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The second term of the semi-classical asymptotic expansion for Feynman path integrals with integrand of polynomial growth

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Abstract. Recently N. Kumano-go [**15**] succeeded in proving that piecewise linear time slicing approximation to Feynman path integral

$$\int F(\gamma)e^{i\nu S(\gamma)}\,\mathscr{D}[\gamma]$$

actually converges to the limit as the mesh of division of time goes to 0 if the functional $F(\gamma)$ of paths γ belongs to a certain class of functionals, which includes, as a typical example, Stieltjes integral of the following form;

$$F(\gamma)=\int_0^T f(t,\gamma(t))\rho(dt), \tag{1}$$

where $\rho(t)$ is a function of bounded variation and $f(t,x)$ is a sufficiently smooth function with polynomial growth as $|x|\to\infty$. Moreover, he rigorously showed that the limit, which we call the Feynman path integral, has rich properties (see also [**10**]).

The present paper has two aims. The first aim is to show that a large part of discussion in [**15**] becomes much simpler and clearer if one uses piecewise classical paths in place of piecewise linear paths.

The second aim is to explain that the use of piecewise classical paths naturally leads us to an analytic formula for the second term of the semi-classical asymptotic expansion of the Feynman path integrals under a little stronger assumptions than that in [**15**]. If $F(\gamma)\equiv 1$, this second term coincides with the one given by G. D. Birkhoff [**1**].

1. Introduction.

Let

$$L(t,\dot{x},x)=\frac{1}{2}|\dot{x}|^2-V(t,x)$$

be the Lagrangian with a smooth time dependent potential $V(t,x)$ on the configuration space $\boldsymbol{R}^d$. A path γ is a continuous or sufficiently smooth map from the time interval

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$[s, s']$ to $\boldsymbol{R}^d$. The action $S(\gamma)$ of a path γ is the functional

$$S(\gamma) = \int_s^{s'} L\Big(t, \frac{d}{dt}\gamma(t), \gamma(t)\Big)dt. \tag{1.1}$$

In [**4**] Feynman introduced the notion of an integral over the path space Ω, which is called Feynman path integral and is often denoted by

$$\int_\Omega e^{i\nu S(\gamma)} F(\gamma)\mathscr{D}[\gamma], \tag{1.2}$$

where $\nu = 2\pi h^{-1}$ with Planck's constant h. It was expected that Feynman path integral could have been defined as a measure theoretic integral if a suitable complex measure on the path space had been defined. However, Cameron [**2**] proved that this is not the case. (cf. also Johnson & Lapidus [**13**].)

Feynman himself gave the meaning to (1.2) as the limit of integrals over finite dimensional spaces. We call this method time slicing approximation method. In this paper as well as in Kumano-go [**15**] we show that his discussion can be made mathematically rigorous if $V(t,x)$ and $F(\gamma)$ satisfy suitable conditions. We assume that $V(t,x)$ is continuous in t and smooth in x. Moreover, we assume that for any non-negative integer m there exists a non-negative constant v_m such that

$$\max_{|\alpha|=m} \sup_{(t,x)\in[0,T]\times\boldsymbol{R}^d} \big|\partial_x^\alpha V(t,x)\big| \le v_m(1+|x|)^{\max\{2-m,0\}}. \tag{1.3}$$

Our assumption is close to that of Pauli [**3**].

We recall time slicing approximation method. Let $[s, s']$ be an interval of time. A path γ is classical if it is a solution to the Euler equation

$$\frac{d^2}{dt^2}\gamma(t) + (\nabla V)(t, \gamma(t)) = 0 \quad \text{for } s < t < s'. \tag{1.4}$$

Here and hereafter ∇ stands for the nabla operator in the configuration space $\boldsymbol{R}^d$.

For arbitrary pair of points $x, y \in \boldsymbol{R}^d$ there exists one and only one classical path γ which satisfies the boundary condition

$$\gamma(s) = y, \quad \gamma(s') = x \tag{1.5}$$

if $|s'-s| \le \mu$ with sufficiently small μ, say for instance (see §3),

$$\frac{\mu^2 d v_2}{8} < 1. \tag{1.6}$$

In this case the action $S(\gamma)$ of γ is a function of (s', s, x, y) and is denoted by $S(s', s, x, y)$, i.e.,

$$S(s', s, x, y) = \int_s^{s'} L\Big(t, \frac{d}{dt}\gamma(t), \gamma(t)\Big)\, dt. \tag{1.7}$$

Let

$$\varDelta : 0 = T_0 < T_1 < \cdots < T_J < T_{J+1} = T \tag{1.8}$$

be a division of the interval $[0, T]$. Then we set $t_j = T_j - T_{j-1}$ and define the mesh $|\varDelta|$ of the division $\varDelta$ by $|\varDelta| = \max_j\{t_j\}$. We always assume that

$$|\varDelta| \leq \mu. \tag{1.9}$$

Let

$$x_j \in \boldsymbol{R}^d, \quad j = 0, 1, \ldots, J, J+1, \tag{1.10}$$

be arbitrary $J+2$ points of the configuration space $\boldsymbol{R}^d$. The piecewise classical path $\gamma_\varDelta$ with vertices $(x_{J+1}, x_J, \ldots, x_1, x_0) \in \boldsymbol{R}^{d(J+2)}$ is the broken path that satisfies Euler's equation

$$\frac{d^2}{dt^2}\gamma_\varDelta(t) + (\nabla V)(t, \gamma_\varDelta(t)) = 0, \tag{1.11}$$

for $T_{j-1} < t < T_j$ $(j = 1, 2, \ldots, J, J+1)$ and boundary conditions

$$\gamma_\varDelta(T_j) = x_j, \quad j = 0, 1, \ldots, J, J+1, \tag{1.12}$$

where $x = x_{J+1}$ and $y = x_0$. When we emphasize the fact that this path $\gamma_\varDelta$ depends on $(x_{J+1}, x_J, \ldots, x_1, x_0)$, we denote it by $\gamma_\varDelta(x_{J+1}, x_J, \ldots, x_1, x_0)$ or $\gamma_\varDelta(t; x_{J+1}, x_J, \ldots, x_1, x_0)$, where t is the time variable.

Let $F(\gamma)$ be a functional defined for paths γ. Its value $F(\gamma_\varDelta)$ at $\gamma_\varDelta$ can be written as a function $F_\varDelta(x_{J+1}, x_J, \ldots, x_1, x_0)$ of $(x_{J+1}, x_J, \ldots, x_1, x_0)$. For example the action functional $S(\gamma_\varDelta)$ of $\gamma_\varDelta$ is given by

$$\begin{aligned} S_\varDelta(x_{J+1}, x_J, \ldots, x_1, x_0) = S(\gamma_\varDelta) &= \int_0^T L\Big(t, \frac{d}{dt}\gamma_\varDelta(t), \gamma_\varDelta(t)\Big)\, dt \\ &= \sum_{j=1}^{J+1} S_j(x_j, x_{j-1}), \end{aligned} \tag{1.13}$$

where we used the abbreviation

$$S_j(x_j, x_{j-1}) = S(T_j, T_{j-1}, x_j, x_{j-1}) = \int_{T_{j-1}}^{T_j} L\Big(t, \frac{d}{dt}\gamma_\varDelta(t), \gamma_\varDelta(t)\Big)\, dt. \tag{1.14}$$

A piecewise classical time slicing approximation to Feynman path integral (1.2) with the integrand $F(\gamma)$ is an oscillatory integral

$$\begin{aligned} &I[F_{\Delta}](\Delta; x, y) \\ &= \prod_{j=1}^{J+1}\left(\frac{\nu}{2\pi i t_j}\right)^{d/2}\int_{\boldsymbol{R}^{dJ}} e^{i\nu S(\gamma_{\Delta})}F(\gamma_{\Delta})\prod_{j=1}^{J}dx_j \\ &= \prod_{j=1}^{J+1}\left(\frac{\nu}{2\pi i t_j}\right)^{d/2}\int_{\boldsymbol{R}^{dJ}} e^{i\nu S_{\Delta}(x_{J+1},x_J,\dots,x_1,x_0)}F_{\Delta}(x_{J+1},x_J,\dots,x_1,x_0)\prod_{j=1}^{J}dx_j, \end{aligned} \tag{1.15}$$

where $x_{J+1}=x$ and $x_0=y$. See Feynman [**4**].

Feynman's definition of path integral (1.2) is

$$\int_{\Omega} e^{i\nu S(\gamma)}F(\gamma)\mathscr{D}[\gamma] = \lim_{|\Delta|\to 0} I[F_{\Delta}](\Delta; x, y), \tag{1.16}$$

if the limit on the right hand side exists.

We remark that Feynman [**4**] used also piecewise linear paths in place of piecewise classical paths. In that case we say piecewise linear time slicing approximation method.

Existence of the limit in (1.16) was proved in the case $F\equiv 1$ by [**5**], [**6**], [**7**], [**12**], [**17**]. Recently N. Kumano-go [**15**] proved the limit in (1.16) exists in the case of more general class of functional F using piecewise linear paths in place of piecewise classical paths.

2. Statement of results.

Although ν is a constant in Physics, we treat it as a parameter satisfying $\nu\ge 1$ in this paper, because our discussion is valid for any $\nu\ge 1$. We assume that the potential satisfies the assumption (1.3) and μ satisfies (1.6). Let Δ be as (1.8) and (1.9). Then the set $\Gamma(\Delta)$ of all piecewise classical paths associated with the division Δ forms a differentiable manifold of dimension $d(J+2)$. The correspondence $\gamma_{\Delta}\to(x_{J+1},\dots,x_0)$ is a global coordinate system. We will describe a basis of the tangent space $T_{\gamma_{\Delta}}\Gamma(\Delta)$ to $\Gamma(\Delta)$ at γ_{Δ}. Let $\{e_k\}_{k=1}^{d}$ be an ortho-normal frame of the configuration space $\boldsymbol{R}^d$, i.e., $x_j=\sum_{k=1}^{d}x_{j,k}e_k$ in our notation (1.10). Let $\eta_{j;k}(t)=\partial_{x_{j,k}}\gamma_{\Delta}(t)$. Then the functions $\{\eta_{j;k}\}_{0\le j\le J+1,1\le k\le d}$ form a basis of the tangent space $T_{\gamma_{\Delta}}\Gamma(\Delta)$. If $j=1,\dots,J$, then for $t\le T_{j-1}$ or $T_{j+1}\le t$,

$$\eta_{j;k}(t)=0, \tag{2.1}$$

and for $T_{j-1}<t<T_j$ or $T_j<t<T_{j+1}$ it satisfies Jacobi equation at γ_{Δ}

$$\frac{d^2}{dt^2}\eta_{j;k}(t)+\nabla\nabla V(t,\gamma_{\Delta}(t))\eta_{j;k}(t)=0, \tag{2.2}$$

and at $t=T_j$ it satisfies the boundary condition:

$$\eta_{j;k}(T_j) = e_k. \tag{2.3}$$

If $j=0$, $\eta_{0;k}(t)=0$ for $T_1 \le t$, (2.2) is satisfied for $0<t<T_1$ and (2.3) is satisfied at $t=T_0$. If $j=J+1$, $\eta_{J+1;k}(t)=0$ for $t \le T_J$, (2.2) is satisfied for $T_J < t < T_{J+1}$ and (2.3) is satisfied at $t=T_{J+1}$.

For a pair of divisions Δ' and Δ we use symbol $\Delta \prec \Delta'$ if Δ' is a refinement of Δ. If $\Delta \prec \Delta'$, then there is a natural inclusion $\Gamma(\Delta) \subset \Gamma(\Delta')$. This inclusion induces inclusion relation of the tangent spaces at γ_Δ, i.e., $T_{\gamma_\Delta}\Gamma(\Delta) \subset T_{\gamma_\Delta}\Gamma(\Delta')$. The set Γ of all piecewise classical paths is the inductive limit of $\{\Gamma(\Delta), \prec\}$, i.e., $\Gamma = \lim_{\to} \Gamma(\Delta)$. Γ is a dense subset of the Sobolev space $H^1([0,T];\boldsymbol{R}^d)$ of order 1 with values in $\boldsymbol{R}^d$ and hence it is also dense in the space $C([0,T];\boldsymbol{R}^d)$ of all continuous paths. Let $\gamma_\Delta \in \Gamma(\Delta)$. Then the tangent space $T_{\gamma_\Delta}\Gamma$ to Γ at γ_Δ is the inductive limit $\lim_{\to} T_{\gamma_\Delta}\Gamma(\Delta')$, which is a dense linear subspace of the Sobolev space $H^1([0,T];\boldsymbol{R}^d)$. (See Lemma 2 bellow).

Let $F(\gamma)$ be a functional defined on Γ. We denote its Fréchet differential at $\gamma \in \Gamma$ by DF_γ if it exists. And $DF_\gamma[\zeta]$ stands for its value at the tangent vector $\zeta \in T_\gamma\Gamma$. For any integer $n>0$ and for $\zeta_j \in T_\gamma\Gamma$ $(j=1,2,\ldots,n)$, we denote by $D^nF_\gamma[\zeta_1 \otimes \zeta_2 \otimes \cdots \otimes \zeta_n]$, the symmetric n-linear form on the tangent space arising from the n-th jet modulo $n-1$-th jet of F at γ.

We assume always in this paper that the functional $F(\gamma)$ satisfies both of the following conditions.

Assumption 1. *Let $m \ge 0$. For any non-negative integer K there exist positive constants A_K and X_K such that for any division Δ of the form* (1.8) *and any integer n_j $(0 \le j \le J+1)$ with $0 \le n_j \le K$*

$$\left| D^{\sum_{j=0}^{J+1} n_j} F_{\gamma_\Delta}\left[\otimes_{j=0}^{J+1} \otimes_{k=1}^{n_j} \zeta_{j,k}\right]\right| \le A_K X_K^{J+2}(1+\|\gamma_\Delta\|+|||\gamma_\Delta|||)^m \prod_{j=0}^{J+1}\prod_{k=1}^{n_j} \|\zeta_{j,k}\|, \tag{2.4}$$

as far as $\zeta_{j,k} \in T_{\gamma_\Delta}\Gamma$ satisfies

$$\operatorname{supp}\zeta_{j,k} \subset \begin{cases} [0,T_1] & \textit{if } \ j=0 \\ [T_{j-1},T_{j+1}] & \textit{if } \ 1 \le j \le J \\ [T_J,T_{J+1}] & \textit{if } \ j=J+1, \end{cases} \tag{2.5}$$

where $\|\zeta\| = \max_{0\le t\le T}|\zeta(t)|$ and $|||\gamma_\Delta|||=$ the total variation of γ_Δ.

Assumption 2 ([**15**] [**10**]). *There exists a positive bounded Borel measure ρ on $[0,T]$ such that with the same A_K, X_K as above*

$$\begin{aligned} &\left| D^{1+\sum_{j=0}^{J+1} n_j} F_{\gamma_\Delta}\left[\eta \otimes \otimes_{j=0}^{J+1} \otimes_{k=1}^{n_j} \zeta_{j,k}\right]\right| \\ &\quad \le A_K X_K^{J+2}(1+\|\gamma_\Delta\|+|||\gamma_\Delta|||)^m \int_{[0,T]} |\eta(t)|\rho(dt) \prod_{j=0}^{J+1}\prod_{k=1}^{n_j} \|\zeta_{j,k}\|, \end{aligned} \tag{2.6}$$

for any division $\varDelta$, integer $n_j \le K$ and $\zeta_{j,k}$ which are the same as in Assumption 1*. And η is an arbitrary element of $T_{\gamma_\varDelta}\varGamma$.*

We can prove the following

Theorem 1. *Assume that the integrand $F(\gamma)$ satisfies Assumption* 1 *and Assumption* 2 *above and T is so small that $|T| \le \mu$, Then the limit of the right hand side of* (1.16) *converges compact-uniformly with respect to $(x, y) \in \boldsymbol{R}^{2d}$.*

We shall make more precise statement. In what follows we always assume that $0 < T \le \mu$. For any fixed $(x, y) \in \boldsymbol{R}^{2d}$ the action $S(\gamma)$ has a unique critical point γ^*, which is the unique classical path starting y at time 0 and reaching x at time T. The critical point is non-degenerate. Similarly, if $0 < T \le \mu$, the function $S_\varDelta(x_{J+1}, x_J, \dots, x_1, x_0)$ of $(x_J, \dots, x_1)$ has only one critical point, which is non-degenerate. Regarding ν as a parameter satisfying $\nu \ge 1$, we can apply stationary phase method to (1.15). Stationary phase method says that $I[F_\varDelta](\varDelta; x, y)$ is of the following form:

$$I[F_\Delta](\varDelta; x, y) = \left(\frac{\nu}{2\pi i T}\right)^{d/2} D(\varDelta; x, y)^{-1/2} e^{i\nu S(\gamma^*)} \big(F(\gamma^*) + \nu^{-1} R_\Delta[F_\Delta](\nu, x, y)\big). \tag{2.7}$$

Here we used the following symbol

$$D(\varDelta; x, y) = \left(\frac{t_{J+1} t_J \dots t_1}{T}\right)^d \det \mathrm{Hess} S(\gamma_\Delta), \tag{2.8}$$

where $\mathrm{Hess} S(\gamma_\Delta)$ denotes the Hessian of $S(\gamma_\Delta)$ with respect to $(x_J, x_{J-1}, \dots, x_1)$ evaluated at the critical point. It is shown in [**8**] and [**15**] that for any non-negative integer K there exist a positive constant C_K and a positive integer $M(K)$ independent of ν and of $\varDelta$ such that

$$\big|\partial_x^\alpha \partial_y^\beta R_\Delta[F_\Delta](\nu, x, y)\big| \le C_K A_{M(K)} T(T + \rho([0, T]))(1 + |x| + |y|)^m, \tag{2.9}$$

as far as $|\alpha|, |\beta| \le K$. Here and hereafter $A_{M(K)}$ is the same constant as appeared in (2.4) and (2.6) with subscript $M(K)$. The function $D(\Delta; x, y)$ is of the form (cf. [**7**])

$$D(\Delta; x, y) = 1 + T^2 d(\Delta; x, y). \tag{2.10}$$

For any multi-indices α, β there exists a positive constant $C_{\alpha,\beta}$ such that

$$\big|\partial_x^\alpha \partial_y^\beta d(\Delta; x, y)\big| \le C_{\alpha,\beta}. \tag{2.11}$$

We also know (cf. [**7**]) that $D(T, x, y) = \lim_{|\Delta| \to 0} D(\varDelta; x, y)$ exists. Moreover, for any multi-indices α, β there exits a non-negative constant $C_{\alpha,\beta}$ such that

$$\big|\partial_x^\alpha \partial_y^\beta (D(T, x, y) - D(\varDelta; x, y))\big| \le C_{\alpha,\beta} |\varDelta| T. \tag{2.12}$$

$D(T,x,y)$ is of the form

$$D(T,x,y) = 1 + T^2 d(T,x,y), \tag{2.13}$$

where $d(T,x,y)$ satisfies the same estimate as (2.11).

The estimate (2.12) was proved earlier in [**7**]. So we do not discuss it here. The function $T^{-d}D(T,x,y)$ is the Morette-VanVleck determinant (cf. [**7**]).

THEOREM 2. *Under the same assumption as in Theorem* 1 *we can write the limit* $\lim_{|\Delta|\to 0} I[F_\Delta](\Delta;x,y)$ *in the following way*:

$$\begin{aligned}\int_\Omega e^{i\nu S(\gamma)}F(\gamma)\mathscr{D}[\gamma] &= \lim_{|\Delta|\to 0} I[F_\Delta](\Delta;x,y)\\ &= \left(\frac{\nu}{2\pi i T}\right)^{d/2} D(T,x,y)^{-1/2}e^{i\nu S(\gamma^*)}\big(F(\gamma^*)+\nu^{-1}R[F](\nu,x,y)\big).\end{aligned} \tag{2.14}$$

Moreover, for any non-negative integer K there exist positive constant C_K and a non-negative integer $M(K)$ independent of ν and of Δ such that

$$\begin{aligned}&\left|\partial_x^\alpha\partial_y^\beta(R[F](\nu,x,y) - R_\Delta[F_\Delta](\nu,x,y))\right|\\ &\quad\le C_K A_{M(K)}|\Delta|\big(\rho([0,T]) + T^2 + T^3 + T^2\rho([0,T]) + T\nu^{-1}\big)(1+|x|+|y|)^m,\end{aligned} \tag{2.15}$$

as far as $|\alpha|\le K$ and $|\beta|\le K$.

COROLLARY 1. *For any non-negative integer K there exist a positive constant C_K and a non-negative integer $M(K)$ independent of ν such that*

$$\left|\partial_x^\alpha\partial_y^\beta R[F](\nu,x,y)\right| \le C_K A_{M(K)} T(T+\rho([0,T]))(1+|x|+|y|)^m. \tag{2.16}$$

It is expected that the following semi-classical asymptotic expansion holds;

$$\int_\Omega e^{i\nu S(\gamma)}F(\gamma)\mathscr{D}[\gamma] = \left(\frac{\nu}{2\pi i T}\right)^{d/2} D(T,x,y)^{-1/2}e^{i\nu S(\gamma^*)}\big(A_0 + \nu^{-1}A_1 + O(\nu^{-2})\big)\quad \text{as } \nu\to\infty. \tag{2.17}$$

Theorem 2 implies $A_0 = F(\gamma^*)$. What is the next term A_1?

In the case $F(\gamma)\equiv 1$ assuming the existence of expansion, Birkhoff gave the answer [**1**]. In fact, he gave even higher order terms of asymptotic expansion. However, if $F(\gamma)\ne$ *constant*, then his method does not apply.

We write down the second term A_1 of (2.17) for general $F(\gamma)$ and prove that the asymptotic expression (2.17) actually holds. For this purpose we introduce a new piece-wise classical path. Let ϵ be an arbitrary small positive number. And $\Delta(t,\epsilon)$ be the division

$$\Delta(t,\epsilon) : 0 = T_0 < t < t+\epsilon < T. \tag{2.18}$$

Let z be an arbitrary point in $\boldsymbol{R}^d$. We abbreviate the piecewise classical path $\gamma_{\Delta(t,\epsilon)}(s;x,\gamma^*(t+\epsilon),z,y)$ associated with the division $\Delta(t,\epsilon)$ by $\gamma_{\{t,\epsilon\}}(s,z)$, i.e., $\gamma_{\{t,\epsilon\}}(s,z)$ is the piecewise classical path which satisfies conditions:

$$\gamma_{\{t,\epsilon\}}(0,z) = y, \quad \gamma_{\{t,\epsilon\}}(t,z) = z, \quad \gamma_{\{t,\epsilon\}}(t+\epsilon,z) = \gamma^*(t+\epsilon), \quad \gamma_{\{t,\epsilon\}}(T,z) = x. \tag{2.19}$$

It is clear that $\gamma_{\{t,\epsilon\}}(s,z)$ coincides with $\gamma^*(s)$ for $t+\epsilon \le s \le T$ independently of z. Therefore, $\partial_z \gamma_{\{t,\epsilon\}}(s,z) = 0$ for $t+\epsilon \le s \le T$.

Lemma 1. *Under Assumption* 1 *and Assumption* 2 *we have bounded convergence of the limit*

$$q(t) = \lim_{\epsilon \to +0} \Big[\Delta_z \big(D(t,z,y)^{-1/2} F(\gamma_{\{t,\epsilon\}}(*,z)) \big)\big|_{z=\gamma^*(t)} \Big], \tag{2.20}$$

where Δ_z stands for the Laplacian with respect to z.

Theorem 3. *In addition to Assumptions* 1 *and* 2 *we further assume that the function $q(t)$ of Lemma* 1 *is Riemannian integrable over* $[0,T]$. *Set*

$$A_1 = \frac{i}{2} \int_0^T D(t,\gamma^*(t),y)^{1/2} q(t) dt. \tag{2.21}$$

Then, there holds the asymptotic formula, as $\nu \to \infty$,

$$\begin{aligned} &\int_\Omega e^{i\nu S(\gamma)} F(\gamma) \mathscr{D}[\gamma] \\ &\quad = \left(\frac{\nu}{2\pi i T} \right)^{d/2} D(T,x,y)^{-1/2} e^{i\nu S(\gamma^*)} \big(A_0 + \nu^{-1} A_1 + \nu^{-2} r(\nu,x,y) \big), \end{aligned} \tag{2.22}$$

where for any α,β the remainder term $r(\nu,x,y)$ satisfies estimate

$$\big| \partial_x^\alpha \partial_y^\beta r(\nu,x,y) \big| \le C_{\alpha,\beta} T^2 (1+|x|+|y|)^m. \tag{2.23}$$

We shall calculate $q(t)$ in more detail for some simple functionals $F(\gamma)$ in §5.

Since our method is completely different from Birkhoff's method, it may be interesting to see that this formula coincides with Birkhoff's result in the case of $F(\gamma) \equiv 1$. This will be done in §6.

Remark 1. In this note the Lagrangian has no vector potential. Kitada-Kumano-go [**14**], Yajima [**17**] and Tshuchida-Fujiwara [**12**] discussed the case of Lagrangian with non zero vector potential. They proved that the limit (1.16) exists and the limit is the fundamental solution of Schrödinger equation if $F(\gamma) \equiv 1$. However we do not know whether the limit (1.16) exists or not if $F(\gamma) \neq$ *constant* and Lagrangian has non-zero vector potential.

3. Proof of Theorem 2.

3.1. Fitting assumptions to stationary phase method.

In order to apply the result of our paper [**11**] to prove theorems stated in the previous section, we show in this subsection that Assumptions 1 and 2 of the present paper imply the assumptions appeared in [**11**]. For that purpose we wish to estimate derivatives of $F_\Delta(x_{J+1}, x_J, \ldots, x_1, x_0)$.

LEMMA 2. *Assume that $|s' - s| \leq \mu$. Let $\gamma(t)$, $s \leq t \leq s'$, be the classical path with end points*:

$$\gamma(s) = y, \quad \gamma(s') = x. \tag{3.1}$$

There exists a positive constant C such that

$$\left|\gamma(t) - \left(\frac{t-s}{s'-s}x + \frac{s'-t}{s'-s}y\right)\right| \leq C|s'-s|^2(1+|x|+|y|),$$
$$\left|\left(\frac{d}{dt}\gamma(t) - \frac{1}{s'-s}(x-y)\right)\right| \leq C|s-s'|(1+|x|+|y|)$$

and for any multi-indices α, β with $|\alpha| + |\beta| > 0$ there exists a positive constant $C_{\alpha,\beta}$ such that

$$\left|\partial_x^\alpha \partial_y^\beta \left(\gamma(t) - \left(\frac{t-s}{s'-s}x + \frac{s'-t}{s'-s}y\right)\right)\right| \leq C_{\alpha,\beta}(s'-s)^2,$$
$$\left|\partial_x^\alpha \partial_y^\beta \left(\frac{d}{dt}\gamma(t) - \frac{1}{s'-s}(x-y)\right)\right| \leq C_{\alpha,\beta}|s'-s|.$$

PROOF. These are consequences of Euler's equation (1.4) and boundary conditions (3.1). See §2 of [**11**] for the details. □

PROPOSITION 1. *Let γ_Δ be the piecewise classical path which satisfies* (1.11) *and* (1.12). *Then there exists a positive constant C such that*

$$|||\gamma_\Delta||| \leq C(1 + |x_{J+1}| + |x_J| + \cdots + |x_1| + |x_0|). \tag{3.2}$$

For any multi-indices α_j and α_{j-1} with $|\alpha_j| + |\alpha_{j-1}| \geq 1$ there exists a positive constant $C_{\alpha_j,\alpha_{j-1}}$ such that

$$\left|\partial_{x_j}^{\alpha_j}\partial_{x_{j-1}}^{\alpha_{j-1}}\gamma_\Delta(t)\right| \leq C_{\alpha_j,\alpha_{j-1}} \quad \textit{for any } t \in [0,T]. \tag{3.3}$$

We have $\mathrm{supp}\partial_{x_0}\gamma_\Delta \subset [T_0, T_1]$, $\mathrm{supp}\partial_{x_{J+1}}\gamma_\Delta \subset [T_J, T_{J+1}]$ *and*

$$\mathrm{supp}\partial_{x_j}\gamma_\Delta \subset [T_{j-1}, T_{j+1}] \quad \textit{if} \quad 0 < j < J+1. \tag{3.4}$$

If $|j-k| \geq 2$, *then*

$$\partial_{x_j}\partial_{x_k}\gamma_\Delta(t) = 0 \quad \textit{for any } t \in [0,T]. \tag{3.5}$$

Proof. As a consequence of the above lemma 2 we have

$$\int_{T_{j-1}}^{T_j} \left|\frac{d}{dt}\gamma_\Delta(t)\right| dt \leq |x_j - x_{j-1}| + Ct_j^2(1 + |x_j| + |x_{j-1}|).$$

Taking summation with respect to j, we have

$$|||\gamma_\Delta||| \leq C(1 + |x_{J+1}| + |x_J| + \cdots + |x_1| + |x_0|).$$

This proves (3.2). Other parts of Proposition are obvious. □

Now we can obtain bounds of derivatives of $F_\Delta(x_{J+1}, x_J, \ldots, x_1, x_0)$.

Proposition 2. *Let $F(\gamma)$ be a functional defined on Γ satisfying Assumption* 1 *and* 2. *Then for any non-negative integer K there exist positive constants A_K and X_K such that for any division Δ of* (1.8) *and for any multi-indices α_j, $j = 0, 1, \ldots, J+1$ satisfying $|\alpha_j| \leq K$ there holds the inequality*

$$\left|\left(\prod_{k=0}^{J+1}\partial_{x_j}^{\alpha_j}\right)F_\Delta\right| \leq A_K X_K^{J+2}(1 + |x_{J+1}| + |x_J| + \cdots + |x_1| + |x_0|)^m. \tag{3.6}$$

Moreover for any $k = 1, 2 \ldots, J$

$$\begin{aligned}&\left|\left(\prod_{j=0}^{J+1}\partial_{x_j}^{\alpha_j}\right)\partial_{x_k}F_\Delta\right| \\ &\quad\leq A_K X_K^{J+2}\rho([T_{k-1}, T_{k+1}])(1 + |x_{J+1}| + |x_J| + \cdots + |x_1| + |x_0|)^m.\end{aligned} \tag{3.7}$$

Proof. We begin with an obvious remark. By completing the tangent space $T_{\gamma_\Delta}\Gamma$ with respect to the maximum norm, we may assume that the inequalities (2.4) and (2.6) hold for $\zeta_{j,k} \in C([0,T], \boldsymbol{R}^d)$ with support conditions (2.5).

Leibnitz' rule and (3.5) give that for any multi-indices $\alpha_{J+1}, \alpha_J, \cdots, \alpha_0$

$$\begin{aligned}\partial_{x_{J+1}}^{\alpha_{J+1}}\partial_{x_J}^{\alpha_J}\cdots\partial_{x_0}^{\alpha_0}F(\gamma_\Delta) = &\sum_{l=1}^{|\alpha_{J+1}|+\cdots+|\alpha_0|}{\sum}' C\big(\{\alpha_j\}_{j=0}^{J+1}, l, \big\{(\beta_{j,n}, \beta'_{j-1,n}), m(j,n)\big\}_{j,n}\big) \\ &\times D^l F_{\gamma_\Delta}\Big[\otimes_{j=1}^{J+1}\otimes_n\big(\partial_{x_j}^{\beta_{j,n}}\partial_{x_{j-1}}^{\beta'_{j-1,n}}\gamma_\Delta\big)^{\otimes m(j,n)}\Big].\end{aligned} \tag{3.8}$$

Here $C(\{\alpha_j\}_{j=0}^{J+1}, l, \{(\beta_{j,n}, \beta'_{j-1,n}), m(j,n)\}_{j,n})$ are non-negative constants. And the summation $\sum'$ ranges over all pairs of multi-indices $\{(\beta_{j,n}, \beta'_{j-1,n})\}$ and natural numbers $\{m(j,n)\}_{j,n}$ such that

$$\begin{aligned} &\sum_n m(j,n)\beta_{j,n} + \sum_n m(j+1,n)\beta'_{j,n} = \alpha_j, \quad \text{for } \ j = 0, \dots, J+1, \\ &\sum_{j,n} m(j,n) = l. \end{aligned} \tag{3.9}$$

Let $K \geq 0$. Then by virtue of relation (3.5) we can easily show by induction on J that there exists a positive constant C_K independent of J such that

$$\sum_{l=1}^{|\alpha_{J+1}|+\cdots+|\alpha_0|} \sum{}' C\big(\{\alpha_j\}_{j=0}^{J+1}, l, \big\{\big(\beta_{j,n}, \beta'_{j-1,n}\big), m(j,n)\big\}_{j,n}\big) \leq C_K^{J+2} \tag{3.10}$$

as far as $|\alpha_j| \leq K$ for $j = 0, \dots, J+1$.

Since (3.4) holds and $\sum_n m(j,n) \leq K$ for $j = 1, 2, \dots, J+1$, we can use the extended form of (2.4) and obtain

$$\begin{aligned} &\Big| D^l F_{\gamma_\Delta}\big[\otimes_{j=1}^{J+1} \otimes_n \big(\partial_{x_j}^{\beta_{j,n}} \partial_{x_{j-1}}^{\beta'_{j-1,n}} \gamma_\Delta\big)^{\otimes m(j,n)}\big]\Big| \\ &\quad \leq A_K X_K^{J+2} (1 + \|\gamma_\Delta\| + |||\gamma_\Delta|||)^m \prod_j \prod_n \big\| \big(\partial_{x_j}^{\beta_{j,n}} \partial_{x_{j-1}}^{\beta'_{j-1,n}} \gamma_\Delta\big)\big\|^{m(j,n)}. \end{aligned}$$

Using Proposition 1, we obtain with some positive constant B_K and C independent of J

$$\begin{aligned} &\Big| D^l F_{\gamma_\Delta}\big[\otimes_{j=1}^{J+1} \otimes_n \big|\big(\partial_{x_j}^{\beta_{j,n}} \partial_{x_{j-1}}^{\beta'_{j-1,n}} \gamma_\Delta\big)^{\otimes m(j,n)}\big]\Big| \\ &\quad \leq A_K X_K^{J+2} (1 + C(1 + |x_{J+1}| + |x_J| + \cdots + |x_1| + |x_0|))^m \prod_j \prod_n B_K^{m(j,n)} \\ &\quad \leq A_K X_K^{J+2} {B_K}^l (1 + C)^m (1 + |x_{J+1}| + |x_J| + \cdots + |x_1| + |x_0|)^m. \end{aligned} \tag{3.11}$$

Since we may assume $B_K \geq 1$, we combine estimate (3.11) with (3.10) and obtain that

$$\begin{aligned} &\Big| \Big(\prod_{k=0}^{J+1} \partial_{x_j}^{\alpha_j} \Big) F_\Delta \Big| \\ &\quad \leq C_K^{J+2} A_K ({B_K}^K X_K)^{J+2} (1 + C)^m (1 + |x_{J+1}| + |x_J| + \cdots + |x_1| + |x_0|)^m \\ &\quad \leq A'_K {X'_K}^{J+2} (1 + |x_{J+1}| + |x_J| + \cdots + |x_1| + |x_0|)^m \end{aligned}$$

with some constants A'_K,X'_K independent of J. Estimate (3.6) is proved.

Now we prove (3.7). If $\alpha_k \neq 0$, then one of $\beta_{k,n}$ or $\beta'_{k,n+1}$ in (3.8) is not 0. If

$\beta_{k,n_0} \neq 0$ with some n_0, then we can use (3.4) and obtain with some $B_K > 0$

$$\int_{[0,T]} \left|\partial_{x_k}^{\beta_{k,n_0}} \partial_{x_{k-1}}^{\beta'_{k-1,n_0}} \gamma_\Delta(t)\right| \rho(dt) \leq B_K \rho([T_{k-1}, T_{k+1}]).$$

Making use of (2.6), we obtain

$$\begin{aligned}&\left| D^l F_{\gamma_\Delta}\left[\otimes_{j=1}^{J+1} \otimes_n \left(\partial_{x_j}^{\beta_{j,n}} \partial_{x_{j-1}}^{\beta'_{j-1,n}} \gamma_\Delta\right)^{m(j,n)}\right]\right| \\ &\quad \leq A_K X_K^{J+2} C(1 + |x_{J+1}| + |x_J| + \cdots + |x_1| + |x_0|)^m B_K^l \rho([T_{k-1}, T_{k+1}]).\end{aligned}$$

Similar estimate holds if $\beta'_{k,n_0+1} \neq 0$. Combining this and (3.10), we obtain

$$\begin{aligned}&\left|\partial_{x_k}\left(\prod_{k=0}^{J+1} \partial_{x_j}^{\alpha_j}\right) F_\Delta\right| \\ &\quad \leq A_K (C_K {B_K}^K X_K)^{J+2} C(1 + |x_{J+1}| + |x_J| + \cdots + |x_1| + |x_0|)^m \rho([T_{k-1}, T_{k+1}]) \\ &\quad \leq A'_K {X'_K}^{J+2} (1 + |x_{J+1}| + |x_J| + \cdots + |x_1| + |x_0|)^m \rho([T_{k-1}, T_{k+1}])\end{aligned}$$

with another constants $A'_K > 0$ and $X'_K > 0$ independent of J. This proves (3.7). Proposition 2 has been proved. □

3.2. Proof of Theorem 2.

Consider an arbitrary division Δ of time interval $[0, T]$ as follows:

$$\Delta : 0 = T_0 < T_1 < \cdots < T_J < T_{J+1} = T. \tag{3.12}$$

First we shall discuss a special simple type of refinement of Δ. Let n be any one of $n = 0, 1, \ldots, J$. We divide the $n+1$-th subinterval $I_n = [T_n, T_{n+1}]$ into smaller sub-subintervals and denote the division of I_n by δ, i.e.,

$$\delta : T_n = T_{n,0} < T_{n,1} < \cdots < T_{n,p_n+1} = T_{n+1}. \tag{3.13}$$

Adding these new division points of δ to Δ and keeping other $[T_j, T_{j+1}]$ unchanged, we get a refinement Δ' of Δ. In other words, Δ' is the same as Δ except for the division δ of $[T_n, T_{n+1}]$. We set $\sigma_k = T_{n,k} - T_{n,k-1}$.

We claim that the following estimate holds: For any non-negative integer K there exist a positive constant C_K and a non-negative integer $M(K)$ such that if $|\alpha|, |\beta| \leq K$ we have

$$\begin{aligned}&\left|\partial_x^\alpha \partial_y^\beta (R_{\Delta'}[F_{\Delta'}](\nu, x, y) - R_\Delta[F_\Delta](\nu, x, y))\right| \\ &\quad \leq C_K A_{M(K)} t_{n+1} \left(\rho([T_n, T_{n+1}]) + t_{n+1}(T + T^2 + T\rho([0, T]) + \nu^{-1})\right)(1 + |x| + |y|)^m.\end{aligned} \tag{3.14}$$

Assuming this claim for the moment, we shall prove Theorem 2. Let

$$\Delta^* : 0 = \tau_0 < \tau_1 < \cdots < \tau_{L+1} = T \tag{3.15}$$

be an arbitrary refinement of Δ. It is clear that Δ^* is obtained from Δ by repeating the above type of division $J+1$ times. In fact, consider for any $j = 1, \ldots, J+1$ a new refinement Δ_j of Δ whose dividing points consist of all of τ_k of (3.15) satisfying $\tau_k \le T_j$ and all T_m of (3.12) satisfying $T_m \ge T_j$. We set $\Delta_0 = \Delta$. Then we have a chain of refinements of Δ:

$$\Delta = \Delta_0 \prec \Delta_1 \prec \Delta_2 \prec \cdots \prec \Delta_{J+1} = \Delta^*.$$

Δ_{n+1} is obtained from Δ_n in the same way as Δ' is obtained from Δ. We apply the claim to $R_{\Delta_{n+1}}[F_{\Delta_{n+1}}](\nu, x, y) - R_{\Delta_n}[F_{\Delta_n}](\nu, x, y)$ and obtain that

$$\begin{aligned}
&\bigl|\partial_x^\alpha \partial_y^\beta (R_{\Delta^*}[F_{\Delta^*}](\nu, x, y) - R_\Delta[F_\Delta](\nu, x, y))\bigr| \\
&\quad = \left| \sum_{n=0}^{J} \bigl(\partial_x^\alpha \partial_y^\beta (R_{\Delta_{n+1}}[F_{\Delta_{n+1}}](\nu, x, y) - R_{\Delta_n}[F_{\Delta_n}](\nu, x, y))\bigr) \right| \\
&\quad \le \sum_{n=0}^{J} C_K A_{M(K)} t_{n+1} \bigl(\rho([T_n, T_{n+1}]) + t_{n+1}(T + T^2 + T\rho([0,T]) + \nu^{-1})\bigr) \\
&\qquad \times (1 + |x| + |y|)^m \\
&\quad \le C_K A_{M(K)} |\Delta| \bigl(\rho([0,T]) + T^2 + T^3 + T^2\rho([0,T]) + T\nu^{-1}\bigr)(1 + |x| + |y|)^m. \qquad (3.16)
\end{aligned}$$

This means that $\{\partial_x^\alpha \partial_y^\beta R_\Delta[F_\Delta](\nu, x, y)\}_\Delta$ forms a Cauchy net. Therefore,

$$\lim_{|\Delta| \to 0} R_\Delta[F_\Delta](\nu, x, y) = R[F](\nu, x, y)$$

exists. Tending $|\Delta^*| \to 0$ of (3.16), we have the estimate (2.15). Theorem 2 is proved up to the proof of the claim.

Now we prove the claim and complete the proof of Theorem 2. We consider an arbitrary piecewise classical path $\gamma_{\Delta'}$ associated with the division Δ' and we write

$$\begin{aligned}
y_k &= \gamma_{\Delta'}(T_{n,k}), \quad \text{for } \ 0 \le k \le p_n + 1, \\
x_j &= \gamma_{\Delta'}(T_j), \quad \text{for } \ 0 \le j \le J + 1,
\end{aligned}$$

where we assume that $y_0 = x_n$, $y_{p_n+1} = x_{n+1}$. We abbreviate the block of variables $(y_{p_n}, \ldots, y_1)$ by $y_{[p_n,1]}$. Similarly for any pair of integers $k \ge j \ge 0$ we denote $(x_k, \ldots, x_j)$ by $x_{[k,j]}$. As a special case we set $x_{[k,k]} = x_k$. Let

$$S_{n,k}(y_k, y_{k-1}) = \int_{T_{n,k-1}}^{T_{n,k}} L\Bigl(t, \frac{d}{dt}\gamma_{\Delta'}(t), \gamma_{\Delta'}(t)\Bigr) dt \quad (k = 1, \ldots, p_n + 1). \tag{3.17}$$

Then the action $S(\gamma_{\Delta'})$ becomes

$$\begin{aligned} S(\gamma_{\Delta'}) &= S_{\Delta'}(x_{[J+1,n+1]}, y_{[p_n,1]}, x_{[n,0]}) \\ &= \sum_{j=1, j\neq n+1}^{J+1} S_j(x_j, x_{j-1}) + \sum_{k=1}^{p_n+1} S_{n,k}(y_k, y_{k-1}). \end{aligned} \tag{3.18}$$

We set the latter term of (3.18) as

$$S_\delta(x_{n+1}, y_{[p_n,1]}, x_n) = \sum_{k=1}^{p_n+1} S_{n,k}(y_k, y_{k-1}). \tag{3.19}$$

The value of the functional $F(\gamma)$ restricted to $\Gamma(\Delta')$ is

$$F_{\Delta'}(x_{[J+1,n+1]}, y_{[p_n,1]}, x_{[n,0]}) = F\big(\gamma_{\Delta'}(x_{[J+1,n+1]}, y_{[p_n,1]}, x_{[n,0]})\big). \tag{3.20}$$

We write $I[F_{\Delta'}](\Delta'; x, y)$ in two different ways and compare results to obtain an expression of the difference $R_{\Delta'}[F_{\Delta'}](\nu, x, y) - R_\Delta[F_\Delta](\nu, x, y)$.

First we integrate it with respect to all the variables $(x_{[J,n+1]}, y_{[p_n,1]}, x_{[n,1]})$ and apply the stationary phase method. Then we have just as (2.7)

$$I[F_{\Delta'}](\Delta'; x, y) = \left(\frac{\nu}{2\pi i T}\right)^{d/2} D(\Delta'; x, y)^{-1/2} e^{i\nu S(\gamma^*)} \big(F(\gamma^*) + \nu^{-1} R_{\Delta'}[F_{\Delta'}](\nu, x, y)\big). \tag{3.21}$$

Next we distinguish two groups of variables $(x_J, \ldots, x_1)$ and $(y_{p_n}, \ldots, y_1)$. Integrating with respect to $(y_{p_n}, \ldots, y_1)$ prior to integration with respect to $(x_J, \ldots, x_1)$, we define $F_{\Delta/\Delta'}(x_{J+1}, x_J, \ldots, x_1, x_0)$ by the equality

$$\begin{aligned} &\left(\frac{\nu}{2\pi i t_{n+1}}\right)^{d/2} e^{i\nu S_{n+1}(x_{n+1}, x_n)} F_{\Delta/\Delta'}(x_{J+1}, x_J, \ldots, x_1, x_0) \\ &= \prod_{k=1}^{p_n+1} \left(\frac{\nu}{2i\pi\sigma_k}\right)^{d/2} \int_{\boldsymbol{R}^{dp_n}} e^{i\nu S_\delta(x_{n+1}, y_{p_n}, \ldots, y_1, x_n)} \\ &\quad \times F_{\Delta'}(x_{[J+1,n+1]}, y_{p_n}, \ldots, y_1, x_{[n,0]}) \prod_{k=1}^{p_n} dy_k. \end{aligned} \tag{3.22}$$

Using (3.18) and (3.19), we have

$$\begin{aligned} I[F_{\Delta'}](\Delta'; x, y) = &\prod_{j=1}^{J+1} \left(\frac{\nu}{2\pi i t_j}\right)^{d/2} \int_{\boldsymbol{R}^{dJ}} e^{i\nu S_\Delta(x_{J+1}, x_J, \ldots, x_1, x_0)} \\ &\times F_{\Delta/\Delta'}(x_{J+1}, x_J, \ldots, x_1, x_0) \prod_{j=1}^{J} dx_j. \end{aligned} \tag{3.23}$$

In other words,

$$I[F_{\Delta'}](\Delta'; x, y) = I[F_{\Delta/\Delta'}](\Delta; x, y). \tag{3.24}$$

Now we shall compare $F_{\Delta}(x_{J+1}, \ldots, x_0)$ with $F_{\Delta/\Delta'}(x_{J+1}, \ldots, , x_0)$. Since Proposition 2 holds, we can apply the result of [**11**] to the integral (3.22) with respect to variables $\{y_{p_n}, \ldots, y_1\}$. We obtain that

$$\begin{aligned} F_{\Delta/\Delta'}(x_{J+1}, x_J, \ldots, x_1, x_0) = D(\delta; x_{n+1}, x_n)^{-1/2}\big(&F_{\Delta'}(x_{[J+1,n+1]}, y^*_{[p_n,1]}, x_{[n,0]}) \\ &+ \nu^{-1} R_\delta[F_{\Delta'}](\nu, x_{J+1}, x_J, \ldots, x_1, x_0)\big). \end{aligned} \tag{3.25}$$

Here as in (2.8), we use the symbol

$$D(\delta; x_{n+1}, x_n) = \left(\frac{\sigma_{p_n+1}\sigma_{p_n}\ldots\sigma_1}{t_{n+1}}\right)^d \det \mathrm{Hess} S_\delta$$

at the critical point $y^*_{[p_n,1]}$ of the phase function

$$y_{[p_n,1]} \longrightarrow S_\delta(x_{n+1}, y_{[p_n,1]}, x_n). \tag{3.26}$$

$F_{\Delta'}(x_{[J+1,n+1]}, y^*_{[p_n,1]}, x_{[n,0]})$ is the main term of (3.25) and the remainder term is $\nu^{-1} R_\delta[F_{\Delta'}](\nu, x_{J+1}, x_J, \ldots, x_1, x_0)$.

The fact that $y^*_{[p_n,1]}$ is the critical point of (3.26) means that it is the solution of the system of equations

$$\partial_{y_k}\big(S_{n,k+1}(y_{k+1}, y_k) + S_{n,k}(y_k, y_{k-1})\big) = 0 \quad \text{for} \ \ 1 \le k \le p_n.$$

This implies that the path $\gamma_{\Delta'}(x_{[J+1,n+1]}, y^*_{[p_n,1]}, x_{[n,0]})$ is not broken at time $T_{n,k}$ for $k = 1, \ldots, p_n$, i.e., $\gamma_{\Delta'}(x_{[J+1,n+1]}, y^*_{[p_n,1]}, x_{[n,0]})$ is smooth for $T_n < t < T_{n+1}$. Therefore,

$$\gamma_{\Delta'}(x_{[J+1,n+1]}, y^*_{[p_n,1]}, x_{[n,0]}) = \gamma_\Delta(x_{J+1}, x_J, \ldots, x_1, x_0).$$

And we have

$$F_{\Delta'}(x_{[J+1,n+1]}, y^*_{[p_n,1]}, x_{[n,0]}) = F_\Delta(x_{J+1}, x_J, \ldots, x_1, x_0).$$

Substituting this in (3.25), we obtain

$$\begin{aligned} F_{\Delta/\Delta'}(x_{J+1}, x_J, \ldots, x_1, x_0) = D(\delta; x_{n+1}, x_n)^{-1/2}\big(&F_\Delta(x_{J+1}, x_J, \ldots, x_1, x_0) \\ &+ \nu^{-1} R_\delta[F_{\Delta'}](\nu, x_{J+1}, x_J, \ldots, x_1, x_0)\big). \end{aligned} \tag{3.27}$$

It follows from this equality (3.27) and (3.24) that

$$\begin{aligned} I[F_{\Delta'}](\Delta';x,y) &= I[F_{\Delta/\Delta'}](\Delta;x,y) \\ &= I[D(\delta;x_{n+1},x_n)^{-1/2}F_\Delta](\Delta;x,y) \\ &\quad + \nu^{-1}I[D(\delta;x_{n+1},x_n)^{-1/2}R_\delta[F_{\Delta'}]](\Delta;x,y). \end{aligned} \tag{3.28}$$

Apply the stationary phase method to the first term of the right hand side and use (2.7). Then we obtain the equality:

$$\begin{aligned} I[F_{\Delta'}](\Delta';x,y) = \left(\frac{\nu}{2\pi iT}\right)^{d/2} & e^{i\nu S(\gamma^*)}D(\Delta;x,y)^{-1/2} \\ \times \big(D(\delta; & x^*_{n+1},x^*_n)^{-1/2}F(\gamma^*) \\ & + \nu^{-1}R_\Delta[D(\delta;x_{n+1},x_n)^{-1/2}F_\Delta](\nu,x,y) + \nu^{-1}a(\Delta;\nu,x,y)\big). \end{aligned} \tag{3.29}$$

Here $x^*_n = \gamma^*(T_n)$, $x^*_{n+1} = \gamma^*(T_{n+1})$ and $a(\Delta;\nu,x,y)$ is defined by

$$\begin{aligned} & I[D(\delta;x_{n+1},x_n)^{-1/2}R_\delta[F_{\Delta'}]](\Delta;x,y) \\ & = \left(\frac{\nu}{2\pi iT}\right)^{d/2} e^{i\nu S(\gamma^*)}D(\Delta;x,y)^{-1/2}a(\Delta;\nu,x,y). \end{aligned} \tag{3.30}$$

It is shown in (cf. [**7**]) that

$$D(\Delta;x,y)D(\delta;x^*_{n+1},x^*_n) = D(\Delta';x,y). \tag{3.31}$$

Taking this equality in mind, we compare (3.21) to (3.29) and obtain that

$$\begin{aligned} & D(\Delta';x,y)^{-1/2}R_{\Delta'}[F_{\Delta'}](\nu,x,y) \\ & \quad = D(\Delta;x,y)^{-1/2}\big(R_\Delta[D(\delta;x_{n+1},x_n)^{-1/2}F_\Delta](\nu,x,y) + a(\Delta;\nu,x,y)\big). \end{aligned} \tag{3.32}$$

Consequently, we obtain

$$\begin{aligned} & R_{\Delta'}[F_{\Delta'}](\nu,x,y) - R_\Delta[F_\Delta](\nu,x,y) \\ & = \big(D(\delta;x^*_{n+1},x^*_n)^{1/2} - 1\big)R_\Delta[F_\Delta](\nu,x,y) \\ & \quad + D(\delta;x^*_{n+1},x^*_n)^{1/2}\big(R_\Delta\big[(D(\delta;x_{n+1},x_n)^{-1/2} - 1)F_\Delta\big](\nu,x,y) + a(\Delta;\nu,x,y)\big). \end{aligned} \tag{3.33}$$

The estimate of the claim (3.14) is a result of this formula. First we know as a consequence of (2.9), (2.10) and (2.11) that for any non-negative integer K there exist a positive constant C_K and a non-negative integer $M(K)$ such that if $|\alpha|, |\beta| \le K$,

$$\begin{aligned}&\left|\partial_x^\alpha\partial_y^\beta\big(D(\delta;x_{n+1}^*,x_n^*)^{1/2}-1\big)R_\Delta[F_\Delta](\nu,x,y)\right|\\&\quad\le C_K A_{M(K)}t_{n+1}^2T(T+\rho([0,T]))(1+|x|+|y|)^m.\end{aligned}\tag{3.34}$$

Next we show that $R_\Delta[(D(\delta;x_{n+1},x_n)^{-1/2}-1)F_\Delta](\nu,x,y)$ is small.

Let $j_0=0<j_1<\cdots<j_s<j_{s+1}=J+1$ be any subsequence of $\{0,1,\ldots,J,J+1\}$. Then $\Delta^\flat: T=T_0<T_{j_1}<\cdots<T_{j_s}<T_{j_{s+1}}=T$ is a division of the interval $[0,T]$ coarser than Δ. We use the symbol $\iota_{\Delta^\flat}^{\Delta}((D(\delta;x_{n+1},x_n)^{-1/2}-1)F_\Delta)(x_{j_{s+1}},x_{j_s},\ldots,x_{j_1},x_0)$ to express the restriction to $\Gamma(\Delta^\flat)$ of $(D(\delta;x_{n+1},x_n)^{-1/2}-1)F_\Delta(x_{J+1},x_J,\ldots,x_1,x_0)$, which is a function defined on the path space $\Gamma(\Delta)$. Since (2.11) and Proposition 2 hold, for any non-negative integer K there exist a positive constant C_K and a non-negative integer $M(K)$ such that if $|\alpha_{j_k}|\le K$

$$\begin{aligned}&\left|\left(\prod_{k=0}^{s+1}\partial_{x_{j_k}}^{\alpha_{j_k}}\right)\iota_{\Delta^\flat}^{\Delta}\big((D(\delta;x_{n+1},x_n)^{-1/2}-1)F_\Delta\big)(x_{j_{s+1}},x_{j_s},\ldots,x_{j_1},x_0)\right|\\&\quad\le C_K t_{n+1}^2A_{M(K)}X_{M(K)}^{s+2}(1+|x_{j_{s+1}}|+\cdots+|x_0|)^m.\end{aligned}$$

The stationary phase method on the space of large dimension (cf. [**8**], [**15**]) yields the following estimate: For any nonnegative integer K there exist a positive constant C_K and a non-negative integer $M(K)$ such that

$$\begin{aligned}&\left|\partial_x^\alpha\partial_y^\beta R_\Delta[(D(\delta;x_{n+1},x_n)^{-1/2}-1)F_\Delta](\nu,x,y)\right|\\&\quad\le C_K A_{M(K)}Tt_{n+1}^2(1+|x|+|y|)^m,\end{aligned}\tag{3.35}$$

if α, β satisfy $|\alpha|\le K$, $|\beta|\le K$.

Finally we show $a(\Delta;\nu,x,y)$ on the right hand side of (3.30) is small. Regarding the variables $x_{[J+1,n+2]}$ and $x_{[n-1,0]}$ of F_{Δ_1} as parameter λ, we know from Proposition 2 that we can apply theorem 1 of [**11**] to the integral (3.22).

In order to recall the result of [**11**], we make a notational convention. For any $k=1,2,\ldots,p_n$ we use the division $\delta(k)$ of the interval $[T_n,T_{n+1}]$ defined by

$$\delta(k): T_n=T_{n,0}<T_{n,k}<\cdots<T_{n,p_n}<T_{n,p_n+1}=T_{n+1}.$$

$\delta(k)$ is coarser than δ. We denote the division of the interval $[T_n,T_{n,k}]$ by $\delta(k)^c: T_n=T_{n,0}<T_{n,1}<\cdots<T_{n,k}$. Theorem 1 of [**11**] says that

$$\begin{aligned}&F_{\Delta/\Delta'}(x_{J+1},\ldots,x_{n+1},x_n,\ldots,x_0)\\&\quad=D(\delta;x_{n+1},x_n)^{-1/2}\left(F_\Delta(x_{J+1},x_J,\ldots,x_1,x_0)+\frac{i}{2\nu}p_\delta(x_{J+1},x_J,\ldots,x_1,x_0)\right)\\&\qquad+\nu^{-1}r_\delta(\nu,x_{J+1},x_J,\ldots,x_1,x_0),\end{aligned}\tag{3.36}$$

where

$$\begin{aligned}
&p_\delta(x_{J+1},x_J,\dots,x_1,x_0)\\
&\quad=\sum_{k=1}^{p_n}\frac{T_{n,k}\sigma_{k+1}}{T_{n,k+1}}\iota^{\delta(k)}_{[T_n,T_{n+1}]}\big(D(\delta(k)^c;y_k,x_n)^{1/2}\\
&\quad\times\Delta_{y_k}\big(D(\delta(k)^c;y_k,x_n)^{-1/2}\iota^{\delta}_{\delta(k)}F_{\Delta'}\big)(x_{J+1},\dots,x_{n+1},y_{p_n},\dots,y_k,x_n,\dots,x_0)\big).
\end{aligned}\tag{3.37}$$

Here $\iota^{\delta}_{\delta(k)}$ means the restriction mapping of functions on path spaces $\Gamma(\delta)$ to $\Gamma(\delta(k))$, Δ_{y_k} stands for the Laplace operator with respect to the variable y_k and $D(\delta(k)^c;y_k,x_n)$ is given by (2.8) for the division $\delta(k)^c$. For any non-negative integer K there exist a positive constant C_K and a non-negative integer $M(K)$ such that

$$\begin{aligned}
&\Big|\Big(\prod_{j=0}^{J+1}\partial^{\alpha_j}_{x_j}\Big)r_\delta(x_{J+1},x_J,\dots,x_1,x_0)\Big|\\
&\quad\le C_K\big(|\delta|t_{n+1}^2+\nu^{-1}(|\delta|t_{n+1}+t_{n+1}^2)\big)A_{M(K)}X^{J+2}_{M(K)}(1+|x_{J+1}|+\cdots+|x_0|)^m\\
&\quad\le C_Kt_{n+1}^2(t_{n+1}+\nu^{-1})A_{M(K)}X^{J+2}_{M(K)}(1+|x_{J+1}|+\cdots+|x_0|)^m
\end{aligned}\tag{3.38}$$

Comparing equality (3.27) and (3.36), we have

$$\begin{aligned}
&D(\delta;x_{n+1},x_n)^{-1/2}R_\delta[F_{\Delta'}](x_{J+1},x_J,\dots,x_1,x_0)\\
&\quad=\frac{i}{2}D(\delta;x_{n+1},x_n)^{-1/2}p_\delta(x_{J+1},x_J,\dots,x_1,x_0)+r_\delta(\nu,x_{J+1},x_J,\dots,x_1,x_0).
\end{aligned}\tag{3.39}$$

Since Proposition 2 and (2.10) hold, for any positive integer K there exist a positive constant C_K and a non-negative integer $M(K)$ such that we have

$$\begin{aligned}
&\bigg|\Big(\prod_{j=0}^{J+1}\partial^{\alpha_j}_{x_j}\Big)\Big(\iota^{\delta(k)}_{[T_n,T_{n+1}]}D(\delta(k)^c;y_k,x_n)^{1/2}\\
&\qquad\qquad\times\Delta_{y_k}\big((D(\delta(k)^c;y_k,x_n)^{-1/2}\iota^{\delta}_{\delta(k)}F_{\Delta'})(x_{[J+1,n+1]},y_{[p_n,k]},x_{[n,0]})\big)\Big)\bigg|\\
&\quad\le C_KA_{M(K)}X^{J+2}_{M(K)}\big((T_{n,k}-T_n)^2+\rho([T_{n,k-1},T_{n,k+1}])\big)(1+|x_{J+1}|+\cdots+|x_0|)^m
\end{aligned}$$

as far as $|\alpha_j|\le K$ for $0\le j\le J+1$. This implies that

$$\begin{aligned}
&\Big|\Big(\prod_{j=0}^{J+1}\partial^{\alpha_j}_{x_j}\Big)p_\delta(x_{J+1},x_J,\dots,x_1,x_0)\Big|\\
&\quad\le C_KA_{M(K)}X^{J+2}_{M(K)}t_{n+1}(t_{n+1}^2+\rho([T_n,T_{n+1}]))(1+|x_{J+1}|+\cdots+|x_0|)^m.
\end{aligned}\tag{3.40}$$

As a consequence of (3.39), (3.40) and (3.38) we have

$$\left|\left(\prod_{j=0}^{J+1}\partial_{x_j}^{\alpha_j}\right)D(\delta;x_{n+1},x_n)^{-1/2}R_\delta[F_{\Delta'}](\nu,x_{J+1},x_J,\dots,x_1,x_0)\right|$$
$$\le C_K A_{M(K)}X_{M(K)}^{J+2}t_{n+1}(\rho([T_n,T_{n+1}])+t_{n+1}^2+t_{n+1}\nu^{-1})(1+|x_{J+1}|+\cdots+|x_0|)^m. \tag{3.41}$$

For any subsequence $j_0=0<j_1<\cdots<j_s<j_{s+1}=J+1$ of $\{0,1,\dots,J,J+1\}$ let $\Delta^\flat$ be the division defined by the subsequence. Then the division $\Delta^\flat$ is coarser than Δ and we similarly have

$$\left|\left(\prod_{k=0}^{s+1}\partial_{x_{j_k}}^{\alpha_{j_k}}\right)\iota_{\Delta^\flat}^{\Delta}\big(D(\delta;x_{n+1},x_n)^{-1/2}R_\delta[F_{\Delta'}]\big)(\nu,x_{j_{s+1}},x_{j_s},\dots,x_{j_1},x_0)\right|$$
$$\le C_K A_{M(K)}X_{M(K)}^{s+2}t_{n+1}(\rho([T_n,T_{n+1}])+t_{n+1}^2+t_{n+1}\nu^{-1})(1+|x_{j_{s+1}}|+\cdots+|x_0|)^m. \tag{3.42}$$

Since (3.41) and(3.42) hold, we may apply the stationary phase method (cf. [**15**] and appendix of [**11**]) to $I[D(\delta;x_{n+1},x_n)^{-1/2}R_\delta[F_{\Delta'}]](x,y)$ and obtain, with another C_K and $M(K)$, that

$$\left|\partial_x^\alpha\partial_y^\beta a(\Delta;\nu,x,y)\right|$$
$$\le C_K A_{M(K)}t_{n+1}\big(\rho([T_n,T_{n+1}])+t_{n+1}^2+t_{n+1}\nu^{-1}\big)(1+|x|+|y|)^m. \tag{3.43}$$

Estimates (3.43), (3.35) and (3.34) give that

$$\Big|\partial_x^\alpha\partial_y^\beta\big((D(\delta;x_{n+1}^*,x_n^*)^{1/2}-1)R_\Delta[F_\Delta](\nu,x,y)$$
$$+D(\delta;x_{n+1}^*,x_n^*)^{1/2}\big(R_\Delta[(D(\delta;x_{n+1},x_n)^{-1/2}-1)F_\Delta](\nu,x,y)+a(\Delta;\nu,x,y)\big)\big)\Big|$$
$$\le C_K A_{M(K)}t_{n+1}(\rho([T_n,T_{n+1}])+t_{n+1}(T+T^2+T\rho([0,T])+\nu^{-1}))(1+|x|+|y|)^m.$$

This together with (3.33) prove the claim. Proof of Theorem 2 is now complete.

4. Semi-classical asymptotic expansion.

4.1. Proof of Lemma 1.

Let γ^* denote the classical path starting y at time 0 and reaching x at time T. We recall the piecewise classical path $\gamma_{\{t,\epsilon\}}(s,z)$ of §2 associated with the division of time interval

$$\Delta(t,\epsilon):0=T_0<t<t+\epsilon<T$$

and satisfying conditions:

$$\gamma_{\{t,\epsilon\}}(0,z)=y,\quad \gamma_{\{t,\epsilon\}}(t,z)=z,\quad \gamma_{\{t,\epsilon\}}(t+\epsilon,z)=\gamma^*(t+\epsilon),\quad \gamma_{\{t,\epsilon\}}(T,z)=x.$$

The path $\gamma_{\{t,\epsilon\}}(s,z)$ always coincides with $\gamma^*(s)$ in the interval $[t+\epsilon,T]$. Moreover, if $z=\gamma^*(t)$ it coincides with $\gamma^*(s)$ in the whole interval $[0,T]$, i.e.,

$$\gamma_{\{t,\epsilon\}}(s,z)\big|_{z=\gamma^*(t)}=\gamma^*(s),\quad \text{for}\ \ s\in[0,T]. \tag{4.1}$$

The first part of $\gamma_{\{t,\epsilon\}}(s,z)$, i.e., the part from $s=0$ to $s=t$ plays an important part in the following. So we denote it by $\gamma_{[0,t]}(s,z)$. This is nothing but the classical path starting $y\in\boldsymbol{R}^d$ at time 0 and reaching $z\in\boldsymbol{R}^d$ at time t.

Now we begin the proof of Lemma 1. Let $z=\sum_{k=1}^d z_k e_k$ be coordinate expression of z with respect to the standard orthonormal basis $\{e_k\}$ of $\boldsymbol{R}^d$. Then clearly we have

$$\begin{aligned}
&\Delta_z(D(t,z,y)^{-1/2}F(\gamma_{\{t,\epsilon\}}(*,z)))\big|_{z=\gamma^*(t)}\\
&\quad=\Delta_z(D(t,z,y)^{-1/2})F(\gamma_{\{t,\epsilon\}}(*,z))\big|_{z=\gamma^*(t)}\\
&\qquad+2\sum_{k=1}^d\partial_{z_k}(D(t,z,y)^{-1/2})\partial_{z_k}F(\gamma_{\{t,\epsilon\}}(*,z))\big|_{z=\gamma^*(t)}\\
&\qquad+D(t,z,y)^{-1/2}\Delta_zF(\gamma_{\{t,\epsilon\}}(*,z))\big|_{z=\gamma^*(t)}.
\end{aligned}\tag{4.2}$$

It is clear from (4.1) that for any small $\epsilon>0$

$$F(\gamma_{\{t,\epsilon\}}(*,z))\big|_{z=\gamma^*(t)}=F(\gamma^*).$$

Therefore it suffices to prove that the following two limits exist:

$$\lim_{\epsilon\to+0}\partial_{z_k}F(\gamma_{\{t,\epsilon\}}(*,z))\big|_{z=\gamma^*(t)}, \tag{4.3}$$

$$\lim_{\epsilon\to+0}\partial_{z_k}^2F(\gamma_{\{t,\epsilon\}}(*,z))\big|_{z=\gamma^*(t)}. \tag{4.4}$$

We prove the limit (4.3) exists. Assumption 1 implies that for any $\eta\in C([0,T];\boldsymbol{R}^d)$

$$|DF_{\gamma^*}[\eta]|\le C(1+\|\gamma^*\|+|||\gamma^*|||)^m\max_{t\in[0,T]}|\eta(s)|.$$

This means that there exists a bounded $\boldsymbol{R}^d$-valued Borel measure σ_{γ^*} such that

$$DF_{\gamma^*}[\eta]=\int_{[0,\,T]}\eta(s)\cdot\sigma_{\gamma^*}(ds),$$

where $\cdot$ means inner product in $\boldsymbol{R}^d$. We have

$$\partial_{z_k} F(\gamma_{\{t,\epsilon\}}(*,z))\big|_{z=\gamma^*(t)} = \int_{[0,\,T]} \partial_{z_k}\gamma_{\{t,\epsilon\}}(s,z)\cdot\sigma_{\gamma^*}(ds). \tag{4.5}$$

We abbreviate $\partial_{z_k}\gamma_{\{t,\epsilon\}}(s,z)$ as $\eta_k^{(\epsilon)}(s)$. This is the piecewise solution of Jacobi's equation: For $s\in(0,t)\cup(t,t+\epsilon)\cup(t+\epsilon,T)$

$$\frac{d^2}{ds^2}\eta_k^{(\epsilon)}(s)+\nabla\nabla V(s,\gamma_{\{t,\epsilon\}}(s,z))\eta_k^{(\epsilon)}(s)=0$$

satisfying boundary conditions

$$\eta_k^{(\epsilon)}(0)=0,\quad \eta_k^{(\epsilon)}(t)=e_k,\quad \eta_k^{(\epsilon)}(t+\epsilon)=0,\quad \eta_k^{(\epsilon)}(T)=0.$$

As $\epsilon\to+0$, $\eta_k^{(\epsilon)}(s)$ converges boundedly to a discontinuous function $\eta_k^{(0)}(s;t)$, where

$$\eta_k^{(0)}(s;t)=\begin{cases}\partial_{z_k}\gamma_{[0,t]}(s,\gamma^*(t)) & \text{for}\ \ s\le t\\ 0 & \text{for}\ \ t<s\le T.\end{cases}$$

Therefore, Lebesgues' bounded convergence theorem applied to (4.5) gives

$$\begin{aligned} a_k(t)&=\lim_{\epsilon\to+0}\partial_{z_k}F(\gamma_{\{t,\epsilon\}}(*,z))\big|_{z=\gamma^*(t)}\\ &=\int_{[0,\,T]}\eta_k^{(0)}(s;t)\cdot\sigma_{\gamma^*}(ds)=\int_{[0,\,t]}\partial_{z_k}\gamma_{[0,t]}(s,\gamma^*(t))\cdot\sigma_{\gamma^*}(ds). \end{aligned}\tag{4.6}$$

This is a function of t continuous at the point where $\sigma_{\gamma^*}(t)$ is continuous. Moreover we obtain from (4.5) and (4.6) that

$$\Big|a_k(t)-\partial_{z_k}F(\gamma_{\{t,\epsilon\}}(*,z))\big|_{z=\gamma^*(t)}\Big|\le C(1+\|\gamma^*\|+|||\gamma^*|||)^m\rho((t,t+\epsilon]). \tag{4.7}$$

Now we prove that the limit of (4.4) exists.

$$\partial_{z_k}^2F(\gamma_{\{t,\epsilon\}}(*,z))\big|_{z=\gamma^*(t)}=D^2F_{\gamma^*}\big[\eta_k^{(\epsilon)}\otimes\eta_k^{(\epsilon)}\big]+DF_{\gamma^*}\big[\partial_{z_k}\eta_k^{(\epsilon)}\big]. \tag{4.8}$$

First we prove that $D^2F_{\gamma^*}[\eta_k^{(\epsilon)}(*)\otimes\eta_k^{(\epsilon)}]$ is a Cauchy sequence. Let $\epsilon,\epsilon'\to+0$.

$$\begin{aligned} &\big|D^2F_{\gamma^*}\big[\eta_k^{(\epsilon)}\otimes\eta_k^{(\epsilon)}\big]-D^2F_{\gamma^*}\big[\eta_k^{(\epsilon')}\otimes\eta_k^{(\epsilon')}\big]\big|\\ &\quad\le\big|D^2F_{\gamma^*}\big[\eta_k^{(\epsilon)}\otimes\big(\eta_k^{(\epsilon)}-\eta_k^{(\epsilon')}\big)\big]\big|+\big|D^2F_{\gamma^*}\big[\big(\eta_k^{(\epsilon)}-\eta_k^{(\epsilon')}\big)\otimes\eta_k^{(\epsilon')}\big]\big|. \end{aligned}$$

Using Assumption 2 we know that

$$
\begin{aligned}
&\big|D^2F_{\gamma^*}\big[\eta_k^{(\epsilon)}\otimes\eta_k^{(\epsilon)}\big]-D^2F_{\gamma^*}\big[\eta_k^{(\epsilon')}\otimes\eta_k^{(\epsilon')}\big]\big|\\
&\quad\le C(1+\|\gamma^*\|+|||\gamma^*|||)^m\big(\|\eta_k^{(\epsilon)}\|+\|\eta_k^{(\epsilon')}\|\big)\int_{[0,T]}\big|\eta_k^{(\epsilon)}(s)-\eta_k^{(\epsilon')}(s)\big|\rho(ds).
\end{aligned}
$$

Since $\eta_k^{(\epsilon)}(s)-\eta_k^{(\epsilon')}(s)\to 0$ boundedly, we have

$$
\int_{[0,T]}\big|\eta_k^{(\epsilon)}(s)-\eta_k^{(\epsilon')}(s)\big|\rho(ds)\to 0
$$

as $\epsilon,\epsilon'\to+0$. Thus we have proved that $D^2F_{\gamma^*}[\eta_k^{(\epsilon)}\otimes\eta_k^{(\epsilon)}]$ is a Cauchy sequence. We know that

$$
b_k(t)=\lim_{\epsilon\to+0}D^2F_{\gamma^*}\big[\eta_k^{(\epsilon)}\otimes\eta_k^{(\epsilon)}\big]
$$

exists and

$$
\big|b_k(t)-D^2F_{\gamma^*}\big[\eta_k^{(\epsilon)}\otimes\eta_k^{(\epsilon)}\big]\big|\le C(1+\|\gamma^*\|+|||\gamma^*|||)^m\rho((t,t+\epsilon]). \tag{4.9}
$$

By Assumption 1 we see that

$$
\big|D^2F_{\gamma^*}\big[\eta_k^{(\epsilon)}\otimes\eta_k^{(\epsilon)}\big]\big|\le C(1+\|\gamma^*\|+|||\gamma^*|||)^m\big\|\eta_k^{(\epsilon)}\big\|^2.
$$

Therefore $D^2F_{\gamma^*}[\eta_k^{(\epsilon)}\otimes\eta_k^{(\epsilon)}]$ converges boundedly.

Next we prove that $DF_{\gamma^*}[\partial_{z_k}\eta_k^{(\epsilon)}]$ converges as $\epsilon\to+0$. The function $\partial_{z_k}\eta_k^{(\epsilon)}(s)$ satisfies equation

$$
\frac{d^2}{ds^2}\partial_{z_k}\eta_k^{(\epsilon)}(s)+\nabla\nabla V(s,\gamma^*(s))\partial_{z_k}\eta_k^{(\epsilon)}(s)+\nabla\nabla\nabla V(s,\gamma^*(s))\eta_k^{(\epsilon)}(s)\cdot\eta_k^{(\epsilon)}(s)=0
$$

for $s\in(0,t)\cup(t,t+\epsilon)\cup(t+\epsilon,T)$ and boundary conditions

$$
\partial_{z_k}\eta_k^{(\epsilon)}(0)=\partial_{z_k}\eta_k^{(\epsilon)}(t)=\partial_{z_k}\eta_k^{(\epsilon)}(t+\epsilon)=\partial_{z_k}\eta_k^{(\epsilon)}(T)=0.
$$

It is clear that $\partial_{z_k}\eta_k^{(\epsilon)}(s)$ converges boundedly to $\partial_{z_k}\eta_k^{(0)}(s)$ as $\epsilon\to+0$, where $\partial_{z_k}\eta_k^{(0)}(s;t)$ is the function

$$
\partial_{z_k}\eta_k^{(0)}(s;t)=\begin{cases}\partial_{z_k}^2\gamma_{[0,t]}(s,\gamma^*(t)) & \text{for}\ \ 0\le s\le t,\\ 0 & \text{for}\ \ t<s\le T.\end{cases}
$$

We have proved that

$$
\lim_{\epsilon\to+0}DF_{\gamma^*}\big[\partial_{z_k}\eta_k^{(\epsilon)}\big]=\int_{[0,T]}\partial_{z_k}\eta_k^0(s;t)\cdot\sigma_{\gamma^*}(ds)=\int_{[0,\,t]}\partial_{z_k}^2\gamma_{[0,t]}(s,\gamma^*(t))\cdot\sigma_{\gamma^*}(ds).
$$

We put

$$c_k(t) = \lim_{\epsilon \to +0} DF_{\gamma^*}\big[\partial_{z_k}\eta_k^{(\epsilon)}\big]. \tag{4.10}$$

Then

$$\big|c_k(t) - DF_{\gamma^*}\big[\partial_{z_k}\eta_k^{(\epsilon)}\big]\big| \le C(1 + \|\gamma^*\| + |||\gamma^*|||)^m \rho((t, t+\epsilon]). \tag{4.11}$$

Consequently, we have obtained

$$\begin{aligned} q(t) &= \Delta_z (D(t,z,y)^{-1/2})\big|_{z=\gamma^*} F(\gamma^*) + 2\sum_{k=1}^{d} \partial_{z_k}(D(t,z,y)^{-1/2})\big|_{z=\gamma^*(t)} a_k(t) \\ &\quad + \sum_{k=1}^{d} D(t, \gamma^*(t), y)^{-1/2}(b_k(t) + c_k(t)). \end{aligned} \tag{4.12}$$

Lemma has been proved.

4.2. Proof of Theorem 3.

Since $q(t)$ is Riemannian integrable the right hand side of (2.21) is meaningful. Let

$$\varDelta : 0 = T_0 < T_1 < \cdots < T_J < T_{J+1} = T$$

be an arbitrary division of the interval $[0, T]$. Then

$$\int_0^T D(t, \gamma^*(t), y)^{1/2} q(t) dt = \lim_{|\varDelta| \to 0} \sum_{j=0}^{J} t_{j+1} D(T_j, \gamma^*(T_j), y)^{1/2} q(T_j). \tag{4.13}$$

$$\begin{aligned} q(T_j) &= \Delta_z(D(T_j, z, y)^{-1/2})\big|_{z=\gamma^*(T_j)} F(\gamma^*) \\ &\quad + 2\bigg(\sum_{k=1}^{d} \partial_{z_k}(D(T_j, z, y)^{-1/2})\big|_{z=\gamma^*(t_j)}\bigg) \cdot a_k(T_j) \\ &\quad + D(T_j, \gamma^*(T_j), y)^{-1/2} \sum_{k=1}^{d} (b_k(T_j) + c_k(T_j)). \end{aligned} \tag{4.14}$$

For any $j = 1, 2, \ldots, J+1$ we set $\varDelta(j)$ the following division of $[0, T]$ coarser than $\varDelta$

$$\varDelta(j) : 0 = T_0 < T_j < T_{j+1} < \cdots < T_{J+1} = T$$

and we denote by $\varDelta(j)^c$ the following division of the interval $[0, T_j]$

$$\varDelta(j)^c : 0 = T_0 < T_1 < \cdots < T_{j-1} < T_j.$$

In particular $\Delta(1) = \Delta$ and $\Delta(J+1) = [0,T]$. Set $F_\Delta(x_{J+1}, x_J, \dots, x_1, x_0) = F(\gamma_\Delta)$. And we shall apply Theorem 1 of [**11**] to the integral $I(F_\Delta)(\Delta, x, y)$. Then

$$\begin{aligned} I[F_\Delta](\Delta; x, y) = \left(\frac{\nu}{2\pi i T}\right)^{d/2} & e^{i\nu S(T,x,y)} \\ & \times \left[D(\Delta; x, y)^{-1/2} \left(F(\gamma^*) + \frac{i}{2\nu} p_\Delta(x, y) \right) + \nu^{-2} r_\Delta(\nu, x, y) \right], \end{aligned}$$

where

$$\begin{aligned} p_\Delta(x, y) = \sum_{j=1}^{J+1} & \frac{T_j t_{j+1}}{T_{j+1}} \iota^{\Delta(j)}_{\Delta(J+1)} \\ & \times \left\{ D(\Delta(j)^c; x_j, x_0)^{1/2} \Delta_{x_j} \left(D(\Delta(j)^c; x_j, x_0)^{-1/2} \iota^{\Delta}_{\Delta(j)} F_\Delta \right)(x_{J+1}, \dots, x_j, x_0) \right\}. \end{aligned}$$

and for any multi-indices α, β

$$\begin{aligned} \limsup_{|\Delta| \to 0} \left| \partial_x^\alpha \partial_y^\beta r_\Delta(\nu, x, y) \right| & \le C_{\alpha,\beta} \limsup_{|\Delta| \to 0} (|\Delta|(T + T^2\nu) + T^2)(1 + |x| + |y|)^m \\ & = C_{\alpha,\beta} T^2 (1 + |x| + |y|)^m. \end{aligned}$$

We claim that $\lim_{|\Delta| \to 0} \frac{i}{2} p_\Delta(x, y)$ exists and equals to A_1 of (2.21). This claim implies (2.23) and Theorem 3.

Now we shall prove the claim. Calculation shows that

$$\begin{aligned} & \iota^{\Delta(j)}_{\Delta(J+1)} \left\{ D(\Delta(j)^c; x_j, x_0)^{1/2} \Delta_{x_j} \left(D(\Delta(j)^c; x_j, x_0)^{-1/2} \iota^{\Delta}_{\Delta(j)} F_\Delta \right)(x_{J+1}, \dots, x_j, x_0) \right\} \\ & \quad = \iota^{\Delta(j)}_{\Delta(J+1)} \left(D(\Delta(j)^c; x_j, x_0)^{1/2} \Delta_{x_j} D(\Delta(j)^c; x_j, x_0)^{-1/2} \right) (\iota^{\Delta}_{\Delta(J+1)} F_\Delta)(x, y) \\ & \qquad + 2 \iota^{\Delta(j)}_{\Delta(J+1)} \left(D(\Delta(j)^c; x_j, x_0)^{1/2} \nabla_{x_j} D(\Delta(j)^c; x_j, x_0)^{-1/2} \right) \\ & \qquad \cdot \iota^{\Delta(j)}_{\Delta(J+1)} \left(\nabla_{x_j} (\iota^{\Delta}_{\Delta(j)} F_\Delta) \right)(x, y) + \iota^{\Delta(j)}_{\Delta(J+1)} \left(\Delta_{x_j} (\iota^{\Delta}_{\Delta(j)} F_\Delta)(x_{J+1}, x_J, \dots, x_j, y) \right), \end{aligned} \tag{4.15}$$

where ∇_{x_j} stands for nabla operator in x_j space. It is clear that

$$(\iota^{\Delta}_{\Delta(J+1)} F_\Delta)(x, y) = F(\gamma^*). \tag{4.16}$$

Applying (2.12) to the division $\Delta(j)^c$ of the interval $[0, T_j]$, we know for any α there exists a constant C_α such that

$$\left| \partial_z^\alpha (D(\Delta(j)^c; z, x_0) - D(T_j, z, x_0)) \right| \le C_\alpha |\Delta(j)^c| T. \tag{4.17}$$

We have

$$\Big|\iota^{\Delta(j)}_{\Delta(J+1)}\big(D(\Delta(j)^c;x_j,x_0)^{1/2}\Delta_{x_j}D(\Delta(j)^c;x_j,x_0)^{-1/2}\big)(\iota^{\Delta}_{\Delta(J+1)}F_\Delta)(x,y)$$
$$-D(T_j,\gamma^*(T_j),y)^{1/2}\Delta_z D(T_j,z,x_0)^{-1/2}\big|_{z=\gamma^*(T_j)}F(\gamma^*)\Big|$$
$$\leq C(1+\|\gamma^*\|+|||\gamma^*|||)^m|\Delta|. \tag{4.18}$$

Let x_{jk} denote the k-th component of x_j. Then for $l=1,2$

$$\iota^{\Delta(j)}_{\Delta(J+1)}\big(\partial^l_{x_{jk}}(\iota^{\Delta}_{\Delta(j)}F_\Delta)(x_{J+1},x_J,\ldots,x_j,y)\big)=\partial^l_{z_k}F(\gamma_{\{T_j,t_{j+1}\}}(*,z))\big|_{z=\gamma^*(T_j)}.$$

Thus we know from (4.7) that

$$\Big|\iota^{\Delta(j)}_{\Delta(J+1)}\big(\partial_{x_{jk}}(\iota^{\Delta}_{\Delta(j)}F_\Delta)(x_{J+1},x_J,\ldots,x_j,y)\big)-a_k(T_j)\Big|$$
$$\leq C(1+\|\gamma^*\|+|||\gamma^*|||)^m\rho((T_j,T_{j+1}]). \tag{4.19}$$

From this we have

$$\Big|\iota^{\Delta(j)}_{\Delta(J+1)}\big(D(\Delta(j)^c;x_j,x_0)^{1/2}\partial_{x_{jk}}D(\Delta(j)^c;x_j,x_0)^{-1/2}\big)\big(\iota^{\Delta(j)}_{\Delta(J+1)}\big(\partial_{x_{jk}}(\iota^{\Delta}_{\Delta(j)}F_\Delta)\big)\big)(x,y)$$
$$-\big(D(T_j,\gamma^*(T_j),y)^{1/2}\partial_{z_k}D(T_j,z,x_0)^{-1/2}\big)\big|_{z=\gamma^*(T_j)}a_k(T_j)\Big|$$
$$\leq C(1+\|\gamma^*\|+|||\gamma^*|||)^m(|\Delta|+\rho((T_j,T_{j+1}])). \tag{4.20}$$

Similarly, it follows from (4.8), (4.9) and (4.11) that

$$\Big|\iota^{\Delta(j)}_{\Delta(J+1)}\big(\partial^2_{x_{jk}}(\iota^{\Delta}_{\Delta(j)}F_\Delta)(x_{J+1},x_J,\ldots,x_j,y)\big)-(b_k(T_j)+c_k(T_j))\Big|$$
$$\leq C(1+\|\gamma^*\|+|||\gamma^*|||)^m\rho((T_j,T_{j+1}]). \tag{4.21}$$

Now combining (4.18), (4.20) and (4.21), we obtain

$$\Big|p_\Delta(x,y)-\sum_{j=0}^{J}t_{j+1}D(T_j,\gamma^*(T_j),y)^{1/2}q(T_j)\Big|$$
$$\leq C\sum_{j=1}^{J+1}t_{j+1}(1+\|\gamma^*\|+|||\gamma^*|||)^m(|\Delta|+\rho((T_j,T_{j+1}]))$$
$$\leq C|\Delta|(1+\|\gamma^*\|+|||\gamma^*|||)^m(T+\rho((0,T])).$$

This together with (4.13) mean that

$$\lim_{|\Delta|\to 0} p_\Delta(x,y) = \int_0^T D(t,\gamma^*(t),y)^{1/2} q(t)dt.$$

Theorem has been proved.

5. Simple examples.

If $F(\gamma)$ is simple we can calculate the second term A_1 of the semi-classical asymptotic expansion more explicitly. We assume $|s'-s| \le \mu$. We treat the following examples, which are treated in [**15**].

Let $f(t,x)$ be a C^∞ function of $(t,x) \in [s,s'] \times \boldsymbol{R}^d$ satisfying

$$|\partial_x^\alpha f(t,x)| \le C_\alpha (1+|x|)^m. \tag{5.1}$$

Then for any multi-indices α, β satisfying $|\alpha|+|\beta| \ge 0$,

$$\partial_x^\alpha \partial_y^\beta f(t,\gamma(t)) = \sum \partial_x^\tau f(t,x)\big|_{x=\gamma(t)} p_{\alpha,\beta;\mu}\big(\{\partial_x^{\alpha'}\partial_y^{\beta'}\gamma(t)\}\big),$$

where $p_{\alpha,\beta;\tau}(\{\partial_x^{\alpha'}\partial_y^{\beta'}\gamma(t)\})$ is a polynomial of degree $|\tau|$ of $\{\partial_x^{\alpha'}\partial_y^{\beta'}\gamma(t); \alpha' \le \alpha, \beta' \le \beta, \alpha'+\beta' \ne 0\}$. Therefore, for any multi-indices α, β there exists a positive constant $C_{\alpha,\beta}$ such that

$$|\partial_x^\alpha \partial_y^\beta f(t,\gamma(t))| \le C_{\alpha,\beta}(1+|\gamma(t)|)^m.$$

This means that for fixed $u \in [0,T]$ the functional

$$\gamma \longrightarrow F_u(\gamma) = f(u,\gamma(u))$$

satisfies our Assumption 1 and Assumption 2.

Now we denote by ∇ the nabla-operator in the configuration space, i.e., $(\nabla f)(t,x) = (\partial_{x_1} f(t,x), \dots, \partial_{x_d} f(t,x)) \in \boldsymbol{R}^d$. Calculation shows that

$$\begin{aligned} a_k(t) &= \lim_{\epsilon\to 0} \partial_{z_k} F_u(\gamma_{\{t,\epsilon\}}(*,z))\big|_{z=\gamma^*(t)} \\ &= \begin{cases} 0 & \text{for } \ 0 \le t < u \le T, \\ (\nabla f)(u,\gamma^*(u)) \cdot \eta_k^{(0)}(u;t) & \text{for } \ 0 \le u \le t \le T. \end{cases} \end{aligned}$$

Here $\cdot$ stands for inner product in $\boldsymbol{R}^d$. Further we obtain

$$\begin{aligned} b_k(t) &= \lim_{\epsilon\to 0} D^2 F_u(\gamma^*)\big[\eta_k^{(\epsilon)} \otimes \eta_k^{(\epsilon)}\big] \\ &= \begin{cases} 0 & \text{for } \ 0 \le t < u \le T \\ (\nabla\nabla f)(u,\gamma^*(u))\eta_k^{(0)}(u;t) \cdot \eta_k^{(0)}(u;t) & \text{for } \ 0 \le u \le t \le T. \end{cases} \end{aligned}$$

$$\begin{aligned} c_k(t) &= \lim_{\epsilon\to 0} DF_u(\gamma^*)\big[\partial_{z_k}\eta_k^{(\epsilon)}\big] \\ &= \begin{cases} 0 & \text{for } \ 0\le t<u\le T \\ (\nabla f)(u,\gamma^*(u))\partial_{z_k}\eta_k^{(0)}(u;t) & \text{for } \ 0\le u\le t\le T. \end{cases} \end{aligned}$$

Thus

$$\begin{aligned} &\lim_{\epsilon\to 0}\partial_{z_k}^2 F_u(\gamma_{\{t,\epsilon\}}(*,z))\big|_{u=\gamma^*(t)} \\ &\quad = \begin{cases} 0 & \text{for } \ 0\le t<u\le T \\ (\nabla\nabla f)(u,\gamma^*)\eta_k^{(0)}(u;t)\cdot\eta_k^{(0)}(u;t) & \\ \quad +(\nabla f)(u,\gamma^*(u))\partial_{z_k}\eta^{(0)}(u;t) & \text{for } \ 0\le u\le t\le T. \end{cases} \end{aligned}$$

Therefore, in case $0\le t<u\le T$

$$q(t) = (\Delta_z D(t,z,y)^{-1/2})\big|_{z=\gamma^*(t)} f(u,\gamma^*(u)) \tag{5.2}$$

in case $0\le u\le t\le T$

$$\begin{aligned} q(t) &= (\Delta_z D(t,z,y)^{-1/2})\big|_{z=\gamma^*(t)} f(u,\gamma^*(u)) \\ &\quad + 2\sum_{k=1}^{d}\partial_{z_k} D(t,z,y)^{-1/2}\big|_{z=\gamma^*(t)}(\nabla f)(u,\gamma^*(u))\cdot\eta_k^0(u;t) \\ &\quad + D(t,\gamma^*(t),y)^{-1/2}\sum_{k=1}^{d}\big((\nabla\nabla f)(u,\gamma^*(u))\eta_k^{(0)}(u;t)\cdot\eta_k^{(0)}(u;t) \\ &\qquad\qquad + (\nabla f)(u,\gamma^*(u))\partial_{z_k}\eta_k^{(0)}(u;t)\big). \end{aligned} \tag{5.3}$$

If $0\le u\le t$, we may write

$$\eta_k^{(0)}(u;t) = \partial_{z_k}\gamma_{[0,t]}(u,z)\big|_{z=\gamma^*(t)}, \quad \partial_{z_k}\eta_k^{(0)}(u;t) = \partial_{z_k}^2\gamma_{[0,t]}(u,z)\big|_{z=\gamma^*(t)}.$$

Next we consider another example. Let $\rho(t)$ be a function of bounded variation on the interval $[0,T]$ and consider the functional defined by Stieltjes integral

$$F(\gamma) = \int_0^T f(t,\gamma(t))\rho(dt). \tag{5.4}$$

That is

$$F(\gamma) = \int_0^T F_u(\gamma)\rho(du).$$

This also satisfies our Assumption 1 and Assumption 2. Using above calculation we know that

$$
\begin{aligned}
q(t) &= (\Delta_z D(t,z,y)^{-1/2})\big|_{z=\gamma^*(t)} \int_0^T f(u,\gamma^*(u))\rho(du) \\
&\quad + 2\sum_{k=1}^d \partial_{z_k} D(t,z,y)^{-1/2}\Big|_{z=\gamma^*(t)} \int_0^t (\nabla f)(u,\gamma^*(u))\cdot\partial_{z_k}\gamma_{[0,t]}(u;\gamma^*(t))\rho(du) \\
&\quad + D(t,\gamma^*(t),y)^{-1/2}\int_0^t \bigg(\sum_{k=1}^d (\nabla\nabla f)(u,\gamma^*(u))\partial_{z_k}\gamma_{[0,t]}(u;\gamma^*(t))\cdot\partial_{z_k}\gamma_{[0,t]}(u;\gamma^*(t)) \\
&\qquad\qquad + (\nabla f)(u,\gamma^*(u))\Delta_z\gamma_{[0,t]}(u,\gamma^*(t))\bigg)\rho(du).
\end{aligned}
$$

6. Birkhoff's formula revisited.

G. D. Birkhoff gave an semi-classical asymptotic expansion of solution to Schrödinger equation in his famous paper [**1**]. In this section we show that the first and second term of his expansion can be explained from our point of view which is completely different from that of Birkhoff.

It is known that the Feynman path integral is the fundamental solution of Schrödinger equation if $F(\gamma)=1$. Let

$$F(\gamma)=1$$

i.e.,

$$F(\gamma)=\int_0^T 1H(dt)$$

where $H(t)$ is Heaviside's function.

We apply our discussion of previous section to this case. Then for any $t\in[0,T]$ (5.2) and (5.3) yield that

$$q(t,x,y)=(\Delta_z D(t,z,y)^{-1/2})\big|_{z=\gamma^*_{[0,T]}(t,x,y)}. \tag{6.1}$$

Therefore, we have

$$
\begin{aligned}
&\int_\Omega e^{i\nu S(\gamma)}\mathscr{D}[\gamma] \\
&\quad = \left(\frac{\nu}{2\pi iT}\right)^{d/2} e^{i\nu S(T,x,y)} D(T,x,y)^{-1/2}\bigg(1+\frac{i}{2\nu}p(T,x,y)+\nu^{-2}r(\nu,T,x,y)\bigg),
\end{aligned}
$$

where

$$p(T,x,y) = \int_0^T D\big(t, \gamma^*_{[0,T]}(t;x,y), y\big)^{1/2} (\Delta_z D(t,z,y)^{-1/2}\big)\Big|_{z=\gamma^*_{[0,T]}(t;x,y)} dt.$$

We set

$$v_0(T,x,y) = \left(\frac{\nu}{2\pi i T}\right)^{d/2} D(T,x,y)^{-1/2},$$

$$v_1(T,x,y) = \frac{-1}{2}\left(\frac{\nu}{2\pi i T}\right)^{d/2} D(T,x,y)^{-1/2} p(T,x,y).$$

Then in accordance with Birkhoff's notation, we can write

$$\int_\Omega e^{i\nu S(\gamma)} \mathscr{D}[\gamma] = e^{i\nu S(T,x,y)} \big(v_0(T,x,y) + (i\nu)^{-1} v_1(T,x,y) + O(\nu^{-2})\big).$$

We already knew in [**9**] that $v_0(T,x,y)$ satisfies the first transport equation:

$$\frac{\delta}{\delta T} v_0(T,x,y) + \frac{1}{2}\Delta_x S(T,x,y) v_0(T,x,y) = 0, \tag{6.2}$$

where the linear differential operator

$$\frac{\delta}{\delta T} = \partial_T + \nabla_x S(T,x,y)\cdot \nabla_x \tag{6.3}$$

is the differentiation along the classical orbit $\gamma^*(t;x,y) = \gamma^*_{[0,T]}(t;x,y)$ starting y at time 0 and passing through x at time T. Note that this equation (6.2) is a consequence of the fact that $D(T,x,y)$ comes from the determinant of Hessian and we do not use Birkhoff's method to prove (6.2) (cf. [**9**]).

We shall prove that $v_1(T,x,y)$ satisfies the second transport equation:

$$\frac{\delta}{\delta T} v_1(T,x,y) + \frac{1}{2}\Delta_x S(T,x,y) v_1(T,x,y) + \frac{1}{2}\Delta_x v_0(T,x,y) = 0. \tag{6.4}$$

Since we can write

$$v_1(T,x,y) = \frac{-1}{2} v_0(T,x,y) p(T,x,y), \tag{6.5}$$

we obtain

$$\frac{\delta}{\delta T} v_1(T,x,y) = \frac{-1}{2}\left(\frac{\delta}{\delta T} v_0(T,x,y) p(T,x,y) + v_0(T,x,y)\frac{\delta}{\delta T} p(T,x,y)\right).$$

It is clear from definition of $p(T,x,y)$ that

$$\begin{aligned}\frac{\delta}{\delta T}p(T,x,y) &= \frac{d}{ds}\int_0^s D(t,\gamma^*_{[0,T]}(t;x,y),y)^{1/2}(\Delta_z D(t,z,y)^{-1/2})\big|_{z=\gamma^*_{[0,T]}(t;x,y)}\,dt\big|_{s=T}\\ &= D(T,\gamma^*_{[0,T]}(T;x,y),y)^{1/2}(\Delta_z D(T,z,y)^{-1/2})\big|_{z=\gamma^*_{[0,T]}(T;x,y)}\\ &= D(T,x,y)^{1/2}\Delta_x D(T,x,y)^{-1/2}.\end{aligned}$$

Thus we obtain

$$v_0(T,x,y)\frac{\delta}{\delta T}p(T,x,y) = \left(\frac{\nu}{2\pi iT}\right)^{d/2}\Delta_x D(T,x,y)^{-1/2} = \Delta_x v_0(T,x,y).$$

Therefore, using the first transport equation (6.2), we obtain

$$\begin{aligned}\frac{\delta}{\delta T}v_1(T,x,y) &= -\frac{1}{2}\bigg(\frac{\delta}{\delta T}v_0(T,x,y)p(T,x,y) + \Delta_x v_0(T,x,y)\bigg)\\ &= -\frac{1}{2}\bigg(-\frac{1}{2}\Delta_x S(T,x,y)v_0(T,x,y)p(T,x,y) + \Delta_x v_0(T,x,y)\bigg)\\ &= -\frac{1}{2}(\Delta_x S(T,x,y)v_1(T,x,y) + \Delta_x v_0(T,x,y)).\end{aligned}$$

We have proved that $v_1(T,x,y)$ satisfies the second transport equation (6.4). Therefore, our v_1 coincides with that of Birkhoff.

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