J. Math. Soc. Japan
Vol. 57, No. 3, 2005

On harmonic function spaces

By Stevo STEVIĆ

(Received Jul. 22, 2003)
(Revised Oct. 7, 2004)

Abstract. In this paper we investigate a-Bloch, Hardy, Bergman, BMO_p and Dirichlet spaces of harmonic functions on the open unit ball in $\boldsymbol{R}^n$, and the boundedness of the Hardy-Littlewood operator on these spaces.

1. Introduction.

Throughout this paper G is a domain in the Euclidean space $\boldsymbol{R}^n, n \geq 1$, $B(a,r) = \{x \in \boldsymbol{R}^n \mid |x-a| < r\}$ denotes the open ball centered at $a \in \boldsymbol{R}^n$ of radius $r > 0$, where $|x|$ denotes the norm of $x \in \boldsymbol{R}^n$ and B is the open unit ball in $\boldsymbol{R}^n$. $S = \partial B = \{x \in \boldsymbol{R}^n \mid |x| = 1\}$ is the boundary of B.

Let dV denote the Lebesgue measure on $\boldsymbol{R}^n$, v_n the volume of B, $d\sigma$ the surface measure on S, σ_n the surface area of S, dV_N the normalized Lebesgue measure on B, $d\sigma_N$ the normalized surface measure on S. Let $\mathscr{H}(B)$ denote the set of complex valued harmonic functions on B.

Let $\boldsymbol{Z}_n^+$ be the set of all ordered n-tuples of nonnegative integers, and for each $\alpha = (\alpha_1, ..., \alpha_n) \in \boldsymbol{Z}_n^+$ let

$$|\alpha| = \alpha_1 + \cdots + \alpha_n, \qquad \alpha! = \alpha_1! \cdots \alpha_n!.$$

For a harmonic function u we denote

$$D^\alpha u = \frac{\partial^{|\alpha|} u}{\partial {x_1}^{\alpha_1} \cdots \partial {x_n}^{\alpha_n}}.$$

Given a function u harmonic on a domain G, and a positive integer m, the gradient of u of order m, $\nabla^m u$, can be defined to be a vector valued function whose components are the derivatives of u of order $|\alpha| = m$, arranged in some fixed order. The norm of $\nabla^m u$ is then uniquely defined by the relation

$$|\nabla^m u(x)| = \left(m! \sum_{\alpha \in \boldsymbol{Z}_n^+, |\alpha| = m} \frac{|D^\alpha u(x)|^2}{\alpha_1! \cdots \alpha_n!} \right)^{1/2}.$$

In particular $|\nabla^1 u| = |\nabla u|$, where ∇u is the usual gradient of u.

2000 *Mathematics Subject Classification.* 31B05.

Key Words and Phrases. harmonic function, weighted Hardy-Littlewood operator, Dirichlet type space, boundedness, Bloch space.

For $p > 0$, $\mathscr{H}^p(B)$ denote the set of harmonic functions u on B such that

$$\|u\|_{\mathscr{H}^p(B)} = \sup_{0<r<1} \left(\int_S |u(r\zeta)|^p d\sigma_N(\zeta) \right)^{1/p} < +\infty.$$

Elements of $\mathscr{H}^p(B)$ theory can be found in [**3**, Chapter VI]. For elements of complex H^p theory see, for example, [**5**].

Let $a > 0$. A function $f \in C^1(B)$ is said to be an a-Bloch function if

$$\|f\|_{\mathscr{B}^a} = \sup_{x\in B}(1-|x|)^a |\nabla f(x)| < +\infty.$$

The space of a-Bloch functions is denoted by $\mathscr{B}^a(B) = \mathscr{B}^a$. If $a = 1$, $\mathscr{B}^a$ just becomes the Bloch space $\mathscr{B}$. Let $\mathscr{H}_{\mathscr{B}^a}(B)$ denote the space which consists of all harmonic a-Bloch functions on the unit ball, i.e., $\mathscr{H}(B) \cap \mathscr{B}^a(B)$. If $a = 1$, we obtain the harmonic Bloch space $\mathscr{H}_{\mathscr{B}}(B)$. Basic results for analytic Bloch functions on the unit disc can be found in [**2**] and for analytic Bloch functions in several variables in [**33**]. For hyperharmonic Bloch functions see [**25**].

Let $p > 0$. A Borel function f, locally integrable in the unit ball B, is said to be a $BMO_p(B)$ function if

$$\|f\|_{BMO_p} = \sup_{B(a,r)\subset B} \left(\frac{1}{V(B(a,r))} \int_{B(a,r)} |f(x) - f_{B(a,r)}|^p dV(x) \right)^{1/p} < +\infty$$

where the supremum is taken over all balls $B(a,r)$ with $\overline{B}(a,r) \subset B$, and $f_{B(a,r)}$ is the mean value of f over $B(a,r)$.

Let $\mathscr{H}_{BMO_p}(B) = \mathscr{H}(B) \cap BMO_p(B)$.

In [**18**] for $p \geq 1$, Muramoto proved that $\mathscr{H}_{\mathscr{B}}(B)$ is isomorphic to $\mathscr{H}(B) \cap BMO_p(B)$ as Banach spaces. In fact he proved the following theorem:

THEOREM A. *Let $p \geq 1$. Then there is a positive constant $c(p,n)$, depending on p and n, such that for every $u \in \mathscr{H}(B)$*

$$\frac{1}{c(p,n)} \|u\|_{BMO_p} \leq \|u\|_{\mathscr{H},n} \leq c(p,n) \|u\|_{BMO_p}$$

where

$$\|u\|_{\mathscr{H},n} = \sup_{x\in B} \frac{1}{2}(1-|x|^2)|\nabla u(x)|.$$

Note that the norms $\|u\|_{\mathscr{H},n}$ and $\|u\|_{\mathscr{B}}$ are equivalent. In the case $n = 2$, this result was essentially obtained by Coifman, Rochberg and Weiss [**4**] and Gotoh [**9**]. In [**20**, Theorems 2 and 3] we proved that Muramoto's result is true also for $p \in (0,1)$. Moreover, by a slight modification of the proof of Theorem 1 in [**20**] we can prove that

$\mathscr{H}_{\mathscr{B}^a}(B) \subset \mathscr{H}_{BMO_p}(B)$ if $a \in (0,1]$ and $p > 0$, or if $1 < a < 1 + \frac{1}{p}$.

This Muramoto's paper inspired us to calculate exactly BMO_p norm for harmonic functions, which is the theme of [**20**]. In the proof of the main result in [**20**], we essentially proved a generalization of the Hardy-Stein identity (see, for example, [**11**, p. 42]). Some further applications of the identity can be found in [**24**] and [**30**]. Among others in [**24**] we proved some results which are closely related to Yamashita's results for analytic functions on the unit disk [**36**], as the main result in [**30**] generalizes the main Yamashita's result in [**34**]. A generalization of the identity on the unit disk can be found in [**17**]. A similar formula for analytic functions on the unit ball in $\boldsymbol{C}^n$ can be found in [**32**].

Let $\omega(r)$, $0 < r < 1$, be a positive weight function which is integrable on $(0,1)$. We extend ω on B by setting $\omega(x) = \omega(|x|)$. We may assume that our weights are normalized so that $\int_B \omega(x) dV(x) = 1$.

For $0 < p < \infty$ the weighted Bergman space $b^p_\omega(B)$ is the space of all harmonic functions u on B such that

$$\|u\|_{\omega,p} = \left(\int_B |u(x)|^p \omega(x) dV(x) \right)^{1/p} < +\infty.$$

If $\omega(r) = (1-r)^\alpha, \alpha > -1$, we denote the norm by $\|u\|_{p,\alpha}$ and the corresponding space by $b^p_\alpha(B)$.

It is easy to see that weights may be modified on intervals $[0,\sigma]$, with $\sigma < 1$ without changing the Bergman space, in fact, the corresponding norms are equivalent. Recently there has been a great interest in studying the weighted Bergman spaces of analytic or harmonic functions with weights other than the classical $\omega(r) = (1-r)^\alpha, \alpha > -1$, see, for example, [**1**], [**14**], [**15**], [**16**], [**19**], [**22**], [**23**], [**26**], [**27**], [**28**] and the references therein.

For $\alpha \in (-1, \infty)$ let $\mathscr{D}^p_\alpha(B) = \mathscr{D}^p_\alpha$ be the class of all harmonic functions u on the unit ball obeying

$$\|u\|^p_{\mathscr{D}^p_\alpha} = |u(0)|^p + \int_B |\nabla u(x)|^p (1-|x|)^\alpha dV(x) < \infty.$$

We say that a locally integrable function f on B possesses $HL-$property, with a constant $c > 0$ if

$$f(a) \le \frac{c}{r^n} \int_{B(a,r)} f(x) dV(x) \quad \text{whenever } \overline{B}(a,r) \subset B.$$

For example, every subharmonic function ([**12**]) possesses $HL-$property when $c = 1/v_n$. In [**10**] Hardy and Littlewood proved that $|u|^p$, $p > 0$, $n = 2$, also possesses $HL-$property whenever u is a harmonic function in B. In the case $n \ge 3$ a generalization was made by Fefferman and Stein [**6**]. Other classes of functions that possess $HL-$property can be found in [**21**], [**29**], [**31**].

In section 2 we prove some auxiliary results which we apply in the sections which follows.

In section 3 we consider the boundedness of the weighted Hardy-Littlewood operator

$$L_g(f)(x) = \int_0^1 f(tx)g(t)dt,$$

on the spaces $\mathscr{H}_{\mathscr{B}^\alpha}(B), \mathscr{H}_{BMO_p}(B), \mathscr{H}^p(B), b^p_\omega(B)$ and $\mathscr{D}^p_\alpha(B)$.

In section 4 we generalize a result of Flett [**7**] and give a short proof of the result. Also we give a new equivalent condition for a harmonic function to be a Bloch function.

In section 5 we improve a local estimate given in [**24**].

In the last section we consider the relationship between the functions which belong to $\mathscr{H}^p(B)$ and $\mathscr{D}^p_\alpha(B)$.

2. Auxiliary results.

In this section we prove some auxiliary results that we use in the sections which follows. The first one is a technical lemma.

For $\alpha \in (-1, \infty)$ and $p > 0$ let $\mathscr{L}^p_\alpha(B) = \mathscr{L}^p_\alpha$ be the class of all measurable functions f obeying

$$\|f\|^p_{\mathscr{L}^p_\alpha} = \int_B |f(x)|^p (1-|x|)^\alpha dV(x) < \infty.$$

Using Fubini's theorem, we can easily show the following lemma:

LEMMA 1. *Let $\alpha \in (0, \infty)$. Suppose that f is a nonnegative measurable function on B. Then*

$$\int_B f(x)(1-|x|)^\alpha dV(x) = \alpha \int_0^1 \left(\int_{rB} f(x)dV(x) \right)(1-r)^{\alpha-1}dr.$$

COROLLARY 1. *Let $p, \alpha \in (0, \infty)$ and $f \in \mathscr{L}^p_\alpha(B)$. Then*

$$\lim_{r \to 1} (1-r)^\alpha \int_{rB} |f(x)|^p dV(x) = 0.$$

PROOF. By Lemma 1 we have

$$\int_0^1 \left(\int_{rB} |f(x)|^p dV(x) \right)(1-r)^{\alpha-1}dr < \infty.$$

Hence, by Cauchy's criterion

$$\lim_{\rho \to 1} \int_\rho^1 \left(\int_{rB} |f(x)|^p dV(x) \right)(1-r)^{\alpha-1}dr = 0.$$

Since the function

$$\int_{rB} |f(x)|^p dV(x)$$

is nondecreasing in r, we obtain

$$\int_{\rho B} |f(x)|^p dV(x) \int_\rho^1 (1-r)^{\alpha-1} dr \le \int_\rho^1 \Big(\int_{rB} |f(x)|^p dV(x) \Big) (1-r)^{\alpha-1} dr,$$

from which the result follows. □

Corollary 2. *Let f be a measurable function on B and $p, \alpha \in (0,\infty)$. Then the following equivalence holds*

$$\|f\|_{\mathscr{L}^p_\alpha} < \infty \Leftrightarrow \int_0^1 \Big(\int_{rB} |f(x)|^p dV(x) \Big) (1-r)^{\alpha-1} dr < \infty.$$

By Corollary 1 we obtain the following growth result.

Corollary 3. *Let $u \in \mathscr{D}^p_\alpha(B)$ and $\alpha \in (0,\infty)$. Then*

$$\lim_{r\to 1} (1-r)^\alpha \int_{rB} |\nabla u(x)|^p dV(x) = 0.$$

Lemma 2. *Let $u \in \mathscr{H}(B)$, α a multi-index and $p > 0$. Then*

$$\Big(|D^\alpha u(x)|\, r^{|\alpha|} \Big)^p \le \frac{C}{r^n} \int_{B(x,r)} |u|^p dV, \tag{1}$$

whenever $B(x,r) \subset B$, where $C = C(p,n,\alpha)$ is a positive constant.

Proof. By Fefferman-Stein Lemma we have

$$|u(x)|^p \le \frac{C}{r^n} \int_{B(x,r)} |u|^p dV, \quad \text{whenever } B(x,r) \subset B$$

and consequently

$$\sup_{y \in B(x,r/2)} |u(y)|^p \le \frac{C2^n}{r^n} \int_{B(x,r)} |u|^p dV, \tag{2}$$

where C is a positive constant depending only on n and p.

On the other hand, by Cauchy's estimate we have

$$|D^\alpha u(x)| \le \Big(\frac{2n|\alpha|}{r} \Big)^{|\alpha|} \sup_{y \in B(x,r/2)} |u(y)| \tag{3}$$

(see, for example, [**8**, p. 23]).

From (3) we obtain

$$|D^\alpha u(x)|^p \le \left(\left(\frac{2n|\alpha|}{r}\right)^{|\alpha|} \sup_{y\in B(x,r/2)} |u(y)|\right)^p. \tag{4}$$

(1) follows from (2) and (4). □

COROLLARY 4. *Let u be a harmonic function on a domain $G \subset \boldsymbol{R}^n$, $p > 0$ and $m \in \boldsymbol{N}$. Then there is a constant $C = C(m,n,p)$ such that*

$$|\nabla^m u(x)| \le \frac{C}{r^m}\left(\frac{1}{V(B(x,r))}\int_{B(x,r)} |u(y)|^p dV(y)\right)^{1/p},$$

for each $B(x,r) \subset G$.

REMARK 1. For $p \ge (n-2)/(m+n-2)$ Lemma 2 was proved in [**7**] by T.M.Flett.

LEMMA 3. *Let u be a harmonic function on a domain G. Then*

$$\Delta^m |u|^2 = 2^m |\nabla^m u|^2.$$

PROOF. Without loss of generality we may assume that u is a real valued harmonic function. We prove the lemma by induction.

Let $m = 1$. Then

$$\Delta|u|^2 = \Delta u^2 = 2(|\nabla u|^2 + u\Delta u) = 2|\nabla u|^2,$$

since $\Delta u = 0$, as desired.

Next, assume that the formula holds for all positive integers $m \le k$. Then for $m = k+1$, we have

$$\Delta^{k+1}u^2 = \Delta(\Delta^k u^2) = \Delta(2^k|\nabla^k u|^2) = 2^k\Delta\left(k! \sum_{\alpha\in \boldsymbol{Z}_n^+, |\alpha|=k} \frac{|D^\alpha u|^2}{\alpha_1!\cdots\alpha_n!}\right)$$
$$= 2^k k! \sum_{\alpha\in \boldsymbol{Z}_n^+, |\alpha|=k} \frac{\Delta|D^\alpha u|^2}{\alpha_1!\cdots\alpha_n!}.$$

Since for every multi-index α, the function $D^\alpha u$ is harmonic, we obtain

$$\Delta|D^\alpha u|^2 = 2|\nabla(D^\alpha u)|^2 = 2\sum_{i=1}^n |D^{\alpha_1\ldots(\alpha_i+1)\ldots\alpha_n} u|^2.$$

Therefore, we obtain that

$$\Delta^{k+1}u^2 = 2^{k+1}k! \sum_{\alpha\in \boldsymbol{Z}_n^+, |\alpha|=k} \frac{1}{\alpha_1!\cdots\alpha_n!} \sum_{i=1}^{n} (|D^{\alpha_1\dots(\alpha_i+1)\dots\alpha_n}u|^2)$$

$$= 2^{k+1}k! \sum_{\alpha\in \boldsymbol{Z}_n^+, |\alpha|=k} \sum_{i=1}^{n} \frac{\alpha_i+1}{\alpha_1!\cdots(\alpha_i+1)!\cdots\alpha_n!} (|D^{\alpha_1\dots(\alpha_i+1)\dots\alpha_n}u|^2).$$

Note that all multi-indices appearing in the above sum are of order $k+1$ and that each multi-index of order $k+1$ appears in the sum. Hence, we can rewrite the sum, summing over multi-indices of order $k+1$. Let β be an arbitrary multi-index of order $k+1$. Set

$$I_\beta = \{i \in \{1,\dots,n\} :\ \beta_i > 0\}$$

and

$$J_\beta = \left\{\alpha \in \boldsymbol{Z}_+^n :\ |\alpha| = k \text{ and } \alpha + e_i = \beta \text{ for some } i \in I_\beta\right\},$$

where

$$e_1 = (1,0,\dots,0),\ e_2 = (0,1,0,\dots,0),\ \dots,\ e_n = (0,\dots,0,1).$$

Then the coefficient standing by $|D^\beta u|^2$ is equal to

$$\sum_{\alpha\in J_\beta} \frac{1}{\alpha_1!\cdots\alpha_n!} = \sum_{i\in I_\beta} \frac{1}{\beta_1!\cdots(\beta_i-1)!\cdots\beta_n!} = \sum_{i\in I_\beta} \frac{\beta_i}{\beta_1!\cdots\beta_i!\cdots\beta_n!}$$

$$= \sum_{i=1}^{n} \frac{\beta_i}{\beta_1!\cdots\beta_i!\cdots\beta_n!}.$$

Thus

$$\Delta^{k+1}u^2 = 2^{k+1}k! \sum_{\beta\in \boldsymbol{Z}_n^+, |\beta|=k+1} |D^\beta u|^2 \sum_{i=1}^{n} \frac{\beta_i}{\beta_1!\cdots\beta_i!\cdots\beta_n!}$$

$$= 2^{k+1}k! \sum_{\beta\in \boldsymbol{Z}_n^+, |\beta|=k+1} \frac{|D^\beta u|^2}{\beta_1!\cdots\beta_n!} \sum_{i=1}^{n} \beta_i$$

$$= 2^{k+1}k! \sum_{\beta\in \boldsymbol{Z}_n^+, |\beta|=k+1} \frac{|D^\beta u|^2}{\beta_1!\cdots\beta_n!} |\beta|$$

$$= 2^{k+1}(k+1)! \sum_{\beta\in \boldsymbol{Z}_n^+, |\beta|=k+1} \frac{|D^\beta u|^2}{\beta_1!\cdots\beta_n!} = 2^{k+1}|\nabla^{k+1}u|^2,$$

finishing the proof. □

LEMMA 4. *Suppose* $0 < p < \infty$ *and* $r \in (0,1)$. *Then there is a constant* $C = C(p,r,n)$ *such that*

$$\int_{|x|<r} |u(x)|^p dV_N(x) \leq C \left(|u(0)|^p + \int_B |\nabla u(x)|^p (1-|x|)^p dV_N(x) \right),$$

for all $u \in \mathscr{H}(B)$.

PROOF. First, notice that

$$\int_{|x|<r} |u(x)|^p dV_N(x) \leq \max_{|x|\leq r} |u(x)|,$$

so, it is enough to estimate $\max_{|x|\leq r} |u(x)|$.

Since

$$u(x_0) - u(0) = \int_0^1 u'(tx_0)dt = \int_0^1 \langle \nabla u(tx_0), x_0 \rangle dt,$$

by elementary inequalities we obtain

$$|u(x_0)|^p \leq c_p \left(|u(0)|^p + |x_0|^p \max_{|x|\leq r} |\nabla u(x)|^p \right),$$

for each $x_0 \in \overline{B(0,r)}$, where $c_p = 1$ for $0 < p < 1$ and $c_p = 2^{p-1}$ for $p \geq 1$.

On the other hand by Fefferman-Stein Lemma we have

$$|D^\alpha u(x)|^p \leq C \int_{B(x,(1-r)/2)} |D^\alpha u(y)|^p dV(y)$$

for each $x \in \overline{B(0,r)}$, every multi-index α of order 1, and for some $C > 0$ independent of $u \in \mathscr{H}(B)$.

This implies

$$|\nabla u(x)|^p \leq C v_n \int_{B(x,(1+r)/2)} |\nabla u(y)|^p dV_N(y)$$

for each $x \in \overline{B(0,r)}$, and consequently

$$\max_{|x|\leq r} |\nabla u(x)|^p \leq C v_n \left(\frac{2}{1-r} \right)^p \int_{B(0,(1+r)/2)} |u(y)|^p (1-|y|)^p dV_N(y).$$

From all above mentioned the result follows. □

3. On the weighted Hardy-Littlewood operator.

Let $g : [0,1] \to \boldsymbol{R}$ be a function. For a measurable complex-valued function f on B, we define the weighted Hardy-Littlewood operator $L_g(f)$ as

$$L_g(f)(x) = \int_0^1 f(tx)g(t)dt,$$

for $x \in B$, provided that the integral exists.

For $g(t) \equiv 1$ and $n = 1$, Hardy proved that this special operator is bounded on $\mathscr{L}^p(0,\infty)$, $p > 1$, moreover $\|L_1\|_{\mathscr{L}^p(0,\infty)} \leq \frac{p}{p-1}$ ([**5**, p. 234]). We are interested in the boundedness of the weighted Hardy-Littlewood operator on $\mathscr{H}_{\mathscr{B}^a}(B), \mathscr{H}_{BMO_p}(B)$, $\mathscr{H}^p(B), b^p_\omega(B)$ and $\mathscr{D}^p_\alpha(B)$.

Theorem 1. *Let $g \in \mathscr{L}[0,1]$ and $a > 0$. Then L_g is a bounded operator from $\mathscr{H}_{\mathscr{B}^a}(B)$ to $\mathscr{H}_{\mathscr{B}^a(B)}$.*

Proof. Let $u \in \mathscr{H}_{\mathscr{B}^a}(B)$. Using Cauchy-Schwarz inequality, we have for $x \in B$

$$\begin{aligned}(1-|x|)^a|\nabla L_g(u)(x)| &= (1-|x|)^a \left|\int_0^1 t\langle(\nabla u)(tx), g(t)\rangle dt\right| \\ &\leq (1-|x|)^a \left(\int_0^1 t^2|g(t)|dt\right)^{1/2} \left(\int_0^1 |(\nabla u)(tx)|^2|g(t)|dt\right)^{1/2} \\ &\leq \left(\int_0^1 t^2|g(t)|dt\right)^{1/2} \left(\int_0^1 (1-|tx|)^{2a}|(\nabla u)(tx)|^2|g(t)|dt\right)^{1/2} \\ &\leq \|u\|_{\mathscr{H}_{\mathscr{B}^a}} \int_0^1 |g(t)|dt.\end{aligned}$$

Taking supremum over $x \in B$ in the obtained inequality, we get the result. □

Remark 2. In the above proof we did not use any special property of harmonic function. Hence we proved the following theorem:

Theorem 1. a). *Let $g \in \mathscr{L}[0,1]$ and $a > 0$. Then L_g is a bounded operator from $\mathscr{B}^a(B)$ to $\mathscr{B}^a(B)$.*

Combining Theorem A and its extension for the case $p \in (0,1)$ ([**18**] and [**20**]), and Theorem 1 for $a = 1$, we obtain the following corollary.

Corollary 5. *Let $p \in (0,\infty)$ and $g \in \mathscr{L}[0,1]$. Then L_g is a bounded operator from $\mathscr{H}_{BMO_p}(B)$ to $\mathscr{H}_{BMO_p}(B)$.*

It is interesting that in the case $p \geq 1$ there is a direct proof of Corollary 5 using definition of $\mathscr{H}_{BMO_p}(B)$. Moreover in this case we obtain a precise estimate of the norm of the operator L_g.

THEOREM 2. *Let $p \geq 1$ and $g \in \mathscr{L}[0,1]$. Then L_g is a bounded operator from $\mathscr{H}_{BMO_p}(B)$ to $\mathscr{H}_{BMO_p}(B)$, moreover*

$$\|L_g\|_{\mathscr{H}_{BMO_p}(B)\to\mathscr{H}_{BMO_p}(B)} \leq \int_0^1 |g(t)|dt.$$

PROOF. Let $u \in \mathscr{H}_{BMO_p}(B)$. Then for any open ball $B(a,r)$ with $\overline{B}(a,r) \subset B$, by Fubini's theorem and the change of variables $tx \to x$ we obtain

$$\begin{aligned} L_g(u)_{B(a,r)} &= \frac{1}{V(B(a,r))}\int_{B(a,r)} (L_g)(u)(x)dV(x) \\ &= \int_0^1 \left(\frac{1}{V(B(a,r))}\int_{B(a,r)} u(tx)dV(x)\right)g(t)dt \\ &= \int_0^1 u_{B(ta,tr)}g(t)dt. \end{aligned}$$

Using this and Minkowski's inequality, we have

$$\begin{aligned} &\|L_g(u)\|_{\mathscr{H}_{BMO_p}(B)} \\ &= \sup_{B(a,r)\subset B}\left(\frac{1}{V(B(a,r))}\int_{B(a,r)} |L_g(u)(x) - L_g(u)_{B(a,r)}|^p dV(x)\right)^{1/p} \\ &= \sup_{B(a,r)\subset B}\left(\frac{1}{V(B(a,r))}\int_{B(a,r)} \left|\int_0^1 (u(tx) - u_{B(ta,tr)})g(t)dt\right|^p dV(x)\right)^{1/p} \\ &\leq \sup_{B(a,r)\subset B}\int_0^1 |g(t)|\left(\frac{1}{V(B(a,r))}\int_{B(a,r)} |u(tx) - u_{B(ta,tr)}|^p dV(x)\right)^{1/p} dt \\ &= \sup_{B(a,r)\subset B}\int_0^1 |g(t)|\left(\frac{1}{V(B(ta,tr))}\int_{B(ta,tr)} |u(x) - u_{B(ta,tr)}|^p dV(x)\right)^{1/p} dt \\ &\leq \|u\|_{\mathscr{H}_{BMO_p}(B)}\int_0^1 |g(t)|dt, \end{aligned}$$

from which the result follows. □

Note that we again did not use any special property of harmonic function. Thus the following theorem holds:

THEOREM 2. a). *Let $p \geq 1$ and $g \in \mathscr{L}[0,1]$. Then L_g is a bounded operator from $BMO_p(B)$ to $BMO_p(B)$. Moreover the operator norm of L_g satisfies the estimate:*

$$\|L_g\|_{BMO_p(B)\to BMO_p(B)} \leq \int_0^1 |g(t)|dt.$$

THEOREM 3. *Let ω be a weight that is non-increasing in $r \in (0,1)$, $p \geq 1$, and $g : [0,1] \to \boldsymbol{R}$ be a function which satisfies the condition*

$$\int_0^1 t^{-n/p} |g(t)| dt < \infty.$$

Then $L_g : b^p_\omega(B) \to b^p_\omega(B)$ is a bounded operator.

PROOF. Using Minkowski's inequality and the change of variables $tx \to x$, we have

$$\begin{aligned}
\|L_g(u)\|_{b^p_\omega(B)} &= \left(\int_B |L_g(u)(x)|^p \omega(x) dV(x)\right)^{1/p} \\
&\leq \int_0^1 \left(\int_B |u(tx)|^p \omega(x) dV(x)\right)^{1/p} |g(t)| dt \\
&\leq \int_0^1 \left(\int_B |u(tx)|^p \omega(tx) dV(x)\right)^{1/p} |g(t)| dt \\
&= \int_0^1 \left(\int_{tB} |u(x)|^p \omega(x) dV(x)\right)^{1/p} t^{-n/p} |g(t)| dt \\
&\leq \|u\|_{b^p_\omega(B)} \int_0^1 t^{-n/p} |g(t)| dt,
\end{aligned}$$

which implies that L_g is bounded on $b^p_\omega(B)$. □

EXAPMPLE 1. The weight $\omega(x) = (1 - |x|)^\alpha$ where $\alpha \geq 0$ is an example of weights that satisfy the condition in Theorem 3.

If we note that $L_g(f)(0) = f(0) \int_0^1 g(t) dt$, we can similarly prove the following result.

THEOREM 4. *Let $\alpha \geq 0$, $p \geq 1$, and $g : [0,1] \to \boldsymbol{R}$ be a function which satisfies the condition*

$$\int_0^1 t^{-n/p} |g(t)| dt < \infty.$$

Then $L_g : \mathscr{D}^p_\alpha(B) \to \mathscr{D}^p_\alpha(B)$ is a bounded operator such that

$$\|L_g\| \leq C \int_0^1 t^{-n/p} |g(t)| dt,$$

where $C = C(n,p)$ is a positive constant.

In the case of $\mathscr{H}^p(B)$ we have the following result.

THEOREM 5. *Let $p \geq 1$ and $g \in \mathscr{L}[0,1]$. Then L_g is a bounded operator from*

$\mathscr{H}^p(B)$ to $\mathscr{H}^p(B)$. Moreover

$$\|L_g\|_{\mathscr{H}^p(B)\to\mathscr{H}^p(B)} \le \int_0^1 |g(t)|dt.$$

PROOF. By Minkowski's inequality we get

$$\begin{aligned}\|L_g(u)\|_{\mathscr{H}^p(B)} &= \sup_{0\le r<1}\left(\int_S |L_g(u)(r\zeta)|^p d\sigma_N(\zeta)\right)^{1/p}\\ &\le \sup_{0\le r<1}\int_0^1\left(\int_S |u(rt\zeta)|^p|g(t)|^p d\sigma_N(\zeta)\right)^{1/p} dt\\ &\le \|u\|_{\mathscr{H}^p(B)}\int_0^1 |g(t)|dt,\end{aligned}$$

as desired. □

4. Growth theorems for harmonic functions.

Throughout the rest of the paper we will use C to denote a positive constant, not necessarily the same on any two occurrences. Any dependence of C on say $p, q, \ldots$ will be denoted by $C(p, q, \ldots)$.

In this section we generalize and give a short proof of the following result of Flett [**7**, Lemma 9]:

THEOREM B. *Let $m \in \boldsymbol{N}, n \ge 2$, and $(n-2)/(m+n-2) \le p \le 1$ (if $n = 2$ we suppose that $0 < p \le 1$). Let also $u \in \mathscr{H}(B)$ such that*

$$I = \int_B |u(x)|^p dV(x) < \infty.$$

Then, for $0 \le r < 1$,

$$\int_{B(0,r)} |\nabla^m u(x)|^p dV(x) \le C(m, n, p) I (1-r)^{-pm}.$$

First we prove a useful inequality.

THEOREM 6. *Let $p > 0, \alpha > -1$ and $m \in \boldsymbol{N}$. Then there is a positive constant $C = C(n, p, \alpha, m)$ such that*

$$\int_B |\nabla^m u(x)|^p (1-|x|)^{pm+\alpha} dV(x) \le C \int_B |u(x)|^p (1-|x|)^\alpha dV(x) \tag{5}$$

for all $u \in b^p_\alpha(B)$.

Proof. By Corollary 4, we have for $x \in B$

$$|\nabla^m u(x)|^p (1-|x|)^{pm} \le \frac{C_1}{(1-|x|)^n} \int_{B(x,\frac{1-|x|}{2})} |u|^p dV, \tag{6}$$

where $C_1 = C_1(n,p,m)$ is a positive constant.

Since $\frac{1}{2}(1-|x|) \le 1-|y| \le \frac{3}{2}(1-|x|)$ for $y \in B\big(x, \frac{1-|x|}{2}\big)$, there is a constant $C_2 = C_2(n,\alpha) > 0$ such that $(1-|x|)^{\alpha-n} \le C_2(1-|y|)^{\alpha-n}$ for $y \in B\big(x, \frac{1-|x|}{2}\big)$. Using this inequality, (6) and Fubini's theorem, we have

$$\begin{aligned}
I &\equiv \int_B |\nabla^m u(x)|^p (1-|x|)^{pm+\alpha} dV(x) \\
&\le C_1 \int_B (1-|x|)^{\alpha-n} dV(x) \int_{B(x,\frac{1-|x|}{2})} |u(y)|^p dV(y) \\
&\le C_1 C_2 \int_B dV(x) \int_{B(x,\frac{1-|x|}{2})} (1-|y|)^{\alpha-n} |u(y)|^p dV(y) \\
&= C_1 C_2 \int_B (1-|y|)^{\alpha-n} |u(y)|^p dV(y) \int_{A(y)} dV(x),
\end{aligned}$$

where

$$A(y) = \left\{ x \in B \,\middle|\, y \in B\left(x, \frac{1-|x|}{2}\right)\right\} \subset \{x \in B \,|\, |x-y| < 1-|y|\} = B(y, 1-|y|).$$

From this the desired result follows:

$$I \le C_1 C_2 v_n \int_B |u(y)|^p (1-|y|)^\alpha dV(y).$$

□

Corollary 6. *Let $u \in b^p_\alpha(B), p > 0, \alpha > -1$ and $pm+\alpha > 0$. Then there is a positive constant $C = C(m,n,p,\alpha)$ such that for $0 \le r < 1$, the following holds*:

(a) $(1-r)^{pm+\alpha} \displaystyle\int_{rB} |\nabla^m u(x)|^p dV(x) \le C \int_B |u(x)|^p (1-|x|)^\alpha dV(x).$

Moreover,

(b) $\displaystyle\lim_{r \to 1-0} (1-r)^{pm+\alpha} \int_{rB} |\nabla^m u(x)|^p dV(x) = 0.$

Proof. Let $I = \int_B |u(x)|^p (1-|x|)^\alpha dV(x)$. By Theorem 6 we have that

$$\int_B |\nabla^m u(x)|^p (1-|x|)^{pm+\alpha} dV(x) \le CI < \infty$$

for some $C = C(m, n, p, \alpha)$. By Corollary 1 for $f = |\nabla^m u|$ and $\alpha \to pm + \alpha$ we obtain the result. □

The main idea in the proof of Theorem 6 motivated us to get another equivalence condition for a harmonic function to be a Bloch function. In order to formulate the result in more complete form we quote several conditions in the following theorem.

THEOREM 7. *Let* $0 < p < \infty$, $k \in \boldsymbol{N}$ *and* $u \in \mathscr{H}(B)$, *then the following conditions are equivalent:*

(a) $u \in \mathscr{H}_{\mathscr{B}}(B)$,

(b) $\sup_{x \in B} (1 - |x|)^2 \Delta(|u|^2(x)) < +\infty$,

(c) $\sup_{x \in B} (1 - |x|)^k |\nabla^k u(x)| < +\infty$,

(d) $\sup_{x \in B} \int_{B\left(x, \frac{1-|x|}{2}\right)} |\nabla^k u(z)|^p (1 - |z|)^{kp-n} dV(z) < +\infty$,

(e) $\|u\|_{BMO_p} < +\infty$.

PROOF. $(a) \Leftrightarrow (b)$ is simple and is based on the formula $\Delta(f^2) = 2f\Delta f + 2|\nabla f|^2$, for any real function f of C^2 class.

$(a) \Leftrightarrow (e)$ was proved in [**18**] and [**20**].

$(a) \Rightarrow (c)$ can be found in [**3**, p. 42].

$(c) \Rightarrow (a)$ this is certainly well known to experts in the field of Bloch space. We include a proof here for completeness and for the lack of a specific reference.

Case $k = 1$ is trivial. Let $k \geq 2$. Take $\alpha \in \boldsymbol{Z}_+^n$ with $|\alpha| = k - 1$. Fix $x \in B$. Since

$$D^\alpha u(x) - D^\alpha u(0) = \int_0^1 \frac{d}{dt}[D^\alpha u(tx)]dt = \int_0^1 \langle \nabla D^\alpha u(tx), x \rangle dt,$$

we have

$$|D^\alpha u(x)| \leq |D^\alpha u(0)| + \int_0^1 |\nabla^k u(tx)|\, |x| dt.$$

Thus

$$\begin{aligned} |D^\alpha u(x)| &\leq |D^\alpha u(0)| + \int_0^1 \frac{|x| dt}{(1 - t|x|)^k} \sup_{y \in B} (1 - |y|)^k |\nabla^k u(y)| \\ &= |D^\alpha u(0)| + \left(\frac{1}{(1 - |x|)^{k-1}} - 1 \right) \frac{1}{k - 1} \sup_{y \in B} (1 - |y|)^k |\nabla^k u(y)| \\ &\leq |D^\alpha u(0)| + \frac{1}{(k - 1)(1 - |x|)^{k-1}} \sup_{y \in B} (1 - |y|)^k |\nabla^k u(y)| \end{aligned}$$

i.e.

$$(1-|x|)^{k-1}|D^\alpha u(x)| \le (1-|x|)^{k-1}|D^\alpha u(0)| + \frac{1}{k-1}\sup_{y\in B}(1-|y|)^k|\nabla^k u(y)|.$$

Since α is an arbitrary multi-index of order $k-1$ and x is an arbitrary point of B, the last inequality and (c) imply that

$$\sup_{x\in B}(1-|x|)^{k-1}|\nabla^{k-1}u(x)| < +\infty.$$

Therefore, by induction the result follows.

$(c) \Rightarrow (d)$ is simple.

Hence the only interesting direction is $(d) \Rightarrow (c)$. Let l be a nonnegative integer. Take $\alpha, \beta \in \boldsymbol{Z}_+^n$ with $|\alpha| = k$ and $|\beta| = l$. Fix $x \in B$.

By Cauchy's estimate and the HL-property of the function $|D^\alpha u|^p$, we have

$$\begin{aligned}
|D^{\alpha+\beta}u(x)|^p &\le \left[\left(\frac{n|\beta|}{4^{-1}(1-|x|)}\right)^{|\beta|}\sup_{y\in B(x,(1-|x|)/4)}|D^\alpha u(y)|\right]^p \\
&\le \left(\frac{4nl}{(1-|x|)}\right)^{lp}\left[\sup_{y\in B(x,(1-|x|)/4)}\frac{C4^n}{(1-|x|)^n}\int_{B(y,(1-|x|)/4)}|D^\alpha u|^p dV\right] \\
&\le \frac{C}{(1-|x|)^{lp+n}}\int_{B(x,(1-|x|)/2)}|D^\alpha u|^p dV \\
&\le C\frac{(1-|x|)^{-kp+n}}{(1-|x|)^{lp+n}}\int_{B(x,(1-|x|)/2)}(1-|y|)^{kp-n}|D^\alpha u(y)|^p dV(y).
\end{aligned}$$

Hence

$$(1-|x|)^{(k+l)p}|D^{\alpha+\beta}u(x)|^p \le C\int_{B(x,(1-|x|)/2)}(1-|y|)^{kp-n}|\nabla^k u(y)|^p dV(y)$$

if $\alpha, \beta \in \boldsymbol{Z}_+^n$, $|\alpha| = k$ and $|\beta| = l$ and $x \in B$. This implies that

$$\sup_{x\in B}(1-|x|)^{k+l}|\nabla^{k+l}u(x)| \le C\left(\sup_{x\in B}\int_{B(x,(1-|x|)/2)}|\nabla^k u(y)|^p(1-|y|)^{kp-n}dV(y)\right)^{1/p},$$

which completes the proof of the theorem. □

5. A local estimate.

In [**24**, Theorems 1 and 2] we proved the following result.

Theorem C. *Let $1 < p < +\infty$. Function $u \in \mathscr{H}(B)$ belongs to $\mathscr{H}^p(B)$ if and only if*

$$\int_B |u(x)|^{p-2}|\nabla u(x)|^2(1-|x|^2)dV_N(x) < +\infty.$$

Moreover if $u \in \mathscr{H}^p(B),\ 1 < p < +\infty$, *then*

$$\|u\|^p_{\mathscr{H}^p} = |u(0)|^p + \frac{p(p-1)}{n(n-2)}\int_B |u(x)|^{p-2}|\nabla u(x)|^2\left(|x|^{2-n}-1\right)dV_N(x) \tag{7}$$

and

$$\|u\|^p_{\mathscr{H}^p} = \int_B |u(x)|^p dV_N(x) + \frac{p(p-1)}{2n}\int_B |u(x)|^{p-2}|\nabla u(x)|^2(1-|x|^2)dV_N(x).$$

Using among others Theorem C we proved in [**24**] the theorem:

Theorem D. *Let* $p \geq 2, n \geq 3$ *and* $u \in \mathscr{H}^p(B)$, *then*

$$|\nabla u(0)|^p \leq \frac{n^{\frac{p}{2}}p(p-1)}{(n-2)n}\int_B |u(x)|^{p-2}|\nabla u(x)|^2(|x|^{2-n}-1)dV_N(x).$$

However the following stronger inequality holds.

Theorem 8. *Let* $p \geq 2, n \geq 3$ *and* $u \in \mathscr{H}^p(B)$, *then*

$$\left(\sum_{m=1}^{\infty}\frac{|\nabla^m u(0)|^2}{m!\prod_{i=0}^{m-1}(n+2i)}\right)^{p/2} \leq \frac{p(p-1)}{(n-2)n}\int_B |u(x)|^{p-2}|\nabla u(x)|^2(|x|^{2-n}-1)dV_N(x).$$

Proof. It is well-known that if $u \in \mathscr{H}(B)$ then $u(x) = \sum_{m=0}^{+\infty} p_m(x)$, where each $p_m(x)$ is a harmonic homogeneous polynomial of degree m. By Hölder inequality we have $\|u\|_{\mathscr{H}^2} \leq \|u\|_{\mathscr{H}^p}$. For $u \in \mathscr{H}^2(B)$, the following formula

$$\|u\|^2_{\mathscr{H}^2} = \sum_{m=0}^{+\infty}\int_S |p_m(\zeta)|^2 d\sigma_N(\zeta) \tag{8}$$

holds, see [**3**, p. 122].

On the other hand, since p_m is a homogeneous polynomial of degree m, it holds that $\langle \nabla p_m(x), x\rangle = mp_m(x)$, $x \in \boldsymbol{R}^n$. From (8) we have

$$\|u\|_{\mathscr{H}^2} - |u(0)|^2 = \sum_{m=1}^{+\infty}\int_S |p_m(\zeta)|^2 d\sigma_N(\zeta).$$

Without loss of generality we may assume that u is a real valued harmonic function. Then p_m is a real homogeneous harmonic polynomial of degree m, and so p_m^2 is a real homogeneous polynomial of degree $2m$. Hence

$$2m\int_S p_m^2(\zeta)d\sigma_N(\zeta) = \int_S \langle \zeta, \nabla p_m^2(\zeta)\rangle d\sigma_N(\zeta) = \frac{1}{n}\int_B \Delta p_m^2 dV_N(x), \tag{9}$$

by the divergence theorem.

Hence

$$\begin{aligned}\int_S p_m^2(\zeta)d\sigma_N(\zeta) &= \frac{1}{2m}\int_0^1\int_S \Delta p_m^2(r\zeta)r^{n-1}d\sigma_N(\zeta)dr \\ &= \frac{1}{2m(2m+n-2)}\int_S \Delta p_m^2(\zeta)d\sigma_N(\zeta).\end{aligned} \tag{10}$$

Note that $\Delta^k p_m^2$, $k = 1, 2, \dots, m$ are homogeneous polynomials of degree $2m - 2k$. Hence we can use (10) m times and obtain

$$\int_S p_m^2(\zeta)d\sigma_N(\zeta) = \frac{1}{(2m)!!n(n+2)\cdots(n+2m-2)}\Delta^m p_m^2(0), \tag{11}$$

since $\Delta^m p_m^2$ is constant.

If h is a harmonic function by Lemma 3 we have

$$\Delta^m |h|^2 = 2^m |\nabla^m h|^2. \tag{12}$$

By easy calculations we obtain

$$|\nabla^m u(0)| = |\nabla^m p_m(0)|. \tag{13}$$

From (10)–(13) we obtain

$$\int_S p_m^2(\zeta)d\sigma_N(\zeta) = \frac{|\nabla^m u(0)|^2}{m!n(n+2)\cdots(n+2m-2)}. \tag{14}$$

Hence

$$\sum_{m=1}^{\infty}\frac{|\nabla^m u(0)|^2}{m!n(n+2)\cdots(n+2m-2)} \le \|u\|^2_{\mathscr{H}^2} - |u(0)|^2 \le \|u\|^2_{\mathscr{H}^p} - |u(0)|^2.$$

By the inequality $(a-b)^q + b^q \le a^q$, $a \ge b > 0$, $q \ge 1$, we obtain

$$\begin{aligned}\left(\sum_{m=1}^{\infty}\frac{|\nabla^m u(0)|^2}{m!n(n+2)\cdots(n+2m-2)}\right)^{p/2} &\le \left(\|u\|^2_{\mathscr{H}^p} - |u(0)|^2\right)^{p/2} \\ &\le \left(\|u\|^p_{\mathscr{H}^p} - |u(0)|^p\right).\end{aligned} \tag{15}$$

From (7), (14) and (15) the result follows. □

6. On Dirichlet type spaces.

In this section we consider the relationship between the functions which belong to $\mathscr{H}^p(B)$ and $\mathscr{D}^p_\alpha(B)$.

THEOREM 9. *Let* $u \in \mathscr{H}(B)$, $p \in [0,\infty)$, $r \in (0,\infty)$, α, $\beta \in (-1,\infty)$, $r < n+\alpha$, $r \le q$, $\alpha \le \beta$ *and*

$$p \le \frac{(\beta-\alpha)r - (n+\alpha)(q-r)}{n+\alpha-r}. \tag{16}$$

Then there is a positive constant C such that

$$\int_B |u(x)|^p |\nabla u(x)|^q (1-|x|)^\beta dV(x) \le C\|u\|^{p+q}_{\mathscr{D}^r_\alpha}. \tag{17}$$

PROOF. Without loss of generality we may assume $u(0) = 0$. Since $\frac{\partial u}{\partial x_i}, i = 1,\dots,n$, are harmonic, for every $r > 0$ the function $|\nabla u(x)|^r$ possesses HL-property. Hence

$$|\nabla u(x)|^r \le \frac{C}{(1-|x|)^{n+\alpha}} \int_{B\left(x,\frac{1-|x|}{2}\right)} |\nabla u(y)|^r (1-|y|)^\alpha dV(y)$$

for some $C > 0$ independent of u and consequently

$$|\nabla u(x)| \le C\frac{\|u\|_{\mathscr{D}^r_\alpha}}{(1-|x|)^{\frac{n+\alpha}{r}}}. \tag{18}$$

On the other hand, from (18) we have

$$\begin{aligned} |u(x)| &= \left|\int_0^1 \langle \nabla u(tx), x\rangle dt\right| \le C|x|\ \|u\|_{\mathscr{D}^r_\alpha} \int_0^1 \frac{dt}{(1-|tx|)^{\frac{n+\alpha}{r}}} \\ &\le C\frac{\|u\|_{\mathscr{D}^r_\alpha}}{(1-|x|)^{\frac{n+\alpha-r}{r}}}. \end{aligned} \tag{19}$$

Let $\varepsilon = q - r$. Then using (18) and (19) we get

$$\begin{aligned} &\int_B |u(x)|^p |\nabla u(x)|^q (1-|x|)^\beta dV(x) \\ &\le C\int_B \frac{\|u\|^p_{\mathscr{D}^r_\alpha}}{(1-|x|)^{\frac{n+\alpha-r}{r}p}} |\nabla u(x)|^r \frac{\|u\|^\varepsilon_{\mathscr{D}^r_\alpha}}{(1-|x|)^{\frac{n+\alpha}{r}\varepsilon}} (1-|x|)^\beta dV(x) \\ &= C\|u\|^{p+\varepsilon}_{\mathscr{D}^r_\alpha} \int_B |\nabla u(x)|^r (1-|x|)^{\alpha+s} dV(x), \end{aligned}$$

where $s = \beta - p\left(\frac{n+\alpha-r}{r}\right) - \frac{n+\alpha}{r}(q-r) - \alpha$. From (16) we have $s \geq 0$. Hence the result follows. □

THEOREM 10. *Let $u \in \mathscr{H}(B)$. If $u \in \mathscr{D}^2_\alpha$ for some $\alpha \in (-1, 1]$ with $n+\alpha > 2$, then $u \in \mathscr{H}^p(B)$ for all $p \in \left(0, \frac{2n-2}{n+\alpha-2}\right]$. Moreover, there is a positive constant $C = C(n, \alpha)$ such that*

$$\|u\|_{\mathscr{H}^p(B)} \leq C\|u\|_{\mathscr{D}^2_\alpha} \tag{20}$$

for all $p \in \left(0, \frac{2n-2}{n+\alpha-2}\right]$.

PROOF. By Theorem 9 we have that there is a positive constant C independent of u such that

$$\int_B |u(x)|^{p-2}|\nabla u(x)|^2(1-|x|)dV(x) \leq C\|u\|^p_{\mathscr{D}^2_\alpha}, \tag{21}$$

for $p \in \left[2, \frac{2n-2}{n+\alpha-2}\right]$.

Hence, by Theorem C, we have $u \in \mathscr{H}^p(B)$ for all $p \in \left[2, \frac{2n-2}{n+\alpha-2}\right]$ and consequently for $p \in \left(0, \frac{2n-2}{n+\alpha-2}\right]$.

To get inequality (20) it remains to show

$$\int_B |u(x)|^p dV_N(x) \leq \|u\|^p_{\mathscr{D}^2_\alpha}.$$

It is well-known that for $u \in \mathscr{H}(B)$ and $p \in (0, \infty)$

$$\int_B |u(x)|^p dV_N(x) \leq C\left(|u(0)|^p + \int_B |\nabla u(x)|^p(1-|x|)^p dV_N(x)\right), \tag{22}$$

for some $C > 0$ independent of u. For example, it is a consequence of [**13**, Theorem 2]. Indeed, taking $\mathscr{D} = B$, $s = 0$, $q = p$, $m = 1$, $x_0 = 0$ and $\varepsilon \in (0, 1)$ in [**13**, Theorem 2], and using the fact that the defining function for the unit ball is $\lambda(x) = |x|^2 - 1$, we get

$$\int_0^\varepsilon M^p_p(u, r)dr \leq C\left(|u(0)|^p + \int_0^\varepsilon r^p M^p_p(\nabla u, r)dr\right) \tag{23}$$

for some $C > 0$ independent of u, where

$$M^p_p(g, r) = \int_S |g(\sqrt{1-r}\,\zeta)|^p d\sigma_N(\zeta).$$

Using the change of variables $\rho = \sqrt{1-r}$ in both integrals in (23), then the polar coordinates and some simple calculation, we obtain

$$\int_{1>|x|\geq\sqrt{1-\varepsilon}} |u(x)|^p dV_N(x) \leq C_1\left(|u(0)|^p + \int_{1>|x|\geq\sqrt{1-\varepsilon}} |\nabla u(x)|^p(1-|x|)^p dV_N(x)\right). \quad (24)$$

On the other hand, Lemma 4 gives

$$\int_{|x|<\sqrt{1-\varepsilon}} |u(x)|^p dV_N(x) \leq C_2\left(|u(0)|^p + \int_B |\nabla u(x)|^p(1-|x|)^p dV_N(x)\right), \quad (25)$$

for some $C_2 > 0$ independent of u.

(24) and (25) together show that (22) holds.

Let $p = \frac{2n-2}{n+\alpha-2}$. Then by (22) and (18) we have

$$\begin{aligned}\int_B |u(x)|^p dV_N(x) &\leq C\left(|u(0)|^p + \|u\|_{\mathscr{D}^2_\alpha}^{p-2}\int_B |\nabla u(x)|^2(1-|x|)^{p-\frac{n+\alpha}{2}(p-2)} dV_N(x)\right)\\ &\leq C\left(|u(0)|^p + \|u\|^p_{\mathscr{D}^2_\alpha}\right).\end{aligned}$$

From this, (18) and Theorem C we obtain

$$\|u\|_{\mathscr{H}^{\frac{2n-2}{n+\alpha-2}}(B)} \leq C\|u\|_{\mathscr{D}^2_\alpha}$$

from which the result follows. □

Corollary 7. *Let $u \in \mathscr{H}(B)$. If $u \in \mathscr{D}^2_\alpha$ for some $\alpha \in (-1, 1]$ with $n+\alpha > 2$, then the function $|u|^p$ admits a harmonic majorant in B for all $p \in \left[1, \frac{2n-2}{n+\alpha-2}\right]$.*

This corollary is a slight generalization of the following result [**35**. Theorem 3]:

Theorem E. *Let $u \in \mathscr{H}(B), n \geq 3$ such that*

$$\mathscr{D}^2_\alpha(u) = \int_B |\nabla u(x)|^2(1-|x|)^\alpha dV(x) < \infty$$

for an $\alpha, 0 \leq \alpha \leq 1$. Then for $p = (2n-2)/(n+\alpha-2)$, the function $|u|^p$ admits a harmonic majorant in B.

References

[1] A. Aleman and A. Siskakis, Integration operators on Bergman spaces, Indiana Univ. Math. J., **46** (1997), 337–356.

[2] S. Axler, The Bergman space, the Bloch space, and commutators of multiplication operators, Duke Math. J., **53** (1986), 315–332.

[3] S. Axler, P. Bourdon and W. Ramey, Harmonic function theory, Springer-Verlag, New York, 1992.

[4] R. R. Coifman, R. Rochberg and G. Weiss, Factorization theorems for Hardy spaces in several complex variables, Ann. of Math., **103** (1976), 611–635.

[5] P. L. Duren, Theory of H^p spaces, Academic Press, New York, 1970.

[6] C. Fefferman and E. M. Stein, H^p spaces of several variables, Acta Math., **129** (1972), 137–193.

[7] T. M. Flett, Inequalities for the pth mean values of harmonic and subharmonic functions with $p \leq 1$, Proc. London Math. Soc. (3), **20** (1970), 249–275.

[8] D. Gilbarg and N. S. Trudinger, Elliptic partial differential equations of second order, Springer-Verlag, Berlin, Heidelberg, New York, Tokyo, 1977.

[9] Y. Gotoh, On BMO functions on Riemann surface, J. Math. Kyoto Univ., **25** (1985), 331–339.

[10] G. H. Hardy and J. E. Littlewood, Some properties of conjugate function, J. Reine Angew. Math., **167** (1931), 405–423.

[11] W. Hayman, Multivalent functions, Cambridge Univ. Press, London, 1958.

[12] W. Hayman and P. B. Kennedy, Subharmonic functions, volume I, Academic Press, London, New York, San Francisco, 1976.

[13] Hu, Zhangjian, Estimates for the integral mean of harmonic functions on bounded domains in $\boldsymbol{R}^n$, Sci. China Ser. A, **38** (1995), 36–46.

[14] T. L. Kriete and B. D. MacCluer, Composition operators on large weighted Bergman spaces, Indiana Univ. Math. J., **41** (1992), 755–788.

[15] P. Lin and R. Rochberg, Hankel operators on the weighted Bergman spaces with exponential weights, Integral Equations Operator Theory, **21** (1995), 460–483.

[16] D. Luecking, Forward and reverse Carleson inequalities for functions in Bergman spaces and their derivatives, Amer. J. Math., **107** (1985), 85–111.

[17] D. Luecking, A new proof of an inequality of Littlewood and Paley, Proc. Amer. Math. Soc., **103** (1988), 887–893.

[18] K. Muramoto, Harmonic Bloch and BMO functions on the unit ball in several variables, Tokyo J. Math., **11** (1988), 381–386.

[19] A. Siskakis, Weighted integrals of analytic functions, Acta Sci. Math., **66** (2000), 651–664.

[20] S. Stević, An equivalent norm on BMO spaces, Acta Sci. Math., **66** (2000), 553–563.

[21] S. Stević, On eigenfunctions of the Laplace operator on a bounded domain, Sci. Ser. A Math. Sci. (N.S.), **7** (2001), 51–55.

[22] S. Stević, A note on weighted integrals of analytic functions, Bull. Greek Math. Soc., **46** (2002), 3–9.

[23] S. Stević, On an area inequality and weighted integrals of analytic functions, Result. Math., **41** (2002), 386–393.

[24] S. Stević, On harmonic Hardy and Bergman spaces, J. Math. Soc. Japan, **54** (2002), 983–996.

[25] S. Stević, On Bloch hyperharmonic functions, Ann. Math. Silesianae, **16** (2002), 57–64.

[26] S. Stević, Weighted integrals of harmonic functions, Studia Sci. Math. Hung., **39** (2002), 87–96.

[27] S. Stević, Weighted integrals of holomorphic functions in $\boldsymbol{C}^n$, Complex Variables, **47** (2002), 821–838.

[28] S. Stević, Composition operators on the generalized Bergman space, J. Indian Math. Soc., **69** (2002), 61–64.

[29] S. Stević, Inequalities for the gradient of eigenfunctions of the Laplace-Beltrami operator, Functiones et Approximatio, **31** (2003), 7–19.

[30] S. Stević, On harmonic Hardy spaces and area integrals, J. Math. Soc. Japan, **56** (2004), 339–347.

[31] S. Stević, On subharmonic behaviour of functions on balls in $\boldsymbol{C}^n$, Archives of Inequalities and Applications (to appear).

[32] M. Stoll, A characterization of Hardy spaces on the unit ball of $\boldsymbol{C}^n$, J. London Math. Soc., **48** (1993), 126–136.

[33] R. M. Timoney, Bloch functions in several complex variables, I, Bull. London Math. Soc., **12** (1980), 241–267.

[34] S. Yamashita, Criteria for functions to be of Hardy class H^p, Proc. Amer. Math. Soc., **75** (1979), 69–72.

[35] S. Yamashita, Dirichlet-finite functions and harmonic majorants, Illinois J. Math., **25** (1981), 626–631.

[36] S. Yamashita, Holomorphic functions and area integrals, Bollettino. U. M. I., **6** (1982), 115–120.

Stevo Stević

Mathematical Institute of Serbian Academy of Science
Knez Mihailova 35/I
11000 Beograd
Serbia
E-mail: sstevic@ptt.yu