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# On Euclidean tight 4-designs

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**Abstract.** A spherical $t$-design is a finite subset $X$ in the unit sphere $S^{n-1} \subset \boldsymbol{R}^n$ which replaces the value of the integral on the sphere of any polynomial of degree at most $t$ by the average of the values of the polynomial on the finite subset $X$. Generalizing the concept of spherical designs, Neumaier and Seidel (1988) defined the concept of Euclidean $t$-design in $\boldsymbol{R}^n$ as a finite set $X$ in $\boldsymbol{R}^n$ for which $\sum_{i=1}^{p}(w(X_i)/(|S_i|))\int_{S_i} f(x)d\sigma_i(x) = \sum_{x\in X} w(x)f(x)$ holds for any polynomial $f(x)$ of $\deg(f) \le t$, where $\{S_i,\ 1 \le i \le p\}$ is the set of all the concentric spheres centered at the origin and intersect with $X$, $X_i = X \cap S_i$, and $w : X \to \boldsymbol{R}_{>0}$ is a weight function of $X$. (The case of $X \subset S^{n-1}$ and with a constant weight corresponds to a spherical $t$-design.) Neumaier and Seidel (1988), Delsarte and Seidel (1989) proved the (Fisher type) lower bound for the cardinality of a Euclidean $2e$-design. Let $Y$ be a subset of $\boldsymbol{R}^n$ and let $\mathscr{P}_e(Y)$ be the vector space consisting of all the polynomials restricted to $Y$ whose degrees are at most $e$. Then from the arguments given by Neumaier-Seidel and Delsarte-Seidel, it is easy to see that $|X| \ge \dim(\mathscr{P}_e(S))$ holds, where $S = \cup_{i=1}^{p} S_i$. The actual lower bounds proved by Delsarte and Seidel are better than this in some special cases. However as designs on $S$, the bound $\dim(\mathscr{P}_e(S))$ is natural and universal. In this point of view, we call a Euclidean $2e$-design $X$ with $|X| = \dim(\mathscr{P}_e(S))$ a tight $2e$-design on $p$ concentric spheres. Moreover if $\dim(\mathscr{P}_e(S)) = \dim(\mathscr{P}_e(\boldsymbol{R}^n))(= \binom{n+e}{e})$ holds, then we call $X$ a Euclidean tight $2e$-design. We study the properties of tight Euclidean $2e$-designs by applying the addition formula on the Euclidean space. Furthermore, we give the classification of Euclidean tight 4-designs with constant weight. It is possible to regard our main result as giving the classification of rotatable designs of degree 2 in $\boldsymbol{R}^n$ in the sense of Box and Hunter (1957) with the possible minimum size $\binom{n+2}{2}$. We also give examples of nontrivial Euclidean tight 4-designs in $\boldsymbol{R}^2$ with nonconstant weight, which give a counterexample to the conjecture of Neumaier and Seidel (1988) that there are no nontrivial Euclidean tight $2e$-designs even for the nonconstant weight case for $2e \ge 4$.

## 1. Introduction.

In the paper of Neumaier-Seidel [**15**], they gave a definition of Euclidean $t$-design. Delsarte and Seidel [**8**] studied more precise properties of Euclidean designs on a union of $p$ concentric spheres centered at the origin. Here we first review these definitions. When they consider Euclidean $t$-designs, they assumed that $0 \notin X$. Since we believe it is better to drop this assumption, we first present the definition of Euclidean $t$-design which is a slight modification of Neumaier-Seidel's definition. Let $X$ be a finite set in $\boldsymbol{R}^n$. We assume $n \ge 2$ unless otherwise stated. Let $\{r_1, r_2, \ldots, r_p\} = \{\|x\| \mid x \in X\}$. Here $\|x\|^2 = (x,x) = \sum_{i=1}^{n} {x_i}^2$ for $x = (x_1, x_2, \ldots, x_n) \in \boldsymbol{R}^n$, and one of $r_i$ may possibly be

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0, that is, $X$ may possibly contain 0. For each $i$ we define $S_i = \{x \in \boldsymbol{R}^n \mid \|x\| = r_i\}$, the sphere of radius $r_i$ centered at 0. We say that $X$ is supported by the $p$ concentric spheres $S_1, \ldots, S_p$. If $r_i = 0$, then $S_i = \{0\}$. It may not be natural to consider $\{0\}$ as a sphere, however we regard it as one of the spheres supporting $X$. Let $X_i = X \cap S_i$. Let $d\sigma(x)$ be a Haar measure on the unit sphere $S^{n-1} \subset \boldsymbol{R}^n$. We consider a Haar measure $d\sigma_i(x)$ on each $S_i$ so that $|S_i| = {r_i}^{n-1}|S^{n-1}|$. Here $|S_i|$ and $|S^{n-1}|$ are the volumes of $S_i$ and the unit sphere $S^{n-1}$ respectively. We associate a positive real valued function $w$ on $X$, which is called a weight of $X$. We define $w(X_i) = \sum_{x \in X_i} w(x)$. Here if $r_i = 0$, then we define $\frac{1}{|S_i|}\int_{S_i} f(x)d\sigma_i(x) = f(0)$ for any function $f(x)$ defined on $\boldsymbol{R}^n$. Let $S = \cup_{i=1}^{p} S_i$. Let $\varepsilon_S \in \{0,1\}$ be defined by

$$\varepsilon_S = 1 \quad \text{if } 0 \in S, \qquad \varepsilon_S = 0 \quad \text{if } 0 \notin S.$$

We give some more definition of symbols we use. Let $\mathscr{P}(\boldsymbol{R}^n) = \boldsymbol{R}[x_1, x_2, \ldots, x_n]$ be the vector space of polynomials in $n$ variables $x_1, x_2, \ldots, x_n$. Let $\mathrm{Hom}_l(\boldsymbol{R}^n)$ be the subspace of $\mathscr{P}(\boldsymbol{R}^n)$ spanned by homogeneous polynomials of degree $l$. Let $\mathscr{P}_l(\boldsymbol{R}^n) = \oplus_{i=0}^{l}\mathrm{Hom}_l(\boldsymbol{R}^n)$. Let $\mathrm{Harm}(\boldsymbol{R}^n)$ be the subspace of $\mathscr{P}(\boldsymbol{R}^n)$ which consists of all the harmonic polynomials. Let $\mathrm{Harm}_l(\boldsymbol{R}^n) = \mathrm{Harm}(\boldsymbol{R}^n) \cap \mathrm{Hom}_l(\boldsymbol{R}^n)$. Let $\mathscr{P}(S)$, $\mathscr{P}_l(S)$, $\mathrm{Hom}_l(S)$, $\mathrm{Harm}(S)$, $\mathrm{Harm}_l(S)$ be the sets of corresponding polynomials restricted to the union $S$ of concentric spheres. For example $\mathscr{P}(S) = \{f|_S \mid f \in \mathscr{P}(\boldsymbol{R}^n)\}$.

The concept of spherical design was given by Delsarte, Goethals and Seidel in [**7**].

Definition 1.1 (Spherical design). Let $X$ be a finite set on the unit sphere $S^{n-1} \subset \boldsymbol{R}^n$. Let $t$ be a natural number. Then, with the notation mentioned above, we say that $X$ is a spherical $t$-design, if the following condition is satisfied:

$$\frac{1}{|S^{n-1}|}\int_{x \in S^{n-1}} f(x)d\sigma(x) = \frac{1}{|X|}\sum_{u \in X} f(u)$$

for any polynomial $f(x)$ of $n$ variables with degree at most $t$.

In [**7**], they proved that a spherical $2e$-design $X$ satisfies the condition

$$|X| \geq \binom{n+e-1}{e} + \binom{n+e-2}{e-1}.$$

The right hand side of the above inequality is the dimension of $\mathscr{P}_e(S^{n-1})$.

Definition 1.2 (Spherical tight $2e$-design). A spherical $2e$-design $X$ satisfying

$$|X| = \binom{n+e-1}{e} + \binom{n+e-2}{e-1}$$

is called a spherical tight $2e$-design.

A generalization of spherical designs was first given by Neumaier-Seidel [**15**] and studied by Delsarte-Seidel [**8**].

Definition 1.3 (Euclidean design). Let $X$ be a finite set with a weight $w$ and let $t$ be a natural number. Then, with the notation mentioned above, we say that $X$ is a Euclidean $t$-design, if the following condition is satisfied:

$$\sum_{i=1}^{p} \frac{w(X_i)}{|S_i|} \int_{x\in S_i} f(x)d\sigma_i(x) = \sum_{u\in X} w(u)f(u)$$

for any polynomial $f(x)$ of $n$ variables with degree at most $t$.

Remark 1. (1) If $0 \notin X$, then $X$ is a Euclidean $t$-design in the sense of Neumaier-Seidel. Also if $0 \in X$ and $X \neq \{0\}$, then $X\backslash\{0\}(= \{x \in X \mid x \neq 0\})$ is a Euclidean $t$-design in the sense of Neumaier-Seidel [**15**].
(2) If $p = 1$, $r_1 = 1$ and $w \equiv 1$ on $X$, then $X$ is a spherical $t$-design on $S^{n-1}$ (Definition 1.1, see also [**7**]).
(3) In the above definition of Euclidean designs, we always implicitly assumed that $n \geq 2$. If $n = 1$ and $r_i > 0$, then $\frac{1}{|S_i|}\int_{S_i} f(x)d\sigma_i(x) = \frac{1}{2}(f(-r_i)+f(r_i))$. (Note that if $r_i = 0$, we put as before, i.e., $\frac{1}{|S_i|}\int_{S_i} f(x)d\sigma_i(x) = f(0)$.) Thus if a finite set $X \subset \boldsymbol{R}$ is symmetric with respect to the origin and the weight function on $X$ is also symmetric with respect to the origin, then $X$ is a Euclidean $t$-design for any $t$. This is one of the reasons why we assume that $n \geq 2$ in this paper.

If $X$ is a Euclidean $2e$-design, then it is well known and easy to see that

$$|X| \geq \dim(\mathscr{P}_e(S))$$

holds ([**15**], [**8**] and [**1**]). We think this lower bound is universal and natural when we consider designs on $S$. At the same time, when we consider a design $X$ on $\boldsymbol{R}^n$ we want $X$ to be something which represents the whole Euclidean space $\boldsymbol{R}^n$. Based on this point of view, we introduce the following definitions for the tightness of the designs.

Definition 1.4 (Euclidean tight $2e$-design). Let $X$ be a Euclidean $2e$-design with weight $w$. If

$$|X| = \binom{n+e}{e}$$

and $\dim(\mathscr{P}_e(S)) = \binom{n+e}{e}$ hold, then we call $X$ a Euclidean tight $2e$-design. Here we note that the value $\binom{n+e}{e}$ is exactly the dimension of $\mathscr{P}_e(\boldsymbol{R}^n)$.

Definition 1.5 (Tight $2e$-design on $p$ concentric spheres). Let $X$ be a Euclidean $2e$-design with weight $w$. If

$$|X| = \dim(\mathscr{P}_e(S))$$

holds, then we call $X$ a tight $2e$-design on $p$ concentric spheres.

REMARK 2. (1) As we will introduce in the next theorem, some better lower bounds for the cardinalities of Euclidean $2e$-designs are given by Neumaier-Seidel and Delsarte-Seidel through the evaluation of $\dim(\mathscr{P}_e(S))$ ([**15**], [**8**]). They called Euclidean $2e$-design $X$ with $|X| = \dim(\mathscr{P}_e(S))$, a tight $2e$-design in $\boldsymbol{R}^n$.
(2) If $t$ is odd, then natural lower bounds for the cardinalities of antipodal Euclidean designs are known ([**8**]). However, the problem is still open for not antipodal ones. In this paper we do not consider the case when $t$ is odd.
(3) The following are known (see [**7**], [**8**], [**11**] and [**1**]):

- If $p \le [\frac{e+\varepsilon_S}{2}]$, then $\dim(\mathscr{P}_e(S)) = \varepsilon_S + \sum_{i=0}^{2(p-\varepsilon_S)-1} \binom{n+e-i-1}{e-i}$ $< \binom{n+e}{e} \big( = \sum_{i=0}^{e} \binom{n+e-i-1}{e-i} \big)$.
- If $p \ge [\frac{e+\varepsilon_S}{2}] + 1$, then $\dim(\mathscr{P}_e(S)) = \binom{n+e}{e}$.

Therefore a Euclidean tight $2e$-design is the same as a tight $2e$-design on $p$ concentric spheres with $p \ge [\frac{e+\varepsilon_S}{2}] + 1$.
(4) Let $X$ be a tight $2e$-design on $p$ concentric spheres. If $p = 1$, $X \ne \{0\}$ and $w$ is constant on $X$, then $X$ is similar to a spherical tight $2e$-design (see Remark 6 given later).

Next theorem was proved by Delsarte and Seidel [**8**].

THEOREM 1.6 (Delsarte-Seidel). *Let $X$ be a Euclidean $2e$-design with weight $w$. Then the following holds*:

$$|X| \ge \varepsilon_S + \sum_{i=0}^{2(p-\varepsilon_S)-1} \binom{n+e-i-1}{e-i}.$$

REMARK 3. In Definition 1.3, $X = \{0\}$ is a Euclidean $t$-design for any $t$ and $n$. Since $\dim(\mathscr{P}_e(\boldsymbol{R}^n)) > 1 = \dim(\mathscr{P}_e(\{0\})$ for any $e \ge 1$ and $n \ge 2$, $X = \{0\}$ is not a Euclidean tight $2e$-design. However if we consider $\{0\}$ as a special case of a sphere, then $X = \{0\}$ is a tight $2e$-design on a special sphere $\{0\}$.

The following proposition was pointed out by the referee.

PROPOSITION 1.7. *Let $X$ be a Euclidean tight $2e$-design. If $0 \in X$, then $e$ is even, $p = \frac{e}{2} + 1$ and $X \backslash \{0\}$ is a tight $2e$-design on $\frac{e}{2}$ concentric spheres.*

PROOF. By assumption $\varepsilon_S = 1$. Then Definition 1.4 and Remark 2 (3) imply $p \ge [\frac{e+1}{2}] + 1$. Since $X\backslash\{0\}$ is a Euclidean $2e$-design on a union of $p-1$ concentric spheres with positive radii, Theorem 1.6 implies

$$|X\backslash\{0\}| \ge \sum_{i=0}^{2(p-1)-1} \binom{n+e-i-1}{e-i}.$$

Therefore $\binom{n+e}{e} - 1 \ge \sum_{i=0}^{2(p-1)-1} \binom{n+e-i-1}{e-i}$. This implies $2(p-1)-1 \le e-1$. Hence $[\frac{e+1}{2}] + 1 \le p \le [\frac{e}{2}] + 1$. Therefore $e$ has to be an even number and $p = \frac{e}{2} + 1$. Since $|X\backslash\{0\}| = \sum_{i=0}^{e-1} \binom{n+e-i-1}{e-i}$, $X\backslash\{0\}$ is a tight $2e$-design on $\frac{e}{2}$ concentric spheres. $\square$

The following theorem is our main result in this paper.

THEOREM 1.8. *Let $n \geq 2$ and $X$ be a Euclidean tight 4-design in $\boldsymbol{R}^n$ whose weight is constant on $X \backslash \{0\}$. Then $0 \in X$ and $X \backslash \{0\}$ is similar to a spherical tight 4-design on $S^{n-1}$.*

REMARK 4. It is known that if spherical tight 4-design in $S^{n-1}$ exists then $n = 2$ or $n = (2m+1)^2 - 3$, where $m$ is an integer (cf. [**1**], [**5**]). The existence of a tight 4-design in $S^1$ and $S^{(2m+1)^2-4}$ is known for $m = 1$ and 2. However, it is generally unknown for $m \geq 3$. Recently, Bannai, Munemasa and Venkov [**5**] proved the non-existence for many values of $m$ including $m = 3$ and 4.

REMARK 5. The concept of a Euclidean 4-design (with constant weight) is equivalent to that of rotatable design of degree 2 in the sense of Box and Hunter (1957) and Kiefer (1960). Therefore, our main result can be regarded as giving the classification of degree 2 rotatable designs in $\boldsymbol{R}^n$ with the possible minimum size $\binom{n+2}{2}$.

Several equivalent definitions of Euclidean $t$-design are known. The following is proved by Neumaier and Seidel [**15**], which is very useful.

THEOREM 1.9 ([**15**]). *Let $X$ be a finite subset which may possibly contain 0 and with a weight $\omega$. Then the following (1) and (2) are equivalent:*

(1) *$X$ is a Euclidean $t$-design.*
(2) *$\sum_{u \in X} w(u) f(u) = 0$ for any polynomial $f \in \|x\|^{2j} \mathrm{Harm}_l(\boldsymbol{R}^n)$ with $1 \leq l \leq t$, $0 \leq j \leq [\frac{t-l}{2}]$.*

A rough sketch of our proof of Theorem 1.8 is as follows.

First we formulate the addition formula on $\boldsymbol{R}^n$ by using the Gegenbauer polynomials (see Theorem 2.3 in §2). Using this addition formula we can prove the following lemma.

LEMMA 1.10. *Let $X$ be a tight $2e$-design on $p$ concentric spheres in $\boldsymbol{R}^n$. Then the following hold:*

(1) *If $\|x\| = \|y\|$, then $w(x) = w(y)$, that is, $w$ is a constant function on each $X_i$.*
(2) *For any $i$, $1 \leq i \leq p$, $X_i$ is an at most $e$-distance set.*
(3) *If $w$ is constant on $X \backslash \{0\}$, then $p - \varepsilon_S \leq e$.*

REMARK 6. Let $X \subset \boldsymbol{R}^n$ be a tight $2e$-design on $p$ concentric spheres. Lemma 1.10 implies that if $p = 1$ and $X \neq \{0\}$, then $X$ is similar to a spherical tight $2e$-design on $S^{n-1}$.

Lemma 1.10 also implies that if $X$ is a Euclidean tight 4-design with constant weight, then Definition 1.4, Remark 2 (3), Proposition 1.7 and Lemma 1.10 (3) imply $p = 2$. Hence one of the following holds:

(1) $0 \notin X$ and $X$ is on 2 concentric spheres.
(2) $0 \in X$ and $X \backslash \{0\}$ is similar to a spherical tight 4-design.

If the case (1) given above occurs, then we may assume that $|X_1| \geq \frac{1}{2}|X| =$

$\frac{(n+2)(n+1)}{4}$. Since $\frac{(n+2)(n+1)}{4} > n+1$ holds for any $n \geq 3$, $X_1$ cannot be a 1-distance set (see [**7**]). Hence Lemma 1.10 implies that $X_1$ is a 2-distance set. Then we can apply the following theorem proved by Larman, Rogers and Seidel ([**14**]).

Theorem 1.11 (Larman-Rogers-Seidel). *Let $X$ be a 2-distance set in $\boldsymbol{R}^n$. If $|X| > 2n+3$, then there exists a natural number $k$ such that the ratio of the two distances of $X$ is given by $\sqrt{k} : \sqrt{k-1}$ and $k \leq \sqrt{\frac{n}{2}} + \frac{1}{2}$.*

We evaluated the ratio of the square of the two distances of $X_1$. It is not difficult to see that we may assume that $S_1$ is the unit sphere ($r_1 = 1$). Let $r = r_2$ and $R = r^2$. Let $\alpha_1$ and $\alpha_2$ be the two distances of the points in $X_1$. Assume $\alpha_1 < \alpha_2$. We define $k$ by

$$\left(\frac{\alpha_1}{\alpha_2}\right)^2 = \frac{k-1}{k}.$$

The number $k$ has to be an integer under the condition of Larman-Rogers-Seidel's Theorem. Instead of $(\alpha_1/\alpha_2)^2$ we consider $\left((\alpha_1^2+\alpha_2^2)/(\alpha_1^2-\alpha_2^2)\right)^2$ $(= (2k-1)^2)$. Then for each $n$ and $|X_1|$, $R$ is a solution of $F(n, |X_1|, R) = 0$, where $F(n, x, T)$ is a polynomial of $n, x$ and $T$, which is of degree 3 with respect to $T$. We can express $\left((\alpha_1^2+\alpha_2^2)/(\alpha_1^2-\alpha_2^2)\right)^2$ as a rational function $G_A(n, |X_1|, R)$ of $n$, $|X_1|$ and $R$. We prove that for any fixed $n$, $G_A(n, |X_1|, R(n, |X_1|))$ is decreasing as a function of $|X_1|$, where $R(n, |X_1|)$ is determined by $F(n, |X_1|, R) = 0$. Using this property we prove that $G_A(n, |X_1|, R(n, |X_1|))$ cannot be the square of an odd integer. That means the ratio of the square of the two distances in $X_1$ does not take the value $k : k-1$ for any integer $k$ which is required by the the theorem of Larman-Rogers-Seidel mentioned above.

In section 2, we give some more related facts. Then we give the addition formula for the Euclidean space using Gegenbauer polynomials and then we give a proof of Lemma 1.10.

In section 3, we discuss Euclidean tight 4-designs with constant weight and we give a proof of Theorem 1.8.

In section 4, we give some examples of Euclidean tight 4-designs whose weight are not constant. This gives a counterexample to Conjecture 3.4 in [**15**], that there exists no nontrivial tight 4-design.

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## 2. Addition formula and Euclidean 2*e*-design.

Let $X$ be a finite subset in $\boldsymbol{R}^n$ with a positive weight $w$. Let $S_1, \ldots, S_p$ be the $p$ concentric spheres defined in section 1 and let $S = \cup_{i=1}^p S_i$. We use the same notation given in section 1.

For any $\varphi, \psi \in \mathrm{Harm}(\boldsymbol{R}^n)$ we define $\langle \varphi, \psi \rangle = \frac{1}{|S^{n-1}|} \int_{\mathbf{x} \in S^{n-1}} \varphi(\mathbf{x}) \psi(\mathbf{x}) d\sigma(\mathbf{x})$. Then the following properties are known (see [**7**], [**8**], [**11**], [**1**]):

PROPOSITION 2.1.

(1) $\mathrm{Harm}(\boldsymbol{R}^n)$ *is a positive definite inner product space under* $\langle -,- \rangle$ *and has the orthogonal decomposition* $\mathrm{Harm}(\boldsymbol{R}^n) = \perp_{i=0}^{\infty} \mathrm{Harm}_i(\boldsymbol{R}^n)$,

(2) $\mathscr{P}_e(\boldsymbol{R}^n) = \bigoplus_{0 \le i+2j \le e} \|x\|^{2j} \mathrm{Harm}_i(\boldsymbol{R}^n)$,

(3) $\mathscr{P}_e(S) = \left\langle \|x\|^{2j} \,|\, 0 \le j \le \min\left\{p-1, \left[\frac{e}{2}\right]\right\}\right\rangle \oplus \Big( \bigoplus_{\substack{1 \le i \le e, \\ 0 \le j \le \min\{p-\varepsilon_S-1, [\frac{e-i}{2}]\}}} \|x\|^{2j} \mathrm{Harm}_i\ (S)\Big)$
*and if* $p \le [\frac{e+\varepsilon_S}{2}]$, *then*

$$\dim(\mathscr{P}_e(S)) = \varepsilon_S + \sum_{i=0}^{2(p-\varepsilon_S)-1} \binom{n+e-i-1}{e-i},$$

*if* $p \ge \left[\frac{e+\varepsilon_S}{2}\right] + 1$, *then*

$$\dim(\mathscr{P}_e(S)) = \binom{n+e}{e},$$

*where* $e$ *is a nonnegative integer.*

Let $h_l = \dim(\mathrm{Harm}_l(\boldsymbol{R}^n))$ and $\varphi_{l,1}, \ldots, \varphi_{l,h_l}$ be an orthonormal basis of $\mathrm{Harm}_l(\boldsymbol{R}^n)$ with respect to the inner product $\langle -,- \rangle$ defined above. Then

$$\left\{ \|x\|^{2j} \,|\, 0 \le j \le \min\left\{p-1, \left[\frac{e}{2}\right]\right\}\right\}$$
$$\cup \left\{ \|x\|^{2j} \varphi_{l,i}(x) \,\middle|\, 1 \le l \le e, 1 \le i \le h_l, 0 \le j \le \min\left\{p-\varepsilon_S-1, \left[\frac{e-l}{2}\right]\right\}\right\}$$

gives a basis of $\mathscr{P}_e(S)$. In the following we are going to construct more convenient basis of $\mathscr{P}_e(S)$ for our purpose. Let $\mathscr{G}(\boldsymbol{R}^n)$ be the subspace of $\mathscr{P}(\boldsymbol{R}^n)$ spanned by $\{\|x\|^{2j} \,|\, j = 0,1,2,\ldots,p-1\}$. Let $\mathscr{G}(X) = \{g|_X \,| g \in \mathscr{G}(\boldsymbol{R}^n)\}$. Then as functions on $X$, $\{\|x\|^{2j} \,|\, j = 0,1,2,\ldots,p-1\}$ is a basis of $\mathscr{G}(X)$. For each $l$ we define an inner product $\langle , \rangle_l$ on $\mathscr{G}(X)$ by

$$\langle f, g \rangle_l = \sum_{x \in X} w(x) \|x\|^{2l} f(x) g(x). \tag{2.1}$$

We apply the Gram-Schmidt's method to the basis $\{\|x\|^{2j} \,|\, j = 0,1,2,\ldots,p-1\}$ and construct an orthonormal basis

$$\{g_{l,1}(x), g_{l,2}(x), \ldots, g_{l,p-1}(x)\}$$

of $\mathscr{G}(X)$ with respect to the inner product $\langle , \rangle_l$. We can construct them so that for any $l$ the following holds:

> $g_{l,j}(x)$ is a linear combination of $1, \|x\|^2, \ldots, \|x\|^{2j}$, with $\deg(g_{l,j}) = 2j$ for $j$, $0 \le j \le p-1$.

For example if $p=2$, then we can express $g_{l,j}(x)$ in the following way.

$$g_{l,0}(x) \equiv \frac{1}{\sqrt{a_l}}, \qquad g_{l,1}(x) = \sqrt{\frac{a_l}{a_l a_{l+2} - {a_{l+1}}^2}}\left(\|x\|^2 - \frac{a_{l+1}}{a_l}\right),$$

where $a_i = \sum_{x\in X} w(x)\|x\|^{2i}$.

Now we are ready to give a new basis of $\mathscr{P}_e(S)$. For any $l$ satisfying $0 \le l \le e$, we consider the following sets of functions:

$$\mathscr{H}_0 = \left\{ g_{0,j} \,|\, 0 \le j \le \min\left\{ p-1, \left[\frac{e}{2}\right]\right\}\right\},$$

$$\mathscr{H}_l = \left\{ g_{l,j}\varphi_{l,i} \,|\, 0 \le j \le \min\left\{ p-\varepsilon_S-1, \left[\frac{e-l}{2}\right]\right\},\ 1 \le i \le h_l\right\} \quad \text{for } l \ge 1.$$

Let $\mathscr{H} = \cup_{l=0}^{e} \mathscr{H}_l$. Then $\mathscr{H}$ is a basis of $\mathscr{P}_e(S)$.

Next we define a matrix which plays an important role in the proof of our main result. Let $M$ be a matrix whose rows and columns are indexed with $X$ and $\mathscr{H}$ respectively. For $(u, g_{l,j}\varphi_{l,i}) \in X \times \mathscr{H}$ the $(u, g_{l,j}\varphi_{l,i})$-entry $M(u, g_{l,j}\varphi_{l,i})$ of $M$ is defined by

$$M(u, g_{l,j}\varphi_{l,i}) = \sqrt{w(u)}\, g_{l,j}(u)\varphi_{l,i}(u). \tag{2.2}$$

Definition 2.2 (Gegenbauer polynomials). Gegenbauer polynomials are a set of orthogonal polynomials $\{Q_l(\alpha) \,|\, l = 0,1,2,\ldots\}$ of one variable $\alpha$. For each $l$, $Q_l(\alpha)$ is a polynomial of degree $l$ and defined in the following manner.

(1) $Q_0(\alpha) \equiv 1$, $Q_1(\alpha) = n\alpha$.
(2) $\alpha Q_l(\alpha) = \lambda_{l+1}Q_{l+1}(\alpha) + (1-\lambda_{l-1})Q_{l-1}(\alpha)$ for $l \ge 1$, where $\lambda_l = \frac{l}{n+2l-2}$.

It is well known that $h_l = Q_l(1) = \binom{n+l-1}{l} - \binom{n+l-3}{l-2}$. The following theorem is also well known (see Erdelyi et al. [**11**], [**7**]).

Theorem 2.3 (Addition formula). *Let $\varphi_{l,1},\ldots,\varphi_{l,h_l}$ be an orthonormal basis of* $\mathrm{Harm}_l(\boldsymbol{R}^n)$. *Then the following hold*:

(1) *If $x, y \in S^{n-1}$, then*

$$\sum_{i=1}^{h_l} \varphi_{l,i}(x)\varphi_{l,i}(y) = Q_l((x,y)),$$

*where* $(x,y) = \sum_{i=1}^{n} x_i y_i$.

(2) *Let $x$ and $y$ be nonzero vectors in $\boldsymbol{R}^n$. Then the following holds*:

$$\sum_{i=1}^{h_l} \varphi_{l,i}(x)\varphi_{l,i}(y) = \|x\|^l\|y\|^l Q_l\left(\frac{(x,y)}{\|x\|\|y\|}\right).$$

From Definition 2.2 it is easy to see that Gegenbauer polynomial $Q_l$ of degree $l$ is of the following form:

$$Q_l(\alpha) = \sum_{j=0}^{[\frac{l}{2}]} \gamma_{l,\ l-2j}\alpha^{l-2j}.$$

In the following, using the facts we explained above, we give some more properties of Euclidean $t$-designs. Theorem 1.9 implies the following proposition.

PROPOSITION 2.4. *Let $X$ be a Euclidean $t$-design with weight $w$. Then the following* (1) *and* (2) *hold*:

(1) *Let $\lambda$ be a positive real number and $X' = \{\lambda u \mid u \in X\}$. Then $X'$ is also a Euclidean $t$-design with weight $w'$ defined by $w'(u') = w\big(\frac{1}{\lambda}u'\big)$, $u' \in X'$.*
(2) *Let $\mu$ be a positive real number and $w'(u) = \mu w(u)$ for any $u \in X$. Then $X$ is also a Euclidean $t$-design with respect to the weight $w'$.*

PROPOSITION 2.5. *Let $X$ be a Euclidean $2e$-design. Let $M$ be the matrix indexed by $X \times \mathscr{H}$ defined in* (2.2). *Then the following holds*:

$${}^tM\,M = I.$$

PROOF. Let us express $g_{l,j}$ by $g_{l,j}(x) = \sum_{k=0}^{j} \alpha_{l,j,k}\|x\|^{2k}$. Then the definition of $g_{l,j}(x)$ implies

$$\begin{aligned}
&\sum_{v\in X} w(v) g_{l,j}(v)\varphi_{l,i}(v) g_{l',j'}(v)\varphi_{l',i'}(v)\\
&= \sum_{v\in X} w(v)\Bigg(\sum_{k=0}^{j}\alpha_{l,j,k}\|v\|^{2k}\Bigg)\Bigg(\sum_{k'=0}^{j'}\alpha_{l',j',k'}\|v\|^{2k'}\Bigg)\varphi_{l,i}(v)\varphi_{l',i'}(v)\\
&= \sum_{k=0}^{j}\sum_{k'=0}^{j'}\alpha_{l,j,k}\alpha_{l',j',k'}\sum_{v\in X} w(v)\|v\|^{2(k+k')}\varphi_{l,i}(v)\varphi_{l',i'}(v)\\
&= \sum_{k=0}^{j}\sum_{k'=0}^{j'}\alpha_{l,j,k}\alpha_{l',j',k'}\frac{\sum_{v\in X} w(v)\|v\|^{2(k+k')+l+l'}}{|S^{n-1}|}\int_{\xi\in S^{n-1}}\|\xi\|^{2(k+k')}\varphi_{l,i}(\xi)\varphi_{l',i'}(\xi)\mathrm{d}\xi\\
&= \delta_{i,i'}\delta_{l,l'}\sum_{k=0}^{j}\sum_{k'=0}^{j'}\alpha_{l,j,k}\alpha_{l,j',k'}\sum_{v\in X} w(v)\|v\|^{2(k+k'+l)}\\
&= \delta_{i,i'}\delta_{l,l'}\sum_{v\in X} w(v)\|v\|^{2l}\Bigg(\sum_{k=0}^{j}\alpha_{l,j,k}\|v\|^{2k}\Bigg)\Bigg(\sum_{k'=0}^{j'}\alpha_{l,j',k'}\|v\|^{2k'}\Bigg)\\
&= \delta_{i,i'}\delta_{l,l'}\sum_{v\in X} w(v)\|v\|^{2l} g_{l,j}(v) g_{l,j'}(v) = \delta_{i,i'}\delta_{l,l'}\delta_{j,j'}. \qquad \square
\end{aligned}$$

Now we are ready to prove Lemma 1.10.

Proof of Lemma 1.10. By the assumption $|X| = |\mathscr{H}|$. Hence $M$ is a square matrix. Therefore Proposition 2.5 implies $M^t = M^{-1}$ and $M^tM = I$ holds. Then for nonzero vectors $u$, $v \in X$, $\frac{(M\ {}^tM)(u,v)}{\sqrt{w(u)w(v)}}$ is given by

$$\sum_{\substack{1\le l\le e,\\ 0\le j\le \min\{p-\varepsilon_S-1,[\frac{e-l}{2}]\},\\ 1\le i\le h_l}} g_{l,j}(u)g_{l,j}(v)\varphi_{l,i}(u)\varphi_{l,i}(v) + \sum_{j=0}^{\min\{p-1,[\frac{e}{2}]\}} g_{0,j}(u)g_{0,j}(v)$$

$$= \sum_{\substack{1\le l\le e,\\ 0\le j\le \min\{p-\varepsilon_S-1,[\frac{e-l}{2}]\}}} g_{l,j}(u)g_{l,j}(v)\sum_{1\le i\le h_l}\varphi_{l,i}(u)\varphi_{l,i}(v) + \sum_{j=0}^{\min\{p-1,[\frac{e}{2}]\}} g_{0,j}(u)g_{0,j}(v)$$

$$= \sum_{\substack{1\le l\le e,\\ 0\le j\le \min\{p-\varepsilon_S-1,[\frac{e-l}{2}]\}}} \|u\|^l\|v\|^l g_{l,j}(u)g_{l,j}(v)Q_l\left(\frac{(u,v)}{\|u\|\,\|v\|}\right) + \sum_{j=0}^{\min\{p-1,[\frac{e}{2}]\}} g_{0,j}(u)g_{0,j}(v).$$

Hence if $u = v$ we have

$$\sum_{\substack{1\le l\le e,\\ 0\le j\le \min\{p-\varepsilon_S-1,[\frac{e-l}{2}]\}}} \|u\|^{2l} g_{l,j}(u)^2Q_l(1) + \sum_{j=0}^{\min\{p-1,[\frac{e}{2}]\}} g_{0,j}(u)^2 = \frac{1}{w(u)}, \tag{2.3}$$

and if $u \ne v$, then we have

$$\sum_{\substack{1\le l\le e,\\ 0\le j\le \min\{p-\varepsilon_S-1,[\frac{e-l}{2}]\}}} \|u\|^l\|v\|^l g_{l,j}(u)g_{l,j}(v)Q_l\left(\frac{(u,v)}{\|u\|\,\|v\|}\right) + \sum_{j=0}^{\min\{p-1,[\frac{e}{2}]\}} g_{0,j}(u)g_{0,j}(v) = 0. \tag{2.4}$$

The left hand side of the equation (2.3) is a polynomial of $\|u\|^2$ which does not depend on the weight of each point. Therefore we have Lemma 1.10 (1). In the equation (2.4), if we let $u$, $v \in X_i$, then $\|u\| = \|v\| = r_i$ and the left hand side is a polynomial of the inner product $(u,v)$ of degree at most $e$. This means that $X_i$ is an at most $e$-distance set. As for the proof of (3), if $w$ is constant on $X\backslash\{0\}$, then $\{{r_i}^2 \,|\, r_i > 0\}$ are roots of the same equation (2.3) of degree at most $e$. Since $p - \varepsilon_S = |\{{r_i}^2 \,|\, r_i > 0\}|$, this implies (3) and completes the proof of Lemma 1.10. □

## 3. Euclidean tight 4-design.

In this section we consider a Euclidean tight 4-design $X \subset \boldsymbol{R}^n$ whose weight is constant on $X\backslash\{0\}$. As we mentioned in section 1 we have either (1) or (2) of the following:

(1) $0 \notin X$ and $X$ is on 2 concentric spheres.
(2) $0 \in X$ and $X \backslash \{0\}$ is a spherical tight 4-design.

Case (2) above is essentially a problem of spherical tight 4-designs. In the following we consider the case (1).

Now we assume $0 \notin X$ and $X$ is on 2 concentric spheres. Let $N = \binom{n+2}{2}$ and $N_i = |X_i|$. We may assume $N_1 \geq N_2$. Then $N_1 \geq \frac{N}{2} = \frac{(n+2)(n+1)}{4}$. By Proposition 2.4 we may assume $r_1 = 1$ and $w = 1$. Let $r_2 = r$. Then the constants $a_i = \sum_{x \in X} w(x) \|x\|^{2i}$ we defined in section 2 are given by

$$a_i = N_1 + (N - N_1) r^{2i}.$$

In particular $a_0 = N$. Then the equation (2.3) corresponding to a point $u \in X_1$ (resp. $u \in X_2$) implies the following (3.1) (resp. (3.2)). Also, the equation (2.4) corresponding to two distinct points $u, v \in X_1$ (resp. $u, v \in X_2$) implies the following (3.3) (resp. (3.4)).

$$\frac{1}{a_0} + \frac{(a_0 - a_1)^2}{a_0(a_0 a_2 - a_1{}^2)} + \frac{n}{a_1} + \frac{(n+2)(n-1)}{2a_2} = 1. \tag{3.1}$$

$$\frac{1}{a_0} + \frac{(a_0 r^2 - a_1)^2}{a_0(a_0 a_2 - a_1{}^2)} + \frac{nr^2}{a_1} + \frac{(n+2)(n-1)r^4}{2a_2} = 1. \tag{3.2}$$

$$\frac{1}{a_0} + \frac{(a_0 - a_1)^2}{a_0(a_0 a_2 - a_1{}^2)} + \frac{n(u,v)}{a_1} + \frac{n(n+2)}{2a_2}\left((u,v)^2 - \frac{1}{n}\right) = 0 \tag{3.3}$$

for any $u \neq v$ with $\|u\| = \|v\| = 1$.

$$\frac{1}{a_0} + \frac{(a_0 r^2 - a_1)^2}{a_0(a_0 a_2 - a_1{}^2)} + \frac{n(u,v)}{a_1} + \frac{n(n+2)}{2a_2}\left((u,v)^2 - \frac{1}{n} r^4\right) = 0 \tag{3.4}$$

for any $u \neq v$ with $\|u\| = \|v\| = r$.

Let us denote $R = r^2$ and substitute $a_0 = N = \frac{(n+2)(n+1)}{2}$, $a_1 = N_1 + (N - N_1)R$, $a_2 = N_1 + (N - N_1)R^2$, in equations (3.1) through (3.4) given above. Then (3.1) and (3.2) give the same equation

$$F(n, N_1, R) = 0, \tag{3.5}$$

where $F(n, x, T)$ is a polynomial defined by

$$\begin{aligned} F(n,x,T) = {} & 4T^3(x-1)(N-x)^2 + 4T^2 x(x-n-1)(N-x) \\ & + 2Tx(2x - n(n+1))(N-x) + 4x^2(x - N + 1). \end{aligned} \tag{3.6}$$

Let $A = \|u - v\|^2$ for $u$, $v \in X_1$ and $B = \|u - v\|^2$ for $u$, $v \in X_2$. Then the equation (3.3) is equivalent to

$$\begin{aligned}&(n+2)nN_1(N_1+R(N-N_1))A^2\\&\quad -4N_1n\big((N-N_1)R^2+(n+2)(N-N_1)R+N_1(n+3)\big)A\\&\quad +8(N-N_1)^2R^3+8N_1(n+1)(N-N_1)R^2\\&\quad +4N_1n(n+1)(N-N_1)R+4nN_1^2(n+3)=0.\end{aligned} \tag{3.7}$$

Similarly, the equation (3.4) is equivalent to

$$\begin{aligned}&n(n+2)(N-N_1)(N_1+R(N-N_1))B^2\\&\quad -4n(N-N_1)\big((n+3)(N-N_1)R^2+(n+2)N_1R+N_1\big)B\\&\quad +4n(n+3)(N-N_1)^2R^3+4N_1n(n+1)(N-N_1)R^2\\&\quad +8N_1(n+1)(N-N_1)R+8N_1^2=0.\end{aligned} \tag{3.8}$$

Equations (3.7) and (3.8) are the special cases of the equation (2.4) in the proof of Lemma 1.10. Thus, as we proved in Lemma 1.10, $X_1$ and $X_2$ are at most 2-distance sets. In the following using the equations (3.5), (3.6), (3.7), (3.8) we prove that the case (1) we explained at the beginning of this section does not occur.

First we investigate the zeros of the polynomial $F(n,x,T)$ defined in (3.6).

Proposition 3.1. *Let $n \geq 2$. Then the following hold*:

(1) $F(n,N-1,T)>0$ *for any* $T>0$.
(2) $F\big(n,\frac{N}{2},T\big)\neq 0$ *for any* $T>0$, *satisfying* $T\neq 1$.
(3) *Let* $\frac{N}{2}<x<N$. *Then* $F(n,x,T)>0$ *for any* $T\geq 1$.
(4) *Let* $\frac{N}{2}<x\leq N-(n+1)$. *Then the following hold*:
  (a) $F(n,x,T)=0$, $T\geq 0$ *has exactly one solution* $T=T(n,x)$ *and it is in the interval* $(0,1)$.
  (b) $\frac{\partial F(n,x,T)}{\partial T}>0$ *for* $T\geq 1-\frac{1}{2n}$.
  (c) $\frac{\partial F(n,x,T)}{\partial x}>0$ *for any* $T$ *satisfying* $1-\frac{1}{2n}\leq T<1$.
  (d) $F\big(n,x,1-\frac{1}{2n}\big)<0<F(n,x,1)$.

Proposition 3.1 immediately implies the next corollary.

Corollary 3.2.

(1) $\frac{N}{2}<N_1<N-1$ *and the radius* $r$ *of* $S_2$ *satisfies* $r<1$.
(2) *Moreover if* $\frac{N}{2}<N_1\leq N-(n+1)$, *then* $1-\frac{1}{2n}<R<1$ *holds.*

Proof of Proposition 3.1.
(1) For any $T>0$, we have

$$\begin{aligned}F(n,N-1,T)&=2T^3(n^2+3n-2)+T^2n(n+3)(n^2+n-2)\\&\quad +2Tn^2(n+3)>0.\end{aligned}$$

(2) $F\left(n, \dfrac{N}{2}, T\right) = -\dfrac{1}{2}N^2(1-T)\big((N-2)T^2 + T(n+2)(n-1) + N - 2\big) \neq 0$

(3) For any $T \neq 1$, we have

$$\begin{aligned}\frac{\partial F(n,x,T)}{\partial T} &= 12T^2(x-1)(N-x)^2 + 8Tx(x-n-1)(N-x)\\ &\quad + 2x(2x - n(n+1))(N-x).\end{aligned}$$

For $T \geq 1$, we have

$$\begin{aligned}\frac{\partial F(n,x,T)}{\partial T} &\geq 12(x-1)(N-x)^2 + 8x(x-n-1)(N-x)\\ &\quad + 2x(2x - n(n+1))(N-x)\\ &= 4(N-x)\big((n^2+2n+4)x - 3N\big)\\ &\geq 2N(N-x)(n^2+2n-2) > 0.\end{aligned}$$

On the other hand

$$F(n,x,1) = 4N(2x-N) > 0.$$

Therefore $F(n,x,T) > 0$, for any $x$, $T$ satisfying $T \geq 1$ and $\frac{N}{2} < x < N$.
(4) (a) We have

$$\frac{\partial^2 F(n,x,T)}{\partial T^2} = 24(x-1)(N-x)^2 T + 8x(x-n-1)(N-x).$$

Since $N > \frac{n(n+1)}{2} \geq x > \frac{N}{2} \geq n+1$, we have

$$\frac{\partial^2 F(n,x,T)}{\partial T^2} > 0,$$

for any $T \geq 0$. Therefore $\frac{\partial F(n,x,T)}{\partial T}$ is strictly increasing for $T \geq 0$ as a function of $T$. Moreover

$$\begin{aligned}\left.\frac{\partial F(n,x,T)}{\partial T}\right|_{T=0} &= 2x(2x-n(n+1))(N-x) \leq 0,\\ \left.\frac{\partial F(n,x,T)}{\partial T}\right|_{T=1} &= 4(x(n^2+2n+4) - 3N)(N-x) > 0,\end{aligned}$$

$F(n,x,0) < 0$ and $F(n,x,1) > 0$ hold. Hence $F(n,x,T) = 0$ has exactly one solution and the solution is in $(0,1)$.

(b) We have

$$\left.\frac{\partial F(n,x,T)}{\partial T}\right|_{T=1-\frac{1}{2n}}$$
$$= \frac{(N-x)}{2n^2}\big((16n-6)x^2+(n-1)(8n^3+12n^2+19n-12)x-6N(2n-1)^2\big). \quad (3.9)$$

In equation (3.9), we have

$$\begin{aligned}&(16n-6)x^2+(n-1)(8n^3+12n^2+19n-12)x-6N(2n-1)^2\\ &\quad > \big((8n-3)N+(n-1)(8n^3+12n^2+19n-12)\big)\frac{N}{2}-6N(2n-1)^2\\ &\quad = \frac{1}{4}N(16n^4+16n^3-61n^2+41n-6)>0. \qquad (3.10)\end{aligned}$$

Hence we have

$$\left.\frac{\partial F(n,x,T)}{\partial T}\right|_{T=1-\frac{1}{2n}} > 0$$

and therefore $\frac{\partial F(n,x,T)}{\partial T}>0$ holds for any $T\geq 1-\frac{1}{2n}$.
(c) We have the following inequalities:

$$\frac{\partial^3 F(n,x,T)}{\partial x^3} = 24(1+T)(1-T)^2>0, \qquad (3.11)$$

$$\left.\frac{\partial^2 F(n,x,T)}{\partial x^2}\right|_{x=\frac{N}{2}} = 4(1-T)\big((N+2)T^2+(n^2+n+2)T+N+2\big)>0. \qquad (3.12)$$

Then (3.11) and (3.12) imply that $\frac{\partial^2 F(n,x,T)}{\partial x^2}>0$ holds for any $x\geq\frac{N}{2}$. Next, we have

$$\left.\frac{\partial F(n,x,T)}{\partial x}\right|_{x=\frac{N}{2}} = (T+1)N\big(-(N-4)T^2+2(N-2)T-(N-4)\big).$$

Then we have

$$\frac{\partial}{\partial T}\big(-(N-4)T^2+2(N-2)T-(N-4)\big) = -2(N-4)T+2(N-2)>4 \qquad (3.13)$$

for any $0<T<1$, and also

$$\big(-(N-4)T^2+2(N-2)T-(N-4)\big)\big|_{T=1-\frac{1}{2n}} = \frac{1}{8n^2}(31n^2-19n+6)>0.$$

Hence we obtain $\frac{\partial F(n,x,T)}{\partial x}\Big|_{x=\frac{N}{2}} > 0$ for any $T \geq 1 - \frac{1}{2n}$. This implies (c).
(d) We have already seen that $F(n,x,1) > 0$ holds. Also we have

$$F\left(n, x, 1-\frac{1}{2n}\right) = \frac{1}{8n^3}\big\{4(4n-1)x^3 + x\big(4x(4n^4+3n^3+4n^2-11n+3) \\ - 2(2n-1)N(4n^3-9n^2+17n-6)\big) - 4N^2(2n-1)^3\big\}. \tag{3.14}$$

By (4) (c) proved above, $F(n,x,1-\frac{1}{2n})$ is increasing as a function of $x$ for $\frac{N}{2} < x \leq \frac{n(n+1)}{2}$. Since

$$F\left(n, \frac{n(n+1)}{2}, 1-\frac{1}{2n}\right) = -\frac{(n+1)^2(7n^4+15n^3-27n^2+13n-2)}{4n^3} < 0,$$

$F(n,x,1-\frac{1}{2n}) < 0$ holds for any $x$ with $\frac{N}{2} < x \leq \frac{n(n+1)}{2}$. □

Next we prove the following proposition.

PROPOSITION 3.3. *If* $\frac{n(n+1)}{2} + 1 \leq N_1 < \frac{n(n+3)}{2}$, *and* $0 < R < 1$, *then the discriminant* $D_B$ *of the quadratic equation* (3.8) *with respect to* $B$ *is negative.*

COROLLARY 3.4. $\frac{N}{2} < N_1 \leq \frac{n(n+1)}{2}$.

PROOF. Proposition 3.1 implies $\frac{N}{2} < N_1 < N-1$ and $0 < R < 1$. □

PROOF OF PROPOSITION 3.3.
The discriminant $D_B$ of the equation (3.8) is given by

$$-16n(N-N_1)\big(d_4R^4 + d_3R^3 + d_2R^2 + d_1R + d_0\big)$$

where

$$\begin{aligned} d_4 &= -n(n+3)(N-N_1)^3, & d_1 &= 4{N_1}^2(n+2)(N-N_1), \\ d_3 &= -2nN_1(n+2)(N-N_1)^2, & d_0 &= {N_1}^2((3n+4)N_1 - Nn), \\ d_2 &= -N_1((n^2+2n+4)N_1 - 4N)(N-N_1). \end{aligned}$$

Since $N_1 \geq \frac{n(n+1)}{2} + 1 > \frac{N}{2}$, we have

$$d_0 > N_1^2\left((3n+4)\frac{N}{2} - nN\right) > 0.$$

Hence $d_1, d_0 > 0$ and $d_2, d_3, d_4 < 0$ hold. Since $0 < R < 1$ and $\frac{n(n+1)}{2} + 1 \leq N_1 < N$, we have

$$d_4R^4 + d_3R^3 + d_2R^2 + d_1R + d_0 > (d_4 + d_3 + d_2 + d_1 + d_0)R^2$$
$$= N^2\big((n+1)(n+4)N_1 - n(n+3)N\big)R^2 \geq 4N^2(n+1)R^2 > 0.$$

This implies $D_B < 0$. □

Corollary 3.2 (2) and Corollary 3.4 imply the following lemma.

Lemma 3.5. *$N_1$ and $R$ satisfy the following inequalities*:

$$\frac{N}{2} < N_1 \leq \frac{n(n+1)}{2} \quad \textit{and} \quad 1 - \frac{1}{2n} < R^2 < 1.$$

It is known that the cardinality of a 1-distance set in $\boldsymbol{R}^n$ is bounded above by $n+1$ (see [**7**]). Since $n \geq 2$ by assumption, $X_1$ must be a 2-distance set. Also, if $n \geq 7$, then $|X_1| \geq \frac{N}{2} + 1 > 2n + 3$ holds. Hence $X_1$ satisfies the condition of Theorem 1.11. In the following we apply Theorem 1.11 to $X_1$.

The solutions of (3.7) are given by

$$\frac{G_{A,1}(n, N_1, R) \pm \sqrt{G_{A,2}(n, N_1, R)}}{G_{A,3}(n, N_1, R)},$$

where $G_{A,1}(n,x,T)$, $G_{A,2}(n,x,T)$, $G_{A,3}(n,x,T)$ are polynomials in $n,x,T$ defined by

$$G_{A,1}(n,x,T) = 2nx\big((N-x)T^2 + (n+2)(N-x)T + x(n+3)\big), \tag{3.15}$$

$$\begin{aligned} G_{A,2}(n,x,T) = {} & 4nx(N-x)^2(3nx + 4x - 4N - 2nN)T^4 - 16nx^2(N-x)^2(n+2)T^3 \\ & - 4nx^2(N-x)\big((n^2+2n+4)x - (n^2+2n)N\big)T^2 \\ & + 8n^2x^3(n+2)(N-x)T + 4n^2x^4(n+3), \end{aligned} \tag{3.16}$$

and

$$G_{A,3}(n,x,T) = nx(n+2)((N-x)T + x).$$

Remark 7. $G_{A,3}(n,x,T) > 0$ for any positive numbers $n,x,T$ satisfying $0 < x \leq N$.

Proposition 3.6. *$G_{A,2}(n,x,T) > 0$ holds for any any positive numbers $n,x,T$ satisfying $\frac{N}{2} \leq x \leq N$ and $0 < T < 1$.*

Proof. Since $0 < T < 1$, we have

$$\begin{aligned} G_{A,2}(n,x,T) > {} & 4nx(N-x)^2(3nx + 4x - 4N - 2nN)T^4 - 16nx^2(N-x)^2(n+2)T^3 \\ & - 4nx^2(N-x)\big((n^2+2n+4)x - (n^2+2n)N\big)T^2 \\ & + 8n^2x^3(n+2)(N-x)T^2 + 4n^2x^4(n+3)T^2 \end{aligned}$$

$$= 4nxN^2\left(\left(x - \frac{N}{2}\right)(n+4)(n+1) + \frac{N}{2}(n^2+n-4)\right)T^4$$
$$+ nx^2\left(4(3n+4)(N-x)\left(x-\frac{N}{2}\right) + 2(7n+12)N\left(x-\frac{N}{2}\right)\right.$$
$$\left. + (4n^2+5n-12)N^2\right)(T^3 - T^4)$$
$$+ 4nx^2\big((n+4)(N-x)^2 + 2(n+2)Nx + (n^2+n-4)N^2\big)(T^2-T^3)$$
$$> 0. \qquad \square$$

Let $k_A(n,x,T)$ be a function defined by,

$$\frac{G_{A,1}(n,x,T) - \sqrt{G_{A,2}(n,x,T)}}{G_{A,1}(n,x,T) + \sqrt{G_{A,2}(n,x,T)}} = \frac{k_A(n,x,T)-1}{k_A(n,x,T)}.$$

Our $X_1$ is a 2-distance set, and

$$\frac{G_{A,1}(n,N_1,R) - \sqrt{G_{A,2}(n,N_1,R)}}{G_{A,1}(n,N_1,R) + \sqrt{G_{A,2}(n,N_1,R)}}$$

gives the ratio of the squares of the two distances of $X_1$. Then we have

$$(2k_A(n,x,T)-1)^2 = \frac{G_{A,1}(n,x,T)^2}{G_{A,2}(n,x,T)}.$$

Let

$$G_A(n,x,T) = \frac{G_{A,1}(n,x,T)^2}{G_{A,2}(n,x,T)}. \tag{3.17}$$

Since $N_1 > 2n+3$ for $n \geq 7$, Theorem 1.11 implies that $k_A(n,N_1,R)$ is a natural number and $G_A(n,N_1,R)$ is the square of a positive odd integer $2k_A(n,N_1,R)-1$ for any $n \geq 7$.

In the following we study the function $G_A(n,x,T)$ under the condition $F(n,x,T) = 0$.

PROPOSITION 3.7. *For any $n$, $x$ and $T$ satisfying $\frac{N}{2} \leq x \leq \frac{n(n+1)}{2}$ and $0 < T < 1$ the following assertions hold*:

(1) $\dfrac{\partial G_A(n,x,T)}{\partial T} > 0.$

(2) $\dfrac{\partial G_A(n,x,T)}{\partial x} < 0.$

Proof.

$$\frac{\partial G_A(n,x,T)}{\partial T} = \frac{-16n^2(n+2)x^2(N-x)TG_{A,1}(n,x,T)G_{A,4}(n,x,T)}{G_{A,2}(n,x,T)^2},$$

where

$$\begin{aligned} G_{A,4}(n,x,T) = (3x-2N)(N-x)^2(n+2)T^3 \\ &+ x(N-x)\big(9(n+2)x-(7n+16)N\big)T^2 \\ &- 9x^2(n+2)(N-x)T - x^2(3(n+2)x - nN). \end{aligned}$$

Since $\frac{N}{2} \leq x \leq \frac{n(n+1)}{2}$ and $0 < T < 1$, we obtain

$$\begin{aligned} G_{A,4}(n,x,T) = & -\frac{N}{2}(N-x)^2(n+2)T^3 + 3\left(x-\frac{N}{2}\right)(N-x)^2(n+2)T^3 \\ & + x(N-x)(9(n+2)x-(7n+16)N)T^2 \\ & - 9x^2(n+2)(N-x)T - x^2(3(n+2)x-nN) \\ < & -\frac{N}{2}(N-x)^2(n+2)T^3 + 3\left(x-\frac{N}{2}\right)(N-x)^2(n+2)T^2 \\ & + x(N-x)(9(n+2)x-(7n+16)N)T^2 \\ & - 9x^2(n+2)(N-x)T^2 - x^2(3(n+2)x-nN)T^2 \\ = & -\frac{N}{2}(N-x)^2(n+2)T^3 - \frac{N}{2}\big((n+2)(3N^2-x^2)+2(n+4)Nx\big)T^2 \\ < & \ 0. \end{aligned}$$

Clearly $G_{A,1}(n,x,T) > 0$ (see equation (3.15)). Hence we have $\frac{\partial G_A(n,x,T)}{\partial T} > 0$.
(2) We have

$$\frac{\partial G_A(n,x,T)}{\partial x} = \frac{16n^2(n+2)NxG_{A,1}(n,x,T)G_{A,5}(n,x,T)T^2}{G_{A,2}(n,x,T)^2},$$

where $G_{A,5}(n,x,T)$ is given by

$$\begin{aligned} G_{A,5}(n,x,T) = & -(N-x)^3\big(T^4+(n+2)T^3\big) \\ & + x(10x+4nx-3(n+3)N)(N-x)T^2 \\ & - 5x^2(n+2)(N-x)T + x^2(nN-(2n+3)x). \end{aligned}$$

Since $nN-(2n+3)x < 0$, we have

$$\begin{aligned}G_{A,5}(n,x,T) <& -(N-x)^3\big(T^4+(n+2)T^3\big)+x(10x+4nx-3(n+3)N)(N-x)T^2\\&-5x^2(n+2)(N-x)T^2+x^2(nN-(2n+3)x)T^2\\=& -(N-x)^3T^4-(n+2)(N-x)^3T^3-(n+3)x\big(x^2+3N(N-x)\big)T^2\\<& \ 0.\end{aligned}$$

This completes the proof of (2). □

PROPOSITION 3.8. *Let $n \geq 2$ and $T = T(n,x)$ be the function defined implicitly by the equation $F(n,x,T) = 0$ and $0 < T < 1$. Then $G_A(n,x,T(n,x))$ is a function of $n$ and $x$. Moreover we have the following inequalities for any $n$ and $x$ satisfying $\frac{N}{2} < x \leq \frac{n(n+1)}{2}$*:

(1) $\dfrac{\partial T(n,x)}{\partial x} < 0.$

(2) $\dfrac{\partial G_A(n,x,T(n,x))}{\partial x} < 0.$

PROOF. (1) By the definition of $T(n,x)$, we have

$$\frac{\partial F(n,x,T)}{\partial T}\frac{\partial T(n,x)}{\partial x}+\frac{\partial F(n,x,T)}{\partial x}=0.$$

Hence we have

$$\frac{\partial T(n,x)}{\partial x}=-\frac{\partial F(n,x,T)}{\partial x}\Big/\frac{\partial F(n,x,T)}{\partial T}.$$

Proposition 3.1 (4)(b) implies that $\frac{\partial F(n,x,T)}{\partial T} > 0$ for any $n,x,T$ satisfying $1-\frac{1}{2n} < T < 1$ and $\frac{N}{2} < x \leq \frac{n(n+1)}{2}$. Proposition 3.1 (4)(d) implies that $1-\frac{1}{2n} < T(n,x) < 1$ for any $n,x$ satisfying $\frac{N}{2} < x \leq \frac{n(n+1)}{2}$. Proposition 3.1 (4)(c) implies that $\frac{\partial F(n,x,T)}{\partial x} > 0$ holds for any $n,x,T$ satisfying $\frac{N}{2} < x \leq \frac{n(n+1)}{2}$ and $1-\frac{1}{2n} < T < 1$. Hence we have (1).
(2) We have

$$\frac{\partial G_A(n,x,T(n,x))}{\partial x}=\frac{\partial G_A(n,x,T)}{\partial x}+\frac{\partial G_A(n,x,T)}{\partial T}\frac{\partial T(n,x)}{\partial x}.$$

(1) and Proposition 3.7 imply (2). □

Proposition 3.8 implies that both $T(n,x)$ and $G_A(n,x,T(n,x))$ decrease as $x$ increases for $\frac{N}{2} < x \leq \frac{n(n+1)}{2}$. Next we estimate the value of $G_A(n,x,T(n,x))$.

PROPOSITION 3.9. *Let $n \geq 7$. Then the following inequalities hold*:

(1) $n+6 > G_A\big(n,\frac{N}{2}+1,T\big(n,\frac{N}{2}+1\big)\big)$.
(2) $n+3 < G_A\big(n,\frac{n(n+1)}{2},T\big(n,\frac{n(n+1)}{2}\big)\big)$.

(3) $G_A\big(n, \frac{n(n+4)}{3}, T\big(n, \frac{n(n+4)}{3}\big)\big) > n+4 > G_A\big(n, \frac{n(n+4)}{3}+1, T\big(n, \frac{n(n+4)}{3}+1\big)\big)$,

(4) $G_A\big(n, \frac{n(n+5)}{4}, T\big(n, \frac{n(n+5)}{4}\big)\big) > n+5 > G_A\big(n, \frac{n(n+5)}{4}+1, T\big(n, \frac{n(n+5)}{4}+1\big)\big)$.

Proof. (1) Let $1 > T > 0$. We have

$$n+6-G_A\left(n, \frac{N}{2}+1, T\right) = -\frac{n(N+2)}{4}\frac{P_1(n,T)}{G_{A,2}\big(n, \frac{N}{2}+1, T\big)}, \tag{3.18}$$

where

$$\begin{aligned}P_1(n,T) = {} & (N-2)^2\big((n^2+11n+12)N-24n-36\big)T^4\\ & +6(N+2)(N-2)^2(n+4)(n+2)T^3\\ & -2(N-2)(N+2)\big((n^2-n-24)N-12\big)T^2\\ & -6n(N-2)(N+2)^2(n+2)T-3n(N+2)^3(n+3).\end{aligned}$$

Since $0 < T < 1$ and $n \geq 7$, we have

$$\begin{aligned}P_1(n,T) < {} & (N-2)^2\big((n^2+11n+12)N-24n-36\big)T^2\\ & +6(N+2)(N-2)^2(n+4)(n+2)T^2\\ & -2(N-2)(N+2)\big((n^2-n-24)N-12\big)T^2\\ & -6n(N-2)(N+2)^2(n+2)T^2-3n(N+2)^3(n+3)T^2\\ = {} & -2(n^4-4n^3-23n^2+14n+48)N^2T^2 < 0.\end{aligned}$$

Then Proposition 3.6 and (3.18) imply

$$G_A\left(n, \frac{N}{2}+1, T\right) < n+6$$

for any $T$ satisfying $0 < T < 1$. In particular (see Proposition 3.1 (4)(d))

$$G_A\left(n, \frac{N}{2}+1, T\left(n, \frac{N}{2}+1\right)\right) < n+6.$$

(2)

$$\begin{aligned}&G_A\left(n, \frac{n(n+1)}{2}, 1-\frac{1}{2n}\right)\\ &\quad = n+3+\frac{(4n^4+20n^3+16n^2-25n+6)(n+2)^2(2n-1)^2}{(4n^9+28n^8+40n^7-80n^6-124n^5+168n^4+64n^3-143n^2+60n-8)}.\end{aligned}$$

Hence we have

$$G_A\left(n, \frac{n(n+1)}{2}, 1-\frac{1}{2n}\right) > n+3.$$

Proposition 3.1 (4) impleis $T(n, \frac{n(n+1)}{2}) > 1 - \frac{1}{2n}$. Therefore Proposition 3.7 (1) implies

$$G_A\left(n, \frac{n(n+1)}{2}, T\left(n, \frac{n(n+1)}{2}\right)\right) > n+3.$$

(3) First we estimate $T(n,x)$ for $x = \frac{n(n+4)}{3}, \frac{n(n+4)}{3}+1$. If $n=7$, then $\frac{n(n+4)}{3} = \frac{77}{3}$ and $1-\frac{2}{n^2} = 1-\frac{2}{7^2} < \frac{96}{100}$. We also have the following equations:

$$F\left(7, \frac{77}{3}, \frac{96}{100}\right) = -\frac{158936656}{421875}, \qquad F\left(7, \frac{77}{3}, \frac{97}{100}\right) = \frac{95460979}{375000}.$$

Hence Proposition 3.1 and Proposition 3.7 imply the following inequalities:

$$1-\frac{2}{7^2} < \frac{96}{100} < T\left(7, \frac{77}{3}\right) < \frac{97}{100} < 1. \tag{3.19}$$

If $n \geq 8$, then we have

$$F\left(n, \frac{n(n+4)}{3}+1, 1-\frac{2}{n^2}\right) = -\frac{(n+1)^2(3n^5-12n^4-62n^3-44n^2-16n+32)}{27n^3} < 0.$$

Therefore we have

$$1-\frac{2}{n^2} < T\left(n, \frac{n(n+4)}{3}+1\right) < T\left(n, \frac{n(n+4)}{3}\right) < 1. \tag{3.20}$$

(i) First we will show that $G_A\big(n, \frac{n(n+4)}{3}, T\big(n, \frac{n(n+4)}{3}\big)\big) > n+4$ holds. We have

$$G_A\left(n, \frac{n(n+4)}{3}, T\right) - (n+4) = \frac{2n^2(n+4)^2}{81} \frac{P_2(n,T)}{G_{A,2}\big(n, \frac{n(n+4)}{3}, T\big)}, \tag{3.21}$$

where

$$\begin{aligned} P_2(n,T) &= 2(2n+3)(n^2+n+6)^2T^4 + n(3n+8)(2+n)(n^2+n+6)^2T^3 \\ &\quad + n^2(n^2+n+6)(n^3+15n^2+48n+52)T^2 \\ &\quad - 2n^3(n+4)(2+n)(n^2+n+6)T - 2n^4(3+n)(n+4)^2. \end{aligned}$$

If $n \geq 7$ and $1-\frac{2}{n^2} < T < 1$, then we have

$$
\begin{aligned}
P_2(n,T) &> 2(2n+3)(n^2+n+6)^2\left(1-\frac{2}{n^2}\right)^4 + n(3n+8)(2+n)(n^2+n+6)^2\left(1-\frac{2}{n^2}\right)^3 \\
&\quad + n^2(n^2+n+6)(n^3+15n^2+48n+52)\left(1-\frac{2}{n^2}\right)^2 \\
&\quad - 2n^3(n+4)(2+n)(n^2+n+6) - 2n^4(3+n)(n+4)^2 \\
&= \frac{2}{n^8}\big(7n^{13}+43n^{12}+13n^{11}-467n^{10}-1320n^9-1140n^8+1364n^7 \\
&\qquad + 4224n^6+3120n^5-2096n^4-5248n^3-2448n^2+1728n+1728\big) > 0.
\end{aligned}
$$

Therefore Proposition 3.6, (3.19), (3.20) and (3.21) imply

$$
G_A\left(n, \frac{n(n+4)}{3}, T\left(n, \frac{n(n+4)}{3}\right)\right) > n+4.
$$

(ii) Next we will prove $G_A\big(n, \frac{n(n+4)}{3}+1, T\big) < n+4$ for any $T$ with $0 < T < 1$.

$$
G_A\left(n, \frac{n(n+4)}{3}+1, T\right) - (n+4) = -\frac{n^2(n+3)(n+1)^4 P_3(n,T)}{81 G_{A,2}\big(n, \frac{n(n+4)}{3}+1, T\big)}, \tag{3.22}
$$

where

$$
\begin{aligned}
P_3(n,T) &= n^2T^4 - 2n(n+3)(n+2)(3n+8)T^3 \\
&\quad - 2(n+3)(n^3+14n^2+46n+48)T^2 + 4n(n+2)(n+3)^2T + 4(n+3)^4.
\end{aligned}
$$

Since $0 < T < 1$, we have

$$
\begin{aligned}
P_3(n,T) &> n^2T^4 - 2n(n+3)(n+2)(3n+8)T^2 - 2(n+3)(n^3+14n^2+46n+48)T^2 \\
&\quad + 4n(n+2)(n+3)^2T^2 + 4(n+3)^4T^2 \\
&= n^2T^4 + 4(n+3)(2n+3)T^2 > 0.
\end{aligned}
$$

Therefore Proposition 3.6 and (3.22) imply $G_A\big(n, \frac{n(n+4)}{3}+1, T\big(n, \frac{n(n+4)}{3}+1\big)\big) < n+4$.
(4) (i) First we will estimate the lower bound of $T\big(n, \frac{n(n+5)}{4}+1\big)$. By Proposition 3.1 (4) (b), $F\big(n, \frac{n(n+5)}{4}+1, T\big)$ is increasing for $T \geq 1 - \frac{1}{2n}$ as a function of $T$. Therefore

$$
F\left(n, \frac{n(n+5)}{4}+1, 1-\frac{1}{n^2}\right) = -\frac{1}{16n^3}(4n^5-8n^4-44n^3-4n^2-3n+5)(n+1)^2 < 0
$$

and $F\big(n, \frac{n(n+5)}{4}+1, T\big) = 0$ imply $T > 1 - \frac{1}{n^2}$. Hence $T\big(n, \frac{n(n+5)}{4}+1\big) > 1 - \frac{1}{n^2}$ holds. By Proposition 3.8, $T(n,x)$ is decreasing as a function of $x$. Hence we have

$$T\left(n,\frac{n(n+5)}{4}\right) > T\left(n,\frac{n(n+5)}{4}+1\right) > 1-\frac{1}{n^2}. \tag{3.23}$$

(ii) Next we will prove $G_A\big(n,\frac{n(n+5)}{4},T\big(n,\frac{n(n+5)}{4}\big)\big) > n+5$.

$$G_A\left(n,\frac{n(n+5)}{4},T\right)-(n+5) = \frac{n^2(n+5)^2}{64}\frac{P_4(n,T)}{G_{A,2}\big(n,\frac{n(n+5)}{4},T\big)}, \tag{3.24}$$

where

$$\begin{aligned}P_4(n,T) = {} & (n^3+2n^2+12n+16)(n^2+n+4)^2T^4\\ & + 2n(3n+10)(n+2)(n^2+n+4)^2T^3\\ & - n^2(n^2+n+4)(n^3-11n^2-52n-76)T^2\\ & - 4(n+5)n^3(n+2)(n^2+n+4)T - 2n^4(n+3)(n+5)^2.\end{aligned}$$

For $T$ with $1 > T > 1-\frac{1}{n^2}$, we have

$$\begin{aligned}P_4(n,T) > {} & (n^3+2n^2+12n+16)(n^2+n+4)^2\left(1-\frac{1}{n^2}\right)^4\\ & + 2n(3n+10)(n+2)(n^2+n+4)^2\left(1-\frac{1}{n^2}\right)^3\\ & + n^2(n^2+n+4)(11n^2+52n+76)\left(1-\frac{1}{n^2}\right)^2 - n^5(n^2+n+4)\\ & - 4(n+5)n^3(n+2)(n^2+n+4) - 2n^4(n+3)(n+5)^2\\ = {} & \frac{1}{n^8}(26n^{13}+182n^{12}+332n^{11}-281n^{10}-1755n^9-2180n^8-5n^7\\ & + 2732n^6+2465n^5-318n^4-1748n^3-752n^2+320n+256) > 0.\end{aligned}$$

Hence Proposition 3.6 and (3.24) imply

$$G_A\left(n,\frac{n(n+5)}{4},T\left(n,\frac{n(n+5)}{4}\right)\right) > n+5.$$

(iii) Next we will prove $G_A\big(n,\frac{n(n+5)}{4}+1,T\big(n,\frac{n(n+5)}{4}+1\big)\big) < n+5$.

$$G_A\left(n,\frac{n(n+5)}{4}+1,T\right)-(n+5) = \frac{n^2(n+4)(n+1)^4P_5(n,T)}{64G_{A,2}\big(n,\frac{n(n+5)}{4}+1,T\big)}, \tag{3.25}$$

where

$$\begin{aligned}P_5(n,T) = & \, n^2(n^2+6n+4)T^4 + 2n(3n+10)(n+4)(n+2)T^3 \\ & - (n+4)(n^3-12n^2-60n-80)T^2 \\ & - 4n(n+2)(n+4)^2T - 2(n+3)(n+4)^3.\end{aligned}$$

For any $T$ with $1 > T > 0$, we have

$$\begin{aligned}P_5(n,T) < & \, n^2(n^2+6n+4)T^2 + 2n(3n+10)(n+4)(n+2)T^2 \\ & - (n+4)(n^3-12n^2-60n-80)T^2 \\ & - 4n(n+2)(n+4)^2T^2 - 2(n+3)(n+4)^3T^2 \\ = & - 16(n+2)^2T^2 < 0.\end{aligned}$$

Hence Proposition 3.6 and (3.25) imply

$$G_A\left(n, \frac{n(n+5)}{4}+1, T\left(n, \frac{n(n+5)}{4}+1\right)\right) < n+5. \qquad \square$$

Proof of Theorem 1.8. Let $X$ be a Euclidean tight 4-design whose weight is constant on $X\backslash\{0\}$. Assume that the case (1) given at the beginning of this section holds. Then by Lemma 3.5 we have $\frac{N}{2} < N_1 \le \frac{n(n+1)}{2}$ and $X_1$ is a 2-distance set. Let $\alpha$ and $\beta$ be the two distances of $X_1$. Assume $\alpha < \beta$. We have $N_1 > \frac{N}{2} > 2n+3$ for any $n \ge 7$. Hence if $n \ge 7$, then there exists a natural number $k$ satisfying $\left(\frac{\alpha}{\beta}\right)^2 = \frac{k-1}{k}$. Let $R = \|u\|^2$, $u \in X_2$. Then $G_A(n, N_1, R) = (2k-1)^2$. By Proposition 3.9, $n+3 < G_A(n,x,T(n,x)) < n+6$ holds for any real number $x$ satisfying $\frac{N}{2}+1 \le x \le \frac{n(n+1)}{2}$ and $G_A(n,x,T(n,x)) = n+4$ for some real number $x$ in the open interval $\left(\frac{n(n+4)}{3}, \frac{(n(n+4))}{3}+1\right)$ and $G_A(n,x,T(n,x)) = n+5$ for some real number $x$ in the open interval $\left(\frac{n(n+5)}{4}, \frac{n(n+5)}{4}+1\right)$. Hence we have either (1) or (2) of the following:

(1) $(2k-1)^2 = n+4$, and $N_1 \in \left(\frac{n(n+4)}{3}, \frac{n(n+4)}{3}+1\right)$.
(2) $(2k-1)^2 = n+5$, and $N_1 \in \left(\frac{n(n+5)}{4}, \frac{n(n+5)}{4}+1\right)$.

Assume (1) holds. Then $n = (2k+1)(2k-3)$ and $\frac{n(n+4)}{3} = \frac{1}{3}(2k+1)(2k-3)(2k-1)^2$. Hence $\frac{n(n+4)}{3}$ is an integer. This contradicts $N_1 \in \left(\frac{n(n+4)}{3}, \frac{n(n+4)}{3}+1\right)$.

Similarly assume (2) holds. Then $n = 4(k^2-k-1)$. Hence $\frac{n(n+5)}{4}$ is an integer. This contradicts $N_1 \in \left(\frac{n(n+5)}{4}, \frac{n(n+5)}{4}+1\right)$.

In the proof of Proposition 3.9 we need the condition $n \ge 7$. Therefore if $n \ge 7$, then the proof of our main theorem is completed. We can prove the nonexistence of a Euclidean tight 4-design satisfying the condition of case (1) for $n \le 6$ by direct calculations. In the following we discuss the cases $2 \le n \le 6$ and give a table of possible distances between the distinct points in $X_1$ and $X_2$. We use one more notation. For a finite subset $Y$ in $\boldsymbol{R}^n$ we define

$$A(Y) = \{\|\mathbf{u} - \mathbf{v}\| \mid \mathbf{u}, \mathbf{v} \in Y, \mathbf{u} \neq \mathbf{v}\}.$$

**Case** $n = 2$**.**
In this case $N = 6$ and $\frac{n(n+1)}{2} = 3 = \frac{N}{2}$. Hence Lemma 3.5 implies that there is no tight 4-design with constant weight on 2 concentric spheres in $\boldsymbol{R}^2$.

**Case** $n = 3$**.**
In this case $N = 10$ and $\frac{n(n+1)}{2} = 6$. Therefore the only possibility is $N_1 = 6$, $N_2 = 4$. Then $X_1$ is a 2-distance set. We have $F(3, 6, R) = 320R^3 + 192R^2 - 432$. Substitute $n = 3$, $N_1 = 6$ in the equation (3.8) we obtain

$$8\left(B - \frac{8R}{3}\right)\big((30R + 45)B - 4(16R^2 + 15R + 9)\big) - \frac{2}{3}F(3, 6, R) = 0.$$

On the other hand, lengths of the edges of a regular tetrahedron on the sphere of radius $r = \sqrt{R}$ are $\sqrt{\frac{8R}{3}}$. Therefore $X_2$ is either a regular tetrahedron on $S_2$, or $X_2$ is a 2-distance set with $A(X_2) = \left\{\sqrt{\frac{8R}{3}}, 2\sqrt{\frac{16R^2+15R+9}{30R+45}}\right\}$.

**Case** $n = 4$**.**
In this case we have $N = 15$, $\frac{n(n+1)}{2} = 10$. Therefore the remaining case is $N_1 = 8, 9, 10$. In these cases $X_1$ is a 2-distance set. If $N_1 = 10$, then $N_2 = 5$. We have $F(4, 10, R) = 900R^3 + 1000R^2 - 1600$ and the equation (3.8) implies

$$(3R + 6)\left(B - \frac{5R}{2}\right)\left(B - \frac{13R^2 + 18R + 8}{6(R + 2)}\right) - \frac{1}{400}F(4, 10, R) = 0.$$

On the other hand, the length of the edges of a regular simplex on the sphere of radius $r = \sqrt{R}$ equals $\sqrt{\frac{5R}{2}}$. Therefore $X_2$ is either a regular simplex or a 2-distance set with $A(X_2) = \left\{\sqrt{\frac{5R}{2}}, \sqrt{\frac{13R^2+18R+8}{6(R+2)}}\right\}$. If $N_1 = 8$ or $9$, then $X_2$ is a 2-distance set.

**Case** $n = 5$**.**
In this case $N = 21$ and $\frac{n(n+1)}{2} = 15$. Hence remaining cases are $N_1 = 11, 12, 13, 14, 15$. Since $2n + 3 = 13$, we can apply the Theorem by Larman-Rogers-Seidel for the case $N_1 \geq 14$. We have

$$\begin{aligned} &8 - G_A(5, x, T) \\ &\quad = -\frac{2940T^2x(21 - x)}{G_{A,2}(5, x, T)}\big((16 - x)(21 - x)T^2 + 2(21 - x)xT + x(x - 5)\big). \end{aligned} \tag{3.26}$$

Since $G_{A,2}(5, x, T) > 0$ by Proposition 3.6 we have $G_A(5, x, T) > 8$ for any $0 < T < 1$. In particular $G_A(5, N_1, R) > 8$. Next, we have

$$11 - G_A(5, x, T) = \frac{120x}{G_{A,2}(5, x, T)}P_6(x, T),$$

where

$$\begin{aligned} P_6(x,T) = (34x-539)(x-21)^2T^4 - 63x(x-21)^2T^3 \\ - 2x(22x-245)(21-x)T^2 + 35x^2(21-x)T + 20x^3. \end{aligned}$$

Since $34x - 539 < 0$ and $0 < T < 1$, we have

$$\begin{aligned} P_6(x,T) &> (34x-539)(x-21)^2T^2 - 63x(x-21)^2T^2 \\ &\quad - 2x(22x-245)(21-x)T^2 + 35x^2(21-x)T^2 + 20x^3T^2 \\ &= 147T^2(137x-1617) > 0. \end{aligned}$$

Hence Proposition 3.6 implies $11 - G_A(5,x,T) > 0$ for any $0 < T < 1$. Therefore $G_A(5,N_1,R) = 9$ or $10$. On the other hand $G_A(5,N_1,R)$ has to be the square of an odd integer. Hence we have $G_A(5,N_1,R) = 9$. We have

$$9 - G_A(5,x,T) = \frac{40x}{G_{A,2}(5,x,T)} P_7(x,T),$$

where

$$\begin{aligned} P_7(x,T) = (83x-1323)(x-21)^2T^4 - 161x(x-21)^2T^3 \\ + 3x(31x-245)(x-21)T^2 - 35x^2(x-21)T + 20x^3. \end{aligned}$$

Since Lemma 3.5 implies $\frac{9}{10} < R < 1$, we obtain

$$\begin{aligned} P_7(14,R) &= -343\big(23R^4 + 322R^3 + 162R^2 - 140R - 160\big) \\ &< -343\left(23\left(\frac{9}{10}\right)^4 + 322\left(\frac{9}{10}\right)^3 + 162\left(\frac{9}{10}\right)^2 - 140 - 160\right) \\ &= -\frac{277995669}{10000} < 0. \end{aligned}$$

Therefore $G(5,14,R) = 9$ is impossible. Similarly $\frac{9}{10} < R < 1$ implies

$$\begin{aligned} P_7(15,R) = -54\Bigg(52\left(R-\frac{9}{10}\right)^4 + \frac{8986}{5}\left(R-\frac{9}{10}\right)^3 + \frac{142493}{25}\left(R-\frac{9}{10}\right)^2 \\ + \frac{1292233}{250}\left(R-\frac{9}{10}\right) + \frac{38317}{625}\Bigg) < 0. \end{aligned}$$

Therefore $G(5,15,R) = 9$ is impossible. Hence the possibilities of $N_1$ are 11, 12 or 13. In those cases both $X_1$ and $X_2$ are 2-distance sets.

**Case $n = 6$.**
In this case $N = 28$ and $\frac{n(n+1)}{2} = 21$. Hence $21 \geq N_1 > \frac{N}{2} = 14$. If $21 \geq N_1 > 15$, then

we can apply the Theorem by Larman-Rogers-Seidel. We have

$$9 - G_A(6, x, T) = -\frac{4608x(28-x)T^2\big((21-x)(28-x)T^2 + 2(28-x)xT + x(x-7)\big)}{G_{A,2}(6, x, T)}.$$

Hence we have $G_A(6, N_1, R) > 9$. Next we have

$$12 - G_A(6, x, T) = \frac{144xP_8(x, T)}{G_{A,2}(6, x, T)},$$

where

$$\begin{aligned} P_8(x, T) = &- (896 - 43x)(28-x)^2T^4 - 80x(28-x)^2T^3 \\ &- 2x(29x - 448)(28-x)T^2 + 48x^2(28-x)T + 27x^3. \end{aligned}$$

Then $0 < T < 1$ implies

$$\begin{aligned} P_8(21, T) &= -82320T^3 + 148176T - 47334T^2 + 343T^4 + 250047 \\ &> -82320 - 47334 + 250047 = 120393 > 0. \end{aligned}$$

If $16 \leq x \leq 20$, then $0 < T < 1$ implies

$$\begin{aligned} P_8(x, T) > &- (896 - 43x)(28-x)^2T - 80x(28-x)^2T \\ &- 2x(29x - 448)(28-x)T + 48x^2(28-x)T + 27x^3T \\ = &\ 784T(59x - 896) > 0. \end{aligned}$$

Therefore Proposition 3.6 implies $12 - G_A(6, N_1, R) > 0$. Hence $G_A(6, N_1, R) = 10$ or 11. Since $G_A(6, N_1, R)$ has to be the square of an integer, this is impossible. Therefore the only possibility for $N_1$ is 15. In this case $X_1$ and $X_2$ are 2-distance sets.

The following table is the list of all the remaining cases for $n \leq 6$.

The remaining cases for $n \leq 6$, no. 1∼no. 10, in the table given above are eliminated by the following arguments. The authors thank Hisakazu Iwai and Makoto Tagami for their help in finishing this calculation. The following explanation was provided by Makoto Tagami.

For a 2-distance set $X$ (of size $m$) in $\boldsymbol{R}^n$, we attach a graph $G = (X, E)$ whose vertex set is $X$ and the edges are the pairs of two vertices with the longer distance. Let $D$ be the adjacency matrix of the graph $G$. For an indeterminate $x$, let $C$ be the $m \times m$ matrix $C = xD + J - I$. Let $L$ be the $(m-1) \times (m-1)$ matrix whose $(i-1, j-1)$-entry is given by $C_{1i} + C_{1j} - C_{ij}$, where $C_{ij}$ means the $(i, j)$-entry of $C$ and $i, j$ are from 2 to $m$. Let us define $D(x) = \det(L)$. The polynomial $D(x)$ is called the *discriminating polynomial.* Then we have the following proposition due to Einhorn and Schoenberg [**9**].

| | $n$ | $N$ | $N_1$ | $r$ | $A(X_1)$ | $A(X_2)$ |
|---|---|---|---|---|---|---|
| no. 1 | 3 | 10 | 6 | 0.9680647814 | 1.261060863, 1.786166652 | $\sqrt{\frac{8}{3}}r,\ 2\sqrt{\frac{16r^4+15r^2+9}{30r^2+45}}$ |
| no. 2 | | | 6 | 0.9680647814 | 1.261060863, 1.786166652 | $\sqrt{\frac{8}{3}}r$ |
| no. 3 | 4 | 15 | 8 | 0.9939261031 | 1.276759120, 1.741496326 | 1.300453366, 1.709766283 |
| no. 4 | | | 9 | 0.9811021675 | 1.254736241, 1.755718569 | 1.333656789, 1.651822070 |
| no. 5 | | | 10 | 0.9657425649 | 1.238414571, 1.765989395 | $\sqrt{\frac{5}{2}}r,\ \sqrt{\frac{13r^4+18r^2+8}{6(r^2+2)}}$ |
| no. 6 | | | 10 | 0.9657425649 | 1.238414571, 1.765989395 | $\sqrt{\frac{5}{2}}r$ |
| no. 7 | 5 | 21 | 11 | 0.9971108543 | 1.271295203, 1.718624969 | 1.282657854, 1.703400226 |
| no. 8 | | | 12 | 0.9911792529 | 1.259432011, 1.726557780 | 1.294778760, 1.679423700 |
| no. 9 | | | 13 | 0.9847128738 | 1.249832383, 1.732878630 | 1.313837759, 1.648459113 |
| no. 10 | 6 | 28 | 15 | 0.9968820164 | 1.265543361, 1.702045669 | 1.278207059, 1.685182835 |

Proposition 3.10 (Einhorn and Schoenberg). *If the graph $G$ is realized in $\boldsymbol{R}^n$ as above as a 2-distnce set in $\boldsymbol{R}^n$ with the 2 distances $\{\alpha, 1\}$, where $\alpha > 1$, then $\alpha^2 - 1$ must be a zero of $D(x)$ with multiplicity $m - n - 1$.*

For any 2-distance set $X_1 \subset \boldsymbol{R}^n$ listed in the table given above, any $n + 2$ $(\leq 8)$ points subset is also a 2-distance set. So we list up all the graphs with at most 8 vertices by using computer software Magma, more precisely by using the library nauty (refer http://cs.anu.edu.au/˜bdm/nauty). For each such graph, we determined the discriminating polynomial explicitly. Then we check that for each possible $\alpha$ which is obtained from each pair of the two distances in $A(X_1)$ in the table given above, we show that $\alpha^2 - 1$ is not a zero of any of such discriminating polynomials $D(x)$. This calculation is rigorous because of the following reason. Our values of $\alpha$ is calculated with the error at most $10^{-8}$, since they are the zeros of very explicit polynomials of degree either 2 or 3. The degree of discriminating polynomial $D(x)$ are at most 7 and the coefficients are at most 280 in absolute values. Therefore in order that $D(\alpha^2 - 1)$ becomes exactly 0, its value must be less than $10^{-4}$. However, this is shown not to be so. This completes the proof of our result for $n \leq 6$.

## 4. Examples of tight 4-designs with nonconstant weight.

So far, we only considered Euclidean tight 4-designs with constant weight. (This was enough to treat tight rotatable 4-designs.) Our method is also applied to study Euclidean tight 4-designs with nonconstant weight, as we have seen in Lemma 1.8. Neumaier and Seidel [**15**] and Delsarte and Seidel [**8**] conjectured that there are no nontrivial Euclidean tight 4-designs even for the nonconstant weight case. (See Conjecture 3.4 in [**15**].) However, we were able to find new nontrivial examples of Euclidean tight 4-designs in $\boldsymbol{R}^2$ with non-constant weight. We will describe these examples below. Currently, we are not aware of other nontrivial examples of tight 4-designs (with nonconstant weight) in $\boldsymbol{R}^n$ for $n \geq 3$, but we suspect that further examples is likely to exist. Anyway, it seems to be very interesting to try to classify Euclidean tight 4-designs also in the case of nonconstant weight. Let $X$ be the set of 6 points in $\boldsymbol{R}^2$ given below:

$$X=\left\{(1,0),\ \left(-\frac{1}{2},\frac{\sqrt{3}}{2}\right),\ \left(-\frac{1}{2},-\frac{\sqrt{3}}{2}\right),\ (-r,0),\ \left(\frac{r}{2},\frac{\sqrt{3}r}{2}\right),\ \left(\frac{r}{2},-\frac{\sqrt{3}r}{2}\right)\right\},$$

where $r$ is any positive real number $r \neq 1$.

$X_1=\left\{(1,0),\ \left(-\frac{1}{2},\frac{\sqrt{3}}{2}\right),\ \left(-\frac{1}{2},-\frac{\sqrt{3}}{2}\right)\right\}$ is the set of vertices of a regular triangle on the unit circle and $X_2=\left\{(-r,0),\ \left(\frac{r}{2},\frac{\sqrt{3}r}{2}\right),\ \left(\frac{r}{2},-\frac{\sqrt{3}r}{2}\right)\right\}$ is the set of vertices of a regular triangle on the circle of radius $r$.

Define a weight function $w$ on $X$ by $w(x)=\begin{cases} 1 & \text{for } x \in X_1 \\ \dfrac{1}{r^3} & \text{for } x \in X_2. \end{cases}$

It is easy to see that $X$ is a 4-design. This means $X$ is a Eucledean tight 4-design. (If $r=1$, then $X$ is also a 4-design. However it is on the unit circle $S^1$.)

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