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Global asymptotics for the damped wave equation with absorption in higher dimensional space

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Abstract. We consider the Cauchy problem for the damped wave equation with absorption

$$u_{tt} - \Delta u + u_t + |u|^{\rho-1}u = 0, \quad (t,x) \in \boldsymbol{R}_+ \times \boldsymbol{R}^N, \tag{$*$}$$

with $N = 3, 4$. The behavior of u as $t \to \infty$ is expected to be the Gauss kernel in the supercritical case $\rho > \rho_c(N) := 1 + 2/N$. In fact, this has been shown by Karch [**12**] (Studia Math., 143 (2000), 175–197) for $\rho > 1 + \frac{4}{N} (N = 1, 2, 3)$, Hayashi, Kaikina and Naumkin [**8**] (preprint (2004)) for $\rho > \rho_c(N) (N = 1)$ and by Ikehata, Nishihara and Zhao [**11**] (J. Math. Anal. Appl., **313** (2006), 598–610) for $\rho_c(N) < \rho \le 1 + \frac{4}{N} (N = 1, 2)$ and $\rho_c(N) < \rho < 1 + \frac{3}{N} (N = 3)$. Developing their result, we will show the behavior of solutions for $\rho_c(N) < \rho \le 1 + \frac{4}{N} (N = 3)$, $\rho_c(N) < \rho < 1 + \frac{4}{N} (N = 4)$. For the proof, both the weighted L^2-energy method with an improved weight developed in Todorova and Yordanov [**22**] (J. Differential Equations, **174** (2001), 464–489) and the explicit formula of solutions are still usefully used. This method seems to be not applicable for $N = 5$, because the semilinear term is not in C^2 and the second derivatives are necessary when the explicit formula of solutions is estimated.

1. Introduction.

We consider the asymptotic behavior of the solution to the Cauchy problem for the semilinear damped wave equation with absorption:

$$\begin{cases} u_{tt} - \Delta u + u_t + |u|^{\rho-1}u = 0, & (t,x) \in \boldsymbol{R}_+ \times \boldsymbol{R}^N, \\ (u, u_t)(0,x) = (u_0, u_1)(x), & x \in \boldsymbol{R}^N. \end{cases} \tag{1.1}$$

Here no restriction of the size of the data is imposed. When $\rho > 1$, the critical exponent $\rho_c(N)$ on the behavior of solutions is expected to be

$$\rho_c(N) = 1 + \frac{2}{N}. \tag{1.2}$$

The behaviors have been shown in some cases, which are as same as those for the semilinear heat equation with absorption, since the damped wave equation has the diffusive

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structure as $t \to \infty$ (Marcati and Nishihara [**14**], Hosono and Ogawa [**9**], Nishihara [**18**], [**19**], Narazaki [**16**], Ikehata-Nishihara [**10**]).

In the subcritical case $1 < \rho < \rho_c(N)$, the solution u to (1.1) is expected to behave as the similarity solution $w_b(t,x) := t^{-1/(\rho-1)} f(x/\sqrt{t})$ to the corresponding heat equation with absorption

$$\phi_t - \Delta\phi + |\phi|^{\rho-1}\phi = 0, \quad (t,x) \in (0,\infty) \times \boldsymbol{R}^N. \tag{1.3}$$

In fact, when $N = 1$, Hayashi, Kaikina and Naumkin [**7**], [**8**] have shown that

$$u(t,x) \sim w_b(t,x) \quad \text{as} \quad t \to \infty, \tag{1.4}$$

provided that ρ is near to $\rho_c(N)$, and, when $N \geq 1$, Nishihara and Zhao [**20**] and Ikehata, Nishihara and Zhao [**11**] showed that

$$\|(u, \nabla u)(t,\cdot)\|_{L^2} = O\big(t^{-\frac{1}{\rho-1}+\frac{N}{4}}, t^{-\frac{1}{\rho-1}+\frac{N}{4}-\frac{1}{2}}\big). \tag{1.5}$$

Here, the similarity solution w_b is given by the ordinary differential equation of $g(r) := f(x/\sqrt{t})$, $r = |x|/\sqrt{t}$:

$$\begin{cases} -g'' - \left(\dfrac{r}{2} + \dfrac{N-1}{r}\right)g' + |g|^{\rho-1}g = \dfrac{1}{\rho-1}g, \quad r \in (0,\infty), \\ g'(0) = 0, \quad \lim_{r\to\infty} r^{\frac{2}{\rho-1}} g(r) = b(\geq 0), \end{cases} \tag{1.6}$$

Note that the decay rates of the similarity solution are

$$\|(w_b, \nabla w_b)(t,\cdot)\|_{L^2} = O\big(t^{-\frac{1}{\rho-1}+\frac{N}{4}}, t^{-\frac{1}{\rho-1}+\frac{N}{4}-\frac{1}{2}}\big), \tag{1.7}$$

and hence the decay rates of (1.5) are sharp in the sense that (1.5) has the same rates as those in (1.7).

In the critical case $\rho = \rho_c(N)$, the solution ϕ to the Cauchy problem for (1.3) satisfies

$$\phi(t,x) \sim \theta_0 G(t,x)(\log t)^{-1/2} \quad \text{as} \quad t \to \infty \tag{1.8}$$

(Galaktionov, Kurdyumov and Samarskii [**4**]), and for (1.1) Hayashi, Kaikina and Naumkin [**6**], [**8**] have shown

$$u(t,x) \sim \theta_0 G(t,x)(\log t)^{-1/2} \quad \text{as} \quad t \to \infty, \tag{1.9}$$

where θ_0 is a suitable constant, G is the one-dimesional Gauss kernel and the N-dimensional Gauss kernel is defined by

$$G(t,x) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{|x|^2}{4t}}, \quad |x| = \sqrt{x_1^2 + x_2^2 + \cdots + x_N^2}. \tag{1.10}$$

When $N \geq 2$, neither the sharp decay orders nor (1.9) even for small data are obtained yet.

In the supercritical case $\rho > \rho_c(N)$, similar to the results by Escobedo and Kavian [**2**] and Escobedo, Kavian and Matano [**3**], it is expected that

$$\begin{aligned} &u(t,x) \sim \theta_0 G(t,x), \quad t \to \infty, \quad \text{with} \\ &\theta_0 = \int_{\boldsymbol{R}^N} (u_0 + u_1)(x)\,dx - \int_0^\infty \int_{\boldsymbol{R}^N} |u|^{\rho-1} u(\tau,x)\,dx\,d\tau. \end{aligned} \tag{1.11}$$

Kawashima, Nakao and Ono [**13**] showed the global existence of solutions for $1 < \rho < 1 + 4/(N-2)$ ($1 < \rho < \infty$ if $N = 1, 2$) and the L^2-decays of the solution including its higher derivatives for $1 + 4/N \leq \rho < 1 + 4/(N-2)$ $(3 \leq N \leq 5)$, $1 + 4/N \leq \rho < \infty$ $(N = 1, 2)$. Based on their results, Karch [**12**] showed (1.11) when $\rho > 1 + 4/N$ with $1 \leq N \leq 3$, and Hayashi, Kaikina and Naumkin [**5**], [**8**] have recently shown (1.11) when $\rho > \rho_c(N) = 3$, $N = 1$. Making use of their results, Ikehata, Nishihara and Zhao [**11**] have extended to the cases

$$\rho_c(N) < \rho \begin{cases} \leq 1 + \dfrac{4}{N} \ (N = 1, 2) \\[2ex] < 1 + \dfrac{3}{N} \ (N = 3). \end{cases} \tag{1.12}$$

Our aim in this paper is, by developing the method in [**11**], to show (1.11) when

$$\rho_c(N) < \rho \begin{cases} \leq 1 + \dfrac{4}{N} \ (N = 3) \\[2ex] < 1 + \dfrac{4}{N} \ (N = 4). \end{cases} \tag{1.13}$$

The same method does not seem to be applicable to show (1.11) in case of $N = 5$. Because the second derivatives of the semilinear term are necessary when we estimate the explicit formula of solutions, and the semilinear term $|u|^{\rho-1}u \notin C^2$ for $\rho < 1 + 4/5$ (See Remark 4.1 below).

For the related works see the references in [**11**], [**8**], etc.

The content of this paper is as follows. Since the proof is following to that in [**11**], we remember the results in [**11**] and its story in Section 2. Our main theorem is also stated. In Section 3 the basic estimates on the solution to the linear damped wave equation are derived. In Section 4 the series of Lemmas will be proved. In the final section 5 the proof of Main Theorem will be completed.

Notations. By $f(x) \sim g(x)$ as $|x| \to a$ we denote $\lim_{|x|\to a} \frac{f(x)}{g(x)} =$ (positive constant). Especially, $f(t,\cdot) \sim g(t,\cdot)$ as $t \to \infty$, $f, g : \boldsymbol{R}_+ \to X$ (Banach space) denotes

$\|f(t,\cdot)-g(t,\cdot)\|_X = o(\|g(t,\cdot)\|_X)$ as $t\to\infty$, so that $g(t,\cdot)$ is an asymptotic profile of $f(t,\cdot)$ as $t\to\infty$. By $C(a,b,\dots), C_{a,b,\dots}$ or $c(a,b,\dots), c_{a,b,\dots}$ we denote several positive constants depending on $a,b,\dots$. Without confusions, we denote them simply by C, c, whose quantities are changed line to line.

By $L^p = L^p(\boldsymbol{R}^N)$ $(1\le p\le\infty)$ we denote a usual Lebesgue space with its norm $\|\cdot\|_{L^p}$. When $p=2$, its suffix L^p is often abbreviated. The Sobolev space $H^m = H^m(\boldsymbol{R}^N) = \{f : \boldsymbol{R}^N \to \boldsymbol{R};\ \partial_x^i f \in L^2 (i=0,1,\dots,m)\}$, and $W^{m,q} = W^{m,q}(\boldsymbol{R}^N) = \{f : \boldsymbol{R}^N \to \boldsymbol{R};\ \partial_x^i f \in L^q (i=0,1,\dots,m)\}$. For $u(t,x) : \boldsymbol{R}_+ \to L^p$, $u \in L^{p,m} = L^{p,m}(\boldsymbol{R}^N)$ means $\big(1+\frac{|\cdot|}{\sqrt{t+1}}\big)^m u(t,\cdot) \in L^p(\boldsymbol{R}^N)$ together with

$$\|u(t,\cdot)\|_{L^{p,m}} = \left(\int_{\boldsymbol{R}^N}\left(1+\frac{|x|}{\sqrt{1+t}}\right)^{pm}|u(t,x)|^p dx\right)^{1/p}.$$

When $t=0$, $L^{p,m}$ becomes a usual weighted L^p space of order m. Often $\|u(t,\cdot)\|_{L^p}$, $\|u(t,\cdot)\|_{L^{p,m}}$ etc. are written simply as $\|u(t)\|_{L^p}$, $\|u(t)\|_{L^{p,m}}$ etc.

2. Known results and the main theorem.

First, we remember the results in [**11**]. By the weighted energy method with the improved weight introduced in [**22**] the following theorem is obtained.

Theorem 2.1 (Theorem 2.1 in [**11**]). *Suppose that* $1<\rho<1+\frac{2}{N-2}$ $(N\ge 3)$, $\rho<\infty$ $(N=1,2)$ *with* $\rho\le 1+\frac{4}{N}$ *and that* $(u_0,u_1)\in H^1(\boldsymbol{R}^N)\times L^2(\boldsymbol{R}^N)$ *with*

$$(1+|x|)^m\big(u_0,\nabla u_0,u_1,|u_0|^{\frac{\rho+1}{2}}\big)\in L^2(\boldsymbol{R}^N). \tag{2.1}$$

Then the solution $u\in C([0,\infty);H^1(\boldsymbol{R}^N))\cap C^1([0,\infty);L^2(\boldsymbol{R}^N))$ *to* (1.1) *uniquely exists, which satisfies for* $t\ge 0$

$$\|u(t,\cdot)\|_{L^{2,m}} \le C(I_0)(1+t)^{-\frac{1}{\rho-1}+\frac{N}{4}} \tag{2.2}$$

$$\|u(t,\cdot)\|_{L^{\rho+1,m}} \le C(I_0)(1+t)^{-\frac{1}{\rho-1}+\frac{N}{2(\rho+1)}} \tag{2.3}$$

$$\|(\nabla u,u_t)(t,\cdot)\|_{L^{2,m}} \le C(I_0)(1+t)^{-\frac{1}{\rho-1}+\frac{N}{4}-\frac{1}{2}} \tag{2.4}$$

together with

$$\int_0^t\Big[(1+\tau)^{\frac{2}{\rho-1}-\frac{N}{2}+\varepsilon}\big\|\big(\nabla u,|u|^{\frac{\rho+1}{2}}\big)(\tau,\cdot)\big\|^2_{L^{2,m}}+(1+\tau)^{\frac{2}{\rho-1}-\frac{N}{2}+1+\varepsilon}\|u_t(\tau,\cdot)\|^2_{L^{2,m}}\Big]d\tau$$
$$\le\begin{cases} C_\varepsilon(I_0) & (\varepsilon<0)\\ C(I_0)\log(2+t) & (\varepsilon=0)\\ C_\varepsilon(I_0)(1+t)^\varepsilon & (\varepsilon>0),\end{cases} \tag{2.5}$$

where

$$I_0 = \left\| (1+|\cdot|)^m \big(u_0, \nabla u_0, u_1, |u_0|^{\frac{\rho+1}{2}}\big)\right\|_{L^2} < \infty \tag{2.6}$$

and

$$m = \frac{2}{\rho-1} - \frac{N-\delta}{2} \ (>0) \tag{2.7}$$

for an arbitrarily fixed constand $\delta > 0$.

Very short sketch of the proof will be given in Section 4 before the proof of Lemma 4.2.

Though (2.1) with (2.7) means $u_0, u_1 \in L^1(\boldsymbol{R}^N)$ in the subcritical case, it is necessary in the supercritical case to assume

$$\delta > 4\left(\frac{N}{2} - \frac{1}{\rho-1}\right) (>0) \quad \text{or} \quad 2m > N. \tag{2.8}$$

Because

$$\begin{aligned} \|u_0\|_{L^1} &= \int_{\boldsymbol{R}^N} (1+|x|)^{-m} \cdot (1+|x|)^m |u_0(x)|\, dx \\ &\le \left(\int_{\boldsymbol{R}^N} (1+|x|)^{-2m} dx \right)^{1/2} \|u_0\|_{L^{2,m}} < \infty. \end{aligned} \tag{2.9}$$

Theorem 2.1 implies the following.

Theorem 2.2 (Corollary 3.1 in [**11**]). *In addition to the assumptions in Theorem 2.1, both $\rho > \rho_c(N)$ and (2.8) are supposed. Then, it holds that*

$$\|u(t)\|_{L^p} \le C(1+t)^{-\frac{1}{\rho-1}+\frac{N}{2p}}, \quad \|(u_t, \nabla u)(t)\|_{L^q} \le C(1+t)^{-\frac{1}{\rho-1}+\frac{N}{2q}-\frac{1}{2}}, \tag{2.10}$$

where

$$\begin{cases} 1 \le p \le \infty & (N=1) \\ 1 \le p < \infty & (N=2) \\ 1 \le p \le \dfrac{2N}{N-2} & (N \ge 3) \end{cases}, \quad 1 \le q \le 2. \tag{2.11}$$

The proof of Theorem 2.2 is based on the estimates like (2.9) and the Gagliardo-Nirenberg inequality.

Lemma 2.1 (Gagliardo-Nireberg). *Let the exponents $s, q, r (1 \le s, q, r \le \infty)$ and $\sigma \in [0,1]$ satisfy*

$$\frac{1}{s} = \sigma\left(\frac{1}{r} - \frac{1}{N}\right) + (1-\sigma)\frac{1}{q} \tag{2.12}$$

with $r \le N$ *except for* $s = \infty$ *or* $r = N$ *when* $N \ge 2$. *Then it hold that*

$$\|u\|_{L^s(\boldsymbol{R}^N)} \le C\|u\|^{1-\sigma}_{L^q(\boldsymbol{R}^N)}\|\nabla u\|^{\sigma}_{L^r(\boldsymbol{R}^N)}. \tag{2.13}$$

In proving the asymptotic behavior (1.11), it is important to show the boundedness of $\|u(t,\cdot)\|_{L^1}$. Following Hayashi, Kaikina and Naumkin [**8**], multiplying (1.1) by $\mathrm{sgn}(u) = 1\,(u>0),\ 0\,(u=0),\ -1\,(u<0)$ and integrating it over $\boldsymbol{R}^N$, we have

$$\frac{d}{dt}\|u(t)\|_{L^1} + \int_{\boldsymbol{R}^N}(-\Delta u\cdot \mathrm{sgn}(u) + |u|^\rho)\,dx = -\int_{\boldsymbol{R}^N} u_{tt}\mathrm{sgn}(u)\,dx$$

and hence

$$\frac{d}{dt}\|u(t)\|_{L^1} \le \|u_{tt}(t)\|_{L^1}$$

and

$$\|u(t)\|_{L^1} \le \|u_0\|_{L^1} + \int_0^t \|u_{tt}(\tau)\|_{L^1}d\tau. \tag{2.14}$$

Moreover, since

$$\|u_{tt}(t)\|_{L^1} \le C(1+t)^{\frac{N}{4}}\|u_{tt}(t)\|_{L^{2,m}} \tag{2.15}$$

in a similar way to (2.9), it is now important to obtain the faster decay estimate of $\|u_{tt}(t)\|_{L^{2,m}}$. To do so, we again use the weighted energy method to

$$(u_t)_{tt} - \Delta(u_t) + (u_t)_t + \rho|u|^{\rho-1}u_t = 0, \tag{2.16}$$

which comes from t-differetiation of (1.1). Though the semilinear term in (1.1) is an absorbing one, the nonlinear term in (2.16) is not absorbed. Hence, to obtain the energy estimates on u_{tt} by (2.16), we need L^∞-estimate of u to treat the last term $\rho|u|^{\rho-1}u_t$. In higher dimensional space we do not have it yet in Theorem 2.2. As in the previous paper [**11**], we have applied the explicit formula $S_N(t)g$ of solutions to

$$\begin{cases} v_{tt} - \Delta v + v_t = 0, & (t,x)\in \boldsymbol{R}_+\times\boldsymbol{R}^N,\\ (v,v_t)(0,x) = (0,g)(x), & x\in\boldsymbol{R}^N. \end{cases} \tag{2.17}$$

Concretely, $S_N(t)g$ for $N = 3,4$ is given by

$$[S_3(t)g](x) = e^{-t/2}\frac{t}{4\pi}\int_{S^2} g(x+t\omega)\,d\omega + \frac{e^{-t/2}}{4\pi}\int_{|z|\le t}\frac{I_1\left(\frac12\sqrt{t^2-|z|^2}\right)}{2\sqrt{t^2-|z|^2}}g(x+z)\,dz \tag{2.18}$$

and

$$[S_4(t)g](x) = \frac{e^{-t/2}}{4\pi^2 t}\partial_t \int_0^t \frac{\cosh\left(\frac{1}{2}\sqrt{t^2-\rho^2}\right)}{\sqrt{t^2-\rho^2}}\rho^3 \int_{S^3} g(x+\rho\omega)\,d\omega\,d\rho, \tag{2.19}$$

where $I_\nu(y)$ is a modified Bessel function of order ν given by

$$I_\nu(y) = \sum_{m=0}^{\infty} \frac{1}{m!\Gamma(m+\nu+1)}\left(\frac{y}{2}\right)^{2m+\nu}$$

with the Gamma function Γ (see e.g. Courant and Hilbert [**1**], Nikiforov and Ouvarov [**17**]. See also Ono [**21**]), S^{N-1} is a unit sphere in $\boldsymbol{R}^N$ and $d\omega$ is its surface element. By the Duhamel principle, the solution u to (1.1) is that of the integral equation

$$u(t,\cdot) = S_N(t)(u_0+u_1) + \partial_t(S_N(t)u_0) - \int_0^t S_N(t-\tau)|u|^{\rho-1}u(\tau)\,d\tau. \tag{2.20}$$

In case of $N=3$, using the explicit formula (2.20) with (2.18) we can show the L^∞-estimate of u when $\rho_c(N) < \rho \le 1+4/N$. On the other hand, in case of $N=4$, combining (2.19)–(2.20) with the weighted L^2-energy estimates on

$$(\nabla u)_{tt} - \Delta(\nabla u) + (\nabla u)_t + \rho|u|^{\rho-1}\nabla u = 0 \tag{2.21}$$

derived by ∇(1.1), we can also show the L^∞-estimate when $\rho_c(N) < \rho < 1+4/N$. More precisely, the following key lemma holds, which will be shown in Section 4. We note that the L^∞-estimate is not optimal. The optimal decay estimates and the asymptotic profile in higher dimensions $N=3,4$ will be obtained by a series of Lemmas.

Lemma 2.2 (Key Lemma). *In addition to the assumptions in Theorem* 2.2, *suppose that* $(u_0,u_1)\in H^2\times H^1$. *Then, if*

$$\rho_c(N) < \rho \le 1+4/N \quad \textit{with} \quad N=3, \tag{2.22}$$

then it holds that

$$\|u(t)\|_{L^\infty} \le C(1+t)^{-\frac{1}{\rho-1}+\mu} \tag{2.23}$$

for an arbitrarily small $\mu>0$. *Moreover, if* $(u_0,u_1)\in H^3\times H^2$ *is assumed and*

$$I_1 := \|(1+|\cdot|)^m(\nabla u_0, \Delta u_0, \nabla u_1)(\cdot)\|_{L^2} < \infty, \tag{2.24}$$

then (2.23) *holds for*

$$\rho_c(N) < \rho < 1+4/N \quad \textit{with} \quad N=4. \tag{2.25}$$

Once we have Lemma 2.2, we return to (2.16) to obtain

LEMMA 2.3. *Under the assumptions in Lemma 2.2 it holds that*

$$\int_0^t (1+\tau)^{\frac{2}{\rho-1}-\frac{N}{2}+3-\varepsilon} \|u_{tt}(\tau)\|_{L^{2,m}}^2 \, d\tau \le C(I_0, I_1). \tag{2.26}$$

From (2.26) and (2.14)–(2.15) we have

$$\begin{aligned} &\int_0^t \|u_{tt}(\tau)\|_{L^1} d\tau \le C\int_0^t (1+\tau)^{\frac{N}{4}} \|u_{tt}(\tau)\|_{L^{2,m}} d\tau \\ &\le C\biggl(\int_0^t (1+\tau)^{\frac{N}{2}-\frac{2}{\rho-1}+\frac{N}{2}-3+\varepsilon} d\tau\biggr)^{1/2} \biggl(\int_0^t (1+\tau)^{\frac{2}{\rho-1}-\frac{N}{2}+3-\varepsilon} \|u_{tt}(\tau)\|_{L^{2,m}}^2 \, d\tau\biggr)^{1/2} \\ &\le C(I_0, I_1) \end{aligned} \tag{2.27}$$

and hence the L^1-boundedness of u. Because

$$\frac{N}{2} - \frac{2}{\rho-1} + \frac{N}{2} - 3 + \varepsilon = -\left(\frac{2}{\rho-1} - \frac{N}{2}\right) - \left(3 - \frac{N}{2}\right) < -1$$

if $0 < \varepsilon \ll 1$, since $N = 3$ or $N = 4$.

LEMMA 2.4. *Under the assumptions in Lemma 2.2, it holds that*

$$\|u(t)\|_{L^1} \le C(I_0, I_1). \tag{2.28}$$

We again apply the L^1-boundedness (2.28) to the integral formula (2.20), then we can obtain the optimal decay rate.

LEMMA 2.5. *Under the assumptions in Lemma 2.2 it holds that*

$$\|u(t)\|_{L^p} \le C(I_0, I_1)(1+t)^{-\frac{N}{2}(1-\frac{1}{p})} \tag{2.29}$$

for $N = 3, 4$.

Moreover, by the integral formula (2.20) of solutions, we can obtain the asymptotic formula $\theta_0 G(t,x)$ of the solution $u(t,x)$, and thus reach to our main theorem.

THEOREM 2.3 (Main Theorem). *Let $N = 3, 4$ and the exponent ρ satisfy (1.13). Suppose that $(u_0, u_1) \in H^2 \times H^1$ $(N = 3)$ or $(u_0, u_1) \in H^3 \times H^2$ $(N = 4)$ with*

$$(1+|\cdot|)^m \bigl(u_0, \nabla u_0, \Delta u_0, |u_0|^{\frac{\rho+1}{2}}, u_1, \nabla u_1\bigr) \in L^2(\boldsymbol{R}^N). \tag{2.30}$$

and (2.8). Then there exists a unique solution $u \in C([0,\infty); H^2(\boldsymbol{R}^N)) \cap C^1([0,\infty); H^1(\boldsymbol{R}^N)) \cap C^2([0,\infty); L^2(\boldsymbol{R}^N))$ to (1.1) satisfying $(u, \nabla u, u_t, \Delta u, \nabla u_t, u_{tt}) \in L^{2,m}(\boldsymbol{R}^N)$. Moreover, for θ_0 given by (1.11) and $1 \le p \le \infty$, it holds that

$$\|u(t,\cdot)-\theta_0 G(t,\cdot)\|_{L^p}=o\big(t^{-\frac{N}{2}(1-\frac{1}{p})}\big) \quad as \quad t\to\infty. \tag{2.31}$$

REMARK 2.1. When $N=4$, the assumption $(u_0,u_1)\in H^3\times H^2$ with (2.29), (2.8) implies that $u_0+u_1\in H^{[\frac{N}{2}]}\cap L^1$ and $u_0\in H^{[\frac{N}{2}]+1}\cap L^1$, and that $\|S_4(t)(u_0+u_1)\|_{L^p}\le C(1+t)^{-2(1-1/p)}$ and $\|\partial_t(S_4(t)u_0)\|_{L^p}\le C(1+t)^{-2(1-1/p)-1}$ (cf. Matsumura [**15**]). The situation in the case $N=3$ is just similar to these. Those properties will be used in the final section.

3. Basic estimates for the linear damped wave equation.

For the proof of the key Lemma 2.2, we analyse the explicit formula (2.18)–(2.19) of the solution to the problem (2.17) for the linear damped wave equation. When $N=3$, we epress the solution $S_3(t)g$ as

$$\begin{aligned}[S_3(t)g](x)&=e^{-t/2}\frac{t}{4\pi}\int_{S^2}g(x+t\omega)\,d\omega+\frac{e^{-t/2}}{4\pi}\int_{|z|\le t}\frac{I_1\big(\frac12\sqrt{t^2-|z|^2}\big)}{2\sqrt{t^2-|z|^2}}g(x+z)\,dz\\ &=: e^{-t/2}[W_{03}(t)g](x)+[J_{03}(t)g](x)\end{aligned}\tag{3.1}$$

and

$$\begin{aligned}[\partial_t(S_3(t)g)](x)&=e^{-t/2}\bigg[\Big(-\frac12+\frac{t}{8}\Big)[W_{03}(t)g](x)+[\partial_t(W_{03}(t)g)](x)\bigg]\\ &\quad+\frac{1}{4\pi}\int_0^t\int_{S^2}\partial_t\bigg[e^{-t/2}I_1\Big(\frac12\sqrt{t^2-|z|^2}\Big)\frac{\rho^2}{2\sqrt{t^2-|z|^2}}\bigg]g(x+\rho\omega)\,d\omega\,d\rho\\ &=: [W_{13}(t)g](x)+[J_{13}(t)g](x).\end{aligned}\tag{3.2}$$

Then we have the following properties.

LEMMA 3.1 ($N=3$). *For $1\le q\le p\le\infty$,*

$$\|J_{03}(t)g\|_{L^p}\le C(1+t)^{-\frac32(\frac1q-\frac1p)}\|g\|_{L^q}, \qquad t\ge 0, \tag{3.3}$$

$$\|J_{03}(t)g-P_3(t)g\|_{L^p}\le t^{-\frac32(\frac1q-\frac1p)-1}\|g\|_{L^q}, \qquad t>0, \tag{3.4}$$

$$\|J_{13}(t)g\|_{L^p}\le C(1+t)^{-\frac32(\frac1q-\frac1p)-1}\|g\|_{L^q}, \quad t\ge 0, \tag{3.5}$$

and, for any constant $\varepsilon>0$,

$$\|W_{03}(t)g\|_{L^\infty}\le C\big(t^{\frac{\varepsilon}{3+\varepsilon}}\|g\|_{L^{3+\varepsilon}}+\|g\nabla g\|_{L^1}^{1/2}\big), \quad t\ge 0, \tag{3.6}$$

$$\|W_{03}(t)g\|_{L^q}\le t\|g\|_{L^q}, \qquad t\ge 0, \tag{3.7}$$

$$\|W_{13}(t)g\|_{L^q}\le C\big[(1+t)^2\|g\|_{L^q}+t\|\nabla g\|_{L^q}\big], \quad t\ge 0, \tag{3.8}$$

where

$$[P_N(t)g](x) = \int_{\boldsymbol{R}^N} \frac{e^{-\frac{|z|^2}{4t}}}{(4\pi t)^{N/2}} g(x+z)\,dz = \int_0^\infty \int_{S^{N-1}} \frac{e^{-\frac{\rho^2}{4t}}}{(4\pi t)^{N/2}} g(x+\rho\omega)\rho^{N-1}\,d\omega\,d\rho. \tag{3.9}$$

REMARK 3.1. By the Gagliardo-Nirenberg inequality, (3.6) with $\varepsilon = 1$ and $\sigma = 3/4$ imply

$$\|W_{03}(t)g\|_{L^\infty} \le C\big(t^{1/4}\|g\|_{L^2}^{1-\sigma}\|\nabla g\|_{L^2}^{\sigma} + \|g\|_{L^2}^{1/2}\|\nabla g\|_{L^2}^{1/2}\big) \le C(1+t)^{1/4}\|g\|_{H^1}.$$

Hence, combining this with (3.7), we have

$$e^{-t/2}\|W_{03}(t)g\|_{L^p} \le Ce^{-\beta t}\|g\|_{H^1}^{1-\frac{q}{p}}\|g\|_{L^q}^{\frac{q}{p}}, \quad 0 < \beta \ll \frac{1}{2}. \tag{3.10}$$

Moreover, the general estimate

$$\|\partial_t(S_N(t)g)\|_{L^\infty} \le C(1+t)^{-\frac{N}{2}\frac{1}{q}-1}\|g\|_{H^{[\frac{N}{2}]+1}\cap L^q}, \quad t \ge 0, \tag{3.11}$$

by Matsumura [**15**] together with (3.5), (3.8) implies

$$\|\partial_t(S_3(t)g)\|_{L^p} \le C(1+t)^{-\frac{3}{2}(\frac{1}{q}-\frac{1}{p})-1}\|g\|_{H^2\cap W^{1,q}}, \quad t \ge 0. \tag{3.12}$$

PROOF OF LEMMA 3.1. The estimates (3.3)–(3.5) are showed in [**18**], and (3.6) is in [**11**]. By (3.1) and (3.2),

$$\|W_{03}(t)g\|_{L^q} \le \frac{t}{4\pi}\int_{S^2}\|g\|_{L^q}d\omega = t\|g\|_{L^q}.$$

and

$$\begin{aligned}\|W_{13}(t)g\|_{L^q} &\le \left(\frac{1}{2}+\frac{t}{8}\right)\frac{t}{4\pi}\int_{S^2}\|g\|_{L^q}d\omega + \frac{1}{4\pi}\int_{S^2}(\|g\|_{L^q} + t\|\nabla g\|_{L^q})d\omega \\ &\le C\big((1+t)^2\|g\|_{L^q} + t\|\nabla g\|_{L^q}\big),\end{aligned}$$

which show (3.7) and (3.8), respectively. □

We want to have the similar estimates in Lemma 3.1 even when $N = 4$. Rewrite (2.19) as

$$\begin{aligned} S_4(t)g &= \frac{e^{-t/2}}{4\pi^2 t}\partial_t\int_0^t \frac{\rho^3}{\sqrt{t^2-\rho^2}}\int_{S^3} g(x+\rho\omega)\,d\omega\,d\rho \\ &\quad + \frac{e^{-t/2}}{4\pi^2 t}\partial_t\int_0^t \frac{\cosh\left(\frac{1}{2}\sqrt{t^2-\rho^2}\right)-1}{\sqrt{t^2-\rho^2}}\rho^3\int_{S^3} g(x+\rho\omega)\,d\omega \\ &=: e^{-t/2}W_{04}(t)g + J_{04}(t)g. \end{aligned} \tag{3.13}$$

By integral by parts

$$\begin{aligned} W_{04}(t)g &= \frac{1}{4\pi^2 t}\partial_t \int_0^t -\partial_\rho\big(\sqrt{t^2-\rho^2}\big)\cdot\rho^2\int_{S^3} g(x+\rho\omega)\,d\omega\,d\rho \\ &= \frac{1}{4\pi^2 t}\partial_t \int_0^t \sqrt{t^2-\rho^2}\left(2\rho\int_{S^3} g(x+\rho\omega)\,d\omega + \rho^2\int_{S^3}\nabla g(x+\rho\omega)\cdot\omega\,d\omega\right)d\rho. \end{aligned}$$

Hence

$$W_{04}(t)g = \frac{1}{4\pi^2}\int_0^t \frac{\rho}{\sqrt{t^2-\rho^2}}\left(2\int_{S^3} g(x+\rho\omega)\,d\omega + \rho\int_{S^3}\nabla g(x+\rho\omega)\cdot\omega\,d\omega\right)d\rho. \quad (3.14)$$

Also, differentiating the integral in $J_{04}(t)g$ with respect to t, we have

$$\begin{aligned} J_{04}(t)g &= \frac{e^{-t/2}}{4\pi^2}\int_0^t \left[\frac{\sinh\left(\frac{1}{2}\sqrt{t^2-\rho^2}\right)}{2(t^2-\rho^2)} - \frac{\cosh\left(\frac{1}{2}\sqrt{t^2-\rho^2}\right)-1}{(t^2-\rho^2)\sqrt{t^2-\rho^2}}\right]\rho^3\int_{S^3} g(x+\rho\omega)\,d\omega\,d\rho \\ &= \begin{cases} \displaystyle\int_0^{\sqrt{t^2-A}} + \int_{\sqrt{t^2-A}}^t & (t \ge \sqrt{A}) \\ \displaystyle 0 + \int_0^t & (t < \sqrt{A}) \end{cases} \\ &=: \bar{J}_{04}(t)g + e^{-t/2}\cdot j_{04}(t)g \end{aligned} \quad (3.15)$$

for a constant $A > 0$. That is, by denoting $(x)_+ = x\,(x>0),\ 0\,(x \le 0)$,

$$\begin{cases} \displaystyle \bar{J}_{04}(t)g = \frac{e^{-t/2}}{4\pi^2}\int_0^{\sqrt{(t^2-A)_+}} \left[\frac{\sinh\left(\frac{1}{2}\sqrt{t^2-\rho^2}\right)}{2(t^2-\rho^2)} - \frac{\cosh\left(\frac{1}{2}\sqrt{t^2-\rho^2}\right)-1}{(t^2-\rho^2)\sqrt{t^2-\rho^2}}\right]\rho^3 \\ \displaystyle \qquad\qquad \cdot\int_{S^3} g(x+\rho\omega)\,d\omega\,d\rho \\ \displaystyle j_{04}(t)g = \frac{1}{4\pi^2}\int_{\sqrt{(t^2-A)_+}}^t \left[\frac{\sinh\left(\frac{1}{2}\sqrt{t^2-\rho^2}\right)}{2(t^2-\rho^2)} - \frac{\cosh\left(\frac{1}{2}\sqrt{t^2-\rho^2}\right)-1}{(t^2-\rho^2)\sqrt{t^2-\rho^2}}\right]\rho^3 \\ \displaystyle \qquad\qquad \cdot\int_{S^3} g(x+\rho\omega)\,d\omega\,d\rho. \end{cases} \quad (3.15)'$$

Also, we put

$$\begin{aligned} S_4(t)g &= e^{-t/2}(W_{04}(t)g + j_{04}(t)g) + \bar{J}_{04}(t)g \\ &=: e^{-t/2}\bar{W}_{04}(t)g + \bar{J}_{04}(t)g \end{aligned} \quad (3.16)$$

By differentiating $S_4(t)g$ in t, we have

$$\partial_t(S_4(t)g) = e^{-t/2}\Big[\Big(-\frac{1}{2}+\partial_t\Big)(W_{04}(t)g + j_{04}(t)g)$$
$$+\frac{1}{4\pi^2}\Big(\frac{\sinh\frac{\sqrt{A}}{2}}{2A} - \frac{\cosh\frac{\sqrt{A}}{2}-1}{A\sqrt{A}}\Big)t(t^2-A)_+\int_{S^3} g\big(x+\sqrt{t^2-A}\omega\big)\,d\omega\Big]$$
$$+\frac{e^{-t/2}}{4\pi^2}\int_0^{\sqrt{(t^2-A)_+}}\Big(-\frac{1}{2}+\partial_t\Big)\Big[\frac{\sinh\big(\frac{1}{2}\sqrt{t^2-\rho^2}\big)}{2(t^2-\rho^2)} - \frac{\cosh\big(\frac{1}{2}\sqrt{t^2-\rho^2}\big)-1}{(t^2-\rho^2)\sqrt{t^2-\rho^2}}\Big]$$
$$\times\rho^3\int_{S^3} g(x+\rho\omega)\,d\omega\,d\rho$$
$$=: e^{-t/2}\bar{W}_{14}(t)g + \bar{J}_{14}(t)g. \tag{3.17}$$

Estimates on $\bar{W}_{i4}(t)g,\ \bar{J}_{i4}(t)g\ (i=0,1)$ are given by the following lemma.

Lemma 3.2 ($N=4$). *For $1\le q\le p\le\infty$,*

$$\|\bar{J}_{04}(t)g\|_{L^p} \le C(1+t)^{-2(\frac{1}{q}-\frac{1}{p})}\|g\|_{L^q}, \qquad t\ge 0, \tag{3.18}$$
$$\|\bar{J}_{04}(t)g - P_4(t)g\|_{L^p} \le Ct^{-2(\frac{1}{q}-\frac{1}{p})-1}\|g\|_{L^q}, \qquad t\ge\sqrt{A}\ge\sqrt{3}, \tag{3.19}$$
$$\|\bar{J}_{14}(t)g\|_{L^p} \le C(1+t)^{-2(\frac{1}{q}-\frac{1}{p})-1}\|g\|_{L^q}, \quad t\ge 0, \tag{3.20}$$

and, for any constants $s,\bar{s}>2$ and $t\ge 0$,

$$\|\bar{W}_{04}(t)g\|_{L^\infty} \le C\big[\big(t^{1-\frac{4}{s}} + t^{2-\frac{2}{s}}\big)\|g\|_{L^s} + t^{2-\frac{4}{\bar{s}}}\|\nabla g\|_{L^{\bar{s}}}\big], \tag{3.21}$$
$$\|\bar{W}_{04}(t)g\|_{L^q} \le C\big[(t+t^2)\|g\|_{L^q} + t^2\|\nabla g\|_{L^q}\big], \qquad q<\infty, \tag{3.22}$$
$$\|\bar{W}_{14}(t)g\|_{L^q} \le C\big[(1+t)^2\|g\|_{L^q} + t^2(1+t)\|\nabla g\|_{L^q} + t^2\|\Delta g\|_{L^q}\big], \quad q<\infty. \tag{3.23}$$

Remark 3.2. Similar to Remark 3.1, even in $N=4$ we have

$$\|\bar{W}_{04}(t)g\|_{L^\infty} \le C\big[(1+t^{3/2})\|g\|_{L^2}^{1-\sigma}\|\nabla g\|_{L^2}^{\sigma} + t\|\nabla g\|_{L^2}^{1-\sigma}\|\Delta g\|_{L^2}^{\sigma}\big]$$
$$\le C(1+t)^{3/2}\|g\|_{H^2}$$

by the Gagliardo-Nirenberg inequality with $s,\bar{s}=4$, $\sigma=1/3$. Combining this with (3.22) implies

$$e^{-t/2}\|\bar{W}_{04}(t)g\|_{L^p} \le e^{-\beta t}\|g\|_{H^2\cap L^q}, \quad 0<\beta\ll 1. \tag{3.24}$$

We also have

$$\|\partial_t(S_4(t)g)\|_{L^p} \le C(1+t)^{-2(\frac{1}{q}-\frac{1}{p})-1}\|g\|_{H^3\cap W^{2,q}} \tag{3.25}$$

by combining (3.11) with (3.20), (3.23).

Proof of Lemma 3.2. Rewriting (3.14) and taking $s, \bar{s} > 2$ with $\frac{1}{s} + \frac{1}{s'} = 1$, $\frac{1}{\bar{s}} + \frac{1}{\bar{s}'} = 1$, we have

$$
\begin{aligned}
|W_{04}(t)g| &= \left| \frac{1}{4\pi^2} \int_{|y|\le t} \left[\frac{2g(x+y)}{\sqrt{t^2 - |y|^2|y|^2}} + \frac{\nabla g(x+y)\cdot y}{\sqrt{t^2-|y|^2|y|^2}} \right] dy \right| \\
&\le C\Big(\int_{|y|\le t} \Big(\frac{1}{\sqrt{t^2-|y|^2|y|^2}} \Big)^{s'} dy \Big)^{1/s'} \|g\|_{L^s} \\
&\quad + C\Big(\int_{|y|\le t} \Big(\frac{1}{\sqrt{t^2-|y|^2|y|}} \Big)^{\bar{s}'} dy \Big)^{1/\bar{s}'} \|\nabla g\|_{L^{\bar{s}}} \\
&\le C\big(t^{1-\frac{4}{s}} \|g\|_{L^s} + t^{2-\frac{4}{\bar{s}}} \|\nabla g\|_{L^{\bar{s}}} \big)
\end{aligned} \tag{3.26}
$$

and

$$
\begin{aligned}
\|W_{04}(t)g\|_{L^q} &\le C \int_{|y|\le t} \frac{dy}{\sqrt{t^2-|y|^2|y|^2}} \cdot \|g\|_{L^q} \\
&\quad + C \int_{|y|\le t} \frac{dy}{\sqrt{t^2-|y|^2|y|}} \cdot \|\nabla g\|_{L^q} \\
&\le C\big(t\|g\|_{L^q} + t^2 \|\nabla g\|_{L^q} \big).
\end{aligned} \tag{3.27}
$$

Moreover, since

$$
\begin{aligned}
\sinh(x) &= x + O(|x|^3), \\
\cosh(x) - 1 &= \frac{x^2}{2!} + O(|x|^4),
\end{aligned} \tag{3.28}
$$

we have

$$
|[j_{04}(t)g](x)| \le C \int_{\sqrt{(t^2-A)_+}}^{t} \frac{\rho^3}{\sqrt{t^2-\rho^2}} \int_{S^3} |g(x+\rho\omega)|\, d\omega\, d\rho
$$

and, hence

$$
\|j_{04}(t)g\|_{L^\infty} \le Ct^{2-\frac{2}{s}} \|g\|_{L^s} \tag{3.29}
$$

and

$$
\|j_{04}(t)g\|_{L^q} \le Ct^2 \|g\|_{L^q}. \tag{3.30}
$$

In fact,

$$\begin{aligned}|[j_{04}(t)g](x)| &\le C\int_{\sqrt{(t^2-A)_+}<|y|<t}\frac{|g(x+y)|}{\sqrt{t^2-|y|^2}}dy\\ &\le C\Big(\int_{\sqrt{(t^2-A)_+}<|y|<t}(t^2-|y|^2)^{-s'/2}dy\Big)^{1/s'}\|g\|_{L^s}\\ &\le C\Big(\int_{\sqrt{(t^2-A)_+}}^t\int_{S^3}(t^2-\rho^2)^{-s'/2}\rho^3d\omega d\rho\Big)^{1/s'}\|g\|_{L^s}\\ &\le Ct^{2/s'}A^{-s'/2+1}\|g\|_{L^s}\le Ct^{2-\frac{2}{s}}\|g\|_{L^s}\end{aligned}$$

and

$$\|j_{04}(t)g\|_{L^q}\le C\int_{\sqrt{(t^2-A)_+}}^t\frac{\rho^3}{\sqrt{t^2-\rho^2}}d\rho\cdot\|g\|_{L^q}\le Ct^2A^{1/2}\|g\|_{L^q}\le Ct^2\|g\|_{L^q}.$$

Thus we have (3.21) and (3.22). Though we need tedious but not difficult culculations, similar to (3.30), we have (3.23).

Noting that $\sqrt{t^2-A}\ge t^{2/3}$ if $t\ge\sqrt{A}\ge\sqrt{3}$, we set

$$\begin{aligned}[\bar{J}_{04}(t)g-P_4(t)g](x) &= \Big(\int_0^{t^{2/3}}+\int_{t^{2/3}}^{\sqrt{t^2-A}}\Big)\frac{e^{-\frac{\rho^2}{4t}}}{(\sqrt{4\pi t})^4}\Big[\frac{2t^2e^{-\frac{t}{2}+\frac{\rho^2}{4t}}\sinh\big(\frac12\sqrt{t^2-\rho^2}\big)}{t^2-\rho^2}-1\\ &\qquad -\frac{4t^2e^{-\frac{t}{2}+\frac{\rho^2}{4t}}\big(\cosh\big(\frac12\sqrt{t^2-\rho^2}\big)-1\big)}{(t^2-\rho^2)\sqrt{t^2-\rho^2}}\Big]\cdot\rho^3\int_{S^3}g(x+\rho\omega)\,d\omega\,d\rho\\ &\quad -\int_{\sqrt{t^2-A}}^\infty\frac{e^{-\frac{\rho^2}{4t}}}{(\sqrt{4\pi t})^4}\cdot\rho^3\int_{S^3}g(x+\rho\omega)\,d\omega\,d\rho\\ &=: (K_{11}+K_{12})+K_2.\end{aligned}\tag{3.31}$$

It is easy to show

$$\|K_2\|_{L^p}=O(e^{-\beta t})\|g\|_{L^q},\quad 0<\beta\ll1.\tag{3.32}$$

For K_{12}, since $\sqrt{A}\le\sqrt{t^2-\rho^2}\le\sqrt{t^2-t^{4/3}}$, $\rho^2/t\ge t^{1/3}$ and

$$-\frac{t}{2}+\frac{\sqrt{t^2-t^{4/3}}}{2}=-\frac{t^{4/3}}{2\big(t+\sqrt{t^2-t^{4/3}}\big)}\le-\frac14t^{1/3},$$

we have

$$\|K_{12}\|_{L^\infty\cap L^q}\le Ce^{-\beta t^{1/3}}\|g\|_{L^q}$$

or

$$\|K_{12}\|_{L^p} \le Ce^{-\beta t^{1/3}}\|g\|_{L^q}, \quad q \le p \le \infty. \tag{3.33}$$

For the main term K_{11}, since $\rho^2/t^2 \le t^{-2/3} \le A^{-2/3}$ and $\rho^2/t^2 \le C(A)\rho/t$,

$$k_1 := \frac{2t^2 e^{-\frac{t}{2}+\frac{\rho^2}{4t}} \sinh\left(\frac{1}{2}\sqrt{t^2-\rho^2}\right)}{t^2-\rho^2} - 1 = \frac{1}{t}O\left(\frac{\rho^2}{t} + \left(\frac{\rho^2}{t}\right)^2\right)$$

$$k_2 := \frac{4t^2 e^{-\frac{t}{2}+\frac{\rho^2}{4t}}\left(\cosh\left(\frac{1}{2}\sqrt{t^2-\rho^2}\right)-1\right)}{(t^2-\rho^2)\sqrt{t^2-\rho^2}} - 1 = \frac{1}{t}O\left(1 + \frac{\rho^2}{t} + \left(\frac{\rho^2}{t}\right)^2\right).$$

In fact, since

$$-\frac{t}{2} + \frac{\rho^2}{4t} + \frac{\sqrt{t^2-\rho^2}}{2} = \frac{\rho^2}{4t} - \frac{\rho^2}{2(t+\sqrt{t^2-\rho^2})} = -\frac{\rho^4}{4t^3}\left(1+\sqrt{1-\frac{\rho^2}{t^2}}\right)^{-2},$$

we have

$$k_1 = \left(e^{-\frac{\rho^4}{4t^3}\left(1+\sqrt{1-\frac{\rho^2}{t^2}}\right)^{-2}} + e^{-\beta t}\right)\left(1-\frac{\rho^2}{t^2}\right)^{-1} - 1 = \frac{1}{t}O\left(\frac{\rho^2}{t} + \left(\frac{\rho^2}{t}\right)^2\right)$$

and

$$k_2 = \frac{4}{t}\left(e^{-\frac{\rho^4}{4t^3}\left(1+\sqrt{1-\frac{\rho^2}{t^2}}\right)^{-2}} - e^{-\beta t}\right)\left(1-\frac{\rho^2}{t^2}\right)^{-3/2} = \frac{1}{t}O\left(1 + \frac{\rho^2}{t} + \left(\frac{\rho^2}{t}\right)^2\right).$$

Thus

$$\begin{aligned}\|K_{11}\|_{L^p} &\le \left\| \int_0^{t^{2/3}} \frac{e^{-\frac{\rho^2}{4t}}}{(\sqrt{4\pi t})^4} \frac{1}{t} O\left(1 + \frac{\rho^2}{t} + \left(\frac{\rho^2}{t}\right)^2\right) \cdot \rho^3 \int_{S^3} |g(x+\rho\omega)|\, d\omega\, d\rho \right\|_{L^p} \\ &\le Ct^{-2(\frac{1}{q}-\frac{1}{p})-1}\|g\|_{L^q}. \end{aligned} \tag{3.34}$$

Here we have used the Hausdorff-Young inequality:

LEMMA 3.3 (Hausdorff-Young). *For p, q, r $(1 \le p, q, r \le \infty)$ satisfying $\frac{1}{q} - \frac{1}{p} = 1 - \frac{1}{r}$, the inequality*

$$\|f * g\|_{L^p} \le C\|f\|_{L^r}\|g\|_{L^q}$$

holds, where $*$ denotes the convolution.

Combining (3.31)–(3.34) we have obtained (3.19). Since $\bar{J}_{04}(t)g \equiv 0$ when $0 \le t \le$

$\sqrt{A}$, both (3.19) and the well-known result

$$\|P_N(t)g\|_{L^p} \le Ct^{-\frac{N}{2}(\frac{1}{q}-\frac{1}{p})}\|g\|_{L^q}, \quad t>0$$

imply (3.18).

For $\bar{J}_{14}(t)g$, after some calculations we have

$$\begin{aligned}\bar{J}_{14}(t)g = \frac{e^{-t/2}}{4\pi^2}\int_0^{\sqrt{(t^2-A)_+}} \Bigg[\Big(-\frac{1}{2}\Big)\Bigg\{ & \frac{\sinh\big(\frac{1}{2}\sqrt{t^2-\rho^2}\big)}{2(t^2-\rho^2)} - \frac{\cosh\big(\frac{1}{2}\sqrt{t^2-\rho^2}\big)-1}{(t^2-\rho^2)\sqrt{t^2-\rho^2}}\Bigg\} \\ & + t\Bigg\{\frac{\cosh\big(\frac{1}{2}\sqrt{t^2-\rho^2}\big)}{4(t^2-\rho^2)\sqrt{t^2-\rho^2}} - \frac{3\sinh\big(\frac{1}{2}\sqrt{t^2-\rho^2}\big)}{2(t^2-\rho^2)^2} \\ & + \frac{3\big(\cosh\big(\frac{1}{2}\sqrt{t^2-\rho^2}\big)-1\big)}{(t^2-\rho^2)\sqrt{t^2-\rho^2}}\Bigg\}\Bigg] \\ \times \rho^3\int_{S^3} g(x+\rho\omega)\,d\omega\,d\rho. \end{aligned} \tag{3.35}$$

Similar to the estimate on K_{12}, when $t \ge \sqrt{A}$,

$$\Big\|\int_{\sqrt{t^2-A}}^t\Big\|_{L^p} \le Ce^{-\beta t^{1/3}}\|g\|_{L^q}. \tag{3.36}$$

Also, similar to K_{11},

$$\begin{aligned}\int_0^{t^{2/3}} = \int_0^{t^{2/3}} \frac{e^{-\frac{\rho^2}{4t}}}{(\sqrt{4\pi t})^4}\cdot e^{-\frac{t}{2}+\frac{\rho^2}{4t}}\Bigg\{ & -\frac{t^2\sinh\big(\frac{1}{2}\sqrt{t^2-\rho^2}\big)}{t^2-\rho^2} + \frac{2t^2\big(\cosh\big(\frac{1}{2}\sqrt{t^2-\rho^2}\big)-1\big)}{(t^2-\rho^2)\sqrt{t^2-\rho^2}} \\ & + \frac{t^3\cosh\big(\frac{1}{2}\sqrt{t^2-\rho^2}\big)}{(t^2-\rho^2)\sqrt{t^2-\rho^2}} - \frac{6t^3\sinh\big(\frac{1}{2}\sqrt{t^2-\rho^2}\big)}{(t^2-\rho^2)^2} \\ & + \frac{12t^3\big(\cosh\big(\frac{1}{2}\sqrt{t^2-\rho^2}\big)-1\big)}{(t^2-\rho^2)^2\sqrt{t^2-\rho^2}}\Bigg\} \\ \times \rho^3\int_{S^3} g(x+\rho\omega)\,d\omega\,d\rho. \end{aligned}$$

When we expand the terms in ρ^2/t^2, there is the cancellation of the terms of order 0, and hence

$$\int_0^{t^{2/3}} = \int_0^{t^{2/3}} \frac{e^{-\frac{\rho^2}{4t}}}{(\sqrt{4\pi t})^4}\frac{1}{t}\Big(O\Big(\frac{\rho^2}{t}+\Big(\frac{\rho^2}{t}\Big)^2\Big)+\frac{1}{t}O(1)\Big)\rho^3\int_{S^3} g(x+\rho\omega)\,d\omega\,d\rho$$

and

$$\left\| \int_0^{t^{2/3}} \right\|_{L^p} \le Ct^{-2(\frac{1}{q}-\frac{1}{p})-1}\|g\|_{L^q}, \quad t \ge \sqrt{A}. \tag{3.37}$$

Thus, noting $\bar{J}_{14}(t)g \equiv 0\,(0 \le t \le \sqrt{A})$, we have (3.20) by (3.36)–(3.37). □

4. Proof of Lemmas 2.2–2.5.

By the explicit formula $S_N(t)g$, the solution $u(t,x)$ to (1.1) is expressed by the integral equation

$$\begin{aligned} u(t,x) = {} & [S_N(t)(u_0+u_1)](x) + [\partial_t(S_N(t)u_0)](x) \\ & - \int_0^t [S_N(t-\tau)|u|^{\rho-1}u(\tau,\cdot)](x)\,d\tau. \end{aligned} \tag{4.1}$$

In the preceding section we have expressed $S_N(t)g$ by

$$S_3(t)g = e^{-t/2}W_{03}(t)g + J_{03}(t)g \tag{3.1}$$

and

$$S_4(t)g = e^{-t/2}\bar{W}_{04}(t)g + \bar{J}_{04}(t)g, \tag{3.16}$$

and obtained the basic estimates in Lemmas 3.1–3.2.

We are now ready to prove the key Lemma 2.2. In $N=3$ we will use the expression (4.1) only, while both (4.1) and the weighted energy method to (2.21) will be applied in the case of $N=4$.

Proof of Lemma 2.2 ($N=3$). If $(u_0,u_1) \in (H^2 \cap W^{1,1}) \times (H^1 \cap L^1)$, then

$$\begin{aligned} \|S_3(t)(u_0+u_1)\|_{L^p} &\le C(1+t)^{-\frac{3}{2}(1-\frac{1}{p})}, \\ \|\partial_t(S_3(t)u_0)\|_{L^p} &\le C(1+t)^{-\frac{3}{2}(1-\frac{1}{p})-1}. \end{aligned} \tag{4.2}$$

by (3.3), (3.10), (3.12). In fact, $I_0 < \infty$ with (2.7) shows $(u_0,u_1) \in W^{1,1} \times L^1$. By (3.1) the inhomogeneous term in (4.1) is written as

$$\begin{aligned} & \int_0^t S_3(t-\tau)|u|^{\rho-1}u(\tau)\,d\tau \\ & = \int_0^t e^{-\frac{t-\tau}{2}}W_{03}(t-\tau)|u|^{\rho-1}u(\tau)\,d\tau + \int_0^t J_{03}(t-\tau)|u|^{\rho-1}u(\tau)\,d\tau \\ & =: w_3(t,\cdot) + h_3(t,\cdot). \end{aligned} \tag{4.3}$$

Using L^∞-L^1 and L^∞-$L^{3/2}$ estimates in (3.3) together with (2.10)–(2.11), we have

$$
\begin{aligned}
\|h_3(t)\|_{L^\infty} &\le C\int_0^{t/2}(1+t-\tau)^{-\frac{3}{2}}\|u(\tau)\|_{L^\rho}^\rho d\tau + C\int_{t/2}^t (1+t-\tau)^{-1}\|u(\tau)\|^\rho_{L^{3\rho/2}}d\tau \\
&\le \int_0^{t/2}(1+t-\tau)^{-\frac{3}{2}}(1+\tau)^{-\frac{\rho}{\rho-1}+\frac{3}{2}}d\tau + C\int_{t/2}^t(1+t-\tau)^{-1}(1+\tau)^{-(\frac{\rho}{\rho-1}-\frac{3}{3})}d\tau \\
&\le C(1+t)^{-\frac{1}{\rho-1}}\log(2+t)
\end{aligned}
$$

and hence

$$
\|h_3(t)\|_{L^\infty} \le C(1+t)^{-\frac{1}{\rho-1}+\mu}, \quad 0<\mu\ll 1. \tag{4.4}
$$

To estimate $w_3(t)$, by setting

$$
M_u(t) = \sup_{0\le\tau\le t}\left[(1+\tau)^{\frac{1}{\rho-1}-\mu}\|u(\tau)\|_{L^\infty}\right] \tag{4.5}
$$

we derive

$$
\|w_3(t)\|_{L^\infty} \le CM_u(t)^{\frac{3}{3+\varepsilon}}\cdot(1+t)^{-(\frac{1}{\rho-1}-\mu)} \tag{4.6}
$$

for a small $\varepsilon>0$. Once (4.6) is available, by (4.1)–(4.6),

$$
M_u(t)\le C+CM_u(t)^{\frac{3}{3+\varepsilon}}, \tag{4.7}
$$

and hence

$$
M_u(t)\le C_\varepsilon, \tag{4.8}
$$

because of $\frac{3}{3+\varepsilon}<1$. Hence the desired estimate (2.23) is proved. To prove (4.6), by (3.6) we estimate $w_3(t)$ as

$$
\|w_3(t)\|_{L^\infty} \le \int_0^t e^{-\frac{t-\tau}{2}}\left[(t-\tau)^{\frac{\varepsilon}{3+\varepsilon}}\|u(\tau)\|^\rho_{L^{(3+\varepsilon)\rho}} + \||u|^{2\rho-1}|\nabla u|(\tau)\|^{1/2}_{L^1}\right]d\tau. \tag{4.9}
$$

Here we have

$$
\begin{aligned}
\|u(\tau)\|^\rho_{L^{(3+\varepsilon)\rho}} &\le \|u(\tau)\|_{L^\infty}^{\frac{3}{3+\varepsilon}}\left(\int_{\boldsymbol{R}^3}|u(\tau,x)|^{(3+\varepsilon)\rho-3}dx\right)^{\frac{1}{3+\varepsilon}} \\
&\le C\left[(1+\tau)^{\frac{1}{\rho-1}-\mu}\|u(\tau)\|_{L^\infty}\right]^{\frac{3}{3+\varepsilon}} \\
&\quad \cdot(1+\tau)^{-(\frac{1}{\rho-1}-\mu)\cdot\frac{3}{3+\varepsilon}}(1+\tau)^{-(\frac{1}{\rho-1}-\frac{3}{2[(3+\varepsilon)\rho-3]})\cdot\frac{(3+\varepsilon)\rho-3}{3+\varepsilon}} \\
&\le M_u(t)^{\frac{3}{3+\varepsilon}}\cdot(1+\tau)^{-(\frac{1}{\rho-1}-\mu)}.
\end{aligned} \tag{4.10}
$$

In fact, to use Theorem 2.2, we need

$$(3+\varepsilon)\rho - 3 \le 6 = \frac{2N}{N-2} \quad \text{or} \quad \rho \le \frac{9}{3+\varepsilon},$$

which is satisfied by $\rho \le 1 + 4/N = 7/3$ if $0 < \varepsilon \ll 1$. Also,

$$\begin{aligned} \text{the exponent of } (1+\tau) = &-\left(\frac{1}{\rho-1} - \mu\right) + \left(\frac{1}{\rho-1} - \mu\right) \cdot \frac{\varepsilon}{3+\varepsilon} \\ &- \left(1 + \frac{\varepsilon}{(\rho-1)(3+\varepsilon)} - \frac{3}{2(3+\varepsilon)}\right) \\ &< -\left(\frac{1}{\rho-1} - \mu\right) \quad \text{if } 0 < \varepsilon \ll 1. \end{aligned}$$

Further,

$$\begin{aligned} \big\| |u|^{2\rho-1} |\nabla u|(\tau) \big\|_{L^1}^{1/2} &\le C \left(\int_{\boldsymbol{R}^3} |u(\tau,x)|^{2(2\rho-1)} dx \right)^{1/4} \|\nabla u(\tau)\|_{L^2}^{1/2} \\ &\le C \|u(\tau)\|_{L^\infty}^{\frac{3}{3+\varepsilon}} \left(\int_{\boldsymbol{R}^3} |u(\tau,x)|^{2(2\rho-1) - \frac{3\cdot 4}{3+\varepsilon}} dx \right)^{1/4} \|\nabla u(\tau)\|_{L^2}^{1/2} \\ &\le C M_u(t)^{\frac{3}{3+\varepsilon}} (1+\tau)^{-(\frac{1}{\rho-1} - \mu)\cdot\frac{3}{3+\varepsilon}} \\ &\quad \cdot (1+\tau)^{-(\frac{1}{\rho-1} - \frac{3}{2[(2(2\rho-1) - \frac{12}{3+\varepsilon}]})\cdot \frac{2(2\rho-1)-\frac{12}{3+\varepsilon}}{4} - (\frac{1}{\rho-1} + \frac{1}{2} - \frac{3}{2\cdot 2})\frac{1}{2}} \\ &\le C M_u(t)^{\frac{3}{3+\varepsilon}} \cdot (1+\tau)^{-(\frac{1}{\rho-1} - \mu)}. \end{aligned} \tag{4.11}$$

We here need

$$2(2\rho-1) - \frac{3\cdot 4}{3+\varepsilon} \le 6 \quad \text{if} \quad \rho \le 2 + \frac{3}{3+\varepsilon},$$

which is satisfied since $\rho \le 7/3$ if $0 < \varepsilon \ll 1$. Also,

$$\begin{aligned} \text{the exponent of } (1+\tau) &= -\left(\frac{1}{\rho-1} - \mu\right) + \left(\frac{1}{\rho-1} - \mu\right)\frac{\varepsilon}{3+\varepsilon} - \left(\frac{1}{2} + \frac{\varepsilon}{(\rho-1)(3+\varepsilon)}\right) \\ &< -\left(\frac{1}{\rho-1} - \mu\right) \quad \text{if } 0 < \varepsilon \ll 1. \end{aligned}$$

Applying (4.10)–(4.11) to (4.9), we have (4.6). □

Proof of Lemma 2.2 ($N = 4$). When $I_0, I_1 < \infty$, $(u_0, u_1) \in (H^3 \cap W^{2,1}) \times (H^2 \cap W^{1,1})$. Hence, by (3.18), (3.24), (3.25),

$$\|S_4(t)(u_0+u_1)\|_{L^p} \le C(1+t)^{-2(1-\frac{1}{p})},$$
$$\|\partial_t(S_4(t)u_0)\|_{L^p} \le C(1+t)^{-2(1-\frac{1}{p})-1}. \tag{4.12}$$

By use of (3.16), the inhomogeneous term is written as

$$\begin{aligned}
&\int_0^t S_4(t-\tau)|u|^{\rho-1}u(\tau)\,d\tau \\
&= \int_0^t e^{-\frac{t-\tau}{2}}\bar{W}_{04}(t-\tau)|u|^{\rho-1}u(\tau)\,d\tau + \int_0^t \bar{J}_{04}(t-\tau)|u|^{\rho-1}u(\tau)\,d\tau \\
&=: \bar{w}_4(t,\cdot)+\bar{h}_4(t,\cdot).
\end{aligned} \tag{4.13}$$

Since $\rho \in (1+\frac{2}{N}, 1+\frac{4}{N}) = (\frac{5}{3}, 2)$, choosing $s, \bar{s} > 2$ as $\rho s \le 4 = \frac{2N}{N-2}$ and $2 < \bar{s} < 4$, we have

$$\begin{aligned}
|\bar{w}_4(t)| \le C\int_0^t e^{-\frac{t-\tau}{2}}\Big[\big\{(t-\tau)^{1-\frac{4}{s}} + (t-\tau)^{2-\frac{2}{s}}\big\}\|u(\tau)\|_{L^{\rho s}}^{\rho} \\
+ (t-\tau)^{2-\frac{4}{\bar{s}}}\big\||u|^{\rho-1}\nabla u(\tau)\big\|_{L^{\bar{s}}}\Big]\,d\tau
\end{aligned} \tag{4.14}$$

by (3.21). Here, by setting

$$M_u(t) = \sup_{0\le\tau\le t}\big[(1+\tau)^{\frac{1}{\rho-1}-\mu}\|u(\tau)\|_{L^\infty}\big] \tag{4.15}$$

and

$$N_{\Delta u}(t) = \sup_{0\le\tau\le t}\big[(1+\tau)^{\frac{1}{\rho-1}-\mu}\|\Delta u(\tau)\|_{L^2}\big], \tag{4.16}$$

we have

$$\|u(\tau)\|_{L^{\rho s}}^{\rho} \le C(1+\tau)^{-(\frac{\rho}{\rho-1}-\frac{4}{2s})} \le C(1+\tau)^{-\frac{1}{\rho-1}} \tag{4.17}$$

and

$$\begin{aligned}
\big\||u|^{\rho-1}\nabla u(\tau)\big\|_{L^{\bar{s}}} &\le C\|u(\tau)\|_{L^\infty}^{\rho-1}\|\nabla u(\tau)\|_{L^2}^{1-\bar{\sigma}}\|\Delta u(\tau)\|_{L^2}^{\bar{\sigma}} \\
&\le CM_u(t)^{\rho-1}N_{\Delta u}(t)^{\bar{\sigma}}\cdot(1+\tau)^{-(\frac{1}{\rho-1}-\mu)(\rho-1+\bar{\sigma})-(\frac{1}{\rho-1}+\frac{1}{2}-\frac{4}{2\cdot 2})(1-\bar{\sigma})} \\
&\le CM_u(t)^{\rho-1}N_{\Delta u}(t)^{\bar{\sigma}}\cdot(1+\tau)^{-\frac{1}{\rho-1}+\mu}
\end{aligned} \tag{4.18}$$

by the Gagliardo-Nirenberg inequality with $\bar{\sigma} = 4(\frac{1}{2}-\frac{1}{\bar{s}})$, $0 < \bar{\sigma} < 1$. Hence (4.14)–(4.18) imply

$$\|\bar{w}_4(t)\|_{L^\infty} \le C(1+t)^{-\frac{1}{\rho-1}} + CM_u(t)^{\rho-1}N_{\Delta u}(t)^{\bar{\sigma}} \cdot (1+t)^{-\frac{1}{\rho-1}+\mu}. \tag{4.19}$$

The L^∞-L^1 and L^∞-L^2 estimates in (3.18) yield

$$\begin{aligned}\|\bar{h}_4(t)\|_{L^\infty} &\le C\int_0^{t/2}(1+t-\tau)^{-2}\|u(\tau)\|_{L^\rho}^\rho d\tau + C\int_{t/2}^t (1+t-\tau)^{-2\cdot\frac{1}{2}}\|u(\tau)\|_{L^{2\rho}}^\rho d\tau \\ &\le C\int_0^{t/2}(1+t-\tau)^{-2}(1+\tau)^{-(\frac{\rho}{\rho-1}-\frac{4}{2})}d\tau \\ &\quad + C\int_{t/2}^t (1+t-\tau)^{-1}(1+\tau)^{-(\frac{\rho}{\rho-1}-\frac{4}{2\cdot 2})}d\tau \\ &\le C(1+t)^{-\frac{1}{\rho-1}+\mu}. \end{aligned} \tag{4.20}$$

Combining (4.12) with $p=\infty$ with (4.19)–(4.20), we have

$$\|u(t)\|_{L^\infty} \le C(1+t)^{-\frac{1}{\rho-1}+\mu} + CM_u(t)^{\rho-1}N_{\Delta u}(t)^{\bar{\sigma}} \cdot (1+t)^{-\frac{1}{\rho-1}+\mu}$$

and hence

$$M_u(t) \le C + CN_{\Delta u}(t)^{\bar{\sigma}} \cdot M_u(t)^{\rho-1} \tag{4.21}$$

We here prepare the following lemma.

Lemma 4.1. *Let the constants α, B, D satisfy $0<\alpha<1, B\le 1, D>0$, respectively. If the inequality*

$$x \le D + Bx^\alpha, \quad x \ge 0$$

holds, then

$$x \le [(D+1)B]^{\frac{1}{1-\alpha}}. \tag{4.22}$$

Proof. Put

$$f(x) = D + Bx^\alpha - x,$$

then

$$f'(x) = \alpha Bx^{\alpha-1} - 1 = \frac{\alpha B}{x^{1-\alpha}} - 1.$$

Hence $f(x)$ has the maximal value at $x = (\alpha B)^{1/(1-\alpha)}$, and $f(x_0)=0$ for a unique value $x_0 > (\alpha B)^{1/(1-\alpha)}$. Since

$$
\begin{aligned}
f\big([(D+1)B]^{\frac{1}{1-\alpha}}\big) &= D + [(D+1)B]^{\frac{\alpha}{1-\alpha}}(B-(D+1)B) \\
&= D\big(1 - B[(D+1)B]^{\frac{\alpha}{1-\alpha}}\big) < 0,
\end{aligned}
$$

$x_0 < [(D+1)B]^{\frac{1}{1-\alpha}}$. Hence $f(x) \geq 0$ means $x \leq x_0$ and (4.22). □

Without loss of generality, $CN_{\Delta u}(t)^{\bar{\sigma}} \geq 1$, and hence (4.21) together with Lemma 4.1 implies

$$
M_u(t) \leq \big[(C+1)CN_{\Delta u}(t)^{\bar{\sigma}}\big]^{\frac{1}{1-(\rho-1)}} \leq CN_{\Delta u}(t)^{\frac{\bar{\sigma}}{2-\rho}}. \tag{4.23}
$$

Note that

$$
\bar{\sigma} = 4\left(\frac{1}{2} - \frac{1}{\bar{s}}\right), \quad 2 < \bar{s} < 4
$$

and

$$
\bar{\sigma} \to 0+ \quad \text{as} \quad \bar{s} \to 2+0. \tag{4.24}
$$

Under the conditions (4.23)–(4.24), we return to

$$
(\nabla u)_{tt} - \Delta(\nabla u) + (\nabla u)_t + \rho|u|^{\rho-1}\nabla u = 0 \tag{2.21}
$$

and apply the weighted energy method to it. Then we have the following estimate on $N_{\Delta u}(t)$.

Lemma 4.2. *Let $N = 4$ with (2.25). If $(u_0, u_1) \in H^3 \times H^2$ satisfy (2.6) and (2.24), then it holds that*

$$
N_{\Delta u}(t) \leq C(I_0, I_1)\big(1 + N_{\Delta u}(t)^{\frac{\rho-1}{2-\rho}\bar{\sigma}}\big). \tag{4.25}
$$

By (4.24) we choose $\bar{\sigma}$ to be small as

$$
\frac{\rho-1}{2-\rho}\bar{\sigma} < 1,
$$

so that

$$
N_{\Delta u}(t) \leq C(I_0, I_1) \ \text{ and } \ M_u(t) \leq C(I_0, I_1). \tag{4.26}
$$

which shows Lemma 2.2 for $N = 4$. □

It is now necessary to prove Lemma 4.2, which is subsequent to the proof of Theorem 2.1. So, we give a short sketch of that in [**11**] before proving Lemma 4.2.

Choose the weight as

$$e^{\psi(t,x)}, \quad \psi(t,x) = \frac{m}{2}\log\left(1+\frac{a|x|^2}{t+t_0}\right) \tag{4.27}$$

with m in (2.7) and the parameters $0 < a \ll 1$, $t_0 \gg 1$. Multiplying (1.1) by $e^{2\psi}u_t$ and $e^{2\psi}u$, we have

$$\begin{aligned}
&\frac{\partial}{\partial t}\left[e^{2\psi}\left(\frac{1}{2}(|u_t|^2+|\nabla u|^2)+\frac{1}{\rho+1}|u|^{\rho+1}\right)\right]\\
&\qquad + e^{2\psi}\left\{\left(1-\frac{|\nabla\psi|^2}{-\psi_t}-\psi_t\right)|u_t|^2+\frac{-2\psi_t}{\rho+1}|u|^{\rho+1}\right\}\\
&\qquad - \nabla\cdot(e^{2\psi}u_t\nabla u)+\frac{e^{2\psi}}{-\psi_t}|\psi_t\nabla u-u_t\nabla\psi|^2\\
&= 0
\end{aligned} \tag{4.28}$$

and

$$\begin{aligned}
&\frac{\partial}{\partial t}\left[e^{2\psi}\left(uu_t+\frac{u^2}{2}\right)\right]+e^{2\psi}\left(|\nabla u|^2-\psi_t u^2+|u|^{\rho+1}\right)\\
&\qquad + e^{2\psi}(-2\psi_t uu_t-|u_t|^2+2u\nabla\psi\cdot\nabla u)-\nabla\cdot(e^{2\psi}u\nabla u)\\
&= 0.
\end{aligned} \tag{4.29}$$

Note that ψ satisfies

$$\begin{aligned}
&0 < -\psi_t < \frac{m}{2}\frac{1}{t+t_0}, \quad \frac{|\nabla\psi|^2}{-\psi_t} = \frac{2am}{1+\frac{a|x|^2}{t+t_0}} \le 2am, \quad \text{and,}\\
&\text{for} \quad \frac{\sqrt{a}|x|}{\sqrt{t+t_0}} \ge K, \quad -\psi_t \ge \frac{m}{2(t+t_0)}\frac{K}{1+K} \to \frac{m}{2(t+t_0)} \ (K\to\infty).
\end{aligned} \tag{4.30}$$

Hence, integrating (4.28)+$\nu\cdot$(4.29), $0<\nu\ll 1$, over $\boldsymbol{R}^N$, we get

$$\begin{aligned}
&\frac{d}{dt}\hat{E}_\psi(t;u)+\hat{H}_\psi(t;u)\\
&:= \frac{d}{dt}\int_{\boldsymbol{R}^N} e^{2\psi}\left(\frac{|u_t|^2}{2}+\nu uu_t+\frac{\nu}{2}u^2+\frac{|\nabla u|^2}{2}+\frac{|u|^{\rho+1}}{\rho+1}\right)dx\\
&\quad + \int_{\boldsymbol{R}^N} e^{2\psi}\left\{\left(1-\frac{|\nabla\psi|^2}{-\psi_t}-\psi_t-\nu\right)|u_t|^2-2\nu\psi_t uu_t+2\nu u\nabla\psi\cdot\nabla u\right.\\
&\qquad\qquad \left. -\nu\psi_t u^2+\nu|\nabla u|^2+\left(\frac{-\psi_t}{\rho+1}+\nu\right)|u|^{\rho+1}\right\}dx\\
&\le 0.
\end{aligned} \tag{4.31}$$

Multiplying (4.31) by $(t+t_0)^{2\alpha(\rho)+\varepsilon}$, $|\varepsilon|\ll 1$, we further have

$$\frac{d}{dt}\big[(t+t_0)^{2\alpha(\rho)+\varepsilon}\hat{E}_\psi(t;u)\big]+(t+t_0)^{2\alpha(\rho)+\varepsilon}\left[\hat{H}_\psi(t;u)-\frac{2\alpha(\rho)+\varepsilon}{t+t_0}\hat{E}_\psi(t;u)\right]\le 0, \quad (4.32)$$

where

$$\alpha(\rho)=\frac{1}{\rho-1}-\frac{N}{4}. \quad (4.33)$$

Again, noting (4.30), we have the crucial estimate

$$\begin{aligned}(t+t_0)^{2\alpha(\rho)+\varepsilon}&\left(\hat{H}_\psi(t;u)-\frac{2\alpha(\rho)+\varepsilon}{t+t_0}\hat{E}_\psi(t;u)\right)\\&\ge c(t+t_0)^{2\alpha(\rho)+\varepsilon}H_\psi(t;u)-C(t+t_0)^{-1+\varepsilon} \quad (4.34)\end{aligned}$$

with

$$\begin{aligned}&\hat{H}_\psi(t;u)\ge cH_\psi(t;u),\quad H_\psi(t;u)=\int_{\boldsymbol{R}^N}e^{2\psi}(|u_t|^2+|\nabla u|^2+|u|^{\rho+1})(t,x)\,dx,\\&CE_\psi(t;u)\ge\hat{E}_\psi(t;u)\ge cE_\psi(t;u),\\&E_\psi(t;u)=\int_{\boldsymbol{R}^N}e^{2\psi}\big(|u_t|^2+|\nabla u|^2+u^2+|u|^{\rho+1}\big)(t,x)\,dx. \quad (4.35)\end{aligned}$$

Thus, we obtain

$$\begin{aligned}&(t+t_0)^{2\alpha(\rho)+\varepsilon}\big(\|u_t(t)\|^2_{L^{2,m}}+\|\nabla u(t)\|^2_{L^{2,m}}+\|u(t)\|^2_{L^{2,m}}+\|u(t)\|^{\rho+1}_{L^{\rho+1,m}}\big)\\&\quad+\int_0^t(\tau+t_0)^{2\alpha(\rho)+\varepsilon}\big(\|u_t(\tau)\|^2_{L^{2,m}}+\|\nabla u(\tau)\|^2_{L^{2,m}}+\|u(\tau)\|^{\rho+1}_{L^{\rho+1,m}}\big)\,d\tau\\&\le C(I_0)+C\int_0^t(\tau+t_0)^{-1+\varepsilon}d\tau. \quad (4.36)\end{aligned}$$

We can now multiply $\int_{\boldsymbol{R}^4}(4.28)\,dx$ by $(t+t_0)^{2\alpha(\rho)+1+\varepsilon}$ using (4.36), so that (2.2)–(2.4) hold for $0<\varepsilon\ll 1$ and (2.5) does for $|\varepsilon|\ll 1$, by re-taking $t_0=1$ and changing the constants.

PROOF OF LEMMA 4.2. Same as the above, operate $e^{2\psi}(\nabla u)_t$, $e^{2\psi}\nabla u$ to (2.21) to have

$$\begin{aligned}&\frac{\partial}{\partial t}\left[\frac{e^{2\psi}}{2}\big(|\nabla u_t|^2+|\Delta u|^2\big)\right]+e^{2\psi}\left[\left(1-\frac{|\nabla\psi|^2}{-\psi_t}-\psi_t\right)|\nabla u_t|^2+\rho e^{2\psi}|u|^{\rho-1}\nabla u\cdot\nabla u_t\right]\\&\quad-\nabla\cdot\big(e^{2\psi}(\Delta u)\nabla u_t\big)+\frac{e^{2\psi}}{-\psi_t}|\psi_t\Delta u-\nabla u_t\cdot\nabla\psi|^2=0 \quad (4.37)\end{aligned}$$

and

$$\frac{\partial}{\partial t}\left[e^{2\psi}\left(\nabla u\cdot\nabla u_t+\frac{|\nabla u|^2}{2}\right)\right]+e^{2\psi}\big(|\Delta u|^2-\psi_t|\nabla u|^2+\rho|u|^{\rho-1}|\nabla u|^2\big)$$
$$+e^{2\psi}\big(-2\psi_t\nabla u\cdot\nabla u_t-|\nabla u_t|^2+2(\nabla u\cdot\nabla\psi)\Delta u\big)-\nabla\cdot\big(e^{2\psi}(\Delta u)\nabla u\big)=0, \quad (4.38)$$

corresponding to (4.28) and (4.29). Noting the estimate on the term in (4.37)

$$\left|-\int_{\boldsymbol{R}^4}\rho e^{2\psi}|u|^{\rho-1}\nabla u\cdot\nabla u_t dx\right|\le\nu\int_{\boldsymbol{R}^4}e^{2\psi}|\nabla u_t|^2dx+C_\nu\|u(t)\|_{L^\infty}^{2(\rho-1)}\int_{\boldsymbol{R}^4}e^{2\psi}|\nabla u|^2dx \quad (4.39)$$

for $0<\nu\ll 1$, we calculate $\int_{\boldsymbol{R}^4}[(4.37)+\nu\cdot(4.38)]\,dx$:

$$\begin{aligned}
&\frac{d}{dt}\hat F_\psi(t;\nabla u)+\hat K_\psi(t;\nabla u)\\
&:=\frac{d}{dt}\int_{\boldsymbol{R}^4}e^{2\psi}\left(\frac{|\nabla u_t|^2}{2}+\nu\nabla u\cdot\nabla u_t+\frac{\nu}{2}|\nabla u|^2+\frac{|\Delta u|^2}{2}\right)dx\\
&\quad\int_{\boldsymbol{R}^4}e^{2\psi}\bigg[\left(1-\frac{|\nabla\psi|^2}{-\psi_t}-\psi_t-2\nu\right)|\nabla u_t|^2-2\nu\psi_t\nabla u\cdot\nabla u_t\\
&\qquad+2\nu(\nabla u\cdot\nabla\psi)\Delta u-\nu\psi_t|\nabla u|^2+\nu|\Delta u|^2+\nu\rho|u|^{\rho-1}|\nabla u|^2\bigg]dx\\
&\le C_\nu\|u(t)\|_{L^\infty}^{2(\rho-1)}\int_{\boldsymbol{R}^4}e^{2\psi}|\nabla u|^2dx.
\end{aligned} \quad (4.40)$$

Like (4.35), for a fixed $\nu(0<\nu\ll 1)$ we have

$$cF_\psi(t;\nabla u)\le\hat F_\psi(t;\nabla u)\le CF_\psi(t;\nabla u) \quad (4.41)$$

with

$$F_\psi(t;\nabla u):=\int_{\boldsymbol{R}^4}e^{2\psi}\big(|\nabla u_t|^2+|\Delta u|^2+|\nabla u|^2\big)(t,x)\,dx, \quad (4.42)$$

and

$$\hat K_\psi(t;\nabla u)\ge c\int_{\boldsymbol{R}^4}e^{2\psi}\big(|\nabla u_t|^2+|\Delta u|^2+(-\psi_t)|\nabla u|^2+|u|^{\rho-1}|\nabla u|^2\big)(t,x)\,dx. \quad (4.43)$$

Therefore, multiplying (4.40) by $(t+t_0)^{2\alpha(\rho)+k-\varepsilon}(\varepsilon>0,\ t_0\gg 1)$ and noting that

$$\begin{aligned}
&\hat K_\psi(t;\nabla u)-\frac{2\alpha(\rho)+k-\varepsilon}{t+t_0}\hat F_\psi(t;\nabla u)\\
&\quad\ge c\int_{\boldsymbol{R}^4}e^{2\psi}\big(|\nabla u_t|^2+|\Delta u|^2\big)\,dx-\frac{C}{t+t_0}\int_{\boldsymbol{R}^4}e^{2\psi}|\nabla u|^2dx,
\end{aligned} \quad (4.44)$$

we get

$$(t+t_0)^{2\alpha(\rho)+k-\varepsilon}F_\psi(t;\nabla u)+\int_0^t(\tau+t_0)^{2\alpha(\rho)+k-\varepsilon}\big(\|\nabla u_t(\tau)\|^2_{L^{2,m}}+\|\Delta u(\tau)\|^2_{L^{2,m}}\big)\,d\tau$$
$$\le CI_1^2+C\int_0^t(\tau+t_0)^{2\alpha(\rho)+k-1-\varepsilon}\|\nabla u(\tau)\|^2_{L^{2,m}}d\tau$$
$$+C\int_0^t(\tau+t_0)^{2\alpha(\rho)+k-\varepsilon}\|u(\tau)\|^{2(\rho-1)}_{L^\infty}\|\nabla u(\tau)\|^2_{L^{2,m}}d\tau. \tag{4.45}$$

Taking $t_0=1$ again and changing the constants C in (4.45) and using

$$\|u(\tau)\|^{2(\rho-1)}_{L^\infty}\le CM_u(t)^{2(\rho-1)}\cdot(1+\tau)^{-(\frac{1}{\rho-1}-\mu)\cdot 2(\rho-1)}$$
$$\le CN_{\Delta u}(t)^{\frac{2(\rho-1)}{2-\rho}\bar\sigma}\cdot(1+\tau)^{-2+2(\rho-1)\mu} \tag{4.46}$$

by (4.15) and (4.23), we obtain

$$(1+t)^{2\alpha(\rho)+k-\varepsilon}F_\psi(t;\nabla u)+\int_0^t(1+\tau)^{2\alpha(\rho)+k-\varepsilon}\big(\|\nabla u_t(\tau)\|^2_{L^{2,m}}+\|\Delta u(\tau)\|^2_{L^{2,m}}\big)\,d\tau$$
$$\le CI_1^2+C\int_0^t(1+\tau)^{2\alpha(\rho)+k-1-\varepsilon}\|\nabla u(\tau)\|^2_{L^{2,m}}d\tau$$
$$+CN_{\Delta u}(t)^{\frac{2(\rho-1)}{2-\rho}\bar\sigma}\int_0^t(1+\tau)^{2\alpha(\rho)+k-2-\varepsilon+2(\rho-1)\mu}\|\nabla u(\tau)\|^2_{L^{2,m}}d\tau. \tag{4.47}$$

Since the integrals in the right hand side of (4.47) with $-\varepsilon<0$ are finite by (2.5) or (4.36) if $k=0,1$, (4.47) is estimated as

$$(1+t)^{2\alpha(\rho)+k-\varepsilon}F_\psi(t;\nabla u)+\int_0^t(1+\tau)^{2\alpha(\rho)+k-\varepsilon}\big(\|\nabla u_t(\tau)\|^2_{L^{2,m}}+\|\Delta u(\tau)\|^2_{L^{2,m}}\big)\,d\tau$$
$$\le C(I_0,I_1)\big(1+N_{\Delta u}(t)^{\frac{2(\rho-1)}{2-\rho}\bar\sigma}\big). \tag{4.48}$$

Return to (4.37) again and multiply $\int_{\boldsymbol{R}^4}(4.37)\,dx$ by $(1+t)^{2\alpha(\rho)+2-\varepsilon}$ to obtain

$$(1+t)^{2\alpha(\rho)+2-\varepsilon}\big(\|\nabla u_t(t)\|^2_{L^{2,m}}+\|\Delta u(t)\|^2_{L^{2,m}}\big)+\int_0^t(1+\tau)^{2\alpha(\rho)+2-\varepsilon}\|\nabla u_t(\tau)\|^2_{L^{2,m}}d\tau$$
$$\le CI_1^2+C\int_0^t(1+\tau)^{2\alpha(\rho)+1-\varepsilon}\big(\|\nabla u_t(\tau)\|^2_{L^{2,m}}+\|\Delta u(\tau)\|^2_{L^{2,m}}\big)\,d\tau$$
$$+CN_{\Delta u}(t)^{\frac{2(\rho-1)}{2-\rho}\bar\sigma}\int_0^t(1+\tau)^{2\alpha(\rho)+2-\varepsilon-2+2(\rho-1)\mu}\|\nabla u(\tau)\|^2_{L^{2,m}}d\tau$$

and hence, by taking $0 < \varepsilon = 2\mu \ll 1$,

$$\begin{aligned}(1+t)^{2\alpha(\rho)+2-2\mu}\big(\|\nabla u_t(t)\|^2_{L^{2,m}} + \|\Delta u(t)\|^2_{L^{2,m}}\big)& \\ + \int_0^t (1+\tau)^{2\alpha(\rho)+2-2\mu}\|\nabla u_t(\tau)\|^2_{L^{2,m}}\,d\tau& \\ \le C(I_0, I_1)\big(1 + N_{\Delta u}(t)^{\frac{2(\rho-1)}{2-\rho}\bar{\sigma}}\big).&\end{aligned} \tag{4.49}$$

Since $\alpha(\rho) + 1 = 1/(\rho - 1)$, the definition (4.16) of $N_{\Delta u}(t)$ and (4.49) yield (4.25). □

REMARK 4.1. Our method used here does not seem to be applicable for the case of $N = 5$. Because, since Lemmas 3.1–3.2 are basic to the key Lemma 2.2, the similar estimates are necesarry. However, some are estimated by $\|\Delta g\|_{L^q}$, not $\|\nabla g\|_{L^q}$. When we apply them to the nonlinear problem, g corresponds to $-|u|^{\rho-1}u$, which is not in C^2 for $\rho_c(N) < \rho < 1 + 4/N\,(N = 5)$. In fact, the solution to (2.17) on $\boldsymbol{R}^5$ is given by

$$S_5(t)g = \frac{e^{-t/2}}{8\pi^2}\left(\frac{1}{t^2}\partial_t^2 - \frac{1}{t^3}\partial_t\right)\int_0^t I_0\left(\frac{1}{2}\sqrt{t^2-\rho^2}\right)\rho^4\int_{S^4} g(x+\rho\omega)\,d\omega\,d\rho \tag{4.50}$$

(see Courant and Hilbert [**1**], Ono [**21**]). Calculating $S_5(t)g$, we set it as

$$\begin{aligned}S_5(t)g &= e^{-t/2}\cdot\frac{1}{8\pi^2}\left\{\left(3t + \frac{1}{8}t^3\right)\int_{S^4} g(x+\rho\omega)\,d\omega + t^2\int_{S^4}\nabla g(x+\rho\omega)\cdot\omega\,d\omega\right\}\\ &\quad + \frac{e^{-t/2}}{8\pi^2}\int_0^t\left(\frac{1}{t^2}\partial_t^2 - \frac{1}{t^3}\partial_t\right)\left[I_0\left(\frac{1}{2}\sqrt{t^2-\rho^2}\right)\right]\rho^4\int_{S^4} g(x+\rho\omega)\,d\omega\,d\rho\\ &=: e^{-t/2}W_{05}(t)g + J_{05}(t)g\end{aligned} \tag{4.51}$$

(Note that the solution to the wave equation without damping is given by

$$\frac{1}{8\pi^2}\left(3t\int_{S^4} g(x+\rho\omega)\,d\omega + t^2\int_{S^4}\nabla g(x+\rho\omega)\cdot\omega\,d\omega\right)$$

for $N = 5$). The second term of $W_{05}(t)g$ is estimated as

$$\begin{aligned}&\left|\frac{t^2}{8\pi^2}\int_{S^4}\nabla g(x+t\omega)\cdot\omega\,d\omega\right|\\ &\quad\le C\left(\int_{S^4} t^4|\nabla g(x+t\omega)|^2 d\omega\right)^{1/2}\\ &\quad\le C\left(\int_0^t \rho^3\int_{S^4}|\nabla g(x+\rho\omega)|^2 d\omega\,d\rho + \int_0^t \rho^4\int_{S^4}|\nabla g|\Delta g(x+\rho\omega)\,d\omega\,d\rho\right)^{1/2}\\ &\quad\le C\big(t^{\frac{1-5\varepsilon}{8}}\|\nabla g\|_{L^{\frac{8}{3+\varepsilon}}} + \||\nabla g||\Delta g|\|_{L^1}\big)^{1/2}\end{aligned} \tag{4.52}$$

for $0 < \varepsilon \ll 1$. Hence we need the second derivative of g.

Sketch of the proof of Lemma 2.3. The proof is completely same as in [**11**] and we give the sketch. For the equation of u_t

$$(u_t)_{tt} - \Delta(u_t) + (u_t)_t + \rho|u|^{\rho-1}u_t = 0, \tag{2.16}$$

instead of (2.21) for ∇u, the estimate corresponding to (4.45) can be obtained:

$$\begin{aligned}
&(1+t)^{2\alpha(\rho)+k-\varepsilon}F_\psi(t;u_t) + \int_0^t (1+\tau)^{2\alpha(\rho)+k-\varepsilon}\left(\|u_{tt}(\tau)\|^2_{L^{2,m}} + \|\nabla u_t(\tau)\|^2_{L^{2,m}}\right)d\tau \\
&\quad \le C(I_0^2 + I_1^2) + C\int_0^t (1+\tau)^{2\alpha(\rho)+k-1-\varepsilon}\|u_t(\tau)\|^2_{L^{2,m}}d\tau \\
&\qquad + C\int_0^t (1+\tau)^{2\alpha(\rho)+k-\varepsilon}\|u(\tau)\|^{2(\rho-1)}_{L^\infty}\|u_t(\tau)\|^2_{L^{2,m}}d\tau
\end{aligned} \tag{4.53}$$

(t_0 is taken to be 1), where F_ψ is defined in (4.42). We already have (2.23) and hence the last term of (4.53) is estimated by

$$C\int_0^t (1+\tau)^{2\alpha(\rho)+k-2-\varepsilon+2(\rho-1)\mu}\|u_t(\tau)\|^2_{L^{2,m}}d\tau. \tag{4.54}$$

Hence we can take $k = 2$ by (2.5) and

$$\begin{aligned}
&(1+t)^{2\alpha(\rho)+2-\varepsilon}F_\psi(t;u_t) + \int_0^t (1+\tau)^{2\alpha(\rho)+2-\varepsilon}\left(\|u_{tt}(\tau)\|^2_{L^{2,m}} + \|\nabla u_t(\tau)\|^2_{L^{2,m}}\right)d\tau \\
&\quad \le C(I_0, I_1).
\end{aligned} \tag{4.55}$$

Therefore, we can multiply the equation corresponding to $\int_{\boldsymbol{R}^4}(4.37)\,dx$ by $(1+t)^{2\alpha(\rho)+3-\varepsilon}$ and use (4.55) with $\mu = \varepsilon/4(\rho-1)$ to get

$$\begin{aligned}
&(1+t)^{2\alpha(\rho)+3-\varepsilon}\left(\|u_{tt}(\tau)\|^2_{L^{2,m}} + \|\nabla u_t(t)\|^2_{L^{2,m}}\right) + \int_0^t (1+\tau)^{2\alpha(\rho)+3-\varepsilon}\|u_{tt}(\tau)\|^2_{L^{2,m}}d\tau \\
&\quad \le C(I_0, I_1) + C\int_0^t (1+\tau)^{2\alpha(\rho)+2-\varepsilon}\left(\|u_{tt}(\tau)\|^2_{L^{2,m}} + \|\nabla u_t(\tau)\|^2_{L^{2,m}}\right)d\tau \\
&\qquad + C\int_0^t (1+\tau)^{2\alpha(\rho)+1-\varepsilon/2}\|u_t(\tau)\|^2_{L^{2,m}}d\tau \\
&\quad \le C(I_0, I_1),
\end{aligned} \tag{4.56}$$

which shows (2.26). □

Proof of Lemma 2.5. When $N = 4$, using (3.18), (3.25), we apply L^∞-L^1 and

L^∞-L^2 estimates to (2.20) and obtain

$$\begin{aligned}\|u(t)\|_{L^\infty} &\le C(1+t)^{-2} + C\int_0^{t/2}(1+t-\tau)^{-2}\|u(\tau)\|_{L^\infty}^{\rho-1}\|u(\tau)\|_{L^1}d\tau \\ &\quad + C\int_{t/2}^{t}(1+t-\tau)^{-2(1-\frac{1}{2})}\|u(\tau)\|_{L^\infty}^{\frac{2\rho-1}{2}}\|u(\tau)\|_{L^1}^{\frac{1}{2}}d\tau \\ &\le C\Big\{(1+t)^{-2} + \int_0^{t/2}(1+t-\tau)^{-2}(1+\tau)^{-1+\mu(\rho-1)}d\tau \\ &\quad + \int_{t/2}^{t}(1+t-\tau)^{-1}(1+\tau)^{-\frac{2\rho-1}{2(\rho-1)}+\frac{2\rho-1}{2}\mu}d\tau\Big\} \\ &\le C(1+t)^{-\frac{1}{\rho-1}\frac{2\rho-1}{2}+(\frac{2\rho-1}{2}+1)\mu}, \end{aligned} \tag{4.57}$$

because

$$1 < \frac{2\rho-1}{2(\rho-1)} < 2 \ \text{ if } \ \rho > \frac{3}{2} = 1 + \frac{2}{N}.$$

The L^∞-estimate (4.57) is not yet optimal. So, applying the estimate (4.57) to (2.20) again, we have

$$\begin{aligned}\|u(t)\|_{L^\infty} &\le C(1+t)^{-2} + C\int_0^{t/2}(1+t-\tau)^{-2}(1+\tau)^{-\frac{2\rho-1}{2}+(\rho-1)(\frac{2\rho-1}{2}+1)\mu}d\tau \\ &\quad + C\int_{t/2}^{t}(1+t-\tau)^{-1}(1+\tau)^{-\frac{1}{\rho-1}(\frac{2\rho-1}{2})^2+(\frac{2\rho-1}{2}+1)\frac{2\rho-1}{2}\mu}d\tau.\end{aligned}$$

Since $\frac{2\rho-1}{2} > 1\,(\rho > \frac{3}{2})$,

$$\|u(t)\|_{L^\infty} \le C(1+t)^{-2} + C(1+t)^{-\frac{1}{\rho-1}(\frac{2\rho-1}{2})^2+((\frac{2\rho-1}{2})^2+\frac{2\rho-1}{2}+1)\mu}.$$

Repeating this procedure yields

$$\|u(t)\|_{L^\infty} \le C(1+t)^{-2} + C(1+t)^{-\frac{1}{\rho-1}(\frac{2\rho-1}{2})^k+((\frac{2\rho-1}{2})^k+\cdots+1)\mu} \tag{4.58}$$

and the choice of suitably large k such as $\frac{1}{\rho-1}(\frac{2\rho-1}{2})^k > 2$ and $0 < \mu \ll 1$ does

$$\|u(t)\|_{L^\infty} \le C(1+t)^{-2}. \tag{4.59}$$

The desired estimate (2.29) easily follows from (2.28) and (4.59) when $N = 4$.

The case $N = 3$ was proved in [**11**] by taking the L^∞-L^1 and L^∞-$L^{3/2}$ estimates. The detail is omitted here. □

5. Completion of the proof of main theorem.

It is shown following the story in Karch [**12**] that $\theta_0 G(t,x)$ is the asymptotic profile of $u(t,x)$ as $t \to \infty$, where θ_0 is defined by (1.11).

First, we write the difference by

$$\begin{aligned} u(t,\cdot) - \theta_0 G(t,\cdot) &= (J_{0N}(t) - P_N(t))(u_0+u_1) + (P_N(t)(u_0+u_1) - \theta_1 G(t,\cdot)) \\ &\quad + e^{-t/2} W_{0N}(t)(u_0+u_1) + \partial_t(S_N(t)u_0) - w_N(t) \\ &\quad + \int_0^{t/2} (J_{0N} - P_N)(t-\tau) f(\tau,\cdot)\, d\tau \\ &\quad + \int_0^{t/2} \left(P_N(t-\tau) - G(t,\cdot) \int_{\boldsymbol{R}^N} f(\tau,y)\, dy \right) d\tau \\ &\quad + \int_{t/2}^t J_{0N}(t-\tau) f(\tau,\cdot)\, d\tau + G(t,\cdot) \int_{t/2}^\infty \int_{\boldsymbol{R}^N} f(\tau,y)\, dy\, d\tau \\ &=: F_1 + F_2 + \cdots + F_9, \end{aligned} \tag{5.1}$$

where $f(t,x) = -|u|^{\rho-1} u(t,x)$ and

$$\begin{aligned} \theta_0 &= \int_{\boldsymbol{R}^N} (u_0+u_1)(x)\, dx + \int_0^\infty \int_{\boldsymbol{R}^N} f(\tau,y)\, dy\, d\tau \\ &=: \theta_1 + \left(\int_0^{t/2} + \int_{t/2}^\infty \right) \int_{\boldsymbol{R}^N} f(\tau,y)\, dy\, d\tau. \end{aligned} \tag{5.2}$$

Also, J_{04}, W_{04} are changed to $\bar{J}_{04}, \bar{W}_{04}$ in (5.1). The L^p norms of first, third, fourth and sixth terms in (5.2) are $o(t^{-2(1-\frac{1}{p})})$ by Lemmas 3.1–3.2. The second term is well-known to be $o(t^{-2(1-\frac{1}{p})})$. Since

$$\int_{\boldsymbol{R}^N} |f(\tau,y)|\, dy = \|u(\tau)\|_{L^\rho}^\rho \le C(1+\tau)^{-\frac{N}{2}(\rho-1)}$$

is integrable, the final term is also $o(t^{-2(1-\frac{1}{p})})$. For the fifth term $-w_N(t)$, when $N=3$, by (3.1)

$$\|W_{03}(t)g\|_{L^\infty} \le Ct\|g\|_{L^\infty}, \quad \|W_{03}(t)g\|_{L^1} \le Ct\|g\|_{L^1},$$

and, by (4.3)

$$\|w_3(t)\|_{L^\infty} \le C \int_0^t e^{-\frac{t-\tau}{2}} (t-\tau) \||u|^\rho(\tau)\|_{L^\infty} d\tau = o(t^{-\frac{3}{2}})$$

and

$$\|w_3(t)\|_{L^1} \le C\int_0^t e^{-\frac{t-\tau}{2}}(t-\tau)\|u(\tau)\|_{L^\rho}^\rho d\tau = o(1),$$

which show $\|w_3(t)\|_{L^p} = o(t^{-2(1-\frac{1}{p})})$. When $N=4$, (4.41) and (3.21)–(3.22) with $s=\infty, \bar{s}=4$ and $q=1$ yield

$$\begin{aligned}\|\bar{w}_4(t)\|_{L^\infty} &\le C\int_0^t e^{-\frac{t-\tau}{2}}(t-\tau)\big(\|u(\tau)\|_{L^\infty}^\rho + \|u(\tau)\|_{L^\infty}^{\rho-1}\|\Delta u(\tau)\|_{L^2}\big)\,d\tau \\ &\le C\int_0^t e^{-\frac{t-\tau}{2}}(t-\tau)\big[(1+\tau)^{-2\rho} + (1+\tau)^{-2(\rho-1)-\frac{1}{\rho-1}+\mu}\big]\,d\tau \\ &= o(t^{-2}),\end{aligned}$$

because $2(\rho-1)+\frac{1}{\rho-1} \ge 2\sqrt{2}$. From (3.22)

$$\begin{aligned}\|\bar{w}_4(t)\|_{L^1} &\le C\int_0^t e^{-\frac{t-\tau}{2}}\big[(t-\tau+(t-\tau)^2)\|u(\tau)\|_{L^\rho}^\rho + (t-\tau)^2\|u(\tau)\|_{L^\infty}^{\rho-1}\|\nabla u(\tau)\|_{L^1}\big]\,d\tau \\ &\le C\int_0^t e^{-\frac{t-\tau}{2}}\big[(t-\tau+(t-\tau)^2)(1+\tau)^{2(\rho-1)} \\ &\qquad + (t-\tau)^2(1+\tau)^{-2(\rho-1)-(\frac{1}{\rho-1}-\frac{4}{2})-\frac{1}{2}}\big]\,d\tau \\ &= o(1).\end{aligned}$$

Thus the case of $N=4$ also holds. The second to the last is estimated as

$$\begin{aligned}\|F_8\|_{L^p} &\le C\int_{t/2}^t \||u|^\rho(\tau)\|_{L^p} d\tau \le C\int_{t/2}^t (1+\tau)^{-2(\rho-\frac{1}{p})} d\tau \\ &= o\big(t^{-2(1-\frac{1}{p})}\big).\end{aligned}$$

Though F_7 is most delicate, the same method in [**12**] is applicable and

$$\|F_7\|_{L^p} = o\big(t^{-2(1-\frac{1}{p})}\big).$$

Thus we have completed the proof of our main Theorem 2.3.

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