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Weighted inequalities for holomorphic functional calculi of operators with heat kernel bounds

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Abstract. Let $\mathscr{X}$ be a space of homogeneous type. Assume that L has a bounded holomorphic functional calculus on $L^2(\Omega)$ and L generates a semigroup with suitable upper bounds on its heat kernels where Ω is a measurable subset of $\mathscr{X}$. For appropriate bounded holomorphic functions b, we can define the operators $b(L)$ on $L^p(\Omega)$, $1 \leq p \leq \infty$. We establish conditions on positive weight functions u, v such that for each p, $1 < p < \infty$, there exists a constant c_p such that

$$\int_\Omega |b(L)f(x)|^p u(x) d\mu(x) \leq c_p \|b\|_\infty^p \int_\Omega |f(x)|^p v(x) d\mu(x)$$

for all $f \in L^p(v d\mu)$.

Applications include two-weight L^p inequalities for Schrödinger operators with non-negative potentials on $\boldsymbol{R}^n$ and divergence form operators on irregular domains of $\boldsymbol{R}^n$.

1. Introduction.

An unbounded linear operator L in a Banach space E is said to be of type ω (where $0 \leq \omega < \pi$) if the spectrum of L is contained in the closed sector $S_\omega = \{\lambda \in \boldsymbol{C} : |\arg\lambda| \leq \omega\}$, and for all $\nu > \omega$, L satisfies the resolvent bounds

$$\|(L - \lambda I)^{-1}\| \leq c_\nu |\lambda|^{-1}, \quad \lambda \notin S_\nu.$$

We can define a functional calculus of L for suitable holomorphic functions b on S_ν^0, the interior of S_ν, when $\nu > \omega$ [**Mc**], [**CDMY**]. Examples of the operators $b(L)$ are the semigroups e^{-zL} and the complex powers L^z (for suitable complex values z). We say that L has a bounded H_∞ functional calculus if the operators $b(L)$ are bounded on E and satisfy the bound

$$\|b(L)\| \leq c\|b\|_\infty$$

for all $b \in H_\infty(S_\nu^0)$ where $H_\infty(S_\nu^0)$ denotes the space of all bounded holomorphic functions on S_ν^0. When the Banach space E is an L^p space, the existence of a bounded

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holomorphic functional calculus of L implies a number of interesting properties of L. These include square function estimates which play an important role in harmonic analysis (see [**CDMY**]), and the maximal regularity property which is useful in estimates for non-linear differential equations [**DV**].

Contributions to understanding the conditions on L for $b(L)$ to be bounded come from the work of Stein [**St1**], Seeley [**Se**], Cowling, Doust, McIntosh, and Yagi [**Mc**], [**CDMY**], [**Y**] and many others. For L an elliptic partial differential operator with smooth coefficients on a domain with smooth boundary, it was known in 1970's that the purely imaginary powers L^{is}, $s \in \boldsymbol{R}$, are bounded on L^p spaces for $1 < p < \infty$ ([**Se**]). Under the same assumptions, L^p boundedness of L^{is} was extended to $b(L)$ for every $b \in H_\infty(S_\nu^0)$ in [**Du**].

Let Ω be a measurable subset of a space of homogeneous type $\mathscr{X}$. If we assume that the operator L has a bounded holomorphic functional calculus in $L^2(\Omega)$ and generates a semigroup with suitable upper bounds on its heat kernels, then L has a bounded holomorphic functional calculus $b(L)$ on $L^p(\Omega)$ for $1 < p < \infty$. See [**DR**], Theorem 3.1 when Ω itself is a space of homogeneous type and [**DM2**], Theorem 6 when Ω is a measurable subset without regularity conditions on its boundary.

Note first that when L has only upper bounds on its heat kernels, the operator $b(L)$ which can be realised as a singular integral operator, may not be a Calderón-Zygmund operator. The reason is that without assumption on the smoothness of the space variables of the heat kernels, the kernel of $b(L)$ may not be Hölder continuous or even may not satisfy the Hörmander condition. Secondly, Ω may not satisfy the doubling condition, hence it is not a space of homogeneous type. To obtain L^p boundedness of $b(L)$, a new method beyond the standard Calderón-Zygmund theory was developed in [**DR**], [**DM2**].

A natural question is to obtain weighted norm inequalities for $b(L)$. See, for examples, Chapter VI of [**GR**] for a discussion on weighted norms of singular operators and [**Ma**] for L^p boundedness of $b(L)$ with A_p weights. For the class of Muckenhoupt A_p weights, see Chapter V of [**St2**]. In this paper, we study the more general problem of the two-weight inequality for $b(L)$, that is, for $1 < p < \infty$,

$$\int_\Omega |b(L)f(x)|^p u(x) d\mu(x) \leq c_p \|b\|_\infty^p \int_\Omega |f(x)|^p v(x) d\mu(x) \tag{1.1}$$

for all $f \in L^p(vd\mu)$ where $u(x), v(x)$ are μ-a.e. positive functions. More precisely, we will give an answer to the following problem:

Find sufficient conditions on $0 \leq v < \infty$ *μ-a.e.* (*resp.* $u > 0$ *μ-a.e.*) *such that* (1.1) *is satisfied by some* $u > 0$ *μ-a.e.* (*resp.* $0 \leq v < \infty$ *μ-a.e.*).

In the case when the heat kernels of L satisfy appropriate pointwise upper bounds and possess Hölder continuity on the space variables, this problem is solved by Theorem 3.4 of [**GM**], because the operators $b(L)$ are standard Calderón-Zygmund operators. See [**Du**] and [**DM1**]. However, the method in Theorem 3.4 of [**GM**] does not work for the operators whose heat kernels satisfy pointwise bounds but not Hölder bounds. We will use the approach which is previously developed in [**DR**], [**DM2**] and [**ADM**] to estimate singular integral operators on L^p spaces in which the usual Hörmander condition was replaced by a weaker condition which involves a generalised approximation to the identity. See Definition 2.1 below. This approach requires no assumption on regularity of space variables and this allows us to obtain the desired results.

The paper is organized as follows. In Section 2 we recall some definitions regarding spaces of homogeneous type, generalised approximations to the identity and singular integral operators. In Section 3 we will obtain certain estimates on the kernel of $b(L)$ and also a representation of $b(L)$. In Section 4, we use a vector-valued theorem of [**ADM**] to prove the weak type (1,1) estimate and L^p boundedness of $b(L)$ for $1 < p < \infty$. Our main result is Theorem 4.8 which gives weighted norm inequalities for $b(L)$. We conclude this article by giving some applications to weighted inequalities for holomorphic functional calculi of Schrödinger operators on $\boldsymbol{R}^n$ and divergence form operators on irregular domains of $\boldsymbol{R}^n$.

2. Preliminaries.

A space of homogeneous type $(\mathscr{X}, d, \mu)$ is a set $\mathscr{X}$ endowed with a quasi-metric d and a non-negative Borel measure μ such that the doubling condition

$$\mu(B(x, 2r)) \leq C_1 \mu(B(x, r)) < \infty$$

holds for all $x \in \mathscr{X}$ and $r > 0$, where $B(x, r) = \{y \in \mathscr{X} : d(y, x) < r\}$ is the ball with center x and radius r. Since d is a quasi-metric, there exists $C_2 \geq 1$ such that

$$d(x, y) \leq C_2 (d(x, z) + d(z, y)) \quad \text{for all } x, y, z \in \mathscr{X}.$$

See, for example, Chapter 3 of [**CW**].

Note that the doubling property implies the following strong homogeneity property,

$$\mu(B(x, \lambda r)) \leq c \lambda^n \mu(B(x, r)) \tag{2.1}$$

for some $c, n > 0$ uniformly for all $\lambda \geq 1$. The parameter n is a measure of the dimension of the space. There also exist c and $N, 0 \leq N \leq n$ so that

$$\mu(B(y, r)) \leq c \left(1 + \frac{d(x, y)}{r}\right)^N \mu(B(x, r)) \tag{2.2}$$

uniformly for all $x, y \in \mathscr{X}$ and $r > 0$. Indeed, the property (2.2) with $N = n$ is a direct consequence of the triangle inequality of the quasi-metric d and the strong homogeneity property. In the cases of Euclidean spaces $\boldsymbol{R}^n$ and Lie groups of polynomial growth, N can be chosen to be 0.

The following concept of generalised approximations to the identity was introduced in [**DM2**].

Definition 2.1. A family of operators $\{A_t, t > 0\}$ is said to be a generalised approximation to the identity if, for every $t > 0$, A_t is represented by kernels $a_t(x, y)$ in the following sense: for every function $f \in L^p(\mathscr{X}), 1 \leq p \leq \infty$,

$$A_t f(x) = \int_{\mathscr{X}} a_t(x,y) f(y) d\mu(y);$$

and the following condition holds:

$$|a_t(x,y)| \le h_t(x,y)$$

for all $x, y \in \mathscr{X}$ where $h_t(x,y)$ is given by

$$h_t(x,y) = \frac{1}{\mu(B(x, t^{1/m}))} g(d(x,y)^m t^{-1}), \tag{2.3}$$

in which m is a positive fixed constant and g is a positive, bounded, decreasing function satisfying

$$\lim_{r\to\infty} r^{n+N+\epsilon} g(r^m) = 0 \tag{2.4}$$

for some $\epsilon > 0$, where n and N are the two constants in (2.1) and (2.2).

In this paper, we will study singular integral operators T satisfying the following conditions:

(A1) T is a bounded operator on $L^q(\mathscr{X})$ for some $q > 1$, with kernel $k(x,y)$ such that for each continuous function f with compact support,

$$T(f)(x) = \int_{\mathscr{X}} k(x,y) f(y) d\mu(y) \quad \text{for } \mu\text{-almost everywhere}, \quad x \notin \mathrm{supp} f.$$

(A2) There exists a generalised approximation of the identity $\{A_t, t > 0\}$ such that the difference operator $T - TA_t$ has an associated kernel $k_t(x,y)$ which satisfies

$$\int_{d(x,y)\ge c_1 t^{1/m}} |k_t(x,y)| d\mu(x) \le c_2, \quad \forall y \in \mathscr{X}$$

for some constants $c_1, c_2 > 0$.

(A3) There exists a generalised approximation of the identity $\{B_t, t > 0\}$ such that the difference operator $T - B_t T$ has an associated kernel $\mathscr{K}_t(x,y)$ which satisfies

$$|\mathscr{K}_t(x,y)| \le c_4 \frac{1}{\mu(B(x; d(x,y)))} \times \frac{t^{\alpha/m}}{d(x,y)^\alpha}, \quad \text{when } d(x,y) \ge c_3 t^{1/m}$$

for some constants $c_3, c_4, \alpha > 0$.

It was proved in [**DM2**] that if T satisfies (A1) and (A2), then T is of weak type (1,1), hence by interpolation, it is bounded on L^p for $1 < p \le q$. When (A3) is also satisfied, the operator T is bounded on $L^p(\mathscr{X})$ for all $1 < p < \infty$.

We note that there is difference in estimates for singular integral operators when the underlying space $\mathscr{X}$ has finite measure and when $\mathscr{X}$ has infinte measure. Throughout this paper, we also assume that when the space $\mathscr{X}$ has infinite measure, all annuli in $\mathscr{X}$ are not empty, that is, for all $x \in \mathscr{X}$ and $0 < r_1 < r_2, B(x, r_2)\backslash B(x, r_1) \neq \varnothing$. Under this assumption, μ satisfies the following reverse doubling property ([**W**]): there exist $\theta > 0$ and $c > 0$ such that

$$\mu(B(x, \lambda r)) \geq c\lambda^{\theta}\mu(B(x, r)) \tag{2.5}$$

uniformly for all $\lambda \geq 1$.

3. Singular integrals on spaces of homogeneous type.

3.1. Definitions.

We give some preliminary definitions regarding the holomorphic functional calculus as introduced by McIntosh [**Mc**].

Let $0 \leq \omega < \nu < \pi$. We define the closed sector in the complex plane $\boldsymbol{C}$

$$S_{\omega} = \{z \in \boldsymbol{C} : |\arg z| \leq \omega\} \cup \{0\}$$

and denote the interior of S_{ω} by S^0_{ω}.

We employ the following subspaces of the space $H(S^0_{\nu})$ of all holomorphic functions on S^0_{ν}:

$$H_{\infty}(S^0_{\nu}) = \{b \in H(S^0_{\nu}) : \|b\|_{\infty} < \infty\},$$

where $\|b\|_{\infty} = \sup\{|b(z)| : z \in S^0_{\nu}\}$, and

$$\Psi(S^0_{\nu}) = \{\psi \in H(S^0_{\nu}) : \exists \ \ s > 0,\ |\psi(z)| \leq c|z|^s(1 + |z|^{2s})^{-1}\}.$$

Let $0 \leq \omega < \pi$. A closed operator L in $L^p(\mathscr{X})$ is said to be of type ω if $\sigma(L) \subset S_{\omega}$, and for each $\nu > \omega$, there exists a constant c_{ν} such that

$$\|(L - \lambda I)^{-1}\|_{p,p} \leq c_{\nu}|\lambda|^{-1}, \quad \lambda \notin S_{\nu}.$$

If L is of type ω and $\psi \in \Psi(S^0_{\nu})$, we define $\psi(L) \in \mathscr{L}(L^p, L^p)$ by

$$\psi(L) = \frac{1}{2\pi i}\int_{\Gamma}(L - \lambda I)^{-1}\psi(\lambda)d\lambda, \tag{3.1}$$

where Γ is the contour $\{\xi = re^{\pm i\theta} : r \geq 0\}$ parametrised clockwise around S_{ω}, and $\omega < \theta < \nu$. Clearly, this integral is absolutely convergent in $\mathscr{L}(L^p, L^p)$, and it is straightforward to show, using Cauchy's theorem, that the definition is independent of the choice of $\theta \in (\omega, \nu.)$ If, in addition, L is one-one and has dense range and if $b \in H_{\infty}(S^0_{\nu})$, then $b(L)$ can be defined by

$$b(L) = [\psi(L)]^{-1}(b\psi)(L), \tag{3.2}$$

where $\psi(z) = z(1+z)^{-2}$. It can be shown that $b(L)$ is a well-defined linear operator in $L^p(\mathscr{X})$. We say that L has a bounded H_∞ calculus in $L^p, 1 < p < \infty$, if there exists $c_{\nu,p} > 0$ such that $b(L) \in \mathscr{L}(L^p, L^p)$, and

$$\|b(L)\|_{p,p} \le c_{\nu,p}\|b\|_\infty \tag{3.3}$$

for $b \in H_\infty(S_\nu^0)$.

To prove a bounded H_∞ functional calculus, we can obtain the bound of $b(L)$ for b in the class $\Psi(S_\nu^0)$, and then extend it to $H_\infty(S_\nu^0)$ by the following Convergence Lemma which appeared in [**Mc**].

Lemma 3.1. *Let $0 \le \omega < \nu \le \pi$ and $1 < p < \infty$. Let L be an operator of type ω which is one-one with dense range. Let $\{b_\alpha\}_\alpha$ be a uniformly bounded net in $H_\infty(S_\nu^0)$. Let $b \in H_\infty(S_\nu^0)$, and suppose, for some $M < \infty$, that*
(i) $\|b_\alpha(L)\|_{p,p} \le M$ *and*
(ii) *for each* $0 < \delta < \Delta < \infty$,

$$\sup\left\{|b_\alpha(\xi) - b(\xi)| : \ \xi \in S_\nu^0 \text{ and } \delta \le |\xi| \le \Delta\right\} \to 0.$$

Then $b(L) \in \mathscr{L}(L^p, L^p)$ and $b_\alpha(L)u \to b(L)u$ for all $u \in L^p(\mathscr{X})$. Hence, $\|b(L)\|_{p,p} \le M$.

3.2. Estimates on singular integrals.

In this subsection, we assume that the space of homogeneous type $\mathscr{X}$ has infinite measure with the reverse doubling property (2.5). Let L be a linear operator of type ω on $L^2(\mathscr{X})$ with $\omega < \pi/2$, so that $(-L)$ generates a holomorphic semigroup $e^{-zL}, 0 \le |\mathrm{Arg}(z)| < \theta, \theta = \pi/2 - \omega$. Assume the following two conditions.

(a) The holomorphic semigroup $e^{-zL}, |\mathrm{Arg}(z)| < \pi/2 - \omega$, is represented by kernels $a_z(x,y)$ which satisfy upper bounds

$$|a_z(x,y)| \le h_{|z|}(x,y)$$

for $x, y \in \mathscr{X}$, $|\mathrm{Arg}(z)| < \pi/2 - \theta$ for $\theta > \omega$, and $h_{|z|}$ is defined on $\mathscr{X} \times \mathscr{X}$ by (2.3).

(b) The operator L has a bounded holomorphic functional calculus in $L^2(\mathscr{X})$. That is, for any $\nu > \omega$ and $b \in H_\infty(S_\nu^0)$, the operator $b(L)$ satisfies

$$\|b(L)\|_{2,2} \le c_\nu \|b\|_\infty.$$

It was proved in [**DM2**] that under the above assumptions (a) and (b), the operator $b(L)$ satisfies (A1), (A2) and (A3) of Section 2 for any $b \in \Psi(S_\nu^0)$, hence $b(L)$ is bounded on $L^p(\mathscr{X})$ for $1 < p < \infty$. The Convergence Lemma allows us to extend L^p boundedness of $b(L)$ to all $b \in H_\infty(S_\nu^0)$, hence the operator L has a bounded holomorphic function calculus in L^p. For the details, we refer the reader to [Theorem 6, **DM2**]. However, it is not clear in [**DM2**] whether $b(L)$ satisfies (A1), (A2) and (A3) of Section 2 when $b \in H_\infty(S_\nu^0)$ but $b \notin \Psi(S_\nu^0)$.

In this section, we will give a positive answer to the above question, i.e., to prove that for any $b \in H_\infty(S_\nu^0)$, the operator $b(L)$ satisfies conditions (A1), (A2) and (A3) of Section 2. We will also give a representation of $b(L)$ in Theorem 3.5.

Given $\omega < \nu < \pi/2$, choose θ and μ such that $\omega < \theta < \mu < \nu$. First, we note that for $b \in \Psi(S_\nu^0)$, we can choose the contour $\gamma = \gamma_+ + \gamma_-$, where $\gamma_+(t) = te^{i\mu}$ if $0 \le t < \infty$; $\gamma_-(t) = -te^{-i\mu}$ if $-\infty < t \le 0$ with $\nu > \mu$, and write

$$b(L) = \frac{1}{2\pi i}\int_\gamma (L - \lambda I)^{-1}\psi(\lambda)d\lambda.$$

Assume $\lambda \in \gamma_+$, then we have

$$(L - \lambda I)^{-1} = \int_{\Gamma_+} e^{\lambda z}e^{-zL}dz$$

where the curve $\Gamma_+(t)$ is defined by $\Gamma_+(t) = te^{i\beta}$ for $t \ge 0$ and $\beta = \pi/2 - \theta$. Let

$$\begin{aligned} b_+(L) &= \frac{1}{2\pi i}\int_{\gamma_+}\left[\int_{\Gamma_+} e^{\lambda z}e^{-zL}dz\right]b(\lambda)d\lambda \\ &= \int_{\Gamma_+}\left[\frac{1}{2\pi i}\int_{\gamma_+} e^{\lambda z}b(\lambda)d\lambda\right]e^{-zL}dz \end{aligned}$$

by a change in the order of integration. Define $\Gamma_-(t) = -te^{-i\beta}$ for $t \le 0$. Similarly, let

$$b_-(L) = \int_{\Gamma_-}\left[\frac{1}{2\pi i}\int_{\gamma_-} e^{\lambda z}b(\lambda)d\lambda\right]e^{-zL}dz,$$

then

$$b(L) = b_+(L) + b_-(L) = \int_{\Gamma_+} e^{-zL}n_+(z)dz + \int_{\Gamma_-} e^{-zL}n_-(z)dz,$$

where

$$n_\pm(z) = \frac{1}{2\pi i}\int_{\gamma_\pm} e^{\lambda z}b(\lambda)d\lambda. \tag{3.4}$$

Therefore, the kernel $\mathscr{G}_b(x,y)$ of $b(L)$ is given by

$$\mathscr{G}_b(x,y) = \int_{\Gamma_+} a_z(x,y)n_+(z)dz + \int_{\Gamma_-} a_z(x,y)n_-(z)dz.$$

Now, for $b \in H_\infty(S_\nu^0)$, we define $\mathscr{G}_b : H_\infty(S_\nu^0) \mapsto L_{\mathrm{loc}}(\mathscr{X} \times \mathscr{X} \backslash \{x \neq y\})$ by

$$\mathscr{G}_b(x,y) = \int_{\Gamma_+} a_z(x,y) n_+(z) dz + \int_{\Gamma_-} a_z(x,y) n_-(z) dz, \tag{3.5}$$

where the contour $\Gamma_\pm(t)$ are defined as above (in the definition of $b_\pm(L)$). The functions $n_\pm(z)$ are given by (3.4). Note that for all x, y in $\mathscr{X}$ and $x \neq y$, $\mathscr{G}_b(x,y)$ is well-defined and independent of θ and μ.

Lemma 3.2. *Given $b \in H_\infty(S_\nu^0)$, there exists a constant $c > 0$ such that*

$$|\mathscr{G}_b(x,y)| \leq c\|b\|_\infty \frac{1}{\mu(B(x, d(x,y)))}$$

for all $x, y \in \mathscr{X}$.

Proof. From (3.4), we get the bound $|n_\pm(z)| \leq c\|b\|_\infty |z|^{-1}$. Hence

$$\begin{aligned}
|\mathscr{G}_b(x,y)| &\leq c\|b\|_\infty \int_0^\infty |a_z(x,y)||z|^{-1} d|z| \\
&\leq c\|b\|_\infty \int_0^\infty \frac{1}{\mu(B(x, t^{1/m}))} g(d(x,y)^m t^{-1}) \frac{dt}{t} \\
&= c\|b\|_\infty \int_0^\infty \frac{1}{\mu(B(x, t^{1/m} d(x,y)))} g(t^{-1}) \frac{dt}{t}.
\end{aligned}$$

It follows from (2.4) that one has $g(t^{-1}) \leq c t^{(n+\epsilon')/m}$ for some $0 < \epsilon' < \epsilon$. Using the properties (2.1) and (2.5), we obtain

$$\begin{aligned}
&\int_0^\infty \frac{1}{\mu(B(x, t^{1/m} d(x,y)))} g(t^{-1}) \frac{dt}{t} \\
&\quad = \left(\int_0^1 + \int_1^\infty \right) \frac{1}{\mu(B(x, t^{1/m} d(x,y)))} g(t^{-1}) \frac{dt}{t} \\
&\quad \leq \frac{c}{\mu(B(x, d(x,y)))} \left(\int_0^1 t^{-n/m} g(t^{-1}) \frac{dt}{t} + \int_1^\infty t^{-\theta/m} g(t^{-1}) \frac{dt}{t} \right) \\
&\quad \leq \frac{c}{\mu(B(x, d(x,y)))} \left(\int_0^1 t^{\epsilon'/m} \frac{dt}{t} + \int_1^\infty t^{-\theta/m} \frac{dt}{t} \right) \\
&\quad \leq c \frac{1}{\mu(B(x, d(x,y)))}.
\end{aligned}$$

In the above estimates, θ is the constant in the reverse doubling property (2.5). The proof of Lemma 3.2 is complete. □

Lemma 3.3. *Given $b \in H_\infty(S_\nu^0)$, we let $b_t(z) = e^{-tz} b(z)$ and $\delta_t(z) = (1 - e^{-tz}) b(z)$*

for $t > 0$. Then, there exist positive constants c, c_1 and c_2 such that

(i) *when $d(x,y) \le c_1 t^{1/m}$, we have*

$$\mathscr{G}_{b_t}(x,y) \le c\|b\|_\infty \frac{1}{\mu(B(x,t^{1/m}))}$$

(ii) *when $d(x,y) \ge c_2 t^{1/m}$, we have*

$$\mathscr{G}_{\delta_t}(x,y) \le c\|b\|_\infty \frac{1}{\mu(B(x,d(x,y)))} \frac{t^{\alpha/m}}{d(x,y)^\alpha}$$

for some $\alpha > 0$.

Proof. We follow p. 262 of [**DM2**] to prove (i). Using the commutative property of functional calculus, we write

$$e^{-tL}b(L) = e^{-tL/2}b(L)e^{-tL/2}.$$

Since e^{-tL} maps $L^1(\mathscr{X})$ into $L^1(\mathscr{X})$ with its operator norm bounded by a constant (independent of t), and e^{-tL} maps $L^1(\mathscr{X})$ into $L^\infty(\mathscr{X})$ with its operator norm less than $(\mu(B(x,t^{-1/m})))^{-1}$, a standard interpolation and duality argument gives

$$\|e^{-tL/2}\|_{L^1\to L^2} = \|e^{-tL/2}\|_{L^2\to L^\infty} \le c(\mu(B(x,t^{-1/m})))^{-1/2}.$$

This estimate, together with the fact that $b(L)$ is bounded on $L^2(\mathscr{X})$, imply that the kernel $\mathscr{G}_{b_t}(x,y)$ of $e^{-tl}b(L)$ satisfies (i).

We now prove (ii). For $d(x,y) \ge c_2 t^{1/m}$, we have

$$|\mathscr{G}_{\delta_t}(x,y)| \le c\|b\|_\infty \int_0^\infty |a_z(x,y)| \int_0^\infty |e^{-z\lambda}(1-e^{-t\lambda})| d|\lambda| d|z|.$$

Observe that $|1-e^{-t\lambda}| \le c$ since $\mathrm{Re}(\lambda) \ge 0$ and $|1-e^{-t\lambda}| \le ct|\lambda| \le c|t\lambda|^\beta$ for $0 < \beta < \min\{\epsilon, 1\}$ when $t|\lambda| \le 1$. Here ϵ is the constant in (2.4). Using the inequality $e^{-s} \le s^{-\beta}$, we then have

$$\begin{aligned}
|\mathscr{G}_{\delta_t}(x,y)| &\le c\|b\|_\infty \int_0^\infty |a_z(x,y)| \left(\int_0^{t^{-1}} |e^{-z\lambda}||t\lambda|^\beta d|\lambda| + \int_{t^{-1}}^\infty |e^{-z\lambda}| d|\lambda| \right) d|z| \\
&\le c\|b\|_\infty \int_0^\infty |a_z(x,y)| \left(\int_0^{t^{-1}} |e^{-z\lambda/2}||t\lambda|^\beta |z\lambda|^{-\beta} d|\lambda| + |z|^{-1}e^{-|z|/t} \right) d|z| \\
&\le c\|b\|_\infty t^\beta \int_0^\infty |a_z(x,y)||z|^{-1-\beta} d|z|.
\end{aligned}$$

Using the heat kernel bounds and elementary integration, as in Lemma 3.2 we obtain

$$|\mathscr{G}_{\delta_t}(x,y)| \le c\|b\|_\infty \frac{1}{\mu(B(x,d(x,y)))} \frac{t^{\alpha/m}}{d(x,y)^\alpha}$$

for $\alpha = m\beta > 0$, for $x, y \in \mathscr{X}$ and $d(x,y) \ge c_2 t^{1/m}$. Hence, the proof of Lemma 3.3 is complete. □

LEMMA 3.4. *Let $1 < p < \infty$. Assume that $b \in \Psi(S_\nu^0)$. Then the kernel $k(x,y)$ of the operator $b(L)$ satisfies the following property: for almost all $x, y \in \mathscr{X}$, $k(x,\cdot) \in L^1(\mathscr{X})$ and $k(\cdot,y) \in L^1(\mathscr{X})$. Moreover,*

$$\|b(L)f\|_p \le \sup_{x\in\mathscr{X}} \left\{ \int_{\mathscr{X}} |k(x,y)| d\mu(y) \right\}^{1/p'} \sup_{y\in\mathscr{X}} \left\{ \int_{\mathscr{X}} |k(x,y)| d\mu(x) \right\}^{1/p} \|f\|_p$$

where $p' = \frac{p}{p-1}$.

PROOF. We note that for $b \in \Psi(S_\nu^0)$, there exist $0 < \beta < 1$ and c_β such that $|b(\lambda)| \le c_\beta |\lambda|^\beta (1 + |\lambda|^{2\beta})^{-1}$ for $\lambda \in S_\nu^0$. By (3.1), the kernel $k(x,y)$ of the operator $b(L)$ can be represented by

$$k(x,y) = \int_{\Gamma_+} a_z(x,y) n_+(z) dz + \int_{\Gamma_-} a_z(x,y) n_-(z) dz,$$

where the contour $\Gamma_\pm(t)$ are defined as in (3.5). The functions $n_\pm(z)$ are given by (3.4). Observe that for $z \in \Gamma_\pm$ and $\lambda \in \gamma_\pm$,

$$|e^{\lambda z}| \le e^{-|\lambda||z|\sin(\theta-\nu)} = e^{-c_{\theta,\nu}|\lambda||z|}.$$

We have

$$\begin{aligned}
\int_{\mathscr{X}} |k(x,y)| d\mu(y) &\le \int_{\mathscr{X}} \int_0^\infty \int_0^\infty |b(\lambda)\|a_z(x,y)||e^{\lambda z}| d|z|\ d|\lambda|\ d\mu(y) \\
&\le c\|b\|_\infty \int_0^\infty \int_0^\infty \frac{|\lambda|^\beta}{(1+|\lambda|^{2\beta})} e^{-c_{\theta,\nu}|\lambda||z|} d|z|\ d|\lambda| \\
&\le c\|b\|_\infty \int_0^\infty \frac{s^\beta}{(1+s^{2\beta})} \frac{ds}{s} \\
&\le c'\|b\|_\infty,
\end{aligned}$$

where c' depends on n, β, and $(\nu - \theta) > 0$.

This shows that for almost all $x \in \mathscr{X}$, $k(x,\cdot) \in L^1(\mathscr{X})$. The same argument shows that for almost all $y \in \mathscr{X}$, $k(\cdot,y) \in L^1(\mathscr{X})$. Using the Hölder inequality, we then obtain for $1 < p < \infty$,

$$\|b(L)f\|_p \leq \sup_{x\in\mathscr{X}} \left\{ \int_{\mathscr{X}} |k(x,y)| d\mu(y) \right\}^{1/p'} \sup_{y\in\mathscr{X}} \left\{ \int_{\mathscr{X}} |k(x,y)| d\mu(x) \right\}^{1/p} \|f\|_p,$$

where $p' = \frac{p}{p-1}$. The proof of Lemma 3.4 is complete. □

We now give a representation of the operator $b(L)$ in $L^p(\mathscr{X}), 1 < p < \infty$.

THEOREM 3.5. *Let $\mathscr{K}$ be the class of all associated kernels $k(x,y)$ of operators which satisfy* (A1), (A2) *and* (A3) *of Section* 2. *We have the following properties*:

(i) *For any function $b \in H_\infty(S_\nu^0)$, we have $\mathscr{G}_b(x,y) \in \mathscr{K}$.*

(ii) *For a fixed arbitrary function $b \in H_\infty(S_\nu^0)$, we denote $k(x,y) = \mathscr{G}_b(x,y)$. Then there exist a sequence of positive functions $\epsilon_j(x)$ and a function $\eta(x) \in L^\infty(\mathscr{X})$ such that $\lim_{j\to\infty} \epsilon_j(x) = 0$ and for $f \in L^p(\mathscr{X})$, $1 < p < \infty$,*

$$b(L)f(x) = \eta(x)f(x) + \lim_{j\to\infty} \int_{d(x,y)\geq\epsilon_j(x)} k(x,y)f(y)d\mu(y) \tag{3.6}$$

for almost every $x \in \mathscr{X}$.

PROOF. We choose $q \in (1,\infty)$ such that $\frac{1}{q'} > \frac{n-m}{n}$, where $\frac{1}{q} + \frac{1}{q'} = 1$, and m is the constant in (2.4). For $t > 0$ and $s > 0$, we define $b_t \in H_\infty(S_\nu^0)$ and $b_{t,s} \in \Psi(S_\nu^0)$ by $b_t(z) = b(z)e^{-tz}$ and $b_{t,s}(z) = z^s(1+z)^{-2s}b(z)e^{-tz}$. Then for fixed $t > 0$, $\lim_{s\to 0} b_{t,s}(z) = b_t(z) = b(z)e^{-tz}$ uniformly in any compact set contained in S_ν^0. Therefore by the Convergence Lemma 3.1,

$$\lim_{s\to 0} b_{t,s}(L)f = b_t(L)f \tag{3.7}$$

in $L^q(\mathscr{X})$ norm for every $f \in L^q(\mathscr{X})$.

Since $b_{t,s} \in \Psi(S_\nu^0)$, the kernel $k_{t,s}(x,y)$ of the operator $b_{t,s}(L)$ equals to $\mathscr{G}_{b_{t,s}}(x,y)$. Hence,

$$k_{t,s}(x,y) = \int_{\Gamma_+} a_z(x,y)n_+(z)dz + \int_{\Gamma_-} a_z(x,y)n_-(z)dz,$$

where

$$n_\pm(z) = \frac{1}{2\pi i} \int_{\gamma_\pm} \frac{\lambda^s}{(1+\lambda)^{2s}} e^{-t\lambda} e^{\lambda z} b(\lambda) d\lambda,$$

where the contour $\gamma_\pm(t)$ and $\Gamma_\pm(t)$ are defined as in (3.4) and (3.5). Note that

$$|n_\pm(z)| \leq c\frac{\|b\|_\infty}{t + |z|}.$$

For any $s > 0$, the same argument as in Lemma 3.2 shows that

$$\begin{aligned}
&\left(\int_{\mathscr{X}}|k_{t,s}(x,y)|^{q'}d\mu(y)\right)^{1/q'}\\
&\quad\le c\|b\|_\infty\int_0^\infty\left(\int_{\mathscr{X}}|a_{|z|}(x,y)|^{q'}d\mu(y)\right)^{1/q'}\frac{d|z|}{t+|z|}\\
&\quad\le c\|b\|_\infty\int_0^\infty(\mu(B(x,|z|^{1/m})))^{(q'-1)/q'}\frac{d|z|}{t+|z|}\\
&\quad\le c\|b\|_\infty\frac{1}{\mu(B(x,1))}\left(\int_0^1\frac{1}{|z|^{n(q'-1)/mq'}}\frac{d|z|}{t}+\int_1^\infty|z|^{-\theta(q'-1)/mq'}\frac{d|z|}{|z|}\right)\\
&\quad\le c_t\|b\|_\infty\frac{1}{\mu(B(x,1))}
\end{aligned}$$

since $0<\frac{n(q'-1)}{mq'}<1$. Here c_t is a positive constant independent of s.

So for any $f\in L^q(\mathscr{X})$, $|k_{t,s}(x,y)f(y)|\le c_{b,t,x}F(y)$ where $F(y)\in L^1(\mathscr{X})$. By Lemma 3.4 and Lebesgue's dominated convergence theorem,

$$\begin{aligned}
b_t(L)f(x)=\lim_{s\to0}b_{t,s}(L)f(x)&=\lim_{s\to0}\int_{\mathscr{X}}k_{t,s}(x,y)f(y)d\mu(y)\\
&=\int_{\mathscr{X}}\lim_{s\to0}k_{t,s}(x,y)f(y)d\mu(y)\\
&=\int_{\mathscr{X}}k_t(x,y)f(y)d\mu(y),
\end{aligned}$$

where

$$k_t(x,y)=\lim_{s\to0}k_{t,s}(x,y)=\mathscr{G}_{b_t}(x,y).$$

Using the Convergence Lemma 3.1 again to the left hand side of the above equality, we have

$$b(L)f(x)=\lim_{t\to0}b_t(L)f(x)=\lim_{t\to0}\int_{\mathscr{X}}k_t(x,y)f(y)d\mu(y),\quad f\in L^q(\mathscr{X}).$$

In particular, by Lebesgue's dominated convergence theorem we have that for each continuous function f with compact support,

$$b(L)f(x)=\lim_{t\to0}\int_{\mathscr{X}}k_t(x,y)f(y)d\mu(y)=\int_{\mathscr{X}}\lim_{t\to0}k_t(x,y)f(y)d\mu(y)$$

for all $x\notin\mathrm{supp}f$, where the convergence is in $L^q(\mathscr{X})$.

Note that for all x,y in $\mathscr{X}$, and $x\ne y$, $\lim_{t\to0}k_t(x,y)=\mathscr{G}_b(x,y)$. Applying Theorem 6 of [**DM2**], the operator $b(L)$ is bounded on $L^q(\mathscr{X})$ such that

$$b(L)f(x) = \int_{\mathscr{X}} k(x,y)f(y)d\mu(y) \quad \forall x \notin \mathrm{supp} f,$$

where $k(x,y)$ satisfies the properties (A1), (A2) and (A3) on $\mathscr{X} \times \mathscr{X} \backslash \{x \neq y\}$. Hence, $k \in \mathscr{K}$ as in Theorem 3.5.

It follows from a standard argument of proving the existence of almost everywhere pointwise limits as a consequence of the corresponding maximal inequality that there exists a sequence of positive functions $\epsilon_j(x)$ such that $\lim_{j\to\infty} \epsilon_j(x) = 0$ and a function $\eta(x) \in L^\infty(\mathscr{X})$ such that for $f \in L^p(\mathscr{X})$ with $1 < p < \infty$,

$$b(L)f(x) = \eta(x)f(x) + \lim_{j\to\infty} \int_{d(x,y)\geq \epsilon_j(x)} k(x,y)f(y)d\mu(y)$$

for almost every $x \in \mathscr{X}$. See, for examples, [**Me**, Chapter 7, Theorem 6] for Euclidean spaces $\mathscr{X} = \boldsymbol{R}^n$, and [**CW**], [Theorem 3, **DM2**] and [**DY**] for spaces of homogeneous type.

Hence, the proof of Theorem 3.5 is complete. □

4. Vector-valued inequalities and weights.

4.1. Boundedness of vector-valued singular integral operators.

In this section, we assume that $\mathscr{X}$ is a space of homogeneous type equipped with a quasi-metric d and a measure μ. In the case $\mu(\mathscr{X}) = \infty$, we assume that the space $\mathscr{X}$ has the reverse doubling property (2.5).

Let $\boldsymbol{A}, \boldsymbol{B}$ be Banach spaces, and $\mathscr{L}(\boldsymbol{A}, \boldsymbol{B})$ the space of bounded linear operators from $\boldsymbol{A}$ to $\boldsymbol{B}$. Let $1 < q < \infty$, and $L^q_{\boldsymbol{A}}(\mathscr{X})$, $L^q_{\boldsymbol{B}}(\mathscr{X})$ be the spaces of L^q integrable functions with values in $\boldsymbol{A}, \boldsymbol{B}$, respectively. Let T be a linear operator mapping boundedly from $L^q_{\boldsymbol{A}}(\mathscr{X})$ into $L^q_{\boldsymbol{B}}(\mathscr{X})$ with an associated kernel $k : \mathscr{X} \times \mathscr{X} \to \mathscr{L}(\boldsymbol{A}, \boldsymbol{B})$, such that, for any $f \in L^q_{\boldsymbol{A}}(\mathscr{X})$,

$$T(f)(x) = \int_{\mathscr{X}} k(x,y)f(y)d\mu(y) \quad \text{for } \ \mu\text{-a.e.} \quad x \in \mathscr{X} \backslash \mathrm{supp} f. \tag{4.1}$$

We assume there exists a class of integral operators $A_t, t > 0$, from $L^q_{\boldsymbol{A}}(\mathscr{X})$ into $L^q_{\boldsymbol{A}}(\mathscr{X})$ which plays the role of approximations to the identity. This means that the operators A_t can be represented by kernels $a_t(x,y) : \mathscr{X} \times \mathscr{X} \to \mathscr{L}(\boldsymbol{A}, \boldsymbol{A})$ in the sense that

$$A_t u(x) = \int_{\mathscr{X}} a_t(x,y)u(y)d\mu(y) \tag{4.2}$$

for every function $u \in L^q_{\boldsymbol{A}}(\mathscr{X}) \cap L^1_{\boldsymbol{A}}(\mathscr{X})$, and the kernels $a_t(x,y)$ satisfy the following conditions:

$$\|a_t(x,y)\|_{\mathscr{L}(\boldsymbol{A},\boldsymbol{A})} \leq h_t(x,y) \tag{4.3}$$

for all $x, y \in \mathscr{X}$, where $h_t(x,y)$ is a function satisfying

$$h_t(x,y) = \frac{1}{\mu(B(x,t^{1/m}))} g(d(x,y)^m t^{-1}) \tag{4.4}$$

in which m is a positive constant and g is a positive, bounded, decreasing function satisfying

$$\lim_{r\to\infty} r^{n+N+\epsilon} g(r^m) = 0$$

for some $\epsilon > 0$, where n and N are two constants in (2.1) and (2.2).

THEOREM 4.1. *Let T be a bounded linear operator from $L^q_{\boldsymbol{A}}(\mathscr{X})$ to $L^q_{\boldsymbol{B}}(\mathscr{X})$ with an associated kernel $k(x,y)$ in the sense of* (4.1)*. Assume there exists a class of operators $A_t, t > 0$, which satisfy the conditions* (4.2), (4.3) *and* (4.4) *so that the composite operators TA_t have associated kernels $k_t(x,y)$ in the sense of* (4.1) *and there exist constants C and $c > 0$ so that*

$$\int_{d(x,y)\ge ct^{1/m}} \|k(x,y) - k_t(x,y)\|_{\mathscr{L}(\boldsymbol{A},\boldsymbol{B})} d\mu(x) \le C \tag{4.5}$$

for all $y \in \mathscr{X}$.

Then the operator T is of weak-type $(1,1)$ *from $L^1_{\boldsymbol{A}}(\mathscr{X})$ into $L^1_{\boldsymbol{B}}(\mathscr{X})$. Hence, T can be extended to a bounded operator from $L^p_{\boldsymbol{A}}(\mathscr{X})$ into $L^p_{\boldsymbol{B}}(\mathscr{X})$ for all $1 < p \le q$.*

PROOF. For the proof, we refer to Theorem 1 in [**ADM**]. □

REMARK 4.2. (i) In [**RRT**], Theorem 4.1 was obtained under the following Hörmander condition:

$$\int_{d(x,y)\ge 2d(y,y')} \|k(x,y) - k(x,y')\|_{\mathscr{L}(\boldsymbol{A},\boldsymbol{B})} d\mu(x) \le C.$$

See also [**BCP**]. In fact, for a suitable generalised approximation of the identity, it is proved in [**DM2**] that in the case of scalar-valued functions, the condition (4.5) is weaker than the above Hörmander condition.

(ii) Theorem 4.1 can be modified so that it is still true when the space of homogeneous type $\mathscr{X}$ is replaced by one of its measurable subsets Ω. In this case, it is sufficient that condition (4.4) on the upper bound $h_t(x,y)$ of the kernel $a_t(x,y)$ is replaced by

$$h_t(x,y) = (\mu(B^{\mathscr{X}}(x,t^{1/m})))^{-1} g(d(x,y)^m t^{-1}), \tag{4.6}$$

where $B^{\mathscr{X}}(x;t^{1/m})$ is the ball of center x, radius $t^{1/m}$ in the space $\mathscr{X}$.

We now apply Theorem 4.1 to obtain vector-valued holomorphic functional calculi of operators with heat kernel bounds. Let L be a linear operator on $L^2(\mathscr{X})$ with $\omega < \pi/2$ so that $(-L)$ generates a holomorphic semigroup $e^{-zL}, 0 \le |\mathrm{Arg}(z)| < \pi/2 - \omega$.

THEOREM 4.3. *Let $1 < p, q < \infty$. Assume the following two conditions.*

(a) *The holomorphic semigroup e^{-zL}, $|\mathrm{Arg}(z)| < \pi/2 - \omega$, is represented by kernels $a_z(x, y)$ which satisfy upper bounds*

$$|a_z(x, y)| \le h_{|z|}(x, y)$$

for $x, y \in \mathscr{X}$, $|\mathrm{Arg}(z)| < \pi/2 - \theta$ for $\theta > \omega$, and $h_{|z|}$ is defined on $\mathscr{X} \times \mathscr{X}$ by (4.4).

(b) *The operator L has a bounded holomorphic functional calculus in $L^2(\mathscr{X})$. That is, for any $\nu > \omega$ and $b \in H_\infty(S_\nu^0)$, the operator $b(L)$ satisfies*

$$\|b(L)\|_{2,2} \le c_\nu \|b\|_\infty.$$

Then, for any $\alpha > 0$ and $f = \{f_j\}_j \in L^q_{L^q_{\boldsymbol{C}}}(\mathscr{X})$ we have

$$\mu\left\{x : \left(\sum_j |b(L)f_j(x)|^q\right)^{1/q} > \alpha\right\} \le c_q \alpha^{-1} \|b\|_\infty \int_{\mathscr{X}} \left(\sum_j |f_j(x)|^q\right)^{1/q} d\mu(x), \tag{4.7}$$

and

$$\left\|\left\{\sum_j |b(L)f_j|^q\right\}^{1/q}\right\|_{L^p(\mathscr{X})} \le c_{p,q} \|b\|_\infty \left\|\left(\sum_j |f_j|^q\right)^{1/q}\right\|_{L^p(\mathscr{X})}. \tag{4.8}$$

PROOF. Let $b \in H_\infty(S_\nu^0)$. For $f = \{f_j\}_j \in L^q_{\ell^q_{\boldsymbol{C}}}(\mathscr{X})$ with compact support, we define

$$\widetilde{b(L)}f(x) = \{b(L)f_j(x)\}_j = \left\{\int_{\mathscr{X}} k_b(x, y) f_j(y) d\mu(y)\right\}_j = \int_{\mathscr{X}} \tilde{k}_b(x, y) f(y) d\mu(y)$$

for $x \in \mathscr{X} \backslash \mathrm{supp} f$, where the kernel $\tilde{k}_b : \mathscr{X} \times \mathscr{X} \to \mathscr{L}(\ell^q_{\boldsymbol{C}}, \ell^q_{\boldsymbol{C}})$ is defined by

$$\tilde{k}_b(x, y)\alpha = \{k_b(x, y)\alpha_j\}_j$$

for any $\alpha = \{\alpha_j\}_j \in \ell^q_{\boldsymbol{C}}$. We take the Banach space $\boldsymbol{A} = \boldsymbol{B} = \ell^q_{\boldsymbol{C}}$. By the argument of the case of scalar-valued functions in Theorem 6 of [**DM2**], Theorem 4.1 gives

$$\begin{aligned}
\|\widetilde{b(L)}f\|^q_{L^q_{\ell^q_{\boldsymbol{C}}}(\mathscr{X})} &= \sum_j \int_{\mathscr{X}} |b(L)f_j(x)|^q d\mu(x) \\
&\le c\|b\|^q_\infty \sum_j \int_{\mathscr{X}} |f_j(x)|^q d\mu(x) \\
&= c\|b\|^q_\infty \|f\|^q_{L^q_{\ell^q_{\boldsymbol{C}}}(\mathscr{X})}.
\end{aligned}$$

See also [**RRT**] and [**BCP**].

We now verify condition (4.5). Let $A_t = e^{-tL}$ and define

$$\widetilde{b(L)}A_t f(x) = \{b(L)A_t f_j(x)\}_j = \left\{ \int_{\mathscr{X}} k_{b,t}(x,y) f_j(y) d\mu(y) \right\}_j = \int_{\mathscr{X}} \tilde{k}_{b,t}(x,y) f(y) d\mu(y)$$

for $x \in \mathscr{X} \backslash \mathrm{supp} f$, where the kernel $\tilde{k}_{b,t} : \mathscr{X} \times \mathscr{X} \to \mathscr{L}(\ell^q_{\boldsymbol{C}}, \ell^q_{\boldsymbol{C}})$ is defined by

$$\tilde{k}_{b,t}(x,y)\alpha = \{k_{b,t}(x,y)\alpha_j\}_j$$

for any $\alpha = \{\alpha_j\}_j \in \ell^q_{\boldsymbol{C}}$. Denote $\delta_t(z) = (1 - e^{-tz})b(z)$ for $t > 0$. Then, estimate (4.5) follows from (ii) of Lemma 3.3 since

$$\|\tilde{k}_b(x,y) - \tilde{k}_{b,t}(x,y)\|_{\mathscr{L}(\ell^q_{\boldsymbol{C}}, \ell^q_{\boldsymbol{C}})} \le |k_b(x,y) - k_{b,t}(x,y)| = |\mathscr{G}_{\delta_t}(x,y)|.$$

So, $\widetilde{b(L)}$ is a vector-valued operator satisfying all conditions in Theorem 4.1. Hence Theorem 4.3 follows from Theorem 4.1 and a standard duality argument. $\square$

As in (ii) of Remark 4.2, our method also works in the case that L is a linear operator of type ω on $L^2(\Omega)$ with $\omega < \pi/2$, where Ω is a measurable subset of a space $\mathscr{X}$ of homogeneous type. We have the following theorem.

THEOREM 4.4. *Let $1 < p, q < \infty$. Assume the following two conditions.*

(a_Ω) *The holomorphic semigroup e^{-zL}, $|\mathrm{Arg}(z)| < \pi/2 - \omega$, is represented by kernels $a_z(x,y)$ which satisfy the estimate*

$$|a_z(x,y)| \le c_\theta h_{|z|}(x,y)$$

for $x, y \in \Omega$, $|\mathrm{Arg}(z)| < \pi/2 - \theta$ for $\theta > \omega$, and $h_{|z|}$ is defined on $\mathscr{X} \times \mathscr{X}$ by (4.4).

(b_Ω) *The operator L has a bounded holomorphic functional calculus in $L^2(\Omega)$. That is, for any $\nu > \omega$ and $b \in H_\infty(S^0_\nu)$, the operator $b(L)$ satisfies*

$$\|b(L)\|_{2,2} \le c_\nu \|b\|_\infty.$$

Then for any $\alpha > 0$ $f = \{f_j\}_j \in L^q_{L^q_{\boldsymbol{C}}}(\mathscr{X})$ we have

$$\mu\left\{ x : \left(\sum_j |b(L)f_j(x)|^q \right)^{1/q} > \alpha \right\} \le c_q \alpha^{-1} \|b\|_\infty \int_\Omega \left(\sum_j |f_j(x)|^q \right)^{1/q} d\mu(x),$$

and

$$\left\| \left\{ \sum_j |b(L)f_j|^q \right\}^{1/q} \right\|_{L^p(\Omega)} \le c_{p,q} \|b\|_\infty \left\| \left(\sum_j |f_j|^q \right)^{1/q} \right\|_{L^p(\Omega)}.$$

4.2. Weighted inequalities for H_∞ functional calculi of operators with heat kernel bounds.

In this section, we assume that Ω is a measurable subset of a space of homogeneous type $(\mathscr{X}, d, \mu)$. Let L be a linear operator of type ω on $L^2(\Omega)$ with $\omega < \pi/2$, so that $(-L)$ generates a holomorphic semigroup $e^{-zL}, 0 \le |\mathrm{Arg}(z)| < \pi/2 - \omega$. We assume that L satisfies the two conditions (a_Ω) and (b_Ω) of Theorem 4.4.

For $1 < p < \infty$, we now study the two-weight inequality for the operator $b(L)$:

$$\int_\Omega |b(L)f(x)|^p u(x) d\mu(x) \le c_p \|b\|_\infty^p \int_\Omega |f(x)|^p v(x) d\mu(x) \tag{4.9}$$

for all $f \in L^p(v d\mu)$ and u, v being μ-a.e. positive functions. Throughout this section we aim to give an answer to the following problem:

Find sufficient conditions on $0 \le v < \infty$ μ-a.e. (resp. $u > 0$ μ-a.e.) such that (4.9) is satisfied by some $u > 0$ μ-a.e. (resp. $0 \le v < \infty$ μ-a.e.).

This problem was studied in [**GR**, pp. 558–562] for Calderón-Zygmund operators in $\boldsymbol{R}^n$. See also [**GM**] for Calderón-Zygmund operators on non-homogeneous spaces. We would like to combine ideas in these papers and Theorem 4.4 to prove similar results for $b(L)$ where $b(L)$ has non-smooth kernels.

The following theorem in [**FT**] establishes the relationship between vector-valued inequalities and weights.

THEOREM 4.5. *Let $(\boldsymbol{Y}, d\nu)$ be a measure space; $\boldsymbol{F}, \boldsymbol{G}$ Banach spaces, and $\{W_k\}_{k\in\boldsymbol{Z}}$ a sequence of pairwise disjoint measurable subsets of $\boldsymbol{Y}$ such that $\boldsymbol{Y} = \bigcup_k W_k$. Consider $0 < s < p < \infty$ and T a sublinear operator which satisfies the following vector-valued inequality*

$$\left\| \left\{ \sum_j \|Tf_j\|_{\boldsymbol{G}}^p \right\}^{1/p} \right\|_{L^s(W_k, d\nu)} \le c_k \left\{ \sum_j \|f_j\|_{\boldsymbol{F}}^p \right\}^{1/p}, \quad k \in \boldsymbol{Z}, \tag{4.10}$$

where, for every $k \in \boldsymbol{Z}$, c_k only depends on $\boldsymbol{F}, \boldsymbol{G}, p$ and s. Then, there exists a positive function $u(x)$ on $\boldsymbol{Y}$ such that

$$\left\{ \int_{\boldsymbol{Y}} \|Tf(x)\|_{\boldsymbol{G}}^p u(x) d\nu(x) \right\}^{1/p} \le c\|f\|_{\boldsymbol{F}}, \tag{4.11}$$

where c depends on $\boldsymbol{F}, \boldsymbol{G}, p$ and s. Moreover, given a sequence of positive numbers $\{a_k\}_{k\in\boldsymbol{Z}}$ with $\sum_k a_k^p < \infty$, the weight u can be found such that $\|u^{-1}\chi_{W_k}\|_{L^{\sigma-1}(W_k, d\nu)} \le (a_k^{-1} c_k)^p$, where $\frac{1}{\sigma} + \frac{s}{p} = 1$.

In our context, we choose $(\boldsymbol{Y}, d\nu) = (\mathscr{X}, d\mu)$. Given $1 < p < \infty$ and some fixed point $x_0 \in \Omega$. Let us recall the definitions of the following classes of weights in Ω ([**GR**], [**GM**]):

$$D_p = \left\{ 0 \le w < \infty \ \mu\text{-a.e.} : \int_\Omega \frac{w(x)^{1-p'}}{(1+\mu(B(x_0, d(x_0,x))))^{p'}} d\mu(x) < \infty \right\}$$

and

$$Z_p = \left\{ 0 \le w < \infty \ \mu\text{-a.e.} : \int_\Omega \frac{w(x)}{(1+\mu(B(x_0, d(x_0,x))))^{p}} d\mu(x) < \infty \right\}.$$

Here p' is the dual of p, i.e., $\frac{1}{p}+\frac{1}{p'}=1$. Note that these classes D_p and Z_p do not depend on the point x_0.

Remark 4.6. When the diameter of the space is finite, there exists R such that $\Omega \subset \mathscr{X} \subset B(x_0, R)$, hence $\mu(\mathscr{X}) < \infty$. In this case, the classes D_p and Z_p are defined as follows:

$$D_p = \left\{ 0 \le w < \infty \ \mu\text{-a.e.} : \int_\Omega w(x)^{1-p'} d\mu(x) < \infty \right\}$$

and

$$Z_p = \left\{ w > 0 \ \mu\text{-a.e.} : \int_\Omega w(x) d\mu(x) < \infty \right\}.$$

We now apply Theorem 4.5 to our operator $b(L)$.

Proposition 4.7. *Take $0 < s < 1 < p < \infty$ and $v \in D_p$. Let L be a linear operator of type ω on $L^2(\Omega)$ with $\omega < \pi/2$. Assume that L satisfies conditions* (a_Ω) *and* (b_Ω).

If the diameter of Ω is infinite, we have

$$\left\| \left\{ \sum_j |b(L)f_j|^p \right\}^{1/p} \right\|_{L^s(S_k, d\mu)} \le c_{s,p}\mu(B_k)^{1/s} \|b\|_\infty \left\{ \sum_j \|f_j\|^p_{L^p(v d\mu)} \right\}^{1/p} \tag{4.12}$$

for $k = 0, 1, \cdots$, where $S_0 = B_0 = \{x : d(x, x_0) \le 1\}$, $S_k = \{x : 2^{k-1} < d(x, x_0) \le 2^k\}$, and $B_k = B(x_0; 2^k)$ for $k = 1, 2, \cdots$.

If the diameter of Ω is finite, we have

$$\left\| \left\{ \sum_j |b(L)f_j|^p \right\}^{1/p} \right\|_{L^s(d\mu)} \le c_{s,p} \|b\|_\infty \left\{ \sum_j \|f_j\|^p_{L^p(v d\mu)} \right\}^{1/p}. \tag{4.13}$$

Proof. Consider the case when Ω has infinite diameter. Fix $k \ge 0$ and set $B_{k+1} = B(x_0, 2^{k+1})$. We write $f = f\chi_{B_{k+1}} + f\chi_{\Omega \backslash B_{k+1}}$. For $x \in S_k$ and $y \in \Omega \backslash B_{k+1}$, we have $2d(x,y) > d(x_0, y)$, and thus

$$\frac{1}{\mu(B(x,d(x,y)))} \le \frac{c}{\mu(B(x_0,d(x_0,y)))}$$

by the doubling property (2.1). By Lemma 3.2,

$$\begin{aligned}
&|b(L)f\chi_{\Omega\setminus B_{k+1}}(x)| \\
&\quad\le c\|b\|_\infty \int_{\Omega\setminus B_{k+1}} \frac{|f(y)|}{\mu(B(x,d(x,y)))} d\mu(y) \\
&\quad\le c\|b\|_\infty \int_{\Omega} \frac{|f(y)|}{(1+\mu(B(x_0,d(x_0,y))))} v(y)^{1/p} v(y)^{-1/p} d\mu(y) \\
&\quad\le c\|b\|_\infty \left\{ \int_\Omega |f(y)|^p v(y) d\mu(y) \right\}^{1/p} \left\{ \int_\Omega \frac{v(y)^{1-p'}}{(1+\mu(B(x_0,d(x_0,y))))^{p'}} d\mu(y) \right\}^{1/p'} \\
&\quad\le c\|b\|_\infty \|f\|_{L^p(vd\mu)}
\end{aligned}$$

by using $v \in D_p$, where $\frac{1}{p}+\frac{1}{p'}=1$. Thus, we have

$$\left\| \left\{ \sum_j |b(L)f_j\chi_{\Omega\setminus B_{k+1}}|^p \right\}^{1/p} \right\|_{L^s(S_k,d\mu)} \le c\mu(B_k)^{1/s}\|b\|_\infty \left\{ \sum_j \|f_j\|^p_{L^p(vd\mu)} \right\}^{1/p}$$

for $k = 0, 1, \cdots$.

We recall that the $L^{1,\infty}$ norm of a function f is given by $\|f\|_{L^{1,\infty}} = \sup_{\lambda>0} \lambda\mu\{x : |f(x)| > \lambda\}$. For $0 < s < 1$, by Kolmogorov's inequality (see [**GR**], page 485) and Theorem 4.4 we obtain

$$\begin{aligned}
&\left\| \left\{ \sum_j |b(L)f_j\chi_{B_{k+1}}|^p \right\}^{1/p} \right\|_{L^s(S_k,d\mu)} \\
&\quad\le c\mu(S_k)^{\frac{1}{s}-1} \left\| \left\{ \sum_j |b(L)f_j\chi_{B_{k+1}}|^p \right\}^{1/p} \right\|_{L^{1,\infty}(S_k,d\mu)} \\
&\quad\le c\mu(S_k)^{1/s-1}\|b\|_\infty \int_{S_k} \left\{ \sum_j |f_j\chi_{B_{k+1}}|^p \right\}^{1/p} v(x)^{1/p} v(x)^{-1/p} d\mu(x) \\
&\quad\le c\mu(S_k)^{1/s-1}\|b\|_\infty \left\{ \int_\Omega \sum_j \|f_j\chi_{B_{k+1}}\|^p_{L^p(vd\mu)} \right\}^{1/p} \left\{ \int_{S_k} v(x)^{1-p'} d\mu(x) \right\}^{1/p'} \\
&\quad\le c\mu(B_k)^{1/s}\|b\|_\infty \left\{ \sum_j \|f_j\|^p_{L^p(vd\mu)} \right\}^{1/p},
\end{aligned}$$

where the last inequality follows from $\frac{1}{s} - 1 > 0$ and

$$\begin{aligned}\Big\{\int_{S_k} v(x)^{1-p'} d\mu(x)\Big\}^{1/p'} &\le c\mu(B_k)\Big\{\int_{S_k} \frac{v(x)^{1-p'}}{(1+\mu(B(x_0,d(x_0,x))))^{p'}} d\mu(x)\Big\}^{1/p'}\\ &\le c\mu(B_k)\end{aligned}$$

by $v \in D_p$. So, (4.12) follows readily by combining the above estimates.

When the space Ω has finite diameter, the measure $\mu(\Omega) < \infty$. We then proceed as with the case of functions $f_j \chi_{B_{k+1}}$. Since $0 < s < 1$, we can apply Kolmogorov's inequality and Theorem 4.4 to obtain

$$\begin{aligned}&\Big\|\Big\{\sum_j |b(L)f_j|^p\Big\}^{1/p}\Big\|_{L^s(d\mu)}\\ &\le c\mu(\Omega)^{1/s-1}\Big\|\Big\{\sum_j |b(L)f_j|^p\Big\}^{1/p}\Big\|_{L^{1,\infty}(d\mu)}\\ &\le c\mu(\Omega)^{1/s-1}\|b\|_\infty \int_\Omega \Big\{\sum_j |f_j|^p\Big\}^{1/p} v(x)^{1/p} v(x)^{-1/p} d\mu(x)\\ &\le c\mu(\Omega)^{1/s-1}\|b\|_\infty \Big\{\int_\Omega \sum_j |f_j|^p v(x) d\mu(x)\Big\}^{1/p}\Big\{\int_\Omega v(x)^{1-p'} d\mu(x)\Big\}^{1/p'}\\ &\le c\|b\|_\infty \Big\{\sum_j \|f_j\|^p_{L^p(vd\mu)}\Big\}^{1/p},\end{aligned}$$

because Ω has finite measure and $v \in D_p$. This completes the proof of Proposition 4.7. □

With these vector-valued estimates, we now prove the main theorem of this paper.

THEOREM 4.8. *Given p, $1 < p < \infty$. Let L be a linear operator of type ω on $L^2(\Omega)$ with $\omega < \pi/2$, which satisfies the conditions* (a_Ω) *and* (b_Ω). *If $u \in Z_p$ (resp. $v \in D_p$), then there exists a weight $0 < v < \infty$ μ-a.e. (resp. $0 < u < \infty$ μ-a.e.) such that*

$$\int_\Omega |b(L)f(x)|^p u(x) d\mu(x) \le c_p \|b\|^p_\infty \int_\Omega |f(x)|^p v(x) d\mu(x) \tag{4.14}$$

for all $f \in L^p(vd\mu)$. Moreover, for $0 < \alpha < 1$, v (resp. u) can be chosen such that $v^\alpha \in Z_p$ (resp. $u^\alpha \in D_p$).

PROOF. First, let us prove the case $v \in D_p$ for Ω with infinite diameter. Fix $0 < \alpha < 1$ and put $q = 1 + \alpha(p' - 1)$. Then $1 < q < p'$ and we can find some $s, 0 < s < 1$, such that $\sigma = (\frac{p}{s})' > q$.

We apply Theorem 4.5 with $(\boldsymbol{Y}, d\nu) = (\Omega, d\mu)$, $\boldsymbol{F} = L^p(vd\mu)$, $\boldsymbol{G} = \boldsymbol{C}$, $\{W_k\}_k = \{S_k\}_0^\infty$, $c_k = \mu(B_k)^{1/s}$ and the sublinear operator $T = b(L)$. Estimate (4.12) of Proposition 4.7 leads to the vector-valued inequality (4.10). Then, there exists a weight

u such that (4.14) holds. Moreover, u can be chosen such that $\|u^{-1}\|_{L^{\sigma-1}(S_k,d\mu)} \leq c(a_k^{-1}\mu(B_k)^{1/s})^p$ with $a_k > 0$ and $\sum_k a_k^p < \infty$. Let $\beta = \frac{\sigma-1}{q-1}$ and β' the conjugate exponent of β. As in [**GR**], by the doubling property we have

$$\begin{aligned}\int_\Omega \frac{u(x)^{1-q}}{(1+\mu(B(x_0,d(x_0,x))))^{p'}}d\mu(x) &= \sum_{k=0}^{\infty}\int_{S_k}\frac{u(x)^{1-q}}{(1+\mu(B(x_0,d(x_0,x))))^{p'}}d\mu(x)\\ &\leq c\sum_{k=0}^{\infty}\mu(B_k)^{-p'}\left\{\int_{S_k}u(x)^{1-\sigma}d\mu(x)\right\}^{1/\beta}\mu(S_k)^{1/\beta'}\\ &\leq c\sum_{k=0}^{\infty}a_k^{-p(q-1)}\mu(B_k)^{(-p'+\frac{p(q-1)}{s}+\frac{1}{\beta'})}.\end{aligned}$$

Note that the first inequality follows from Hölder's inequality with exponent $\beta = \frac{\sigma-1}{q-1} > 1$. Observe that

$$-p' + \frac{p(q-1)}{s} + \frac{1}{\beta'} = q - p' < 0.$$

Hence, we can choose $\epsilon > 0$ such that $q - p' + \epsilon < 0$. The sequence $\{a_k\}$ can be chosen to satisfy $a_k^{-p(q-1)} = \mu(B_k)^\epsilon$. Using the reverse doubling property (2.5), we have $\mu(B_k) \geq c2^{k\theta}$ for all $0 \leq k < \infty$, where θ is the constant in (2.5). Therefore,

$$\sum_{k=0}^{\infty}a_k^p = \sum_{k=0}^{\infty}\mu(B_k)^{-\frac{\epsilon}{q-1}} \leq c\sum_{k=0}^{\infty}2^{-\frac{k\theta\epsilon}{q-1}} < \infty$$

and

$$\begin{aligned}\int_\Omega \frac{u(x)^{1-q}}{(1+\mu(B(x_0,d(x_0,x))))^{p'}}d\mu(x) &\leq c\sum_{k=0}^{\infty}\mu(B_k)^{(q-p'+\epsilon)}\\ &\leq c\sum_{k=0}^{\infty}2^{k\theta(q-p'+\epsilon)}\\ &\leq c' < \infty.\end{aligned}$$

To finish the proof, we note that $\alpha = \frac{1-q}{1-p'}$ and thus $u^\alpha \in D_p$.

When the space Ω has finite diameter, we proceed analogously and the proof is even simpler because we do not have to decompose the space. We leave the details to the reader.

If $u \in Z_p$, then $\tilde{u} = u^{1-p'} \in D_{p'}$. It follows that there exists some weight $\tilde{v}$ with $0 < \tilde{v} < \infty$ μ-a.e., such that the adjoint operator $g(L)^*$ satisfies

$$\int_\Omega |b(L)^*f(x)|^{p'}\tilde{v}(x)d\mu(x) \leq c_{p'}\|b\|_\infty^{p'}\int_\Omega |f(x)|^{p'}\tilde{u}(x)d\mu(x). \tag{4.15}$$

Take v so that $\tilde{v} = v^{1-p'}$. Since $0 < v < \infty$ μ-a.e., a standard duality argument shows that (4.15) implies (4.14). Furthermore, we can choose $\tilde{v}$ such that $\tilde{v}^\alpha \in D_{p'}$, provided that $0 < \alpha < 1$. That is, we can find v such that $v^\alpha \in Z_p$. The proof of Theorem 4.8 is complete. □

4.3. Applications.

Theorem 4.8 gives new results when we do not assume smoothness of heat kernels in the space variables, or when Ω is a measurable set with no assumptions on smoothness of its boundary. We give examples of operators L which satisfy the assumptions of Theorem 4.8.

(a) Let V be a nonnegative function on $\boldsymbol{R}^n$. The Schrödinger operator with potential V is defined by

$$L = -\triangle + V(x).$$

The Trotter formula shows that the kernel $p_t(x,y)$ of the semigroup e^{-tL} satisfies a Gaussian upper bound, that is, for some constants $c_1, c_2 > 0$,

$$0 < p_t(x,y) \le \frac{c_1}{t^{n/2}} e^{-c_2 \frac{|x-y|^2}{t}}$$

for $x, y \in \boldsymbol{R}^n$ and all $t > 0$. However, unless V satisfies certain additional conditions, $p_t(x,y)$ can be a discontinuous function of the space variables and the Hölder continuity estimates may fail to hold.

(b) Let

$$Lf = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} a_{ij}(x) \frac{\partial}{\partial x_j} f$$

be an elliptic divergence form operator of real, symmetric coefficients with Dirichlet boundary conditions on a domain Ω of $\boldsymbol{R}^n$ which is defined by the variational method. More precisely, L is the positive self-adjoint operator associated with the form

$$Q(f,g) = \int_\Omega \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial}{\partial x_j} f(x) \cdot \frac{\partial}{\partial x_i} \bar{g}(x) dx$$

on $V \times V$ by $\langle Lf, g \rangle = Q(f,g)$, where V is the Sobolev space $H_0^1(\Omega)$. It is known that the operator L has Gaussian heat kernel bounds without any conditions on smoothness of the boundary of Ω.

More general operators on open domains of $\boldsymbol{R}^n$ which possess Gaussian bounds can be found in [**Da**], [**DM1**] and [**DM2**].

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