Classification of all Jacobian elliptic fibrations on certain $K3$ surfaces

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Abstract. In this paper we classify all configurations of singular fibers of elliptic fibrations on the double cover of $\boldsymbol{P}^2$ ramified along six lines in general position.

1. Introduction.

In this article we work over a field of characteristic 0, although most (but not all) of the results hold over fields of any characteristic, not 2 or 3. For example, in Corollary 1.4, Remark 5.9 and Section 6 we need to assume that we work over a subfield of the complex numbers.

In this article we classify all possible bad fiber configurations on Jacobian elliptic fibrations on the $K3$ surface X, which is the minimal model of the double cover of $\boldsymbol{P}^2$ ramified along six lines in 'general position'. When we say that six lines are in 'general position' we mean that the rank of the Néron-Severi group $NS(X)$ is 16.

The strategy we use is purely geometric, and very similar to Oguiso's classification of Jacobian elliptic fibrations on the Kummer surface of the product of two non-isogenous elliptic curves ([**6**]). It seems possible to extend this method to the case of $K3$ surfaces, which are birational to a double cover of $\boldsymbol{P}^2$ ramified along a sextic curve, such that the Néron-Severi group of the $K3$ surface is generated by the (reduced) components of the pull-back of the branch divisor together with all divisors obtained by resolving singularities on the double cover and the pull-back of a general line in $\boldsymbol{P}^2$.

Here we give the full list of possibilities.

THEOREM 1.1. *Let X be as before. Suppose $\pi : X \to \boldsymbol{P}^1$ be an elliptic fibration with positive Mordell-Weil rank. Then the configuration of singular fibers is contained in the following list.*

Class	Configuration of singular fibers		MW-rank
1.1	$I_{10} \, I_2 \, aII \, bI_1$	$2a + b = 12$	4
1.2	$I_8 \, I_4 \, aII \, bI_1$	$2a + b = 12$	4
1.3	$2I_6 \, aII \, bI_1$	$2a + b = 12$	4
1.4	$IV^* \, I_4 \, aII \, bI_1$	$2a + b = 12$	5

Conversely, for each class there exist a, b such that these fibrations occur.

Class	Configuration	$MW(\pi)$	$iii + i_2$	$3iii + 2i_2 + 2ii + i_1$
2.1	II^*	1	6	14
2.2	III^*	$\mathbf{Z}/2\mathbf{Z}$	7	15
2.3	$III^* I_0^*$	1	3	9
2.4	I_6^*	1	4	12
2.5	I_4^*	$\mathbf{Z}/2\mathbf{Z}$	6	14
2.6	$I_4^* I_0^*$	1	2	8
2.7	I_2^*	$(\mathbf{Z}/2\mathbf{Z})^2$	8	16
2.8	$I_2^* I_0^*$	$\mathbf{Z}/2\mathbf{Z}$	4	8
2.9	$2I_2^*$	1	2	8
2.10	$I_2^* \, 2I_0^*$	1	0	8
2.11	$2I_0^*$	$(\mathbf{Z}/2\mathbf{Z})^2$	6	12
2.12	$3I_0^*$	$\mathbf{Z}/2\mathbf{Z}$	2	6

Table 1.   List of possible configurations.

We did not establish the possible values of a and b. By Proposition 6.1 we know that in characteristic zero, generically $a \le 4$ in $1.1, 1.2$ and 1.3, $a \le 5$ in 1.4. We believe that for each of the classes $1.1 - 1.4$ we have that generically $a = 0$. A consequence of Theorem 1.1 is:

COROLLARY 1.2.   *Suppose $\pi_i : X \to \mathbf{P}^2, i = 1, 2$ are morphisms, obtained by taking a minimal resolution of double cover ramified along six lines $\ell_j^{(i)}$ in general position. Then there is a birational automorphism of $\mathbf{P}^2$ mapping $\{\ell_j^{(1)}\}$ to $\{\ell_j^{(2)}\}$.*

This corollary is proved in Section 8.

THEOREM 1.3.   *Let X be as before. Let $\pi : X \to \mathbf{P}^1$ be a Jacobian elliptic fibration with finite Mordell-Weil group. Then the configuration of singular fibers is contained in Table 1. (In this table we list all fiber types, different from III, I_2, II, I_1, the structure of $MW(\pi)$ and two quantities, namely $iii + i_2$ and $3iii + 2i_2 + 2ii + i_1$, where iii means the number of fibers of type III etc.) Conversely, for each class there exists a Jacobian elliptic fibration $\pi : X \to \mathbf{P}^1$ with the given configuration of singular fibers.*

COROLLARY 1.4.   *Let X be as before, and let $\pi : X \to \mathbf{P}^1$ be an elliptic fibration with finite Mordell-Weil group. If the six lines are sufficiently general then the configuration of singular fibers of π is contained in Table 2. Conversely, for each class there exists a Jacobian elliptic fibration $\pi : X \to \mathbf{P}^1$ with the given configuration of singular fibers.*

This corollary is an immediate consequence from Theorem 1.3 and Proposition 6.1.

REMARK 1.5.   Note that in the cases $2.7, 2.8, 2.10$ and 2.11, the fiber configuration in Theorem 1.3 and Corollary 1.4 are the same.

The generality condition is very essential in our strategy: one of the key tools in this article uses the fact that the involution σ on X induced by the double-cover involution acts trivially on the Néron-Severi group $NS(X)$. Our definition of general position

Class	Configuration	$MW(\pi)$	i_2	i_1
2.1	II^*	1	6	2
2.2	III^*	$\boldsymbol{Z}/2\boldsymbol{Z}$	7	1
2.3	$III^*I_0^*$	1	3	3
2.4	I_6^*	1	4	4
2.5	I_4^*	$\boldsymbol{Z}/2\boldsymbol{Z}$	6	2
2.6	$I_4^*I_0^*$	1	2	4
2.7	I_2^*	$(\boldsymbol{Z}/2\boldsymbol{Z})^2$	8	0
2.8	$I_2^*I_0^*$	$\boldsymbol{Z}/2\boldsymbol{Z}$	4	0
2.9	$2I_2^*$	1	2	4
2.10	$I_2^* \, 2I_0^*$	1	0	8
2.11	$2I_0^*$	$(\boldsymbol{Z}/2\boldsymbol{Z})^2$	6	0
2.12	$3I_0^*$	$\boldsymbol{Z}/2\boldsymbol{Z}$	2	2

Table 2. List of possible configurations on a general X.

implies that σ acts trivial on $NS(X)$.

The organization of this article is as follows. In Section 2 we introduce some basic definitions and notation. In Section 3 we start with recalling some facts about curves on $K3$ surfaces. In Section 4 we recall some standard facts concerning singular fibers of elliptic fibrations and some special results on elliptic surfaces. In Section 5 we study the Néron-Severi group of a double cover of $\boldsymbol{P}^2$, ramified along six lines in general position. In Section 6 we give a variant of [**4**, Proposition 2.4.6], which implies that Corollary 1.4 follows from Theorem 1.3. In Section 7 we list all types of possible singular fibers for an elliptic fibration on X and we count the number of pre-images of the six lines contained in each fiber type. In Section 8 we classify all fibrations in which all special rational curves (the pre-images of the six branch lines) are contained in the singular fibers, thereby proving Theorem 1.1. In Section 9 we prove Theorem 1.3.

Every proof of the actual existence of a fibration presented in this article runs as follows: First we give an effective divisor D, with $D^2 = 0$ and such that there exists an irreducible curve $C \subset X$ with $D.C = 1$. It is easy to see that then $|D|$ defines an elliptic fibration $\pi : X \to \boldsymbol{P}^1$ (see Lemma 3.2), that $C \cong \boldsymbol{P}^1$, and that $(\pi|_C)^{-1} : \boldsymbol{P}^1 \to X$ is a section.

In this article all fibrations, sections and components of singular fibers are defined over the field of definition of the six lines.

2. Definitions and Notation.

DEFINITION 2.1.   An *elliptic surface* is a triple (π, X, C) with X a surface, C a curve, π is a morphism $X \to C$, such that almost all fibers are irreducible genus 1 curves and X is relatively minimal, i.e., no fiber of π contains an irreducible rational curve D

with $D^2 = -1$.

We denote by $j(\pi) : C \to \boldsymbol{P}^1$ the rational function such that $j(\pi)(P)$ equals the j-invariant of $\pi^{-1}(P)$, whenever $\pi^{-1}(P)$ is non-singular.

A *Jacobian elliptic surface* is an elliptic surface together with a section $\sigma_0 : C \to X$ to π. The set of sections of π is an abelian group, with σ_0 as the identity element. Denote this group by $MW(\pi)$.

By an *elliptic fibration* on X we mean that we give a surface X a structure of an elliptic surface.

Let $NS(X)$ denote the group of divisors modulo algebraic equivalence. We call $NS(X)$ the Néron-Severi group of X. Denote by $\rho(X)$ the rank of $NS(X)$. We call $\rho(X)$ the *Picard number* of X.

Recall the following theorem.

THEOREM 2.2 (Shioda-Tate ([**8**, Theorem 1.3 and Corollary 5.3])). *Let $\pi : X \to C$ be a Jacobian elliptic surface, such that at least one of the fibers of π is singular. Then the Néron-Severi group of X is generated by the classes of $\sigma_0(C)$, a non-singular fiber, the components of the singular fibers not intersecting $\sigma_0(C)$, and the generators of the Mordell-Weil group. Moreover, let S be the set of points P such that $\pi^{-1}(P)$ is singular. Let $m(P)$ be the number of irreducible components of $\pi^{-1}(P)$, then*

$$\rho(X) = 2 + \sum_{P \in S} (m(P) - 1) + \operatorname{rank}(MW(\pi)).$$

3. Curves on $K3$ surfaces.

In this section we give some elementary, well-known results on curves on $K3$ surfaces. (E.g., see [**1**] or [**9**].)

LEMMA 3.1.   *Suppose D be a smooth curve on a $K3$ surface X. Then*

$$g(D) = 1 + \frac{D^2}{2}.$$

PROOF.   Since the canonical bundle K_X is trivial, the adjunction formula for a divisor on a $K3$ surface is $2p_a(D) - 2 = D^2$ (see [**2**, Proposition V.1.5]). Since $p_a(D) = g(D)$ for a smooth curve, this implies the result.   □

LEMMA 3.2.   *Suppose D is an effective divisor on a $K3$ surface X with $p_a(D) = 1$. Then $|D|$ defines an elliptic fibration $\pi : X \to \boldsymbol{P}^1$. Every effective connected divisor D' such that $D \cdot D' = D'^2 = 0$ is a multiple of a fiber of π.*

PROOF.   From the adjunction formula [**2**, Proposition V.1.5] it follows that $D^2 = 0$. Applying Riemann-Roch [**2**, Theorem V.1.6] yields

$$\dim H^0(X, \mathscr{O}_X(D)) + \dim H^0(X, \mathscr{O}_X(-D)) \geq 2.$$

Since $\dim H^0(X, \mathscr{O}_X(-D)) = 0$ (D is effective), we obtain that $\dim H^0(X, \mathscr{O}_X(D)) > 1$.

This, combined with the fact that $D^2 = 0$, implies that $|D|$ is base-point-free. So we can apply Bertini's theorem [**2**, Theorem II.8.18 and Remark III.7.91], hence there is an irreducible curve F linearly equivalent to D. By Lemma 3.1 this curve has genus 1.

The exact sequence

$$0 \to \mathscr{O}_X \to \mathscr{O}_X(F) \to \mathscr{O}_F \to 0,$$

together with the facts $H^1(X, \mathscr{O}_X) = H^2(X, \mathscr{O}_X(F)) = 0$ and $\dim H^2(X, \mathscr{O}_X) = 1$, gives that $H^1(X, \mathscr{O}_X(F)) = 0$, whence by Riemann-Roch [**2**, Theorem V.1.6] we obtain that the dimension of $H^0(X, \mathscr{O}_X(F))$ equals 2, so "the" morphism associated to $|D|$, $\pi : X \to \boldsymbol{P}^1$ is an elliptic fibration.

Since D' is effective and $D' \cdot D = 0$, we obtain that no irreducible component of D' is intersecting D. Since D' is connected we obtain that $\pi(D')$ is a point. Since $D'^2 = 0$ and $\pi(D')$ is a point, it follows from Zariski's Lemma [**1**, Lemma III.8.2] that D' is a multiple of a fiber. $\qquad\square$

4. Kodaira's classification of singular fibers.

In this section we describe the possible fibers for a minimal elliptic surface $\pi : X \to \boldsymbol{P}^1$. The dual graph associated to a (singular, reducible) curve has a vertex for each irreducible component of the curve and has an edge between two vertices if and only if the two corresponding components intersect.

THEOREM 4.1 (Kodaira). *Let $\pi : X \to \boldsymbol{P}^1$ be an elliptic surface. Then the following types of fibers are possible*:

- I_0: *A smooth elliptic curve.*
- I_1: *A nodal rational curve.* (*The dual graph is $\tilde{A}_0$.*)
- $I_\nu, \nu \geq 2$: *A ν-gon of smooth rational curves.* (*The dual graph is $\tilde{A}_{\nu-1}$.*)
- *II*: *A cuspidal rational curve.* (*The dual graph is $\tilde{A}_0$.*)
- *III*: *Two rational curves, intersecting in exactly one point with multiplicity 2.* (*The dual graph is $\tilde{A}_1$.*)
- *IV*: *Three concurrent lines.* (*The dual graph is $\tilde{A}_2$.*)
- $I_\nu^*, \nu \geq 0$: *The dual graph is of type $\tilde{D}_{4+\nu}$.*
- IV^*, III^*, II^*: *The dual graph is of type $\tilde{E}_6, \tilde{E}_7, \tilde{E}_8$.*

For more on this see for example [**5**, Lecture I], [**10**, Appendix C, Theorem 15.2] or [**11**, Theorem IV.8.2].

In Figure 1 we give the dual graph for some of the fiber types. For a resolution $X \to Y$ of a fixed double covering $\varphi' : Y \to \boldsymbol{P}^2$ ramified along R we define "special" curve as the strict transform of a component of $\varphi'^{-1}(R)$, and a curve is called "ordinary" otherwise. This notion depends heavily on our situation: it gives information on the behavior of the double cover involution on the fiber components. We prove in Section 8 that the notion of special curves does not depend on the choice of a morphism $\pi : X \to \boldsymbol{P}^2$.

Moreover, for all fiber types, except *III* and I_2, one knows which components are special rational curves. In the dual graphs given here a vertex is drawn as a circle if
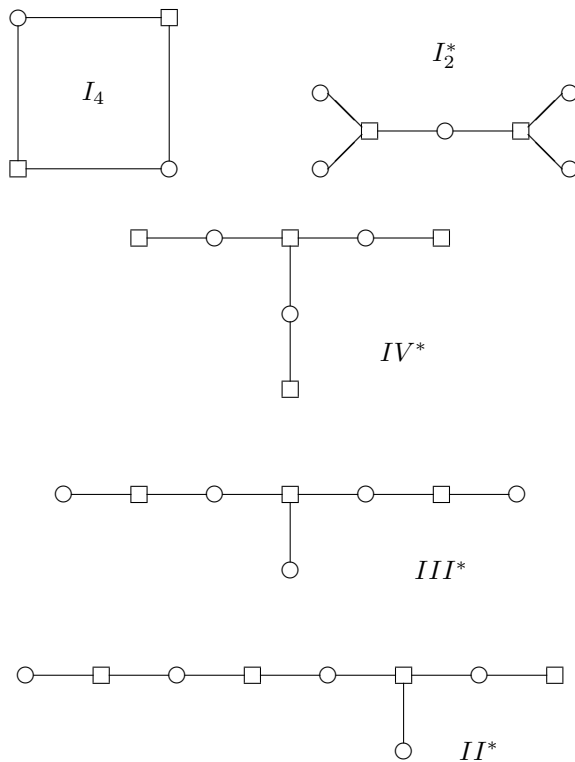
Figure 1.   Examples of dual graphs of singular fibers.

the component is a ordinary rational curve; a vertex is drawn as a square then the corresponding component is a special rational curve.

We now give some invariants associated to the singular fibers of an elliptic surface $\pi : X \to \boldsymbol{P}^1$.

DEFINITION 4.2.   Let $\pi : X \to \boldsymbol{P}^1$ be an elliptic surface. Let P be a point of $\boldsymbol{P}^1$. Define $v_P(\Delta_P)$ as the valuation at P of the minimal discriminant of the Weierstrass model, which equals the topological Euler characteristic of $\pi^{-1}(P)$.

PROPOSITION 4.3.   Let $\pi : X \to \boldsymbol{P}^1$ be an elliptic surface, not birational to a product $E \times \boldsymbol{P}^1$, with E an elliptic curve. Then

$$\sum_{P \in C} v_P(\Delta_P) = 12(p_g(X) + 1).$$

PROOF.   This follows from Noether's formula (see [1, p. 20]). The precise reasoning can be found in [5, Section III.4].   □

REMARK 4.4.   If P is a point on $\boldsymbol{P}^1$, such that $\pi^{-1}(P)$ is singular then $j(\pi)(P)$ and $v_p(\Delta_p)$ behave as in Table 3. For proofs of these facts see [1, p. 150], [11, Theorem IV.8.2] or [5, Lecture 1].

Kodaira type of fiber over P	$j(\pi)(P)$	$v_p(\Delta_p)$	number of components
I_0^*	$\neq \infty$	6	1
I_ν $(\nu > 0)$	∞	ν	$\nu + 1$
I_ν^* $(\nu > 0)$	∞	$6 + \nu$	$\nu + 5$
II	0	2	1
IV	0	4	3
IV^*	0	8	7
II^*	0	10	9
III	1728	3	2
III^*	1728	9	8

Table 3.   Invariants of singular fiber types.

5.   Divisors on double covers of $\boldsymbol{P}^2$ ramified along six lines.

In this section we study the Néron-Severi group of a double cover of $\boldsymbol{P}^2$ ramified along six lines. Most of the results are probably well known to the experts and are likely to be found somewhere in the literature.

NOTATION 5.1.   Fix six distinct lines $L_i \subset \boldsymbol{P}^2$, such that no three of them are concurrent. Denote by $P_{i,j}$ the point of intersection of L_i and L_j.

Let $\varphi' : Y \to \boldsymbol{P}^2$ be the double cover ramified along the six lines L_i. Then Y has 15 double points of type A_1. Resolving these points gives a surface X, with fifteen exceptional divisors and a rational map $\varphi : X \to \boldsymbol{P}^2$. Denote by σ the involution on X induced by the double-cover involution on Y associated to φ'.

Let $\tilde{\boldsymbol{P}}$ be the blow-up of $\boldsymbol{P}^2$ at the points $P_{i,j}$. Then $X/\langle \sigma \rangle = \tilde{\boldsymbol{P}}$, and $\varphi : X \to \boldsymbol{P}^2$ factors through $\psi : X \to \tilde{\boldsymbol{P}}$. Then ψ is a degree 2 cover with branch locus $\tilde{B}$, the strict transform of the six lines L_i.

Denote by $\ell_{i,j} \subset X$ the divisor obtained by blowing up the point of Y above $P_{i,j}$ ($i < j$). Let ℓ_i be such that $2\ell_i$ is the strict transform of $\varphi'^*(L_i)$.

Let $M_{k,m}^{i,j}$ be the line connecting $P_{i,j}$ and $P_{k,m}$ ($i < j, k < m, i < k$ and i, j, k, m pairwise distinct). Let $\mu_{k,m}^{i,j} \subset X$ be the strict transform of $\varphi'^* M_{k,m}^{i,j}$.

With these notations we have the following intersection results.

LEMMA 5.2.   *We have $\ell_i \cdot \ell_j = -2\delta_{i,j}$, $\ell_{i,j}\ell_{k,m} = -2\delta_{i,k}\delta_{j,m}$ and $\ell_i \cdot \ell_{k,m} = \delta_{i,k} + \delta_{i,m}$.*

LEMMA 5.3.   *The curve $\mu_{k,m}^{i,j}$ is irreducible if and only if $M_{k,m}^{i,j}$ does not intersect any of the 13 points $P_{i',j'}$ with $(i', j') \neq (i, j), (k, m)$.*

PROOF.   The curve $\mu_{k,m}^{i,j}$ is reducible if and only if the strict transform $\tilde{M}$ of $M_{k,m}^{i,j}$ on $\tilde{\boldsymbol{P}}$ intersects the branch locus $\tilde{B}$ of $\psi : X \to \tilde{P}$ at every point of intersection with even multiplicity.

Let $n, n' \in \{1, \ldots, 6\} \setminus \{i, j, k, m\}$ such that $n' \neq n$. Let P be the intersection point of $M_{k,m}^{i,j}$ and L_n. Since the intersection multiplicity of $M_{k,m}^{i,j}$ and B at P is even this implies that P is also on $L_{n'}$, hence $P = P_{n,n'}$. Conversely, if $M_{k,m}^{i,j}$ passes through $P_{n,n'}$

then $\tilde{M}$ and $\tilde{B}$ are disjoint, hence the degree morphism from $\mu_{k,m}^{i,j}$ to the rational curve $\varphi'^*(M_{k,m}^{i,j})$ is unramified, hence $\mu_{k,m}^{i,j}$ is reducible. $\qquad\square$

LEMMA 5.4.   *Let N be the subgroup of $NS(X)$ generated by the ℓ_i's, the $\ell_{i,j}$'s and the $\mu_{k,m}^{i,j}$'s. Then $N \cong \mathbf{Z}^{16}$ as abelian groups. Moreover ℓ_1 and the $\ell_{i,j}$'s form a basis for $N \otimes \mathbf{Q}$.*

PROOF.   First we show that rank $N \leq 16$. Since $2\ell_i + \sum_j \ell_{i,j}$ is linearly equivalent to the pull-back of a general line $L \subset \mathbf{P}^2$, we obtain that $\varphi^* L \in N$. Since $\varphi^* L$ is linear equivalent to $\mu_{k,m}^{i,j} + \sum a_{k,m}\ell_{k,m}$ for some choice of $a_{k,m}$, we obtain that the $\ell_2, \ldots, \ell_6, \mu_{k,m}^{i,j}$ are contained in

$$(\mathbf{Z}[\ell_1] \oplus \langle [\ell_{i,j}] \rangle) \otimes \mathbf{Q},$$

hence rank$(N) \leq 16$.

Since all $\ell_{i,j}$ are disjoint and $\ell_{i,j}^2 = -2$, they form a subgroup N' of rank 15 of the Néron-Severi group. An easy computation shows that ℓ_1 is not linearly equivalent to a divisor contained in $N' \otimes \mathbf{Q}$, proving that rank $N = 16$. Since N can be identified with a subgroup of the torsion-free group $H^2(X, \mathbf{Z})$, it is torsion-free. $\qquad\square$

LEMMA 5.5.   *The action of σ on N is trivial.*

PROOF.   This follows from the fact that for all i, j the automorphism $\sigma|_{\ell_i}$ is the identity and σ fixes the curves $\ell_{i,j}$. $\qquad\square$

LEMMA 5.6.   *Let $L \subset \mathbf{P}^2$ be a smooth rational curve, $L \neq L_i$. Suppose $\varphi'^{-1}(L)$ has two components. Then $\rho(X) \geq 17$.*

PROOF.   Let $r_i, i = 1, 2$ be the strict transforms on X of the components of $\varphi'^{-1}(L)$. It suffices to show that $r_1 \notin N \otimes \mathbf{Q}$. Since $\sigma(r_1) = r_2$ and σ acts trivial on N, we obtain that r_1 is linearly equivalent to r_2. From Lemma 3.1 it follows that $r_1^2 = -2$. We have that any irreducible effective divisor $r \neq r_1$ linear equivalent to r_1 satisfies $\#r \cap r_1 = r.r_1 = -2$, hence the only effective irreducible divisor linearly equivalent to r_1 is r_1 itself. From this it follows that $r_1 = r_2$, contradicting our assumption. $\qquad\square$

LEMMA 5.7.   *Suppose that there exists a permutation $\tau \in S_6$ such that $P_{\tau(1),\tau(2)}$, $P_{\tau(3),\tau(4)}$, $P_{\tau(5),\tau(6)}$ are collinear. Then $NS(X)$ has rank at least 17.*

PROOF.   Using Lemma 5.3 the assumption implies that at least one of the $\mu_{k,m}^{i,j}$ is reducible. Hence by Lemma 5.6 we have that rank $NS(X) \geq 17$. $\qquad\square$

ASSUMPTION 5.8.   For the rest of this article assume that the six lines in $\mathbf{P}^2$ are chosen in such a way that rank $NS(X) = 16$. (Hence the $\mu_{k,m}^{i,j}$ are reduced and irreducible.)

REMARK 5.9.   Choosing 6 distinct lines in $\mathbf{P}^2$ gives 4 moduli. Hence the family G of $K3$ surfaces that can be obtained as a double cover of six lines in $\mathbf{P}^2$ is 4-dimensional. From the Torelli theorem for $K3$ surfaces ([7]) it follows that a generic element $X \in G$ satisfies $\rho(X) \leq 16$. By Lemma 5.4 any $X \in G$ has Picard number at least 16. Hence a generic $X \in G$ satisfies $\rho(X) = 16$. Therefore the above assumption makes sense.

DEFINITION 5.10. Let C be an irreducible curve on X different from the ℓ_i's, with $C^2 = -2$. Then C is a called an *ordinary rational curve*. If C is one of the ℓ_i's then C is called a *special rational curve*.

A smooth rational curve C satisfies $p_a(C) = 0$. From Lemma 3.1 it follows that $C^2 = -2$, which explains one of the conditions in the above definition.

Let $B = \ell_1 + \ell_2 + \cdots + \ell_6$. Since $\psi : X \to \tilde{P}$ is ramified along the strict transforms of the L_i, we obtain that the fixed locus X^σ of σ equals B.

LEMMA 5.11. *Let D be an ordinary rational curve. Then $D \cdot B = 2$.*

PROOF. Assumption 5.8 and Lemma 5.4 imply that the involution σ maps D to a linear equivalent divisor. Since $D^2 = -2$ the only effective irreducible divisor linear equivalent to D, it D itself. Hence σ acts non-trivially on $D \cong \mathbf{P}^1$, from which it follows that there are two fixed points. $\square$

LEMMA 5.12. *Let D_1 and D_2 be two ordinary rational curves. Then $D_1 \cdot D_2 \equiv 0 \bmod 2$.*

PROOF. The proof of [**6**, Lemma 1.6] carries over in this case. $\square$

6. A special result.

PROPOSITION 6.1. *Suppose X is a double cover of $\mathbf{P}^2$ ramified along six lines. If the position of the six lines is sufficiently general then for every elliptic fibration $\pi : X \to \mathbf{P}^1$ the total number of fibers of π of type II, III or IV is at most $\operatorname{rank}(MW(\pi))$.*

PROOF. As explained in Remark 5.9 the general member of the family of all double covers of $\mathbf{P}^2$ ramified along 6 lines has Picard number 16, hence

$$20 = h^{1,1}(X) = 4 + \rho_{tr}(\pi) + \operatorname{rank} MW(\pi).$$

Using [**4**, Proposition 4.6] we obtain that

$$4 \leq \dim\{[\psi : X \to \mathbf{P}^1] \in \mathscr{M}_2 | C(\psi) = C(\pi)\}$$
$$= h^{1,1} - \rho_{tr}(\pi) - \#\{\text{fibers of type } II, III, IV\}$$
$$\leq 4 + \operatorname{rank} MW(\pi) - \#\{\text{fibers of type } II, III, IV\},$$

which gives the desired inequality. $\square$

REMARK 6.2. This proposition can be used to determine the number of fibers of type I_1 and I_2 in several cases of Oguiso's classification [**6**]. However, Oguiso classified all Jacobian elliptic fibrations on the Kummer surface of the product of two non-isogenous elliptic curves, while if one wants to apply Proposition 6.1 one obtains only the classification of Jacobian elliptic fibrations on a Kummer surface of a product of two general elliptic curves.

7.   Possible singular fibers.

In this section we classify all elliptic fibrations on the double cover of $\boldsymbol{P}^2$ ramified along six fixed lines ℓ_i in general position, where general position means that $\rho(X) = 16$.

DEFINITION 7.1.   By a *simple component D* of a fiber F we mean an irreducible component D of F such that D occurs with multiplicity one in F.

PROPOSITION 7.2.   *Let X be as before. Let $\pi : X \to \boldsymbol{P}^1$ be an elliptic fibration with a section. Then the Kodaira type of the singular fiber is contained in the following list. For each Kodaira type we list the number of components which are special rational curves, the number of simple components which are special rational curves, and the number of simple components which are ordinary rational curves, are contained in the following list*:

Type	#Cmp. special rational curves	#Simple cmp. special rational curves	#Simple cmp. ordinary rational curves
I_1	0	0	0
I_2	0 *or* 1	0 *or* 1	2 *or* 1
I_4	2	2	2
I_6	3	3	3
I_8	4	4	4
I_{10}	5	5	5
I_0^*	1	0	4
I_2^*	2	0	4
I_4^*	3	0	4
I_6^*	4	0	4
II	0	0	0
III	0	0	2
IV^*	4	3	0
III^*	3	0	2
II^*	4	0	1.

SKETCH OF PROOF.   First of all we prove that no fiber F of type $I_{2k+1}, k > 0$ exists. Such a fiber F is a $2k + 1$-gon of rational curves. Since two special rational curves do not intersect, and two ordinary rational curves have even intersection number (Lemma 5.12), it follows that every ordinary rational curve intersects two special rational curves, and each special rational curve intersects two ordinary rational curves. This forces the number of components in an n-gon to be even. Hence I_{2k+1} does not occur. For the same reasons no fiber of type IV or type I_{2k+1}^* occurs.

Since there at most 6 special nodal curves, no fiber of type $I_{2k}, k > 6$ or of type I_{2k}^*, $k > 5$ occurs.

We prove now that no fiber F of type I_{12} exists. If it would exist, then this fiber would contain all special rational curves. Hence the zero section is an ordinary rational curve Z. From Lemma 5.11 it follows that then $1 = Z \cdot F \geq Z \cdot B = 2$, a contradiction. The non-existence of I_{10}^* follows similarly. A fiber of type I_8^* has four ordinary rational curves R_i with the property that R_i intersects only one other fiber component. Moreover,

the R_i are the only simple components. Hence the zero-section Z intersects one of the R_i, say R_1, and Z has to be a special rational curve, by Lemma 5.12. The curve $R_i, i \neq 1$ intersect a special rational curve not contained in the fiber and different from Z, hence there are at most 4 special rational curves contained in F. Using Lemma 5.12 one obtains easily that F contains at least 5 special nodal curves, a contradiction.

Let D be a rational curve intersecting three other disjoint rational curves $D_i, i = 1, \ldots, 3$. If D were ordinary, then by Lemma 5.12 the curves D_i would be special. This would imply that $D \cdot B \geq 3$, contradicting Lemma 5.11. Hence D is a special rational curve. This observation is the key ingredient for determining the number of special components in a singular fiber of $*$-type.

We prove that a fiber F of type III does not contain special rational curves. Suppose D would be a component of F, which is a special rational curve. Let R be other component of F. Then R is an ordinary rational curve. Since $C.D = 2$ and $C.B = 2$, we have that $\operatorname{Supp} C \cap B$ is the unique intersection point of C and D. This would imply that the fixed locus of the involution σ is one point, this contradicts the fact that this number is two, hence D is not a special rational curve. $\qquad\square$

8. Possible configurations I.

In this section we study all Jacobian elliptic fibrations $\pi : X \to \boldsymbol{P}^1$ having the property that all special rational curves are fiber components. This section yields the proofs of Theorem 1.1 and Corollary 1.2.

PROPOSITION 8.1. *Let X be as before, in particular* $\operatorname{rank} NS(X) = 16$. *Let $\pi : X \to \boldsymbol{P}^1$ be a Jacobian elliptic fibration. Suppose all special rational curves are contained in the fibers of π. Then one of the following occurs*:

Singular fibers		Mordell-Weil rank
$I_{10}\ I_2\ aII\ bI_1$	$2a + b = 12$	4
$I_8\ I_4\ aII\ bI_1$	$2a + b = 12$	4
$2I_6\ aII\ bI_1$	$2a + b = 12$	4
$IV^*\ I_4\ aII\ bI_1$	$2a + b = 12$	5

Conversely, for each case there exist a, b such that these fibration do occur.

PROOF. Since the zero section is a rational curve and all special rational curves are contained in some fibers, we have that the zero section Z is an ordinary rational curve. From Lemma 5.12 it follows that if Z intersects a reducible fiber, then it intersects in a simple component, which is a special rational curve. From Lemma 5.11 and the fact that special rational curves are smooth, it follows that there are precisely two reducible fibers. Using Proposition 7.2 we obtain that the possible reducible fibers are IV^* and I_{2k}, with $1 \leq k \leq 5$.

Since there are six special rational curves, the above possibilities are the only ones.

The quantity $2a + b$ can be determined using Noether's formula (Theorem 4.3), the Mordell-Weil rank can be obtained using the Shioda-Tate formula (Theorem 2.2).

It remains to prove the existence of the remaining four cases.

Let $k \in \{3, 4, 5\}$. To prove the existence of $I_{2k}I_{12-2k}aIIbI_1$, take $D = \ell_1 + \ell_{1,2} +$

$\ell_2 + \ell_{2,3} + \cdots + \ell_k + \ell_{1,k}$. If $k = 3, 4$ then $D_1 = \ell_{k+1} + \ell_{k+1,k+2} + \cdots + \ell_6 + \ell_{k+1,6}$ is an effective divisor with $D_1^2 = 0$ and $D.D_1 = 0$. From Lemma 3.2 it follows that the fibration associated to $|D|$ has D and D_1 as fibers. They are of type I_{2k} and I_{12-2k}.

If $k = 5$, then from Lemma 3.2 it follows that $|D|$ defines a fibration with a I_{10} fiber, which proves the existence of the first case.

To prove the existence of the fibration with the IV^* and I_4 fiber, take $D = \ell_1 + 2\ell_{1,2} + 3\ell_2 + 2\ell_{2,3} + \ell_3 + 2\ell_{2,4} + \ell_4$. Then the fibration associated to $|D|$ has a fiber of type IV^*, yielding the final case.

It remains to prove that the above fibrations are Jacobian. In all cases one easily shows that $D.\ell_{1,6} = 1$, hence $\ell_{1,6}$ is a section. $\square$

The following result is a consequence of Theorem 1.1.

COROLLARY 8.2.   *The notion of special curves does not depend on the morphism* $X \to \boldsymbol{P}^2$.

PROOF.   We will use our classification of Jacobian elliptic fibrations to identify the six special rational curves.

By Theorem 1.1 there is a fibration on X having a fiber F of type IV^*. By a reasoning as in the proof of Proposition 7.2 the three "end-components" E_i have to be special rational curves (see also Figure 1). There exists a unique component E_4 of F not intersecting the E_i and by Proposition 7.2 and Lemma 5.12 this component is also a special rational curve.

There is a section, which image S is not a special rational curve. By Lemma 5.12 this curve S intersects the fiber F' of type I_4 in a special rational curve E_5. There is a unique component E_6 of F' not intersecting E_5. Using Lemma 5.12 again we obtain that E_6 is a special rational curve. $\square$

This enables to prove Corollary 1.2:

PROOF OF COROLLARY 1.2.   Corollary 8.2 implies that the morphism from X to $\tilde{\boldsymbol{P}}$, the blow-up of $\boldsymbol{P}^2$ in 15 points, is unique up to a geometric automorphism. Combining this with the fact that any two choices of blow-down morphism $\tilde{\boldsymbol{P}} \to \boldsymbol{P}^2$ differ by a birational automorphism of $\boldsymbol{P}^2$, yields the proof. $\square$

9.   Possible configurations II.

In this section we consider fibrations on X such that at least one of the special rational curves is not a component of a singular fiber. This section yields the proof of Theorem 1.3.

Let $\pi : X \to \boldsymbol{P}^1$ be a Jacobian elliptic fibration, such that at least one of the special rational curves is not contained in a fiber.

LEMMA 9.1.   *The group* $MW(\pi)$ *is finite.*

PROOF.   The proof of [**6**, Lemma 2.4] carries over. $\square$

LEMMA 9.2.   *Suppose one of the singular fibers of* π *is of type* I_{2k}^*, III^* *or* II^*. *Then all sections are special rational curves.*

PROOF. A section intersects a reducible fiber in a simple component with multiplicity one. By Proposition 7.2 we know that all simple components in the above mentioned singular fibers are ordinary rational curves. From Lemma 5.12 we know that two ordinary rational curves intersect with even multiplicity, hence every section is a special rational curve. □

LEMMA 9.3. *Let* $\pi : X \to \boldsymbol{P}^1$ *be as above. Then the only fibers of type* I_ν *are of type* I_1 *and* I_2. *Moreover no fiber of type* I_2 *or type III contains a special rational curve as a component.*

PROOF. From Proposition 7.2 we know that every fiber of type I_ν, with $\nu > 1$, is of type I_{2k}, $k \geq 1$.

Without loss of generality, we may assume that ℓ_1 is not contained in a fiber of π, hence ℓ_1 intersects every fiber. Let F be a singular fiber of type $I_{2k}, k \geq 1$ or of type III, containing a special rational curve. We know that ℓ_1 intersects a reducible fiber in an ordinary rational curve, say D. If F is of type I_{2k}, $k > 1$ then D intersects two other components, and by Lemma 5.12 these components are special rational curves. Let $B = \sum \ell_i$. Then $D \cdot B \geq 3$, contradicting Lemma 5.11. Hence it is not possible to have a fiber of type I_{2k}, $k > 1$ containing a special rational curve. By Proposition 7.2 every singular fiber of type I_{2k}, $k > 1$ contains a special nodal curve, hence such a fiber does not occur.

If $k = 1$ or the fiber is of type III, then D intersects the other component twice, and, by assumption, this component is special. Hence $D \cdot B \geq 3$, contradicting Lemma 5.11. □

From now on we study the possibilities for the fibration π. We distinguish eight cases. In each case we suppose that π has a fiber F of a certain given Kodaira type. In each case we study which other singular fibers can occur. Then we prove the existence of the configuration by giving a divisor D such that the linear system $|D|$ gives the desired fibration. First, we determine all the types of fiber containing a special rational curve. Observe that by Proposition 7.2 all singular fibers, not containing special rational curves, are of type III, I_2, II or I_1. Using the Shioda-Tate formula (Theorem 2.2) and Noether's formula (Theorem 4.3) we can determine the quantities $iii + i_2$ and $3iii + 2i_2 + 2ii + i_1$.

For existence proofs we use Proposition 7.2.

DEFINITION 9.4. An *end-component* C of a fiber F is a component C intersecting the support of $\overline{F \setminus C}$ transversally in one point.

In the sequel we use that if an end-component of a fiber is a ordinary rational curve, then this component has to intersect a special rational curve, not contained in the fiber. This follows immediately from Lemma 5.11.

In order to determine the Mordell-Weil group, we use Lemma 9.2.

9.1. II^*. In this case four special rational curves are contained in F. Using Proposition 7.2 it follows that there are three end-components E_1, E_2, E_3, of which E_1 and E_2 are ordinary rational curves, and E_1 is a simple component. This means that the zero-section is a special nodal curve, say D_1, and E_2 intersects a special nodal curve, not contained in F and different from D_1, say D_2. This yields that $F \cdot D_1 = 1$, $F \cdot D_2 = 3$.

All other fibers are of type II, III, I_1, I_2, yielding case 2.1. For example take $D = \ell_{1,5} + 2\ell_1 + 3\ell_{1,2} + 4\ell_2 + 5\ell_{2,3} + 6\ell_3 + 4\ell_{3,4} + 2\ell_4 + 3\ell_{3,6}$. Then ℓ_5 is a section and ℓ_6 is a trisection.

9.2. III^*. In this case three special rational curves are contained in F. At least two special rational curves have positive intersection number with F.

Suppose that two special rational curves are sections, then the third special rational curve is a multisection and all other fibers are of type II, III, I_1, I_2. For example, take $D = \ell_{3,4} + 2\ell_3 + 3\ell_{1,3} + 4\ell_1 + 2\ell_{1,5} + 3\ell_{1,2} + 2\ell_2 + \ell_{2,6}$. Then ℓ_4 and ℓ_6 are sections and ℓ_5 is a two-section. This gives the case 2.2.

If precisely one special rational curve is a section, then there is a special rational which is a multisection, and one special rational curve which is contained in some singular fiber. Since the multisection and the section intersect the fiber four times, this fiber cannot be of type III or I_2, so it is of type I_0^*. All other fibers are of type II, III, I_1, I_2. For example take $D = \ell_{3,4} + 2\ell_3 + 3\ell_{1,3} + 4\ell_1 + 2\ell_{1,5} + 3\ell_{1,2} + 2\ell_2 + \ell_{2,5}$. This gives case 2.3. In this case ℓ_4 is a section and ℓ_5 is a tri-section.

9.3. IV^*. In this case F contains four special rational curves. The other two special rational curves do not intersect this fiber, so they are components of other singular fibers, hence all special rational curves are components, contradicting our assumptions.

9.4. I_6^*. In this case F has four special rational curves as components. One special rational curve is a section and one a multisection. All other fibers are of type III, I_2, II and I_1. For example take $D = \ell_{1,5} + \ell_{1,6} + 2\ell_1 + 2\ell_{1,2} + 2\ell_2 + 2\ell_{2,3} + 2\ell_3 + 2\ell_{3,4} + 2\ell_4 + \ell_{4,5} + \mu_{2,6}^{1,3}$. The curve ℓ_6 is a section, the curve ℓ_5 is a tri-section. This gives case 2.4.

9.5. I_4^*. In this case F has three special rational curves as components. Either three or two of the other special curves intersect any fiber.

If all three special curves intersect any fiber then all other singular fibers are of type III, I_2, II, I_1. For example take $D = \ell_{1,5} + \ell_{1,4} + 2\ell_1 + 2\ell_{1,2} + 2\ell_2 + 2\ell_{2,3} + 2\ell_3 + \ell_{3,5} + \ell_{3,6}$. In this case ℓ_4 and ℓ_6 are sections and ℓ_5 is a two-section. This gives case 2.5.

If two special curves intersect any fiber, then one of the special curves is again a component of a singular fiber. Since the two other special rational curves intersect any fiber four times, this fiber cannot be of type I_2 or III, so it is a fiber of type I_0^*. For example take $D = \ell_{1,5} + \ell_{1,4} + 2\ell_1 + 2\ell_{1,2} + 2\ell_2 + 2\ell_{2,3} + 2\ell_3 + \ell_{3,5} + \mu_{2,4}^{1,6}$. In this case, ℓ_4 is a section and ℓ_5 is a trisection. The curve ℓ_6 is a component of the I_0^*-fiber. This gives case 2.6.

9.6. I_2^*. In this case F has 2 special curves as components.

If 4 special curves intersect any fiber then all other singular fibers are of type III, I_2, II, I_1. For example take

$$D = \ell_{1,3} + \ell_{1,4} + 2\ell_1 + 2\ell_{1,2} + 2\ell_2 + \ell_{2,5} + \ell_{2,6}.$$

Then $\ell_i, i = 3, \ldots, 6$ are sections. From [**5**, Corollary VII.3.3] it follows that $MW(\pi) \not\cong \mathbf{Z}/4\mathbf{Z}$. This gives case 2.7.

If 3 special curves intersect any fiber then there is one special rational curve D_1

not intersecting the fiber and not contained in the F. From this it follows that D_1 is a component of a fiber of type I_0^*. All other fibers are of type II, III, I_1, I_2. For example take $D = \ell_{1,3} + \ell_{1,4} + 2\ell_1 + 2\ell_{1,2} + 2\ell_2 + \ell_{2,4} + \ell_{2,5}$. Then ℓ_3 and ℓ_5 are sections. The curve ℓ_4 is a two-section and ℓ_6 is a component of the I_0^*-fiber. This gives case 2.8.

If 2 special curves intersect any fiber then there are two remaining special rational curves D_1, D_2, with $F \cdot D_1 = F \cdot D_2 = 0$. If D_1 and D_2 are components of the same fiber, then this fiber is of type I_2^*. For example take $D = \ell_{1,3} + \ell_{1,4} + 2\ell_1 + 2\ell_{1,2} + 2\ell_2 + \ell_{2,4} + \mu_{3,6}^{1,5}$. Then ℓ_5, ℓ_6 and $\ell_{5,6}$ are components of another singular fiber, which has to be of type I_2^*. Then ℓ_3 is a section and ℓ_4 is a tri-section. This gives case 2.9.

If D_1 and D_2 are in different fibers then they are both components of fibers of type I_0^*. For example take $D = \ell_{1,3} + \ell_{1,4} + 2\ell_1 + 2\ell_{1,2} + 2\ell_2 + \ell_{2,4} + C'$, with C' the strict transform of $\varphi'^{-1}(C)$, with C the conic through $P_{1,3}P_{1,5}P_{2,3}P_{4,6}P_{5,6}$. The curves $\ell_{3,5}, \ell_{3,6}, \ell_5, \ell_6$ are components of some singular fibers. Since $F \cdot \ell_3 = 1$, they are components of two distinct fibers. The curve ℓ_3 is a section, the curve ℓ_4 is a tri-section. This gives case 2.10.

9.7. I_0^*. In this case F contains only one special rational curve. Each ordinary component intersects only one special rational curve not contained in F. Hence there are at most 4 special curves intersecting F.

If there are four special rational curves D_i, with $F \cdot D_i > 0$, then the remaining special rational curve is a component of a fiber, which has to be of type I_0^*. There are still six components of fibers left. The only way to arrange them is with 6 I_2 fibers. For example take $D = 2\ell_1 + \sum_{k=2}^{5} \ell_{1,k}$, then $D' = 2\ell_6 + \sum_{k=2}^{5} \ell_{k,6}$ is another singular fiber. The rational curves $\ell_k, k = 2, \ldots, 5$ are sections. From [**5**, Corollary VII.3.3] it follows that $MW(\pi) \not\cong \mathbf{Z}/4\mathbf{Z}$. This gives case 2.11.

If there are three special rational curves D_i, with $F \cdot D_i > 0$, then the two other special rational curves are components of some singular fiber. If both components are in the same fiber then that fiber is of type I_2^* (which is handled above), otherwise the fibers containing the special rational curves are of type I_0^*. So we have in total 3 fibers of type I_0^*. All other fibers are of type II, III, I_1, I_2. For example take $D = \mu_{5,6}^{2,3} + \ell_{1,4} + \ell_{1,5} + \ell_{1,6} + 2\ell_1$. Then none of $\ell_2, \ell_{2,4}, \ell_{2,5}, \ell_{2,6}, \ell_3, \ell_{3,4}, \ell_{3,5}, \ell_{3,6}$ intersect a fiber. Hence they are components of two I_0^* fibers. The curves ℓ_5 and ℓ_6 are sections. The curve ℓ_4 is a two-section. This gives case 2.12.

If there are two special rational curves D_i, with $F \cdot D_i > 0$, then the three other special curves D_i' are components of some singular fibers.

If all D_i''s are contained in the same fiber, then that fiber is either of type III^* (which we already handled) or of type I_4^* (which we also handled above).

If all D_i''s are contained in two singular fibers then we obtain one fiber of type I_2^* and one of I_0^*. This case we handled above.

If all D_i''s are contained in three singular fibers then all three fibers are of type I_0^*, which is impossible, since then the Picard number of X would be at least 18.

9.8. ONLY I_2 AND III. From the Shioda-Tate formula (Theorem 2.2) it follows that there are $\rho(X) - 2 - \text{rank}\, MW(\pi) = 14$ singular fibers of type I_2 or III. From Noether's formula 4.3 it follows that then $24 = 12p_g(X) + 12 = \sum v_p(\Delta_p) \geq 2 \cdot 14 = 28$. A contradiction. Hence this does not occur.

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