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Lagrangian calculus on Dirac manifolds

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Abstract. We define notions of isotropic, coisotropic and lagrangian submanifolds of Dirac manifolds. Notion of Dirac manifolds, Dirac maps and Dirac relations are defined. Extending the isotropic calculus on presymplectic manifolds and the coisotropic calculus on Poisson manifolds to Dirac manifolds, we construct the lagrangian calculus on Dirac manifolds as an extension of the one on symplectic manifolds. We see that there are three natural categories of Dirac manifolds.

1. Introduction.

In symplectic geometry the following facts are well known.

(a) A map between symplectic manifolds (M_1, Ω_1) and (M_2, Ω_2) is a symplectic map if and only if the graph of the map is a canonical relation, i.e., a lagrangian submanifold of $M_1 \times M_2^-$, where M_2^- is the symplectic manifold with symplectic structure $-\Omega_2$.

(b) If (N_1, N_2) is a very clean pair ([**14**]) of canonical relations of $M_1 \times M_2$ and $M_2 \times M_3$ respectively then the composition $N_1 \circ N_2$ is a canonical relation of $M_1 \times M_3$, where

$$N_1 \circ N_2 := \{(x, z) | {}^{\exists} y \ : (x, y) \in N_1, (y, z) \in N_2\}.$$

A. Weinstein ([**14**]) defined the concept of coisotropic submanifold and Poisson relation as an extension of canonical relation. A submanifold N of a Poisson manifold (M, π) is coisotropic if $\pi(TN^\perp, TN^\perp) = 0$ by definition, i.e., $TN^\perp$ is isotropic for the Poisson structure, $\tilde{\pi}(TN^\perp) \subset TN$, where $\tilde{\pi}$ is the bundle map $T^*M \to TM$ inducing the Poisson bivector π defined by $\pi(a, b) = \langle \tilde{\pi}(a), b \rangle$, $a, b \in T^*M$. A Poisson relation from (M_2, π_2) to (M_1, π_1) is defined as a coisotropic submanifold of product Poisson manifold $(M_1 \times M_2, \pi_1 \oplus (-\pi_2))$. Two facts (a), (b) above are extended to Poisson manifolds by replacing lagrangian submanifolds with coisotropic submanifolds.

These notion and facts form a mathematical system which is useful in the calculus of Fourier integral operators on manifolds, and then these are called "lagrangian calculus" in Weinstein [**13**]. The coisotropic calculus is a basic theory of Poisson groupoid (cf. [**14**]), and similarly, one can construct the isotropic calculus on presymplectic manifolds.

The purpose of this paper is to extend the lagrangian (resp. isotropic, resp. coisotropic) calculus on symplectic (resp. presymplectic, resp. Poisson) manifolds to that on Dirac manifolds. More precisely, we will construct both an isotropic calculus and a

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coisotropic calculus on Dirac manifold, which contain the isotropic calculus on presymplectic manifold and the coisotropic calculus on Poisson manifold respectively as special cases. By similar usage of terms as Weinstein [**13**], this paper supplies two "Dirac categories". One is the category of Dirac manifolds with isotropic submanifolds of product Dirac manifolds as morphisms, and another is the category of Dirac manifolds with coisotropic submanifolds of product Dirac manifolds as morphisms. Furthermore, it will be shown that lagrangian calculus on Dirac manifold is also possible as an "intersection" of two calculus, and then there exists the third Dirac category whose morphisms are given by lagrangian submanifolds.

First we will give definitions of isotropic submanifold and of coisotropic submanifold of Dirac manifold. The notion of Dirac vector space or Dirac manifold is introduced by T. Courant and A. Weinstein ([**6**]) and studied by T. Courant ([**5**]).

A Dirac structure L on a manifold M (resp. vector space V) is a subbundle of $TM \oplus T^*M$ (resp. subspace of $V \oplus V^*$) which is maximally isotropic with respect to the bilinear form $\langle\cdot,\cdot\rangle_+$, and whose sections are closed under the Courant bracket:

$$[(X,\alpha),(Y,\beta)] = ([X,Y], \mathfrak{L}_X\beta - \mathfrak{L}_Y\alpha + d\langle(X,\alpha),(Y,\beta)\rangle_-),$$
$$(X,\alpha),(Y,\beta) \in \Gamma(TM \oplus T^*M), \tag{1}$$

where the bilinear forms $\langle\cdot,\cdot\rangle_\pm$ are respectively given by

$$\langle(u,a),(v,b)\rangle_\pm = \frac{1}{2}(\langle a,v\rangle \pm \langle b,u\rangle), {}^\forall(u,a),(v,b) \in TM \oplus T^*M.$$

A Dirac structure is a natural extension of symplectic, presymplectic and Poisson structures: It is well-known that (M,Ω) (resp. (M,π)) is a presymplectic manifold (resp. Poisson manifold) if and only if the graph L_Ω (resp. L_π) is a Dirac structure. Here the graphs are given by

$$L_\Omega := \{(x, \tilde{\Omega}(x)) | x \in TM\}, \tag{2}$$
$$L_\pi := \{(\tilde{\pi}(a), a) | a \in T^*M\}, \tag{3}$$

where $\tilde{\Omega}$ is the bundle map inducing the 2-form Ω defined by $\Omega(x,y) = \langle y, \tilde{\Omega}(x)\rangle$, $x,y \in TM$. In general, a Dirac manifold (M,L) has the involutive distribution $\rho(L)$ (see Theorem 2.3.6, [**5**]), where $\rho : TM \oplus T^*M \to TM$ is the canonical projection, and every leaf has a closed 2-form Ω_L (see (2.3.3), [**5**]). Hence the foliation consists of presymplectic leaves.

In a Poisson (resp. presymplectic) manifold, a coisotropic (resp. isotropic) submanifold can be rewritten in terms of Dirac structure as

$$L_\pi(TN^\perp) \subset TN, \quad (resp.\ L_\Omega^{-1}(TN) \subset TN^\perp),$$

where $L_\pi(TN^\perp) \ni x \iff {}^\exists a \in TN^\perp$ such that $(x,a) \in L_\pi$. We notice that a subspace C of a Poisson vector space(V,π) is coisotropic if and only if $C \cap \mathrm{Im}\tilde{\pi}$ is coisotropic in the symplectic vector space (Im $\tilde{\pi}, \pi_s$) (see, Weinstein [**14**]).

Definitions of coisotropic subspaces and isotropic subspaces of Dirac vector space (V, L) are given by the same phrase as above. Namely, using the well-known fact that $\rho(L)$ has the natural presymplectic structure Ω_L, we define a subspace of V to be coisotropic (resp. isotropic) when the induced space is coisotropic (resp. isotropic) in the quotient symplectic vector space $\rho(L)/\ker \tilde{\Omega}_L$. We also define a lagrangian subspace in a Dirac vector space, as a subspace both isotropic and coisotropic (Definition 3.1). Regarding a certain isotropic (resp. coisotropic) subspace of a product Dirac vector spaces as a relation, we will obtain the notion of isotropic (resp. coisotropic) Dirac relation.

We can extend these notions to submanifolds of Dirac manifolds, for example, a submanifold is isotropic if a tangent space is an isotropic subspace of the tangent space of the Dirac manifold for every point. Coisotropic submanifolds, lagrangian submanifolds, etc, are also defined by the similar manner.

For the construction of the calculi on Dirac manifolds, it is crucial to consider the composition of these relations. Different from the Poisson or the presymplectic manifolds, some difficulty arises for Dirac manifolds. Roughly speaking, the difficulty is that a composition $A \circ B$ of two isotropic (resp. coisotropic) subspaces of Dirac vector spaces does not necessarily make an isotropic (resp. coisotropic) subspace, and then we need to characterize a certain class of isotropic (resp. coisotropic) subspaces where the compositions are closed.

The point of this paper is that the auther found the (uniquely determined) subspace D such that the composition $A \circ D \circ B$ is an isotropic (resp. coisotropic) subspaces, although the composition $A \circ B$ is not necessarily isotropic (resp. coisotropic). Thus, the first main result of this paper is as follows:

(0) Let V_1, V_2 and V_3 be Dirac vector spaces and let $S : V_1 \leftarrow V_2$ and $T : V_2 \leftarrow V_3$ be isotropic (resp. coisotropic). Then $S \circ D_2 \circ T : V_1 \leftarrow V_3$ is isotropic (resp. coisotropic). Thus, if S and T are lagrangian, then $S \circ D_2 \circ T$ is also lagrangian (Theorem 0).

Moreover, we can define an equivalence relation for lagrangian subspaces (see Definition 3.9). For any two lagrangian subspaces, the equivalence is described by means of the subspace D. These equivalence classes form a category which can be regarded as an extension of symplectic category (see, Theorem 4.6, Remarks 4.7, 4.17).

We remark here that for the presymplectic (resp. Poisson) vector space case, the space D makes no trouble because D contains (resp. is contained in) the diagonal relation. In section 4.2, we analyze the compositions of the forms $A \circ D \circ B$ and $A \circ B$ in detail. We will also give a sufficient condition for isotropic, coisotropic submanifolds under which the submanifolds are closed with respect to the composition $\circ$ without using D, i.e., the composition of the form $A \circ B$ (Corollary 4.5 in section 4.5 and Theorem 4.11 in section 4.3).

In section 4.3, we will give definitions of isotropic (resp. coisotropic) Dirac maps and the definitions of isotropic (resp. coisotropic) Dirac relations. Our main results are as follows:

(A) The graph of a map $f : M_1 \to M_2$ between two Dirac manifolds is a canonical (resp. isotropic Dirac, resp. coisotropic Dirac) relation in $M_1 \times M_2^-$ if and only if f is a Dirac (resp. isotropic Dirac, resp. coisotropic Dirac) map (Theorem A).

(B) Under suitable clean intersection assumptions, the composition of canonical (resp. isotropic Dirac, resp. coisotropic Dirac) relations is a canonical (resp. isotropic Dirac, resp. coisotropic Dirac) relation (Theorem B).

Our formulation is based on the graphs of maps; the isotropic (resp. coisotropic) Dirac maps are the maps whose graphs are isotropic (resp. coisotropic) Dirac relations. These definitions can be regarded as direct extensions of the presymplectic maps between presymplectic manifolds and the Poisson maps between Poisson manifolds.

On the other hand, we can see a germ of the concept of Dirac map in Drinfeld ([**7**]) and Liu, Weinstein and Xu ([**9**]). The notion of Dirac map was also clearly given by Bursztyn and Radko ([**2**]). In [**2**], a linear map $\phi : V \to W$ is called forward (resp. backward) Dirac map when ϕ transfers the Dirac structure of V (resp. W) to the Dirac structure of W (resp. V), for the detail see Remark 4.20. We see that the notion of coisotropic (resp. isotropic) Dirac map is equal to the notion of forward (resp. backward) Dirac map (Remark 4.20). In addition, we refer some recent works of coisotropic, Lagrangian submanifolds [**8**], [**3**].

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2. Notations and Technical lemmas.

We consider vector spaces V_1, V_2, V_3. Let S, T be subspaces of $V_1 \times V_2$ and $V_2 \times V_3$ respectively. Suppose U is a subspace of V_3, and let V_i^* be the dual space of V_i $(i = 1, 2, 3)$.

Definition 2.1. We set maps

$$- : (x, y) \in V_1 \times V_2 \mapsto (x, -y) \in V_1 \times V_2,$$

$$^{-1} : (x, y) \in V_1 \times V_2 \mapsto (y, x) \in V_2 \times V_1,$$

and subspaces

$$S^\dagger := -(S^\perp), \quad \text{where } S^\perp \subset V_1^* \times V_2^* \text{ is the annihilator space of } S.$$

$$T(U) := \{x \in V_2 | ^\exists y \in V_3; (x, y) \in T, y \in U\},$$

$$S \circ T := \{(x, z) \in V_1 \times V_3 | ^\exists y \in V_2; (x, y) \in S, (y, z) \in T\}.$$

Remark 2.2. We regard a subspace $S \subset V_1 \times V_2$ as a relation. The $\circ$-product defines the category on the class of subspaces in product spaces: The objects are vector spaces, morphisms are subspaces, and compositions of morphisms are given by $\circ$-product (cf. Weinstein [**13**]).

By definition, we have $-(-S) = S$, $(S^{-1})^{-1} = S$, $(S \circ T)^{-1} = T^{-1} \circ S^{-1}$ and $(S \circ T)(U) = S(T(U))$, where the subspaces $-S$ and S^{-1} are the images of S by $-$ and $^{-1}$ respectively. Further we obtain

LEMMA 2.3.

(a) $(-S)^{-1} = -(S^{-1})$.
(b) $(-S)^{\perp} = -(S^{\perp})$.
(c) $(S^{-1})^{\perp} = (S^{\perp})^{-1}$.
(d) $(S^{\dagger})^{\dagger} = S$.
(e) $S \circ (-T) = (-S) \circ T = -(S \circ T)$.
(f) $(S \circ T)^{\dagger} = S^{\dagger} \circ T^{\dagger}$ *and* $(T(U))^{\perp} = T^{\dagger}(U^{\perp}) = T^{\perp}(U^{\perp})$.

PROOF. The dual maps of $-$ and $^{-1}$ are $(-)^* = -$ and $(^{-1})^* = {}^{-1}$ respectively, and hence we have (b) and (c), and then (d) follows. We notice

$$
\begin{aligned}
{}^{-1} \circ - &: (x, y) \mapsto (x, -y) \mapsto (-y, x), \\
- \circ^{-1} &: (x, y) \mapsto (y, x) \mapsto (y, -x) = (-1) \times (-y, x)
\end{aligned}
$$

which gives (a). For $(x, y) \in S \circ (-T)$, there are $(x, z) \in S, (z, y) \in -T$. Hence $(x, -z) \in -S$ and $(-z, y), (z, -y) \in T$, which yields $(x, y) \in (-S) \circ T$ and $(x, -y) \in S \circ T$. Thus we obtain $S \circ (-T) \subset (-S) \circ T$ and $S \circ (-T) \subset -(S \circ T)$. Similarly we obtain $S \circ (-T) \supset (-S) \circ T$ and $S \circ (-T) \supset -(S \circ T)$. Thus we have (e).

For (f), let $pr : V_1 \times V_2 \times V_2 \times V_3 \to V_1 \times V_3$ and δ_{V_2} denote the natural projection and the diagonal subspace of $V_2 \times V_2$ respectively. We have $S \circ T = pr(S \times T \cap V_1 \times \delta_{V_2} \times V_3)$ and by an elementary argument we see

$$
(pr(S \times T \cap V_1 \times \delta_{V_2} \times V_3))^{\perp} = (pr^*)^{-1}((S \times T \cap V_1 \times \delta_{V_2} \times V_3)^{\perp}).
$$

Since $(S \times T \cap V_1 \times \delta_{V_2} \times V_3)^{\perp} = (S^{\perp} \times T^{\perp}) + (0 \times (-\delta_{V_2^*}) \times 0)$ and $pr^*(a, b) = (a, 0, 0, b)$, this implies $(S \circ T)^{\perp} = S^{\dagger} \circ T^{\perp}$. Then applying "-" to the both sides, we have $(S \circ T)^{\dagger} = S^{\dagger} \circ T^{\dagger}$. We identify U with $U \subset V_3 \times 0$. Then the second identity follows. □

LEMMA 2.4. *Let W be a subspace of V_3. If $(0, W) \subset T$ then $T(U) = T(U + W)$.*

PROOF. $T(U) \subset T(U + W)$ is trivial. For x of $T(U + W)$.

$$
T(U + W) \ni x \iff (x, y) \in T, y \in U + W.
$$

We put $y = u + w$, where $u \in U$ and $w \in W$. Since $(0, w) \in T$ from the assumption, we have

$$
(x, y) - (0, w) = (x, y - w) = (x, u) \in T.
$$

Hence $x \in T(U)$. □

Next we prepare the functorial lemmas of Dirac structures. Let (V, L) be a Dirac vector space and $(\rho(L), \Omega_L)$ be the induced presymplectic vector space, where $\rho : V \oplus V^* \to V$ be the canonical projection. Let $i : \rho(L) \hookrightarrow V$ be the inclusion map and $p : \rho(L) \to \rho(L)/\ker \tilde{\Omega}_L$ be the natural projection, where Ω_L is the skew 2-form on $\rho(L)$. The next lemma gives nice functorial relations between a Dirac vector space and the induced presymplectic, symplectic vector space.

Lemma 2.5. *The diagram* (6) *below commutes, i.e.,*

$$L = I^{-1} \circ L_{\Omega_L} \circ I^{\dagger}, \tag{4}$$

$$L_{\Omega_s} = P^{-1} \circ L_{\Omega_L} \circ P^{\dagger}, \tag{5}$$

where $I := \{(x, i(x)) | x \in \rho(L)\}$, $P := \{(x, p(x)) | x \in \rho(L)\}$, *and* $(\rho(L)/\ker \tilde{\Omega}_L, \Omega_s)$ *is the quotient symplectic vector space.*

$$\begin{tikzcd} V^* \arrow[r, "i^*"] \arrow[d, "L"'] & (\rho(L))^* \arrow[d, "L_{\Omega_L}"'] & (\rho(L)/\ker \tilde{\Omega}_L)^* \arrow[l, "p^*"'] \arrow[d, "L_{\Omega_s}"'] \\ V & \rho(L) \arrow[l, "i"] \arrow[r, "p"] & \rho(L)/\ker \tilde{\Omega}_L \end{tikzcd} \quad . \tag{6}$$

Proof. Since a Dirac structure L is nothing but a presymplectic subspace $(\rho(L), \Omega_L)$ of $V \times V^*$ ([**5**]), we have

$$L = \{(x, a) | a \in V^*, a|_{\rho(L)} = \tilde{\Omega}_L(x), x \in \rho(L)\}. \tag{7}$$

For x of $\rho(L)$ we have

$$\begin{aligned} I^{-1} \circ L_{\Omega_L} \circ I^{\dagger} \ni (x, a) &\iff (x, x) \in I^{-1}, (x, \tilde{\Omega}_L(x)) \in L_{\Omega_L}, (\tilde{\Omega}_L(x), a) \in I^{\dagger} \\ &\iff \tilde{\Omega}_L(x) = i^*(a) \\ &\iff a|_{\rho(L)} = \tilde{\Omega}_L(x). \end{aligned}$$

Hence we obtain (4). Since $\tilde{\Omega}_s := (p^*)^{-1} \circ \tilde{\Omega}_L \circ p^{-1}$, we obtain (5). □

Let (V, L) be a Dirac vector space with a Dirac structure L. We introduce a subspace D of $V \times V$ which plays an important role to define $\circ D \circ$-composition in the next sections. Consider the diagram:

$$V \xleftarrow[i]{} \rho(L) \xrightarrow{p} \rho(L)/\ker \tilde{\Omega}_L.$$

From this diagram we naturally obtain a relation $I^{-1} \circ P$ between V and $\rho(L)/\ker \tilde{\Omega}_L$, where I, P are the graphs given in Lemma 2.5. Our D is given by

$$D := I^{-1} \circ P \circ P^{-1} \circ I. \tag{8}$$

By identifying x with $i(x)$, $x \in \rho(L)$, we obtain $D \ni (x,y) \iff x - y \in \ker \tilde{\Omega}_L$, $x, y \in \rho(L)$, and we see also $D = P \circ P^{-1}$. The formula (a) in the next lemma is useful.

LEMMA 2.6.

(a) $D = L \circ L^{-1}$.
(b) $L^{\perp} = L^{-1}$, *where* $L^{\perp} \subset V^* \times V^{**} (\cong V^* \times V)$ *is the annihilator space of* L.
(c) $D^{\dagger} = L^{-1} \circ L$.
(d) $D \circ L = L \circ D^{\dagger} = L$ *and* $D^{\dagger} \circ L^{-1} = L^{-1} \circ D = L^{-1}$. *Hence* $D \circ D = D$.

PROOF. (b) follows from the definition of Dirac structure. We show (a). If $(x,y) \in D$ then $(x,b) \in L$ since $x \in \rho(L)$. Since $x - y \in \ker \Omega_L$ and $(\ker \Omega_L, 0) \subset L$, we have $(x,b) + (y-x, 0) = (y,b) \in L$. Thus $(x,y) \in L \circ L^{-1}$. Conversely let $(x,y) \in L \circ L^{-1}$. Then $x, y \in \rho(L)$ and $(x,b), (y,b) \in L$, which yields $x - y \in \ker \tilde{\Omega}_L$. (c) is given by (a), (b) and Lemma 2.3(f). (d) is given by (c) and the definition of $\circ$-product. □

We set a symplectic vector space $(c(L), \Omega_s)$ as follows:

$$c(L) := \rho(L)/\ker \tilde{\Omega}_L, \quad \text{with the symplectic structure } \Omega_s. \tag{9}$$

We call the space $(c(L), \Omega_s)$ *symplectic core* or simply the *core* of Dirac vector space (V, L).

LEMMA 2.7. *Let* (V_1, L_1) *and* (V_2, L_2) *be any Dirac vector spaces and* W *be a subspace of* $V_1 \times V_2$. *Then subspaces* L_1' *and* L_2' *given by*

$$L_1' := W \circ L_2 \circ (W^{\dagger})^{-1}, \quad L_2' := W^{-1} \circ L_1 \circ W^{\dagger} \tag{10}$$

are Dirac structures of V_1 *and* V_2 *respectively.*

PROOF. First, we show that L_1' is a skew relation of $V_1 \times V_1^*$, i.e., $L_1' \ni (x,a) : \langle x, a \rangle = 0$. Remark that

$$L_1' \ni (x,a) \iff (x,y) \in W, (y,b) \in L_2, (b,a) \in (W^{\dagger})^{-1}.$$

By definition of †, we have $\langle x, a \rangle - \langle y, b \rangle = 0$. Since $(y,b) \in L_2$, we have $\langle x, a \rangle = 0$, which implies that L_1' is isotropic with respect to the bilinear form $\langle \cdot, \cdot \rangle_+$.

Next we show $\dim L_1' = \dim V_1$. By Lemma 2.3, we have

$$(L_1')^{\dagger} = (W)^{\dagger} \circ (L_2)^{\dagger} \circ W^{-1}.$$

We apply "-" to the both sides and we have

$$(L_1')^{\perp} = (W)^{\dagger} \circ (L_2)^{\perp} \circ W^{-1}.$$

Since $L_2^{\perp} = L_2^{-1}$ (Lemma 2.6(b)), it follows

$$(L_1')^{\perp} = (W)^{\dagger} \circ (L_2)^{-1} \circ W^{-1} = (W \circ L_2 \circ (W^{\dagger})^{-1})^{-1}.$$

Hence we have $(L_1')^{\perp} = (L_1')^{-1}$. This implies that $\dim L_1' = \dim V_1$. In a similar way, we can see L_2' is a Dirac structure. □

3. Isotropic, coisotropic and Lagrangian subobjects.

In this section, we will give definitions of isotropic submanifolds and coisotropic submanifolds of Dirac manifolds. We will also show several basic properties for these submanifolds.

We recall Lemma 2.5. A Dirac structure L induces the presymplectic structure Ω_L on $\rho(L)$ and the symplectic structure Ω_s on the symplectic core $c(L) = \rho(L)/\ker\tilde{\Omega}_L$. Since every subspace W of Dirac vector space (V, L) is sent into the core $(c(L), \Omega_s)$ along the diagram

$$(V, L) \supset W \xrightarrow{\cap\rho(L)} (\rho(L), \Omega_L) \supset W \cap \rho(L) \xrightarrow{\quad p \quad} (c(L), \Omega_s) \supset p(W \cap \rho(L)). \tag{11}$$

We are permitted to define the following notions.

Definition 3.1.

(I-1) A subspace W of a Dirac vector space (V, L) is isotropic if $p(W \cap \rho(L))$ is an isotropic subspace of the symplectic core $c(L)$.

(I-2) A submanifold C of a Dirac manifold (M, L) is isotropic if the tangent space $W = T_xC$ is isotropic in the Dirac vector space (T_xM, L_x) for every $x \in C$.

(C-1) A subspace W of a Dirac vector space (V, L) is coisotropic if $p(W \cap \rho(L))$ is a coisotropic subspace of the symplectic core $c(L)$.

(C-2) A submanifold C of a Dirac manifold (M, L) is coisotropic if the tangent space $W = T_xC$ is coisotropic in the Dirac vector space (T_xM, L_x) for every $x \in C$.

(L-1) A subspace W of a Dirac vector space (V, L) is lagrangian if W is isotropic and coisotropic.

(L-2) A submanifold C of a Dirac manifold (M, L) is lagrangian if C is isotropic and coisotropic.

From (2) of Introduction, if a Dirac structure is given by a presymplectic structure Ω then the induced presymplectic structure Ω_L is Ω. Since $(\rho(L), \Omega_L)$ is a presymplectic vector space, a subspace $U \subset \rho(L)$ is isotropic (resp. coisotropic) if and only if the projection $p(U)$ is isotropic (resp. coisotropic) in the symplectic core $c(L)$. Thus, by the Definition 3.1, we easily have the following:

Corollary 3.2. *Let W be a subspace of a Dirac vector space (V, L). The following conditions are equivalent*:

(a) *W is isotropic (resp. coisotropic) in V.*
(b) *$W \cap \rho(L)$ is isotropic (resp. coisotropic) in V.*

(c) *$W \cap \rho(L)$ is isotropic (resp. coisotropic) in $\rho(L)$ with respect to L_{Ω_L}, where Ω_L is the presymplectic structure on $\rho(L)$.*

(d) *$p(W \cap \rho(L))$ is isotropic (resp. coisotropic) in the symplectic core $c(L)$.*

The first purpose of this section is to rewrite Definition 3.1 to useful forms in terms of functorial properties of Dirac structures.

LEMMA 3.3. *A subspace W of a Dirac vector space (V, L) is isotropic (resp. coisotropic) if and only if it satisfies the condition*

$$L^{-1}(W) \subset W^{\perp} + (\rho(L))^{\perp}. \tag{12}$$

$$(\textit{resp.}\ \ L(W^{\perp}) \subset W + \ker \tilde{\Omega}_L) \tag{13}$$

PROOF. We show the coisotropic case. The isotropic case is proved by the similar manner. For a subspace $U \subset \rho(L)$, we denote by $U^{\perp}$ the annihilator space of U in V^*, and by U^0 the annihilator space of U in $(\rho(L))^*$, respectively. Also we denote by $(p(U))^a$ the annihilator space of $p(U)$ in the core $(c(L))^*$. Then we have $U^{\perp} = (i^*)^{-1}(U^0)$ and $(p(U))^a = (p^*)^{-1}(U^0)$.

First step. The subspace $p(U)$ of the symplectic core $(c(L), \Omega_s)$ is coisotropic if and only if

$$\tilde{\Omega}_s^{-1}((p(U))^a) \subset p(U). \tag{14}$$

The condition (14) is rewritten as $L_{\Omega_s}((p(U))^a) \subset p(U)$. From Lemma 2.5, we have

$$\begin{aligned} L_{\Omega_s}((p(U))^a) &= P^{-1} \circ L_{\Omega_L} \circ P^{\dagger}((p(U))^a) \\ &= P^{-1} \circ L_{\Omega_L}(p^*((p(U))^a)) \\ &= p(L_{\Omega_L}(p^*((p(U))^a))). \end{aligned}$$

Since $(p(U))^a = (p^*)^{-1}(U^0)$, we have $p^*((p(U))^a) = U^0 \cap \mathrm{Im}\ p^*$. Since $\mathrm{Im}\ p^* = \mathrm{Im}\ \tilde{\Omega}_L$ and $L_{\Omega_L}(U^0 \cap \mathrm{Im}\ \tilde{\Omega}_L) = L_{\Omega_L}(U^0)$, we have

$$L_{\Omega_s}((p(U))^a) = p(L_{\Omega_L}(U^0 \cap \mathrm{Im}\ \tilde{\Omega}_L)) = p(L_{\Omega}(U^0)).$$

Hence we have

$$\begin{aligned} L_{\Omega_s}((p(U))^a) \subset p(U) &\iff p(L_{\Omega_L}(U^0)) \subset p(U) \\ &\iff L_{\Omega_L}(U^0) \subset U + \ker p. \end{aligned} \tag{15}$$

The equivalence (15) together with the fact $\ker p = \ker \tilde{\Omega}_L$ yields

$$\tilde{\Omega}_s^{-1}((p(U))^a) \subset p(U) \iff L_{\Omega_L}(U^0) \subset U + \ker \tilde{\Omega}_L. \tag{16}$$

Second step. By Lemma 2.5, we have $L = I^{-1} \circ L_{\Omega_L} \circ I^{\dagger}$. Then applying $U^{\perp}$ to the both sides from the right, we have

$$L(U^{\perp}) = I^{-1} \circ L_{\Omega_L} \circ I^{\dagger}(U^{\perp}) = I^{-1} \circ L_{\Omega_L}(i^*(U^{\perp})) = I^{-1} \circ L_{\Omega_L}(U^0),$$

where we use $U^{\perp} = (i^*)^{-1}(U^0)$. Hence we obtain

$$L(U^{\perp}) \subset U + \ker \tilde{\Omega}_L \iff L_{\Omega_L}(U^0) \subset U + \ker \tilde{\Omega}_L. \tag{17}$$

Now, we set $U = W \cap \rho(L)$. Since $U^{\perp} = W^{\perp} + \rho(L)^{\perp}$ and $(0, \rho(L)^{\perp}) \subset L$, we have the following from Lemma 2.4.

$$L(W^{\perp}) \subset W + \ker \tilde{\Omega}_L \iff L_{\Omega_L}(U^0) \subset U + \ker \tilde{\Omega}_L.$$

The conditions (14), (16) and (17) show the desired result. □

We can write Definition 3.1 into the following equivalent forms.

Proposition 3.4. *Let (V, L) be a Dirac vector space and let W be its subspace. Then the following conditions are equivalent.*

(A-1) *W is isotropic, i.e., $L^{-1}(W) \subset W^{\perp} + (\rho(L))^{\perp}$.*
(A-2) *$L^{-1}(W \cap \rho(L)) \subset (W \cap \rho(L))^{\perp}$.*
(A-3) *$\rho(W \oplus W^{\perp} \cap L) = W \cap \rho(L)$.*
(A-4) *$D(W) \subset L(W^{\perp})$.*

Similarly, the following conditions are equivalent.

(B-1) *W is coisotropic, i.e., $L(W^{\perp}) \subset W + \ker \tilde{\Omega}_L$.*
(B-2) *$L((W + \ker \tilde{\Omega}_L)^{\perp}) \subset W + \ker \tilde{\Omega}_L$.*
(B-3) *$\rho^*(W \oplus W^{\perp} \cap L) = W^{\perp} \cap \rho^*(L)$, where $\rho^* : V \oplus V^* \to V^*$ is the canonical projection.*
(B-4) *$L(W^{\perp}) \subset D(W)$.*

Proof. Notice that $L^{-1}(W) = L^{-1}(W \cap \rho(L))$. Then we obtain the equivalence of (A-1) and (A-2). Similarly the equivalence of (B-1) and (B-2) follows from the fact $(\ker \tilde{\Omega}_L)^{\perp} = \rho^*(L)$ ([**5**]).

We remark (A-3) and (B-3) are equivalent to the following (18) and (19) respectively,

$${}^{\forall}x \in W \cap \rho(L), {}^{\exists} a \in W^{\perp} : (x, a) \in L, \tag{18}$$

$${}^{\forall}a \in W^{\perp} \cap \rho^*(L), {}^{\exists} x \in W : (x, a) \in L. \tag{19}$$

(A-1) $\iff$ (18): We assume (A-1). For $x \in W \cap \rho(L)$, we take $b \in V^*$ such that $(x, b) \in L$. Then by (A-2) we have $b \in W^{\perp} + (\rho(L))^{\perp}$. We put $b = a + f$, where $a \in W^{\perp}$ and $f \in (\rho(L))^{\perp}$. Since $(0, (\rho(L))^{\perp}) \subset L$, we have $(x, a) = (x, b) - (0, f) \in L$. This yields (18). Conversely, we assume (18). For an arbitrary element $a \in L^{-1}(W)$, there is $x \in W$

such that $(a, x) \in L^{-1}$. Here $x \in W \cap \rho(L)$ and $(x, a) \in L$. Then by the assumption (18), there exists $b \in W^{\perp}$ such that $(x, b) \in L$. Hence $(0, a-b) = (x, a) - (x, b) \in L$ which gives $a - b \in (\rho(L))^{\perp}$. Thus $a = b + (a-b) \in W^{\perp} + (\rho(L))^{\perp}$ which yields (A-1).
(B-1) $\Longleftrightarrow$ (19): We assume (B-1). For an element $a \in W^{\perp} \cap \rho^*(L)$, there is z such that $(z, a) \in L$ by definition. Then the condition (B-2) yields that $z \in W + \ker \tilde{\Omega}_L$. We put $z = x + e$, where $x \in W$ and $e \in \ker \tilde{\Omega}_L$. Since $(\ker \tilde{\Omega}_L, 0) \subset L$, we have $(x, a) = (z, a) - (e, 0) \in L$. This implies (19). Conversely, we assume (19). For an arbitrary $x \in L(W^{\perp})$, there is $a \in W^{\perp}$ such that $(x, a) \in L$ by definition, hence $a \in W^{\perp} \cap \rho^*(L)$. On the other hand, the assumption (19) gives $y \in W$ such that $(y, a) \in L$. Hence we have $(x - y, 0) = (x, a) - (y, a) \in L$. This implies $x - y \in \ker \tilde{\Omega}_L$. Thus $x = y + (x - y) \in W + \ker \tilde{\Omega}_L$ and we obtain (B-1).
(A-1) $\Longleftrightarrow$ (A-4): We assume (A-1). Apply L to the both sides of (A-1). Then Lemma 2.4 gives

$$D(W) \subset L(W^{\perp} + (\rho(L))^{\perp}) = L(W^{\perp}),$$

since $(0, (\rho(L))^{\perp}) \subset L$. Then we have (A-4). Conversely we assume (A-4). Apply L^{-1} to (A-4). Then Lemma 2.6(d) gives

$$L^{-1}(W) \subset D^{\dagger}(W^{\perp}),$$

and then $(L^{-1}(W))^{\perp} \supset (D^{\dagger}(W^{\perp}))^{\perp}$. From Lemma 2.3(f), this is equivalent to $L(W^{\perp}) \supset D(W)$, where we use $L^{\perp} = L^{-1}$. From the definition of D, we obtain $D(W) = W \cap \rho(L) + \ker \tilde{\Omega}_L$, and then we have

$$L(W^{\perp}) \supset W \cap \rho(L) + \ker \tilde{\Omega}_L.$$

Again we take the annihilator space and we see

$$L^{-1}(W) \subset (W^{\perp} + \rho(L)^{\perp}) \cap (\ker \tilde{\Omega}_L)^{\perp}.$$

This implies (A-1). In a similar way we have the equivalence of (B-1) and (B-4). □

Since Definition 3.1 is made as a naive and natural extension of the notions in presymplectic and Poisson categories, we have also the following:

COROLLARY 3.5.

(a) *In presymplectic (V, L_{Ω}) case, a subspace W is isotropic if and only if $\tilde{\Omega}(W) \subset W^{\perp}$, and W is coisotropic if and only if* $\mathrm{Im}\, \tilde{\Omega} \cap W^{\perp} \subset \tilde{\Omega}(W)$.
(b) *In Poisson (V, L_{π}) case, a subspace W is coisotropic if and only if $\tilde{\pi}(W^{\perp}) \subset W$, and W is isotropic if and only if* $\mathrm{Im}\, \tilde{\pi} \cap W \subset \tilde{\pi}(W^{\perp})$.

PROOF. For (a) (resp. (b)), we notice $\rho(L) = V$ and $\tilde{\Omega}_L = \tilde{\Omega}$ (resp. $\rho(L) = \mathrm{Im}\, \tilde{\pi}$ and $\ker \tilde{\Omega}_L = 0$). □

The conditions (A-4), (B-4) of Proposition 3.4 yield

Corollary 3.6. *Let W_1 and W_2 be subspaces of a Dirac vector space (V, L). We assume $W_1 \subset W_2$. If W_1 is coisotropic then W_2 is coisotropic and if W_2 is isotropic then W_1 is isotropic.*

Proof. We assume W_2 is isotropic. By Proposition 3.4 we have

$$D(W_1) \subset D(W_2) \subset L(W_2^{\perp}) \subset L(W_1^{\perp}),$$

where we use $W_2^{\perp} \subset W_1^{\perp}$. This implies that W_1 is isotropic. The coisotropic case is also proved by the similar way. □

Remark 3.7. Corollary 3.5(b) shows that a submanifold N of a Poisson manifold M is lagrangian if and only if

$$T_x\Sigma \cap T_xN = \tilde{\pi}(T_xN^{\perp}), \ {}^{\forall}x \in \Sigma \cap N,$$

where Σ is a symplectic leaf. It is to be noticed that the definition of lagrangian submanifolds is the same as in ([**11**]) when M is a Poisson manifold.

As a main theorem of this section, from (A-4) and (B-4) of Proposition 3.4 we have the following criterion of a lagrangian subspace.

Theorem 3.8. *Let (V, L) be a Dirac vector space and let W be a subspace.*

(1) *W is lagrangian if and only if the equation $L(W^{\perp}) = D(W)$ holds.*
(2) *If $L(W^{\perp}) = W$ then W is lagrangian.*

We notice here, lagrangian submanifolds of a Dirac manifold may have various dimensions. In fact, if W is lagrangian then $W \cap \rho(L)$ and $W + \ker \tilde{\Omega}_L$ are also lagrangian and thus, from Corollary 3.6, we have many lagrangian objects between $W \cap \rho(L)$ and $W + \ker \tilde{\Omega}_L$.

$$W \cap \rho(L) \subset \cdots \subset W \subset \cdots \subset W + \ker \tilde{\Omega}_L.$$

We next introduce an equivalence relation and the set of lagrangian subspaces. Let W_1 and W_2 be lagrangian subspaces of a Dirac vector space (V, L). Then, along the diagram (11), we obtain two lagrangian subspaces $p(W_i \cap \rho(L))$, $(i = 1, 2)$ in the symplectic core $c(L)$. We define an equivalence relation $W_1 \sim W_2$ as follows.

Definition 3.9. We say two lagrangian subspaces W_1 and W_2 are equivalent $W_1 \sim W_2$, if the relation $p(W_1 \cap \rho(L)) = p(W_2 \cap \rho(L))$, or equivalently, $W_1 \cap \rho(L) + \ker \tilde{\Omega}_L = W_2 \cap \rho(L) + \ker \tilde{\Omega}_L$ holds.

From the definition, we obtain directly

Proposition 3.10. *Let (V, L) be a Dirac vector space. If a subspace $W_c \subset c(L)$ is lagrangian in the symplectic core $c(L)$ then the fiber space $p^{-1}(W_c) \subset \rho(L) \subset V$ is lagrangian in the Dirac vector space (V, L). Thus, the set of the equivalence classes* *of lagrangian subspaces in V is bijective to the set of all lagrangian subspaces in the symplectic core $c(L)$.*

From the definition of D, we have $D(W) = W \cap \rho(L) + \ker \tilde{\Omega}_L$. Then we obtain

PROPOSITION 3.11. *Let (V, L) be a Dirac vector space. Any two lagrangian subspaces W_1 and W_2 are equivalent if and only if $D(W_1) = D(W_2)$.*

COROLLARY 3.12. *Let W be a lagrangian subspace of a Dirac vector space (V, L). Then $W \sim D(W)$.*

DEFINITION 3.13. In a Dirac manifold (M, L), we can call subbundle $E \subset TM$ is isotropic (resp. coisotropic, resp. lagrangian) if a fiber E_x is isotropic (resp. coisotropic, resp. lagrangian) on the Dirac vector space (T_xM, L_x) for every $x \in M$ similarly to Poisson manifold case ([**14**]). The equivalence relation above is also defined for lagrangian vector bundles.

4. Dirac relations.

4.1. Product Dirac structure.

Let (M_i, L_i), $(i = 1, 2)$ be any Dirac manifolds. Then, it is obvious that $-L_2$ is a Dirac structure on M_2. We next show that $L_1 \times L_2$ is a Dirac structure on $M_1 \times M_2$. We easily see that the bilinear form $\langle \cdot, \cdot \rangle_+$ on $T(M_1 \times M_2) \oplus T^*(M_1 \times M_2)$ vanishes, and hence $L_1 \times L_2$ is maximally isotropic. The Courant bracket (2) satisfies one of the axioms of Lie algebroid (Leibniz rule) for a pair of the elements of $\Gamma(L_1 \times L_2)$ ([**5**], [**9**]). The remains to check is that the Courant bracket (2) is closed on $\Gamma(L_1 \times L_2)$. We put $\xi_i \in \Gamma L_1$ and $\eta_i \in \Gamma L_2$ and $f, g \in C^\infty(M_1 \times M_2)$ then, by the assumption, we have $[\xi_1, \xi_2] \in \Gamma L_1$ and $[\eta_1, \eta_2] \in \Gamma L_2$ and $[\xi_i, \eta_j] = 0$ and further

$$\begin{aligned}
[f\xi_i, g\xi_j] &= fg[\xi_i, \xi_j] + f\rho(\xi_i)(g)\xi_j - g\rho(\xi_j)(f)\xi_i, \\
[f\eta_i, g\eta_j] &= fg[\eta_i, \eta_j] + f\rho(\eta_i)(g)\eta_j - g\rho(\eta_j)(f)\eta_i, \\
[f\xi_i, g\eta_j] &= f\rho(\xi_i)(g)\eta_j - g\rho(\eta_j)(f)\xi_i,
\end{aligned}$$

where $\rho : T(M_1 \times M_2) \oplus T^*(M_1 \times M_2) \to T(M_1 \times M_2)$ is the canonical projection. Thus the Courant bracket is closed on $\Gamma(L_1 \times L_2)$. Then $L_1 \times L_2$ is a Dirac structure.

If L_1 and L_2 are Dirac structures on M_1 and M_2 respectively then $M_1 \times M_2$ is a Dirac manifold with Dirac structure $L_1 \times (-L_2)$. We denote the Dirac manifold $(M_1 \times M_2, L_1 \times (-L_2))$ by $M_1 \times M_2^-$. We use the same notations for Dirac vector spaces.

REMARK 4.1. We can easily see that if L corresponds to $(\rho(L), \Omega_L)$ then $-L$ corresponds to $(\rho(L), -\Omega_L)$ and the following properties are satisfied

$$\begin{aligned}
&\rho(L) = \rho(-L), \ \rho^*(L) = \rho^*(-L), \\
&\rho(L_1 \times (-L_2)) = \rho(L_1 \times L_2), \ \rho^*(L_1 \times (-L_2)) = \rho^*(L_1 \times L_2).
\end{aligned}$$

4.2. Lagrangian relations.

When N is a submanifold (resp. subspace) of a Dirac manifold (resp. Dirac vector space) $M_1 \times M_2^-$ (resp. $V_1 \times V_2^-$), we sometimes write as $N : M_1 \leftarrow M_2$ (resp. $N : V_1 \leftarrow V_2$).

For a subspace $S \subset V_1 \times V_2$, we notice $D_1 \times D_2(S) = D_1 \circ S \circ D_2$ and $D \circ L = L$ by Lemma 2.6(d). From Definition 3.1, Theorem 3.8 and Corollary 3.12 we obtain

Proposition 4.2. *Let (V_1, L_1) and (V_2, L_2) be Dirac vector spaces and S be a subspace of $V_1 \times V_2$. Then the following conditions are equivalent.*

(E-1) *$S : V_1 \leftarrow V_2$ is a lagrangian subspace.*
(E-2) *$L_1 \circ S^\dagger \circ D_2^\dagger = D_1 \circ S \circ L_2$, where $D_i := L_i \circ L_i^{-1}$.*
(E-3) *$D_1 \circ S \circ D_2 : V_1 \leftarrow V_2$ is a lagrangian subspace.*

Remark 4.3. It is easy to see (E-3) $\iff$ $S \circ D_2 : V_1 \leftarrow V_2$ is a lagrangian subspace $\iff$ $D_1 \circ S : V_1 \leftarrow V_2$ is a lagrangian subspace.

Similar to Proposition 4.2, we have the next proposition, which is easily seen by Definition 3.1 and Proposition 3.4.

Proposition 4.4. *Under the assumption of Proposition 4.2, $S : V_1 \leftarrow V_2$ is an isotropic (resp. a coisotropic) subspaces if and only if $L_1 \circ S^\dagger \circ D_2^\dagger \supset$ (resp. $\subset$) $D_1 \circ S \circ L_2$.*

The identity (E-2) implies that the following diagram is commutative.

$$\begin{tikzcd}
& V_1^* \arrow[dl, "L_1"'] \arrow[d, dash, dotted, "L_1"] & V_2^* \arrow[l, "S^\dagger"'] \arrow[d, dash, dotted, "L_2"'] & V_2^* \arrow[l, "D_2^\dagger"'] \arrow[dl, "L_2"] \\
V_1 & V_1 \arrow[l, "D_1"] & V_2 \arrow[l, "S"] &
\end{tikzcd} \tag{20}$$

We notice here that $L_1 \circ S^\dagger = S \circ L_2$ does not necessarily hold. Next lemma plays an important role in the next section to define compositions without using D.

Corollary 4.5. *Under the assumption of Proposition 4.2, if $L_1 \circ S^\dagger \subset$ (resp. $\supset$) $S \circ L_2$ then $S : V_1 \leftarrow V_2$ is coisotropic (resp. isotropic). Thus if $L_1 \circ S^\dagger = S \circ L_2$ then $S : V_1 \leftarrow V_2$ is lagrangian.*

Proof. We assume $L_1 \circ S^\dagger \subset S \circ L_2$. Then applying $D_2^\dagger$ and D_1 from the right hand side and the left hand side respectively, we have

$$D_1 \circ L_1 \circ S^\dagger \circ D_2^\dagger \subset D_1 \circ S \circ L_2 \circ D_2^\dagger.$$

From $L_2 = L_2 \circ D_2^\dagger$, $L_1 = D_1 \circ L_1$ by Lemma 2.6(d) and Proposition 4.4, we have the desired result. In a similar way, the isotropic case is shown. □

Now we consider the composition of these subspaces. As is mentioned in the introduction, compositions $A \circ B$ of isotropic (resp. coisotropic, resp. lagrangian) subspaces are not necessarily isotropic (resp. coisotropic, resp. lagrangian) in general. However, the next theorem show that $\circ D\circ$-composition is well defined. Indeed, this is the first main theorem of this paper.

THEOREM 0. *Let V_1, V_2 and V_3 be Dirac vector spaces and let $S : V_1 \leftarrow V_2$ and $T : V_2 \leftarrow V_3$ be isotropic (resp. coisotropic). Then $S \circ D_2 \circ T : V_1 \leftarrow V_3$ is isotropic (resp. coisotropic). Thus, if S and T are lagrangian, then $S \circ D_2 \circ T$ is also lagrangian.*

PROOF. First, we consider the coisotropic case and we show that $S \circ D_2 \circ T$ is coisotropic. Although the proof is given by a direct calculation by using Proposition 4.2(E-2) and Lemma 2.3(f), (see for the isotropic case below), we give a "geometric" proof, through which one can see why the space D_2 naturally comes in.

We put $S' := S \cap \rho(L_1 \times (-L_2))$ and $T' := T \cap \rho(L_2 \times (-L_3))$ where the canonical projection $p_i : \rho(L_i) \to c(L_i)$, $(i = 1, 2, 3)$. By the assumption and Corollary 3.2, S' and T' are coisotropic in $\rho(L_1) \times (\rho(L_2))^-$ and $\rho(L_2) \times (\rho(L_3))^-$, respectively. Also $p_1 \times p_2(S')$ and $p_2 \times p_3(T')$ are coisotropic in $c(L_1) \times c(L_2)^-$ and $c(L_2) \times c(L_3)^-$, respectively. Here $\rho(L_i)^-$ and $c(L_i)^-$ are anti-presymplectic and anti-symplectic vector spaces for $\rho(L_i)$ and $c(L_i)$, respectively. Since $c(L_i)$ is a symplectic vector space, $p_1 \times p_2(S') \circ p_2 \times p_3(T')$ is coisotropic in $c(L_1) \times c(L_3)^-$ ([**14**], [**1**]). We put $P_i := \{(x, p_i(x)) | x \in \rho(L_i)\}$. Then $p_1 \times p_2(S') \circ p_2 \times p_3(T')$ is equal to

$$\begin{aligned} P_1^{-1} \times P_2^{-1}(S') \circ P_2^{-1} \times P_3^{-1}(T') &= P_1^{-1} \circ S' \circ P_2 \circ P_2^{-1} \circ T' \circ P_3 \\ &= P_1^{-1} \times P_3^{-1}(S' \circ P_2 \circ P_2^{-1} \circ T'). \end{aligned}$$

Hence $p_1 \times p_3(S' \circ P_2 \circ P_2^{-1} \circ T')$ is coisotropic in $c(L_1) \times c(L_3)^-$. By Corollary 3.2, $S' \circ P_2^{-1} \circ P_2 \circ T'$ is coisotropic in $\rho(L_1) \times (\rho(L_3))^-$, and again by Corollary 3.2 it is coisotropic in $V_1 \times V_3^-$ too. By the definition of D, we obtain $P_2 \circ P_2^{-1} = D_2$. Hence we have that $S' \circ D_2 \circ T'$ is coisotropic in $V_1 \times V_3^-$. Since $S' \subset S$ and $T' \subset T$, we have $S' \circ D_2 \circ T' \subset S \circ D_2 \circ T$. By Corollary 3.6, we obtain that $S \circ D_2 \circ T$ is coisotropic.

Next, for the isotropic case we show directly that $S \circ D_2 \circ T$ is isotropic by the diagram (20). Since S and T are isotropic, by Proposition 4.4 we have

$$D_1 \circ S \circ L_2 \subset L_1 \circ S^\dagger \circ D_2^\dagger, \quad D_2 \circ T \circ L_3 \subset L_2 \circ T^\dagger \circ D_3^\dagger. \tag{21}$$

Then we have

$$D_1 \circ S \circ D_2 \circ T \circ L_3 \subset D_1 \circ S \circ L_2 \circ T^\dagger \circ D_3^\dagger \subset L_1 \circ S^\dagger \circ D_2^\dagger \circ T^\dagger \circ D_3^\dagger.$$

By Lemma 2.3(f), $S^\dagger \circ D_2^\dagger \circ T^\dagger = (S \circ D_2 \circ T)^\dagger$. Hence we obtain

$$D_1 \circ (S \circ D_2 \circ T) \circ L_3 \subset L_1 \circ (S \circ D_2 \circ T)^\dagger \circ D_3^\dagger,$$

which gives the desired result by Proposition 4.4. □

Let $[S]$ be the equivalence class of S by the equivalence relation of Definition 3.9. Proposition 3.11 yields

$$[S] \ni S' \iff D_1 \circ S \circ D_2 = D_1 \circ S' \circ D_2.$$

By Theorem 0 we obtain the following:

Theorem 4.6. *Let V_1, V_2 and V_3 be Dirac vector spaces and let $S : V_1 \leftarrow V_2$ and $T : V_2 \leftarrow V_3$ be lagrangian subspaces. Let $[S]$ and $[T]$ denote equivalence classes of S and T respectively. Then the composition $[S]\bar{\circ}[T]$ given by $[S]\bar{\circ}[T] := [S \circ D_2 \circ T]$ is well-defined. The product $\bar{\circ}$ among the equivalence classes gives a category on Dirac vector spaces such that identity morphism is $[D]$.*

Proof. We show the product $[S]\bar{\circ}[T] := [S \circ D_2 \circ T]$ is well-defined. Let $S \sim S'$ and $T \sim T'$. Then by Proposition 3.11 we have

$$D_1 \circ S \circ D_2 = D_1 \circ S' \circ D_2, \quad D_2 \circ T \circ D_3 = D_2 \circ T' \circ D_3.$$

Since $D = D \circ D$, we have

$$\begin{aligned} D_1 \circ S \circ D_2 \circ T \circ D_3 &= D_1 \circ S \circ D_2 \circ D_2 \circ T \circ D_3 \\ &= D_1 \circ S' \circ D_2 \circ D_2 \circ T' \circ D_3 \\ &= D_1 \circ S' \circ D_2 \circ T' \circ D_3. \end{aligned}$$

By definition, we obtain $S \circ D_2 \circ T \sim S' \circ D_2 \circ T'$. Thus $[S]\bar{\circ}[T] = [S']\bar{\circ}[T']$. Since $D_1 \circ S \circ D_2 = D_1 \circ S \circ D_2 \circ D_2$, we can easily see $[S]\bar{\circ}[D_2] = [S]$ and $[D_2]\bar{\circ}[T] = [T]$. $\square$

Remark 4.7. We obtain categories [Dir] and Sym, where objects, morphisms of [Dir] (resp. Sym) are Dirac vector spaces and equivalence classes $[S]$ of lagrangian subspaces (resp. symplectic vector spaces and canonical relations) respectively. Recall Proposition 3.10, we obtain a full and faithful functor $\mathfrak{C} : [\mathrm{Dir}] \to \mathrm{Sym}$.

Let $[S]$, $[T]$ be equivalence classes in Theorem 4.6. We define $\mathfrak{C}$ by $\mathfrak{C}([S]) := p_1 \times p_2(S \cap \rho(L_1 \times L_2))$ and $\mathfrak{C}(V_i) = c(L_i)$, i.e., the symplectic core, where $p_i : \rho(L_i) \to c(L_i)$ is the canonical projection. We can easily show that $\mathfrak{C}([S]\bar{\circ}[T]) = \mathfrak{C}([S]) \circ \mathfrak{C}([T])$ and $\mathfrak{C}([D_i])$ is a diagonal subspace of $c(L_i) \times c(L_i)$.

Theorem 0 implies that the space D naturally appears when we consider the composition of lagrangian subspaces and also shows that the lagrangian subspaces are closed under $\circ D \circ$-product. In general, we do not know the characterization of class of lagrangian subspaces which are closed under the $\circ$-product. However, if the intermediate space is a "good" space, lagrangian subspaces are closed under the $\circ$-product as it will be seen below.

In Theorem 0, if (V_2, L_2) is a Dirac vector space given by a presymplectic (resp. Poisson) structure, then it holds that $\delta_{V_2} \subset D_2$ (resp. $\delta_{V_2} \supset D_2$), where δ_{V_2} is the diagonal subspace of $V_2 \times V_2$. Hence we have the following by Corollary 3.6 and Theorem 0.

Corollary 4.8. *Let M_2 be a symplectic (resp. presymplectic, resp. Poisson) manifold and M_1, M_3 be any Dirac manifolds. Suppose $S : M_1 \leftarrow M_2$ and $T : M_2 \leftarrow M_3$ be lagrangian (resp. isotropic, resp. coisotropic) such that (S, T) is a very clean pair* ([**14**])*. Then $S \circ T : M_1 \leftarrow M_3$ is a lagrangian (resp. isotropic, resp. coisotropic).*

In the next section we will give a certain class of lagrangian submanifolds which are closed under $\circ$-product without using D.

4.3. Dirac relations and Dirac maps.

In this section, we give definitions of Dirac relations and Dirac maps which are essential to main results (A), (B). We recall the diagram (20) and Corollary 4.5. Let (V_1, π_1) and (V_2, π_2) be any Poisson vector spaces and let $S \subset V_1 \times V_2$ be a subspace. Since $L_{\pi_i} \circ (L_{\pi_i})^{-1} \subset \delta_{V_i}$, Proposition 4.4 and Lemma 2.6(d) yield that the following conditions (22) and (23) are mutually equivalent and gives a criterion of that S is a Poisson relation ([**14**]).

$$L_1 \circ S^\dagger \circ D_2^\dagger \subset D_1 \circ S \circ L_2, \tag{22}$$

$$L_1 \circ S^\dagger \subset S \circ L_2, \tag{23}$$

where $L_i = L_{\pi_i}$. However, these conditions are not equivalent in our Dirac case, but (23) induces (22) (cf. Proposition 4.4 and Corollary 4.5). The condition (22) implies that S is coisotropic, and the condition (23) implies the commutativity of the following diagram:

$$\begin{tikzcd} V_1^* \arrow[d, "L_1"'] & V_2^* \arrow[l, "S^\dagger"'] \arrow[d, "L_2"] \\ V_1 & V_2 \arrow[l, "S"] \end{tikzcd} .$$

The condition (23) is useful for the construction of our calculus.

Definition 4.9.

(1) Let $(V_1, L_1), (V_2, L_2)$ be Dirac vector spaces and let S be a subspace of $V_1 \times V_2$. S is called an isotropic (resp. coisotropic) Dirac relation when

$$L_1 \circ S^\dagger \subset S \circ L_2 \tag{24}$$

$$\text{(resp. } L_1 \circ S^\dagger \supset S \circ L_2). \tag{25}$$

In particular, S is called a canonical relation when it is both an isotropic and a coisotropic Dirac relation.

(2) Let (M_1, L_1), (M_2, L_2) be Dirac manifolds and let N be a submanifold of $M_1 \times M_2$. N is called an isotopic (resp. coisotropic) Dirac relation when every tangent space satisfies (25) (resp. (24)). In particular, N is called a canonical relation when it is an isotropic and a coisotropic Dirac relation, i.e.,

$$L_1 \circ TN^{\dagger} = TN \circ L_2. \tag{26}$$

Remark 4.10. From Corollary 4.5, an isotropic (coisotropic) Dirac relation is an isotropic (resp. a coisotropic) subspace. Thus a canonical relation is a lagrangian subspace. However it is to be remarked that in Dirac case, a lagrangian subspace of $V_1 \times V_2^-$ is not necessary a canonical relation since (22) and (23) are not equivalent.

We can see a geometrical meaning of an isotropic (resp. a coisotropic) Dirac relation in Corollary 4.13 below.

Theorem 0 means that the compositions of lagrangian subspaces are "closed" under the $\circ D \circ$-product. However, for this product $\circ D \circ$, lagrangian subspaces have no identity element in general and hence this system does not form a category. However such a weak point is covered by the next theorem where the Dirac relations are used:

Theorem 4.11. *Let V_1, V_2 and V_3 be Dirac vector spaces and let $S \subset V_1 \times V_2$, $T \subset V_2 \times V_3$ be canonical* (*resp. isotropic Dirac, resp. coisotropic Dirac*) *relations. Then $S \circ T \subset V_1 \times V_3$ is a canonical* (*resp. isotropic Dirac, resp. coisotropic Dirac*) *relation.*

Proof. By the assumption and Definition 4.9, we have

$$L_1 \circ S^{\dagger} = S \circ L_2, \ L_2 \circ T^{\dagger} = T \circ L_3. \tag{27}$$

Then by Lemma 2.3(f) we have

$$L_1 \circ (S \circ T)^{\dagger} = L_1 \circ S^{\dagger} \circ T^{\dagger} = S \circ L_2 \circ T^{\dagger} = S \circ T \circ L_3,$$

which shows that $S \circ T$ is a canonical relation. □

In a similar way, we obtain a more general result.

Proposition 4.12. *Let V_1, V_2 and V_3 be Dirac vector spaces, and let $S \subset V_1 \times V_2$ be a canonical relation* (*resp. a lagrangian subspace*) *and $T : V_2 \leftarrow V_3$ be a lagrangian subspace* (*resp. canonical relation*). *Then $S \circ T : V_1 \leftarrow V_3$ is a lagrangian subspace.*

Proof. By the assumption, we have

$$L_1 \circ S^{\dagger} = S \circ L_2, \tag{28}$$

$$L_2 \circ T^{\dagger} \circ D_3^{\dagger} = D_2 \circ T \circ L_3. \tag{29}$$

It follows that

$$L_1 \circ S^{\dagger} \circ T^{\dagger} \circ D_3^{\dagger} = S \circ L_2 \circ T^{\dagger} \circ D_3^{\dagger} = S \circ D_2 \circ T \circ L_3. \tag{30}$$

Apply L_2^{-1} from the right to (28) and from the left to (29). Then we have $L_1 \circ S^{\dagger} \circ L_2^{-1} = S \circ D_2$ and $D_2^{\dagger} \circ T^{\dagger} \circ D_3^{\dagger} = L_2^{-1} \circ T \circ L_3$ respectively. The equation (30) then becomes

$$L_1 \circ S^\dagger \circ T^\dagger \circ D_3^\dagger = L_1 \circ S^\dagger \circ L_2^{-1} \circ T \circ L_3 = L_1 \circ S^\dagger \circ D_2^\dagger \circ T^\dagger \circ D_3^\dagger.$$

This implies $D_1^\dagger \circ S^\dagger \circ T^\dagger \circ D_3^\dagger = D_1^\dagger \circ S^\dagger \circ D_2^\dagger \circ T^\dagger \circ D_3^\dagger$. Applying † to the both sides, we have

$$D_1 \circ S \circ T \circ D_3 = D_1 \circ S \circ D_2 \circ T \circ D_3. \tag{31}$$

Here $S \circ D_2 \circ T$ is lagrangian. By Proposition 4.2, the right hand side above is lagrangian, and thus the left hand side is lagrangian also. Again from Proposition 4.2, we obtain $S \circ T$ is lagrangian. □

The condition (31) in Proposition 4.12 implies that

Corollary 4.13. *If $S \subset V_1 \times V_2$ is a canonical relation and $T : V_2 \leftarrow V_3$ is lagrangian then $S \circ T \sim S \circ D_2 \circ T$, i.e, $[S]\bar{\circ}[T] = [S \circ T]$.*

Remark 4.14. For a vector space case, we have easily an example of canonical relations. If $S : V_1 \leftarrow V_2$ is a lagrangian subspace then (E-3) and Lemma 2.6(d) show that $D_1 \circ S \circ D_2$ is a canonical relation.

Now we have (B) in the introduction:

Theorem B. *Let $S \subset M_1 \times M_2$ be a canonical relation (resp. a lagrangian submanifold) and $T : M_2 \leftarrow M_3$ be a lagrangian submanifold (resp. canonical relation) respectively. If (S, T) is a very clean pair then $S \circ T : M_1 \leftarrow M_3$ is a lagrangian submanifold. In particular if S, T are canonical relations then $S \circ T$ is a canonical relation.*

Remark 4.15. Proposition 4.12 and Theorem B are valid for isotropic, coisotropic cases: if S is an isotropic Dirac relation (resp. isotropic submanifold) and T is an isotropic submanifold (resp. isotropic Dirac relation) of very clean pair then $S \circ T$ is an isotropic submanifold. In particular if S, T are isotropic Dirac relations of very clean pair then $S \circ T$ is an isotropic Dirac relation. In a coisotropic case, we obtain the similar results.

Remark 4.16. From Theorem 4.11 and Remark 4.15, we obtain that the set of canonical relations, the set of isotropic Dirac relations and the set of coisotropic Dirac relations are closed under the $\circ$-product respectively. The identity morphism is the diagonal set for these objects. Thus we have three categories of Dirac manifolds whose morphisms are canonical relations, isotropic Dirac relations and coisotropic Dirac relations respectively.

We recall Remark 4.7. Here, in term of category theory, we give again the following remark:

Remark 4.17. We obtain a category Dir whose objects and morphisms are Dirac vector spaces and canonical relations respectively. We notice the identity morphisms of Dir are diagonal subspaces δ_V for Dirac vector spaces V. Recalling Corollary 4.13, we have a functor $[\mathfrak{C}] : \mathrm{Dir} \to [\mathrm{Dir}]$ such that $[\mathfrak{C}](V) := V$, $[\mathfrak{C}](S) := [S]$ (see a diagram (32) below). In fact, by Corollary 4.13, we obtain that if $[\mathfrak{C}](S) = [\mathfrak{C}](T)$ then $[\mathfrak{C}](S \circ T) = [\mathfrak{C}](S)\bar{\circ}[\mathfrak{C}](T)$, and further $[\mathfrak{C}](\delta_V) = [D]$.

$$\mathrm{Dir} \ni S \xrightarrow{[\mathfrak{C}]} [\mathrm{Dir}] \ni [S] \xrightarrow{\mathfrak{C}} \mathrm{Sym} \ni p_1 \times p_2(S \cap \rho(L_1 \times L_2)). \tag{32}$$

Now we define the notion of Dirac map. First we consider a Poisson map. Let (M_1, π_1), (M_2, π_2) be Poisson manifolds. A map $f : M_1 \to M_2$ is a Poisson map if and only if the following diagram commutes

$$\begin{tikzcd} T^*M_1 \arrow[d, "\tilde{\pi}_1"'] & T^*M_2 \arrow[l, "(Tf)^*"'] \arrow[d, "\tilde{\pi}_2"'] \\ TM_1 \arrow[r, "Tf"] & TM_2. \end{tikzcd}$$

If we rewrite this diagram in terms of the graph of f, we obtain the identity similar to (34) below. For presymplectic manifolds and a presymplectic map, we also obtain the identity similar to (33) below. We generalize these identities directly to Dirac manifolds and we give the following definitions.

Definition 4.18. Let (M_1, L_1), (M_2, L_2) be any Dirac manifolds and let $f : M_1 \to M_2$ be a C^∞ map. We call f an isotropic (resp. coisotropic) Dirac map if the graph of f satisfies the following condition (33) (resp. (34)) at each point. In particular, we call f a Dirac map if f is an isotropic and a coisotropic Dirac map.

$$L_1 = TF \circ L_2 \circ (TF^\dagger)^{-1} \tag{33}$$

$$(\text{resp. } L_2 = TF^{-1} \circ L_1 \circ TF^\dagger), \tag{34}$$

where $F := \{(x, f(x)) | x \in M_1\}$ and TF is the tangent bundle of F. We give analogous definitions for the case of vector spaces.

Similarly to the property (a) of introduction, we can characterize the isotropic Dirac maps and the coisotropic Dirac maps by the graphs as follows.

Proposition 4.19. *Let (V_1, L_1) and (V_2, L_2) be any Dirac vector spaces and $f : V_1 \to V_2$ be a linear map. We put $F := \{(x, f(x)) | x \in V_1\}$. Then the following conditions are equivalent*:

(G-1) $F \circ L_2 \subset L_1 \circ F^\dagger$, *i.e., F is an isotropic Dirac relation.*
(G-2) $L_1 = F \circ L_2 \circ (F^\dagger)^{-1}$, *i.e., f is an isotropic Dirac map.*

Similarly, the following conditions are equivalent.

(H-1) $L_1 \circ F^\dagger \subset F \circ L_2$, *i.e., F is a coisotropic Dirac relation.*
(H-2) $L_2 = F^{-1} \circ L_1 \circ F^\dagger$, *i.e., f is a coisotropic Dirac map.*

Proof. First we show the equivalence of (H-1) and (H-2). We assume (H-1). Then applying F^{-1} to (H-1), we have $F^{-1} \circ L_1 \circ F^\dagger \subset F^{-1} \circ F \circ L_2$. Since $F^{-1} \circ F \subset \delta_{V_2}$, we have $F^{-1} \circ L_1 \circ F^\dagger \subset L_2$. From Lemma 2.7, $\dim(F^{-1} \circ L_1 \circ F^\dagger) = \dim L_2$ and then we obtain (H-2). Conversely, we assume (H-2). We apply F to (H-2) and we have $F \circ L_2 = F \circ F^{-1} \circ L_1 \circ F^\dagger$. Since $\delta_{V_1} \subset F \circ F^{-1}$, we have (H-1). The equivalence of (G-1) and (G-2) is proved by a similar manner. □

From Proposition 4.19, we have (A) in the introduction:

THEOREM A. *Let f be a map between Dirac manifolds. Then f is a Dirac* (*resp. isotropic Dirac, resp. coisotropic Dirac*) *map if and only if its graph is a canonical* (*resp. isotropic Dirac, resp. coisotropic Dirac*) *relation.*

REMARK 4.20. The Definition 4.18 is essentially the same as the definition in Bursztyn-Radko [**2**], Liu, Weinstein and Xu [**9**]. In [**2**], a Dirac map is defined as a map between Dirac vector spaces which induces the correspondence between two Dirac structures given as follows:

Let (V, L_v) and (W, L_w) be any Dirac vector spaces and $\phi : V \to W$ be a linear map. Consider the subspaces

$$\mathfrak{F}\phi := \{(\phi(x), \eta, x, \phi^*\eta) | x \in V, \eta \in W^*\}, \quad \mathfrak{B}\phi := \{(x, \phi^*\eta, \phi(x), \eta) | x \in V, \eta \in W^*\}.$$

Then we have the two maps given by composition $\mathfrak{F}\phi(L_v)$ and $\mathfrak{B}\phi(L_w)$ such that

$$L_v \mapsto \mathfrak{F}\phi(L_v), \ L_w \mapsto \mathfrak{B}\phi(L_w).$$

If $\mathfrak{F}\phi(L_v) = L_w$ (resp. $\mathfrak{B}\phi(L_w) = L_v$), ϕ is called a forward (resp. backward) Dirac map. Since $\mathfrak{F}\phi(L_v) = \Phi^{-1} \circ L_v \circ \Phi^\dagger$ and $\mathfrak{B}\phi(L_w) = \Phi \circ L_w \circ (\Phi^\dagger)^{-1}$, our coisotropic (resp. isotropic) Dirac map is the same as the forward (resp. backward) Dirac map. See [**2**] and [**4**] for the fundamental properties of Dirac maps.

5. Examples.

In this last section we give several examples of lagrangian submanifolds, canonical relations and Dirac maps. These objects are naturally seen in Poisson geometry.

EXAMPLE 5.1. It is well known that the quotient space $V/\ker \tilde{\Omega}_L$ of a Dirac vector space (V, L) has a Poisson structure π_L induced from L ([**5**]). Here we obtain two relations between L and L_{π_L}, which is similar to (4).

$$L_{\pi_L} = Pr^{-1} \circ L \circ Pr^\dagger, \quad L = Pr \circ L_{\pi_L} \circ (Pr^\dagger)^{-1},$$

where Pr is the graph of the canonical projection $pr : V \to V/\ker \tilde{\Omega}_L$. Thus the projection pr is a Dirac map. In general, we consider a quotient space V/W for an arbitrary subspace $W \subset V$. By Lemma 2.7, we define a Dirac structure on V/W : $L^w := P_w^{-1} \circ L \circ P_w^\dagger$, where $pw : V \to V/W$ is the canonical projection and P_w is the graph of the map. From the definition it is easy to see that the map pw is a coisotropic Dirac map. If $W \subset \ker \tilde{\Omega}_L$ then pw is a Dirac map. In fact, since $P_w \circ P_w^{-1} = \delta_V + (W, 0)$, when the condition is satisfied we obtain $P_w \circ L^w = P_w \circ P_w^{-1} \circ L \circ P_w^\dagger = L \circ P_w^\dagger$. Conversely if pw is a Dirac map then the condition $W \subset \ker \tilde{\Omega}_L$ is satisfied, since $(W, 0) \subset P_w$.

EXAMPLE 5.2. Let M be a foliated manifold with involutive subbundle H. On M, a Dirac structure is defined by $L := H \oplus H^\perp$. Then every submanifold of M is lagrangian.

Hence such a Dirac structure can be regarded as the "zero" Dirac structure. Similarly, we have the notion of the zero Poisson structure or the zero presymplectic structure.

Let (V_i, L_i), $(i = 1, 2)$ be any Dirac vector spaces such that $L_i = H_i \oplus H_i^{\perp}$. Here H_i be a subspace of V_i. We consider Dirac maps between these Dirac vector spaces. Let $f : V_1 \to V_2$ be a linear map. From the assumption, we obtain $L_1 \circ F^{\dagger} = H_1 \oplus (F^{\dagger})^{-1}(H_1^{\perp})$, $F \circ L_2 = F(H_2) \oplus H_2^{\perp}$, where F is the graph of the map f. From Proposition 4.19, we obtain the following:

(I) the map f is an isotropic Dirac map if and only if $H_1 \supset F(H_2)$ and $(F^{\dagger})^{-1}(H_1^{\perp}) \supset H_2^{\perp}$. These two conditions are simplified by $H_1 = F(H_2)$, or equivalently $H_1 = f^{-1}(H_2)$.
(C) the map f is an coisotropic Dirac map if and only if $H_1 \subset F(H_2)$ and $(F^{\dagger})^{-1}(H_1^{\perp}) \subset H_2^{\perp}$. In a similar way, these two conditions are simplified by $F^{-1}(H_1) = H_2$, or equivalently $f(H_1) = H_2$.

Thus we obtain the following: Let f be a diffeomorphism between foliated manifolds. If, by the map f, a foliation is transformed to the other foliation then the map is a Dirac map.

Example 5.3. Let (M, L) be a Dirac manifold with an isotropic submanifold N, and let B be a closed 2-form. Assume that N is isotropic under the 2-form B, i.e., $\tilde{B}(TN) \subset TN^{\perp}$. By (A-3) of Proposition 3.4, we have $\rho(TN \oplus TN^{\perp} \cap L) = TN \cap \rho(L)$. We remember the gauge transformation $\tau_B : (x, a) \mapsto (x, a + \tilde{B}(x))$ on the bundle $TM \oplus T^*M$ (see [**10**]). From the assumption, we obtain $\tau_B(TN \oplus TN^{\perp}) = TN \oplus TN^{\perp}$. Since $\tau_B^{-1} = \tau_{-B}$ and $\rho \circ \tau_{\pm B} = \rho$, we have

$$\begin{aligned}\rho(TN \oplus TN^{\perp} \cap L) &= \rho \circ \tau_{-B} \circ \tau_B(TN \oplus TN^{\perp} \cap L)\\ &= \rho \circ \tau_{-B}(TN \oplus TN^{\perp} \cap \tau_B L)\\ &= \rho(TN \oplus TN^{\perp} \cap \tau_B L) = TN \cap \rho(\tau_B L).\end{aligned}$$

It is well-known that $\tau_B L$ is also a Dirac structure. Thus N is an isotropic submanifold of the Dirac manifold $(M, \tau_B L)$.

Let W be a subspace of a Dirac vector space (V, L). Then W becomes a Dirac vector space with the induced Dirac structure isomorphic to $(L \cap (W \oplus V^*))/(L \cap (0 \oplus W^{\perp}))$ ([**5**]).

Example 5.4. Let W be a subspace of a Dirac vector space (V, L). Then $L_W := I \circ L \circ (I^{\dagger})^{-1}$ gives a Dirac structure on W induced by L, where $I := \{(x, i(x)) | x \in W\}$. An inclusion map $i : W \to V$ gives an isotropic Dirac map. We can easily see that

$$L_W = I \circ L \circ (I^{\dagger})^{-1} \cong \frac{L \cap (W \oplus V^*)}{L \cap (0 \oplus W^{\perp})}. \tag{35}$$

Especially, the map i is a Dirac map if and only if a condition $\rho(L) \subset W$ is satisfied. Since $\ker i^* = W^{\perp}$, we have $(I^{\dagger})^{-1} \circ I^{\dagger} = \delta_{V^*} + (W^{\perp}, 0)$. The condition $\rho(L) \subset W$ is equivalent to $(0, W^{\perp}) \subset L$. We assume the condition $\rho(L) \subset W$. We have $L_W \circ I^{\dagger} = I \circ L \circ (I^{\dagger})^{-1} \circ I^{\dagger}$. Since $(0, W^{\perp}) \subset I \circ L$, we obtain $I \circ L \circ (I^{\dagger})^{-1} \circ I^{\dagger} = I \circ L$. This implies that i is a Dirac map. The converse is easily checked. When the condition satisfied, we can see $L_W \cong L/W^{\perp}$ by (35).

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