

## Uniqueness theorems for parabolic equations and Martin boundaries for elliptic equations in skew product form

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**Abstract.** We give a method to determine Martin boundaries of product domains for elliptic equations in skew product form via Widder type uniqueness theorems for parabolic equations. It is shown that the fiber of the Martin boundary at infinity of the base space degenerates into one point if any nonnegative solution to the Dirichlet problem for a corresponding parabolic equation with zero initial and boundary data is identically zero. We apply it in a unified way to several concrete examples to explicitly determine Martin boundaries for them.

### 1. Introduction.

This paper is a continuation of [64] and [65], and its aim is to explicitly determine Martin boundaries for elliptic equations in skew product form via Widder type uniqueness theorems for parabolic equations.

The Widder type uniqueness theorem for a parabolic equation asserts that its nonnegative solution with zero initial (and boundary) value must be identically zero; while the Martin representation theorem for an elliptic equation says that any positive solution of it is represented by an integral of the Martin kernel with respect to a finite Borel measure on the Martin boundary. During the last few decades, Widder type and related uniqueness theorems have been investigated to a satisfactory extent (cf. [9], [15]–[18], [21], [23], [25], [26], [28], [32], [34], [42]–[44], [46], [48], [57]–[61], [65], [69], [76], [78], [80]), and there has been a significant progress in determining explicitly Martin boundaries in many important cases (cf. [4]–[7], [10]–[13], [20], [29]–[31], [36], [38]–[40], [49]–[51], [53]–[56], [63], [64], [66], [72], [73]). Among others, Ishige and Murata [44] showed that under a general and sharp condition, any nonnegative solution to the Cauchy problem for a parabolic equation is determined uniquely by its initial value; while Murata [64] constructed Martin boundaries for a wide class of elliptic equations in skew product form.

The purpose of this paper is twofold: (1) to determine Martin boundaries for elliptic equations in skew product form via Widder type uniqueness theorems for parabolic equations on the basis of general results on Martin boundaries given in [64]; and (2) to apply it, together with results given in [44] and [65], in a unified way to several concrete examples to explicitly determine Martin boundaries for them. We are deeply motivated by concrete examples.

We consider positive solutions of an elliptic equation in skew product form

$$Lu \equiv (L_1 + W_1 L_2)u = 0 \quad \text{in } D = D_1 \times D_2 \subset M = M_1 \times M_2. \quad (1.1)$$

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Here  $L_1$  and  $W_1$  are an elliptic operator and a positive measurable function on a Riemannian manifold  $M_1$ , respectively,  $L_2$  is an elliptic operator on a domain  $D_2$  of a Riemannian manifold  $M_2$ ,  $D_1$  and  $D = D_1 \times D_2$  are non-compact domains of  $M_1$  and the product Riemannian manifold  $M = M_1 \times M_2$ , respectively. We assume that  $(L, D)$  is subcritical, i.e. there exists a minimal positive Green function of  $L$  on  $D$ . In order to determine explicitly the Martin boundary  $\partial_M D$  of  $D$  with respect to  $L$ , we study uniqueness of nonnegative solutions to the initial and boundary value problem for a parabolic equation

$$(\partial_t + W_1^{-1}L_1)v = 0 \quad \text{in } D_1 \times (0, \infty), \tag{1.2}$$

$$v(x, 0) = 0 \quad \text{on } D_1, \tag{1.3}$$

$$v(x, t) = 0 \quad \text{on } \partial D_1 \times (0, \infty). \tag{1.4}$$

(It is needless to say that when the boundary  $\partial D_1$  of  $D_1$  in  $M_1$  is empty, the condition (1.4) is redundant, and the problem reduces to the initial value problem.) We shall show from the uniqueness of nonnegative solutions that the fiber of  $\partial_M(D_1 \times D_2)$  at infinity of the base space  $D_1$  reduces into one point.

Now, in order to state our main results, we fix notations and recall several notions and facts. For  $i = 1$  or  $2$ , let  $M_i$  be a connected separable  $n_i$ -dimensional smooth manifold with Riemannian metric of class  $C^0$ . With  $N = M_1$  or  $M_2$ ,  $T_x N$  and  $TN$  denote the tangent space to  $N$  at  $x \in N$  and the tangent bundle, respectively. We denote by  $\text{End}(T_x N)$  and  $\text{End}(TN)$  the set of endomorphisms in  $T_x N$  and the corresponding bundle, respectively. The inner product on  $TN$  is denoted by  $\langle X, Y \rangle$ , where  $X, Y \in TN$ ; and  $|X| = \langle X, X \rangle^{1/2}$ . The divergence and gradient with respect to the metric on  $N$  are denoted by  $\text{div}$  and  $\nabla$ , respectively. Let  $L_1$  be an elliptic differential operator on  $M_1$  of the form

$$L_1 u = -m_1^{-1} \text{div}(m_1 A_1 \nabla u - m_1 u C_1) - \langle B_1, \nabla u \rangle + V_1 u, \tag{1.5}$$

where  $m_1$  is a positive measurable function on  $M_1$  such that

$$m_1 \text{ and } m_1^{-1} \text{ are bounded on any compact subset of } M_1, \tag{1.6}$$

$A_1$  is a symmetric measurable section on  $M_1$  of  $\text{End}(TM_1)$ ,  $B_1$  and  $C_1$  are measurable vector fields on  $M_1$ , and  $V_1$  is a real-valued measurable function on  $M_1$ . We assume that  $L_1$  is locally uniformly elliptic on  $M_1$ , i.e., for any compact set  $K$  in  $M_1$  there exists a positive constant  $\lambda$  such that

$$\lambda |\xi|^2 \leq \langle (A_1)_x \xi, \xi \rangle \leq \lambda^{-1} |\xi|^2, \quad x \in K, (x, \xi) \in TM_1. \tag{1.7}$$

Denote by  $\nu_1$  the Riemannian measure on  $M_1$ , and put  $d\mu_1 = m_1 d\nu_1$ . For  $1 \leq p \leq \infty$ , denote by  $L_{\text{loc}}^p(M_1) = L_{\text{loc}}^p(M_1, d\mu_1)$  the set of complex-valued functions on  $M_1$  locally  $p$ -th integrable with respect to  $d\mu_1$ . We assume that

$$|B_1|^2, |C_1|^2, V_1 \in L_{\text{loc}}^p(M_1, d\mu_1), \quad \text{for some } p > \max\left(\frac{n}{2}, 1\right). \tag{1.8}$$

Here  $n = n_1 + n_2$  is the dimension of  $M = M_1 \times M_2$ . Let  $W_1$  be a positive measurable function on  $M_1$  such that

$$W_1, W_1^{-1} \in L^\infty_{\text{loc}}(M_1, d\mu_1). \tag{1.9}$$

Let  $L_2$  be an elliptic differential operator on a domain  $D_2 \subset M_2$  of the form

$$L_2 u = -m_2^{-1} \operatorname{div}(m_2 A_2 \nabla u - m_2 u B_2) - \langle B_2, \nabla u \rangle + V_2 u, \tag{1.10}$$

where  $m_2, A_2, B_2$ , and  $V_2$  satisfy the conditions (1.6), (1.7) and (1.8) with obvious modifications. Note the  $L_2$  is formally selfadjoint with respect to the measure  $d\mu_2$ . We assume that the generalized principal eigenvalue  $\lambda_0$  of  $L_2$  on  $D_2$  is finite, i.e., with  $\Lambda$  being the set of all real numbers  $\lambda$  such that the equation  $(L_2 - \lambda)u = 0$  in  $D_2$  has a positive solution,

$$\lambda_0 \equiv \sup \Lambda > -\infty. \tag{1.11}$$

We denote by  $\mathcal{L}_2$  the Dirichlet realization of  $L_2$  on  $D_2$ , i.e., the selfadjoint operator on  $L^2(D_2, d\mu_2)$  associated with  $L_2$  on  $D_2$  (cf. Subsection 2.2 of [64]).

We assume the hypothesis **(SMI2)** for  $(L_2, D_2)$ , which is composed of three conditions (S), (M), and (I), i.e., semismallness, minimality and intrinsic ultracontractivity for  $(L_2, D_2)$ . Let us state the conditions (S), (M) and (I). We say that the semigroup  $e^{-t\mathcal{L}_2}$  generated by  $-\mathcal{L}_2$  is IU (i.e., intrinsically ultracontractive) when  $\lambda_0$  is the first eigenvalue of  $\mathcal{L}_2$ , and there exists a positive continuous decreasing function  $C(t)$  on  $(0, \infty)$  such that

$$p_2(x_2, y_2, t) \leq C(t)e^{-\lambda_0 t} \phi_0(x_2)\phi_0(y_2), \quad x_2, y_2 \in D_2, \quad t > 0, \tag{1.12}$$

where  $\phi_0$  is a normalized positive eigenfunction associated with  $\lambda_0$ ,  $p_2(x_2, y_2, t)$  is the integral kernel of the semigroup  $e^{-t\mathcal{L}_2}$ . For IU, see [22], [24], [64] and references therein. We assume the following condition (I).

(I) The semigroup  $e^{-t\mathcal{L}_2}$  is IU and the function  $C(t)$  in (1.12) satisfies

$$\lim_{t \rightarrow 0} t \log C(t) = 0. \tag{1.13}$$

For example, when  $D_2$  is compact this condition is satisfied with  $C(t) = \alpha t^{-n_2/2}$  for some positive constant  $\alpha$  (cf. Example 9.2 of [64]). The condition (I) implies that the spectrum of  $\mathcal{L}_2$  consists of discrete eigenvalues with finite multiplicity. Let  $\lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots$  be the eigenvalues of  $\mathcal{L}_2$  repeated according to multiplicity. Let  $\phi_j$  be an eigenfunction associated with  $\lambda_j$  ( $j = 0, 1, 2, \dots$ ) such that  $\{\phi_j\}_{j=0}^\infty$  is a complete orthonormal system of  $L^2(D_2, d\mu_2)$ . It follows from (I) that  $\phi_j/\phi_0 \in L^\infty(D_2)$  for any  $j \geq 1$ . We assume the following condition (S).

(S) The constant function 1 is a semismall perturbation of  $L_2 - \lambda$  on  $D_2$  for some  $\lambda < \lambda_0$ . This condition means that for any  $\varepsilon > 0$  there exists a compact subset  $K$  of  $D_2$  such that

$$\int_{D_2 \setminus K} g(x_2^0, z)g(z, y_2)d\mu_2(z) \leq \varepsilon g(x_2^0, y_2), \quad y_2 \in D_2 \setminus K, \tag{1.14}$$

where  $x_2^0$  is a reference point in  $D_2$ , and  $g$  is the Green function of  $L_2 - \lambda$  on  $D_2$  with respect to the measure  $d\mu_2$  (cf. [62]). When  $D_2$  is compact, the condition (S) is redundant. When  $D_2$  is non-compact, we denote by  $D_2^*$  and  $\partial_M D_2$  the Martin compactification and Martin boundary of  $D_2$  with respect to  $L_2 - \lambda$ , respectively (cf. [14], [49], [64], [73], [79] and references therein). We recall that for any  $\eta \in \partial_M D_2$  there exists a sequence  $\{y_2^j\}_j$  in  $D_2$  such that  $\{y_2^j\}_j$  has no point of accumulation in  $D_2$  and the sequence  $\{g(\cdot, y_2^j)/g(x_2^0, y_2^j)\}_j$  of functions on  $D_2$  converges locally uniformly to the Martin kernel  $h(\cdot, \eta)$ , which is a positive solution of the equation  $(L_2 - \lambda)u = 0$  in  $D_2$ . We also recall a positive solution  $u$  is said to be minimal if another positive solution satisfies  $v \leq u$ , then  $v = cu$  for some constant  $c$ . When  $D_2$  is non-compact, we assume the following condition (M).

(M) For any  $\eta \in \partial_M D_2$ , the Martin kernel  $h(\cdot, \eta)$  for  $(L_2 - \lambda, D_2)$  is minimal.

When  $D_2$  is compact, we put  $D_2^* = D_2$  and  $\partial_M D_2 = \emptyset$  as convention. The condition (S) implies that for any  $j = 1, 2, \dots$ , the function  $\phi_j/\phi_0$  has a continuous extension  $[\phi_j/\phi_0]$  up to the Martin boundary  $\partial_M D_2$  (cf. Theorem 6.3 of [71] and Theorem 5.12 of [64]). The condition (M) together with (I) and (S) implies that the family  $\{[\phi_j/\phi_0]; j = 0, 1, 2, \dots\}$  separates finite Borel measures on  $D_2^*$  (cf. Proposition 9.7 of [64]). Throughout the present paper we assume the hypothesis (SMI2):

(SMI2) The conditions (S), (M) and (I) are satisfied for  $(L_2, D_2)$ .

For example, (SMI2) holds when  $D_2$  is compact or a relatively compact Lipschitz domain and  $L_2$  is an elliptic operator on the whole space  $M_2$  (cf. Examples 9.2 and 9.3 of [64]). Here  $D_2$  is said to be a Lipschitz domain of  $M_2$  when for any boundary point  $z \in \partial D_2$ , the domain  $D_2$  in a coordinate neighborhood of  $z$  is the upper side of a Lipschitz continuous graph.

Let  $D_1$  be a non-compact domain of  $M_1$ . We assume that either  $D_1 = M_1$  or  $D_1$  is a Lipschitz domain of  $M_1$ . Consider (weak) solutions of the Dirichlet problem (1.2), (1.3) and (1.4). The boundary condition (1.4) means that for any  $\psi \in C_0^\infty(M_1)$  and  $T > 0$ ,

$$\psi v \in L^\infty((0, T); L^2(D_1, d\mu_1)) \cap L^2((0, T); H_0^1(D_1, d\mu_1)),$$

where  $H_0^1(D_1, d\mu_1)$  is the closure of  $C_0^\infty(D_1)$  in the Sobolev space  $H^1(D_1, d\mu_1)$  of order 1. We introduce the following condition (U1), i.e., uniqueness for the positive Dirichlet problem for  $(\partial_t + W_1^{-1}L_1, D_1)$ .

(U1) Any nonnegative solution of the problem (1.2), (1.3) and (1.4) must be identically zero.

Let  $L = L_1 + W_1L_2$  and  $D = D_1 \times D_2$ . We assume that  $(L, D)$  is subcritical, i.e., there exists the (minimal positive) Green function  $G$  of  $L$  on  $D$ . This implies that  $(L_1 + \lambda_j W_1, D_1)$  are also subcritical for any  $j = 0, 1, \dots$  (cf. Theorem 7.4 of [64]). Denote by  $H_j$  the Green function for  $(L_1 + \lambda_j W_1, D_1)$ . Fix a reference point  $x^0 \in D$ . Denote by

$$D^*, \partial_M D, \partial_m D, \quad \text{and} \quad K(x, \xi)$$

the Martin compactification, Martin boundary, minimal Martin boundary, and Martin kernel for  $(L, D)$ , respectively. Similarly,

$$D_1^*, \partial_M D_1, \partial_m D_1 \quad \text{and} \quad k_0(x_1, \xi_1)$$

denote those for  $(L_1 + \lambda_0 W_1, D_1)$ . It is known that the closure  $\overline{D_1}$  of  $D_1$  in  $M_1$  is continuously embedded into  $D_1^*$  and  $\partial D_1 \subset \partial_m D_1$  (cf. Theorem 2.1 of [55]). We put

$$\Gamma_1 = \partial_M D_1 \setminus \partial D_1.$$

Our main results are the following Theorem 1.1 and new examples such as Theorem 1.3(ii), Theorem 6.2, Examples 8.1–8.4 to be stated later on. Theorem 1.1 gives a method to determine Martin boundaries for elliptic equations via Widder type uniqueness theorems for parabolic equations.

**THEOREM 1.1.** *Assume the conditions (SMI2) and (U1). Then the following (i)–(vi) hold true:*

(i) *With  $d_2$  being an ideal point outside of  $D_2^*$ , the Martin boundary  $\partial_M D$  is equal to the disjoint union of  $\Gamma_1 \times \{d_2\}$ ,  $\partial D_1 \times D_2^*$ , and  $D_1 \times \partial_M D_2$ :*

$$\partial_M D = \Gamma_1 \times \{d_2\} \sqcup \partial D_1 \times D_2^* \sqcup D_1 \times \partial_M D_2. \tag{1.15}$$

Furthermore,

$$\partial_m D = (\Gamma_1 \cap \partial_m D_1) \times \{d_2\} \sqcup \partial D_1 \times D_2^* \sqcup D_1 \times \partial_M D_2. \tag{1.16}$$

In particular,  $\partial_m D = \partial_M D$  if and only if  $\Gamma_1 \subset \partial_m D_1$ , i.e.,  $\partial_m D_1 = \partial_M D_1$ .

(ii) *For  $\xi_1 \in \Gamma_1$ , a subset  $U$  of  $D^*$  is a neighborhood of  $\check{\xi}_1 = (\xi_1, d_2)$  if and only if there exists a neighborhood  $U_1$  of  $\xi_1$  in  $D_1^*$  such that*

$$U \supset (U_1 \cap \Gamma_1) \times \{d_2\} \cup (U_1 \cap \overline{D_1}) \times D_2^*. \tag{1.17}$$

(iii) *For  $\xi \in \partial D_1 \times D_2^* \cup D_1 \times \partial_M D_2$ , a subset  $U$  of  $D^*$  is a neighborhood of  $\xi$  if and only if there exist neighborhoods  $U_1$  and  $U_2$  of  $\xi_1$  and  $\xi_2$  in  $\overline{D_1}$  and  $D_2^*$ , respectively, such that  $U_1 \times U_2 \subset U$ .*

(iv) *For  $\xi \in \Gamma_1 \times \{d_2\}$ ,*

$$K(x, \xi) = k_0(x_1, \xi_1) \phi_0(x_2) / \phi_0(x_2^0), \quad x \in D. \tag{1.18}$$

(v) *For  $\xi \in \partial D_1 \times D_2^*$ ,*

$$K(x, \xi) = k(x, \xi) / k(x^0, \xi), \quad x \in D, \tag{1.19}$$

where  $k(\cdot, \xi)$  is a positive solution of (1.1) defined by

$$k(x, \xi) = \sum_{j=0}^{\infty} k_j(x_1, \xi_1) \phi_j(x_2) [\phi_j / \phi_0](\xi_2), \quad x \in D, \tag{1.20}$$

$$k_j(x_1, \xi_1) = \lim_{D_1 \ni y_1 \rightarrow \xi_1} H_j(x_1, y_1) / H_0(x_1^0, y_1), \quad j = 0, 1, 2, \dots. \tag{1.21}$$

Here the series on the right hand side of (1.20) converges uniformly on  $(F \times E) \times (\partial D_1 \times D_2^*)$  for any relatively compact domains  $F \subset D_1$  and  $E \subset D_2$ . It also converges in  $L^\infty(\partial D_1 \times D_2^*; L^2(F \times D_2))$  for any relatively compact domain  $F$  in  $D_1$ . Furthermore,  $k(x, \xi)$  is continuous on  $D \times (\partial D_1 \times D_2^*)$ , and  $k_j(\cdot, \xi_1)$  is a positive solution of  $(L_1 + \lambda_j W_1)u = 0$  in  $D_1$  for any  $j = 0, 1, 2, \dots$ .

(vi) For  $\xi \in D_1 \times \partial_M D_2$ ,

$$K(x, \xi) = H(x, \xi)/H(x^0, \xi), \quad x \in D, \tag{1.22}$$

where  $H(\cdot, \xi)$  is a positive solution of (1.1) determined by

$$H(x, \xi) = \sum_{j=0}^{\infty} H_j(x_1, \xi_1) \phi_j(x_2) [\phi_j/\phi_0](\xi_2), \quad x \in (D_1 \setminus \{\xi_1\}) \times D_2. \tag{1.23}$$

Here the series on the right hand side of (1.23) converges uniformly on any compact subset of  $(D_1 \setminus \{\xi_1\}) \times D_2$ . It also converges in  $L^2(F \times D_2)$  for any relatively compact domain  $F$  in  $D_1 \setminus \{\xi_1\}$ . Furthermore,  $H$  is continuous on  $D \times (D_1 \times \partial_M D_2)$ .

Theorem 1.1 says that the uniqueness for a parabolic equation implies that the fiber of  $\partial_M(D_1 \times D_2)$  at infinity of the base space  $D_1$  reduces into one point. This theorem will be proved in Section 5. The condition (U1) in Theorem 1.1 implies that for  $\xi_1 \in \Gamma_1$  the limit  $k_j(x_1, \xi_1) = 0$  for any  $j \geq 1$  (see Lemma 5.3 in Section 5). This means that the perturbation  $W_1$  of the operator  $L_1 + \lambda_0 W_1$  on  $D_1$  is big in some sense, since the Green function  $H_j$  of  $L_1 + \lambda_j W_1$  on  $D_1$  becomes smaller as the positive function  $W_1$  becomes bigger. Now, we introduce the following condition (S1), i.e., semismallness of  $W_1$ , which is complementary to the condition (U1).

(S1)  $W_1$  is a semismall perturbation of  $L_1 + \lambda_0 W_1$  on  $D_1$ .

This condition means that for any  $\varepsilon > 0$  there exists a compact subset  $K$  of  $D_1$  such that

$$\int_{D_1 \setminus K} H_0(x_1^0, z) W_1(z) H_0(z, y_1) d\mu_1(z) \leq \varepsilon H_0(x_1^0, y_1), \quad y_1 \in D_1 \setminus K,$$

where  $x_1^0$  is a reference point in  $D_1$ . By Theorem 3.1 and Remark 3.4 to be stated in Section 3, both the conditions (S1) and (U1) do not hold together. Interestingly, in several important cases, either (S1) or (U1) holds.

When (S1) holds, the Martin compactification  $(D_1 \times D_2)^*$  of  $D_1 \times D_2$  with respect to  $L$  is extremely simple. In this case,  $(D_1 \times D_2)^*$  is regular:  $(D_1 \times D_2)^* = D_1^* \times D_2^*$ . The following theorem is a special case of Theorem 9.1 of [64] (see Theorem 4.2 in Section 4).

**THEOREM 1.2.** *Assume the conditions (SMI2) and (S1). Then the following (i)–(iii) hold true.*

(i) *The Martin compactification  $D^*$  of  $D$  with respect to  $L$  is homeomorphic to  $D_1^* \times D_2^*$ . In particular,*

$$\partial_M D = \Gamma_1 \times D_2^* \sqcup \partial D_1 \times D_2^* \sqcup D_1 \times \partial_M D_2. \tag{1.24}$$

Furthermore,

$$\partial_m D = (\Gamma_1 \cap \partial_m D_1) \times D_2^* \sqcup \partial D_1 \times D_2^* \sqcup D_1 \times \partial_M D_2. \tag{1.25}$$

In particular,  $\partial_m D = \partial_M D$  if and only if  $\Gamma_1 \subset \partial_m D_1$ , i.e.,  $\partial_m D_1 = \partial_M D_1$ .

(ii) The assertion (v) of Theorem 1.1 holds with  $\partial D_1$  replaced by  $\Gamma_1 \cup \partial D_1$ . In particular, the Martin kernel  $K(x, \xi)$  for  $\xi \in (\Gamma_1 \cup \partial D_1) \times D_2^*$  is given by (1.19).

(iii) The assertion (vi) of Theorem 1.1 holds.

This theorem says that “smallness” of  $W_1$  implies the regularity of  $(D_1 \times D_2)^*$ , while Theorem 1.1 says that “bigness” of  $W_1$  implies the degeneration of the fiber at infinity.

Here, as an application of Theorems 1.1 and 1.2, we give a simple example concerning positive harmonic functions on horn-shaped domains in  $\mathbf{R}^{N+1}$ ,  $N \geq 2$ . Further examples will be given in Section 8.

**THEOREM 1.3.** *Let  $\alpha$  and  $\beta$  be Lipschitz continuous functions on  $[1, \infty)$  such that  $\alpha > \beta$  and  $(\alpha(r) - \beta(r))/r$  is decreasing. Let*

$$D_1 = \{(r, s) \in \mathbf{R}^2; \alpha(r) > s > \beta(r), 1 < r < \infty\}.$$

Let  $D_2$  be a Lipschitz domain in the unit sphere  $S^{N-1}$  of  $\mathbf{R}^N$  or the whole space  $S^{N-1}$ , where  $N \geq 2$ . Let  $L = -\Delta$  on  $\mathbf{R}^{N+1}$  and

$$D = \{(z, s) \in \mathbf{R}^N \times \mathbf{R}^1; \alpha(|z|) > s > \beta(|z|), |z| > 1, z/|z| \in D_2\}.$$

(i) Suppose that

$$\int_1^\infty (\alpha(r) - \beta(r))r^{-2}dr < \infty. \tag{1.26}$$

Then  $D^*$  is homeomorphic to  $D_1^* \times \overline{D_2}$ , where  $D_1^* = \overline{D_1} \cup \{\infty\}$  is the closure of  $D_1$  in the one-point compactification of  $\mathbf{R}^2$ . Furthermore,  $\partial_M D = \partial_m D = \partial_M D_1 \times \overline{D_2} \cup D_1 \times \partial D_2$  and  $\partial_m D_1 = \partial_M D_1 = \partial D_1 \cup \{\infty\}$ .

(ii) Suppose that

$$\int_1^\infty (\alpha(r) - \beta(r))r^{-2}dr = \infty. \tag{1.27}$$

Then  $D^*$  is homeomorphic to  $(\overline{D_1} \times \overline{D_2}) \sqcup \{(\infty, d_2)\}$ , where a fundamental neighborhood system of the ideal point  $(\infty, d_2)$  is given by the family

$$\{(\{(r, s) \in \overline{D_1}; \varepsilon^{-1} < r < \infty\} \times \overline{D_2}) \cup \{(\infty, d_2)\}\}_{0 < \varepsilon < 1}.$$

Furthermore,  $\partial_M D = \partial_m D = \{(\infty, d_2)\} \cup \partial D_1 \times \overline{D_2} \cup D_1 \times \partial D_2$  and  $\partial_M D_1 = \partial_m D_1 = \partial D_1 \cup \{\infty\}$ .

The assertion (i) of Theorem 1.3 was shown by Aikawa and Murata [4] (see also Theorem 6.3 in Section 6). The assertion (ii) will be proved in Section 6. A special case of this theorem was shown under more stringent condition by Ioffe and Pinsky [40], and related results were announced by Maz'ya [50].

The remainder of this paper is organized as follows. Sections 2, 3 and 4 are preliminaries.

In Section 2, we recall a uniqueness theorem for a parabolic equation in [44], and give a concrete example related to Theorem 1.3. In Section 3, we recall criteria for non-h-bigness, and observe that the Widder type uniqueness theorem implies h-bigness. In Section 4, we recall general results on Martin boundaries for elliptic equations in skew product form given in [64]. We prove Theorem 1.1 in Section 5. Theorem 1.3(ii) is proved in Section 6. There we also give a theorem on small perturbation, and generalize the assertion (i) of Theorem 1.3. In Section 7, we give a generalization of Theorems 1.1 and 1.2. By applying it, we give several concrete examples in Section 8.

**2. Uniqueness theorems for parabolic equations.**

In this section we recall for readability a typical uniqueness theorem of Widder type in [44], and give a simple example related to Theorem 1.3. Let  $N$  be a connected separable smooth manifold with Riemannian metric of class  $C^0$ . We assume that the Riemannian manifold  $N$  is complete. Let  $P$  be an elliptic operator on  $N$  of the form

$$Pu = -w^{-1} \operatorname{div}(wa \nabla u - wuc) - \langle b, \nabla u \rangle + qu, \tag{2.1}$$

where  $w, a, b, c, q$  satisfy the conditions (1.6), (1.7) and (1.8) with obvious modifications. We further assume that  $P$  is uniformly elliptic on  $N$ , i.e., there exists a positive constant  $\kappa$  such that

$$\kappa |\xi|^2 \leq \langle a_x \xi, \xi \rangle \leq \kappa^{-1} |\xi|^2, \quad (x, \xi) \in TN. \tag{2.2}$$

We denote by  $\nu$  the Riemannian measure on  $N$ , and put  $d\lambda = w d\nu$ . Consider the Cauchy problem

$$\mathcal{P}u = 0 \quad \text{in } N \times (0, \infty), \tag{2.3}$$

$$u(x, 0) = u_0(x) \quad \text{on } N, \tag{2.4}$$

where  $\mathcal{P} = \partial_t + P$  and  $u_0 \in L^2_{\text{loc}}(N, d\lambda)$ . In order to state a Widder type uniqueness theorem, we need two conditions. Put  $q^\pm = \max(\pm q, 0)$ . Fix a point  $O$  in  $N$ , and let  $d(x) = \operatorname{dist}(O, x)$  be the Riemannian distance between  $O$  and  $x \in N$ . Put  $B(O, R) = \{x \in N; d(x) < R\}$  for  $R > 0$ . Let  $\rho$  be a positive continuous increasing function on  $[0, \infty)$ . Then the condition [RB- $\rho$ ] (i.e., relative boundedness with scale function  $\rho$ ) to be imposed on  $b, c, q^-$  is as follows.

[RB- $\rho$ ] There exist constants  $\alpha_1 > 0, 0 \leq \beta_1 < 1, 0 < \beta_2 < 1, 0 < \beta_3 < 1$  such that  $\beta_1 + \beta_2 + \beta_3 < 1$  and

$$\begin{aligned} & \int_{B(O,R)} \left[ \frac{1}{4\beta_2} \langle a^{-1}b, b \rangle + \frac{1}{4\beta_3} \langle a^{-1}c, c \rangle + q^- \right] \psi^2 d\lambda \\ & \leq \beta_1 \int_{B(O,R)} \langle a \nabla \psi, \nabla \psi \rangle d\lambda + \alpha_1 \rho(R)^2 \int_{B(O,R)} \psi^2 d\lambda \end{aligned} \tag{2.5}$$

for any  $R > 1$  and  $\psi \in C_0^\infty(B(O, R))$ .

The second condition to be imposed on  $P$  is the following condition [PHP- $\rho$ ], i.e., the parabolic Harnack principle with scale function  $\rho$ .



[PHP- $\rho$ ] There exists a positive constant  $\alpha_2$  such that for any

$$(x, t) \in N \times (0, \infty), 0 < r \leq \frac{1}{\rho(d(x))},$$

any nonnegative solution  $u$  of the equation

$$\mathcal{P}u = 0 \quad \text{in } Q = B(x, r) \times (t - r^2, t + r^2) \tag{2.6}$$

satisfies the inequality

$$\sup_{Q^-} u \leq \alpha_2 \inf_{Q^+} u, \tag{2.7}$$

where

$$Q^- = B\left(x, \frac{r}{2}\right) \times \left(t - \frac{3}{4}r^2, t - \frac{1}{4}r^2\right),$$

$$Q^+ = B\left(x, \frac{r}{2}\right) \times \left(t + \frac{1}{4}r^2, t + \frac{3}{4}r^2\right).$$

For the parabolic Harnack inequality (2.7), see [15], [27], [33], [41], [44], [48], [52], [75], [77] and references therein. We are now ready to state a Widder type uniqueness theorem, which is a time independent elliptic operator case of Theorem 2.2 in [44].

**THEOREM 2.1.** *Suppose that the conditions [RB- $\rho$ ] and [PHP- $\rho$ ] hold with  $\rho$  satisfying*

$$\int_1^\infty \frac{dr}{\rho(r)} = \infty. \tag{2.8}$$

*Then a nonnegative solution  $u$  of the Cauchy problem (2.3) and (2.4) is determined uniquely by the initial data  $u_0$ .*

As for analogous theorems concerning the Dirichlet problem, see Theorem 4.4 of [65] and references therein.

Here, we give a simple example related to Theorem 1.3.

**THEOREM 2.2.** *Let  $\gamma \geq -2$ . Let  $\alpha$  and  $\beta$  be Lipschitz continuous functions on  $[1, \infty)$  such that  $\alpha > \beta$  and  $(\alpha(r) - \beta(r))r^{\gamma/2}$  is decreasing. Let*

$$\Omega = \{(r, s) \in \mathbf{R}^2; 1 < r < \infty, \alpha(r) > s > \beta(r)\}.$$

*Let  $P = -r^{-\gamma}(\partial^2/\partial r^2 + \partial^2/\partial s^2)$ . Consider the Dirichlet problem*

$$(\partial_t + P)u = 0 \quad \text{in } \Omega \times (0, \infty), \tag{2.9}$$

$$u(x, 0) = u_0(x) \quad \text{on } \Omega, \tag{2.10}$$

$$u(x, t) = 0 \quad \text{on } \partial\Omega \times (0, \infty), \tag{2.11}$$

where  $u_0$  satisfies  $\eta u_0 \in L^2(\Omega, r^\gamma dr ds)$  for any  $\eta \in C_0^\infty(\{(r, s) \in \mathbf{R}^2; r > 0\})$ . Suppose that

$$\int_1^\infty (\alpha(r) - \beta(r)) r^\gamma dr = \infty. \tag{2.12}$$

Then a nonnegative solution of the Dirichlet problem (2.9), (2.10) and (2.11) is determined uniquely by the initial data  $u_0$ .

We omit the proof, since the theorem can be shown along the line given in the proof of Theorem 5.6 of [65].

REMARK 2.3. Actually, the condition (2.12) is also a necessary condition for the Widder type uniqueness theorem to hold. Indeed, suppose that

$$\int_1^\infty (\alpha(r) - \beta(r)) r^\gamma dr < \infty.$$

Apply Theorem 6.1 in Section 6 with  $v_1(r) = r$  and  $\Phi(t_1) = t_1^\gamma$ . Then we obtain that  $r^\gamma$  is a small perturbation of  $-\Delta$  on  $\Omega$ . Thus Remark 3.5 and Theorem 3.1 in Section 3 show that there exists a positive solution of (2.9), (2.10) and (2.11) with  $u_0 = 0$ .

### 3. h-big perturbations.

In this section we recall a non-uniqueness theorem in [65], and observe that the Widder type uniqueness theorem implies h-bigness.

Let  $N$  be a connected separable smooth manifold with Riemannian metric of class  $C^0$ . Let  $L$  be an elliptic operator on  $N$  of the form

$$Lu = -m^{-1} \operatorname{div}(mA \nabla u - mCu) - \langle B, \nabla u \rangle + Vu, \tag{3.1}$$

where  $m, A, B, C, V$  satisfy the conditions (1.6), (1.7) and (1.8), with obvious modifications. Let  $W$  be a positive measurable function on  $N$  such that  $W, W^{-1} \in L_{\text{loc}}^\infty(N, d\lambda), d\lambda = m dv$ , where  $v$  is the Riemannian measure on  $N$ . Let  $\Omega$  be a domain of  $N$ . We consider the Dirichlet problem

$$(\partial_t + W^{-1}L)u = 0 \quad \text{in } \Omega \times (0, \infty), \tag{3.2}$$

$$u(x, 0) = 0 \quad \text{on } \Omega, \tag{3.3}$$

$$u(x, t) = 0 \quad \text{on } \partial\Omega \times (0, \infty). \tag{3.4}$$

Suppose that  $(L, \Omega)$  is subcritical, i.e., there exists the Green function  $G$  of  $L$  on  $\Omega$ . Let  $h$  be a positive solution of the Dirichlet problem

$$Lv = 0 \quad \text{on } \Omega, \tag{3.5}$$

$$v = 0 \quad \text{on } \partial\Omega. \tag{3.6}$$

Here, the boundary condition (3.6) means  $v \in H_{0,\text{loc}}^1(\Omega)$ . Following [35], we say that  $W$  is h-big

(on  $\Omega$ ) when any function  $v$  satisfying

$$(L + W)v = 0 \quad \text{and} \quad 0 \leq v \leq h \quad \text{on } \Omega \tag{3.7}$$

must be identically zero. Otherwise,  $W$  is said to be non-h-big (on  $\Omega$ ). Theorem 2.5 of [65] partially reads as follows.

**THEOREM 3.1.** *The following are equivalent:*

- (i)  $W$  is non-h-big.
- (ii) There exist a non-empty domain  $E \subset \Omega$  and a positive solution  $f$  of the Dirichlet problem

$$Lf = 0 \quad \text{on } E, \quad f = 0 \quad \text{on } \partial E$$

such that  $0 < f \leq h$  on  $E$  and

$$\int_E G_E(x, y)W(y)f(y)d\lambda(y) < \infty, \quad x \in E, \tag{3.8}$$

where  $G_E$  is the Green function of  $L$  on  $E$  with respect to the measure  $d\lambda$ .

- (iii) There exists a solution  $u$  of (3.2), (3.3) and (3.4) such that  $0 < u(x, t) \leq h(x)$  on  $\Omega \times (0, \infty)$ .

We should mention here that the statement of the assertion (ii) is slightly different from that of the assertion (II) of Theorem 2.5 in [65], but they are equivalent because a nonnegative solution of an elliptic equation on a connected open set is positive or identically zero.

The following is a direct consequence of Theorem 3.1 but a key observation in proving Theorem 1.1.

**PROPOSITION 3.2.** *Suppose that the Dirichlet problem (3.2), (3.3) and (3.4) has no non-negative solution which is not identically zero. Then  $W$  is h-big for any positive solution  $h$  of (3.5) and (3.6).*

**REMARK 3.3.** When a positive solution  $h$  satisfies an appropriate growth condition at infinity, a Täcklind type uniqueness theorem (cf. [44], [65]) can be used also as a sufficient condition of h-bigness.

We conclude this section by remarking facts on semismall perturbations (cf. Section 5 of [64]).

**REMARK 3.4.** If  $W$  is a semismall perturbation of  $L$  on  $\Omega$ , then  $W$  is non-h-big for any positive solution  $h$  of (3.5) (cf. Remark 5.9 of [64]).

**REMARK 3.5.** We say that  $W$  is a small perturbation of  $L$  on  $\Omega$  when for any  $\varepsilon > 0$  there exists a compact subset  $K$  of  $\Omega$  such that

$$\int_{\Omega \setminus K} G(x, z)W(z)G(z, y)d\lambda(z) \leq \varepsilon G(x, y), \quad x, y \in \Omega \setminus K.$$

It is known (cf. [62]) that if  $W$  is a small perturbation, then it is a semismall perturbation. Thus, if  $W$  is a small perturbation of  $L$  on  $\Omega$ , then  $W$  is non-h-big for any positive solution  $h$  of (3.5).

### 4. Martin boundaries for elliptic equations in skew product form.

In this section we recall general results in [64], from which Theorem 1 is derived. Consider the equation (1.1). For  $(L_2, D_2)$ , we assume the same conditions as in Section 1; so  $L_2$  is the operator (1.10) on  $D_2$  satisfying the hypothesis (SMI2). But, in this section, we treat  $L_1$  and  $D_1$  under more general conditions although we use the same notations as in Section 1. Let  $D_1$  be a non-compact domain of  $M_1$ . Let  $L_1$  be an elliptic operator on  $D_1$  of the form (1.5), where  $m_1, A_1, B_1, C_1, V_1$  satisfy the conditions (1.6), (1.7) and (1.8) with  $M_1$  replaced by  $D_1$ . Let  $W_1$  be a positive measurable function on  $D_1$  such that  $W_1, W_1^{-1} \in L^\infty_{loc}(D_1, d\mu_1)$ . Let  $L = L_1 + W_1 L_2$  and  $D = D_1 \times D_2$ . We assume that  $(L, D)$  is subcritical. We denote by  $D_1^*, \partial_M D_1, \partial_m D_1$ , and  $k_0$  the Martin compactification, Martin boundary, minimal Martin boundary, and Martin kernel for  $(L_1 + \lambda_0 W_1, D_1)$ , respectively. For an open set  $\Omega \subset D_1$ , we denote by  $\Omega^*$  the closure of  $\Omega$  in  $D_1^*$ ; while  $\bar{\Omega}$  denotes the closure  $\Omega$  in the relative topology of  $D_1$ . We denote by  $L_1^*$  the formal adjoint operator of  $L_1$  with respect to  $d\mu_1$ . For an elliptic operator  $P$  on an open set  $\Omega \subset D_1$ , a subset  $F$  of  $\Omega$  with  $\bar{F} \cap \Omega = F$ , and a family  $\mathcal{F}$  of positive solutions of  $Pu = 0$  in  $\Omega$ , we put  $\mathcal{S} = (\mathcal{F}, P, \Omega, F)$ . We say that CP (i.e., the comparison principle) holds for  $\mathcal{S}$  when there exists a positive constant  $c$  such that for any  $u$  and  $v$  in  $\mathcal{F}$

$$c \frac{v(x)}{v(y)} \leq \frac{u(x)}{u(y)} \leq c^{-1} \frac{v(x)}{v(y)}, \quad x, y \in F. \tag{4.1}$$

We impose on  $\{(L_1 + \lambda_j W_1, D_1)\}_{j=0}^\infty$  the following condition (ZCS1), i.e., zero limit, comparison principle and semismallness.

(ZCS1) There exist subsets  $\Xi_0$  and  $\Xi_\infty$  of  $\partial_M D_1$  such that  $\Xi_0 \cup \Xi_\infty = \partial_M D_1$  and the following conditions (ZC) and (CS) are satisfied.

(ZC) For any  $\xi_1 \in \Xi_0$ , there exist domains  $U_i$  ( $i = 1, 2, 3, 4$ ) of  $D_1$  such that

$$\bar{U}_i \subset U_{i+1} \text{ for } i = 1, 2, 3, \quad \xi_1 \in U_1^* \cap \partial_M D_1, \quad x_1^0 \in U_3 \setminus \bar{U}_1, \tag{4.2}$$

$$\lim_{U_3 \ni y_1 \rightarrow \xi_1} h_1(x_1, y_1) / h_0(x_1^0, y_1) = 0, \quad x_1 \in U_3, \tag{4.3}$$

where  $h_1$  (resp.  $h_0$ ) is the Green function of  $L_1 + \lambda_1 W_1$  (resp.  $L_1 + \lambda_0 W_1$ ) on  $U_4$ . Furthermore, CP holds for  $\mathcal{S}$  and  $\mathcal{R}$ , where

$$\mathcal{S} = (\{H_0(\cdot, y_1); y_1 \in \bar{U}_1 \cup (D_1 \setminus U_3)\}, L_1 + \lambda_0 W_1, U_3 \setminus \bar{U}_1, \partial U_2), \tag{4.4}$$

$$\mathcal{R} = (\{H_1(x_1^0, \cdot), h_1(x_1^0, \cdot)\}, L_1^* + \lambda_1 W_1, U_4 \setminus \{x_1^0\}, \partial U_3).$$

(CS) For any  $\xi_1 \in \Xi_\infty$ , there exist domains  $E_i$  ( $i = 1, \dots, 8$ ) of  $D_1$  such that

$$\bar{E}_i \subset E_{i+1} \text{ for } i = 1, \dots, 7, \quad \xi_1 \in E_1^* \cap \partial_M D_1, \quad x_1^0 \in E_6 \setminus \bar{E}_5, \tag{4.5}$$

$$W_1 \text{ is a semismall perturbation of } L_1 + \lambda_0 W_1 \text{ on } E_8, \tag{4.6}$$

and CP holds for  $\mathcal{S}_i$  ( $i = 1, 2, 3$ ),  $\mathcal{T}_j$  and  $\mathcal{U}_j$  ( $j = 0, 1, \dots$ ), where

$$\mathcal{S}_i = (\{H_0(\cdot, y_1); y_1 \in \overline{E_{2i-1}} \cup (D_1 \setminus E_{2i+1})\}, L_1 + \lambda_0 W_1, E_{2i+1} \setminus \overline{E_{2i-1}}, \partial E_{2i}), \quad (4.7)$$

$$\mathcal{T}_j = (\{H_j(\cdot, y_1); y_1 \in \overline{E_6}\} \cup \{h_j(\cdot, y_1); y_1 \in \overline{E_6}\}, L_1 + \lambda_j W_1, E_8 \setminus \overline{E_6}, \partial E_7),$$

$$\mathcal{U}_j = (\{H_j(x_1^0, \cdot), h_j(x_1^0, \cdot)\}, L_1^* + \lambda_j W_1, E_8 \setminus \{x_1^0\}, \partial E_6), \quad j = 0, 1, 2, \dots$$

Here  $h_j$  is the Green function of  $L_1 + \lambda_j W_1$  on  $E_8$ .

This condition (ZCS1) always holds when  $D_1$  is one dimensional (cf. [64]). The semi-localized condition (4.3) and (4.6) are useful in treating domains having several connected components at infinity. Note that CP holds for (4.7), for example, if  $\overline{E_8} \setminus E_1$  is a compact subset of  $D_1$ .

We are now ready to state Theorem 9.1 of [64] except for the case where  $D_1$  is compact.

**THEOREM 4.1.** *Assume the conditions (SMI2) and (ZCS1). Then the following (i)–(iv) hold true:*

(i) *With  $d_2$  being an ideal point outside of  $D_2^*$ , the Martin boundary  $\partial_M D$  is equal to the disjoint union of  $\Xi_0 \times \{d_2\}$ ,  $\Xi_\infty \times D_2^*$  and  $D_1 \times \partial_M D_2$ :*

$$\partial_M D = \Xi_0 \times \{d_2\} \sqcup \Xi_\infty \times D_2^* \sqcup D_1 \times \partial_M D_2. \quad (4.8)$$

Furthermore,

$$\partial_m D = (\Xi_0 \cap \partial_m D_1) \times \{d_2\} \sqcup (\Xi_\infty \cap \partial_m D_1) \times D_2^* \sqcup D_1 \times \partial_M D_2. \quad (4.9)$$

In particular,  $\partial_m D = \partial_M D$  if and only if  $\partial_m D_1 = \partial_M D_1$ .

- (ii) *The assertions (ii) and (iv) of Theorem 1.1 hold with  $\Gamma$  replaced by  $\Xi_0$ .*
- (iii) *The assertions (iii) and (v) of Theorem 1.1 hold with  $\partial D_1$  replaced by  $\Xi_\infty$ .*
- (iv) *The assertion (vi) of Theorem 1.1 holds.*

A special case of this theorem is worth stating.

**THEOREM 4.2.** *Assume (SMI2). Suppose that  $W_1$  is a semismall perturbation of  $L_1 + \lambda_0 W_1$  on  $D_1$ . Then the Martin compactification  $D^*$  of  $D$  with respect to  $L$  is homeomorphic to  $D_1^* \times D_2^*$ , and all the assertions of Theorem 4.1 hold with  $\Xi_0 = \emptyset$  and  $\Xi_\infty = \partial_M D_1$ .*

### 5. Proof of Theorem 1.1.

In this section we prove Theorem 1.1 by applying Theorem 4.1 in the last section. We use the notations in Section 1, and assume the conditions (SMI2) and (U1). We start with a lemma concerning small perturbation and the boundary Harnack principle for elliptic equations. For definition of small perturbation, see Remark 3.5 in Section 3. As for the boundary Harnack principle, see [3], [5], [19], [20], [38], [39], [55], [81].

**LEMMA 5.1.** *The condition (CS) of the hypothesis (ZCS1) holds with  $\Xi_\infty$  and  $E_1^*$  replaced by  $\partial D_1$  and  $\overline{E_1}$ , respectively.*

Here and in what follows we abuse notations as follows:  $\overline{E_1}$  and  $\partial E_1$  in this section mean the closure and boundary of  $E_1$  in  $M_1$ , respectively; so  $\overline{E_1} \cap D_1$  and  $\partial E_1 \cap D_1$  are equal to the

symbols  $\overline{E_1}$  and  $\partial E_1$  in the hypothesis (ZCS1).

PROOF. Let  $\xi_1 \in \partial D_1$ . Since  $D_1$  is a Lipschitz domain, we can choose a local coordinate system  $(U, z)$  such that

$$z(U \cap D_1) = \{z = (z', z_N) \in \mathbf{R}^N; |z'| < R, 0 < z_N - f(z') < R\}, \tag{5.1}$$

$$z(U \cap \partial D_1) = \{z \in \mathbf{R}^N; |z'| < R, z_N = f(z')\}, \tag{5.2}$$

and  $z(\xi_1) = (0, 0)$ , where  $N = n_1, R$  is a positive number and  $f$  is a Lipschitz continuous function on  $\mathbf{R}^{N-1}$  with  $f(0) = 0$ . We denote the right hand side of (5.1) by  $E(R)$ . For  $0 < r < R/8$  with  $x_1^0 \notin \overline{E(5r)}$ , choose a Lipschitz curve  $\gamma$  in  $D_1 \setminus \overline{E(5r)}$  such that  $\gamma(0) = x_1^0$  and  $\gamma(1) = (0, f(0) + 6r)$ . For  $s > 0$ , put

$$F(s) = \{p \in M_1; \inf_{0 \leq t \leq 1} \text{dist}(p, \gamma(t)) < s\}.$$

Choose  $s$  so small that  $F(8s) \subset D_1 \setminus \overline{E(5r)}$ . For  $i = 6, 7, 8$ , put  $E_i = E(ir) \cup F(is)$ . Modifying  $F(is)$  if necessary, we may assume that  $E_i$  are relatively compact Lipschitz domain of  $D_1$ . For  $i = 1, \dots, 5$ , put  $E_i = E(ir)$ . Then  $\overline{E_i} \cap D_1 \subset E_{i+1}$  for  $i = 1, \dots, 7$ ,  $\xi_1 \in \overline{E_1} \cap \partial D_1$ , and  $x_1^0 \in E_6 \setminus \overline{E_5}$ . In the coordinate system  $(U, z)$  the operator  $L_1 + \lambda_0 W_1$  has the form

$$w(L_1 + \lambda_0 W_1)u = - \sum_{1 \leq i, j \leq N} \partial_i(a_{ij} \partial_j u) - \sum_{1 \leq j \leq N} b_j \partial_j u + \sum_{1 \leq j \leq N} \partial_j(c_j u) + qu, \tag{5.3}$$

where  $w$  is a positive measurable function with  $w, w^{-1} \in L^\infty(E(R))$  and  $a_{i,j}, b_j, c_j, q$  satisfy the condition (1.7) and (1.8) with obvious modifications. Thus, re-choosing  $r$  and  $s$  sufficiently small if necessary, we can show by Theorem 9.1', Proposition 9.2 and the proof of Corollary 6.1 of [8] that  $W_1$ , which is bounded on  $E_8$ , is a small perturbation of  $L_1 + \lambda_0 W_1$  on  $E_8$  (see also [1], [62]). This implies (4.6). Let  $i = 1, 2, 3$ . By the boundary Harnack principle, there exists a positive constant  $c$  such that

$$c \frac{v(x)}{v(y)} \leq \frac{u(x)}{u(y)} \leq c^{-1} \frac{v(x)}{v(y)}, \quad x, y \in \partial E_{2i} \cap D_1, \tag{5.4}$$

for any positive solutions  $u$  and  $v$  of the equation  $(L_1 + \lambda_0 W_1)u = 0$  in  $E_{2i+1} \setminus \overline{E_{2i-1}}$  such that

$$u = v = 0 \quad \text{on } \{z \in \mathbf{R}^N; (2i-1)r < |z'| < (2i+1)r, z_n = f(z')\}$$

(cf. Theorem 1.3 of [55]). (We have abused notations:  $\partial E_{2i}$  in (5.4) is the boundary of  $E_{2i}$  in  $M_1$ , and so  $\partial E_{2i} \cap D_1$  is the boundary of  $E_{2i}$  in  $D_1$  which is equal to  $\partial E_{2i}$  in (4.7).) Here, let us give another proof of (5.4). Denote by  $P$  the operator on the right hand side of (5.3), and put

$$Qu = - \sum_{1 \leq i, j \leq N} \partial_i(a_{ij} \partial_j u).$$

Choose a relatively compact Lipschitz domain  $E \subset D_1$  such that  $\overline{E} \cap D_1 \subset E_{2i+1} \setminus \overline{E_{2i-1}}$  and  $E \supset U \cap D_1$  for some neighborhood  $U$  of  $\overline{\partial E_{2i} \cap D_1}$ . Let  $u$  and  $v$  be positive solutions of the equation  $Pu = 0$  in  $E$  such that they are continuous up to the boundary and vanish on  $\{z \in \partial E; z_n = f(z')\}$ . Let  $\hat{u}$  be a positive solution of the equation  $Q\hat{u} = 0$  in  $E$  with  $\hat{u} = u$  on  $\partial E$ . Denote by  $\mu_x$  and  $\nu_x$ ,  $x \in E$ , the harmonic measures for  $P$  and  $Q$ , respectively. Then there exists a positive constant  $c_1$  such that  $c_1\mu_x \leq \nu_x \leq c_1^{-1}\mu_x$ ,  $x \in E$  (cf. Proposition 8.3 and the comment after Theorem 9.1' of [8]). Thus  $c_1u(x) \leq \hat{u}(x) \leq c_1^{-1}u(x)$ ,  $x \in E$ . Similarly,  $c_1v(x) \leq \hat{v}(x) \leq c_1^{-1}v(x)$ ,  $x \in E$ . By Theorem 1.4 of [19], there exists a positive constant  $c_2$  such that

$$c_2 \frac{\hat{v}(x)}{\hat{v}(y)} \leq \frac{\hat{u}(x)}{\hat{u}(y)} \leq c_2^{-1} \frac{\hat{v}(x)}{\hat{v}(y)}, \quad x, y \in \partial E_{2i} \cap D_1.$$

This implies (5.4). Now, for  $y_1 \in \overline{E_{2i-1}} \cup (D_1 \setminus E_{2i+1})$ ,  $H_0(\cdot, y_1)$  is a positive solution of the equation  $(L_1 + \lambda_0 W_1)u = 0$  in  $E_{2i+1} \setminus \overline{E_{2i-1}}$  which vanishes on  $\{z \in \partial(E_{2i+1} \setminus \overline{E_{2i-1}}); z_N = f(z')\}$ . Hence CP holds for  $\mathcal{S}_i$  given by (4.7). Similarly, CP holds for  $\mathcal{T}_j$  and  $\mathcal{U}_j$  given by (4.7). This completes the proof of the lemma.  $\square$

The following lemma is essentially Lemma 5.8 of [64]. It is a simple observation, but plays a crucial role in proving Theorem 1.1.

LEMMA 5.2. *Let  $h(x_1) = k_0(x_1, \xi_1)$  for some  $\xi_1 \in \Gamma_1 = \partial_M D_1 \setminus \partial D_1$ , where  $k_0$  is the Martin kernel for  $(L_1 + \lambda_0 W_1, D_1)$ . If  $W_1$  is  $h$ -big, then*

$$\lim_{D_1 \ni y_1 \rightarrow \xi_1} H_1(x_1, y_1) / H_0(x_1^0, y_1) = 0, \quad x_1 \in D_1. \quad (5.5)$$

PROOF. We give a proof, since it is simple. Suppose that (5.5) does not hold. Then there exists a positive solution  $v$  of the equation  $(L_1 + \lambda_1 W_1)v = 0$  in  $D_1$  satisfying  $0 < v \leq h$ . This is a contradiction, since  $(\lambda_1 - \lambda_0)W_1$  is also  $h$ -big (cf. Propositions 7.16 and 3.7 of [35]).  $\square$

LEMMA 5.3. *Let  $h(x_1) = k_0(x_1, \xi_1)$  for some  $\xi_1 \in \Gamma_1$ . Then (5.5) holds.*

PROOF. By the a priori estimates near boundary points,  $h$  is a positive solution in  $H_{0, \text{loc}}^1(D_1)$  of the equation  $(L_1 + \lambda_0 W_1)h = 0$  in  $D_1$ . It follows from the assumption (U1) that any nonnegative solution of the problem

$$\begin{aligned} (\partial_t + W_1^{-1}(L_1 + \lambda_0 W_1))v &= 0 && \text{in } D_1 \times (0, \infty), \\ v(x, 0) &= 0 && \text{on } D_1, \\ v(x, t) &= 0 && \text{on } \partial D_1 \times (0, \infty) \end{aligned}$$

must be identically zero. Thus, by Proposition 3.2 in Section 3,  $W_1$  is  $h$ -big. Hence Lemma 5.2 implies (5.5).  $\square$

We are now ready to complete the proof of Theorem 1.1.

PROOF OF THEOREM 1.1. We claim that the condition (ZC) of the hypothesis (ZCS1) holds with  $\Xi_0$  replaced by  $\Gamma_1$ . Choose domains  $U_i$  ( $i = 1, \dots, 4$ ) such that  $D_1 \setminus U_1$  is a compact subset of  $D_1$ ,  $U_4 = D_1$ ,  $\overline{U_i} \cap D_1 \subset U_{i+1}$  for  $i = 1, 2, 3$ , and  $x_1^0 \in U_3 \setminus \overline{U_1}$ . Then (4.2) holds. By

Lemma 5.3, (4.3) holds. By the Harnack inequality, (CP) holds for  $\mathcal{S}$  and  $\mathcal{R}$  given by (4.4). This proves the claim, which together with Lemma 5.1 implies that the hypothesis (ZCS1) holds with  $\Xi_0 = \Gamma_1$  and  $\Xi_\infty = \partial D_1$ . Hence Theorem 4.1 in the last section shows Theorem 1.1.  $\square$

**6. Martin boundaries of horn-shaped domains.**

In this section we show the assertion (ii) of Theorem 1.3, and generalize the assertion (i) of Theorem 1.3.

**6.1. Small perturbations.**

In this subsection we give a theorem on small perturbation, Theorem 6.1. By using it we also give an improvement of Theorem 4 of [4], Theorem 6.2, from which the assertion (i) of Theorem 1.3 follows. Theorems 6.1 and 6.2 are of independent interest.

Let  $\Omega$  be a domain in  $\mathbf{R}^2$  such that  $(-\Delta, \Omega)$  is subcritical, i.e., there exists the Green function  $H$  of  $-\Delta$  on  $\Omega$  (cf. Theorem 8.33 of [37]). Let  $\Phi(t_1, \dots, t_l)$  be a nonnegative Borel measurable function on  $(0, \infty]^l$ . Define  $\Psi(t_1, \dots, t_l)$  by

$$\Psi(t_1, \dots, t_l) = \sup_{4^{-1} < c_1, \dots, c_l < 4} \Phi(c_1 t_1, \dots, c_l t_l).$$

Let  $v_j$  ( $j = 1, \dots, l$ ) be  $(0, \infty]$ -valued continuous superharmonic functions on  $\Omega$ . Put

$$W(z) = \Phi(v_1(z), \dots, v_l(z)).$$

Then we have the following

**THEOREM 6.1.** *Suppose that*

$$\int_{\Omega} \Psi(v_1(z), \dots, v_l(z)) dz < \infty, \tag{6.1}$$

where  $dz$  is the Lebesgue measure on  $\mathbf{R}^2$ . Then  $W$  is a small perturbation of  $-\Delta$  on  $\Omega$ .

**PROOF.** Let  $\partial_\infty \Omega$  be the boundary of  $\Omega$  in the one point compactification of  $\mathbf{R}^2$ . Let  $F$  be the set of points in  $\partial_\infty \Omega$  which are irregular with respect to the Dirichlet problem for harmonic functions on  $\Omega$ . Then there exists a positive superharmonic function  $v$  on  $\Omega$  such that  $\lim_{z \rightarrow x} v(z) = \infty$  for all  $x \in F$  (cf. Lemmas 9.18 and 9.19 of [37]). For an interval  $I$  in  $(0, \infty]$ , denote by  $\chi_I$  the characteristic function of  $I$ . For  $\delta > 0$ , put

$$\phi_\delta(z) = \chi_{(0, \delta)}(H(z, y^0)) + \chi_{[\delta^{-1}, \infty)}(v(z)),$$

where  $y_0$  is a point fixed in  $\Omega$ . Then there exists a positive constant  $c_l$  depending only on  $l$  such that

$$\begin{aligned} & \frac{1}{H(x, y)} \int_{\Omega} H(x, z) H(z, y) \phi_\delta(z) W(z) dz \\ & \leq c_l \int_{\Omega} \phi_{4\delta}(z) \Psi(v_1(z), \dots, v_l(z)) dz, \quad x, y \in \Omega, \end{aligned} \tag{6.2}$$



(cf. [2], Theorem 1 of [4] and the remark which follows it). Since  $v(z) < \infty$  for a.e.  $z \in \Omega$  and  $H(z, y^0) > 0$  for any  $z \in \Omega$ , we have

$$\lim_{\delta \rightarrow 0} \phi_{4\delta}(z) = 0 \quad \text{for a.e. } z \in \Omega.$$

By the Lebesgue dominated convergence theorem, for any  $\varepsilon > 0$  there exists  $\delta > 0$  such that the right hand side of (6.2) is less than  $\varepsilon$ . Put  $K = \{z \in \Omega; \phi_\delta(z) = 0\}$ . Since  $\lim_{z \rightarrow \zeta} H(z, y^0) = 0$  for any regular boundary point  $\zeta$  in  $\partial_\infty \Omega$ ,  $\phi_\delta(z) \geq 1$  on a neighborhood of  $\partial_\infty \Omega$ . Thus  $K$  is a relatively compact subset of  $\Omega$ . We have

$$\begin{aligned} & \frac{1}{H(x, y)} \int_{\Omega \setminus K} H(x, z) H(z, y) W(z) dz \\ & \leq \frac{1}{H(x, y)} \int_{\Omega} H(x, z) H(z, y) \phi_\delta(z) W(z) dz < \varepsilon \end{aligned}$$

for any  $x, y \in \Omega$ . Hence  $W$  is a small perturbation of  $-\Delta$  on  $\Omega$ . □

The following is an improvement of Theorem 4 of [4].

**THEOREM 6.2.** *Let  $D_1$  be a domain in  $\{(r, s) \in \mathbf{R}^2; r > 0\}$ . Let  $D_2$  be a Lipschitz domain in  $S^{N-1}$  or the whole space  $S^{N-1}$ , where  $N \geq 2$ . Let  $L = -\Delta$  on  $\mathbf{R}^{N+1}$  and*

$$D = \{(z, s) \in \mathbf{R}^N \times \mathbf{R}^1; (|z|, s) \in D_1, z/|z| \in D_2\}. \tag{6.3}$$

Suppose that

$$\iint_{D_1} \frac{dr ds}{r^2} < \infty. \tag{6.4}$$

Then the Martin compactification  $D^*$  for  $(L, D)$  is homeomorphic to  $D_1^* \times \overline{D_2}$ , where  $D_1^*$  is the Martin compactification for  $(-\Delta, D_1)$ . In particular,  $\partial_M D$  is homeomorphic to  $(D_1 \times \partial D_2) \cup (\partial_M D_1 \times \overline{D_2})$ . Furthermore,

$$\partial_m D = (D_1 \times \partial D_2) \cup (\partial_m D_1 \times \overline{D_2}).$$

**PROOF.** We show the theorem by applying Theorem 4.2. In the polar coordinates of  $\mathbf{R}^N$ ,

$$L = -\frac{\partial^2}{\partial r^2} - \frac{N-1}{r} \frac{\partial}{\partial r} - \frac{\Lambda}{r^2} - \frac{\partial^2}{\partial s^2}, \tag{6.5}$$

where  $\Lambda$  is the Laplace-Beltrami operator on the sphere  $S^{N-1}$ . With

$$L_1 = -\frac{\partial^2}{\partial r^2} - \frac{\partial^2}{\partial s^2} - \frac{N-1}{r} \frac{\partial}{\partial r}, \quad W_1 = \frac{1}{r^2}, \quad L_2 = -\Lambda, \tag{6.6}$$

we have  $L = L_1 + W_1 L_2$ . For  $(L_2, D_2)$ , the hypothesis (SMI2) holds with  $\lambda_0 \geq 0$  (cf. Examples 9.2 and 9.3 of [64]). Let us show that  $W_1$  is a small perturbation of  $L_1 + \lambda_0 W_1$  on  $D_1$ . We have

$$\begin{aligned}
 P &\equiv r^{-(N-1)/2} \circ (L_1 + \lambda_0 W_1) \circ r^{(N-1)/2} \\
 &= -\frac{\partial^2}{\partial r^2} - \frac{\partial^2}{\partial s^2} + \left[ \lambda_0 + \frac{(N-1)(N-3)}{4} \right] \frac{1}{r^2}.
 \end{aligned}
 \tag{6.7}$$

Apply Theorem 6.1 with  $\Phi(t_1) = t_1^{-2}$  and  $v_1(z) = z_1$ . Then it follows from (6.4) that  $W_1 = r^{-2}$  is a small perturbation of  $-\Delta$  on  $D_1$ . Thus the Green function  $g$  of  $P$  on  $D_1$  is comparable with the Green function  $H$  of  $-\Delta$  on  $D_1$ , i.e.,  $cg \leq H \leq c^{-1}g$  for some positive constant  $c$  (cf. [62]). This together with Theorem 6.1 shows that  $W_1$  is a small perturbation of  $P$  on  $D_1$ . Denote by  $H_0(r, s; \tilde{r}, \tilde{s})$  and  $g(r, s; \tilde{r}, \tilde{s})$  the Green functions of  $L_1 + \lambda_0 W_1$  and  $P$  on  $D_1$ . Then

$$g(r, s; \tilde{r}, \tilde{s}) = (r/\tilde{r})^{(N-1)/2} H_0(r, s; \tilde{r}, \tilde{s}).
 \tag{6.8}$$

Thus

$$\frac{g(r, s; z_1, z_2)g(z_1, z_2; \tilde{r}, \tilde{s})}{g(r, s; \tilde{r}, \tilde{s})} = \frac{H_0(r, s; z_1, z_2)H_0(z_1, z_2; \tilde{r}, \tilde{s})}{H_0(r, s; \tilde{r}, \tilde{s})}.$$

It follows from this that  $W_1$  is a small perturbation of  $L_1 + \lambda_0 W_1$  on  $D_1$ . In view of Theorem 4.2, it remains to show that the Martin compactification  $D_{1, L_1 + \lambda_0 W_1}^*$  of  $D_1$  with respect to  $L_1 + \lambda_0 W_1$  is homeomorphic to Martin compactification  $D_{1, -\Delta}^*$  of  $D_1$  with respect to  $-\Delta$ . We have

$$\frac{H_0(r, s; \tilde{r}, \tilde{s})}{H_0(r^0, s^0; \tilde{r}, \tilde{s})} = \left( \frac{r}{r_0} \right)^{(1-N)/2} \frac{g(r, s; \tilde{r}, \tilde{s})}{g(r^0, s^0; \tilde{r}, \tilde{s})},$$

where  $(r^0, s^0)$  is a reference point in  $D_1$ . Thus  $D_{1, L_1 + \lambda_0 W_1}^*$  is homeomorphic to  $D_{1, P}$  which is homeomorphic to  $D_{1, -\Delta}^*$ , since  $r^{-2}$  is a small perturbation of  $-\Delta$  on  $D_1$ . Hence  $D_{1, L_1 + \lambda_0 W_1}^*$  is homeomorphic to  $D_{1, -\Delta}^*$ .  $\square$

In Theorem 4 of [4], it was assumed that every boundary point of  $D_1$  is regular with respect to the Dirichlet problem. Theorem 6.2 removes this regularity assumption.

The following is a special case of Theorem 6.2 and a generalization of the assertion (i) of Theorem 1.3.

**THEOREM 6.3.** *Let  $\alpha$  and  $\beta$  be continuous functions on  $[1, \infty)$  such that  $\alpha > \beta$ . Let  $D_1 = \{(r, s) \in \mathbf{R}^2; \alpha(r) > s > \beta(r), 1 < r < \infty\}$ . Let  $D_2, D$  and  $L$  be as in Theorem 1.3. Then the assertion (i) of Theorem 1.3 holds.*

**PROOF.** By virtue of Theorem 6.2, it suffices to show that  $D_1^* = \overline{D_1} \cup \{\infty\}$  and  $\partial_M D_1 = \partial_m D_1 \cup \{\infty\}$ . But this follows from the Carathéodory theorem (cf. [74]) which says that there exists a homeomorphism from  $\overline{D_1} \cup \{\infty\}$  onto the closed unit disc which is conformal in  $D_1$ .  $\square$

**6.2. Proof of Theorem 1.3 (ii).**

In this subsection we show the assertion (ii) of Theorem 1.3 by applying Theorem 1.1.

LEMMA 6.4. *Let  $D_1$  be as in Theorem 1.3. Then the Martin compactification  $D_1^*$  of  $D_1$  with respect to  $L_1 + \lambda_0 W_1$  (see (6.6)) is homeomorphic to  $\overline{D_1} \cup \{\infty\}$  which is the closure of  $D_1$  in the one point compactification of  $\mathbf{R}^2$ . Furthermore,  $\partial_M D_1 = \partial_m D_1 = \partial D_1 \cup \{\infty\}$ .*

PROOF. We give only an outline of the proof. Let  $\mathcal{F}$  be the set of all positive solutions  $u$  of  $(L_1 + \lambda_0 W_1)u = 0$  in  $D_1$  such that  $u = 0$  on  $\partial D_1$  and  $u(x_1^0) = 1$ . Then the boundary Harnack principle and the scaling argument as in the proof of Theorem A of [55] show that  $\mathcal{F}$  consists of one element. It follows from this that  $D_1^* = \overline{D_1} \cup \{\infty\}$  and  $\partial_M D_1 = \partial_m D_1$ .  $\square$

PROOF OF THEOREM 1.3 (ii). Recall that the hypothesis (SMI2) for  $(L_2, D_2)$  holds (cf. Examples 9.2 and 9.3 of [64]). By Lemma 6.4 and Theorem 1.1, it suffices to show that the condition (U1) holds. Consider the equation

$$(\partial_t + W_1^{-1}L_1)v = 0 \quad \text{in } D_1 \times (0, \infty)$$

(see (6.6) and (6.7)). We have

$$r^{-(N-1)/2} \circ (\partial_t + W_1^{-1}L_1) \circ r^{(N-1)/2} = \partial_t - r^2 \Delta + (N-1)(N-3)/4.$$

Thus Theorem 2.2 and the assumption (1.27) show that any nonnegative solution of (1.2), (1.3) and (1.4) must be identically zero, i.e., (U1) holds.  $\square$

**7. Generalization.**

In this section we slightly generalize Theorems 1.1 and 1.2 for giving more concrete examples.

Let  $L = L_1 + W_1 L_2$  and  $D = D_1 \times D_2$  be as in Section 1. Assume (SMI2) for  $(L_2, D_2)$ . Suppose that

$$D_1 = \bigcup_{j=0}^N E_j, \tag{7.1}$$

where  $N$  is a natural number,  $E_j$  ( $j = 1, \dots, N$ ) are Lipschitz domains in  $M_1$  or the whole space  $M_1$  such that  $\overline{E_j} \cap \overline{E_k} = \emptyset$  for  $j \neq k$ , and  $E_0$  is a relatively compact Lipschitz domain in  $M_1$  or an empty set. Here  $\overline{E_j}$  is the closure of  $E_j$  in  $M_1$ , while  $E_j^*$  denote the closure of  $E_j$  in  $D_1^*$ . For  $j = 1, \dots, N$ , consider the Dirichlet problem

$$(\partial_t + W_1^{-1}L_1)v = 0 \quad \text{in } E_j \times (0, \infty), \tag{7.2}$$

$$v(x, 0) = 0 \quad \text{on } E_j, \tag{7.3}$$

$$v(x, t) = 0 \quad \text{on } \partial E_j \times (0, \infty). \tag{7.4}$$

We introduce the following condition.

(US1) There exists an integer  $l$  such that (i)  $0 \leq l \leq N$ , (ii) for  $1 \leq j \leq l$ , any nonnegative solution of (7.2), (7.3) and (7.4) must be identically zero, and (iii) for  $l < j \leq N$ ,  $W_1$  is a semismall perturbation of  $L_1 + \lambda_0 W_1$  on  $E_j$ .

**THEOREM 7.1.** *Assume the conditions (SMI2) and (US1). Put*

$$\Xi_0 = \bigcup_{j=1}^l (E_j^* \cap \partial_M D_1) \setminus \partial D_1, \quad \Xi_\infty = \bigcup_{j=l+1}^N (E_j^* \cap \partial_M D_1) \cup \partial D_1.$$

*Then all the conclusions of Theorem 4.1 hold true. Furthermore,  $E_j^* \cap E_k^* = \emptyset$  for  $j, k = 1, \dots, N$  with  $j \neq k$ .*

This theorem can be shown as Theorems 1.1 and 1.2. For proving the last assertion, use Lemma 1.5 of [64].

**8. Examples.**

In this section we give several concrete examples as applications of Theorem 7.1.

**EXAMPLE 8.1.** Let  $L = -\Delta$  on  $\mathbf{R}^{2+m}$ . Let  $D_2$  be a bounded Lipschitz domain in  $\mathbf{R}^m$ . Let  $D_1$  be a Lipschitz domain in  $\mathbf{R}^2$  of the form

$$D_1 = \bigcup_{j=0}^N E_j,$$

where  $N$  is a natural number and  $E_j$  are Lipschitz domains defined as follows: For  $j = 1, \dots, N$ , let  $f_j$  be a Lipschitz continuous positive function on  $[1, \infty)$  such that it is decreasing and  $f_j(1) < 1/2$ ; and let

$$E_j = \{(r, s) \in \mathbf{R}^2; |s - j| < f_j(r), r > 1\}, \quad j = 1, \dots, N,$$

$$E_0 = \bigcup_{j=1}^N \{(r, s) \in \mathbf{R}^2; |s - j| < f_j(r), 1 \leq r < 2\} \cup (0, 1) \times (0, N + 1).$$

Let  $0 \leq l \leq N$  be an integer. Suppose that

$$\int_1^\infty f_j(r) dr = \infty, \quad 1 \leq j \leq l,$$

$$\int_1^\infty f_j(r) dr < \infty, \quad l < j \leq N.$$

For  $j = 1, \dots, N$ , let  $\eta_j$  be the point at infinity of the one point compactification of  $\overline{E_j}$ ; and set  $\eta_j \neq \eta_k$  for  $j \neq k$ . Put

$$\Xi_0 = \{\eta_j; j = 1, \dots, l\}, \quad \Xi_\infty = \{\eta_j; j = l + 1, \dots, N\} \cup \partial D_1.$$

Let  $D = D_1 \times D_2$ . Then the Martin boundary  $\partial_M D$  for  $(L, D)$  is homeomorphic to

$$\Xi_0 \times \{d_2\} \cup \Xi_\infty \times \overline{D_2} \cup D_1 \times \partial D_2.$$

Furthermore,  $\partial_M D = \partial_m D$ . Indeed, by Theorems 2.2 and 6.1, the hypothesis (US1) holds. We see that  $\partial_M D_1 = \partial_m D_1 = \partial D_1 \cup \{\eta_j; j = 1, \dots, N\}$ . Thus Theorem 7.1 shows the assertion.

EXAMPLE 8.2. Let  $D = \{x \in \mathbf{R}^n; |x| > 1\}$ . Let  $V$  be a locally bounded measurable real-valued function on  $[1, \infty)$ . Let  $L = -\Delta + V(|x|)$ . Suppose that  $(L, D)$  is subcritical. Then it is known that  $\partial_M D = \partial_m D \supset \partial D$  and the set

$$\Gamma = \partial_M D \setminus \partial D$$

is homeomorphic to the unit sphere  $S^{n-1}$  or one point.

(i) Suppose that

$$\int_1^\infty \left( \sup_{1 \leq s \leq r} s^2 |V(s)| + 1 \right)^{-1/2} \frac{dr}{r} = \infty. \tag{8.1}$$

Then  $\Gamma$  consists of one point.

(ii) Suppose that  $r^2 V(r) + \alpha \geq 1$  on  $[1, \infty)$  for some positive constant  $\alpha$ . Assume that

$$\int_1^\infty (r^2 V(r) + \alpha)^{-1/2} \frac{dr}{r} < \infty. \tag{8.2}$$

Then  $\Gamma$  is homeomorphic to the unit sphere  $S^{n-1}$ .

For results related to (i) and (ii), see [45], [53], [55], [56], [67], [72], and Example 10.1 of [64].

Let us show the assertion (i) by applying Theorem 7.1. In the polar coordinates of  $\mathbf{R}^n$ ,

$$L = -r^{1-n} \frac{\partial}{\partial r} \left( r^{n-1} \frac{\partial}{\partial r} \right) + V(r) - \frac{\Lambda}{r^2}, \tag{8.3}$$

where  $\Lambda$  is the Laplace-Beltrami operator on the sphere  $S^{n-1}$ . Let  $D_1 = (1, \infty)$ ,  $D_2 = S^{n-1}$ ,

$$L_1 = -r^{1-n} (\partial/\partial r)(r^{n-1} \partial/\partial r) + V(r), \quad W_1 = r^{-2}, \quad L_2 = -\Lambda.$$

Then  $L = L_1 + W_1 L_2$  on  $D = D_1 \times D_2$ . Put  $E_0 = (2, 9)$ ,  $E_1 = (e^2, \infty)$  and  $E_2 = (1, 3)$ . Then  $D_1 = E_0 \cup E_1 \cup E_2$ . We see that  $W_1$  is a small perturbation of  $L_1$  on  $E_2$  (cf. Theorem 6.3 of [64]). Consider the Dirichlet problem (7.2)–(7.4) for  $j = 1$ . We claim that any nonnegative solution of it must be identically zero. Change the variable  $r$  to  $z = \log r$ . Then (7.2) becomes

$$\left[ \frac{\partial}{\partial t} - \frac{\partial^2}{\partial z^2} - (n-2) \frac{\partial}{\partial z} - e^{2z} V(e^z) \right] \hat{v} = 0 \quad \text{in } (2, \infty) \times (0, \infty), \tag{8.4}$$

where  $\hat{v}(z, t) = v(r, t)$ . Let us show that there exists a positive continuous increasing function  $\rho$  on  $[0, \infty)$  such that

$$\int_1^\infty \frac{dR}{\rho(R)} = \infty, \quad \sup_{0 \leq z \leq R} e^{2z}|V(e^z)| \leq \rho(R)^2.$$

For  $R \geq 0$ , put

$$\phi(R) = \left( \sup_{0 \leq z \leq R} e^{2z}|V(e^z)| + 1 \right)^{1/2}. \tag{8.5}$$

For any natural number  $k$ , denote by  $\chi_k$  the characteristic function of the interval

$$I_k = \{R \geq 0; 2^{k-1} \leq \phi(R) < 2^k\}.$$

Put  $\psi = \sum_{k=1}^\infty 2^k \chi_k$ . Then the step function  $\psi$  is increasing, and satisfies  $\phi \leq \psi \leq 2\phi$ . By (8.5) and (8.1),

$$\sum_{k \in J} 2^{-k} |I_k| = \int_0^\infty \frac{dR}{\psi(R)} = \infty,$$

where  $|I_k|$  is the length of  $I_k$  and  $J = \{k \in \mathbf{N}; |I_k| \neq 0\}$ . Choose a series of intervals  $\{I'_k\}_{k \in J}$  and a piecewise linear continuous increasing function  $\rho$  on  $[0, \infty)$  such that

$$I'_k \subset I_k, \quad \sum_{k \in J} 2^{-k} |I'_k| = \infty, \quad \rho = \psi \text{ on } \bigcup_{k \in J} I'_k, \quad \rho \geq \psi \text{ on } [0, \infty).$$

Then  $\rho$  has the desired properties. Now, let  $\hat{v}$  be a nonnegative solution of (8.4) with  $\hat{v}(z, 0) = 0$  on  $(2, \infty)$  and  $\hat{v}(2, t) = 0$  on  $(0, \infty)$ . Then Theorem 4.4 of [65] together with the scaling argument as in the proof of Theorem 6.2 of [44] shows that  $\hat{v} = 0$ . Thus the claim holds, and so the condition (US1) is satisfied. Hence Theorem 7.1 shows the assertion (i).

Let us show the assertion (ii). We claim that  $W_1$  is a small perturbation of  $L_1 + \alpha W_1$  on  $D_1$ . Let  $f$  and  $g$  be positive solutions of the equation  $(L_1 + \alpha W_1)f = 0$  in  $D_1$  and  $(L_1 + (\alpha + 1)W_1)g = 0$  in  $D_1$  with  $f(1) = g(1) = 0$  and  $f'(1) = g'(1) = 1$ , respectively. Change the variable  $r$  to  $z = \log r$ , and put  $\hat{f}(z) = f(e^z)$  and  $\hat{g}(z) = g(e^z)$ . Then the equation becomes

$$P\hat{f} \equiv \left[ \frac{d^2}{dz^2} + (n-2) \frac{d}{dz} - e^{2z}V(e^z) - \alpha \right] \hat{f} = 0 \quad \text{in } (0, \infty),$$

$$(P-1)\hat{g} = 0 \quad \text{in } (0, \infty),$$

with  $\hat{f}(0) = \hat{g}(0) = 0$  and  $\hat{f}'(0) = \hat{g}'(0) = 1$ . By (8.2),

$$\int_0^\infty (e^{2z}V(e^z) + \alpha)^{-1/2} dz < \infty.$$

Then the same argument as in the proof of Lemma 2 of [58] shows that  $\lim_{z \rightarrow \infty} \hat{g}(z)/\hat{f}(z) < \infty$ . Thus

$$\lim_{r \rightarrow \infty} \frac{g(r)}{f(r)} < \infty.$$

We see that the Martin boundary for  $(L_1 + \alpha W_1, D_1)$  is  $\{1, \infty\}$  and the Martin kernel  $k_0(x_1, \infty)$  is a constant multiple of  $f$  (cf. Appendix of [53]). Thus, by Theorem 6.3 of [64],  $W_1 \chi_{(2, \infty)}$  is a small perturbation of  $L_1 + \alpha W_1$  on  $D_1$ . Since  $W_1 \chi_{(1, 2]}$  is a small perturbation of  $L_1 + \alpha W_1$  on  $D_1$ , this implies that  $W_1$  is a small perturbation of  $L_1 + \alpha W_1$  on  $D_1$ , i.e., the claim holds. Thus the Green functions of  $L_1 + \alpha W_1$  on  $D_1$  and  $L_1$  on  $D_1$  are comparable. Hence  $W_1$  is a small perturbation of  $L_1$  on  $D_1$ . Hence Theorem 7.1 (or Theorem 1.2) shows the assertion (ii).

**EXAMPLE 8.3.** Let  $D_1$  be a bounded Lipschitz domain in  $\mathbf{R}^n$ , and put  $\delta_1(x_1) = \text{dist}(x_1, \partial D_1)$ . Let  $L_1 = -\delta_1(x_1)^\gamma \Delta_1$ , where  $\gamma$  is a real number and  $\Delta_1$  is the Laplacian on  $\mathbf{R}^n$ . Let  $D_2 = M_2$  be a compact manifold. Let  $L_2 = -\Delta_2$ , where  $\Delta_2$  is the Laplacian on  $M_2$ . Let  $L = L_1 + L_2$  and  $D = D_1 \times D_2$ . Let  $\partial_M D$  and  $\partial_m D$  be the Martin boundary and minimal Martin boundary for  $(L, D)$ . Then we have the following:

- (i) For  $\gamma \geq 2$ ,  $\partial_m D = \partial_M D = \partial D_1 \times \{d_2\}$ .
- (ii) For  $\gamma < 2$ ,  $\partial_m D = \partial_M D = \partial D_1 \times D_2$ .

Let us show the assertions. We see that the Martin compactification  $D_1^*$  of  $D_1$  with respect to  $L_1$  is homeomorphic to  $\overline{D_1}$ , and  $\partial_m D_1 = \partial_M D_1 = \partial D_1$ . Suppose that  $\gamma \geq 2$ . Then, by Theorem 7.8 of [44], any nonnegative solution of the Cauchy problem

$$(\partial_t + L_1)u = 0 \quad \text{in } D_1 \times (0, \infty), \quad u(x, 0) = 0 \quad \text{on } D_1$$

must be identically zero. Thus the assumption (U1) of Theorem 1.1 is satisfied with  $D_1 = M_1$ . Hence the assertion (i) follows from Theorem 1.1. Next, suppose that  $\gamma < 2$ . Then, by Theorem 9.1 of [8], 1 is a small perturbation of  $L_1$  on  $D_1$ . Thus the assumption (S1) of Theorem 1.2 is satisfied. Hence the assertion (ii) follows from Theorem 1.2.

**EXAMPLE 8.4.** Let  $D_1 = \mathbf{R}^n$  and  $L_1 = -\Delta_1 + V_1$ , where  $\Delta_1$  is the Laplacian on  $\mathbf{R}^n$  and  $V_1$  is the function on  $\mathbf{R}^n$  such that  $V_1(z) = 1$  for  $z_n > 0$  and  $V_1(z) = 2$  for  $z_n \leq 0$ . Let  $W_1(x_1) = \langle x_1 \rangle^\gamma$ , where  $\gamma$  is a real number and  $\langle x_1 \rangle = (1 + |x_1|^2)^{1/2}$ . Let  $D_2 = \mathbf{R}^m$  and

$$L_2 = \langle x_2 \rangle^\alpha (-\Delta_2 + 1) - \beta,$$

where  $\alpha > 2$ ,  $\Delta_2$  is the Laplacian on  $\mathbf{R}^m$ , and  $\beta$  is a positive constant such that  $\lambda_0 = 0$ , i.e., 0 is the first eigenvalue of the selfadjoint operator  $\mathcal{L}_2$  associated with  $L_2$  on  $D_2$ . Let  $L = L_1 + W_1 L_2$  and  $D = D_1 \times D_2$ . It is known (cf. [54]) that the Martin boundary  $\partial_M D_1$  and the minimal Martin boundary  $\partial_m D_1$  for  $(L_1, D_1)$  are homeomorphic to the set  $\Sigma$  and  $\sigma$  defined by

$$\begin{aligned} \sigma &= \{\omega \in \mathbf{R}^n; |\omega| = 1, \omega_n \geq 0\} \cup \{\omega \in \mathbf{R}^n; |\omega| = \sqrt{2}, \omega_n \leq -1\}, \\ \Sigma &= \sigma \cup \{(\omega', -\theta) \in \mathbf{R}^n; |\omega'| = 1, 0 < \theta < 1\}, \end{aligned}$$

respectively, i.e.,  $\partial_M D_1 \cong \Sigma$  and  $\partial_m D_1 \cong \sigma$ . Furthermore, 1 is a small perturbation of  $\langle x_2 \rangle^\alpha (-\Delta_2 + 1)$  on  $\mathbf{R}^m$  (cf. Theorem 5.1 of [64]); the Martin boundary  $\partial_M D_2$  for  $(\langle x_2 \rangle^\alpha (-\Delta_2 + 1), \mathbf{R}^m)$  is homeomorphic to the unit sphere  $S^{m-1}$  at infinity (cf. [53]), i.e.,  $\partial_M D_2 = S^{m-1\infty}$ ;  $D_2^* = \mathbf{R}^m \sqcup S^{m-1\infty}$ ; the hypothesis (SMI2) for  $(L_2, D_2)$  is satisfied (cf. Example 9.4 of [64]); and  $D_1 \times \partial_M D_2 \subset \partial_m D$ . Put

$$\Gamma = \partial_M D \setminus (D_1 \times \partial_M D_2).$$

Then we have the following:

- (i) For  $\gamma \geq -1$ ,  $\Gamma \cong \Sigma$  and  $\Gamma \cap \partial_m D \cong \sigma$ .
- (ii) For  $\gamma < -1$ ,  $\Gamma \cong \Sigma \times D_2^*$  and  $\Gamma \cap \partial_m D \cong \sigma \times D_2^*$ .

Let us show the assertions. Suppose that  $\gamma < -1$ . Then  $W_1(x_1) = \langle x_1 \rangle^\gamma$  is a small perturbation of  $L_1$  on  $\mathbf{R}^n$  (cf. Theorem 5.1 of [64]). Thus Theorem 1.2 shows the assertion (ii). Next, suppose that  $\gamma \geq -1$ . Consider the Cauchy problem

$$(\partial_t + \langle z \rangle^{-\gamma} L_1)v = 0 \quad \text{in } \mathbf{R}^n \times (0, \infty), \quad v(z, 0) = 0 \quad \text{on } \mathbf{R}^n. \tag{8.6}$$

In order to show that the Cauchy problem (8.6) allows no positive solution, we introduce a Riemannian metric  $g = (g_{ij})$  on  $\mathbf{R}^n$  by  $g_{ii} = \langle z \rangle^\gamma$  and  $g_{ij} = 0$  for  $i \neq j$ . Then  $M_1 = \mathbf{R}^n$  becomes a complete Riemannian manifold with this metric  $g$ . The associated gradient  $\nabla$  and divergence  $\text{div}$  are written as

$$\nabla = \langle z \rangle^{-\gamma} \nabla^0, \quad \text{div} = \langle z \rangle^{-n\gamma/2} \text{div}^0 \langle z \rangle^{n\gamma/2},$$

where  $\nabla^0$  and  $\text{div}^0$  are the standard gradient and divergence on  $\mathbf{R}^n$ . Put  $m_1(z) = \langle z \rangle^{(1-n/2)\gamma}$ . Then

$$\langle z \rangle^{-\gamma} L_1 v = -m_1^{-1} \text{div}(m_1 \nabla v) + \langle z \rangle^{-\gamma} V_1(z)v. \tag{8.7}$$

For  $z$  with  $|z| > 1$ , denote by  $d(z)$  the Riemannian distance from 0 to  $z$ . Then  $d(z)$  is comparable with  $|z|^{(\gamma/2+1)}$ . Thus

$$|\langle z \rangle^{-\gamma} V_1(z)| \leq C d(z)^{-\gamma/(\gamma/2+1)} \leq C d(z)^2, \quad |z| > 1,$$

for some constant  $C > 0$ . We see from this that the assumption [PHP- $\rho$ ] of Theorem 2.1 is satisfied with  $\rho(R) = C(R + 1)$  for a sufficiently large positive constant  $C$  (cf. the proof of Theorem 6.2 of [44]). By Theorem 2.1, any nonnegative solution of (8.6) must be identically zero. Thus the assumption (U1) of Theorem 1.1 is satisfied. Hence Theorem 1.1 shows the assertion (i).

### References

- [ 1 ] H. Aikawa, Norm estimate of Green operator, perturbation of Green function and integrability of superharmonic functions, *Math. Ann.*, **312** (1998), 289–318.
- [ 2 ] H. Aikawa, Generalized Cranston-McConnell inequalities for discontinuous superharmonic functions, *Potential Anal.*, **8** (1998), 127–135.



- [ 3 ] H. Aikawa, Boundary Harnack principle and Martin boundary for a uniform domain, *J. Math. Soc. Japan*, **53** (2001), 119–145.
- [ 4 ] H. Aikawa and M. Murata, Generalized Cranston-McConnell inequalities and Martin boundaries of unbounded domains, *J. Anal. Math.*, **69** (1996), 137–152.
- [ 5 ] A. Ancona, Principe de Harnack à frontière et théorème de Fatou pour un opérateur elliptique dans un domaine lipschitzien, *Ann. Inst. Fourier (Grenoble)*, **28** (1978), 169–213.
- [ 6 ] A. Ancona, Negatively curved manifolds, elliptic operators and the Martin boundary, *Ann. of Math.*, **121** (1987), 429–461.
- [ 7 ] A. Ancona, Théorie du potentiel sur les graphes et les variétés, *Lecture Notes in Math.* Vol. 1427, Springer, Berlin (1988), 1–23.
- [ 8 ] A. Ancona, First eigenvalues and comparison of Green's functions for elliptic operators on manifolds or domains, *J. Anal. Math.*, **72** (1997), 45–92.
- [ 9 ] A. Ancona and J. C. Taylor, Some remarks on Widder's theorem and uniqueness of isolated singularities for parabolic equations, In: *Partial Differential Equations with Minimal Smoothness and Applications*, (eds. Dahlberg et al.) Springer, New York, 1992, 15–23.
- [ 10 ] M. T. Anderson and R. Schoen, Positive harmonic functions on complete manifolds of negative curvature, *Ann. of Math.*, **121** (1985), 429–461.
- [ 11 ] J.-P. Anker and L. Ji, Heat kernel and Green function estimates on noncompact symmetric space, *Geom. Funct. Anal.*, **9** (1999), 1035–1091.
- [ 12 ] J.-P. Anker and L. Ji, Heat kernel and Green function estimates on noncompact symmetric space II, In: *Topics in Probability and Lie Groups: Boundary Theory*, (ed. J. C. Taylor), CRM Proc. Lecture Notes, Vol. 28, Amer. Math. Soc., 2001, 153–202.
- [ 13 ] H. Arai, Degenerate elliptic operators, Hardy spaces and diffusions on strongly pseudoconvex domains, *Tohoku Math. J.*, **46** (1994), 469–498.
- [ 14 ] D. H. Armitage and S. J. Gardiner, *Classical Potential Theory*, Springer, London, 2001.
- [ 15 ] D. G. Aronson, Non-negative solutions of linear parabolic equations, *Ann. Sci. Norm. Sup. Pisa*, **22** (1968), 607–694.
- [ 16 ] D. G. Aronson and P. Besala, Uniqueness of solutions of the Cauchy problem for parabolic equations, *J. Math. Anal. Appl.*, **13** (1966), 516–526.
- [ 17 ] D. G. Aronson and P. Besala, Uniqueness of positive solutions of parabolic equations with unbounded coefficients, *Colloq. Math.*, **18** (1967), 126–135.
- [ 18 ] R. Azencott, Behavior of diffusion semi-groups at infinity, *Bull. Soc. Math. France*, **102** (1974), 193–240.
- [ 19 ] L. A. Caffarelli, E. Fabes, S. Mortola and S. Salsa, Boundary behavior of nonnegative solutions of elliptic operators in divergence form, *Indiana Univ. Math. J.*, **30** (1981), 621–640.
- [ 20 ] B. E. Dahlberg, Estimates of harmonic measure, *Arch. Ration. Mech. Anal.*, **65** (1977), 275–282.
- [ 21 ] E. B. Davies,  $L^1$  properties of second order elliptic operators, *Bull. London Math. Soc.*, **17** (1985), 417–436.
- [ 22 ] E. B. Davies, *Heat Kernels and Spectral Theory*, Cambridge Univ. Press, Cambridge, 1989.
- [ 23 ] E. B. Davies, Heat kernel bounds, conservation of probability and the Feller property, *J. Anal. Math.*, **58** (1992), 99–119.
- [ 24 ] E. B. Davies and B. Simon, Ultracontractivity and the heat kernel for Schrödinger operators and Dirichlet Laplacians, *J. Funct. Anal.*, **59** (1984), 335–395.
- [ 25 ] H. Donnelly, Uniqueness of the positive solutions of the heat equation, *Proc. Amer. Math. Soc.*, **99** (1987), 353–356.
- [ 26 ] D. Eidus and S. Kamin, The filtration equation in a class of functions decreasing at infinity, *Proc. Amer. Math. Soc.*, **120** (1994), 825–830.
- [ 27 ] E. B. Fabes and D. W. Stroock, A new proof of Moser's parabolic Harnack inequality using the old ideas of Nash, *Arch. Ration. Mech. Anal.*, **96** (1986), 327–338.
- [ 28 ] W. Feller, The parabolic differential equations and the associated semi-groups of transformations, *Ann. of Math.*, **55** (1952), 468–519.
- [ 29 ] A. Freire, On the Martin boundary of Riemannian products, *J. Differential Geom.*, **33** (1991), 215–232.
- [ 30 ] S. J. Gardiner, The Martin boundary of NTA strips, *Bull. London Math. Soc.*, **22** (1990), 163–166.
- [ 31 ] S. Giulini and W. Woes, The Martin compactification of the Cartesian product of two hyperbolic spaces, *J. Reine Angew. Math.*, **444** (1993), 17–28.
- [ 32 ] A. Grigor'yan, On stochastically complete manifolds, *Soviet Math. Dokl.*, **34** (1987), 310–313.
- [ 33 ] A. Grigor'yan, The heat equation on non-compact Riemannian manifolds, *Math. USSR Sb.*, **72** (1992), 47–77.

- [34] A. Grigor'yan, Analytic and geometric background of recurrence and non-explosion of the Brownian motion on Riemannian manifolds, *Bull. Amer. Math. Soc.*, **36** (1999), 135–249.
- [35] A. Grigor'yan and W. Hansen, A Liouville property for Schrödinger operators, *Math. Ann.*, **312** (1998), 659–716.
- [36] Y. Guivarc'h, L. Ji and J. C. Taylor, *Compactifications of Symmetric Spaces*, Birkhäuser, Boston, 1998.
- [37] L. L. Helms, *Introduction to Potential Theory*, Wiley-Interscience, New York, 1969.
- [38] R. A. Hunt and R. L. Wheeden, On the boundary values of harmonic functions, *Trans. Amer. Math. Soc.*, **132** (1968), 307–322.
- [39] R. A. Hunt and R. L. Wheeden, Positive harmonic functions on Lipschitz domains, *Trans. Amer. Math. Soc.*, **147** (1970), 505–527.
- [40] D. Ioffe and R. Pinsky, Positive harmonic functions vanishing on the boundary for the Laplacian in unbounded horn-shaped domains, *Trans. Amer. Math. Soc.*, **342** (1994), 773–791.
- [41] K. Ishige, On the behavior of the solutions of degenerate parabolic equations, *Nagoya Math. J.*, **155** (1999), 1–26.
- [42] K. Ishige, An intrinsic metric approach to uniqueness of the positive Dirichlet problem for parabolic equations in cylinders, *J. Differential Equations*, **158** (1999), 251–290.
- [43] K. Ishige and M. Murata, An intrinsic metric approach to uniqueness of the positive Cauchy problem for parabolic equations, *Math. Z.*, **227** (1998), 313–335.
- [44] K. Ishige and M. Murata, Uniqueness of nonnegative solutions of the Cauchy problem for parabolic equations on manifolds or domains, *Ann. Scuola Norm. Sup. Pisa*, **30** (2001), 171–223.
- [45] M. Kawamura and M. Nakai, A test of Picard principle for rotation free densities, II, *J. Math. Soc. Japan*, **28** (1976), 323–342.
- [46] R. Z. Khas'minskii, Ergodic properties of recurrent diffusion processes and stabilization of the solution to the Cauchy problem for parabolic equations, *Theory Probab. Appl.*, **5** (1960), 179–196.
- [47] A. Koranyi and J. C. Taylor, Minimal solutions of the heat equations and uniqueness of the positive Cauchy problem on homogeneous spaces, *Proc. Amer. Math. Soc.*, **94** (1985), 273–278.
- [48] P. Li and S. T. Yau, On the parabolic kernel of the Schrödinger operator, *Acta Math.*, **156** (1986), 153–201.
- [49] R. S. Martin, Minimal positive harmonic functions, *Trans. Amer. Math. Soc.*, **49** (1941), 137–172.
- [50] V. G. Maz'ya, On the relationship between Martin and Euclidean topologies, *Soviet Math. Dokl.*, **18** (1977), 283–286.
- [51] R. Mazzeo and A. Vasy, Resolvents and Martin boundaries of product spaces, *Geom. Funct. Anal.*, **12** (2002), 1018–1079.
- [52] J. Moser, A Harnack inequality for parabolic differential equations, *Comm. Pure Appl. Math.*, **17** (1964), 101–134.
- [53] M. Murata, Structure of positive solutions to  $(-\Delta + V)u = 0$  in  $\mathbf{R}^n$ , *Duke Math. J.*, **53** (1986), 869–943.
- [54] M. Murata, Martin compactification and asymptotics of Green functions for Schrödinger operators with anisotropic potentials, *Math. Ann.*, **288** (1990), 211–230.
- [55] M. Murata, On construction of Martin boundaries for second order elliptic equations, *Publ. Res. Inst. Math. Sci.*, **26** (1990), 585–627.
- [56] M. Murata, Positive harmonic functions on rotationary symmetric Riemannian manifolds, In: *Potential Theory*, (ed. M. Kishi), Walter de Gruyter, Berlin, 1992, 251–259.
- [57] M. Murata, Uniform restricted parabolic Harnack inequality, separation principle, and ultracontractivity for parabolic equations, In: *Functional Analysis and Related Topics, 1991, Lecture Notes in Math.*, Vol. 1540, Springer, Berlin, 1993, 277–288.
- [58] M. Murata, Sufficient condition for non-uniqueness of the positive Cauchy problem for parabolic equations, *Adv. Stud. Pure Math.*, **23** (1994), 275–282.
- [59] M. Murata, Non-uniqueness of the positive Cauchy problem for parabolic equations, *J. Differential Equations*, **123** (1995), 343–387.
- [60] M. Murata, Uniqueness and non-uniqueness of the positive Cauchy problem for the heat equation on Riemannian manifolds, *Proc. Amer. Math. Soc.*, **123** (1995), 1923–1932.
- [61] M. Murata, Non-uniqueness of the positive Dirichlet problem for parabolic equations in cylinders, *J. Funct. Anal.*, **135** (1996), 456–487.
- [62] M. Murata, Semismall perturbations in the Martin theory for elliptic equations, *Israel J. Math.*, **102** (1997), 29–60.
- [63] M. Murata, Structure of positive solutions to Schrödinger equations, *Sugaku Expositions*, **11** (1998), 101–121.
- [64] M. Murata, Martin boundaries of elliptic skew products, semismall perturbations, and fundamental solutions of parabolic equations, *J. Funct. Anal.*, **194** (2002), 53–141.
- [65] M. Murata, Heat escape, *Math. Ann.*, **327** (2003), 203–226.

- [66] M. Murata and T. Tsuchida, Asymptotics of Green functions and Martin boundaries for elliptic operators with periodic coefficients, *J. Differential Equations*, **195** (2003), 82–118.
- [67] M. Nakai and T. Tada, Monotoneity and homogeneity of Picard dimensions for signed radial densities, *Hokkaido Math. J.*, **26** (1997) 253–296.
- [68] M. A. Perel'muter and YU. A. Semenov, Elliptic operators preserving probability, *Theory Probab. App.*, **32** (1987), 718–721.
- [69] Y. Pinchover, On existence of any  $\lambda_0$ -invariant positive harmonic function, a counter example to Stroock's conjecture, *Comm. Partial Differential Equations*, **20** (1995), 1831–1846.
- [70] Y. Pinchover, On uniqueness and nonuniqueness of positive Cauchy problem for parabolic equations with unbounded coefficients, *Math. Z.*, **232** (1996), 569–586.
- [71] Y. Pinchover, Maximum and anti-maximum principles and eigenfunctions estimates via perturbation theory of positive solutions of elliptic equations, *Math. Ann.*, **314** (1999), 555–590.
- [72] R. G. Pinsky, A new approach to the Martin boundary via diffusions conditioned to hit a compact set, *Ann. Probab.*, **21** (1993), 453–481.
- [73] R. G. Pinsky, *Positive Harmonic Functions and Diffusion*, Cambridge Univ. Press, Cambridge, 1995.
- [74] C. Pommerenke, *Univalent functions*, Vandenhoeck & Ruprecht, Göttingen, 1975.
- [75] L. Saloff-Coste, Uniform elliptic operators on Riemmanian manifolds, *J. Differential Geom.*, **36** (1992), 417–450.
- [76] K.-Th. Sturm, Analysis on local Dirichlet spaces -I. Recurrence, conservativeness and  $L^p$ -Liouville properties, *J. Reine Angew. Math.*, **456** (1994), 173–196.
- [77] K.-Th. Sturm, Analysis on local Dirichlet spaces -III. Poincaré and parabolic Harnack inequality, *J. Math. Pures Appl.* **IX**, Ser. 75 (1996), 273–297.
- [78] S. Täcklind, Sur les classes quasianalytiques des solutions des équations aux dérivées partielles du type parabolique, *Nova Acta Regiae Soc. Scien. Upsaliensis*, Ser. IX, **10** (1936), 1–57.
- [79] J. C. Taylor, The Martin compactification associated with a second order strictly elliptic partial differential operator on a manifold M, In: *Topics in Probability and Lie Groups: Boundary Theory*, CRM Proc. Lecture Notes, Vol. 28, Amer. Math. Soc., Providence, RI, 2001, 153–202.
- [80] D. V. Widder, Positive temperatures on an infinite rod, *Trans. Amer. Math. Soc.*, **55** (1944), 85–95 .
- [81] J.-M. G. Wu, Comparisons of kernel functions, boundary Harnack principle and relative Fatou theorem on Lipschitz domains, *Ann. Inst. Fourier (Grenoble)*, **28** (1978), 147–167.

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