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On a variational problem for soap films with gravity and partially free boundary

By Miyuki KOISO and Bennett PALMER

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Abstract. We pose a variational problem for surfaces whose solutions are a geometric model for thin films with gravity which is partially supported by a given contour. The energy functional contains surface tension, a gravitational energy and a wetting energy, and the Euler-Lagrange equation can be expressed in terms of the mean curvature of the surface, the curvatures of the free boundary and a few other geometric quantities. Especially, we study in detail a simple case where the solutions are vertical planar surfaces bounded by two vertical lines. We determine the stability or instability of each solution.

1. Introduction.

We study a variational problem whose solutions are a geometric model for thin films subject to a vertical gravitational force.

We will consider immersed surfaces $\mathscr{F}$ in the three-dimensional Euclidean space which are partially supported by a given curve Γ. We represent $\mathscr{F}$ as an immersion $X = (x^1, x^2, x^3) : \Sigma \to \boldsymbol{R}^3$ of a two-dimensional orientable compact connected C^∞ manifold Σ into $\boldsymbol{R}^3$. Our problem is to investigate critical points of an energy functional which is the sum of three terms:

- The length of the free boundary of $\mathscr{F}$. This can be considered as a type of adhesion energy for the film to air interface. We assume that this energy is proportional to the arc length of the free boundary and normalize the constant of proportionality to be one.
- A "gravitational potential energy" for $\mathscr{F}$ which arises from a vertical gravitational force. We consider the acceleration due to gravity to be constant.
- A "wetting energy" for $\mathscr{F}$ which is proportional to the length of the part of the fixed boundary which comes into contact with the film.

We also impose the realistic assumption that the area of $\mathscr{F}$ is preserved under deformations. Thus, our energy functional can be expressed as,

$$\mathscr{E}(X) := L(X) + G_\gamma(X) + W_\beta(X) = \int_\rho |dX| + \gamma \int_\Sigma x^3 d\Sigma + \beta \int_\sigma |dX|,$$

where $d\Sigma$ is the volume element of Σ induced by X, $\partial\Sigma = \sigma \cup \rho$, $X(\sigma) \subset \Gamma$, ρ is the free boundary of X, γ and β are given constants which depend on the material of the film and the surrounding media. Our objective is to study the geometry and stability of equilibria for the

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functional $\mathscr{E}$. For additional explanation about the components of the energy functional, we refer the reader to [**3**].

Let us denote by $\nu = (\nu^1, \nu^2, \nu^3)$ the Gauss map of X. The Euler-Lagrange equation for the energy yields that the mean curvature H of X satisfies the equation $2H(\gamma x^3 + c) - \gamma \nu^3 = 0$ for some constant c, the normal curvature of the free boundary ρ vanishes, the geodesic curvature of ρ is the linear function $\gamma x^3 + c$ of the vertical coordinate x^3 and that the free boundary intersects Γ in complementary angles at its endpoints.

In this paper, we concentrate on the simplest case where Γ consists of two vertical rays and the horizontal segment connecting them. We assume from the outset that the surface is planar when it is in equilibrium and the free boundary component consists of a smooth embedded curve C whose two endpoints are constrained to lie on each of the vertical lines (see Figure 1 on page 339). Clearly the configuration of the surface is completely determined by the curve C. In this case, the Euler-Lagrange equation for the energy yields that the curvature of C is a linear function of the vertical coordinate in the interior of C and that C intersects the each of the vertical lines in complementary angles. The problem that we consider is to determine all stable configurations which can occur. The critical curves C are found explicitly in terms of elliptic functions and the stability of each type is analyzed. Conceptually, this simple problem is a one-dimensional version of the type of free boundary problem considered in [**6**], [**8**], [**7**] with the addition to the energy of the gravitational term.

The paper is organized as follows. In the second section, we will formulate the variational problem and derive first variation formulas for the general setting. In the remainder of the paper, we will restrict our attention to the simple case of a planar film mentioned above. In the third section, we derive the Euler-Lagrange equation which characterizes critica of the variational problem. In the fourth section, we derive the second variation of the energy functional and define the notion of stability. In the fifth section, by studying the ordinary differential equation comes from the Euler-Lagrange equation, we derive some geometric properties of the critical curves. We also show that the Euler-Lagrange equation for the critical curve is equivalent to the pendulum equation. The main results of the paper are in the sixth section where we determine the stability or instability of each critical point. At the end of the paper, we will give some pictures of examples of critical curves (Figure 2).

We wish to thank Professor Oscar Garay for pointing out to us that the critical curves which we study as the simplest case are exactly the elastic curves in the plane. We note however that the variational problem we study is very different from the one for elastica and that, in particular, the stability analysis differs greatly for the two problems.

2. Formulation of problem and first variation formulas.

Let Γ be a piecewise-smooth closed curve without self-intersections in $\boldsymbol{R}^3$. Also let Σ be a two-dimensional orientable compact connected C^∞ manifold with boundary $\partial\Sigma$ which is homeomorphic to S^1. We divide $\partial\Sigma$ into two connected parts as follows.

$$\partial\Sigma = \sigma \cup \rho,$$

where $\sigma \cap \rho$ consists of two points:

$$\sigma \cap \rho = \{\zeta_1, \zeta_2\}.$$

Consider a smooth immersion

$$X = (x^1, x^2, x^3) : \Sigma \to \boldsymbol{R}^3$$

whose restriction $X|_\sigma$ to σ is an injection into Γ. Denote by $\nu = (\nu^1, \nu^2, \nu^3) : \Sigma \to S^2$ the Gauss map of X. We assign to X the following four quantities:

$$A(X) := \int_\Sigma d\Sigma, \qquad L(X) := \int_\rho |dX|,$$
$$G_\gamma(X) := \gamma \int_\Sigma x^3 d\Sigma, \qquad W_\beta(X) := \beta \int_\sigma |dX|,$$

where $d\Sigma$ is the volume element of Σ induced by X, and γ and β are constants. Then $A(X)$ represents the area of X, $L(X)$ represents the length of the free boundary $C := X(\rho)$, $G_\gamma(X)$ and $W_\beta(X)$ represents the (gravitational) potential energy and the wetting energy of X respectively. (One can find physical examples corresponding to γ, β with any sign. For example, when we think of soap film $\mathscr{F}$ partially supported by such a special Γ as in §3., it is natural to assume that $\gamma \geq 0$ and $\beta \leq 0$ hold.)

Consider a smooth variation $X_\varepsilon : \Sigma \to \boldsymbol{R}^3$ of X satisfying the boundary condition

$$X_\varepsilon(\sigma) \subset \Gamma.$$

We will call such a variation X_ε an *admissible variation* of X.

For simplicity, we will write X instead of X_ε. We will often denote by ‘ $'$ ’ the partial derivative with respect to ε. Denote by n the exterior normal of X along $\partial\Sigma$. Let $\tilde{\rho} : [\alpha_1, \alpha_2] \to \rho$ be a parametrization of ρ such that $\tilde{\rho}(\alpha_i) = \zeta_i,\ i = 1, 2$, and $\dot{X} \times \nu = |\dot{X}| n$ hold, where

$$\dot{X} = \dot{X}|_\rho := \frac{\partial (X|_\rho \circ \tilde{\rho})}{\partial t},$$

and t is the parameter in $[\alpha_1, \alpha_2] \subset \boldsymbol{R}$. Similarly, let $\tilde{\sigma} : [\delta_1, \delta_2] \to \sigma$ be a parametrization of σ such that $\tilde{\sigma}(\delta_1) = \zeta_2,\ \tilde{\sigma}(\delta_2) = \zeta_1$, and $\dot{X} \times \nu = |\dot{X}| n$ hold, where

$$\dot{X} = \dot{X}|_\sigma := \frac{\partial (X|_\sigma \circ \tilde{\sigma})}{\partial t},$$

and t is the parameter in $[\delta_1, \delta_2] \subset \boldsymbol{R}$. In general, we will denote by ‘ $\dot{}$ ’ the partial derivative with respect to t. We set

$$\xi = \frac{\partial X}{\partial \varepsilon}, \quad f = \langle \xi, \nu \rangle,$$
$$\psi = \langle \xi, n \rangle \quad \text{on } \partial\Sigma,$$

where $\langle\ ,\ \rangle$ is the usual Euclidean inner product in $\boldsymbol{R}^3$.

We will denote by H the mean curvature of X. Also, we will denote by κ_n, κ_g the normal curvature and the geodesic curvature of $X|_{\partial\Sigma}$, respectively.

Notice that for any admissible variation,

$$\psi = 0 \text{ and } f = 0 \quad \text{on } \sigma \tag{1}$$

holds.

PROPOSITION 2.1 (First variation formula). *For each admissible variation, the following first variation formulas hold.*

$$A' = -2\int_{\Sigma} H f d\Sigma + \int_{\rho} \psi ds, \tag{2}$$

$$G'_{\gamma} = \gamma \int_{\Sigma} (-2Hx^3 + \nu^3) f d\Sigma + \gamma \int_{\rho} x^3 \psi ds, \tag{3}$$

$$L' = |\xi(\zeta_2)|\cos\omega_2 - |\xi(\zeta_1)|\cos\omega_1 - \int_{\rho} \big(\langle \ddot{X}, \nu\rangle f + \langle \ddot{X}, n\rangle \psi\big)|\dot{X}|^{-2} ds \tag{4}$$

$$= |\xi(\zeta_2)|\cos\omega_2 - |\xi(\zeta_1)|\cos\omega_1 - \int_{\rho} (\kappa_n f + \kappa_g \psi) ds \tag{5}$$

$$W'_{\beta} = \beta\big(|\xi(\zeta_1)|\cos\eta_2 - |\xi(\zeta_2)|\cos\eta_1\big),$$

where ω_i denotes the angle between $\dot{X}|_{\rho}(\alpha_i)$ and $\xi(\zeta_i)$, $i = 1,\ 2$, η_1 denotes the angle between $\dot{X}|_{\sigma}(\delta_1)$ and $\xi(\zeta_2)$, and η_2 denotes the angle between $\dot{X}|_{\sigma}(\delta_2)$ and $\xi(\zeta_1)$. Consequently,

$$\cos\eta_i \in \{1, -1\} \quad i = 1, 2,$$

holds since $\dot{X}|_{\sigma}(\delta_1)$ (resp. $\dot{X}|_{\sigma}(\delta_2)$) is proportional to $\xi(\zeta_2)$ (resp. $\xi(\zeta_1)$).

The proof of Proposition 2.1 will be given after Proposition 2.2.

By virtue of Proposition 2.1, we immediately observe the following:

PROPOSITION 2.2 (Euler-Lagrange equation). *Denote by θ_1, θ_2 the angles from $\dot{X}|_{\sigma}(\delta_2)$ to $\dot{X}|_{\rho}(\alpha_1)$, from $\dot{X}|_{\rho}(\alpha_2)$ to $\dot{X}|_{\sigma}(\delta_1)$, respectively. Then, $(L + G_{\gamma} + W_{\beta})'(0) = 0$ for all area-preserving admissible variations if and only if*

$$2H(\gamma x^3 + c) - \gamma\nu^3 = 0 \quad \text{on } \Sigma,$$

$$\kappa_n = 0 \quad \text{and} \quad \kappa_g = \gamma x^3 + c \quad \text{on } \rho,$$

and

$$\cos\theta_1 = \cos\theta_2 = \beta$$

for some constant $c \in \boldsymbol{R}$.

PROOF OF PROPOSITION 2.1. (2) is derived directly from the well-known first variation formula for the area functional.

Let us prove (3). Let(u^1, u^2) be local coordinates in Σ. Set

$$X_i = \frac{\partial X}{\partial u^i}, \quad etc.,$$

$$\xi = \xi^T + f\nu = V^1 X_1 + V^2 X_2 + f\nu.$$

Denote by $\langle\ ,\ \rangle_g$ the Riemannian inner product in the metric induced by X.

$$G_\gamma{}' = \gamma \int_\Sigma (x^3)' d\Sigma + \gamma \int_\Sigma x^3 (d\Sigma)', \tag{6}$$

$$\int_\Sigma x^3 (d\Sigma)' = \int_\Sigma x^3 (-2Hf + \mathrm{div}\xi^T) d\Sigma. \tag{7}$$

By using the divergence theorem, we see that

$$\begin{aligned}
\int_\Sigma x^3 \mathrm{div}\xi^T d\Sigma &= \int_\Sigma \mathrm{div}(x^3 \xi^T) d\Sigma - \int_\Sigma \langle \nabla x^3, \xi^T \rangle_g d\Sigma \\
&= \int_{\partial\Sigma} \langle x^3 \xi^T, n \rangle ds - \int_\Sigma \langle \nabla x^3, \xi^T \rangle_g d\Sigma \\
&= \int_{\partial\Sigma} \langle x^3 \xi, n \rangle ds - \int_\Sigma \langle \nabla x^3, \xi^T \rangle_g d\Sigma \\
&= \int_\rho x^3 \langle \xi, n \rangle ds - \int_\Sigma \langle \nabla x^3, \xi^T \rangle_g d\Sigma,
\end{aligned} \tag{8}$$

where we have used $\langle \xi, n \rangle|_\sigma = 0$. On the other hand, since

$$\xi^T = V^1 X_1 + V^2 X_2 = \xi - \langle \xi, \nu \rangle \nu,$$

we obtain

$$\langle \nabla x^3, \xi^T \rangle_g = V^1 (x^3)_1 + V^2 (x^3)_2 = \xi^3 - \langle \xi, \nu \rangle \nu^3.$$

Therefore,

$$\int_\Sigma \langle \nabla x^3, \xi^T \rangle_g d\Sigma = \int_\Sigma \big(\xi^3 - \langle \xi, \nu \rangle \nu^3 \big) d\Sigma. \tag{9}$$

From (6), (7), (8), and (9), we get (3).

Next, we will derive the formula for L'. We see that

$$\begin{aligned}
L(X) &= \int_\rho ds = \int_{\alpha_1}^{\alpha_2} |\dot{X}| dt, \\
L' &= \int_{\alpha_1}^{\alpha_2} \frac{\partial}{\partial \varepsilon} |\dot{X}| dt = \int_{\alpha_1}^{\alpha_2} |\dot{X}|^{-1} \left\langle \frac{\partial X}{\partial t}, \frac{\partial^2 X}{\partial t \partial \varepsilon} \right\rangle dt \\
&= \int_{\alpha_1}^{\alpha_2} \langle |\dot{X}|^{-1} \dot{X}, \dot{\xi} \rangle dt \\
&= [\langle |\dot{X}|^{-1} \dot{X}, \xi \rangle]_{\alpha_1}^{\alpha_2} - \int_{\alpha_1}^{\alpha_2} \left\langle \frac{\partial}{\partial t} (|\dot{X}|^{-1} \dot{X}), \xi \right\rangle dt \\
&= |\xi(\zeta_2)| \cos\omega_2 - |\xi(\zeta_1)| \cos\omega_1 - \int_{\alpha_1}^{\alpha_2} \left\langle \frac{\partial}{\partial t} (|\dot{X}|^{-1} \dot{X}), \xi \right\rangle dt \\
&= |\xi(\zeta_2)| \cos\omega_2 - |\xi(\zeta_1)| \cos\omega_1 - \int_\rho \left\langle -|\dot{X}|^{-2} \langle \dot{X}, \ddot{X} \rangle \dot{X} + \ddot{X},\ \xi \right\rangle |\dot{X}|^{-2} ds \\
&= |\xi(\zeta_2)| \cos\omega_2 - |\xi(\zeta_1)| \cos\omega_1 - \int_\rho \langle X_{ss}, \xi \rangle ds,
\end{aligned}$$

which implies (4) and (5).

The derivation of the formula for W_β' can be handled similarly and will be omitted. □

3. Euler-Lagrange equation for the simplest case.

In the following, we will consider the special case depicted in Figure 1. We will write (x,y,z) instead of (x^1,x^2,x^3). Let a,x_0,x_1 $(x_0<x_1)$ be constants. Set

$$l_1 = \{(x_0,0,z)|z \le a\},$$

$$l_2 = \{(x,0,a)|x_0 \le x \le x_1\},$$

$$l_3 = \{(x_1,0,z)|z \le a\},$$

$$\Gamma = l_1 \cup l_2 \cup l_3.$$

We assume that $X(\Sigma)$ is contained in the xz plane. Moreover, we will assume that the endpoints of the curve $C(s)=(x(s),z(s))$ are constrained to lie on distinct vertical rays l_1, l_3. For the boundary curve $C(s)=(x(s),z(s))$, $s=$ arc length, we let: $L=L[C]=$ length of C, $x(0)=x_0$, and $x(L)=x_1$. Note that C has no self-intersections and $x_0<x(s)<x_1$ holds for all $s\in(0,L)$. Set $E_3=(0,1)$. Since $\operatorname{div}(zE_3)=1$ and $\operatorname{div}((z^2/2)E_3)=z$, we can write the area, gravitational and wetting energies as

$$A[C]:=A(X)=-\int_C z\,dx+c_1, \quad c_1:=(x_1-x_0)a,$$
$$G_\gamma[C]:=G_\gamma(X)=(-\gamma/2)\int_C z^2dx+c_2, \quad c_2:=(\gamma/2)(x_1-x_0)a^2,$$
$$W_\beta[C]:=W_\beta(X)=\beta[(a-z(0))+(a-z(L))+(x_1-x_0)].$$

For a function f on $[0,L]$, set $[[f]]=f(L)-f(0)$.

We consider a variation $C_\varepsilon:[0,L]\to\boldsymbol{R}^2$ of C with variation vector field $\delta C(s)=((\delta x)(s),(\delta z)(s)):=(\partial C_\varepsilon(s)/\partial\varepsilon)|_{\varepsilon=0}$, where ε is the variation parameter and $C_0=C$. Because the endpoints stay on the vertical lines, we have:

$$(\delta x)(0)=0=(\delta x)(L). \tag{10}$$

In this case, these variation comprise the admissible ones.

Denote by n the unit normal to C which points out of the surface $X(\Sigma)$, and by κ the curvature of C with respect to n. We remark that this is not the usual sign convention for the curvature.

For any admissible variation C_ε of C, we get the following first variation formulas.

$$\delta L[C]:=\left.\frac{\partial L[C_\varepsilon]}{\partial\varepsilon}\right|_{\varepsilon=0}=-\int_C \kappa\langle\delta C,n\rangle\,ds+[[(\delta z)z']],$$
$$\delta A[C]=\int_C\langle\delta C,n\rangle\,ds,$$
$$\delta G_\gamma[C]=\gamma\int_C z\langle\delta C,n\rangle\,ds,$$
$$\delta W_\beta[C]=-\beta((\delta z)(L)+(\delta z)(0)).$$

We are using "prime" here to denote differentiation with respect to s which is a departure from its usage in section 2. These formulas are obtained from Proposition 2.1 by letting $f = 0$. Consequently, we get

PROPOSITION 3.1 (Euler-Lagrange equation). *Assume that* $|\beta| \leq 1$. *Then,*

$$\delta(L + G_\gamma + W_\beta) = 0$$

holds for all area-preserving admissible variations if and only if

$$\kappa = \gamma z + \kappa_0 \tag{11}$$

holds for some constant $\kappa_0 \in \boldsymbol{R}$ *and*

$$-z'(0) = \cos\theta_1 = \beta = \cos\theta_2 = z'(L) \tag{12}$$

holds, where θ_1 *is the angle from* $-E_3$ *to* $C'(0)$, θ_2 *is the angle from* $C'(L)$ *to* E_3.

The Euler-Lagrange equation implies that

$$|\beta| \leq 1$$

necessarily holds for all critical points, so from now on we will only consider values of β in this range.

From (12), it follows that

$$|\theta_1| = |\theta_2| \qquad (\mathrm{mod}\ 2\pi).$$

Since C has no self-intersections and $x_0 < x(s) < x_1$ holds for all $s \in (0, L)$, $x'(0) \geq 0$, $x'(L) \geq 0$ holds. Therefore, we may set

$$\theta := \theta_1 = \theta_2 \in [0, \pi],$$

and we have

$$x'(0) = x'(L) = \sin\theta \geq 0. \tag{13}$$

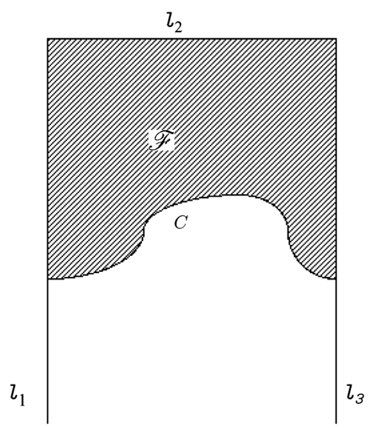

Figure 1.

4. Second variation and definition of stability.

From now on, we assume that the curve C satisfies the Euler-Lagrange equations (11) and (12). We will derive the second variation

$$\delta^2(L+G_\gamma+W_\beta) := \frac{\partial^2}{\partial\varepsilon^2}\bigg|_{\varepsilon=0}\big(L[C_\varepsilon]+G_\gamma[C_\varepsilon]+W_\beta[C_\varepsilon]\big)$$

for any area-preserving admissible variation

$$C_\varepsilon(s) := C(\varepsilon,s) = C(s)+\varepsilon\big(p(s)n(s)+q(s)C'(s)\big)+\mathscr{O}(\varepsilon^2).$$

For simplicity, we suppress the subscript ε although all quantities below are assumed to depend on it.

At $\varepsilon=0$, by using (11) and (12), we see that

$$\begin{aligned}
&\delta^2(L+G_\gamma+W_\beta) = \delta^2(L+G_\gamma+W_\beta+\kappa_0 A)\\
&= \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0}\left[\int_C(-\kappa+\gamma z+\kappa_0)\langle\delta C,n\rangle ds+[[\langle\delta C,C'\rangle]]-\beta\big((\delta z)(L)+(\delta z)(0)\big)\right]\\
&= \int_C\left(\frac{\partial}{\partial\varepsilon}\bigg|_{\varepsilon=0}(-\kappa+\gamma z+\kappa_0)\right)\cdot\langle\delta C,n\rangle ds+\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0}\left[[[\langle\delta C,C'\rangle]]-\beta[[\langle\delta C,\bar{C}'\rangle]]\right]\\
&=: I+II,
\end{aligned}$$

where

$$\bar{C}'(L) := E_3 = (0,1), \quad \bar{C}'(0) := -E_3 = (0,-1).$$

First we observe

LEMMA 4.1. *At $\varepsilon=0$,*

$$I = \int_C\big\{-p''-(\kappa^2+\gamma x')p\big\}p\,ds$$

holds.

PROOF. Elementary calculations give at $\varepsilon=0$

$$\delta\kappa = \kappa^2 p+p''+\gamma q z', \quad \delta z = -px'+qz', \quad \langle\delta C,n\rangle = p. \qquad \square$$

On the other hand, we see

LEMMA 4.2. *At $\varepsilon=0$,*

$$II = [[p(p'+\kappa q)]]$$

holds. If $|\beta|<1$, then

$$II = \left[\left[p\left(p'-\frac{\kappa\beta}{\sqrt{1-\beta^2}}\vec{z}'p\right)\right]\right] = [[p(p'-\kappa\cot\theta\cdot\vec{z}'p)]],$$

where

$$\vec{z}'(0) := -1, \quad \vec{z}'(L) := 1.$$

PROOF. At $\varepsilon = 0$, by using (12), we see that

$$\begin{aligned} II &= \delta[[\langle \delta C, C' - \beta \overline{C'} \rangle]] = [[\big\langle \delta C, \delta C' - C' \langle C', \delta C' \rangle \big\rangle]] \\ &= [[\big\langle \delta C, \langle \delta C', n \rangle n \big\rangle]] = [[\big\langle \delta C, (p' + \kappa q) n \big\rangle]] \\ &= [[p(p' + \kappa q)]]. \end{aligned}$$

Assume that $|\beta| < 1$. Since

$$0 = \delta x = pz' + qx',$$

we observe that

$$q = -pz'/x' = -p \cot\theta \cdot \vec{z}'. \qquad \square$$

From Lemmas 4.1, 4.2, we get the following

PROPOSITION 4.1 (Second variation formula). *Assume that the curve C satisfies the Euler-Lagrange equations* (11), (12). *Set $\theta := \theta_1 = \theta_2$. The second variation $\delta^2(L + G_\gamma + W_\beta)$ for any area-preserving admissible variation of C depends only on the normal component pn of the variation vector field, and it is given by the following formulas.*

(i) *If $|\beta| < 1$, then*

$$\delta^2(L + G_\gamma + W_\beta) = \int_C \big\{ -p'' - (\kappa^2 + \gamma x')p \big\} p\, ds + [[(p' - \kappa \cot\theta \cdot \vec{z}' p)p]]$$

holds, where

$$\vec{z}'(0) := -1, \quad \vec{z}'(L) := 1.$$

(ii) *If $|\beta| = 1$, then*

$$\delta^2(L + G_\gamma + W_\beta) = \int_C \big\{ -p'' - (\kappa^2 + \gamma x')p \big\} p\, ds$$

holds.

The following lemma is proved by a modification of the proof of the existence of volume-preserving variations fixing the boundary given by Barbosa-do Carmo [**1**].

LEMMA 4.3. *Suppose that a curve $C : [0, L] \to \boldsymbol{R}^2$ satisfies the Euler-Lagrange equations* (11), (12).

(i) *Assume that $|\beta| < 1$. Let $p : [0, L] \to \boldsymbol{R}$ be a C^∞ function with $\int_0^L p\, ds = 0$. Then, there exists an area-preserving admissible variation C_ε of C such that $\langle \delta C, n \rangle = p$.*

(ii) *Assume that $|\beta| = 1$. Let $p : [0, L] \to \boldsymbol{R}$ be a C^∞ function with $\int_0^L p\, ds = 0$ and $p|_{\partial[0,L]} = 0$. Then, there exists an area-preserving admissible variation C_ε of C such that $\langle \delta C, n \rangle = p$.*

In view of the above lemma, we define function spaces $F_0,\ F$ for a curve C satisfying (11), (12) as follows.

$$F_0 := \begin{cases} C^\infty([0,L]), & |\beta| < 1, \\ \{p \in C^\infty([0,L]) \,|\, p(0) = p(L) = 0\}, & |\beta| = 1, \end{cases}$$

$$F := \left\{ p \in F_0 \,\middle|\, \int_0^L p\,ds = 0 \right\}.$$

Set

$$\mathscr{I}(p) := \begin{cases} \int_C \{-p'' - (\kappa^2 + \gamma x')p\} p\,ds + [[(p' - \kappa \cot\theta \cdot \overline{z}' p)p]], & |\beta| < 1, \\ \int_C \{-p'' - (\kappa^2 + \gamma x')p\} p\,ds, & |\beta| = 1. \end{cases}$$

We define the stability as follows:

DEFINITION (Stability). Suppose that the boundary $C : [0,L] \to \boldsymbol{R}^2$ of a film $\mathscr{F}$ satisfies the Euler-Lagrange equations (11), (12). Then, $\mathscr{F}$ will be called *stable* if $\mathscr{I}(p) \geq 0$ holds for all $p \in F$, and $\mathscr{F}$ will be called unstable otherwise. $\mathscr{F}$ is said to be strongly stable if $\mathscr{I}(p) \geq 0$ holds for all $p \in F_0$.

5. Some geometry of critical curves.

In this section, we study geometry of critical curves and their global extensions.

Integrating $x'' = \kappa z'$ over C and using (11) and (13), gives

$$0 = [[x']] = \frac{\gamma}{2}[[z^2]] + \kappa_0[[z]].$$

Therefore, we have the following:

LEMMA 5.1. *In the case of a critical curve, at least one of the following holds*:

$$z(L) = z(0) \tag{14}$$

and/or

$$\gamma(z(L) + z(0)) = -2\kappa_0. \tag{15}$$

Moreover, (14) *is equivalent to*

$$\kappa(L) = \kappa(0),$$

and (15) *is equivalent to*

$$\kappa(L) + \kappa(0) = 0.$$

Next, we will study the shapes of the global extensions of critical curves. The equation (11) can be integrated explicitly. The explicit representation of the critical curves appears to be well-known and classical. We will include the computation for the reader's convenience. It will turn out that in almost all cases, the solutions are periodic curves which, except for circles and a particular other one-parameter family of curves, are invariant under certain parallel translations in the x direction.

Let $C=(x,z): I\to \boldsymbol{R}^2,\ I\subset \boldsymbol{R}$, be a solution of (11) which is parametrized by the arc-length. Here we will take I to be the largest possible interval.

When $\gamma=0$, the curve is a horizontal straight line or a circle. So, we will assume that $\gamma\neq 0$. The equation $C''=\kappa n$ gives

$$x''=\kappa z', \qquad z''=-\kappa x'. \tag{16}$$

By setting $\cos\sigma := x'$, $\sin\sigma = z'$, we have a solution to the system:

$$z'=\sin\sigma, \quad \sigma' = -\kappa = -(\gamma z+\kappa_0).$$

These equations can be combined to show that σ satisfies the *pendulum equation*

$$\sigma'' = -\gamma \sin\sigma. \tag{17}$$

Conversely, it is easy to show that a solution of this equation generates a solution of (16) with $\kappa=\gamma z+\text{constant}$.

Define

$$\rho := |\gamma|, \qquad \omega := \begin{cases} \sigma, & \gamma>0,\\ \sigma+\pi, & \gamma<0.\end{cases}$$

Then in either case, we obtain a solution of the pendulum equation

$$\omega'' = -\rho\sin\omega \tag{18}$$

with $\rho>0$.

Each solution of (18) admits a first integral

$$(\omega')^2 - 2\rho\cos\omega \equiv \text{constant} =: a. \tag{19}$$

The geometric meaning of a is given by

$$a = (1/L)\left(\int_0^L \kappa^2\,ds - 2\gamma(x(L)-x(0))\right) \tag{20}$$

for all $L>0$.

We may rewrite (19) as

$$(\omega')^2 = 4\rho\left\{\left(\frac{a+2\rho}{4\rho}\right) - \sin^2\left(\frac{\omega}{2}\right)\right\}.$$

In the case where $\omega\equiv 0$, the solution curve is a horizontal straight line. Excluding this case, we have

$$0<\frac{a+2\rho}{4\rho} =: \chi^2, \quad \chi>0.$$

By replacing the parameter s by $s+\text{constant}$ if necessary, the solutions of (19) are expressed as follows (cf. Lawden [**5**, Chapter 5]).

LEMMA 5.2. *Case*(i) $\chi^2 < 1$. *The solution can be expressed in terms of elliptic functions*:

$$\sin(\omega/2) = \chi \mathrm{sn}(\sqrt{\rho}s, \chi), \qquad \cos(\omega/2) = \mathrm{dn}(\sqrt{\rho}s, \chi).$$

Case(ii) $\chi^2 > 1$. *The solution can be expressed in terms of elliptic functions*:

$$\sin(\omega/2) = \mathrm{sn}(\sqrt{\rho}s/\chi, \chi), \qquad \cos(\omega/2) = \mathrm{cn}(\sqrt{\rho}s/\chi, \chi).$$

Note that, $a > 2\rho$ holds, so by (19) *ω' never vanishes and so the curve C has nowhere vanishing curvature and is therefore convex.*
Case(iii) $\chi^2 = 1$. *The solutions are expressed as*

$$\sin(\omega/2) = \tanh(\sqrt{\rho}s), \qquad \cos(\omega/2) = \mathrm{sech}(\sqrt{\rho}s).$$

REMARK 5.1. Since $\rho = |\gamma|$, using (20), we have

$$(a-2\rho)L = \begin{cases} \int_0^L \kappa^2\, ds - 2\gamma(x(L) - x(0) + L), & \gamma > 0, \\ \int_0^L \kappa^2\, ds - 2\gamma(x(L) - x(0) - L), & \gamma < 0. \end{cases}$$

Therefore, if $\gamma < 0$, then $(a-2\rho)L > 0$ holds. Hence, in this case $\chi^2 > 1$ holds, which corresponds to case(ii) above. On the other hand, for $\gamma > 0$, all cases(i)–(iii) can occur.

LEMMA 5.3. *If C is a solution curve, then*
(i) $I = \boldsymbol{R}$.
(ii) *The function z is bounded.*
(iii) *Except in case*(iii) *of the previous lemma, the curve C is periodic. More precisely, there exists some $d_0 > 0$, such that*

$$x(s+d_0) = x(s) + a_0, \quad z(s+d_0) = z(s), \quad \forall s \in \boldsymbol{R}$$

holds, where

$$a_0 := x(d_0) - x(0).$$

(iv) $z'(s_1) = 0$ *for some* s_1.
(v) *If $z'(s_1) = 0$ holds for some s_1, then C is symmetric with respect to* $\{x = x(s_1)\}$.
(vi) *If $\kappa(s_2) = 0$ holds for some s_2, then C is symmetric with respect to* $C(s_2)$.

PROOF. The statement (i) follows since the explicit values of $\cos\omega = \cos^2(\omega/2) - \sin^2(\omega/2)$ and $\sin\omega = 2\cos(\omega/2)\sin(\omega/2)$ given above are defined for all values of s.

The boundedness of z follows since (19) can be written as $(\gamma z + \kappa_0)^2 = 2|\gamma|\cos\omega + a$.

In cases(i) and (ii) of Lemma 5.2, the periodicity of $x' = \cos\sigma$ and $z' = \sin\sigma$ follow from the periodicity of the elliptic functions sn, cn and dn. In fact, the period for these functions is given by $4K$, where $\mathrm{sn}(K) = 1$. Since $\gamma z = x''/z'$ holds, the periodicity of z follows from that of x' and z'. The periodicity of x as described above follows from the periodicity of x'.

In cases(i) and (ii) of the previous lemma, the statement (iv) follows immediately from the periodicity. In case(iii), it follows since

$$|z'(0)| = |\sin(\sigma(0))| = 2|\sin(\sigma(0)/2)\cos(\sigma(0)/2)| = 2|\tanh(0)\mathrm{sech}(0)| = 0.$$

Next we prove (v). By a translation of the parameter s and coordinates, we may assume that

$$s_1 = 0, \quad x(0) = 0, \quad \kappa = \gamma z$$

hold. Set

$$\tilde{C} := (\tilde{x}, \tilde{z}), \quad \tilde{x}(s) := -x(-s), \quad \tilde{z}(s) := z(-s).$$

Then, in view of (16), we get

$$\tilde{x}'' = (\gamma \tilde{z})\tilde{z}', \quad \tilde{z}'' = -(\gamma \tilde{z})\tilde{x}'.$$

Moreover, we see

$$\tilde{C}(0) = (0, z(0)) = C(0),$$

$$\tilde{C}'(0) = (1,0) = C'(0),\ \tilde{C}''(0) = (-x''(0), z''(0)) = (0, z''(0)) = C''(0).$$

Therefore, C and $\tilde{C}$ satisfy the same ordinary differential equations and have the same initial values, which implies that $C \equiv \tilde{C}$ and we have proved (v).

We prove (vi). By a translation of the parameter s and coordinates, we may assume that

$$s_2 = 0, \quad C(0) = (0,0), \quad \kappa = \gamma z$$

hold. Set

$$\tilde{C} := (\tilde{x}, \tilde{z}), \quad \tilde{x}(s) := -x(-s), \quad \tilde{z}(s) := -z(-s).$$

By a similar argument to the above, we see that $C \equiv \tilde{C}$ holds and we get (vi). $\square$

6. Stability of critical points.

Assume that the curve C satisfies the Euler-Lagrange equations (11), (12), and set $\theta := \theta_1 = \theta_2$. We define the Jacobi and boundary operators by

$$J[p] = p'' + (\kappa^2 + \gamma x')p,$$
$$B[p] = \begin{cases} p' - \kappa \cot(\theta)\overline{z}' p, & |\beta| < 1, \\ p, & |\beta| = 1. \end{cases}$$

Then, by Proposition 4.1, the second variation $\delta^2(L + G_\gamma + W_\beta)$ for any area-preserving admissible variation of C can be expressed as

$$\mathscr{I}(p) = -\int_C pJ[p]\, ds + [[p \cdot B[p]]], \quad p := \langle \delta C, n \rangle.$$

Here we remark that if $|\beta| = 1$, then, from the boundary condition (10), $p(0) = p(L) = 0$ holds.

Recall that $C'' = \kappa n$ gives

$$x'' = \kappa z', \qquad z'' = -\kappa x'.$$

From this and the boundary values (12), (13) for x' and z', we obtain

$$
\begin{aligned}
&J[x'] = \gamma, \qquad B[x']|_{\partial C} = 0, \\
&J[z'] = 0, \qquad B[z']|_{\partial C} = \begin{cases} -\kappa/\sqrt{1-\beta^2}, & |\beta| < 1, \\ \pm 1, & |\beta| = 1. \end{cases}
\end{aligned} \tag{21}
$$

Since J is a Sturm-Liouville operator, the eigenvalue problem

$$
J[p] = -\lambda p \quad \text{in } [0,L], \qquad B[p] = 0 \quad \text{on } \partial[0,L] \tag{22}
$$

has a discrete spectrum, all eigenvalues are real, and the multiplicity of each eigenvalue is one. We denote these eigenvalues by $\lambda_1 < \lambda_2 < \lambda_3 < \cdots$. We remark that each eigenfunction belonging to λ_1 has a definite sign. Denote by E_i the eigenspace belonging to λ_i, and by $E_i^\perp$ the orthogonal complement of E_i in $L^2([0,L])$. We can choose eigenfunctions $\varphi_i \in E_i$ so that $\{\varphi_i\}_i$ is an orthonormal basis in $L^2([0,L])$. By a standard method, we see that

$$
\begin{aligned}
\lambda_1 &= \mathscr{I}(\varphi_1) = \min\left\{ \mathscr{I}(p) \,|\, p \in C^\infty([0,L]), \int_C p^2\, ds = 1 \right\}, \\
\lambda_i &= \mathscr{I}(\varphi_i) = \min\left\{ \mathscr{I}(p) \,|\, p \in C^\infty([0,L]), \int_C p^2\, ds = 1, \int_C p\varphi_j\, ds = 0 \ (j = 1, \cdots, i-1) \right\}
\end{aligned}
$$

(cf. [**2**]). From these properties, the following lemma is proved by a modification of the proof of Theorem 1.3 in [**4**].

LEMMA 6.1. (I) *If $\lambda_1 \geq 0$, then $\mathscr{F}$ is strongly stable.*
(II) *If $\lambda_1 < 0 < \lambda_2$, then there exists a uniquely determined function $u \in C^\infty([0,L])$ satisfying $J[u] = 1$ and $B[u]|_{\partial C} = 0$, and the following* (II-1) *and* (II-2) *hold.*
(II-1) *If $\int_C u\, ds \geq 0$, then $\mathscr{F}$ is stable.*
(II-2) *If $\int_C u\, ds < 0$, then $\mathscr{F}$ is unstable.*
(III) *If $\lambda_2 = 0$, then the following* (III-A) *and* (III-B) *hold*:
(III-A) *If $\int_C \varphi_2\, ds \neq 0$, then $\mathscr{F}$ is unstable.*
(III-B) *If $\int_C \varphi_2\, ds = 0$, then there exists a uniquely determined function $u \in E_2^\perp \cap C^\infty([0,L])$ satisfying $J[u] = 1$ and $B[u]|_{\partial C} = 0$, and the following* (III-B1) *and* (III-B2) *hold.*
(III-B1) *If $\int_C u\, ds \geq 0$, then $\mathscr{F}$ is stable.*
(III-B2) *If $\int_C u\, ds < 0$, then $\mathscr{F}$ is unstable.*
(IV) *If $\lambda_2 < 0$, then $\mathscr{F}$ is unstable.*

THEOREM 6.1. *If $\gamma = 0$, then C is a part of a round circle or a horizontal straight line segment, $\lambda_1 = 0$, and so the film $\mathscr{F}$ is strongly stable.*

PROOF. If $\gamma = 0$, then it is clear that C is a part of a round circle or a horizontal straight line. Hence, from assumption (13), C is a graph and $x' > 0$ in the interior of C. Therefore, from (21), x' is an eigenfunction belonging to the first eigenvalue zero. □

PROPOSITION 6.1. *If $\lambda_1 < 0 \leq \lambda_2$ holds and $\mathscr{F}$ is stable, then $\gamma > 0$ holds. Conversely, if $\lambda_1 < 0 < \lambda_2$ and $\gamma > 0$ hold, then $\mathscr{F}$ is stable.*

PROOF. If $\gamma = 0$, then, from Theorem 6.1, it follows that $\lambda_1 = 0$. Therefore, if $\lambda_1 < 0$, then $\gamma \neq 0$. Set $u := x'/\gamma$. Then, by (21), we have $J[u] = 1$, $B[u]|_{\partial C} = 0$. Moreover, we see that

$$\int_C u\,ds = \gamma^{-1}\int_C x'\,ds = \gamma^{-1}(x(L)-x(0)) = \gamma^{-1}(x_1-x_0).$$

By these observations and Lemma 6.1, we get the desired result. □

THEOREM 6.2. *Let $\mathscr{F}$ be a film bounded by a horizontal line segment C.*
(i) *If $\gamma \le 0$, then $\mathscr{F}$ is strongly stable.*
(ii) *If $\gamma > 0$, then*

$$\mathscr{F} \text{ is stable.} \iff \lambda_2 \ge 0. \iff L \le \pi/\sqrt{\gamma}.$$

PROOF. We have already considered the case where $\gamma = 0$ in Theorem 6.1. So we assume that $\gamma \ne 0$. We have

$$0 \equiv \kappa = \gamma z + \kappa_0,$$

so that

$$z \equiv -\kappa_0/\gamma.$$

The second variation of the energy for this film is given by

$$\mathscr{I}(p) = \int_0^L (p'(x))^2 - \gamma(p(x))^2\,dx,$$

so that $\mathscr{F}$ is strongly stable for $\gamma < 0$.

Next, assume that $\gamma > 0$ holds. The n'th eigenvalue λ_n of the problem

$$J[p] = -\lambda p \quad \text{in } [0,L], \qquad B[p] = p' = 0 \quad \text{on } \partial[0,L]$$

is given by $\{(n-1)\pi/L\}^2 - \gamma$, and the corresponding eigenfunctions are $c\cos\big((n-1)\pi s/L\big)$. Note that $J[1] = \gamma$, $B[1]|_{\partial[0,L]} = 0$. Now we see that, by Lemma 6.1, the film $\mathscr{F}$ is stable if and only if $\lambda_2 \ge 0$ holds, and we have

$$\lambda_2 \ge 0. \iff L \le \pi/\sqrt{\gamma}.$$ □

THEOREM 6.3. *Assume that the free boundary C of a film $\mathscr{F}$ is not a straight line segment, and $\gamma < 0$. Then*
(i) *κ has a definite sign on C.*
(ii) *C is a graph over an interval of the x-axis.*
(iii) *$\mathscr{F}$ is strongly stable.*

PROOF. First we prove (i) and (ii). As in Section 5, we set $\cos\sigma := x'$, $\sin\sigma = z'$. From (12), (13), and the embeddedness of C, we may assume that

$$-\pi/2 \le \sigma(0) \le \pi/2, \quad -\pi/2 \le \sigma(L) \le \pi/2, \quad \sigma(L) = -\sigma(0). \tag{23}$$

Moreover, from (12), there exists some $s_2 \in (0,L)$ such that z takes either its maximum or its minimum value at $s = s_2$. First, assume that $z(s_2)$ is the maximum of z. Then, $n(s_2) = (0,-1)$ and $\kappa(s_2) > 0$ hold. Therefore, if $\gamma < 0$, then

$$\kappa = \gamma z + \kappa_0 > 0$$

on $[0,L]$. Since $\sigma' = -\kappa$ holds, σ is strictly decreasing on $[0,L]$. Hence, from (23), it follows that

$$-\pi/2 \le \sigma(L) < \sigma(s) < \sigma(0) \le \pi/2, \quad \forall s \in (0,L).$$

Consequently, we have

$$x'(s) = \cos\sigma(s) > 0, \quad \forall s \in (0,L),$$

which implies that C is a graph over an interval of the x-axis. Also in the case where $z(s_2)$ is the minimum of z, by a similar argument, we can show (i) and (ii).

Next, we prove (iii). Suppose $|\beta| < 1$. If C is given by $z = z(x)$, then $ds = \sqrt{1+(dz/dx)^2}\,dx$ and so

$$x'(s) = \frac{1}{\sqrt{1+(dz/dx)^2}} > 0$$

holds.

Note that for sufficiently smooth functions ζ, q on C, we have

$$\begin{aligned}
-\int_C \zeta q J[\zeta q]\,ds + [[\zeta q B[\zeta q]\,]] &= -\int_C \zeta^2 q J[q] + q^2\zeta\zeta'' + 2\zeta q\zeta' q'\,ds \\
&\quad + [[\zeta q(\zeta B[q] + q\zeta')]] \\
&= -\int_C \zeta^2 q J[q]\,ds + \int_C q^2(\zeta')^2\,ds + [[\zeta^2 q B[q]\,]].
\end{aligned}$$

Note that since $x' > 0$ holds, we can write an arbitrary smooth function on C as $\zeta x'$. Applying the previous formula with $q = x'$, gives

$$\mathscr{I}(\zeta x') = -\int_C \zeta^2(\gamma x')\,ds + \int_C (x')^2(\zeta')^2\,ds \ge 0,$$

which implies that $\mathscr{F}$ is strongly stable. For the case $|\beta| = 1$, a similar proof with a little modification derives the desired result. $\square$

We have determined all cases for $\gamma \le 0$. So, from now on, we will assume that $\gamma > 0$ holds.

Let C be the boundary curve of a film $\mathscr{F}$. For the case where C is a straight line segment, we have already observed its stability (Theorem 6.2). So, we will assume that C is not a straight line segment.

LEMMA 6.2. *Assume that C is not a straight line segment. Then, the situation is divided into the following five cases.*

Case(I) *C has no inflection point.*

Case(II) *C has only one inflection point.*

Case(III) *Both endpoints of C are inflection points, and C has no inflection point in its interior.*

Case(IV) *C has exactly two inflection points and only one zero of z' in its interior.*

Case(V) *C has at least three zeros of z' in its interior.*

Before proving the above lemma, we prepare another lemma.

LEMMA 6.3. *Assume that C is not a straight line segment. Then $z''(s) = z'(s) = 0$ does not hold for any s.*

PROOF. If $z''(s) = z'(s) = 0$ holds, then, from $z'' = -\kappa x'$, $\kappa(s) = 0$ must hold. Now, by the symmetry (Lemma 5.3 (v), (vi)), C must be a straight line, which is a contradiction. □

PROOF OF LEMMA 6.2. Since $\kappa' = \gamma z'$ holds, there exists at least one zero of z' between any two inflection points. Moreover, since $z''(s) = z'(s) = 0$ does not hold for any s, from (12), it follows that the number of zeros of z' is odd. From these observations and Lemma 5.1, it is easy to see that the statement follows. □

We will prove the following result.

THEOREM 6.4. *Assume that C is not a straight line segment, and $\gamma > 0$. Then, for each case mentioned in Lemma 6.2, the stability of the film $\mathscr{F}$ is determined as follows.*
Case(I) $\lambda_2 > 0$ *and* $\mathscr{F}$ *is stable.*
Case(II) $\mathscr{F}$ *is unstable.*
Case(III) $\lambda_2 = 0$ *and* $\mathscr{F}$ *is stable.*
Case(IV) $\mathscr{F}$ *is unstable.*
Case(V) $\mathscr{F}$ *is unstable.*

PROOF. We will prove the result under the assumption that $|\beta| < 1$. Also for the case $|\beta| = 1$, a similar argument works. As in the proof of Theorem 6.3, we set $\cos\sigma := x'$, $\sin\sigma = z'$.

Case(I): Since $\sigma' = -\kappa \neq 0$ holds on C, $\sigma(s)$ is monotone. From the embeddedness of C, as in the proof of Theorem 6.3, we may assume that

$$-\pi/2 < \sigma(s) < \pi/2, \quad \forall s \in (0, L).$$

Therefore, z' has a unique zero $s_1 \in (0, L)$. Hence, z assumes either its maximum or minimum at $s = s_1$. Remark that $n(s_1) = (0, -1)$ and $\sigma(s_1) = 0$ hold. If $z(s_1)$ is the maximum of z, then $\kappa(s_1) > 0$, and therefore $z'(0) > 0$ holds. So we get $z'(0)\kappa(0) > 0$. By a similar argument, we can show, in both cases where $z(s_1)$ is the maximum or the minimum, the following inequalities hold:

$$z'(0)\kappa(0) > 0, \quad z'(L)\kappa(L) < 0. \tag{24}$$

For $\hat{s} \in (0, L)$, denote by $\lambda_1^0([0, \hat{s}])$ the first eigenvalue of the problem

$$J[p] = -\lambda p \text{ in } [0, \hat{s}], \qquad B[p]|_{s=0} = 0, \qquad p(\hat{s}) = 0.$$

And denote by $\lambda_1^L([\hat{s}, L])$ the first eigenvalue of the problem

$$J[p] = -\lambda p \text{ in } [\hat{s}, L], \qquad p(\hat{s}) = 0, \qquad B[p]|_{s=L} = 0.$$

From the min-max principle, we have

$$\lambda_1^0([0,\hat{s}]) = \min\left\{ \left(-\int_0^{\hat{s}} pJ[p]\,ds\right)\left(\int_0^{\hat{s}} p^2\,ds\right)^{-1} \middle|\, p \in C^\infty([0,\hat{s}]) - \{0\},\right.$$
$$\left. B[p]|_{s=0} = 0,\ p(\hat{s}) = 0 \right\}$$
$$= \min\left\{ \left(-\int_0^{\hat{s}} pJ[p]\,ds\right)\left(\int_0^{\hat{s}} p^2\,ds\right)^{-1} \middle|\, p \in C^0([0,\hat{s}]) - \{0\},\right.$$
$$p \text{ is piecewise } C^1 \text{ and piecewise } C^2 \text{on } [0,\hat{s}],$$
$$\left. B[p]|_{s=0} = 0,\ p(\hat{s}) = 0 \right\} \tag{25}$$

(cf. [**2**]). From (25) and the corresponding result on λ_1^L, we can observe the monotonicity of λ_1^0 and λ_1^L in the following sense.

CLAIM 1. If $0 < \sigma_0 < \sigma_1 < L$, then

$$\lambda_1^0([0,\sigma_0]) > \lambda_1^0([0,\sigma_1]), \tag{26}$$
$$\lambda_1^L([\sigma_0,L]) < \lambda_1^L([\sigma_1,L])$$

hold.

On the other hand, on s_1, we can show the following:

CLAIM 2.

$$\lambda_1^0([0,s_1]) > 0, \tag{27}$$
$$\lambda_1^L([s_1,L]) > 0. \tag{28}$$

PROOF OF CLAIM 2. Let $p \in C^\infty([0,s_1])$ be a non-constant function satisfying

$$B[p]|_{s=0} = 0, \qquad p(s_1) = 0.$$

Since $z''(s_1) \neq 0$ (cf. Lemma 6.3), we can define a function $\varphi = p/z'$ on $[0,s_1]$. Hence we can write $p = \varphi z'$. We compute

$$-\int_0^{s_1} pJ[p]ds = \int_0^{s_1} (z')^2(\varphi')^2 ds - [(z')^2\varphi\varphi']_0^{s_1}$$
$$= \int_0^{s_1} (z')^2(\varphi')^2 ds + (z')^2\varphi\varphi'|_{s=0}.$$

Note that

$$0 = pB[p]|_{s=0} = (z')^2\varphi\varphi'|_{s=0} + \varphi^2 z' B[z']|_{s=0}.$$

Therefore, we get, by using (24),

$$-\int_0^{s_1} pJ[p]ds = \int_0^{s_1} (z')^2(\varphi')^2 ds - \varphi^2 z' B[z']|_{s=0}$$
$$= \int_0^{s_1} (z')^2(\varphi')^2 ds + \varphi^2 z' \kappa/\sqrt{1-\beta^2}|_{s=0}$$
$$> 0.$$

Therefore, (27) holds. (28) is proved by a similar way. □

Suppose that $\lambda_2 \leq 0$ holds. We will derive a contradiction. Let e be an eigenfunction belonging to λ_2, that is,

$$J[e] = -\lambda_2 e, \quad B[e]|_{\partial C} = 0$$

hold. Then, $e(s_0) = 0$ holds only for a unique $s_0 \in (0,L)$. Since e does not vanish in $(0,s_0)$, $e|_{[0,s_0]}$ is an eigenfunction belonging to $\lambda_1^0([0,s_0])$. Therefore,

$$\lambda_1^0([0,s_0]) = \lambda_2 \leq 0. \tag{29}$$

Similarly,

$$\lambda_1^L([s_0,L]) = \lambda_2 \leq 0. \tag{30}$$

Assume that $s_0 < s_1$ holds. Then, from (27), (29) and (26), we get a contradiction. Similarly, $s_0 > s_1$ does not hold. Therefore, $s_0 = s_1$ holds. (27) – (30) leads to a contradiction, and we have proved that $\lambda_2 > 0$. Consequently, by Proposition 6.1, $\mathscr{F}$ is stable.

Case(II): From Lemma 6.3, $z''(s) = z'(s) = 0$ does not hold. Therefore, since $\kappa = \gamma z + \kappa_0$ and $z'' = -\kappa x'$ hold, κ is monotone near the inflection point. Hence, from Lemma 5.1, it follows that

$$\kappa(L) + \kappa(0) = 0, \ \kappa(0) \neq 0. \tag{31}$$

So, κ has the unique zero at some $s_2 \in (0,L)$.

CLAIM 3.

$$\int_C z'\,ds \neq 0, \quad \int_C z'\kappa^3\,ds = 0. \tag{32}$$

PROOF OF CLAIM 3. From Lemma 5.3 (vi) and $z'(L) = -z'(0)$, One of the following Case(II-1) – (II-3) holds.

Case(II-1) $C([0,s_2])$ and $C([s_2,L])$ are symmetric to each other with respect to the point $C(s_2)$.

Case(II-2) There exists some $s_3 \in (0,s_2)$ such that $C([s_3,s_2])$ and $C([s_2,L])$ are symmetric to each other with respect to $C(s_2)$.

Case(II-3) There exists some $s_3 \in (s_2,L)$ such that $C([0,s_2])$ and $C([s_2,s_3])$ are symmetric to each other with respect to $C(s_2)$.

First we assume that

$$z'(s) \neq 0, \quad \forall s \in (0,L) \tag{33}$$

holds. Then Case(II-1) must occur. In fact, if Case(II-2) occurs, then $\kappa(0) = \kappa(s_3)$ holds. Therefore, since $\kappa = \gamma z + \kappa_0$, we have $z(0) = z(s_3)$. So $z'(s) = 0$ holds for some $s \in (0,s_3)$, which contradicts the assumption (33). Similarly, Case(II-3) does not occur. Therefore, Case(II-1) occurs, and therefore we obtain (32).

Next, we assume that $z'(s) = 0$ holds for some $s \in (0, L)$. We observed that the number of zeros of z' is odd in the proof of Lemma 6.2. Let s_1 be the middle zero of z'. From (31), $\kappa(0) \neq \kappa(L)$, and so $z(0) \neq z(L)$. Therefore, by Lemma 5.3 (v), either the following (A) or (B) holds.

(A) There exists some $s_4 \in (s_1, L)$ such that $C([0, s_1])$ and $C([s_1, s_4])$ are symmetric to each other with respect to the line $\{x = x(s_1)\}$.

(B) There exists some $s_4 \in (0, s_1)$ such that $C([s_4, s_1])$ and $C([s_1, L])$ are symmetric to each other with respect to the line $\{x = x(s_1)\}$.

Assume that (A) holds. Then $\kappa(s_4) = -\kappa(L)$ holds. Moreover, $\kappa' = \gamma z'$ does not vanish in (s_4, L). Therefore, Case(II-2) holds and $s_4 = s_3$. Hence, we have

$$\int_C z'\,ds = \int_{s_4}^L z'\,ds \neq 0,$$
$$\int_C z'\kappa^3\,ds = \int_0^{s_4} z'\kappa^3\,ds + \int_{s_4}^L z'\kappa^3\,ds = 0 + 0 = 0.$$

Similarly, in the case (B), we obtain (32). □

Note that

$$\kappa'' = -\gamma\kappa x'.$$

Then, we get

$$J[z'] = 0, \quad J[\kappa] = \kappa^3, \tag{34}$$

$$B[z'] = -\kappa/x', \quad B[\kappa] = z'(\gamma - \kappa^2/x') \quad \text{on } \partial[0, L]. \tag{35}$$

Noting (32), we define a number a as

$$a = -\left(\int_C \kappa\,ds\right)\left(\int_C z'\,ds\right)^{-1},$$

and set

$$u = az' + \kappa.$$

Then

$$\int_C u\,ds = 0. \tag{36}$$

We will compute $\mathscr{I}(u)$. In view of (34), we get

$$-uJ[u] = -az'\kappa^3 - \kappa^4.$$

Therefore, from (32), we have

$$-\int_C uJ[u]\,ds = -\int_C \kappa^4\,ds < 0. \tag{37}$$

On the other hand,

$$uB[u] = (az' + \kappa)(aB[z'] + B[\kappa]).$$

Since, from (12), (31) and (35), it follows that

$$(az' + \kappa)|_{s=0} = -(az' + \kappa)|_{s=L},$$

$$B[z']|_{s=0} = -B[z']|_{s=L}, \qquad B[\kappa]|_{s=0} = -B[\kappa]|_{s=L}.$$

we get

$$[[uB[u]]] = 0. \tag{38}$$

From (37) and (38), we see

$$\mathscr{I}[u] = -\int_C \kappa^4\, ds < 0.$$

Recalling (36), we obtain the instability of the film.

Case(III): Since $\kappa' = \gamma z'$ holds, there exists at least one zero of z' in $(0,L)$. From the symmetry of C (Lemma 5.3 (v)) and the assumption of Case(III), we observe that z' has a unique zero at some $s_1 \in (0,L)$, and that C is symmetric with respect to $\{x = x(s_1)\}$. Since

$$J[z'] = 0, \qquad B[z']|_{\partial C} = 0$$

hold and z' vanishes at only one interior point of C, zero is the second eigenvalue of the problem (22). Note that

$$J[x'] = \gamma, \quad B[x']|_{\partial C} = 0, \quad \int_C (x'/\gamma)\, ds > 0.$$

Recall that the multiplicity of each eigenvalue is one. Moreover, we have, by symmetry of C, $\int_0^L z' ds = 0$ and $\int_0^L x'z' ds = 0$. Therefore, by Lemma 6.1, we see that the film $\mathscr{F}$ is stable.

Case(IV): Let s_1 be the unique zero of z'. From $z'' = -\kappa x'$, $x'(0) > 0$, $x'(L) > 0$, and Lemma 6.3, we observe that C is symmetric with respect to $\{x = x(s_1)\}$, and

$$z'(0)\kappa(0) < 0, \quad z'(L)\kappa(L) > 0$$

hold. Therefore,

$$\mathscr{I}(z') = [[z'B[z']\,]] = -\frac{\beta}{\sqrt{1-\beta^2}}\big(z'(L)\kappa(L) - z'(0)\kappa(0)\big) < 0$$

and

$$\int_C z' ds = 0$$

hold, which implies that the film $\mathscr{F}$ is unstable.

Case(V): Let $C_0 \subset\subset C$ be a curve such that $z' = 0$ at both of the end points and exactly one interior point of C_0. Then, z' is an eigenfunction belonging to the second eigenvalue 0 for the Dirichlet boundary condition:

$$J[p] = -\lambda p \text{ in } C_0, \qquad p|_{\partial C_0} = 0.$$

Therefore, from the monotonicity of the eigenvalue with respect to the domain, for the fixed boundary value problem, the second eigenvalue of C is negative, and therefore C is unstable. In particular, C is unstable for our free boundary variational problem. □

REMARK 6.1. The stability results of this section can be applied to curves arising from solutions of the pendulum equation as described in the previous section, as follows.

Because of Remark 5.1, Theorem 6.3 can be applied to curves defined by solutions of the pendulum equation as in (ii) of Lemma 5.2. Lemma 6.2 (I) and Theorem 6.4 (I) may apply to solutions arising from any of the cases of Lemma 5.2. Lemma 6.2 (II)–(V) and Theorem 6.4 (II)–(V) may apply to solutions arising from case(i) of Lemma 5.2.

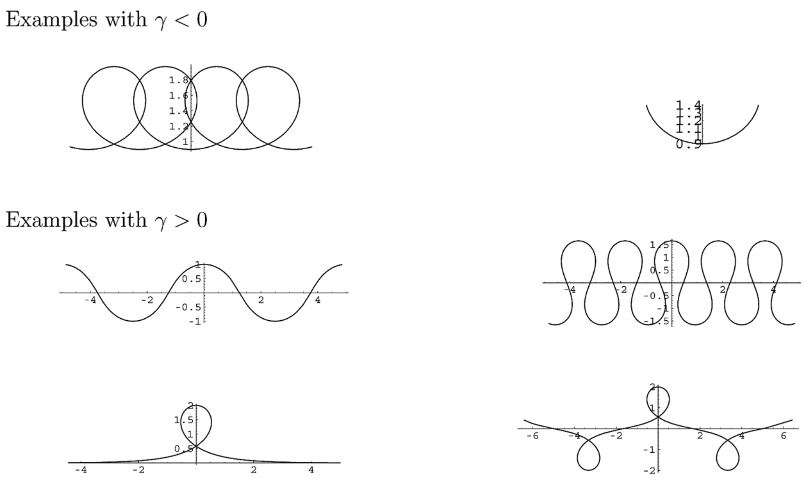

Figure 2. Examples of critical curves.

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Miyuki KOISO
Department of Mathematics
Nara Women's University
Nara 630-8506
Japan
E-mail: koiso@cc.nara-wu.ac.jp

Bennett PALMER
Department of Mathematics
Idaho State University
Pocatello, ID 83209
U.S.A.
E-mail: palmbenn@isu.edu