$

From (2.2) we have

$$A_{i,0}(0,y)=B_{j,0}(0,y)=0,\quad A_{i,0}(x,y),B_{j,0}(x,y)=O((|x|+|y|)^2). \tag{2.9}$$

The coefficients of $P_0(x,y,\partial_x,\partial_y)$ vanish on $\{x=0\}$. By (2.6), (2.9) and $F_0(0,0,0)=0$ it follows from Proposition 7.1 that there exists a unique holomorphic solution $u_0(x,y)$ with $u_0(0,0)=0$ (see also [**8**]). By considering $v(x,y,t)=u(x,y,t)-u_0(x,y)$ as an unknown, we have $P(x,y,t,\partial_x,\partial_y,\partial_t)v=G(x,y,t,v)$, where $G(x,y,t,v)=F(x,y,t,v+u_0)-P(x,y,t,\partial_x,\partial_y,\partial_t)u_0$ and $G(x,y,0,0)=F(x,y,0,u_0)-P_0(x,y,\partial_x,\partial_y)u_0=0$. By denoting $v(x,y,t)$ $(G(x,y,t,u))$ by $u(x,y,t)$ $(resp.\ F(x,y,t,u))$ again, from the beginning we may assume in (2.4)

$$F(x,y,0,0)(=F_0(x,y,0))=0. \tag{2.10}$$

Before showing the existence of a solution $u(x,y,t)\in\mathscr{C}(V)\{t\}_{\gamma,\theta_0}$ of (2.1), we give

LEMMA 2.2. *Suppose that $\sum_{i=1}^{d_0}m_i\lambda_i-F_1(0,0,0)\neq 0$ holds for all $m=(m_1,\cdots,m_{d_0})\in\boldsymbol{N}^{d_0}$. Then there exist θ and $\delta,K_0>0$ such that for ξ with $|\arg\xi-\theta|<\delta$*

$$\left|\sum_{k=1}^{d_0}m_k\lambda_k+\gamma C(0,0,0)\xi^{\gamma}-F_1(0,0,0)\right|\geq K_0(|m|+|\xi|^{\gamma}+1) \tag{2.11}$$

holds for all $m\in\boldsymbol{N}^{d_0}$.

PROOF. Set $c_0=C(0,0,0)$. Since the convex hull of $\{\lambda_i\}_{i=1}^{d-1}$ does not contain the origin, there exists θ' such that $\Re\lambda_k e^{-i\theta'}>0$ for all k. We may assume $\theta'=0$. So there are $\theta_\pm$ such that $-\pi/2<\theta_-<\theta_+<\pi/2$ and $\theta_-<\arg\lambda_k<\theta_+$ for all k. Let $0<\varepsilon_0<\pi-(\theta_+-\theta_-)/2$. Suppose that φ satisfies $\theta_+-\pi+\varepsilon_0<\gamma\varphi+\arg c_0<\theta_-+\pi-\varepsilon_0$. Then (d_0+1) points $\lambda_1,\cdots,\lambda_{d_0}$ and $c_0e^{i\gamma\varphi}$ are contained in a half plane divided by a line through the origin, so there is a constant $C_{\varepsilon_0}>0$ such that

$$\left|\sum_{k=1}^{d_0}m_k\lambda_k+\gamma c_0r^{\gamma}e^{i\gamma\varphi}\right|\geq C_{\varepsilon_0}(|m|+r^{\gamma})$$

holds for $r\geq 0$ and all $m=(m_1,\cdots,m_{d_0})\in\boldsymbol{N}^{d_0}$. Hence there exist $C,R_0>0$ such that for $|m|+r^{\gamma}\geq R_0$

$$\left|\sum_{k=1}^{d_0} m_k\lambda_k + \gamma c_0 r^\gamma e^{i\gamma\varphi} - F_1(0,0,0)\right| \geq C(|m| + r^\gamma + 1).$$

Suppose $|m| \leq R_0$ and consider (r,φ) satisfying

$$\sum_{k=1}^{d_0} m_k\lambda_k + \gamma c_0 r^\gamma e^{i\gamma\varphi} - F_1(0,0,0) = 0, \tag{2.12}$$

where $r \geq 0$ and $\theta_+ - \pi + \varepsilon_0 < \gamma\varphi + \arg c_0 < \theta_- + \pi - \varepsilon_0$. However we have $r \neq 0$ from the assumption. Hence there are at most finite $\{(r_i,\varphi_i)\}_{1\leq i\leq \ell}$ satisfying (2.12) for some $m \in \boldsymbol{N}^{d_0}$ with $|m| \leq R_0$. Now let θ with $\theta_+ - \pi + \varepsilon_0 < \gamma\theta + \arg c_0 < \theta_- + \pi - \varepsilon_0$ such that $\theta \neq \varphi_i$ for all $1 \leq i \leq \ell$. Then there exists $\delta > 0$ such that (2.11) holds for ξ with $|\arg\xi - \theta| < \delta$ and all $m = (m_1,\cdots,m_{d_0}) \in \boldsymbol{N}^{d_0}$. □

We have the following existence of a solution of (2.5) on a sectorial region.

THEOREM 2.3. *Suppose that there are constants $\delta_0, K_0 > 0$ such that for ξ with $|\arg\xi - \theta_0| < \delta_0$ and all $m = (m_1,\cdots,m_{d_0}) \in \boldsymbol{N}^{d_0}$*

$$\left|\sum_{i=1}^{d_0} m_i\lambda_i + \gamma C(0,0,0)\xi^\gamma - F_1(0,0,0)\right| \geq K_0(|m| + |\xi|^\gamma + 1). \tag{2.13}$$

Then there exists uniquely $u(x,y,t) \in \mathscr{O}(V)\{t\}_{\gamma,\theta_0}$ with $u(0,0,0) = 0$ for a polydisk V about $(x,y) = (0,0)$ such that $Pu = F(x,y,t,u)$.

If $\sum_{i=1}^{d_0} m_i\lambda_i - F_1(0,0,0) \neq 0$ for all $m = (m_1,\cdots,m_{d_0}) \in \boldsymbol{N}^{d_0}$, then by Lemma 2.2 there exists θ_0 satisfying the assumption of Theorem 2.3. The existence of a solution $u(x,y,t) \in \mathscr{O}(V)\{t\}_{\gamma,\theta_0}$ means the existence of a formal solution $\tilde{u}(x,y,t) \in \mathscr{O}(V)[[t]]_{1/\gamma}$ with $\tilde{u}(x,y,t) = u(x,y,t)$ as a formal series. We give a modification of Theorem 2.3, which can be used to show Theorem 1.7. We further assume the equation (2.5) contains neither ∂_y nor variables y, so $d_2 = 0$ and $d_0 = d - 1$, and P is of the form

$$P(x,t,\partial_x,\partial_t) = \sum_{i=1}^{d-1} (\lambda_i x_i + A_i(x,t))\partial_{x_i} + t^{\gamma+1}C(x,t)\partial_t, \tag{2.14}$$

where $\lambda_i \neq 0$, $C(0,0) \neq 0$ and

$$A_i(0,t) = 0,\ A_i(x,t) = O((|x| + |t|)^2) \quad \text{for: } 1 \leq i \leq d-1. \tag{2.15}$$

The coefficients of ∂_{x_i} of P vanish on $\{x = 0\}$. Let $F(x,t,u) \in \mathscr{O}(U \times U_0)\{t\}_{\gamma,\theta_0}$ satisfying

$$F(x,t,0) = O(|x|^N) \ \text{ for some } N \in \boldsymbol{N} - \{0\}. \tag{2.16}$$

Let us consider under the above assumptions

$$Pu = F(x,t,u). \tag{2.17}$$

THEOREM 2.4. *Suppose that there are constants $\delta_0, K_0 > 0$ such that for ξ with $|\arg \xi - \theta_0| < \delta_0$ and all $m = (m_1, \cdots, m_{d_0}) \in \boldsymbol{N}^{d_0}$ with $|m| \geq N$*

$$\left| \sum_{i=1}^{d_0} m_i \lambda_i + \gamma C(0,0)\xi^{\gamma} - \frac{\partial F}{\partial u}(0,0,0) \right| \geq K_0(|m| + |\xi|^{\gamma} + 1). \tag{2.18}$$

Then there exists uniquely a solution $u(x,t) \in \mathscr{O}(V)\{t\}_{\gamma,\theta_0}$ of $Pu = F(x,t,u)$ with $u(x,t) = O(|x|^N)$ for a polydisk V about $x = 0$.

We can find a formal solution $\tilde{u}(x,t) = \sum_{n=0}^{\infty} u_n(x)t^n$ of (2.17) with $u_n(x) = O(|x|^N)$. As remarked after Proposition 2.1, by considering $v(x,t) = u(x,t) - u_0(x)$ as an unknown, we may further assume in (2.16)

$$F(x,0,0) = 0. \tag{2.19}$$

As stated in Introduction, the proofs of Theorems of 2.3 and 2.4 are given in Section 6. We show in Section 3 Theorem 1.5 by using Theorem 2.3, and in Section 4 Theorems 1.7 and 1.8 by using Theorem 2.4.

3. Coordinates transformations by holomorphic functions.

Let $L = \sum_{i=1}^{d} X_i(z)\partial_{z_i}$ be a singular vector field satisfying C.1, C.2 and $d_1 = d_0 + 1$. In this section firstly we show that there exists a holomorphic local coordinates system $(x,y,t) = (x_1(z), \cdots, x_{d_0}(z), y_1(z), \cdots, y_{d_2}(z), t(z))$, where $d_2 = d - d_1$, such that L is represented in the form (3.9) in Proposition 3.5 (see also (1.15) and (2.1)). Holomorphic coordinates transformations used here are the same as those in [**18**] except for the last one, which appears in Proposition 3.5 and makes L much simpler than the form transformed in [**18**]. Secondly we give the proof of Lemma 1.4 and a remark about condition C.3-(b) (Lemma 3.7). Finally we show that Theorem 1.5 follows from Theorem 2.3.

Let us show how to change coordinates step by step. By a nonsingular linear transformation we have

LEMMA 3.1. *We can find a holomorphic coordinates system $z = (z_1, \cdots, z_d)$ such that L is of the form*

$$L = \sum_{i=1}^{d_0} (\lambda_i z_i + \mu_{i-1} z_{i-1} + a_i(z))\partial_{z_i} + \sum_{i=d_0+1}^{d} a_i(z)\partial_{z_i}, \tag{3.1}$$

with $a_i(\overbrace{0, \cdots, 0}^{d_1 = d_0+1}, z_{d_1+1}, \cdots, z_d) = 0$ and $a_i(z) = O(|z|^2)$ for $i = 1, \cdots, d$.

The proof is not difficult. We assume L is of the form in Lemma 3.1. Set $z' = (z_1, z_2, \cdots, z_{d_0})$ and $z'' = (z_{d_1+1}, \cdots, z_d)$, so $z = (z', z_{d_1}, z'')$. The next transformation is constructed by using a holomorphic solution of some singular nonlinear partial differential equation.

LEMMA 3.2. *There exist holomorphic functions* $\phi(z',z'')$ *and* $r(z)$ *in a neighborhood of the origin such that* $\phi(0,z'')=0$, $r(0)=0$ *and* $L(z,\partial_z)(z_{d_1}-\phi(z',z''))=r(z)(z_{d_1}-\phi(z',z''))$.

PROOF. Let us consider

$$\sum_{i=1}^{d_0}(\lambda_i z_i+\mu_{i-1}z_{i-1}+a_i(z',\phi,z''))\partial_{z_i}\phi(z',z'') + \sum_{j=d_1+1}^{d} a_i(z',\phi,z'')\partial_{z_j}\phi(z',z'') = a_{d_1}(z',\phi,z''). \tag{3.2}$$

We note $a_i(z',\phi,z'')|_{z'=\phi=0}=0$ and $a_i(z',\phi,z'')=O((|z'|+|z''|+|\phi|)^2)$. So it follows from Proposition 7.2 that there exists a unique holomorphic solution $\phi(z',z'')$ of (3.2) with $\phi(0,z'')=0$. Then

$$\begin{aligned} &L(z,\partial_z)(z_{d_1}-\phi(z',z''))\\ &=a_{d_1}(z)-\sum_{i=1}^{d_0}(\lambda_i z_i+\mu_{i-1}z_{i-1}+a_i(z))\partial_{z_i}\phi(z',z'')-\sum_{j=d_1+1}^{d}a_j(z)\partial_{z_j}\phi(z',z'')\\ &=a_{d_1}(z)-a_{d_1}(z',\phi,z'')+\sum_{i=1}^{d_0}(a_i(z',\phi,z'')-a_i(z))\partial_{z_i}\phi(z',z'')\\ &\quad+\sum_{j=d_1+1}^{d}(a_j(z',\phi,z'')-a_j(z))\partial_{z_j}\phi(z',z''), \end{aligned}$$

which vanishes on $\{z_{d_1}=\phi(z',z'')\}$. Hence there exists $r(z)$ such that $L(z,\partial_z)(z_{d_1}-\phi(z',z''))=r(z)(z_{d_1}-\phi(z',z''))$ and $r(0)=0$ by $a_i(z)=O(|z|^2)$. □

LEMMA 3.3. *There exists a holomorphic local coordinates system* $(z',z'',\tau)\in \boldsymbol{C}^{d_0}\times\boldsymbol{C}^{d_2}\times\boldsymbol{C}$ *in a small polydisk* D *about the origin such that* $\Sigma\cap D=\{z'=\tau=0\}$ *and*

$$L=\sum_{i=1}^{d_0}(\lambda_i z_i+\mu_{i-1}z_{i-1}+a'_i(z',z'',\tau))\partial_{z_i}+\sum_{j=1}^{d_2}b'_j(z',z'',\tau)\partial_{z_{d_1+j}}+c'(z',z'',\tau)\partial_\tau, \tag{3.3}$$

where

$$\begin{aligned} &a'_i(0,z'',0)=0,\ a'_i(z',z'',\tau)=O((|z'|+|z''|+|\tau|)^2),\\ &b'_j(0,z'',0)=0,\ b'_j(z',z'',\tau)=O((|z'|+|z''|+|\tau|)^2),\\ &c'(z',z'',0)=0,\ c'(z',z'',\tau)=O((|z'|+|z''|+|\tau|)^2). \end{aligned} \tag{3.4}$$

PROOF. Set $z_i=z_i$ for $i\neq d_1$, and $\tau=z_{d_1}-\phi(z',z'')$, where $\phi(z',z'')$ is that in Lemma 3.2. Then L is transformed to

$$L=\sum_{i=1}^{d_0}\big(\lambda_i z_i+\mu_{i-1}z_{i-1}+a_i(z',\tau+\phi(z',z''),z'')\big)\partial_{z_i}$$
$$+\sum_{j=1}^{d_2}a_{d_1+j}(z',\tau+\phi(z',z''),z'')\partial_{z_{d_1+j}}+(L\tau)\partial_\tau.$$

We have $a_i'(z',z'',\tau)=a_i(z',\tau+\phi(z',z''),z'')$, $b_j'(z',z'',\tau)=a_{d_1+j}(z',\tau+\phi(z',z''),z'')$ and from Lemma 3.2 $c'(z',z'',\tau)=L\tau=r(z',\tau+\phi(z',z''),z'')\tau$, which satisfy (3.4). ☐

Here we give a remark on the coefficients $b_j'(z',z'',\tau)$ $(1\le j\le d_2)$ and $c'(z',z'',\tau)$ in (3.3). It follows from the definition of σ (see (1.11)) that there are holomorphic functions $\{c_{j,i}(z',z'',\tau);1\le i\le d_0,0\le j\le d_0\}$ such that

$$\begin{aligned}b_j'(z',z'',\tau)&=\sum_{i=1}^{d_0}c_{j,i}(z',z'',\tau)(\lambda_i z_i+\mu_{i-1}z_{i-1}+a_i'(z',z'',\tau))+O(|\tau|^\sigma)\\ c'(z',z'',\tau)&=\sum_{i=1}^{d_0}c_{0,i}(z',z'',\tau)(\lambda_i z_i+\mu_{i-1}z_{i-1}+a_i'(z',z'',\tau))+O(|\tau|^\sigma).\end{aligned}\tag{3.5}$$

We assume L is an operator of the form (3.3) satisfying (3.4) and (3.5).

LEMMA 3.4. *There exists a holomorphic local coordinates system $(x,y,\tau)\in\boldsymbol{C}^{d_0}\times\boldsymbol{C}^{d_2}\times\boldsymbol{C}$ in a small polydisk D about the origin such that $\Sigma\cap D=\{x=\tau=0\}$ and*

$$L=\sum_{i=1}^{d_0}(\lambda_i x_i+\mu_{i-1}x_{i-1}+A_i'(x,y,\tau))\partial_{x_i}+\sum_{j=1}^{d_2}B_j'(x,y,\tau)\partial_{y_j}+\tau C'(x,y,\tau)\partial_\tau,\tag{3.6}$$

where the coefficients satisfy

$$\begin{aligned}&A_i'(0,y,\tau)=O(|\tau|^\sigma),\ A_i'(x,y,\tau)=O((|x|+|y|+|\tau|)^2),\\ &B_j'(0,y,\tau)=O(|\tau|^\sigma),\ B_j'(x,y,\tau)=O((|x|+|y|+|\tau|)^2),\\ &C'(0,y,\tau)=O(|\tau|^{\sigma-1}),\ C'(x,y,\tau)=O(|x|+|y|+|\tau|).\end{aligned}\tag{3.7}$$

PROOF. Let us return to (3.3). Set $x_i=\lambda_i z_i+\mu_{i-1}z_{i-1}+a_i'(z',z'',\tau)$ for $1\le i\le d_0$, $y_j=z_{d_1+j}$ for $1\le j\le d_2$ and $\tau=\tau$. Then $z_i=z_i(x,y,\tau)$ with $z_i(0,y,0)=0$ for $1\le i\le d_0$ and $L=\sum_{i=1}^{d_0}(Lx_i)\partial_{x_i}+\sum_{j=1}^{d_2}B_j'(x,y,\tau)\partial_{y_j}+\tau C'(x,y,\tau)\partial_\tau$, where $B_j'(x,y,\tau)=b_j'(z'(x,y,\tau),y,\tau)$ and $C'(x,y,\tau)=c(z'(x,y,\tau),y,\tau)/\tau$. Then it follows from (3.4) and (3.5) that

$$\begin{aligned}&B_j'(0,y,\tau)=O(|\tau|^\sigma),\ B_j'(x,y,\tau)=O((|x|+|y|+|\tau|)^2),\\ &C'(0,y,\tau)=O(|\tau|^{\sigma-1}),\ C'(x,y,\tau)=O(|x|+|y|+|\tau|).\end{aligned}\tag{3.8}$$

Set $A_i'(x,y,\tau)=(La_i')$. Then $Lx_i=\lambda_i x_i+\mu_{i-1}x_{i-1}+A_i'(x,y,\tau)$ and we have

$$A'_i(x,y,\tau) = \left(\sum_{i=1}^{d_0} x_i\partial_{z_i} + \sum_{j=1}^{d_2} B'_j(x,y,\tau)\partial_{z_{d_1+j}} + \tau C'(x,y,\tau)\partial_\tau\right) a'_i(z',z'',\tau),$$

$A'_i(x,y,\tau) = O((|x|+|y|+|\tau|)^2)$ and $A'_i(0,y,\tau) = O(|\tau|^\sigma)$ by (3.8). Hence we have (3.6) and (3.7). □

In [**18**] formal solutions are studied, after transforming L to the form in Lemma 3.4. In our discussions we further employ a new change of coordinates which transforms L much simpler. We assume L is of the form (3.6) with (3.7).

PROPOSITION 3.5. *There exists a holomorphic local coordinates system $(x,y,t) \in \boldsymbol{C}^{d_0} \times \boldsymbol{C}^{d_2} \times \boldsymbol{C}$ in a small polydisk D about the origin such that $\Sigma \cap D = \{x = t = 0\}$ and*

$$L = \sum_{i=1}^{d_0}(\lambda_i x_i + \mu_{i-1}x_{i-1} + A_i(x,y,t))\partial_{x_i} + \sum_{j=1}^{d_2} B_j(x,y,t)\partial_{y_j} + t^\sigma C(x,y,t)\partial_t, \tag{3.9}$$

where the coefficients satisfy

$$\begin{aligned} &A_i(0,y,t) = O(|t|^\sigma),\ A_i(x,y,t) = O((|x|+|y|+|t|)^2), \\ &B_j(0,y,t) = O(|t|^\sigma),\ B_j(x,y,t) = O((|x|+|y|+|t|)^2). \end{aligned} \tag{3.10}$$

Proposition 3.5 implies that there exists a holomorphic local coordinates system (x,y,t) such that the coefficient of ∂_t vanishes with order σ with respect to t.

PROOF. Let us return to (3.6). Set $A'_{i,0}(x,y,\tau) = A'_i(x,y,\tau) - A'_i(0,y,\tau)$, $B'_{j,0}(x,y,\tau) = B'_j(x,y,\tau) - B'_j(0,y,\tau)$ and $C'_0(x,y,\tau) = C'(x,y,\tau) - C'(0,y,\tau)$. Then by (3.7)

$$\begin{aligned} &A'_{i,0}(x,y,\tau),\ B'_{j,0}(x,y,\tau) = O(|x|(|x|+|y|+|\tau|)) \\ &C'_0(x,y,\tau) = O(|x|). \end{aligned} \tag{3.11}$$

Define L_0 by

$$L_0 = \sum_{i=1}^{d_0}(\lambda_i x_i + \mu_{i-1}x_{i-1} + A'_{i,0}(x,y,\tau))\partial_{x_i} + \sum_{j=1}^{d_2} B'_{j,0}(x,y,\tau)\partial_{y_j} + \tau C'_0(x,y,\tau)\partial_\tau.$$

By (3.11) all the coefficients of L_0 vanish on $\{x=0\}$ and $A'_{i,0}(x,y,\tau), B'_{j,0}(x,y,\tau)$ and $\tau C'_0(x,y,\tau)$ vanish with order 2 at the origin. Consider

$$L_0(\tau(1+T(x,y,\tau))) = 0. \tag{3.12}$$

Then $T := T(x,y,\tau)$ satisfies

$$L_0 T + C'_0(x,y,\tau)T + C'_0(x,y,\tau) = 0,$$

where $C_0'(x,y,\tau) = O(|x|)$ by (3.11). It follows from Proposition 7.1 that there is a holomorphic solution $T(x,y,\tau)$ with $T(0,y,\tau) = 0$ in neighborhood of the origin. Set $x = x$, $y = y$ and $t = \tau(1+T(x,y,\tau))$. Then we have $\tau = \tau(x,y,t)$ with $\tau(x,y,0) = 0$ and $\tau(0,y,t) = t$, and

$$L = \sum_{i=1}^{d_0}(\lambda_i x_i + \mu_{i-1}x_{i-1} + A_i(x,y,t))\partial_{x_i} + \sum_{j=1}^{d_2} B_j(x,y,t)\partial_{y_j} + C^*(x,y,t)\partial_t,$$

where $A_i(x,y,t) = A_i'(x,y,\tau)|_{\tau=\tau(x,y,t)}$ and $B_j(x,y,t) = B_j'(x,y,\tau)|_{\tau=\tau(x,y,t)}$. By (3.7) $A_i(0,y,t) = A_i'(0,y,\tau(0,y,t)) = O(|t|^\sigma)$ and $B_j(0,y,t) = O(|t|^\sigma)$. We have $C^*(x,y,t) = L(\tau(1+T(x,y,\tau))) = L(\tau(1 + T(x,y,\tau))) - L_0(\tau(1 + T(x,y,\tau))) = \left(\sum_{i=1}^{d_0} A_i'(0,y,\tau)\partial_{x_i} + \sum_{j=1}^{d_2} B_j'(0,y,\tau)\partial_{y_j} + \tau C'(0,y,\tau)\partial_\tau\right)\tau(1+T(x,y,\tau)) = O(|\tau|^\sigma)$ by (3.7). Hence $C^*(x,y,t) = t^\sigma C(x,y,t)$ and L is of the form (3.9) with (3.10). □

Now we assume that L is of the form (3.9) with (3.10). Let us give the proof of Lemma 1.4. Before the proof of Lemma 1.4 we have

LEMMA 3.6. *Set $X_i = \lambda_i x_i + \mu_{i-1}x_{i-1} + A_i(x,y,t)$ for $1 \le i \le d_0$, $X_{d_0+j} = B_j(x,y,t)$ for $1 \le j \le d_2$, $X_d = t^\sigma C(x,y,t)$. Then*

$$Lt = t^\sigma C(x,y,t), \tag{3.13}$$

$$LX_i(x,y,t) \equiv O(|t|^{2\sigma-1}) \mod \mathscr{I}(X_1,\cdots,X_{d_0}). \tag{3.14}$$

PROOF. Set $X' = (X_1,\cdots,X_{d_0})$ and $\mathscr{I}(X') = \mathscr{I}(X_1,\cdots,X_{d_0})$. (3.13) is obvious. It follows from $A_i(x,y,t) = (A_i(x,y,t) - A_i(0,y,t)) + A_i(0,y,t)$ and (3.10) that $x_i \equiv O(|t|^\sigma) \mod \mathscr{I}(X')$ and

$$A_i(x,y,t), \partial_{y_k}A_i(x,y,t) \equiv O(|t|^\sigma), \partial_t A_i(x,y,t) \equiv O(t^{\sigma-1}) \mod \mathscr{I}(X').$$

In the same way we also have

$$B_j(x,y,t), \partial_{y_k}B_j(x,y,t) \equiv O(|t|^\sigma), \partial_t B_j(x,y,t) \equiv O(t^{\sigma-1}) \mod \mathscr{I}(X').$$

We have

$$LX_k \equiv \sum_{j=1}^{d_2} B_j(x,y,t)\partial_{y_j}X_k + t^\sigma C(x,y,t)\partial_t X_k \mod \mathscr{I}(X').$$

So for $1 \le k \le d_0$

$$\begin{aligned} LX_k \equiv& \sum_{j=1}^{d_2} B_j(x,y,t)\partial_{y_j}A_k(x,y,t) + t^\sigma C(x,y,t)\partial_t A_k(x,y,t) \\ \equiv& O(|t|^{2\sigma}) + O(|t|^{2\sigma-1}) \equiv O(|t|^{2\sigma-1}) \mod \mathscr{I}(X'). \end{aligned}$$

We also have $LX_k \equiv O(|t|^{2\sigma-1}) \mod \mathscr{I}(X')$ for $d_0 < k \le d$. □

PROOF OF LEMMA 1.4. By setting $\phi(x,y,t) = t$ and $\rho(x,y,t) = C(x,y,t)$, Lemma 1.4 follows from (3.13). □

We give a remark about condition C.3-(b). In the next lemma $\{X_i\}_{i=1}^{d}$ are those in Lemma 3.6, $X' = (X_1, \cdots, X_{d_0})$ and $\rho(x,y,t) := C(x,y,t)$ is defined above.

LEMMA 3.7. *Further assume $\sigma \geq 2$. Let $\varphi(x,y,t) \in \mathscr{I}(\Sigma)$ and $\varrho(x,y,t)$ be holomorphic functions in a neighborhood of the origin such that*

$$\begin{cases} \dim\{(\text{grad}\,\varphi)(0), (\text{grad}\,X_1)(0), \cdots, (\text{grad}\,X_d)(0)\} = d_0 + 1, \\ L\varphi(x,y,t) = \varrho(x,y,t)\varphi(x,y,t)^{\sigma}. \end{cases} \tag{3.15}$$

Then there is a holomorphic function $k(x,y,t)$ in a neighborhood of the origin such that $k(0,0,0) \neq 0$ and $\varrho(0,y,0) = k(0,y,0)\rho(0,y,0)$.

PROOF. There are holomorphic functions $\{c_j(x,y,t)\}_{j=1}^{d_1}$ with $c_{d_1}(0,0,0) \neq 0$ such that $\varphi(x,y,t) = \sum_{j=1}^{d_0} c_j(x,y,t)X_j(x,y,t) + c_{d_1}(x,y,t)t$. We have $\varphi^{\sigma}(x,y,t) \equiv c_{d_1}(x,y,t)^{\sigma}t^{\sigma}$ mod $\mathscr{I}(X')$. The coefficients of L belong to $\mathscr{I}(X', t^{\sigma})$. By (3.14) and $\sigma \geq 2$ we have $LX_j(x,y,t) \equiv O(|t|^{\sigma+1})$ mod $\mathscr{I}(X')$, so

$$\begin{aligned} L\varphi(x,y,t) = &\sum_{j=1}^{d_0} X_j(x,y,t)Lc_j(x,y,t) + \sum_{j=1}^{d_0} c_j(x,y,t)LX_j(x,y,t) \\ &+ tLc_{d_1}(x,y,t) + c_{d_1}(x,y,t)Lt \\ \equiv &c_{d_1}(x,y,t)\rho(x,y,t)t^{\sigma} + O(|t|^{\sigma+1}) \quad \text{mod } \mathscr{I}(X'). \end{aligned}$$

We have from (3.15)

$$\begin{aligned} L\varphi(x,y,t) &= \varrho(x,y,t)\varphi(x,y,t)^{\sigma} \equiv \varrho(x,y,t)(c_{d_1}(x,y,t)t)^{\sigma} \\ &\equiv c_{d_1}(x,y,t)\rho(x,y,t)t^{\sigma} + O(t^{\sigma+1}) \quad \text{mod } \mathscr{I}(X'). \end{aligned} \tag{3.16}$$

Assume $X_1(x,y,t) = \cdots = X_{d_0}(x,y,t) = 0$. Then $x_i = x_i(X', y, t)$ $(1 \leq i \leq d_0)$ with $x_i(0,y,t) = O(|t|^{\sigma})$. Set $k(x,y,t) = c_{d_1}(x,y,t)^{1-\sigma}$. Then by (3.16)

$$\varrho(x(0,y,t),y,t) = c_{d_1}(x(0,y,t),y,t)^{1-\sigma}\rho(x(0,y,t),y,t) + O(|t|),$$

hence $\varrho(0,y,0) = k(0,y,0)\rho(0,y,0)$ and $k(0,0,0) \neq 0$. □

PROOF OF THEOREM 1.5. Let L be an operator satisfying the conditions C.1, C.2 and C.3 and (1.14). Set $\sigma = \gamma + 1 \geq 2$. Then it follows from Proposition 3.5 and $\rho(0) = C(0,0,0) \neq 0$ that L is represented in the form (2.1) with (2.2) by a suitable holomorphic local coordinates system (x,y,t). By (1.14) and Lemma 2.2 the equation $Lu = F(x,y,t,u)$ satisfies the assumptions in Theorem 2.3. Therefore Theorem 1.5 follows from it. □

4. Normal forms of some singular vector fields by transformations with holomorphic functions on a sectorial region.

In this section we consider L satisfying conditions C.1' and C.2'. The assumptions imply $d_2 = 0$, $d_1 = d = d_0 + 1$. Firstly we give the proof of Lemma 1.6.

PROOF OF LEMMA 1.6. It follows from Proposition 3.5 that there exists a holomorphic local coordinates system $(x,t) \in \boldsymbol{C}^{d-1} \times \boldsymbol{C}$ such that

$$L = \sum_{i=1}^{d-1} (\lambda_i x_i + A_i(x,t))\partial_{x_i} + t^\sigma C(x,t)\partial_t, \tag{4.1}$$

where $A_i(x,t) = O((|x|+|t|)^2)$, $A_i(0,t) = O(|t|^\sigma)$. Set $X_i = \lambda_i x_i + A_i(x,t)$ $(1 \le i \le d-1)$ and $X_d = t^\sigma C(x,t)$. Then $x_i \equiv O(|t|^\sigma) \mod \mathscr{I}(X_1,\cdots,X_{d-1})$, so $C(x,t) = C(0,t) + \sum_{i=1}^{d-1} x_i C_i(x,t) \equiv C(0,t) + O(|t|^\sigma) \mod \mathscr{I}(X_1,\cdots,X_{d-1})$. Hence $t^\sigma C(x,t) \equiv t^\sigma(C(0,0) + O(|t|)) \mod \mathscr{I}(X_1,\cdots,X_{d-1})$, and $C(0,0) \neq 0$ by the definition of σ (see (1.9) and (1.10)). From C.2' we have $\sigma \ge 2$. By setting $\varphi(t) = t$ and $\rho(x,t) = C(x,t)$, the assertions hold. $\square$

Now we further transform L satisfying C.1' and C.2' in order to obtain a normal form of L.

PROPOSITION 4.1. *There exists a holomorphic local coordinates system $(x,t) \in \boldsymbol{C}^{d-1} \times \boldsymbol{C}$ in a small polydisk D about the origin such that $\Sigma \cap D = \{x = t = 0\}$ and*

$$L = \sum_{i=1}^{d-1} \left(a_i^0(t) + \sum_{j=1}^{d-1} a_{i,j}^1(t) x_j + a_i^2(x,t) \right) \partial_{x_i} + t^\sigma c(x,t)\partial_t, \tag{4.2}$$

where

$$\begin{aligned} &a_i^0(t) = O(|t|^\sigma), \quad a_{i,j}^1(t) = \lambda_i(t)\delta_{i,j} + O(|t|^\sigma), \\ &a_i^2(x,t) = O(|x|^2), \quad c(0,0) = 1, \end{aligned} \tag{4.3}$$

and $\lambda_i(t)$ is a polynomial with degree $\le \sigma - 1$ and $\lambda_i(0) = \lambda_i$.

PROOF. L is of the form (4.1) with

$$A_i(x,t) = O((|x|+|t|)^2),\ A_i(0,t) = O(|t|^\sigma),\ C(0,0) \neq 0. \tag{4.4}$$

By replacing $C(0,0)^{1/(\sigma-1)}t$ by t, we may assume $C(0,0) = 1$. We have $A_i(x,t) = A_i(0,t) + \sum_{j=1}^{d-1} A_{i,j}^1(t) x_j + A_i^2(x,t)$ with $A_{i,j}^1(0) = 0$, $A_i^2(x,t) = O(|x|^2)$. Set $(d-1)\times(d-1)$ matrix $A'(t) = (A'_{i,j}(t))_{1 \le i,j \le d-1}$, $A'_{i,j}(t) = \lambda_i \delta_{i,j} + A_{i,j}^1(t)$. Since λ_i's are non zero and distinct and $A_{i,j}^1(0) = 0$, there is a nonsingular matrix $S(t) = (S_{i,j}(t))$ with holomorphic elements at $t = 0$ such that $S(t)A'(t)S^{-1}(t) = diagonal\ (\Lambda_1(t),\cdots,\Lambda_{d-1}(t))$ with $\Lambda_i(0) = \lambda_i$. Consider the transformation $\tilde{x}_p = \sum_{i=1}^{d-1} S_{p,i}(t) x_i$, $\tau = t$. Then $L = \sum_{p=1}^{d-1} (L\tilde{x}_p)\partial_{\tilde{x}_p} + (L\tau)\partial_\tau$, where

$$L\tilde{x}_p = \sum_{i=1}^{d-1} S_{p,i}(t)A_i(0,t) + \sum_{i,j=1}^{d-1} S_{p,i}(t)A'_{i,j}(t)x_j + \sum_{i=1}^{d-1} S_{p,i}(t)A_i^2(x,t) + t^\sigma C(x,t)\partial_t\tilde{x}_p$$
$$= \sum_{i=1}^{d-1} S_{p,i}(t)A_i(0,t) + \big(\Lambda_p(t)\tilde{x}_p + t^\sigma C(x,t)\partial_t\tilde{x}_p\big) + \sum_{i=1}^{d-1} S_{p,i}(t)A_i^2(x,t).$$

Set $a_p^0(t) = \sum_{i=1}^{d-1} S_{p,i}(t)A_i(0,t)$, $\sum_{j=1}^{d-1} a_{p,j}^1(t)\tilde{x}_j = \Lambda_p(t)\tilde{x}_p + t^\sigma C(0,t)\partial_t\tilde{x}_p$ and $a_p^2(\tilde{x},t) = t^\sigma(C(x,t)-C(0,t))\partial_t\tilde{x}_p + \sum_{i=1}^{d-1} S_{p,i}(t)A_i^2(x,t)|_{x=S^{-1}(t)\tilde{x}}$. Then $a_p^0(t) = O(|t|^\sigma)$ by (4.4), $a_{p,j}^1(t) = \lambda_p(t)\delta_{p,j} + O(|t|^\sigma)$ and $a_p^2(\tilde{x},t) = O(|\tilde{x}|^2)$. By setting $c(\tilde{x},t) = C(S^{-1}(t)\tilde{x},t)$ and denoting $\tilde{x}$ by x again, L is of the form (4.2) with (4.3). □

Thus L satisfying C.1' and C.2' is transformed to (4.2) with (4.3) by holomorphic transformations in a full neighborhood of the origin, hence, we restart by assuming L is represented in the form (4.2) with (4.3). From now on we transform L by coordinates transformations with holomorphic functions on a sectorial region, that is, functions in $\boldsymbol{C}\{x\}\{t\}_{\sigma-1,\theta}$, and obtain a normal form of L, which implies Theorems 1.7 and 1.8. In constructing transformations we need the existence of solutions with asymptotic expansion of singular differential equations on sectorial region (Theorem 2.4 and Proposition 7.3). It follows from C.1', C.2' and (1.18) that there is a constant $C > 0$ such that

$$\left|\sum_{i=1}^{d-1} m_i\lambda_i\right| \geq C(|m|+1) \quad \textit{for} \;\; |m| \geq 1, \tag{4.5}$$
$$\left|\sum_{i=1}^{d-1} m_i\lambda_i - \lambda_k\right| \geq C(|m|+1) \quad \textit{for} \;\; |m| \geq 2. \tag{4.6}$$

From (4.5), (4.6) and $c(0,0) = 1$ we have in the same way as Lemma 2.2

LEMMA 4.2. *There are θ and $\delta, K_0 > 0$ such that for ξ with $|\arg\xi - \theta| < \delta$*

$$\left|\sum_{i=1}^{d_0} m_i\lambda_i + \gamma\xi^\gamma\right| \geq K_0(|m| + |\xi|^\gamma + 1) \quad \textit{for} \;\; |m| \geq 1, \tag{4.7}$$
$$\left|\sum_{i=1}^{d_0} m_i\lambda_i - \lambda_k + \gamma\xi^\gamma\right| \geq K_0(|m| + |\xi|^\gamma + 1) \quad \textit{for} \;\; |m| \geq 2. \tag{4.8}$$

We assume in the following of this section that θ satisfies (4.7) and (4.8). Let us proceed to find transformations. For this purpose we introduce conditions (Θ_0) and (Θ_1) on θ,

(Θ_0) $\quad (\sigma-1)\theta \not\equiv arg\lambda_i \mod 2\pi$ for all $1 \leq i \leq d-1$,
(Θ_1) $\quad (\sigma-1)\theta \not\equiv arg(\lambda_j - \lambda_k) \mod 2\pi$ for all $1 \leq j,k \leq d-1$ with $j \neq k$.

We remark that there are many θ satisfying (4.7), (4.8), (Θ_0) and (Θ_1). Now consider a system of ordinary differential equations derived from (4.2)

$$t^\sigma c(\Psi(t),t)\frac{d\psi_i(t)}{dt} = a_i^0(t) + \sum_{j=1}^{d-1} a_{i,j}^1(t)\psi_j(t) + a_i^2(\Psi(t),t) \quad (1 \leq i \leq d-1), \tag{4.9}$$

where $\boldsymbol{\Psi}(t) = (\psi_1(t), \cdots, \psi_{d-1}(t))$.

LEMMA 4.3. *Suppose that θ satisfies (Θ_0). Then there exists a solution $\boldsymbol{\Psi}(t) = (\psi_1(t), \cdots, \psi_{d-1}(t))$ of* (4.9) *such that $\psi_i(t) \in \boldsymbol{C}\{t\}_{\sigma-1,\theta}$ and $\psi_i(t) = O(|t|^\sigma)$ for all $1 \le i \le d-1$.*

PROOF. We write (4.9) in another form. Set $L_{1/c} = c(x,t)^{-1}L$. Then by $c(x,t)^{-1} = 1 + O(|x| + |t|)$ we have

$$L_{1/c} = \sum_{i=1}^{d-1} \left(r_i^0(t) + \sum_{j=1}^{d-1} r_{i,j}^1(t)x_j + r_i^2(x,t) \right) \partial_{x_i} + t^\sigma \partial_t, \tag{4.10}$$

where

$$r_i^0(t) = O(|t|^\sigma), \quad r_{i,j}^1(t) = \mu_i(t)\delta_{i,j} + O(|t|^\sigma), \quad r_i^2(x,t) = O(|x|^2), \tag{4.11}$$

and $\mu_i(t)$ is a polynomial in t with $\mu_i(0) = \lambda_i$. Set $\gamma = \sigma - 1$ and consider

$$t^{\gamma+1} \frac{d\psi_i(t)}{dt} = r_i^0(t) + \sum_{j=1}^{d-1} r_{i,j}^1(t)\psi_j(t) + r_i^2(\boldsymbol{\Psi}(t),t) \quad (1 \le i \le d-1). \tag{4.12}$$

The system (4.9) is equivalent to (4.12). By (Θ_0) and Proposition 7.3 there exists a solution $\boldsymbol{\Psi}(t) \in (\boldsymbol{C}\{t\}_{\sigma-1,\theta})^{d-1}$ with $\psi_i(t) = O(|t|^\sigma)$. □

LEMMA 4.4. *Suppose that θ satisfies (Θ_0). Let $\boldsymbol{\Psi}(t) = (\psi_1(t), \cdots, \psi_{d-1}(t))$, $\psi_i(t) \in \boldsymbol{C}\{t\}_{\sigma-1,\theta}$, be a solution of* (4.9) *whose existence is assured by Lemma* 4.3. *By transformation $w_i = x_i - \psi_i(t)$ $(1 \le i \le d-1)$ and $t = t$, L is transformed to*

$$L = \sum_{i=1}^{d-1} \left(\lambda_i(t)w_i + \sum_{j=1}^{d-1} A_{i,j}^1(t)w_j + A_i^2(w,t) \right) \partial_{w_i} + t^\sigma C(w,t)\partial_t, \tag{4.13}$$

where $A_{i,j}^1(t), A_i^2(w,t), C(w,t) \in \boldsymbol{C}\{w\}\{t\}_{\sigma-1,\theta}$ satisfying

$$A_{i,j}^1(t) = O(|t|^\sigma),\ A_i^2(w,t) = O(|w|^2),\ C(0,0) = 1. \tag{4.14}$$

PROOF. We have

$$\begin{aligned} Lw_i &= a_i^0(t) + \sum_{j=1}^{d-1} a_{i,j}^1(t)(w_j + \psi_j(t)) + a_i^2(w + \boldsymbol{\Psi}(t),t) - t^\sigma c(w + \boldsymbol{\Psi}(t),t)\psi_i'(t) \\ &= \sum_{j=1}^{d-1} a_{i,j}^1(t)w_j + g_i(w,t), \end{aligned}$$

where $g_i(w,t) = a_i^2(w + \boldsymbol{\Psi}(t),t) - a_i^2(\boldsymbol{\Psi}(t),t) + t^\sigma \big(c(\boldsymbol{\Psi}(t),t) - c(w + \boldsymbol{\Psi}(t),t) \big) \psi_i'(t)$. It follows from $g_i(0,t) = 0$, $a_i^2(w,t) = O(|w|^2)$ and $\psi_i(t) = O(|t|^\sigma)$ that

$$g_i(w,t) = \sum_{j=1}^{d-1} \partial_{w_j} g_i(0,t) w_j + g_i^2(w,t),$$

where $\partial_{w_j} g_i(0,t) = O(|t|^\sigma)$ and $g_i^2(w,t) = O(|w|^2)$. So there exist $A_{i,j}^1(t) = O(|t|^\sigma)$ $(1 \le i, j \le d-1)$ such that $a_{i,j}^1(t) + \partial_{w_j} g_i(0,t) = \lambda_i(t)\delta_{i,j} + A_{i,j}^1(t)$. By setting $A_i^2(w,t) = g_i^2(w,t)$ and $C(w,t) = c(w + \Psi(t), t)$ we have (4.13). □

Now let us assume that L is of the form (4.13) with (4.14). Define a polynomial $C_0(t)$ with degree $\le \sigma - 1$ by

$$C_0(t) = \sum_{n=0}^{\sigma-1} \frac{1}{n!} \left(\frac{d}{dt}\right)^n C(0,t)|_{t=0} t^n \tag{4.15}$$

and set $C_1(t) = C(0,t) - C_0(t) = O(|t|^\sigma)$. Consider

$$L\big(t\phi(w,t)\big) = (t\phi(w,t))^\sigma C_0(t\phi(w,t)). \tag{4.16}$$

We apply Theorem 2.4 to construct a solution of (4.16) in next lemma.

LEMMA 4.5. *There exists a solution* $\phi(w,t) \in \boldsymbol{C}\{w\}\{t\}_{\sigma-1,\theta}$ *of* (4.16) *with* $\phi(0,0) = 1$.

PROOF. First consider

$$C(0,t)(t\phi_0(t))' = \phi_0(t)^\sigma C_0(t\phi_0), \ \phi_0(0) = 1.$$

Set $\phi_0(t) = 1 + \varphi(t)$. Then $\varphi(0) = 0$ and

$$C(0,t)(t\varphi(t))' = (1+\varphi(t))^\sigma C_0(t + t\varphi(t)) - C(0,t).$$

Set $G(t,u) = C(0,t)^{-1}\big((1+u)^\sigma C_0(t+tu) - C_0(t)\big)$ and $g(t) = -C(0,t)^{-1} C_1(t) = O(|t|^\sigma)$. Then $(t\varphi(t))' = G(t,\varphi) + g(t)$. By $G(t,0) = 0$ and $G_u(0,0) = \sigma$ we have $G(t,u) = \sigma u + G_2(t,u)$, where $G_2(t,u) = O(|u|(|t| + |u|))$. Hence $\varphi(t)$ satisfies $(t\varphi(t))' - \sigma\varphi(t) = g(t) + G_2(t,\varphi(t))$. Set $\varphi(t) = t^{\sigma-1}\psi(t)$. Then

$$\psi(t)' = t^{-\sigma} g(t) + t^{-\sigma} G_2(t, t^{\sigma-1}\psi(t)). \tag{4.17}$$

Since $g(t) = O(|t|^\sigma)$ and $G_2(t, t^{\sigma-1}u) = O(|t|^\sigma)$, there exists a solution $\psi(t) \in \boldsymbol{C}\{t\}_{\sigma-1,\theta}$ of (4.17) with $\psi(0) = 0$ (see Proposition 7.3). Hence $\phi_0(t) = 1 + \varphi(t) = 1 + t^{\sigma-1}\psi(t) \in \boldsymbol{C}\{t\}_{\sigma-1,\theta}$ exists. Let us proceed to solve (4.16). Set $\phi_1(w,t) = \phi(w,t) - \phi_0(t)$. Then

$$\begin{aligned}
L(t\phi_1(w,t)) &= L(t\phi(w,t)) - L(t\phi_0(t)) \\
&= t^\sigma(\phi_0(t) + \phi_1(w,t))^\sigma C_0\big(t\phi_0(t) + t\phi_1(w,t)\big) - t^\sigma C(w,t)(t\phi_0(t))' \\
&= t^\sigma(\phi_0(t) + \phi_1(w,t))^\sigma C_0\big(t\phi_0(t) + t\phi_1(w,t)\big) - (t\phi_0(t))^\sigma C(w,t) C(0,t)^{-1} C_0(t\phi_0(t)).
\end{aligned}$$

Set

$$H(w,t,u) = t^{\sigma-1}\Big(\big(\phi_0(t)+u\big)^{\sigma} C_0(t\phi_0(t)+tu) \\ - \phi_0(t)^{\sigma} C(w,t)C(0,t)^{-1}C_0(t\phi_0(t)) - C(w,t)u\Big).$$

Then we have

$$L\phi_1 = H(w,t,\phi_1(t)), \tag{4.18}$$

where $H(w,t,0) = O(|w|)$ and $\partial_u H(0,0,0) = 0$ hold. We assume θ satisfies (4.7), so the equation (4.18) satisfies assumptions in Theorem 2.4 with $N = 1$. Hence there is a solution $\phi_1(w,t) \in \boldsymbol{C}\{w\}\{t\}_{\sigma-1,\theta}$ with $\phi_1(w,t) = O(|w|)$ of (4.18). Thus $\phi(w,t) = \phi_0(t) + \phi_1(w,t)$ is a solution of (4.16). □

LEMMA 4.6. *Let $\phi(w,t) \in \boldsymbol{C}\{w\}\{t\}_{\sigma-1,\theta}$ be that in Lemma 4.5. By transformation $\tau = t\phi(w,t)$, $w_i = w_i$ $(1 \le i \le d-1)$, L is transformed to*

$$L = \sum_{i=1}^{d-1}\left(\Lambda_i(\tau)w_i + \sum_{j=1}^{d-1} A^1_{i,j}(\tau)w_j + A^2_i(w,\tau)\right)\partial_{w_i} + \tau^{\sigma}C(\tau)\partial_{\tau}, \tag{4.19}$$

where $\Lambda_i(\tau)$ $(1 \le i \le d-1)$ and $C(\tau)$ are polynomials in τ with degree $\le \sigma - 1$ with $\Lambda(0) = \lambda_i$ and $C(0) = 1$, and $A^1_{i,j}(\tau) \in \boldsymbol{C}\{\tau\}_{\sigma-1,\theta}$, $A^2_i(w,\tau) \in \boldsymbol{C}\{w\}\{\tau\}_{\sigma-1,\theta}$ with

$$A^1_{i,j}(\tau) = O(|\tau|^{\sigma}),\ A^2_i(w,\tau) = O(|w|^2). \tag{4.20}$$

PROOF. Let $t = \varphi(w,\tau) \in \boldsymbol{C}\{w\}\{\tau\}_{\sigma-1,\theta}$, be the inverse function of $\tau = t\phi(w,t)$. Then $\varphi(w,\tau) = \tau(1 + O(|w| + |\tau|))$. Recall that we assume L is the form (4.13) with (4.14). Since $L\tau = \tau^{\sigma}C_0(\tau)$, L is transformed to

$$L = \sum_{i=1}^{d-1}\left(\lambda_i(\varphi(w,\tau))w_i + \sum_{j=1}^{d-1} A^1_{i,j}(\varphi(w,\tau))w_j + A^2_i(w,\varphi(w,\tau))\right)\partial_{w_i} + \tau^{\sigma}C_0(\tau)\partial_{\tau}.$$

Note the coefficient of ∂_{w_i}. Set $\Lambda_i(\tau) = \sum_{h=0}^{\sigma-1}(d/d\tau)^h\lambda_i(\varphi(0,\tau))|_{\tau=0}\tau^h/h!$ and $A'_{i,j}(\tau) = (\lambda_i(\varphi(0,\tau)) - \Lambda_i(t))\delta_{i,j} + A^1_{i,j}(\varphi(0,\tau))$. Then $A'_{i,j}(\tau) = O(|\tau|^{\sigma})$ holds. Set $A''_i(w,\tau) = \big(\lambda_i(\varphi(w,\tau)) - \lambda_i(\varphi(0,\tau))\big)w_i + \sum_{j=1}^{d-1}\big(A^1_{i,j}(\varphi(w,\tau)) - A^1_{i,j}(\varphi(0,\tau))\big)w_j + A^2_i(w,\varphi(w,\tau))$. We denote again $A'_{i,j}(\tau)$ ($A''_i(w,\tau)$,$C_0(\tau)$) by $A^1_{i,j}(\tau)$ (*resp.* $A^2_i(w,\tau)$, $C(\tau)$). Then they satisfy (4.20). □

We assume L is of the form (4.19) with (4.20). Moreover let us consider a singular partial differential equation for each fixed $k \in \{1,2,\cdots,d-1\}$

$$L\phi_k(w,\tau) = \Lambda_k(\tau)\phi_k(w,\tau). \tag{4.21}$$

We apply again Theorem 2.4 to find a solution of (4.7) in next lemma.

LEMMA 4.7. *Suppose that* θ *satisfies* (Θ_1). *Then there exists a solution* $\phi_k(w,\tau) \in \boldsymbol{C}\{w\}\{\tau\}_{\sigma-1,\theta}$ *of* (4.21) *with* $\phi_k(w,\tau) = w_k + O(|w|(|w|+|\tau|))$.

PROOF. Set $\phi_k(w,\tau) = w_k + \psi_k(w,\tau)$. Then $\psi_k(w,\tau)$ satisfies

$$\sum_{i=1}^{d-1}\left(\Lambda_i(\tau)w_i + \sum_{j=1}^{d-1}A^1_{i,j}(\tau)w_j + A^2_i(w,\tau)\right)\frac{\partial\psi_k}{\partial w_i} + \tau^\sigma C(\tau)\frac{\partial\psi_k}{\partial\tau}$$
$$= \Lambda_k(\tau)\psi_k - \sum_{j=1}^{d-1}A^1_{k,j}(\tau)w_j - A^2_k(w,\tau). \tag{4.22}$$

Let us consider an auxiliary equation to solve (4.22)

$$\sum_{i=1}^{d-1}\left(\Lambda_i(\tau)w_i + \sum_{j=1}^{d-1}A^1_{i,j}(\tau)w_j\right)\frac{\partial\psi^1_k}{\partial w_i} + \tau^\sigma C(\tau)\frac{\partial\psi^1_k}{\partial\tau} = \Lambda_k(\tau)\psi^1_k - \sum_{j=1}^{d-1}A^1_{k,j}(\tau)w_j. \tag{4.23}$$

We show there exists a solution $\psi^1_k(w,\tau) = \sum_{j=1}^{d-1}\psi^1_{k,j}(\tau)w_j$ of (4.23). We have

$$\sum_{i=1}^{d-1}\left(\Lambda_i(\tau)w_i + \sum_{j=1}^{d-1}A^1_{i,j}(\tau)w_j\right)\psi^1_{k,i} + \sum_{j=1}^{d-1}\tau^\sigma C(\tau)\frac{d\psi^1_{k,j}}{d\tau}w_j = \sum_{j=1}^{d-1}\left(\Lambda_k(\tau)\psi^1_{k,j}(\tau) - A^1_{k,j}(\tau)\right)w_j,$$

hence, a system of linear differential equations of unknowns $\{\psi^1_{k,j}(\tau)\}_{j=1}^{d-1}$

$$\tau^\sigma C(\tau)\frac{d\psi^1_{k,j}(\tau)}{d\tau} = (\Lambda_k(\tau) - \Lambda_j(\tau))\psi^1_{k,j}(\tau) - \sum_{i=1}^{d-1}A^1_{i,j}(\tau)\psi^1_{k,i}(\tau) - A^1_{k,j}(\tau). \tag{4.24}$$

If $j \neq k$, then $\Lambda_j(0) - \Lambda_k(0) = \lambda_j - \lambda_k \neq 0$. If $j = k$, then

$$\tau^\sigma C(\tau)\frac{d\psi^1_{k,k}(\tau)}{d\tau} = -\sum_{i=1}^{d-1}A^1_{i,k}(\tau)\psi^1_{k,i}(\tau) - A^1_{k,k}(\tau). \tag{4.25}$$

The right hand side of (4.25) vanishes at $\tau = 0$ with order σ, hence by dividing by τ^σ it becomes of normal type with respect to $(d/d\tau)$. It follows from (Θ_1) and Proposition 7.3 that there exist $\{\psi^1_{k,j}(\tau)\}_{j=1}^{d-1} \in \boldsymbol{C}\{\tau\}_{\sigma-1,\theta}$ with $\psi^1_{k,j}(0) = 0$ satisfying (4.24), so $\psi^1_k(w,\tau) = \sum_{j=1}^{d-1}\psi^1_{k,j}(\tau)w_j = O(|w||\tau|)$ is a solution of (4.23). Set $\psi^2_k(x,\tau) = \psi_k(x,\tau) - \psi^1_k(x,\tau)$. By (4.22) and (4.23)

$$\sum_{i=1}^{d-1}\left(\Lambda_i(\tau)w_i + \sum_{j=1}^{d-1}A^1_{i,j}(\tau)w_j + A^2_i(w,\tau)\right)\frac{\partial\psi^2_k}{\partial w_i} + \tau^\sigma C(\tau)\frac{\partial\psi^2_k}{\partial\tau}$$
$$= \Lambda_k(\tau)\psi^2_k - A^2_k(w,\tau) - \sum_{i=1}^{d-1}A^2_i(w,\tau)\frac{\partial\psi^1_k}{\partial w_i}, \tag{4.26}$$

where $A_k^2(w,\tau)-\sum_{i=1}^{d-1}A_i^2(w,\tau)(\partial\psi_k^1/\partial w_i)=O(|w|^2)$. We assume θ satisfies (4.8), so the equation (4.26) satisfies assumptions in Theorem 2.4 with $N=2$. Hence there exists $\psi_k^2(w,\tau)\in \boldsymbol{C}\{w\}\{\tau\}_{\sigma-1,\theta}$ with $\psi_k^2(w,\tau)=O(|w|^2)$. Thus $\phi_k(x,\tau)=w_k+\psi_k^1(x,\tau)+\psi_k^2(x,\tau)$ is a solution of (4.21). □

PROOF OF THEOREM 1.7. We can choose θ so that (4.7), (4.8), (Θ_0) and (Θ_1) hold. Consider transformation $\zeta_k=\phi_k(w,\tau)$ $(1\le k\le d-1)$, $\eta=\tau$, where $\{\phi_k(w,\tau)\}_{k=1}^{d-1}$ are those in Lemma 4.7. Then L is transformed to

$$L=\sum_{i=1}^{d-1}(L\phi_i)\frac{\partial}{\partial\zeta_i}+\eta^\sigma C(\eta)\frac{\partial}{\partial\eta}=\sum_{i=1}^{d-1}\Lambda_i(\eta)\zeta_i\frac{\partial}{\partial\zeta_i}+\eta^\sigma C(\eta)\frac{\partial}{\partial\eta}.$$

By setting $\lambda_i(\eta)=\Lambda_i(\eta)$ and $c(\eta)=C(\eta)$, which are polynomials in η with degree $\le\sigma-1$ we have (1.19). It follows from the process of transformations in the above Lemmas that (1.20) holds. □

PROOF OF THEOREM 1.8. Let us return to the proof of Lemma 4.3. Set $h(x,t)=1/c(x,t)$. Then $L_h=h(x,t)L$ is of the form (4.10), so the coefficient of ∂_t becomes t^σ. By repeating the above process, but without Lemmas 4.5 and 4.6, we have (1.21) and (1.22). □

5. Borel and Laplace transforms, convolution and majorant functions.

One of the aims of this section is to study Laplace transform, Borel transform and convolution for holomorphic functions on some sectorial regions. We refer for the details of these topics and the proofs of some Lemmas to [**2**] and [**3**]. The other is to introduce majorant functions. The solution $u(x,y,t)$ in Theorems 2.3 or 2.4 is constructed by Laplace integral

$$u(x,y,t)=\int_0^{\infty e^{i\theta_0}}\exp\left(-\left(\frac{\xi}{t}\right)^\gamma\right)\hat{u}(x,y,\xi)d\xi^\gamma \tag{5.1}$$

in Section 6, where we use the results in this section.

Let U be a neighborhood of the origin in $\boldsymbol{C}^n$ and we denote by $x=(x_1,\cdots,x_n)$ its coordinates. Given θ and $\delta>0$, set $S^*(\theta,\delta):=\{\xi\neq 0;|\arg\xi-\theta|<\delta\}$ and $S^*_{\{0\}}(\theta,\delta):=\{\xi\in S^*(\theta,\delta);0<|\xi|<\rho(\arg\xi)\}$, where $\rho(\cdot)$ is some positive continuous function on $(\theta-\delta,\theta+\delta)$. Let $\gamma>0$ be a constant. Let $\phi(x,\xi)\in\mathscr{O}(U\times S^*(\theta,\delta))$ satisfying for $(x,\xi)\in U\times S^*(\theta,\delta)$

$$\begin{aligned}&|\phi(x,\xi)|\le A\exp(c|\xi|^\gamma)\quad\text{for}\quad|\xi|\ge 1,\\&|\phi(x,\xi)|\le A|\xi|^{\varepsilon-\gamma}\ (\varepsilon>0)\quad\text{for}\quad 0<|\xi|<1.\end{aligned}\tag{5.2}$$

Then we can define γ-Laplace transform $(\mathscr{L}_{\gamma,\theta}\phi)(x,t)$ by

$$(\mathscr{L}_{\gamma,\theta}\phi)(x,t)=\int_0^{\infty e^{i\theta}}\exp\left(-\left(\frac{\xi}{t}\right)^\gamma\right)\phi(x,\xi)d\xi^\gamma.\tag{5.3}$$

$(\mathscr{L}_{\gamma,\theta}\phi)(x,t)$ is holomorphic in $U\times S_{\{0\}}(\theta,\pi/2\gamma+\delta)$, where $S_{\{0\}}(\theta,\pi/2\gamma+\delta)$ is a sectorial region in t-space defined in Section 1. Let $\psi(x,t)$ be holomorphic in $U\times S_{\{0\}}(\theta,\pi/2\gamma+\delta)$ and $|\psi(x,t)|\le C|t|^{\varepsilon}$ for some $\varepsilon>0$. Let $\xi\neq 0$ with $|\arg\xi-\theta|<\delta$. Let $\mathscr{C}$ be a contour in $S_{\{0\}}(\theta,\pi/2\gamma+\delta)$ from $0\exp(i(\theta'+\arg\xi))$ to $0\exp(i(-\theta'+\arg\xi))$ with $\pi/2\gamma<\theta'<\pi/2\gamma+\min\{\theta+\delta-\arg\xi,\ \arg\xi-\theta+\delta,\pi/2\gamma\}$. Then we define γ-Borel transform $(\mathscr{B}_{\gamma,\theta}\psi)(x,\xi)$ by

$$(\mathscr{B}_{\gamma,\theta}\psi)(x,\xi)=\frac{1}{2\pi i}\int_{\mathscr{C}}\exp\left(\left(\frac{\xi}{t}\right)^{\gamma}\right)\psi(x,t)dt^{-\gamma}. \tag{5.4}$$

Let $\phi_i(x,\xi)\in\mathscr{O}(U\times S^*_{\{0\}}(\theta,\delta))$ $(i=1,2)$ satisfying $|\phi_i(x,\xi)|\le C|\xi|^{\varepsilon-\gamma}$ $(\varepsilon>0)$. Then γ-convolution of $\phi_1(x,\xi)$ and $\phi_2(x,\xi)$ is defined by

$$(\phi_1\underset{\gamma}{*}\phi_2)(x,\xi)=\int_0^{\xi}\phi_1(x,(\xi^{\gamma}-\eta^{\gamma})^{1/\gamma})\phi_2(x,\eta)d\eta^{\gamma}\quad \xi\in S^*_{\{0\}}(\theta,\delta). \tag{5.5}$$

The following relations hold.

LEMMA 5.1. *Suppose that* $\phi_i(x,\xi)\in\mathscr{O}(U\times S^*(\theta,\delta))$ $(i=0,1,2)$ *satisfy the estimates* (5.2). *Then*

$$\mathscr{B}_{\gamma,\theta}\mathscr{L}_{\gamma,\theta}\phi_0=\phi_0, \tag{5.6}$$

$$(\mathscr{L}_{\gamma,\theta}\phi_1)(\mathscr{L}_{\gamma,\theta}\phi_2)=\mathscr{L}_{\gamma,\theta}(\phi_1\underset{\gamma}{*}\phi_2). \tag{5.7}$$

We have a characterization of $f(x,t)\in\mathscr{O}(U)\{t\}_{\gamma,\theta}$ by its γ-Borel transform $(\mathscr{B}_{\gamma,\theta}f)(x,\xi)$.

PROPOSITION 5.2. *Suppose that* $f(x,t)\in\mathscr{O}(U)\{t\}_{\gamma,\theta}$ *and its asymptotic expansion is* $\sum_{m=k}^{\infty}f_m(x)t^m$ *with* $k\ge 1$. *Then for any* $V\Subset U$ *there is a positive constant* $\hat{\xi}_0>0$ *such that* $(\mathscr{B}_{\gamma,\theta}f)(x,\xi)$ *is holomorphic in* $\{0<|\xi|<\hat{\xi}_0\}$ *and*

$$(\mathscr{B}_{\gamma,\theta}f)(x,\xi)=\sum_{m=k}^{\infty}\frac{f_m(x)}{\Gamma(m/\gamma)}\xi^{m-\gamma} \tag{5.8}$$

holds. Moreover it is holomorphically extensible to $S^*(\theta,\delta)$ *for some* $\delta>0$ *such that*

$$|(\mathscr{B}_{\gamma,\theta}f)(x,\xi)|\le C|\xi|^{k-\gamma}\exp(c|\xi|^{\gamma}) \tag{5.9}$$

in $V\times(\{0<|\xi|<\hat{\xi}_0\}\cup S^*(\theta,\delta))$ *and*

$$f(x,t)=\int_0^{\infty e^{i\theta}}\exp\left(-\left(\frac{\xi}{t}\right)^{\gamma}\right)(\mathscr{B}_{\gamma,\theta}f)(x,\xi)d\xi^{\gamma}. \tag{5.10}$$

Next let us study majorant functions. For formal power series of n variables $A(x)=\sum_{\alpha}A_{\alpha}x^{\alpha}$ and $B(x)=\sum_{\alpha}B_{\alpha}x^{\alpha}$, $A(x)\ll B(x)$ means $|A_{\alpha}|\le B_{\alpha}$ for all $\alpha\in\boldsymbol{N}^n$. $A(x)\gg 0$ means $A_{\alpha}\ge 0$ for all $\alpha\in\boldsymbol{N}^n$. Let $\theta(X)$ be a power series of one variable X defined by

$$\theta(X) = c\sum_{k=0}^{+\infty}\frac{X^k}{(k+1)^2} \qquad c>0, \tag{5.11}$$

which is used in [**10**] and [**17**]. By $\theta(X)\theta(X) = c^2\sum_{k=0}^{\infty}(\sum_{l+m=k}1/((l+1)^2(m+1)^2))X^k \ll c^2C\sum_{k=0}^{\infty}X^k/(k+1)^2$ for some $C>0$, we choose $c>0$ so that $\theta(X)\theta(X)\ll\theta(X)$ and fix it. Set $\Phi(r;X)=\theta(X/r)$ for $r>0$ and we denote $(d/dX)^s\Phi(r;X)=r^{-s}\theta^{(s)}(X/r)$ by $\Phi^{(s)}(r;X)$.

LEMMA 5.3. *Let $0<r<R\le 1$. The following estimates hold.*

$$\begin{aligned}(s+1)\Phi^{(s)}(r;X) &\ll 4\Phi^{(s+1)}(r;X),\\ \Phi^{(s')}(r;X)\Phi^{(s'')}(r;X) &\ll \Phi^{(s'+s'')}(r;X),\\ \Phi^{(s)}(R;X) &\ll (r/R)^s\Phi^{(s)}(r;X).\end{aligned} \tag{5.12}$$

PROOF. Since $\theta^{(s+1)}(X) = \sum_{k=0}^{\infty}((k+s+1)(k+s)\cdots(k+1)/(k+s+2)^2)X^k$ and $(k+s+1)^3/(k+s+2)^2\ge(s+1)/4$, we have $(s+1)\theta^{(s)}(X)\ll 4\theta^{(s+1)}(X)$ and the first estimate. By differentiating $\Phi(r;X)\Phi(r;X)\ll\Phi(r;X)$ $(s'+s'')$-times, we have the second. By $\theta^{(s)}(X/R)\ll\theta^{(s)}(X/r)$ for $0<r<R$, we have $\Phi^{(s)}(R;X)=R^{-s}\theta^{(s)}(X/R)\ll R^{-s}\theta^{(s)}(X/r)\ll(r/R)^s r^{-s}\theta^{(s)}(X/r)=(r/R)^s\Phi^{(s)}(r;X)$. □

PROPOSITION 5.4. (1) *Let i and n be positive integers. Then*

$$\sum_{\substack{n_1,n_2\cdots,n_i\in\boldsymbol{N}\\ n_1+n_2+\cdots+n_i=n}}\frac{\Phi^{(n_1)}(r;X)\Phi^{(n_2)}(r;X)\cdots\Phi^{(n_i)}(r;X)}{n_1!n_2!\cdots n_i!}\ll\frac{\Phi^{(n)}(r;X)}{n!}. \tag{5.13}$$

(2) *Let N be a positive integer and $m\in\boldsymbol{N}^N$. Then*

$$\sum_{\substack{p,q\in\boldsymbol{N}^N\\ p+q=m}}\frac{\Phi^{(|p|+s')}(r;X)\Phi^{(|q|+s'')}(r;X)}{p!q!}\ll\frac{\Phi^{(|m|+s'+s'')}(r;X)}{m!}. \tag{5.14}$$

PROOF. By differentiating $\overbrace{\Phi(r;X)\cdots\Phi(r;X)}^{i}\ll\Phi(r;X)$ n-times, we have (5.13). We show (5.14). For $N=1$ we have from (5.13)

$$\sum_{\substack{p_1,q_1\in\boldsymbol{N}\\ p_1+q_1=m_1}}\frac{\Phi^{(p_1)}(r;X)\Phi^{(q_1)}(r;X)}{p_1!q_1!}\ll\frac{\Phi^{(p_1+q_1)}(r;X)}{m_1!}$$

and by differentiating $(s'+s'')$-times we have

$$\sum_{\substack{p_1,q_1\in\boldsymbol{N}\\ p_1+q_1=m_1}}\frac{\Phi^{(p_1+s')}(r;X)\Phi^{(q_1+s'')}(r;X)}{p_1!q_1!}\ll\frac{\Phi^{(m_1+s'+s'')}(r;X)}{m_1!}.$$

Assume

$$\sum_{\substack{p',q'\in \boldsymbol{N}^{N-1}\\ p'+q'=m'}} \frac{\Phi^{(|p'|+s')}(r;X)\Phi^{(|q'|+s'')}(r;X)}{p'!q'!} \ll \frac{\Phi^{(|m'|+s'+s'')}(r;X)}{m'!}.$$

Then, by differentiating m_N-times, we have (5.14). □

LEMMA 5.5. *Let* $x=(x_1,\cdots,x_n)\in \boldsymbol{C}^n$. *Let* $\varphi_i(\xi,x), i=1,2$, *be holomorphic functions in* x *in a neighborhood of* $x=0$ *and continuous in* ξ *on* $\arg\xi=\theta$. *Put* $X=\sum_{i=1}^n x_i$. *Suppose that there are* $\Psi_i(X)\gg 0$, $s_i>0$ *and* $\gamma>0$ *such that*

$$\varphi_i(\xi,x)\ll \frac{A_i e^{c|\xi|^\gamma}|\xi|^{s_i-\gamma}}{\Gamma(s_i/\gamma)}\Psi_i(X). \tag{5.15}$$

Then for ξ *with* $\arg\xi=\theta$

$$\varphi_1(\xi,x)\underset{\gamma}{*}\varphi_2(\xi,x)\ll \frac{A_1A_2 e^{c|\xi|^\gamma}|\xi|^{s_1+s_2-\gamma}}{\Gamma((s_1+s_2)/\gamma)}\Psi_1(X)\Psi_2(X). \tag{5.16}$$

PROOF. We have

$$\begin{aligned}\varphi_1(\xi,x)\underset{\gamma}{*}\varphi_2(\xi,x)&=\int_0^{|\xi|e^{i\theta}}\varphi_1((\xi^\gamma-\eta^\gamma)^{1/\gamma},x)\varphi_2(\eta,x)d\eta^\gamma\\ &=\int_0^{|\xi|}\varphi_1(((|\xi|^\gamma-r^\gamma)^{1/\gamma}e^{i\theta},x)\varphi_2(re^{i\theta},x)e^{i\gamma\theta}dr^\gamma\\ &\ll \frac{A_1A_2e^{c|\xi|^\gamma}}{\Gamma(s_1/\gamma)\Gamma(s_2/\gamma)}\left(\int_0^{|\xi|}(|\xi|^\gamma-r^\gamma)^{s_1/\gamma-1}r^{s_2-\gamma}dr^\gamma\right)\Psi_1(X)\Psi_2(X)\\ &\ll \frac{A_1A_2e^{c|\xi|^\gamma}|\xi|^{s_1+s_2-\gamma}}{\Gamma((s_1+s_2)/\gamma)}\Psi_1(X)\Psi_2(X).\end{aligned}$$

□

LEMMA 5.6. *Let* $(x,y)\in \boldsymbol{C}^{d_0}\times \boldsymbol{C}^{d_2}$ *and* $f(x,y)=\sum_{m\in \boldsymbol{N}^{d_0}} f_m(y)x^m$ *be a holomorphic function in a neighborhood of* $(x,y)=(0,0)$. *Set* $X=\sum_{i=1}^{d_0}x_i$ *and* $Y=\sum_{i=1}^{d_2}y_i$. *Then the following estimates hold.*

(1) *If* $f(x,y)\ll C\Phi^{(s)}(R;X+Y)$, *then* $f_m(y)\ll C\Phi^{(s+|m|)}(R;Y)/m!$.

(2) *If* $f(x,y)\ll CX\Phi^{(s+1)}(R;X+Y)$, *then* $f_m(y)\ll C|m|\Phi^{(s+|m|)}(R;Y)/m!$.

PROOF. The first assertion follows from $\partial_x^m f(0,y)\ll C\Phi^{(s+|m|)}(R;Y)$. The second follows from $\partial_x^m f(0,y)\ll C|m|\Phi^{(s+|m|)}(R;Y)$. □

6. Proofs of Theorems 2.3 and 2.4.

The proofs of Theorems 2.3 and 2.4 are almost the same. So we give the proof of Theorem 2.3 in detail. We sum up shortly the assumptions of Theorem 2.3. Set $U=\{(x,y)\in \boldsymbol{C}^{d_0}\times$

$\boldsymbol{C}^{d_2}; |x| < R, |y| < R\}$ and $U_0 = \{u \in \boldsymbol{C}; |u| < R_0\}$. $P = P(x,y,t,\partial_x,\partial_y,\partial_t)$ is a vector field with coefficients in $\mathscr{O}(U)\{t\}_{\gamma,\theta_0}$,

$$P = \sum_{i=1}^{d_0}(\lambda_i x_i + \mu_{i-1}x_{i-1} + A_i(x,y,t))\partial_{x_i} + \sum_{j=1}^{d_1} B_j(x,y,t)\partial_{y_j} + t^{\gamma+1}C(x,y,t)\partial_t, \tag{6.1}$$

where γ is a positive integer and the coefficients satisfy (see (2.2))

$$\begin{cases} A_i(0,y,t) = O(|t|^{\gamma+1}), \quad A_i(x,y,t) = O\big((|x|+|y|+|t|)^2\big) \\ B_j(0,y,t) = O(|t|^{\gamma+1}), \quad B_j(x,y,t) = O\big((|x|+|y|+|t|)^2\big) \\ C(0,0,0) \neq 0. \end{cases} \tag{6.2}$$

It follows from (6.2) that the coefficients $A_i(x,y,t), B_j(x,y,t)$ and $C(x,y,t)$ can be represented in the following form

$$\begin{cases} A_i(x,y,t) = a_i(x,y) + a_{i,0}(y,t) + a_{i,1}(x,y,t), \\ B_j(x,y,t) = b_j(x,y) + b_{j,0}(y,t) + b_{j,1}(x,y,t), \\ C(x,y,t) = c_0(x,y) + c_1(x,y,t), \end{cases} \tag{6.3}$$

where

$$\begin{cases} a_i(0,y) = 0, & a_i(x,y) = O\big((|x|+|y|)^2\big), \\ b_j(0,y) = 0, & b_j(x,y) = O\big((|x|+|y|)^2\big), \\ a_{i,0}(y,t) = O(|t|^{\gamma+1}), & a_{i,1}(0,y,t) = a_{i,1}(x,y,0) = 0, \\ b_{j,0}(y,t) = O(|t|^{\gamma+1}), & b_{j,1}(0,y,t) = b_{i,1}(x,y,0) = 0, \\ c_0(0,0) \neq 0, & c_1(x,y,0) = 0. \end{cases} \tag{6.4}$$

As for $A_i(x,y,t)$, by setting $a_i(x,y) = A_i(x,y,0)$, $a_{i,0}(y,t) = A_i(0,y,t)$ and $a_{i,1}(x,y,t) = A_i(x,y,t) - a_i(x,y) - a_{i,0}(y,t)$, we have $A_i(x,y,t) = a_i(x,y) + a_{i,0}(y,t) + a_{i,2}(x,y,t)$ with (6.4). Thus we assume P is of the form

$$\begin{aligned} P(x,y,t,\partial_x,\partial_y,\partial_t) = &\sum_{i=1}^{d_0}(\lambda_i x_i + \mu_{i-1}x_{i-1} + a_i(x,y) + a_{i,0}(y,t) + a_{i,1}(x,y,t))\partial_{x_i} \\ &+ \sum_{j=1}^{d_2}(b_j(x,y) + b_{j,0}(y,t) + b_{j,1}(x,y,t))\partial_{y_j} + t^{\gamma+1}(c_0(x,y) + c_1(x,y,t))\partial_t \end{aligned} \tag{6.5}$$

and the coefficients satisfy (6.4). As for the nonlinear term, $F(x,y,t,u) \in \mathscr{O}(U \times U_0)\{t\}_{\gamma,\theta_0}$ with

$F(x,y,0,0)=0$ (see (2.10)), hence

$$\begin{cases} F(x,y,t,u)=\sum_{i=0}^{+\infty} F_i(x,y,t)u^i, \quad F_0(x,y,0)=0, \\ F_i(x,y,t)\in\mathscr{O}(U)\{t\}_{\gamma,\theta_0}. \end{cases} \tag{6.6}$$

An important assumption is (2.13), that is, for ξ with $|\arg\xi-\theta_0|<\delta_0$

$$\left|\sum_{i=1}^{d_0} m_i\lambda_i+\gamma c_0(0,0)\xi^\gamma-F_1(0,0,0)\right|\geq K_0(|m|+|\xi|^\gamma+1) \tag{6.7}$$

holds for all $m=(m_1,\cdots,m_{d_0})\in\boldsymbol{N}^{d_0}$.

Our aim is to construct a solution $u(x,y,t)$ of $Pu=F(x,y,t,u)$ by Laplace integral (see (5.1)). We denote by $\hat{g}(x,y,\xi)$ γ-Borel transform of $g(x,y,t)$ with respect to t, so $g(x,y,t)=\int_0^{\infty e^{i\theta_0}}\exp(-(\xi/t)^\gamma)\hat{g}(x,y,\xi)d\xi^\gamma$. Now let us proceed to find the equation that $\hat{u}(x,y,\xi)$ satisfies. The coefficients of P belong to $\mathscr{O}(U)\{t\}_{\gamma,\theta_0}$, so we can represent them by γ-Laplace transform. By shrinking U if necessary, it follows from Proposition 5.2 that there is a constant $\hat{\xi}_0>0$ such that γ-Borel transforms of the coefficients are holomorphic in $(x,y,\xi)\in U\times\Xi_0^*$,

$$\Xi_0^*=\{0<|\xi|<\hat{\xi}_0\}\cup S^*(\theta_0,\delta_0), \tag{6.8}$$

and by (6.4) there are constants C_0 and c_0 such that

$$\begin{cases} |\hat{a}_{i,0}(y,\xi)|,\ |\hat{b}_{j,0}(y,\xi)|\leq C_0|\xi|\exp(c_0|\xi|^\gamma), \\ |\hat{a}_{i,1}(x,y,\xi)|,\ |\hat{b}_{j,1}(x,y,\xi)|\leq C_0|x||\xi|^{1-\gamma}\exp(c_0|\xi|^\gamma), \\ |\hat{c}_1(x,y,\xi)|\leq C_0|\xi|^{1-\gamma}\exp(c_0|\xi|^\gamma) \end{cases} \tag{6.9}$$

(see (5.9)), hence there is a constant $c_0>0$ such that

$$\begin{cases} \hat{a}_{i,0}(y,\xi),\ \hat{b}_{j,0}(y,\xi)\ll C_0|\xi|\exp(c_0|\xi|^\gamma)\Phi(R;Y), \\ \hat{a}_{i,1}(x,y,\xi),\ \hat{b}_{j,1}(x,y,\xi)\ll C_0|\xi|^{1-\gamma}\exp(c_0|\xi|^\gamma)X\Phi(R;X+Y), \\ \hat{c}_1(x,y,\xi)\ll C_0|\xi|^{1-\gamma}\exp(c_0|\xi|^\gamma)\Phi(R;X+Y), \end{cases} \tag{6.10}$$

where the majorant function $\Phi(R;X)$ is that defined in Section 5 and $X=\sum_{i=1}^{d_0}x_i$, $Y=\sum_{i=1}^{d_2}y_i$ and $R>0$ is some constant. Let us apply γ-Borel transform to the equation $Pu=F(x,y,t,u)$. Set

$$F_{i,0}(x,y)=F_i(x,y,0), \quad F_{i,1}(x,y,t)=F_i(x,y,t)-F_i(x,y,0) \tag{6.11}$$

(see (6.6)) and define

$$\begin{aligned}\mathscr{P}(x,y,\xi,\partial_x,\partial_y) = &\sum_{i=1}^{d_0}(\lambda_i x_i + \mu_{i-1}x_{i-1} + a_i(x,y))\partial_{x_i}\\ &+\sum_{j=1}^{d_2} b_j(x,y)\partial_{y_j} + \gamma c_0(x,y)\xi^\gamma - F_{1,0}(x,y),\end{aligned} \tag{6.12}$$

$$\begin{aligned}\mathscr{Q}(x,y,\xi,\partial_x,\partial_y) = &\sum_{i=1}^{d_0}\big(\hat{a}_{i,0}(y,\xi) + \hat{a}_{i,1}(x,y,\xi)\big) \underset{\gamma}{*} (\partial_{x_i}\cdot\,)\\ &+\sum_{j=1}^{d_2}\big(\hat{b}_{j,0}(y,\xi) + \hat{b}_{j,1}(x,y,\xi)\big) \underset{\gamma}{*} (\partial_{y_j}\cdot\,) + \hat{c}_1(x,y,\xi) \underset{\gamma}{*} (\gamma\xi^\gamma\cdot\,).\end{aligned} \tag{6.13}$$

$\mathscr{P}(x,y,\xi,\partial_x,\partial_y)$ is a singular linear partial differential operator with a parameter ξ and $\mathscr{Q}(x,y,\xi,\partial_x,\partial_y)$ is a linear partial differential convolution operator. Let $\mathscr{F}(x,y,\xi,v)$ be a nonlinear convolution operator defined by

$$\mathscr{F}(x,y,\xi,v) = \sum_{i=2}^{\infty} F_{i,0}(x,y)(\overbrace{v \underset{\gamma}{*} v \underset{\gamma}{*} \cdots \underset{\gamma}{*} v}^{i}) + \sum_{i=1}^{\infty} \hat{F}_{i,1}(x,y,\xi) \underset{\gamma}{*} \overbrace{v \underset{\gamma}{*} v \underset{\gamma}{*} \cdots \underset{\gamma}{*} v}^{i}. \tag{6.14}$$

Putting $u = (\mathscr{L}_{\gamma,\theta_0}\hat{u})(x,y,t)$, we have

$$\begin{aligned}&(P - F_{1,0}(x,y))u = \mathscr{L}_{\gamma,\theta_0}\big(\mathscr{P}(x,y,\xi,\partial_x,\partial_y)\hat{u} + \mathscr{Q}(x,y,\xi,\partial_x,\partial_y)\hat{u}\big),\\ &F(x,y,u) - F_{1,0}(x,y)u = \mathscr{L}_{\gamma,\theta_0}\big(\hat{F}_0(x,y,\xi) + \mathscr{F}(x,y,\xi,\hat{u})\big).\end{aligned}$$

So $\hat{u}$ satisfies

$$\mathscr{P}(x,y,\xi,\partial_x,\partial_y)\hat{u} + \mathscr{Q}(x,y,\xi,\partial_x,\partial_y)\hat{u} = \hat{F}_0(x,y,\xi) + \mathscr{F}(x,y,\xi,\hat{u}). \tag{6.15}$$

Hence we have

$$\mathscr{P}(x,y,\xi,\partial_x,\partial_y)v + \mathscr{Q}(x,y,\xi,\partial_x,\partial_y)v = \hat{F}_0(x,y,\xi) + \mathscr{F}(\xi,x,y,v), \tag{6.16}$$

which is the equation to be solved. We shall show the existence of a solution $v(x,y,\xi)$ of (6.16) and get its estimate. The coefficients of (6.16) are holomorphic in $(x,y,\xi) \in U \times \Xi_0^*$ and the estimates (6.9) and (6.10) hold. Set

$$h(x,y,\xi) = \gamma c_0(x,y)\xi^\gamma - F_{1,0}(x,y). \tag{6.17}$$

Then it follows from (6.7) that for a small neighborhood U there is a constant $K > 0$ such that for $(0,y,\xi) \in U \times \Xi_0^*$ and $m \in \boldsymbol{N}^{d_0}$

$$\left(\sum_{i=1}^{d_0} m_i\lambda_i + h(0,y,\xi)\right)^{-1} \ll \frac{K\Phi(R;Y)}{(1+|\xi|)^\gamma + |m|} \tag{6.18}$$

holds. Under the above assumptions, we have

THEOREM 6.1. *There exists a solution* $v(x,y,\xi)$ *of* (6.16) *which is holomorphic in* $(x,y,\xi)\in V\times\Xi_0^*$, *where* $V=\{(x,y);|x|<r,|y|<r\}$ *for some* $r>0$, *and has the following properties.*
(1) $\xi^{-1+\gamma}v(x,y,\xi)$ *is holomorphic in* $V\times\Xi_0^*$.
(2) *There exist positive constants C and c such that*

$$|v(x,y,\xi)|\le C|\xi|^{1-\gamma}\exp(c|\xi|^{\gamma})\quad \textit{for}\ \ (x,y,\xi)\in V\times\Xi_0^*. \tag{6.19}$$

After completing the proof of Theorem 6.1, we show Theorem 2.3. The proof of Theorem 6.1 consists of several steps, so we give lemmas and propositions. We use in the following discussions Lemmas 5.3, 5.5 and 5.6 and Proposition 5.4. Set $h_1(x,y,\xi)=h(x,y,\xi)-h(0,y,\xi)$ and

$$\mathscr{P}_0(x,y,\xi,\partial_x)=\sum_{i=1}^{d_0}(\lambda_i x_i+\mu_{i-1}x_{i-1})\partial_{x_i}+h(0,y,\xi)$$
$$\mathscr{P}_1(x,y,\partial_x,\partial_y)=h_1(x,y,\xi)+\sum_{i=1}^{d_0}a_i(x,y)\partial_{x_i}+\sum_{j=1}^{d_2}b_j(x,y)\partial_{y_j}. \tag{6.20}$$

Then $\mathscr{P}(x,y,\xi,\partial_x,\partial_y)=\mathscr{P}_0(x,y,\xi,\partial_x)+\mathscr{P}_1(x,y,\partial_x,\partial_y)$. Here we give a remark on the constants $\{\mu_i\}_{i=1}^{d_0-1}$. Set $w_i=c^i x_i, c>0$, for $1\le i\le d_0$. Then $\mathscr{P}_0(x,y,\xi,\partial_x)=(\lambda_i w_i+c\mu_{i-1}w_{i-1})\partial_{w_i}+h(0,y,\xi)$, so μ_i changes to $c\mu_i$. Hence, by choosing small $c>0$, μ_i becomes as small as possible.

LEMMA 6.2. *Let* $f(x,y,\xi)$ $(\xi\in\Xi_0^*)$ *be a holomorphic function satisfying* $f(x,y,\xi)\ll M(|\xi|)\Phi^{(s)}(r;X+Y)$ *or* $f(x,y,\xi)\ll M(|\xi|)X\Phi^{(s+1)}(r;X+Y)$. *Then there is a unique holomorphic solution* $v(x,y,\xi)$ *of*

$$\mathscr{P}_0(x,y,\xi,\partial_x)v(x,y,\xi)=f(x,y,\xi) \tag{6.21}$$

with the following estimate. There are constants r_0 *and* C_0 *such that for* $0<r\le r_0$ *if* $f(x,y,\xi)\ll M(|\xi|)\Phi^{(s)}(r;X+Y)$,

$$v(x,y,\xi)\ll\frac{C_0M(|\xi|)}{(1+|\xi|)^{\gamma}}\Phi^{(s)}(r;X+Y) \tag{6.22}$$

and if $f(x,y,\xi)\ll M(|\xi|)X\Phi^{(s+1)}(r;X+Y)$,

$$v(x,y,\xi)\ll C_0M(|\xi|)\Phi^{(s)}(r;X+Y). \tag{6.23}$$

PROOF. Set $\mathscr{P}_{0,0}(x,y,\xi,\partial_x)=\sum_{i=1}^{d_0}\lambda_i x_i\partial_{x_i}+h(0,y,\xi)$ and first consider

$$\mathscr{P}_{0,0}(x,y,\xi,\partial_x)v(x,y,\xi)=f(x,y,\xi). \tag{6.24}$$

Let $v(x,y,\xi)=\sum_{m\in\boldsymbol{N}^{d_0}} v_m(y,\xi)x^m$ and $f(x,y,\xi)=\sum_{m\in\boldsymbol{N}^{d_0}} f_m(y,\xi)x^m$. Then

$$\left(\sum_{i=1}^{d_0} m_i\lambda_i + h(0,y,\xi)\right)v_m(y,\xi) = f_m(y,\xi).$$

Hence we can uniquely determine $v_m(y,\xi)$. Let us estimate them. Assume $f(x,y,\xi) \ll M(|\xi|)\Phi^{(s)}(r;X+Y)$. Then there is a constant C such that $v_m(y,\xi) \ll CM(|\xi|)(1+|\xi|)^{-\gamma}\Phi^{(s+|m|)}(r;Y)/m!$ by (6.18) and Lemma 5.6. Hence $v(x,y,\xi) \ll CM(|\xi|)(1+|\xi|)^{-\gamma}\Phi^{(s)}(r;X+Y)$. Next assume $f(x,y,\xi) \ll M(|\xi|)X\Phi^{(s+1)}(r;X+Y)$. Then $f_m(y,\xi) \ll M(|\xi|)|m|\Phi^{(s+|m|)}(r;Y)/m!$ by Lemma 5.6, hence, $v_m(y,\xi) \ll CM(|\xi|)\Phi^{(s+|m|)}(r;Y)/m!$ for some $C>0$ and we have $v(x,y,\xi) \ll CM(|\xi|)\Phi^{(s)}(r;X+Y)$.

Now let us solve (6.21). Consider

$$\begin{aligned}&\mathscr{P}_{0,0}(x,y,\xi,\partial_x)v^0(x,y,\xi) = f(x,y,\xi)\\ &\mathscr{P}_{0,0}(x,y,\xi,\partial_x)v^n(x,y,\xi) + \left(\sum_{i=1}^{d_0}\mu_{i-1}x_{i-1}\partial_{x_i}\right)v^{n-1}(x,y,\xi) = 0,\end{aligned} \tag{6.25}$$

where we may assume that $C(\sum_{i=2}^{d_0}\mu_{i-1}) < 1/2$ by the above remark. We can determine successively $v^n(x,y,\xi)$. Let us show the convergence. Suppose $f(x,y,\xi) \ll M(|\xi|)\Phi^{(s)}(r;X+Y)$. Then $v^0(x,y,\xi) \ll CM(|\xi|)(1+|\xi|)^{-\gamma}\Phi^{(s)}(r;X+Y)$. Assume $v^{n-1}(x,y,\xi) \ll C2^{-n+1}M(|\xi|)(1+|\xi|)^{-\gamma}\Phi^{(s)}(r;X+Y)$. Then

$$\left(\sum_{i=1}^{d_0}\mu_{i-1}x_{i-1}\partial_{x_i}\right)v^{n-1}(x,y,\xi) \ll \frac{2^{-n}M(|\xi|)}{(1+|\xi|)^{\gamma}}X\Phi^{(s+1)}(r;X+Y)$$

and we have $v^n(x,y,\xi) \ll C2^{-n}M(|\xi|)(1+|\xi|)^{-\gamma}\Phi^{(s)}(r;X+Y)$ and $v(x,y,\xi)=\sum_{n=0}^{\infty}v^n(x,y,\xi) \ll 2CM(|\xi|)(1+|\xi|)^{-\gamma}\Phi^{(s)}(r;X+Y)$. In the other case $f(x,y,\xi) \ll M(|\xi|)X\Phi^{(s+1)}(r;X+Y)$ we can show the existence of a solution $v(x,y,\xi)$ with (6.23) in the same way. □

LEMMA 6.3. *Let $v(x,y,\xi)$ $(\xi \in \Xi_0^*)$ be a holomorphic function with $v(x,y,\xi) \ll M_1(|\xi|)\Phi^{(s)}(r;X+Y)$. Then for any $\varepsilon>0$ there is $r_1>0$ such that for $0<r\le r_1$*

$$\mathscr{P}_1(x,y,\partial_x,\partial_y)v(x,y,\xi) \ll \varepsilon M_1(|\xi|)\big((1+|\xi|)^{\gamma}\Phi^{(s)}(r;X+Y) + X\Phi^{(s+1)}(r;X+Y)\big). \tag{6.26}$$

PROOF. It follows from (6.4) and $h_1(0,y,\xi)=0$ that for any $\varepsilon>0$ there is $R>0$ such that $a_i(x,y), b_j(x,y) \ll (\varepsilon/(d_0+d_2))X\Phi(R;X+Y)$, and $h_1(x,y,\xi) \ll \varepsilon(1+|\xi|)^{\gamma}\Phi(R;X+Y)$. So for $0<r\le r_1<R$

$$\begin{aligned}&\sum_{i=1}^{d_0}a_i(x,y)\partial_{x_i}v(x,y,\xi) + \sum_{j=1}^{d_2}b_j(x,y)\partial_{y_j}v(x,y,\xi) \ll \varepsilon M_1(|\xi|)X\Phi^{(s+1)}(r;X+Y)\\ &h_1(x,y,\xi)v(x,y,\xi) \ll \varepsilon(1+|\xi|)^{\gamma}M_1(|\xi|)\Phi^{(s)}(r;X+Y),\end{aligned}$$

hence, we have (6.26). □

Thus we have the solvability of $\mathscr{P}(x,y,\xi,\partial_x,\partial_y)v(x,y,\xi) = f(x,y,\xi)$.

PROPOSITION 6.4. *Let $f(x,y,\xi)$ ($\xi \in \Xi_0^*$) be a holomorphic function with $f(x,y,\xi) \ll M(|\xi|)\Phi^{(s)}(r;X+Y)$ or $f(x,y,\xi) \ll M(|\xi|)X\Phi^{(s+1)}(r;X+Y)$. Then there is a constant $r_0 > 0$ such that for $0 < r \le r_0$ there exists a unique holomorphic solution $v(x,y,\xi)$ of*

$$\mathscr{P}(x,y,\xi,\partial_x,\partial_y)v(x,y,\xi) = f(x,y,\xi). \tag{6.27}$$

Furthermore there exists a constant C such that if $f(x,y,\xi) \ll M(|\xi|)\Phi^{(s)}(r;X+Y)$,

$$v(x,y,\xi) \ll \frac{CM(|\xi|)}{(1+|\xi|)^{\gamma}}\Phi^{(s)}(r;X+Y) \tag{6.28}$$

and if $f(x,y,\xi) \ll M(|\xi|)X\Phi^{(s+1)}(r;X+Y)$,

$$v(x,y,\xi) \ll CM(|\xi|)\Phi^{(s)}(r;X+Y). \tag{6.29}$$

PROOF. Let us show the existence of a solution by iteration. We define $v^n(x,y,\xi)$ ($n = 0,1,\cdots$) as follows:

$$\mathscr{P}_0(x,y,\xi,\partial_x)v^0(x,y,\xi) = f(x,y,\xi),$$
$$\mathscr{P}_0(x,y,\xi,\partial_x)v^n(x,y,\xi) + \mathscr{P}_1(x,y,\partial_x,\partial_y)v^{n-1}(x,y,\xi,x,y) = 0.$$

We can determine successively $v^n(x,y,\xi)$ by Lemma 6.2. Let us show the convergence of $\sum_{n=0}^{\infty} v^n(x,y,\xi)$. Assume $f(x,y,\xi) \ll M(|\xi|)\Phi^{(s)}(r;X+Y)$. Then by Lemma 6.2 $v^0(x,y,\xi) \ll C_0M(\xi)(1+|\xi|)^{-\gamma}\Phi^{(s)}(r;X+Y)$. Assume $v^{n-1}(x,y,\xi) \ll C_0 2^{-n+1}M(|\xi|)(1+|\xi|)^{-\gamma}\Phi^{(s)}(r;X+Y)$. Then we have by Lemma 6.3

$$\begin{aligned}\mathscr{P}_1(x,y,\partial_x,\partial_y)v(x,y,\xi) &\ll C_0 2^{-n+1}\varepsilon M(|\xi|)\\ &\quad\times\bigl(\Phi^{(s)}(r;X+Y) + (1+|\xi|)^{-\gamma}X\Phi^{(s+1)}(r;X+Y)\bigr).\end{aligned}$$

By Lemma 6.2 $v^n(x,y,\xi) \ll 2C_0^2\varepsilon 2^{-n+1}M(|\xi|)(1+|\xi|)^{-\gamma}\Phi^{(s)}(r;X+Y)$. Now choose ε so small such that $0 < C_0\varepsilon < 1/4$. Then $2C_0^2\varepsilon 2^{-n+1} \le C_0 2^{-n}$ and $v^n(x,y,\xi) \ll C_0 2^{-n}M(|\xi|)(1+|\xi|)^{-\gamma}\Phi^{(s)}(r;X+Y)$. So $v(x,y,\xi) = \sum_{n=0}^{\infty} v^n(x,y,\xi)$ converges and (6.28) holds. If $f(x,y,\xi) \ll M(|\xi|)X\Phi^{(s+1)}(r;X+Y)$, we also have $v^n(x,y,\xi) \ll C_0 2^{-n}M(|\xi|)\Phi^{(s)}(r;X+Y)$ in the same way as above. If $f(x,y,\xi) \equiv 0$, there exists $M_1(|\xi|)$ such that $v(x,y,\xi) \ll C_0 2^{-n}M_1(|\xi|)\Phi^{(s)}(r;X+Y)$ holds for any n, from which the uniqueness follows. □

Now let us proceed to construct a solution of (6.16). Define $v_n = v_n(x,y,\xi)$ ($n \in \boldsymbol{N}$) inductively, $v_0 = 0$ and by

$$\begin{aligned}
&\mathscr{P}(x,y,\xi,\partial_x,\partial_y)v_1 = \hat{F}_0(x,y,\xi),\\
&\mathscr{P}(x,y,\xi,\partial_x,\partial_y)v_n + \mathscr{Q}(x,y,\xi,\partial_x,\partial_y)v_{n-1}\\
&\quad = \sum_{i=2}^{\infty} F_{i,0}(x,y)\Bigg(\sum_{\substack{(n_1,n_2,\cdots,n_i)\in \boldsymbol{N}^i\\ n_1+n_2+\cdots+n_i=n}} \overbrace{v_{n_1} \underset{\gamma}{*} v_{n_2} \underset{\gamma}{*} \cdots \underset{\gamma}{*} v_{n_i}}^{i}\Bigg)\\
&\qquad + \sum_{i=1}^{\infty} \hat{F}_{i,1}(x,y,\xi) \underset{\gamma}{*} \Bigg(\sum_{\substack{(n_1,n_2,\cdots,n_i)\in \boldsymbol{N}^i\\ n_1+n_2+\cdots+n_i=n-1}} \overbrace{v_{n_1} \underset{\gamma}{*} v_{n_2} \underset{\gamma}{*} \cdots \underset{\gamma}{*} v_{n_i}}^{i}\Bigg). \qquad (6.30)
\end{aligned}$$

Since $v_0 = 0$, we have in (6.30)

$$\sum_{i=2}^{\infty} F_{i,0}(x,y)\Bigg(\sum_{\substack{(n_1,\cdots,n_i)\in \boldsymbol{N}^i\\ n_1+\cdots+n_i=n}} \overbrace{v_{n_1} \underset{\gamma}{*} v_{n_2} \underset{\gamma}{*} \cdots \underset{\gamma}{*} v_{n_i}}^{i}\Bigg) = \sum_{i=2}^{n} F_{i,0}(x,y)\Bigg(\sum_{\substack{(n_1,\cdots,n_i)\in \boldsymbol{N}^i\\ n_1+\cdots+n_i=n}} \cdots\Bigg),$$

$$\sum_{i=1}^{\infty} \hat{F}_{i,1}(x,y,\xi) \underset{\gamma}{*} \Bigg(\sum_{\substack{(n_1,\cdots,n_i)\in \boldsymbol{N}^i\\ n_1+\cdots+n_i=n-1}} \overbrace{v_{n_1} \underset{\gamma}{*} v_{n_2} \underset{\gamma}{*} \cdots \underset{\gamma}{*} v_{n_i}}^{i}\Bigg) = \sum_{i=1}^{n-1} \hat{F}_{i,1}(x,y,\xi) \underset{\gamma}{*} \Bigg(\sum_{\substack{(n_1,\cdots,n_i)\in \boldsymbol{N}^i\\ n_1+\cdots+n_i=n-1}} \cdots\Bigg). \qquad (6.31)$$

Let us show the existence of $v_n(x,y,\xi)$ $(n \geq 1)$ with estimate

$$v_n(x,y,\xi) \ll \frac{AB^{n-1}|\xi|^{n-\gamma} e^{c_0|\xi|^\gamma}}{\Gamma(n/\gamma)n!} \boldsymbol{\Phi}^{(n)}(r;X+Y), \qquad (6.32)$$

which is holomorphic in Ξ_0^* and $\xi^{-1+\gamma} v_n(x,y,\xi)$ is holomorphic at $\xi = 0$. We note that ξ is a holomorphic parameter in (6.16). In the following estimates $0 < r < R \leq 1$ and r and R are small, if necessary. It follows from $F_0(x,y,0) = 0$ and Proposition 5.2 that $\xi^{-1+\gamma}\hat{F}_0(x,y,\xi)$ is holomorphic at $\xi = 0$ and

$$\hat{F}_0(x,y,\xi) \ll \frac{C_0|\xi|^{1-\gamma} e^{c_0|\xi|^\gamma}}{\Gamma(1/\gamma)} \boldsymbol{\Phi}^{(1)}(R;X+Y). \qquad (6.33)$$

By Proposition 6.4 there exists $v_1(x,y,\xi)$ with (6.32) such that $v_1(x,y,\xi) = \sum_{i=1}^{\infty} v_{1,i}(x,y)\xi^{i-\gamma}$ in a neighborhood of $\xi = 0$. Assume that there exist $v_p(x,y,\xi)$ $(1 \leq p \leq n-1)$ with the above properties. Set

$$\begin{aligned}
I_{n-1}^0(x,y,\xi) &:= \sum_{i=2}^{n} F_{i,0}(x,y)\Bigg(\sum_{\substack{(n_1,n_2,\cdots,n_i)\in \boldsymbol{N}^i\\ n_1+n_2+\cdots+n_i=n}} \overbrace{v_{n_1} \underset{\gamma}{*} v_{n_2} \underset{\gamma}{*} \cdots \underset{\gamma}{*} v_{n_i}}^{i}\Bigg)\\
I_{n-1}^1(x,y,\xi) &:= \sum_{i=1}^{n-1} \hat{F}_{i,1}(x,y,\xi) \underset{\gamma}{*} \Bigg(\sum_{\substack{(n_1,n_2,\cdots,n_i)\in \boldsymbol{N}^i\\ n_1+n_2+\cdots+n_i=n-1}} \overbrace{v_{n_1} \underset{\gamma}{*} v_{n_2} \underset{\gamma}{*} \cdots \underset{\gamma}{*} v_{n_i}}^{i}\Bigg).
\end{aligned}$$

Then $\mathscr{P}(x,y,\xi,\partial_x,\partial_y)v_n + \mathscr{Q}(x,y,\xi,\partial_x,\partial_y)v_{n-1} = I^0_{n-1}(x,y,\xi) + I^1_{n-1}(x,y,\xi)$ by (6.30) and (6.31). Let us estimate $\{I^i_{n-1}(x,y,\xi)\}_{i=0,1}$ and $\mathscr{Q}(x,y,\xi,\partial_x,\partial_y)v_{n-1}$ for $n \geq 2$.

LEMMA 6.5. *For some constant $C_1 > 0$ the following estimates hold*:

$$I^0_{n-1}(x,y,\xi), I^1_{n-1}(x,y,\xi) \ll \frac{C_1 AB^{n-2}|\xi|^{n-\gamma}e^{c_0|\xi|^\gamma}}{\Gamma(n/\gamma)n!}\Phi^{(n)}(r;X+Y). \tag{6.34}$$

PROOF. We estimate $I^0_{n-1}(x,y,\xi)$. By (5.13) in Proposition 5.4 and Lemma 5.5 we have

$$\sum_{\substack{(n_1,n_2,\cdots,n_i)\in \boldsymbol{N}^i\\ n_1+n_2+\cdots+n_i=n}} \overbrace{v_{n_1} \underset{\gamma}{*} v_{n_2} \underset{\gamma}{*} \cdots \underset{\gamma}{*} v_{n_i}}^{i} \ll \frac{A^iB^{n-i}|\xi|^{n-\gamma}e^{c_0|\xi|^\gamma}}{\Gamma(n/\gamma)n!}\Phi^{(n)}(r;X+Y).$$

Hence from $F_{i,0}(x,y) \ll B_1^{i-1}\Phi(R;X+Y)$ for $i \geq 2$ we have

$$\begin{aligned}&\sum_{i=2}^n F_{i,0}(x,y)\Bigg(\sum_{\substack{(n_1,n_2,\cdots,n_i)\in \boldsymbol{N}^i\\ n_1+n_2+\cdots+n_i=n}} \overbrace{v_{n_1} \underset{\gamma}{*} v_{n_2} \underset{\gamma}{*} \cdots \underset{\gamma}{*} v_{n_i}}^{i}\Bigg)\\ &\qquad \ll \left(\sum_{i=2}^n A^iB^{n-i}B_1^{i-1}\right)\frac{|\xi|^{n-\gamma}e^{c_0|\xi|^\gamma}}{\Gamma(n/\gamma)n!}\Phi^{(n)}(r;X+Y).\end{aligned}$$

Choose B with $B \geq 2AB_1$. Then $\sum_{i=2}^n A^iB^{n-i}B_1^{i-1} \leq A^2B_1B^{n-2}\sum_{i=0}^{n-2}(AB_1/B)^i \leq 2A^2B_1B^{n-2}$. So by choosing $C_1 \geq 2AB_1$, (6.34) holds. The estimate for $I^1_{n-1}(x,y,\xi)$ is obtained in the same way. By (5.13) in Proposition 5.4

$$\sum_{\substack{(n_1,n_2,\cdots,n_i)\in \boldsymbol{N}^i\\ n_1+n_2+\cdots+n_i=n-1}} \overbrace{v_{n_1} \underset{\gamma}{*} v_{n_2} \underset{\gamma}{*} \cdots \underset{\gamma}{*} v_{n_i}}^{i} \ll \frac{A^iB^{n-1-i}|\xi|^{n-1-\gamma}e^{c_0|\xi|^\gamma}}{\Gamma((n-1)/\gamma)(n-1)!}\Phi^{(n-1)}(r;X+Y)$$

holds and by $F_{i,1}(x,y,0) = 0$ we have $\hat{F}_{i,1}(x,y,\xi) \ll \big((B_1^i|\xi|^{1-\gamma}e^{c_0|\xi|^\gamma})/\Gamma(1/\gamma)\big)\Phi(R;X+Y)$ for $i \geq 1$. Hence by (5.12)

$$\begin{aligned}&\sum_{i=1}^{n-1} \hat{F}_{i,1}(x,y,\xi) \underset{\gamma}{*}\Bigg(\sum_{\substack{(n_1,n_2,\cdots,n_i)\in \boldsymbol{N}^i\\ n_1+n_2\cdots+n_i=n-1}} \overbrace{v_{n_1} \underset{\gamma}{*} v_{n_2} \underset{\gamma}{*} \cdots \underset{\gamma}{*} v_{n_i}}^{i}\Bigg)\\ &\qquad \ll C'\left(\sum_{i=1}^{n-1} A^iB^{n-1-i}B_1^i\right)\frac{|\xi|^{n-\gamma}e^{c_0|\xi|^\gamma}}{\Gamma(n/\gamma)n!}\Phi^{(n)}(r;X+Y)\end{aligned}$$

holds. For B with $B \geq 2AB_1$, $\sum_{i=1}^{n-1} A^iB^{n-1-i}B_1^i \leq AB_1B^{n-2}\sum_{i=0}^{n-2}(AB_1/B)^i \leq 2AB_1B^{n-2}$ holds, so (6.34) holds with $C_1 \geq 2C'B_1$. □

Set

$$\begin{aligned}
I^2_{n-1}(x,y,\xi) &:= -\left(\sum_{i=1}^{d_0} \hat{a}_{i,0}(y,\xi) \underset{\gamma}{*} \partial_{x_i} v_{n-1} + \sum_{i=1}^{d_2} \hat{b}_{i,0}(y,\xi) \underset{\gamma}{*} \partial_{y_i} v_{n-1}\right),\\
I^3_{n-1}(x,y,\xi) &:= -\left(\sum_{i=1}^{d_0} \hat{a}_{i,1}(x,y,\xi) \underset{\gamma}{*} \partial_{x_i} v_{n-1} + \sum_{i=1}^{d_2} \hat{b}_{i,1}(x,y,\xi) \underset{\gamma}{*} \partial_{y_i} v_{n-1}\right),\\
I^4_{n-1}(x,y,\xi) &:= -\hat{c}_1(x,y,\xi) \underset{\gamma}{*} (\gamma\xi^\gamma v_{n-1}). \qquad (6.35)
\end{aligned}$$

Then $-\mathscr{Q}(x,y,\xi,\partial_x,\partial_y)v_{n-1} = \sum_{i=2}^4 I^i_{n-1}(x,y,\xi)$ and we have

LEMMA 6.6. *For some constant* $C_2 > 0$ *the following estimates hold*:

$$I^2_{n-1}(x,y,\xi) \ll \frac{C_2 AB^{n-2}}{n!} \frac{|\xi|^n e^{c_0|\xi|^\gamma}}{\Gamma(n/\gamma)} \Phi^{(n)}(r;X+Y), \qquad (6.36)$$

$$I^3_{n-1}(x,y,\xi) \ll \frac{C_2 AB^{n-2}}{(n-1)!} \frac{|\xi|^{n-\gamma} e^{c_0|\xi|^\gamma}}{\Gamma(n/\gamma)} X\Phi^{(n)}(r;X+Y), \qquad (6.37)$$

$$I^4_{n-1}(x,y,\xi) \ll \frac{C_2 AB^{n-2}}{n!} \frac{|\xi|^n e^{c_0|\xi|^\gamma}}{\Gamma(n/\gamma)} \Phi^{(n)}(r;X+Y). \qquad (6.38)$$

We use bounds (6.10) to obtain the above estimates.

PROOF OF (6.36). We show the estimate of $\hat{a}_{i,0}(y,\xi) \underset{\gamma}{*} \partial_{x_i} v_{n-1}$ and can estimate other terms in the same way. By $\hat{a}_{i,0}(y,\xi) \ll C_0|\xi|e^{c_0|\xi|^\gamma}\Phi(R;Y)$ we have

$$\begin{aligned}
&\hat{a}_{i,0}(y,\xi) \underset{\gamma}{*} \partial_{x_i} v_{n-1}\\
&\quad \ll \frac{C'AB^{n-2}}{(n-1)!}\left(\frac{|\xi|e^{c_0|\xi|^\gamma}}{\Gamma((\gamma+1)/\gamma)} \underset{\gamma}{*} \frac{|\xi|^{n-1-\gamma}e^{c_0|\xi|^\gamma}}{\Gamma((n-1)/\gamma)}\right)\Phi(R;Y)\Phi^{(n)}(r;X+Y)\\
&\quad \ll \frac{C'AB^{n-2}}{(n-1)!}\frac{|\xi|^n e^{c_0|\xi|^\gamma}}{\Gamma((n/\gamma)+1)}\Phi^{(n)}(r;X+Y) \ll \frac{C_2AB^{n-2}}{n!}\frac{|\xi|^n e^{c_0|\xi|^\gamma}}{\Gamma(n/\gamma)}\Phi^{(n)}(r;X+Y).
\end{aligned}$$

PROOF OF (6.37). We estimate $\hat{a}_{i,1}(x,y,\xi) \underset{\gamma}{*} \partial_{x_i} v_{n-1}$ and other terms are estimated in the same way. By $\hat{a}_{i,1}(x,y,\xi) \ll C_0|\xi|^{1-\gamma}e^{c_0|\xi|^\gamma}X\Phi(R;X+Y)$ we have

$$\begin{aligned}
&\hat{a}_{i,1}(x,y,\xi) \underset{\gamma}{*} \partial_{x_i} v_{n-1}\\
&\quad \ll \frac{C'AB^{n-2}}{(n-1)!}\left(\frac{|\xi|^{1-\gamma}e^{c_0|\xi|^\gamma}}{\Gamma(1/\gamma)} \underset{\gamma}{*} \frac{|\xi|^{n-1-\gamma}e^{c_0|\xi|^\gamma}}{\Gamma((n-1)/\gamma)}\right)X\Phi(R;X+Y)\Phi^{(n)}(r;X+Y)\\
&\quad \ll \frac{C_2AB^{n-2}}{(n-1)!}\frac{|\xi|^{n-\gamma}e^{c_0|\xi|^\gamma}}{\Gamma(n/\gamma)}X\Phi^{(n)}(r;X+Y).
\end{aligned}$$

PROOF OF (6.38). By $\hat{c}_1(x,y,\xi) \ll C_0|\xi|^{1-\gamma}e^{c_0|\xi|^\gamma}\Phi(R;X+Y)$ we have

$$\begin{aligned}
&\hat{c}_1(x,y,\xi) \underset{\gamma}{*} (\gamma\xi^\gamma v_{n-1}) \\
&\ll \frac{C'AB^{n-2}}{(n-1)!}\left(\frac{|\xi|^{1-\gamma}e^{c_0|\xi|^\gamma}}{\Gamma(1/\gamma)} \underset{\gamma}{*} \frac{|\xi|^{n-1}e^{c_0|\xi|^\gamma}}{\Gamma((n-1)/\gamma)}\right)\Phi(R;X+Y)\Phi^{(n-1)}(r;X+Y) \\
&\ll \frac{C'AB^{n-2}}{(n-1)!}\frac{\Gamma(((n-1)/\gamma)+1)}{\Gamma((n-1)/\gamma)}\frac{|\xi|^n e^{c_0|\xi|^\gamma}}{\Gamma((n/\gamma)+1)}\Phi^{(n-1)}(r;X+Y) \\
&\ll \frac{C''AB^{n-2}}{(n-1)!}\frac{|\xi|^n e^{c_0|\xi|^\gamma}}{\Gamma(n/\gamma)}\Phi^{(n-1)}(r;X+Y) \ll \frac{C_2AB^{n-2}}{n!}\frac{|\xi|^n e^{c_0|\xi|^\gamma}}{\Gamma(n/\gamma)}\Phi^{(n)}(r;X+Y).
\end{aligned}$$

□

EXISTENCE OF A SOLUTION $v_n(x,y,\xi)$ OF (6.30).
We have $\mathscr{P}(x,y,\xi,\partial_x,\partial_y)v_n = \sum_{i=0}^4 I_{n-1}^i(x,y,\xi)$. In order to solve it, consider

$$\mathscr{P}(\xi,x,y,\partial_x,\partial_y)v_n^i(\xi,x,y) = I_{n-1}^i(\xi,x,y). \tag{6.39}$$

For $i = 0,1,2,4$ it follows from Proposition 6.4 and Lemmas 6.5 and 6.6 that there exists $v_n^i(\xi,x,y)$ with

$$v_n^i(\xi,x,y) \ll \frac{C'AB^{n-2}}{n!}\frac{|\xi|^{n-\gamma}e^{c_0|\xi|^\gamma}}{\Gamma(n/\gamma)}\Phi^{(n)}(r;X+Y). \tag{6.40}$$

For $i = 3$ by Proposition 6.4 and Lemma 6.6 that there exists $v_n^3(x,y,\xi)$ with

$$\begin{aligned}
v_n^3(x,y,\xi) &\ll \frac{C'AB^{n-2}}{(n-1)!}\frac{|\xi|^{n-\gamma}e^{c_0|\xi|^\gamma}}{\Gamma(n/\gamma)}\Phi^{(n-1)}(r;X+Y) \\
&\ll \frac{C'AB^{n-2}}{n!}\frac{|\xi|^{n-\gamma}e^{c_0|\xi|^\gamma}}{\Gamma(n/\gamma)}\Phi^{(n)}(r;X+Y).
\end{aligned} \tag{6.41}$$

Thus $v_n(x,y,\xi) = \sum_{i=0}^4 v_n^i(x,y,\xi)$ is a solution of (6.30) and holomorphic in Ξ_0^* with (6.32) and $\xi^{-1+\gamma}v_n(\xi,x,y)$ is holomorphic at $\xi = 0$.

EXISTENCE OF A SOLUTION $v(x,y,\xi)$ OF (6.16).
Set $V = \{(x,y); \sum_{i=1}^{d_0}|x_i| + \sum_{i=1}^{d_1}|y_i| \le r/2\}$. Then there exists a constant C_1 such that $|\Phi^{(n)}(r;X+Y)| \le C_1^{n+1}n!$ for $(x,y) \in V$, hence, for some constant $c > c_0$

$$\sum_{n=1}^\infty |v_n(x,y,\xi)| \le \sum_{n=1}^\infty \frac{A'(BC_1)^{n-1}|\xi|^{n-\gamma}e^{c_0|\xi|^\gamma}}{\Gamma(n/\gamma)} \le C|\xi|^{1-\gamma}\exp(c|\xi|^\gamma),$$

which means the convergence of $v(x,y,\xi) = \sum_{n=0}^\infty v_n(x,y,\xi)$ in $V \times \Xi_0^*$ and $\xi^{-1+\gamma}v(\xi,x,y)$ is holomorphic at $\xi = 0$. We also have by Lemma 6.5

$$\sum_{n=2}^{\infty}|I^0_{n-1}(x,y,\xi)|, \sum_{n=2}^{\infty}|I^1_{n-1}(x,y,\xi)|$$
$$\leq \sum_{n=2}^{\infty}\frac{A'(BC_1)^{n-2}|\xi|^{n-\gamma}e^{c_0|\xi|^\gamma}}{\Gamma(n/\gamma)} \leq C|\xi|^{2-\gamma}\exp(c|\xi|^\gamma). \tag{6.42}$$

We show $v(x,y,\xi)$ satisfies (6.16). Set $v^N(x,y,\xi) = \sum_{n=1}^N v_n(x,y,\xi)$. Then we have from (6.30)

$$\mathscr{P}(x,y,\xi,\partial_x,\partial_y)v^N + \mathscr{Q}(x,y,\xi,\partial_x,\partial_y)v^{N-1} = \hat{F}_0(x,y,\xi) + \sum_{n=2}^{N}(I^0_{n-1}(x,y,\xi) + I^1_{n-1}(x,y,\xi)). \tag{6.43}$$

On the other hand from (6.14)

$$\mathscr{F}(x,y,\xi,v) = \sum_{i=2}^{\infty}F_{i,0}(x,y)\Bigg(\overbrace{\sum_{n=1}^{\infty}v_n \underset{\gamma}{*} \sum_{n=1}^{\infty}v_n \underset{\gamma}{*}\cdots\underset{\gamma}{*}\sum_{n=1}^{\infty}v_n}^{i}\Bigg)$$
$$+\sum_{i=1}^{\infty}\hat{F}_{i,1}(x,y,\xi)\underset{\gamma}{*}\overbrace{\sum_{n=1}^{\infty}v_n \underset{\gamma}{*} \sum_{n=1}^{\infty}v_n \underset{\gamma}{*}\cdots\underset{\gamma}{*}\sum_{n=1}^{\infty}v_n}^{i}.$$

It follows from (6.31) and (6.42) that

$$\sum_{i=2}^{\infty}F_{i,0}(x,y)\Bigg(\overbrace{\sum_{n_1=1}^{\infty}v_{n_1} \underset{\gamma}{*} \sum_{n_2=1}^{\infty}v_{n_2} \underset{\gamma}{*}\cdots\underset{\gamma}{*}\sum_{n_i=1}^{\infty}v_{n_i}}^{i}\Bigg)$$
$$= \sum_{i=2}^{\infty}F_{i,0}(x,y)\Bigg(\sum_{n=i}^{\infty}\Bigg(\sum_{\substack{(n_1,n_2,\cdots,n_i)\in \boldsymbol{N}^i\\ n_1+n_2+\cdots+n_i=n}}\overbrace{v_{n_1}\underset{\gamma}{*}v_{n_2}\underset{\gamma}{*}\cdots\underset{\gamma}{*}v_{n_i}}^{i}\Bigg)\Bigg)$$
$$= \sum_{n=2}^{\infty}\Bigg(\sum_{i=2}^{n}F_{i,0}(x,y)\Bigg(\sum_{\substack{(n_1,n_2,\cdots,n_i)\in \boldsymbol{N}^i\\ n_1+n_2+\cdots+n_i=n}}\overbrace{v_{n_1}\underset{\gamma}{*}v_{n_2}\underset{\gamma}{*}\cdots\underset{\gamma}{*}v_{n_i}}^{i}\Bigg)\Bigg)$$
$$= \sum_{n=2}^{+\infty}(I^0_{n-1}(x,y,\xi))$$

and

$$\sum_{i=1}^{\infty}\hat{F}_{i,1}(x,y,\xi)\underset{\gamma}{*}\overbrace{\sum_{n=1}^{\infty}v_n \underset{\gamma}{*} \sum_{n=1}^{\infty}v_n \underset{\gamma}{*}\cdots\underset{\gamma}{*}\sum_{n=1}^{\infty}v_n}^{i} = \sum_{n=2}^{+\infty}I^1_{n-1}(x,y,\xi).$$

Hence by letting $N \to +\infty$ in (6.43),

$$\mathscr{P}(x,y,\xi,\partial_x,\partial_y)v + \mathscr{Q}(x,y,\xi,\partial_x,\partial_y)v = \hat{F}_0(x,y,\xi) + \mathscr{F}(x,y,\xi,v).$$

Thus the proof of Theorem 6.1 is completed.

PROOF OF THEOREM 2.3. Let $v(x,y,\xi)$ be a solution of (6.16) assured by Theorem 6.1. Define

$$u(x,y,t)=\int_0^{\infty e^{i\theta_0}}\exp\left(-\left(\frac{\xi}{t}\right)^{\gamma}\right)v(x,y,\xi)\,d\xi^{\gamma}.$$

Then it follows from Proposition 5.2 that $u(x,y,t)\in\mathscr{O}(V)\{t\}_{\gamma,\theta_0}$. Since $v(x,y,\xi)$ satisfies (6.16), $u(x,y,t)$ is a solution of $Pu=F(x,y,t,u)$. □

We give a comment about uniqueness of solutions of (6.16), which is not stated in Theorem 6.1. The solution of $Pu=F(x,y,t,u)$ is unique in $\mathscr{O}(V)\{t\}_{\gamma,\theta_0}$, so the uniqueness of (6.16) satisfying (1) and (2) in Theorem 6.1 follows from it.

Let us proceed to show Theorem 2.4. The proof is similar to that of Theorem 2.3. We sum up shortly the assumptions of Theorem 2.4. Set $U=\{x\in\boldsymbol{C}^{d_0};|x|<R\}$ and $U_0=\{u\in\boldsymbol{C};|u|<R_0\}$. $P=P(x,t,\partial_x,\partial_t)$ is a vector field with coefficients in $\mathscr{O}(U)\{t\}_{\gamma,\theta_0}$ and has the form

$$P=\sum_{i=1}^{d_0}(\lambda_i x_i+A_i(x,t))\partial_{x_i}+t^{\gamma+1}C(x,t)\partial_t, \tag{6.44}$$

where the coefficients satisfy (see (2.15))

$$A_i(0,t)=0,\ A_i(x,t)=O\big((|x|+|t|)^2\big),\ C(0,0)\neq 0. \tag{6.45}$$

We have from (6.45)

$$A_i(x,t)=a_i(x)+a_{i,1}(x,t),\ C(x,t)=c_0(x)+c_1(x,t) \tag{6.46}$$

with

$$\begin{cases} a_i(x)=O(|x|^2), \quad a_{i,1}(0,t)=a_{i,1}(x,0)=0,\\ c_0(0)\neq 0, \quad c_1(x,0)=0. \end{cases} \tag{6.47}$$

Hence we assume P is of the form

$$P(x,t,\partial_x,\partial_t)=\sum_{i=1}^{d_0}(\lambda_i x_i+a_i(x)+a_{i,1}(x,t))\partial_{x_i}+t^{\gamma+1}(c_0(x)+c_1(x,t))\partial_t. \tag{6.48}$$

As for the nonlinear term $F(x,t,u)$

$$F(x,t,u)=\sum_{i=0}^{+\infty}F_i(x,t)u^i, \quad F_i(x,t)\in\mathscr{O}(U)\{t\}_{\gamma,\theta_0} \tag{6.49}$$

and it follows from the assumptions $F(x,0,0)=0$ and $F(x,t,0)=O(|x|^N)$ for some $N\in\boldsymbol{N}-\{0\}$ (see (2.19) and (2.16)) that $F_0(x,0)=0$ and $\partial_x^\alpha F_0(0,t)=0$ for $|\alpha|\le N-1$. Set

$$\mathscr{P}(x,\xi,\partial_x)=\sum_{i=1}^{d_0}(\lambda_i x_i+a_i(x))\partial_{x_i}+\gamma c_0(x)\xi^\gamma-F_1(x,0), \tag{6.50}$$

$$\mathscr{Q}(x,\xi,\partial_x)=\sum_{i=1}^{d_0}\hat{a}_{i,1}(x,\xi)\underset{\gamma}{*}(\partial_{x_i}\cdot\,)+\hat{c}_1(x,\xi)\underset{\gamma}{*}(\gamma\xi^\gamma\cdot\,) \tag{6.51}$$

and

$$h(x,\xi)=\gamma c_0(x)\xi^\gamma-F_1(x,0). \tag{6.52}$$

It follows from (2.18) that

$$\left|\sum_{i=1}^{d_0}m_i\lambda_i+h(0,\xi)\right|^{-1}\le\frac{K\Phi(R;0)}{(1+|\xi|)^\gamma+|m|} \tag{6.53}$$

holds for $\xi\in\Xi_0^*$ and $m\in\boldsymbol{N}^{d_0}$ with $|m|\ge N$. Set

$$F_{i,0}(x)=F_i(x,0),\quad F_{i,1}(x,t)=F_i(x,t)-F_i(x,0) \tag{6.54}$$

and let $\mathscr{F}(x,\xi,v)$ be a nonlinear convolution operator defined by

$$\mathscr{F}(x,\xi,v)=\sum_{i=2}^{\infty}F_{i,0}(x)\Bigl(\overbrace{v\underset{\gamma}{*}v\underset{\gamma}{*}\cdots\underset{\gamma}{*}v}^{i}\Bigr)+\sum_{i=1}^{\infty}\hat{F}_{i,1}(x,\xi)\underset{\gamma}{*}\overbrace{v\underset{\gamma}{*}v\underset{\gamma}{*}v\underset{\gamma}{*}\cdots\underset{\gamma}{*}v}^{i}. \tag{6.55}$$

Consider

$$\mathscr{P}(x,\xi,\partial_x)v+\mathscr{Q}(x,\xi,\partial_x)v=\hat{F}_0(x,\xi)+\mathscr{F}(x,\xi,v). \tag{6.56}$$

It follows from the assumptions on $F_0(x,t)$ that for $(x,\xi)\in U\times\Xi_0^*$

$$|\hat{F}_0(x,\xi)|\le C|x|^N|\xi|^{1-\gamma}\exp(c_0|\xi|^\gamma). \tag{6.57}$$

We have under the above assumptions the following proposition, which corresponds to Proposition 6.4.

PROPOSITION 6.7. *Let $f(x,\xi)$ be a holomorphic function on $U\times\Xi_0^*$ with $\partial_x^\alpha f(0,\xi)=0$ for $|\alpha|\le N-1$. Suppose that $f(x,\xi)$ satisfies $f(x,\xi)\ll M(|\xi|)\Phi^{(s)}(r;X)$ or $f(x,\xi)\ll M(|\xi|)X\Phi^{(s+1)}(r;X)$. Consider*

$$\mathscr{P}(x,\xi,\partial_x)v(x,\xi)=f(x,\xi). \tag{6.58}$$

Then there is $r_0 > 0$ such that for $0 < r < r_0$ there exists a unique holomorphic solution $v(x,\xi)$ of (6.58) *satisfying $\partial_x^\alpha v(0,\xi) = 0$ for $|\alpha| \le N-1$ and the following estimates, if $f(x,\xi) \ll M(|\xi|)\Phi^{(s)}(r;X)$,*

$$v(x,\xi) \ll \frac{CM(|\xi|)}{(1+|\xi|)^\gamma}\Phi^{(s)}(r;X) \tag{6.59}$$

and if $f(x,\xi) \ll M(|\xi|)X\Phi^{(s+1)}(r;X)$,

$$v(x,\xi) \ll CM(|\xi|)\Phi^{(s)}(r;X). \tag{6.60}$$

By repeating the arguments similar to the proofs of Lemmas 6.2 and 6.3 and Proposition 6.4, we can show Proposition 6.7. Let us solve (6.56) by the iteration such as (6.30). Define $v_n = v_n(x,\xi)$ $(n \in \boldsymbol{N})$ inductively, $v_0 = 0$ and by

$$\begin{aligned}
&\mathscr{P}(x,\xi,\partial_x)v_1 = \hat{F}_0(x,\xi),\\
&\mathscr{P}(x,\xi,\partial_x)v_n + \mathscr{Q}(x,\xi,\partial_x)v_{n-1}\\
&\quad = \sum_{i=2}^{\infty} F_{i,0}(x)\Bigg(\sum_{\substack{(n_1,n_2,\cdots,n_i)\in \boldsymbol{N}^i\\ n_1+n_2+\cdots+n_i=n}} \overbrace{v_{n_1} \underset{\gamma}{*} v_{n_2} \underset{\gamma}{*} \cdots \underset{\gamma}{*} v_{n_i}}^{i}\Bigg)\\
&\qquad + \sum_{i=1}^{\infty} \hat{F}_{i,1}(x,\xi) \underset{\gamma}{*} \Bigg(\sum_{\substack{(n_1,n_2,\cdots,n_i)\in \boldsymbol{N}^i\\ n_1+n_2+\cdots+n_i=n-1}} \overbrace{v_{n_1} \underset{\gamma}{*} v_{n_2} \underset{\gamma}{*} \cdots \underset{\gamma}{*} v_{n_i}}^{i}\Bigg).
\end{aligned} \tag{6.61}$$

The right hand side of (6.61) and $\mathscr{Q}(x,\xi,\partial_x)v_{n-1}$ are $O(|x|^N)$. Hence the existence of $v_n(x,\xi)$ with $v_n(x,\xi) = O(|x|^N)$ follows from Proposition 6.7. By estimating $\{v_n(x,\xi)\}_{n\in\boldsymbol{N}}$, we can show in the same as way as the preceding case that $v(x,\xi) = \sum_{n=0}^{\infty} v_n(x,\xi)$ converges and it is a solution of (6.56). Set $u(x,t) = \mathscr{L}_{\gamma,\theta_0}v$. Then $u(x,t) \in \mathscr{O}(V)\{t\}_{\gamma,\theta_0}$ is a solution of $Pu = F(x,y,u)$ satisfying $u(x,t) = O(|x|^N)$ and we have Theorem 2.4.

7. Existence of solutions of singular differential equations.

In this section we give results about the existence of solutions of some singular differential equations. We have applied them to finding coordinates transformations that simplify singular vector fields in the preceding sections. As for the topics in this section we refer to [**6**], [**8**] and [**18**]. The book [**8**] is concerned with singular partial differential equations in complex variables, in particular, in which partial differential equations of Fuchsian type and those of Briot-Bouquet type in higher dimension are investigated and several results about existence of solutions are given.

Let $z = (z_1,\cdots,z_p,\cdots,z_{p+q})$, $z' = (z_1,\cdots,z_p)$ and $z'' = (z_{p+1},\cdots,z_{p+q})$. Set

$$K(z,\partial_z) = \sum_{i=1}^{p}(\lambda_i z_i + \mu_{i-1}z_{i-1} + c_i(z))\partial_{z_i} + \sum_{i=p+1}^{p+q} c_i(z)\partial_{z_i},$$

where $\lambda_i \neq 0$, $\mu_i = 0$ or 1 and $\{c_i(z)\}_{i=1}^{p+q}$ are holomorphic in a neighborhood of $z = 0$ such that $c_i(0,z'') = 0$ for all $1 \leq i \leq p+q$ and $c_i(z) = O(|z|^2)$ for $1 \leq i \leq p$. Let $f(z,u)$ be a holomorphic function in a neighborhood of $(z,u) = (0,0)$ with $f(0,0) = 0$. Consider

$$K(z,\partial_z)u(z) = f(z,u). \tag{7.1}$$

PROPOSITION 7.1. *Let $N \in \boldsymbol{N}$ such that $\partial_{z'}^{\alpha'} f(0,z'',0) = 0$ for $|\alpha'| \leq N-1$. Suppose that there exists $C > 0$ such that for all $m = (m_1,\cdots,m_p) \in \boldsymbol{N}^p$ with $|m| \geq N$*

$$\left|\sum_{i=1}^{p} m_i\lambda_i - \frac{\partial f}{\partial u}(0,0)\right| \geq C(|m|+1). \tag{7.2}$$

Then there exists a unique holomorphic solution $u(z)$ of (7.1) with $u(z) = O(|z'|^N)$ in a neighborhood of $z = 0$.

For the proof we refer to [**8**]. Next consider

$$\sum_{i=1}^{p}(\lambda_i z_i + \mu_{i-1}z_{i-1} + a_i(z,\phi))\partial_{z_i}\phi(z) + \sum_{j=p+1}^{p+q} a_j(z,\phi)\partial_{z_j}\phi(z) = a_0(z,\phi), \tag{7.3}$$

where $\lambda_i \neq 0$, μ_i =0 or 1. $\{a_i(z,u)\}_{i=0}^{p+q}$ are holomorphic in a neighborhood of $(z,u) = (0,0)$ such that $a_i(0,z'',0) = 0$ and $a_i(z,u) = O((|z|+|u|)^2)$ for all $0 \leq i \leq p+q$.

PROPOSITION 7.2. *Suppose that the convex hull of $\{\lambda_i\}_{i=1}^{p}$ in the complex plane does not contain the origin. Then there exists a unique holomorphic solution $\phi(z)$ of (7.3) with $\phi(0,z'') = 0$ in a neighborhood of $z = 0$.*

Systems of singular partial differential equations including (7.3) as a special case were studied in [**18**] and the existence of solutions was shown. The equation (7.3) is not a system but a single one, so the proof is simpler than systems. Here we give how to construct $\phi(z)$ briefly and remarks. Set $a_i(z',z'',u) = \sum_{|m|+k\geq 1} a_{i,m,k}(z'')z'^m u^k$, where $a_{i,m,k}(0) = 0$ for $|m|+k = 1$, and $\phi(z',z'') = \sum_{|m|\geq 1}\phi_m(z'')z'^m$. Let $e_i = (0,\cdots,\overset{i}{1},\cdots,0) \in \boldsymbol{N}^p$. Then

$$\begin{aligned}
&\left(\sum_{i=1}^{p} m_i\lambda_i\right)\phi_m(z'') + \sum_{i=2}^{p}\mu_{i-1}(m_i+1)\phi_{m+e_i-e_{i-1}}(z'')\\
&\quad + \sum_{i=1}^{p}\Bigg(\sum_{\substack{m^0+m^1=m\\ |m_0|=1}} a_{i,m^0,0}(z'')(m_i^1+1)\phi_{m^1+e_i}(z'')\\
&\qquad + \sum_{m^1+m^2=m} a_{i,0,1}(z'')\phi_{m^1}(z'')(m_i^2+1)\phi_{m^2+e_i}(z'')\Bigg) - a_{0,0,1}(z'')\phi_m(z'')\\
&= -\sum_{i=1}^{p}\sum_{|m_0|+k\geq 2}\Bigg(\sum_{m^0+\cdots+m^{k+1}=m} a_{i,m^0,k}(z'')(m_i^{k+1}+1)\overbrace{\phi_{m^1}(z'')\cdots\phi_{m^k}(z'')\phi_{m^{k+1}+e_i}(z'')}^{k+1}\Bigg)
\end{aligned}$$

$$- \sum_{j=p+1}^{p+q} \sum_{|m^0|+k\geq 1} \left(\sum_{m^0+\cdots+m^{k+1}=m} a_{j,m^0,k}(z'') \overbrace{\phi_{m^1}(z'')\cdots\phi_{m^k}(z'')\partial_{z_j}\phi_{m^{k+1}}(z'')}^{k+1} \right)$$
$$+ \sum_{k\geq 2} \sum_{m^0+\cdots+m^k=m} a_{0,m^0,k}(z'') \overbrace{\phi_{m^1}(z'')\cdots\phi_{m^k}(z'')}^{k}$$
$$+ \sum_{m_0\neq 0} a_{0,m_0,1}(z'')\phi_{m-m^0}(z'') + a_{0,m,0}(z''), \tag{7.4}$$

where $m=(m_1,\cdots,m_p)$, $m^i=(m^i_1,\cdots,m^i_p)\in \boldsymbol{N}^p$. If $|m|=1$, then

$$\lambda_h\phi_{e_h}(z'')+\mu_h\phi_{e_{h+1}}(z'')+\sum_{i=1}^{p}(a_{i,e_h,0}(z'')\phi_{e_i}(z'')+a_{i,0,1}(z'')\phi_{e_h}(z'')\phi_{e_i}(z''))$$
$$= a_{0,0,1}(z'')\phi_{e_h}(z'')+a_{0,e_h,0}(z''). \tag{7.5}$$

It follows from the assumption on $\{\lambda_i\}_{i=1}^p$ that $|\sum_{i=1}^p \lambda_i m_i|\geq C|m|$ $(C>0)$ holds for all $m\in \boldsymbol{N}^p$. Since $a_{i,m,k}(0)=0$ for $|m|+k=1$, we can determine $\phi_m(z'')$ with $\phi_m(0)=0$ for $|m|=1$ by (7.5) and inductively $\{\phi_m(z'')\}_{|m|\geq 2}$ by (7.4). We can show the convergence of $\phi(z',z'')=\sum_{|m|\geq 1}\phi_m(z'')z'^m$, by estimating $\{\phi_m(z'')\}_{|m|\geq 1}$. The estimation is done by the majorant functions in section 5. In [**18**] the convergence is shown by other majorant functions.

Finally consider a system of ordinary differential equations with unknowns $\boldsymbol{\Psi}(t)=(\psi_1(t),\cdots,\psi_n(t))$

$$\begin{cases} t^{\gamma+1}\dfrac{d\psi_i(t)}{dt}=\nu_i\psi_i(t)+H_i(\boldsymbol{\Psi}(t),t)+h_i(t) \quad \textit{for}\ \ 1\leq i\leq n_0, \\ \qquad\ \dfrac{d\psi_i(t)}{dt}=H_i(\boldsymbol{\Psi}(t),t)+h_i(t) \quad \textit{for}\ \ n_0<i\leq n, \end{cases} \tag{7.6}$$

where γ is a positive integer, $\nu_i\neq 0$, $H_i(w,t)\in \boldsymbol{C}\{w\}\{t\}_{\gamma,\theta}$ and $h_i(t)\in \boldsymbol{C}\{t\}_{\gamma,\theta}$. The origin $t=0$ is irregular singular. The equations such as (7.6) appear in the proofs of Lemmas 4.3, 4.5 and 4.7, where $n=n_0=d-1$ in (4.12), $n=1, n_0=0$ in (4.17) and $n=d-1, n_0=d-2$ in (4.24). The main result in [**6**] is much concerned with the existence of solutions of (7.6) in $\boldsymbol{C}\{t\}_{\gamma,\theta}$. It was shown in [**6**] that formal power series solutions of system of nonlinear ordinary differential equations were multisummable, which means the existence of solutions with asymptotic expansion in a wider sectorial region, and (7.6) is a special case studied there. We have from [**6**]

Proposition 7.3. *Let p be a positive integer. For $1\leq i\leq n_0$, assume $\gamma\theta\not\equiv \arg\nu_i$* mod 2π, *$H_i(w,t)=O(|w|(|w|+|t|))$ and $h_i(t)=O(|t|^p)$. For $n_0<i\leq n$, assume $H_i(0,t)=0$ and $h_i(t)=O(|t|^{p-1})$. Then there exists uniquely $\boldsymbol{\Psi}(t)=(\psi_1(t),\cdots,\psi_n(t))\in(\boldsymbol{C}\{t\}_{\gamma,\theta})^n$ satisfying* (7.6) *with $\psi_i(t)=O(|t|^p)$.*

Let $h_i(t)=h_{i,p}t^p+O(|t|^{p+1})$ for $1\leq i\leq n_0$ and $h_i(t)=h_{i,p-1}t^{p-1}+O(|t|^p)$ for $n_0<i\leq n$. Set $\boldsymbol{\Psi}_p=(-h_{1,p}/\nu_1,\cdots,-h_{n_0,p}/\nu_{n_0},h_{n_0+1,p-1}/p,\cdots,h_{n,p-1}/p)$. It is not difficult under the assumptions of Proposition 7.3 to show that there exists uniquely formal series $\tilde{\boldsymbol{\Psi}}(t)\in(t^p\boldsymbol{C}[[t]])^n$ satisfying (7.6) with $\tilde{\boldsymbol{\Psi}}(t)=\boldsymbol{\Psi}_p t^p+\cdots$. Set $\boldsymbol{\Psi}(t)=t^p(\boldsymbol{\Phi}(t)+\boldsymbol{\Psi}_p)$, $\boldsymbol{\Phi}(t)=(\phi_1(t),\cdots,\phi_n(t))$.

Then for $1 \le i \le n_0$

$$\begin{aligned} t^{\gamma+1}\frac{d\psi_i(t)}{dt} &= t^{\gamma+p+1}\frac{d\phi_i(t)}{dt} + pt^{p+\gamma}(\phi_i(t) - h_{i,p}/\nu_i) \\ &= t^p(\nu_i\phi_i(t) - h_{i,p}) + H_i(t^p\boldsymbol{\Phi}(t) + t^p\boldsymbol{\Psi}_p, t) + h_i(t) \end{aligned}$$

and for $n_0 < i \le n$

$$\frac{d\psi_i(t)}{dt} = t^p\frac{d\phi_i(t)}{dt} + t^{p-1}(p\phi_i(t) + h_{i,p-1}) = H_i(t^p\boldsymbol{\Phi}(t) + t^p\boldsymbol{\Psi}_p, t) + h_i(t).$$

Hence

$$\begin{cases} t^{\gamma+1}\dfrac{d\phi_i(t)}{dt} = \nu_i\phi_i(t) + G_i(\boldsymbol{\Phi}, t) \quad \textit{for } \ 1 \le i \le n_0, \\ \quad t\dfrac{d\phi_i(t)}{dt} = -p\phi_i(t) + G_i(\boldsymbol{\Phi}(t), t) \quad \textit{for } \ n_0 < i \le n, \end{cases} \tag{7.7}$$

where for $1 \le i \le n_0$

$$G_i(w,t) = -pt^{\gamma}w_i + H_i(t^p(w + \boldsymbol{\Psi}_p), t)/t^p + (h_i(t) - h_{i,p}t^p)/t^p + pt^{\gamma}h_{i,p}/\nu_i,$$

and for $n_0 < i \le n$

$$G_i(w,t) = H_i(t^p w + t^p\boldsymbol{\Psi}_p, t)/t^{p-1} + (h_i(t) - h_{i,p-1}t^{p-1})/t^{p-1}.$$

From the assumptions we have $G_i(w,0) = 0$ for all i. Thus it follows from Theorem 1 in [**6**] that there exists a solution $\boldsymbol{\Phi}(t) \in (t\boldsymbol{C}\{t\}_{\gamma,\theta})^n$ of (7.7), hence $\boldsymbol{\Psi}(t) = t^p(\boldsymbol{\Psi}_p + \boldsymbol{\Phi}(t))$ is a solution of (7.6) with $\boldsymbol{\Psi}(t) \sim \tilde{\boldsymbol{\Psi}}(t)$.

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