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A construction of non-regularly orbicular modules for Galois coverings

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Abstract. For a given finite dimensional k-algebra A which admits a presentation in the form R/G, where G is an infinite group of k-linear automorphisms of a locally bounded k-category R, a class of modules lying out of the image of the "push-down" functor associated with the Galois covering $R \to R/G$, is studied. Namely, the problem of existence and construction of the so called non-regularly orbicular indecomposable R/G-modules is discussed. For a G-atom B (with a stabilizer G_B), whose endomorphism algebra has a suitable structure, a representation embedding $\Phi^{B(f,s)}|: I_n\text{-spr}_{\,l(s)}(kG_B) \to \operatorname{mod}(R/G)$, which yields large families of non-regularly orbicular indecomposable R/G-modules, is constructed (Theorem 2.2). An important role in consideration is played by a result interpreting some class of R/G-modules in terms of Cohen-Macaulay modules over certain skew grup algebra (Theorem 3.3). Also, Theorems 4.5 and 5.4, adapting the generalized tensor product construction and Galois covering scheme, respectively, for Cohen-Macaulay modules context, are proved and intensively used.

Introduction.

The last thirty years have been a period of a great and permanent progress of representation theory of finite-dimensional algebras. Many deep problems and classical conjectures have been solved in that time. In the meantime also new, challenging and stimulating questions, phrased already in a language of modern notions and concepts, have been appeared.

The essential reason of that progress was an appearance of several fresh, original ideas. After short time they brought an inventing and development of completely new, efficient research tools, transformed afterwards to powerful methods of contemporary representation theory. Galois covering techniques ([**27**], [**19**], [**3**], [**21**], [**12**], [**11**], [**13**], [**4**], [**7**]) have remained one of them. It is usually used to reduce a problem for modules over an algebra to an analogous one, often much simpler, for its cover category. This kind of treatment allows to answer many interesting theoretical questions and obtain classifications for various classes of algebras (respectively, matrix problems) in representation-finite or tame case ([**34**], [**35**], [**36**], [**37**], [**20**], [**38**], [**39**], [**40**], [**16**], [**17**], [**18**], [**22**], [**28**], [**29**], [**30**], [**15**], [**10**]).

In the last decade the main interest in the coverings topics was concentrated, for obvious reasons, on "Galois covering tame-conjecture". Roughly speaking it asserts that the base algebra R/G is of tame representation type provided so is its cover category R

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(see [**12**], [**11**], [**13**], [**4**], [**7**] for partial solutions, also unpublished preprint [**14**]). But also other more detailed questions concerning the tame case were studied. For example closely related to the previous one, the so-called "stabilizer conjecture" is affirmatively solved in [**8**] for a representation-tame locally bounded category R over an algebraically closed field (the stabilizers G_B of infinite G-atoms B with respect to a free action of a torsion-free group G on R are infinite cyclic groups).

Recently, a behaviour of the category $\mathrm{mod}\,(R/G)$ of R/G-modules that is quite different from that one in the tame case was studied. The investigations concern the notion of non-orbicular module introduced in [**9**]. Recall that an indecomposable module X in $\mathrm{mod}\,(R/G)$ is called *orbicular* (respectively, *non-orbicular*) if the "pull-up" $F_\bullet X$ of X, with respect to the Galois covering $F : R \to R/G$, decomposes into a direct sum of indecomposable locally finite-dimensional modules which belong (respectively, do not belong) to one G-orbit (see 1.3). According to the conjecture formulated long time ago, all indecomposable R/G-modules in the tame case (studied in terms of Galois covering F) are supposed to be always orbicular (with respect to G). Moreover, they are expected to be formed by use of the standard functorial construction $\Phi^B = - \otimes_{kG_B} F_\lambda B : \mathrm{mod}\, kG_B \to \mathrm{mod}\,(R/G)$, defined by periodic G-atoms B (see 1.3). In [**9**] the problem of existence of non-orbicular indecomposable modules was discussed. It is presented there a construction of a representation embedding into the category $\mathrm{mod}\,(R/G)$ whose image contains a large, usually wild subcategory consisting of non-orbicular indecomposable R/G-modules. The construction is based on the notion of generalized tensor product with respect to a suitable sequence of periodic G-atoms.

In this paper we consider an analogous question for the so-called non-regularly orbicular R/G-modules. An indecomposable orbicular R/G-module X in $\mathrm{mod}\,(R/G)$ is called *non-regularly orbicular* provided there exists no R-action of the stabilizer G_B on a periodic G-atom B such that $X \simeq \Phi^{(B,\nu)}(V)$, for some (indecomposable) V in $\mathrm{mod}\,(kG_B)^{\mathrm{op}}$ (see 2.1). In a discussion of our problem we use the generalized tensor product for the sequences $B(f,s)$ consisting of several copies of the same periodic G-atom B, dependent on some endomorphism $f \in \mathrm{End}_R(B)$ and certain finite sequence $s = (s_2, \dots s_n)$ of positive integers. Applying this construction, we define the functors $\Phi^{B(f,s)} : I_n\text{-spr}\,(kG_B) \to \mathrm{mod}\,(R/G)$, where $I_n\text{-spr}\,(kG_B)$ denotes the category of finite-dimensional n-filtered kG_B-modules. We study a behavior of these functors with respect to possibility of creating indecomposable non-regularly orbicular R/G-modules. The main result of the paper, Theorem 2.2, asserts that under specific assumptions expressed in terms of certain conditions on the structure of the endomorphism algebra $\mathrm{End}_R(B)$, the restrictions of $\Phi^{B(f,s)}$ to some subcategories of $I_n\text{-spr}\,(kG_B)$ are representation embeddings (in the sense of [**32**], see also 1.3). Moreover, they furnish large, usually wild, families of searched modules. The idea used in the proof relies on the replacement of R/H-modules with a fixed direct summand support (see 1.3, they seem to form to a narrow class for making internally some covering construction), by more friendly world of the so-called maximal Cohen-Macaulay modules over skew group algebras (see Theorem 3.3). We study this class of modules by considering its analogue for weakly locally bounded categories with a trivial action of a fixed group (we introduce the corresponding notions in 4.1 and 4.4). We adopt into that context the generalized tensor product construction (see Theorem 4.5) and formulate a variant of the classical Galois covering scheme (see Theorem 5.4). Finding a common platform which allowed for simultaneous applying and efficient combining of these two concepts, is a crucial point of the proof.

The paper is organized as follows. In Section 1 we recall basic definitions and fix notation used in the paper. Section 2 is devoted to a discussion of properties of the functors $\Phi^{B(f,s)}$ for various sequences s. There a precise definition of non-regularly orbicular module is given and the main result of the paper, Theorem 2.2, is formulated. In Section 3, Theorem 3.3 about interpretation of the category $\operatorname{mod}_{\mathscr{B}_{\mathrm{o}}}(R/H)$ in term of the category of maximal Cohen-Macaulay modules over the skew group algebra EH is proved, where E is the endomorphism algebra of direct sum of all G-atoms from $\mathscr{B}_{\mathrm{o}}$. In Section 4 the notion of a weakly locally bounded k-category is introduced. It is shown there that indecomposable maximal Cohen-Macaulay modules over $\mathscr{E}H$ (H is a group operating trivially on objects of a weakly locally bounded category $\mathscr{E}$) have local endomorphisms rings (see Theorem 4.2). Moreover, the construction of generalized tensor product is adopted into a context of the category $\mathrm{CM}\,(\mathscr{E}H)$ (see Theorem 4.5). Section 5 is devoted to Galois coverings for weakly locally bounded categories equipped with an action of a fixed group H that acts trivially on objects. Theorem 5.4 shows that the "push-down" functor associated to a Galois covering behaves nicely in restriction to categories of maximal Cohen-Macaulay modules. Also the concept of the Galois coverings associated to suitable gradings is discussed there. Section 6 is devoted to the proof of Theorem 2.2. The properties of the induction functor and the left Kan extension functor in a context of maximal Cohen-Macaulay modules are studied there (see Lemma 6.1 and Proposition 6.2).

1. Preliminaries.

Throughout the paper we use the notation and definitions established in [**5**], [**7**], [**9**]. Nevertheless, for a benefit of the reader, we briefly recall the general situation and notions we deal with in the paper.

For basic information concerning representation theory of algebras (respectively, rings and modules, notions of theory of categories) we refer to [**31**] (respectively, [**1**], [**23**]).

1.1.

Let k be a field (not necessarily algebraically closed) and R be a k-category, that is, each set $R(x,y)$ of morphisms from x to y in R, $x,y \in \operatorname{ob} R$, is the k-linear spaces and composition in R is k-bilinear. By an *R-module* we mean a contravariant k-linear functor from R to the category of all k-vector spaces. We denote by $\mathrm{MOD}\,R$ the category of all R-modules. We denote by $\mathscr{J}_R$ the Jacobson radical of the category $\mathrm{MOD}\,R$.

Let R be a *locally bounded k-category*, that is, all objects of R have local endomorphism rings, the different objects are nonisomorphic, and the sums $\sum_{y\in R}\dim_k R(x,y)$ and $\sum_{y\in R}\dim_k R(y,x)$ are finite for each $x \in R$. An R-module M is *locally finite-dimensional* (respectively, *finite-dimensional*) if $\dim_k M(x)$ is finite for each $x \in R$ (respectively, the *dimension* $\dim_k M = \sum_{x\in R}\dim_k M(x)$ of M is finite). We denote by $\mathrm{Mod}\,R$ (respectively, $\mathrm{mod}\,R$) the full subcategory of all locally finite-dimensional (respectively, finite-dimensional) R-modules and by $\mathrm{Ind}\,R$ (respectively, $\mathrm{ind}\,R$) the full subcategory of all indecomposable R-modules in $\mathrm{Mod}\,R$ (respectively, $\mathrm{mod}\,R$). By the *support* of an object M in $\operatorname{MOD} R$ we mean the full subcategory $\operatorname{supp} M$ of R formed by the set $\{x \in R : M(x) \neq 0\}$.

For any k-algebra A we denote analogously by $\operatorname{MOD} A$ (respectively, $\operatorname{mod} A$) the category of all (respectively, all finite-dimensional) right A-modules and by $J(A)$ the Jacobson radical of A. Clearly, $\operatorname{MOD} A$ can be always interpreted as $\operatorname{MOD} R(A)$, where $R(A)$ is a k-category consisting of one object with the endomorphism k-algebra equal to A.

To any finite full subcategory C of R we can attach the finite-dimensional algebra $A(C) = \bigoplus_{x,y \in \operatorname{ob} C} R(x,y)$ endowed with the multiplication given by the composition in R. It is well known that the mapping $M \mapsto \bigoplus_{x \in \operatorname{ob} C} M(x)$ yields an equivalence

$$\operatorname{mod} C \simeq \operatorname{mod} A(C)\,.$$

1.2.

Let G be a group acting by k-linear automorphisms on a k-category R. Then G acts on the category $\operatorname{MOD} R$ by translations ${}^{g}(-)$, which assign to each M in $\operatorname{MOD} R$ the R-module ${}^{g}M = M \circ g^{-1}$ and to each $f : M \to N$ in $\operatorname{MOD} R$ the R-homomorphism ${}^{g}f : {}^{g}M \to {}^{g}N$ given by the family $(f(g^{-1}(x)))_{x \in R}$ of k-linear maps.

Given M in $\operatorname{MOD} R$, the subgroup

$$G_M = \{g \in G : {}^{g}M \simeq M\}$$

of G is called the *stabilizer* of M.

Let R be a locally bounded k-category. Assume that G acts freely on the objects of R (that is the stabilizer G_x is trivial for every $x \in \operatorname{ob} R$) so it can be regarded as a subgroup of $\operatorname{Aut}_{k-\mathrm{cat}}(R)$. Then the orbit category R/G of the action of G on R is again a locally bounded k-category (see [**19**]). One can study the module category $\operatorname{mod}(R/G)$ in terms of the category $\operatorname{Mod} R$ using the pair of functors

$$\operatorname{MOD} R \underset{F_\bullet}{\overset{F_\lambda}{\rightleftarrows}} \operatorname{MOD}(R/G)$$

where $F_\bullet : \operatorname{MOD}(R/G) \to \operatorname{MOD} R$ is the "pull-up" functor associated with the canonical Galois covering functor $F : R \to R/G$, assigning to each X in $\operatorname{MOD}(R/G)$ the R-module $X \circ F$, and the "push-down" functor $F_\lambda : \operatorname{MOD} R \to \operatorname{MOD}(R/G)$ is the left adjoint to $F_\bullet$.

The classical results from [**19**] asserts that if G acts freely on $(\operatorname{ind} R)/\simeq$ (that is $G_M = \{\mathrm{id}_R\}$ for every M in $\operatorname{ind} R$) then F_λ induces an embedding of the set $((\operatorname{ind} R)/\simeq)/G$ of the G-orbits of isoclasses of objects in $\operatorname{ind} R$ into $(\operatorname{ind}(R/G))/\simeq$.

Let H be a subgroup of the stabilizer G_M of a given M in $\operatorname{MOD} R$. By an *R-action* of H on M we mean a family

$$\mu = (\mu_g : M \to {}^{g^{-1}}M)_{g \in H}$$

of R-homomorphisms such that $\mu_e = \mathrm{id}_M$, where $e = \mathrm{id}_R$ is the unit of H, and ${}^{g_1^{-1}}\mu_{g_2} \cdot \mu_{g_1} = \mu_{g_2 g_1}$ for all $g_1, g_2 \in H$ (see [**19**]). We note that if H is a free group then M admits an R-action of H (see [**4**, Lemma 4.1]).

For any subgroup H of G we denote by $\mathrm{MOD}^H R$ (respectively, $\mathrm{Mod}^H R$) the category consisting of pairs (M, μ), where M is an R-module (respectively, a locally finite-dimensional R-module) and μ an R-action of H on M. For any $M = (M, \mu)$ and $N = (N, \nu)$ in $\mathrm{MOD}^H R$ (respectively, $\mathrm{Mod}^H R$) the space of morphisms from M to N in $\mathrm{MOD}^H R$ (respectively, $\mathrm{Mod}^H R$) consists of all $f \in \mathrm{Hom}_R(M, N)$ such that ${}^{g^{-1}}f \cdot \mu_g = \nu_g \cdot f$, for every $g \in H$, and is denoted by $\mathrm{Hom}_R^H(M, N)$. Note that $\mathrm{Hom}_R^H(M, N)$ is the set of H-invariant elements in $\mathrm{Hom}_R(M, N)$ with respect to the action $\mathrm{Hom}_R(\mu, \nu) : H \times \mathrm{Hom}_R(M, N) \to \mathrm{Hom}_R(M, N)$, given by the mapping $(g, f) \mapsto {}^g\nu_g {}^g f\ \mu_{g^{-1}}$, $g \in H$, $f \in \mathrm{Hom}_R(M, N)$. By $\mathrm{Mod}_{\mathrm{f}}^H R$ we denote the full subcategory of the category $\mathrm{Mod}^H R$ formed by all (M, μ) such that $\operatorname{supp} M$ is contained in the union of a finite number of H-orbits in R (see [**19**], [**13**], [**4**]). Then the functor $F_\bullet$, associating with any X in $\mathrm{mod}\,(R/G)$ the R-module $F_\bullet X$ endowed with the natural R-action of G, yields an equivalence

$$\mathrm{mod}\,(R/G) \simeq \mathrm{Mod}_{\mathrm{f}}^G R\,.$$

The main notions of this paper refer to the structure of objects from $\mathrm{Mod}_{\mathrm{f}}^G R$ (consequently, $\mathrm{mod}\,(R/G)$) based on the concept of G-atoms. Following [**4**], an indecomposable R-module B in $\mathrm{Mod}\,R$ (with local endomorphism ring) is called a *G-atom* (over R) provided $\operatorname{supp} B$ is contained in the union of a finite number of G_B-orbits in R. The G-atom B is said to be *finite* (respectively, *infinite*) if G_B (equivalently $\operatorname{supp} B$) is finite (respectively, infinite).

Denote by $\mathscr{A}$ a fixed set of representatives of isoclasses of all G-atoms in $\mathrm{Mod}\,R$, by $\mathscr{A}_{\mathrm{o}}$ a fixed set of representatives of G-orbits of the induced action of G on $\mathscr{A}$ and for any $B \in \mathscr{A}_{\mathrm{o}}$ by S_B a fixed set of representatives of left cosets of G_B in G, containing the unit id_R of the group G. One can show that the category $\mathrm{mod}\,(R/G)$ is equivalent via $F_\bullet$ to the full subcategory of $\mathrm{Mod}_{\mathrm{f}}^G R$ formed by all possible pairs (M_n, μ), where $n = (n_B)_{B \in \mathscr{A}_{\mathrm{o}}}$ is a sequence of natural numbers, such that almost all n_B are zeros, M_n an R-module given by the formula

$$M_n = \bigoplus_{B \in \mathscr{A}_{\mathrm{o}}} \Big(\bigoplus_{g \in S_B} {}^g(B^{n_B}) \Big)$$

and μ an arbitrary R-action of G on M_n. Therefore to any module X in $\mathrm{mod}\,(R/G)$ one can attach the *direct summand support* $\mathrm{dss}(X)$ of X which is the finite set consisting of all $B \in \mathscr{A}_{\mathrm{o}}$ such that $n_B \neq 0$.

For any $\mathscr{U} \subseteq \mathscr{A}_{\mathrm{o}}$ one can study the full subcategory $\mathrm{mod}_{\mathscr{U}}(R/G)$ of $\mathrm{mod}\,(R/G)$ consisting of all X in $\mathrm{mod}\,(R/G)$ such that $\mathrm{dss}(X) \subseteq \mathscr{U}$.

1.3.

Following [**9**], an indecomposable module X in $\mathrm{mod}\,(R/G)$ is called *orbicular* (cf. [**19**]) provided $\mathrm{dss}(X) = \{B\}$, for some $B \in \mathscr{A}_{\mathrm{o}}$. This condition simply means that in a decomposition of the R-module $F_\bullet X$ into a direct sum of indecomposables occur only G-atoms contained, up to isomorphism, in one orbit of G in $\mathscr{A}$. The module X is called *non-orbicular* if X is not orbicular. The additive closure of all indecomposable orbicular R/G-modules can be presented as a splitting union

$$\bigvee_{B\in\mathscr{A}_o} \operatorname{mod}_B(R/G)\,,$$

in the sense of [**9**], whereas, indecomposable non-orbicular modules are those objects of $\operatorname{ind}(R/G)$, which lay out of $\bigvee_{B\in\mathscr{A}_o}\operatorname{mod}_B(R/G)$, where $\operatorname{mod}_B(R/G)=\operatorname{mod}_{\{B\}}(R/G)$.

The category of orbicular modules forms an essential part of the category $\operatorname{mod}(R/G)$. Recall that if R/G is representation-finite then all R/G-modules are orbicular (see [**19**], [**24**]). According to a general conjecture, all R/G-modules would be orbicular in the tame case (especially those which belong to 1-parameter families). Roughly speaking all R/G-modules which occurred up to now in the Galois covering context (in representation-finite and tame case) were orbicular. Moreover, they were described by use of the following construction.

Suppose that a G-atom B admits an R-action ν_B of G_B on itself (this is always the case if the group G_B is free). Then $F_\lambda B$ carries the structure of a kG_B-R/G-bimodule, which is finitely generated free as a left kG_B-module, where kG_B is the group algebra of G_B over k (see [**13**, 3.6]). This bimodule induces a functor

$$\Phi^{(B,\,\nu_B)} = -\otimes_{kG_B} F_\lambda B : \operatorname{mod} kG_B \to \operatorname{mod}_B(R/G)$$

which is a representation embedding, provided the field $\operatorname{End}_R(B)/J(\operatorname{End}_R(B))$ is equal to k (see [**5**, Proposition 2.3]). (Following [**32**], a k-linear functor $T:\operatorname{mod}A_1\to\operatorname{MOD}A_2$, between module categories of finitely generated k-algebras A_1 and A_2, is a representation embedding, provided it is exact and induces an injection between the sets of isomorphism classes of indecomposable modules.) Note that if G_B is trivial then $kG_B\simeq k$ and if G_B is an infinite cyclic group then kG_B is isomorphic to the algebra $k[t,t^{-1}]$ of Laurent polynomials. We refer to [**13**], [**4**], [**7**], [**9**] for more details about the functors

$$\{\Phi^{(B,\,\nu_B)}\}_{B\in\mathscr{U}} : \coprod_{B\in\mathscr{U}} \operatorname{mod} kG_B \to \operatorname{mod}(R/G)$$

where $\mathscr{U}$ consists of cyclic G-atoms.

1.4.

Let H be a group. Then left modules over the group k-algebra kH are just k-representations, so each module V in $\operatorname{MOD}(kH)^{\mathrm{op}}$ is uniquely represented by a pair (V,μ), where V is a k-vector space and $\mu: H\to\operatorname{Aut}_k(V)$ is a group homomorphism.

For any $n\in\boldsymbol{N}$, we denote by I_n-$\operatorname{spr}(kH)$ the chain category whose objects are sequences of the form

$$V: \qquad V_1\subseteq V_2\subseteq\cdots\subseteq V_{n-1}\subseteq V_n$$

where V_i, $i = 1, \ldots, n-1$, are kH-submodules of a left finite-dimensional kH-module V_n, and the set of morphisms from V to V' consists of all kH-homomorphisms $f : V_n \to V'_n$ such that $f(V_i) \subseteq V'_i$ for every $i = 1, \ldots, n-1$ (see [**9**], [**33**]).

Suppose we are given a sequence

$$B: \qquad B_1 \overset{\beta_2}{\leftarrow} B_2 \leftarrow \cdots \leftarrow B_{n-1} \overset{\beta_n}{\leftarrow} B_n \tag{$*$}$$

in $\mathrm{Mod}^H R$, that is all objects $B_i = (B_i, \nu_i)$ are in $\mathrm{Mod}^H R$ (B_i is an R-module and ν_i is an R-action of H on B_i) and all R-homomorphisms β_i are morphisms in $\mathrm{Mod}^H R$ (β_i are compatible with the actions). We denote by $\beta = \beta(B)$ the family $(\beta_{i,j}(h) = (\nu_i)_h \cdot \beta_{i,j} : B_j \to {}^{h^{-1}}B_i)_{1 \le i,j \le n,\, h \in H}$, of R-homomorphisms, where the R-homomorphisms $(\beta_{i,j} : B_j \to B_i)_{1 \le i,j \le n}$ are defined as follows:

$$\beta_{i,j} = \begin{cases} \beta_{i+1} \cdot \cdots \cdot \beta_j & \text{if } \ i < j\,, \\ \mathrm{id}_{B_i} & \text{if } \ i = j\,, \\ 0 & \text{if } \ i > j\,. \end{cases}$$

Then for any object

$$V: \qquad V_1 \subseteq V_2 \subseteq \cdots \subseteq V_{n-1} \subseteq V_n$$

in I_n-spr$\,(kH)$, $V_n = (V_n, \mu)$, one constructs an object $\underline{V} \otimes_k B = (\underline{V} \otimes_k B, \underline{\mu} \otimes_k \beta)$ in $\mathrm{Mod}^H R$ [**9**], where $\underline{V} = (\underline{V}_i)_{\,i=1,\ldots,n}$ is a fixed *sequence of complementary direct summands* for V ($\underline{V}$ is a sequence of subspaces $\underline{V}_i$ of V such that $\underline{V}_1 = V_1$ and $V_i = V_{i-1} \oplus \underline{V}_i$ for $i = 2, \ldots, n$). We set

$$\underline{V} \otimes_k B = \bigoplus_{i=1}^{n} \underline{V}_i \otimes_k B_i\,,$$

and $\underline{\mu} \otimes_k \beta = ((\underline{\mu} \otimes_k \beta)_h : \underline{V} \otimes_k B \to {}^{h^{-1}}(\underline{V} \otimes_k B))_{h \in H}$, where R-homomorphisms $(\underline{\mu} \otimes_k \beta)_h$, $h \in H$, are defined as the matrix R-homomorphisms

$$[\,\mu(h)_{i,j} \otimes_k \beta_{i,j}(h)]_{\,1 \le i,j \le n} : \bigoplus_{j=1}^{n} \underline{V}_j \otimes_k B_j \to \bigoplus_{i=1}^{n} {}^{h^{-1}}(\underline{V}_i \otimes_k B_i)$$

($\underline{\mu}(h) = [\,\mu(h)_{i,j}]_{\,1 \le i,j \le n}$ is the matrix presentation of the k-automorphism $\mu(h) : \bigoplus_{j=1}^{n} \underline{V}_j \to \bigoplus_{i=1}^{n} \underline{V}_i$, for $h \in H$). It is not hard to check that the data $(\underline{V} \otimes_k B, \underline{\mu} \otimes_k \beta)$ defines correctly an object $\underline{V} \otimes_k B$ in $\mathrm{Mod}^H R$. Moreover, $\underline{V} \otimes_k B$ belongs to $\mathrm{Mod}_{\mathrm{f}}^H R$ provided so do all objects $B_i = (B_i, \nu_i)$, $i = 1, \ldots, n$.

The above construction can be extended to a functor

$$- \otimes_k B : I_n\text{-spr}\,(kH) \to \mathrm{Mod}^H R$$

(we set $V \otimes_k B = \underline{V} \otimes_k B$), called *the generalized tensor product* functor. So in the case all objects B_i belong to $\mathrm{Mod}_{\mathrm{f}}^{H} R$, we can define the composite functor

$$\Phi^B : I_n\text{-spr}\,(kH) \to \mathrm{mod}\,(R/G)$$

which is given by the composition

$$I_n\text{-spr}\,(kH) \xrightarrow{-\otimes_k B} \mathrm{Mod}_{\mathrm{f}}^{H} R \xrightarrow{\theta} \mathrm{Mod}_{\mathrm{f}}^{G} R \xrightarrow{F_\bullet^{-1}} \mathrm{mod}\,(R/G)\,,$$

where $F_\bullet^{-1}$ is a fixed quasi-inverse for $F_\bullet : \mathrm{mod}\,(R/G) \to \mathrm{Mod}_{\mathrm{f}}^{G} R$, and

$$\theta = \theta_H^G : \mathrm{Mod}_{\mathrm{f}}^{H} R \to \mathrm{Mod}_{\mathrm{f}}^{G} R$$

is the induction functor assigning to any $M = (M, \mu)$ in $\mathrm{Mod}_{\mathrm{f}}^{H} R$, the object $\theta(M) = (\oplus_{g_1 \in S_H} {}^{g_1}M, \mu^G)$ in $\mathrm{Mod}_{\mathrm{f}}^{G} R$. (Here S_H is a fixed set of representatives of G/H containing $e = \mathrm{id}_R$, μ^G a standard R-action of G induced by μ, consisting of the R-isomorphisms $\mu_g^G : \oplus_{g_1 \in S_H} {}^{g_1}M \to \oplus_{g_2 \in S_H} {}^{g^{-1} g_2}M$, $g \in G$, given by the families ${}^{g_1}\mu_h : {}^{g_1}M \to {}^{g^{-1} g_2}M$, $g_1 \in S_H$, where $g_2 \in S_H$ and $h \in H$ are determined by the equality $gg_1 = g_2 h$; see [**9**, 3.1] for the precise definitions).

The functors Φ^B introduced in [**9**] were used for studying non-orbicular modules, where the sequences B were formed by pairwise different periodic G-atoms with a common stabilizer H. This construction was, in a natural way, an extension of that described in 1.3. More precisely, $\Phi^B = \Phi^{(B_1, \nu_1)}$ for $n = 1$ (see [**4**, Proposition 2.3]).

2. The main result.

2.1.
In the paper we study a certain class of indecomposable orbicular R/G-modules distinguished in internal terms of coverings, for a given Galois covering $F : R \to R/G$.

Definition. Let B be a G-atom in $\mathrm{Mod}\, R$. An indecomposable (orbicular) R/G-module X in $\mathrm{mod}_B(R/G)$ is called *regularly orbicular* provided there exists an R-action of G_B on B such that

$$X \simeq \Phi^{(B,\nu)}(V)$$

for some (indecomposable) V in $\mathrm{mod}\,(kG_B)^{\mathrm{op}}$.

An indecomposable (orbicular) R/G-module X in $\mathrm{mod}_B(R/G)$ is called *non-regularly orbicular* if it is not regularly orbicular.

Note that preselection of a G-atom B in the definition does not restrict generality and does not cause any problems, since each indecomposable orbicular R/G-module belongs to precisely one subcategory $\mathrm{mod}_B(R/G)$, $B \in \mathscr{A}_{\mathrm{o}}$.

We present a construction which shows (under some circumstances) an existence of indecomposable non-regularly orbicular R/G-modules appearing in large, usually wild families.

2.2.

Let $B = (B, \nu)$ be a periodic G-atom together with a selected R-action of G_B on itself. For simplicity we denote by H the stabilizer G_B of B. For any H-invariant endomorphism $f \in \mathrm{End}_R(B)$ and sequence $s = (s_2, \dots s_n)$ of positive integers, $n \geq 2$, we denote by $B(f, s)$ a sequence

$$\mathbb{B}: \qquad B_1 \overset{\beta_2}{\leftarrow} B_2 \leftarrow \cdots \leftarrow B_{n-1} \overset{\beta_n}{\leftarrow} B_n$$

of objects and morphisms in the category in $\mathrm{Mod}^H R$ such that all objects $B_i = (B_i, \nu_i)$ are equal to B and $\beta_i = f^{s_i}$, for $i = 2, \dots, n$. We obtain the functor

$$\Phi^{\mathbb{B}} = \Phi^{B(f,s)} : I_n\text{-spr}\,(kH) \to \mathrm{mod}_B(R/G)\,.$$

We assume that $f^{s_2+\cdots+s_n} \neq 0$. Note that if $f \in \mathrm{End}_R(B)$ is nilpotent and $r = r(f) \in \boldsymbol{N}$ is a nilpotency degree of f then the longest possible sequence $\bar{s} = \bar{s}(f)$, with that property, has the form $\bar{s} = (\bar{s}_2, \dots, \bar{s}_r)$, where $\bar{s}_2 = \cdots = \bar{s}_r = 1$.

Before we formulate the main result of this paper we recall some definitions and fix a notation.

To any $V = (V_1 \subseteq V_2 \subseteq \cdots \subseteq V_n)$ in I_n-spr (kH) we attach the *coordinate vector*

$$\mathrm{cdn}(V) = (d_1, \dots, d_n)$$

in $\boldsymbol{N}^n$, given by $d_i = \dim_k V_i/V_{i-1}$ ($V_0 = 0$), and the *coordinate support* csupp(V) which by definition is the increasing sequence consisting of all $i \in \{1, \dots, n\}$ such that $d_i \neq 0$. We say that the object V is *sincere* if all coordinates d_i, $i = 1, \dots, n$, are nonzero. By I_n-spr$\,'(kH)$ (respectively, I_n-spr$_{\,1}(kH)$) we denote the additive closure of the full subcategory formed by all indecomposables V in I_n-spr (kH) such that cdn(V) has at least two nonzero coordinates (respectively, the first coordinate of cdn(V) is nonzero). By I_n-spr$'_1(kH)$ we denote the additive closure of the full subcategory formed by all indecomposable V in I_n-spr (kH) lying simultaneously in I_n-spr$\,'(kH)$ and I_n-spr$_{\,1}(kH)$. For a fixed sequence s as above, by I_n-spr$_{\,l(s)}(kH)$ (respectively, I_n-spr$'_{\,l(s)}(kH)$) we denote the additive closure of the full subcategory formed by all indecomposables V in I_n-spr (kH) whose coordinate support csupp$(V) = (u_1, \dots, u_m)$, $m \leq n$, has the property that there exists no $u' = (u'_1, \dots, u'_m) \in \boldsymbol{N}^m$, $1 \leq u'_1 < \cdots < u'_m \leq n$, satisfying the condition

$$\begin{cases} s_{u'_1+1} + \dots + s_{u'_2} = s_{u_1+1} + \dots + s_{u_2} \\ \qquad \vdots \qquad\qquad\qquad\qquad \vdots \\ s_{u'_1+1} + \dots + s_{u'_m} = s_{u_1+1} + \dots + s_{u_m} \end{cases} \tag{$*$}$$

or equivalently,

$$\begin{cases} s_{u'_1+1} + \dots + s_{u'_2} = s_{u_1+1} + \dots + s_{u_2} \\ \qquad \vdots \qquad\qquad\qquad\qquad \vdots \\ s_{u'_{m-1}+1} + \dots + s_{u'_m} = s_{u_{m-1}+1} + \dots + s_{u_m} \end{cases} \tag{$*'$}$$

such that $u'_1 < u_1$ (respectively, all indecomposables lying simultaneously in I_n-spr$'(kH)$ and I_n-spr$_{l(s)}(kH)$). Clearly, always I_n-spr$_1(kH)$ is contained in I_n-spr$_{l(s)}(kH)$. Note that if $s_2 = \cdots = s_n = 1$ then I_n-spr$_{l(s)}(kH) = I_n$-spr$_1(kH)$ since for any V in I_n-spr(kH) with $u_1 > 1$, csupp$(V) = u = (u_1, \ldots, u_m)$, one can take for u' any of the sequences $u[i] = (u_1 - i, \ldots, u_m - i)$, where $u_m - n \le i \le u_1 - 1$ (we discuss the meaning of this effect in 2.4).

Theorem. *Let $B = (B, \nu)$ be a periodic G-atom with a fixed R-action of the stabilizer $H = G_B$, such that* $\mathrm{End}_R(B)/J(\mathrm{End}_R(B)) \simeq k$. *Assume that the algebra* $\mathrm{End}_R(B)$ *admits a grading* $\mathrm{End}_R(B) = \bigoplus_{\gamma\in\Gamma} E_\gamma$ *by an infinite cyclic group $\Gamma = \boldsymbol{Z}$, satisfying the following conditions*:

(a) *each E_γ, $\gamma \in \Gamma$, is an H-invariant subspace and $E_\gamma = 0$, for almost all $\gamma \in \Gamma$,*

(b) *there exists a homogeneous element* $f \in \mathrm{End}_R^H(B) \cap J(\mathrm{End}_R(B))$ *which admits a surjective H-invariant algebra homomorphism* $\pi : \mathrm{End}_R(B) \to A$, *where* $A = k[f] = \bigoplus_{i=0}^{r-1} kf^i$, $r = r(f)$, *is the subalgebra of* $\mathrm{End}_R(B)$ *generated by f, such that* $\pi_{|A} = \mathrm{id}_A$ *and*

$$\pi(E_\gamma) = \begin{cases} kf^\gamma & \text{if } \gamma \ge 0\,, \\ 0 & \text{if } \gamma < 0\,. \end{cases}$$

Then for any f as above and a sequence $s = (s_2, \ldots, s_n)$ of positive integers, $n \ge 2$, such that $s_2 + \cdots + s_n < r$, the restriction of the functor

$$\Phi^{B(f,s)} : I_n\text{-spr}\,(kH) \to \mathrm{mod}\,(R/G)$$

to the category I_n-spr$_{l(s)}(kH)$ is a representation embedding. Moreover, the all indecomposables from $\mathrm{mod}_B(R/G)$ *lying in the image of the restriction of $\Phi^{B(f,s)}$ to I_n-spr$'_{l(s)}(kH)$ are non-regularly orbicular indecomposable R/G-modules. In particular, the full subcategory formed by all indecomposable non-regularly orbicular modules from* $\mathrm{mod}_B(R/G)$ *is wild, provided $n \ge 2$ and H has a factor which is an infinite cyclic group (respectively, a cyclic p-group of order greater than* 7, *if* $\mathrm{char}\, k = p > 0$).

Note that all maps f^i, $i = 1, \ldots, n$, are morphisms in $\mathrm{Mod}^H R$ (see 1.2 and 3.2) and that f is nilpotent since $\mathrm{End}_R(B)$ is semiprimary (see [**7**, Theorem 2.9]).

A complete proof of the theorem (together with closer explanation of the real meaning of the assumptions) is given in Section 6. It needs several preparatory results.

2.3.

We start by analyzing quite general problem when an R/G-module of the form $\Phi^{B(f,s)}(V)$ can be indecomposable regularly orbicular. We keep the notation introduced in 2.2 (without assumptions of Theorem 2.2 on the structure of $\mathrm{End}_R(B)$).

Let $B' = (B', \nu')$ be another (cf. 2.2) periodic G-atom together with a selected R-action of $G_{B'}$ on itself. Assume that $\mathrm{End}_R(B')/J(\mathrm{End}_R(B')) \simeq k$. Then by

$$\Psi^{(B',\nu')} : \mathrm{mod}\,(R/G) \to \mathrm{mod}\,(kG_{B'})^{\mathrm{op}}$$

we denote the composite functor

$$\operatorname{mod}(R/G) \xrightarrow{F_\bullet} \operatorname{Mod}_{\mathrm{f}}^G \xrightarrow{\bar{\mathscr{H}}} \operatorname{mod}(kG_{B'})^{\mathrm{op}}$$

where $\bar{\mathscr{H}} = \operatorname{Hom}_R(B', -)/\mathscr{J}(B', -)$. Recall that the structure of $kG_{B'}$-module on the k-linear space $\operatorname{Hom}_R(B', M)/\mathscr{J}(B', M)$, for $M = (M, \mu)$ in $\operatorname{Mod}_{\mathrm{f}}^G$, is induced by the k-linear action $\operatorname{Hom}_R(\nu', \mu)$ of the stabilizer $G_{B'}$ on the space $\operatorname{Hom}_R(B', M)$. We have at our disposal the formula

$$\Psi^{(B',\nu')} \circ \Phi^{(B,\nu)} = \begin{cases} - \otimes_k k_{(\nu',\nu)} & \text{if } \ B = B' \\ 0 & \text{if } \ B \not\simeq B' \end{cases} \tag{$**$}$$

where $k_{(\nu',\nu)} = \Psi^{(B,\nu')}(B,\nu)$ is the representation of G_B in the space $\operatorname{End}_R(B)/J(\operatorname{End}_R(B))\,(\simeq k)$ with the action of G_B induced by $\operatorname{Hom}_R(\nu', \nu)$. In particular, we have (see [**5**])

$$\Psi^{(B',\nu')} \circ \Phi^{(B',\nu')} \simeq \mathrm{id}_{\operatorname{mod} kG_{B'}}$$

and the functor

$$\Phi^{(B',\nu')} : \operatorname{mod} kG_{B'} \to \operatorname{mod}_{B'}(R/G)$$

is a representation embedding in the sense of [**32**].

Let

$$Gr : I_n\text{-spr}\,(kH) \to \operatorname{mod}(kH)^{\mathrm{op}}$$

be the classical functor attaching to a filtered module $V = (V_1 \subseteq V_2 \subseteq \cdots \subseteq V_n)$ in I_n-spr (kH), the associated graded one, given by the formula

$$Gr(V) = \bigoplus_{i=1}^{n} V_i/V_{i-1}\,.$$

Note that $Gr(V)$ is a decomposable R/G-module whenever V belongs to I_n-spr$'(kH)$.

We can formulate the following important, for the proof of Theorem 2.2, result.

Lemma. *Assume that $B' = B$ in $\operatorname{Mod} R$ and that f belongs to $J(\operatorname{End}_R(B))$. Then the endofunctors*

$$(- \otimes_k k_{(\nu',\nu)}) \circ Gr,\ \Psi^{(B,\nu')} \circ \Phi^{B(f,s)} : I_n\text{-spr}\,(kH) \to I_n\text{-spr}\,(kH)$$

are isomorphic.

Proof. Fix an object $V = (V_1 \subseteq V_2 \subseteq \cdots \subseteq V_n)$ in I_n-spr(kH). Then we have the sequence of kH-isomorphisms

$$\begin{aligned}\Psi^{(B,\nu')} \circ \Phi^{B(f,s)}(V) &\simeq \bar{\mathscr{H}}\Big(B, \bigoplus_{g\in S_H} {}^g\Big(\bigoplus_{i=1}^n \underline{V}_i \otimes_k B\Big)\Big) \\ &\simeq \bar{\mathscr{H}}\Big(B, \bigoplus_{i=1}^n \underline{V}_i \otimes_k B\Big) \oplus \bar{\mathscr{H}}\Big(B, \bigoplus_{e\neq g\in S_H} {}^g\Big(\bigoplus_{i=1}^n \underline{V}_i \otimes_k B\Big)\Big) \\ &= \bar{\mathscr{H}}\Big(B, \bigoplus_{i=1}^n \underline{V}_i \otimes_k B\Big).\end{aligned}$$

Note that, by the definition of S_H, the decomposition

$$\bigoplus_{g\in S_H} {}^g\Big(\bigoplus_{i=1}^n \underline{V}_i \otimes_k B\Big) = \Big(\bigoplus_{i=1}^n \underline{V}_i \otimes_k B\Big) \oplus \Big(\bigoplus_{e\neq g\in S_H} {}^g\Big(\bigoplus_{i=1}^n \underline{V}_i \otimes_k B\Big)\Big)$$

of R/G-modules (see also definition of the induced R-action of G), and the equality

$$\bar{\mathscr{H}}\Big(B, \bigoplus_{e\neq g\in S_B} {}^g\Big(\bigoplus_{i=1}^n \underline{V}_i \otimes_k B\Big)\Big) = 0$$

hold. Moreover, the standard k-isomorphism

$$\bar{\mathscr{H}}\Big(B, \bigoplus_{i=1}^n \underline{V}_i \otimes_k B\Big) \simeq \bigoplus_{i=1}^n \bar{\mathscr{H}}(B, \underline{V}_i \otimes_k B)$$

yields in fact a decomposition into a direct sum of kH-modules (the action of H on $\underline{V}_i$ is given by the family $\underline{\mu}_{i,i} = (\mu_{i,i}(h))_{h\in H}$) since all but the diagonal components of the matrices $(\underline{\mu} \otimes_k B(f,s))(h)$, $h \in H$, defining the R-action of H on the R-module $\underline{V} \otimes_k B(f,s)$, belong to the Jacobson radical $\mathscr{J}$. Finally, for $B' = (B', \nu')$ with $B' = B$ (as R-modules), the k-linear maps $\underline{V}_i \otimes_k \mathrm{Hom}_R(B', B) \to \mathrm{Hom}_R(B', \underline{V}_i \otimes_k B)$, given by the mapping $v \otimes t \mapsto v \otimes t(-)$, $v \in \underline{V}_i$, $t \in \mathrm{Hom}_R(B', B)$, are H-invariant with respect to the standard actions induced by $\underline{\mu}_{i,i}$ and $\mathrm{Hom}_R(\nu', \nu)$. They yield the kH-isomorphisms

$$\underline{V}_i \otimes_k k_{(\nu',\nu)} \simeq \bar{\mathscr{H}}(B', \underline{V}_i \otimes_k B)$$

$i = 1, \ldots, n$; in consequence, the kH-isomorphism

$$Gr(V) \otimes_k k_{(\nu',\nu)} \simeq \bigoplus_{i=1}^n \underline{V}_i \otimes_k k_{(\nu',\nu)} \simeq \bigoplus_{i=1}^n \bar{\mathscr{H}}(B', \underline{V}_i \otimes_k B)\,.$$

It is easy to check that the composite kH-isomorphism

$$\Psi^{(B,\nu')} \circ \Phi^{B(f,s)}(V) \simeq Gr(V) \otimes_k k_{(\nu',\nu)}$$

is natural with respect to V in I_n-spr$\,(kH)$. □

COROLLARY. *For any V in I_n-spr$'(kH)$, $\Phi^{B(f,s)}(V)$ is not an indecomposable regularly orbicular R/G-module.*

PROOF. Suppose that $\Phi^{B(f,s)}(V)$ is a regularly orbicular R/G-module, where V is as above. Since $\Phi^{B(f,s)}(V)$ belongs to $\mathrm{mod}_B(R/G)$, we have $\Phi^{B(f,s)}(V) \simeq \Phi^{(B',\nu')}(W)$ for some indecomposable! kH-module W, where $B' = B$ and ν' is an R-action of H on B. Then by the lemma and the formula $(**)$ it follows that

$$W \simeq Gr(V) \otimes_k k_{(\nu',\nu)}\,.$$

On the other hand the kH-module $Gr(V)$ has the decomposition $Gr(V) = \bigoplus_{i=1}^n V_i/V_{i-1}\,$, therefore $Gr(V) \otimes_k k_{(\nu',\nu)} \simeq \bigoplus_{i=1}^n (V_i/V_{i-1} \otimes_k k_{(\nu',\nu)})\,$, and the kH-module $Gr(V) \otimes_k k_{(\nu',\nu)}$ is not indecomposable since V belongs to I_n-spr$'(kH)$, a contradiction. Consequently, $\Phi^{B(f,s)}(V)$ can not be an indecomposable regularly orbicular R/G-module. □

2.4.

Next we discuss the problem how can the fibers of the functor $\Phi^{B(f,s)}$ look like.

For any $m \leq n$ and a sequence $u = (u_1, \dots, u_m) \in \boldsymbol{N}^m$ such that $1 \leq u_1 < u_2 < \cdots < u_m \leq n$, we denote by I_n^u-spr$\,(kH)$ the full subcategory of I_n-spr$\,(kH)$ formed by all $V = (V_1 \subseteq \cdots \subseteq V_n)$ such that csupp(V) is contained in u (regarded as a set). Moreover, we denote by

$$\varepsilon_n^u : I_m\text{-spr}\,(kH) \hookrightarrow I_n\text{-spr}\,(kH)$$

the full embedding given by $(V_1 \subseteq \cdots \subseteq V_m) \mapsto (V_1' \subseteq \cdots \subseteq V_n')$, where $V_j' = 0$ for $j < u_1$, $V_j' = V_i$ for $u_i \leq j < u_{i+1}$, $i = 1, \dots, n-1$, and $V_j' = V_m$ for $j \geq u_m$ (in particular $\mathrm{cdn}(\varepsilon_n^u(V))_j = \mathrm{cdn}(V)_i$ if $j = u_i$ for some i and $\mathrm{cdn}(\varepsilon_n^u(V))_j = 0$ otherwise). It is clear that ε_n^u yields the equivalence

$$I_m\text{-spr}\,(kH) \simeq I_n^u\text{-spr}\,(kH)$$

of categories. Consequently, for any increasing sequences $u, u' \in \boldsymbol{N}^m$ of positive integers as above, the functors ε_n^u and $\varepsilon_n^{u'}$ induce the equivalence

$$\varepsilon_n^{u,u'} : I_n^{u'}\text{-spr}\,(kH) \to I_n^u\text{-spr}\,(kH)$$

of categories; in particular, I_n^u-spr$\,(kH) \simeq I_n^{u[i]}$-spr$\,(kH)$ for any i, $u_m - n \leq i \leq u_1 - 1$.

Assume we are given a sequence $s = (s_2, \dots, s_n)$ of positive integers such that

$s_2+\cdots+s_n < r = r(f)$. For any $u \in \boldsymbol{N}^m$, $1 \le u_1 < u_2 < \cdots < u_m \le n$, we denote by $s(u)$ the sequence $s(u) = (s(u)_2, \ldots, s(u)_m)$, where $s(u)_i = s_{u_{i-1}+1} + \cdots + s_{u_i}$, $i = 2, \ldots, m$, and by $\bar{u}^s$ the minimal with respect to u'_1 increasing sequence $u' \in \boldsymbol{N}^m$ of positive integers, satisfying together with u the condition $(*)$ (generally $1 \le u'_1 \le u_1$, but it can happen $1 < u'_1$ or even $u'_1 = u_1$). Then the equalities $(*')$ mean exactly $s(u) = s(u')$. It is also clear that indecomposable V in I_n-spr$\,(kH)$ belongs to I_n-spr$_{l(s)}(kH)$ if and only if $u = \bar{u}^s$, where $u = \operatorname{csupp}(V)$.

Lemma. (a) *Let $u \in \boldsymbol{N}^m$ be a sequence as above. Then the functors*

$$\Phi^{B(f,s)} \circ \varepsilon_n^u,\ \Phi^{B(f,s(u))} : I_m\text{-spr}\,(kH) \to \operatorname{mod}(R/G)$$

are isomorphic.

(b) *Let $u = (u_1, \ldots, u_m)$ and $u' = (u'_1, \ldots, u'_m)$ be a pair of sequences in $\boldsymbol{N}^m$ such that $1 \le u_1 < u_2 < \cdots < u_m \le n$ and $1 \le u'_1 < u'_2 < \cdots < u'_m \le n$. Assume that u and u' satisfy the equalities $(*)$ for s. Then the functors*

$$\Phi^{B(f,s)} \circ \varepsilon_n^u,\ \Phi^{B(f,s)} \circ \varepsilon_n^{u'} : I_m\text{-spr}\,(kH) \to \operatorname{mod}(R/G)$$

are isomorphic; *equivalently, the functors*

$$\Phi^{B(f,s)}{}_{|\,I_n^{u'}\text{-}spr(kH)} \circ \varepsilon_n^{u',u},\ \Phi^{B(f,s)}{}_{|\,I_n^{u}\text{-}spr(kH)} : I_n^u\text{-spr}\,(kH) \to \operatorname{mod}(R/G)$$

are isomorphic.

(c) *Let V and V' be a pair of objects in I_n-spr$\,(kH)$ with* $\operatorname{csupp}(V) = u$ *and* $\operatorname{csupp}(V') = u'$, *where $u = (u_1, \ldots, u_m)$ and $u' = (u'_1, \ldots, u'_m)$. Assume that u and u' satisfy $(*)$ for s (in particular, this is the case if $u' = \bar{u}^s$). If there exists a sincere object V'' in I_m-spr$\,(kH)$ such that $V = \varepsilon_n^u(V'')$ and $V' = \varepsilon_n^{u'}(V'')$ then the R/G-modules $\Phi^{B(f,s)}(V)$ and $\Phi^{B(f,s)}(V')$ are isomorphic.*

(d) *The full subcategories of* $\operatorname{mod}(R/G)$ *formed by all modules in the images of the functors $\Phi^{B(f,s)}$ and $\Phi^{B(f,s)}{}_{|\,I_n\text{-spr}_{l(s)}(kH)}$ are equivalent. Moreover, for any V and V' in I_n-spr$\,(kH)$ with coordinate supports u and u', respectively, we have $\Phi^{B(f,s)}(V) \simeq \Phi^{B(f,s)}(V')$ if and only if $\Phi^{B(f,s)}(\bar{V}) \simeq \Phi^{B(f,s)}(\bar{V}')$, where $\bar{V} = \varepsilon_n^{\bar{u}^s,u}(V)$ and $\bar{V}' = \varepsilon_n^{\bar{u}'^s,u'}(V')$.*

Proof. (a) Let $V = (V_1 \subseteq V_2 \subseteq \cdots \subseteq V_m)$, $V_m = (V_m, \mu)$, be an object in I_m-spr$\,(kH)$. We set $V' = \varepsilon_n^u(V)$. V' is the object of I_n-spr$\,(kH)$ given by $V'_1 \subseteq V'_2 \subseteq \cdots \subseteq V'_n$ defined as above, and $V'_n = (V'_n, \mu') = (V_m, \mu)$. Fix a sequence of complementary direct summands $\underline{V} = (\underline{V}_i)_{i=1,\ldots,m}$ for V. Then the sequence $\underline{V}' = (\underline{V}'_j)_{j=1,\ldots,n}$, given by $\underline{V}'_{u_i} = \underline{V}_i$ for $i = 1, \ldots, m$, and $\underline{V}'_j = 0$ for $j \in \{1, \ldots, n\} \setminus \{u_1, \ldots, u_m\}$, is a sequence of complementary direct summands for V'. Therefore, we can assume that $V' \otimes_k B(f,s)$ is equal to

$$\underline{V}' \otimes_k B(f,s) = \Big(\bigoplus_{j=1}^{n} \underline{V}'_j \otimes_k B, \underline{\mu}' \otimes_k \beta\Big)$$

where $\beta = \beta(B(f,s))$ and the R-homomorphisms $(\underline{\mu}' \otimes_k \beta)_h : \bigoplus_{j=1}^n \underline{V}'_j \otimes_k B \to \bigoplus_{j'=1}^n {}^{h^{-1}}(\underline{V}_{j'} \otimes_k B)$, $h \in H$, defining the R-action $\underline{\mu}' \otimes_k \beta$ of H, have the upper-triangular matrix form, with the components $\mu'(h)_{j',j} \otimes_k (\nu_h\, f^{s_{j'+1}+\cdots+s_j})$, for $1 \le j' \le j \le n$ (see 1.4). Since $\underline{V}_j = 0$ for $j \ne u_1, \ldots, u_m$, we have the canonical R-isomorphism

$$\bigoplus_{j=1}^{n} \underline{V}'_j \otimes_k B \simeq \bigoplus_{i=1}^{m} \underline{V}_i \otimes_k B \tag{***}$$

and under this identification $(\underline{\mu} \otimes_k \beta)_h$ corresponds to the upper-triangular matrix R-homomorphism with the components $\mu(h)_{i',i} \otimes_k (\nu_h\, f^{s_{u_{i'}+1}+\cdots+s_{u_i}})$, for $1 \le i' \le i \le m$ (compare components of $\underline{\mu}(h)$ and $\underline{\mu}'(h)$). On the other hand $V \otimes_k B(f, s(u))$ is given by

$$\underline{V} \otimes_k B(f, s(u)) = \left(\bigoplus_{i=1}^{m} \underline{V}_i \otimes_k B,\ \underline{\mu} \otimes_k \beta^u \right)$$

where $\beta^u = \beta(B(f, s(u)))$ and the R-homomorphisms $(\underline{\mu} \otimes_k \beta^u)_h : \bigoplus_{i=1}^m \underline{V}_i \otimes_k B \to \bigoplus_{i'=1}^m {}^{h^{-1}}(\underline{V}_{i'} \otimes_k B)$, $h \in H$, defining the R-action $\underline{\mu} \otimes_k \beta^u$ of H, are given by the upper-triangular matrix with the components $\mu(h)_{i',i} \otimes_k (\nu_h\, f^{s(u)_{i'+1}+\cdots+s(u)_i})$, for $1 \le i' \le i \le m$. By the definition of $s(u)$ we have $s(u)_{i'+1} + \cdots + s(u)_i = s_{u_{i'}+1} + \cdots + s_{u_i}$. Consequently, $(***)$ yields the isomorphism $\underline{V}' \otimes_k B(f,s) \simeq \underline{V} \otimes_k B(f, s(u))$ in $\mathrm{Mod}^H_{\mathrm{f},B} R$ and induces the R/G-isomorphism $\eta(V) : \Phi^{B(f,s)} \varepsilon^u_n(V) \to \Phi^{B(f,\, s(u))}(V)$. It is easy to check that $(\eta(V))_{V \in \mathrm{ob}\, I_m\text{-spr}\,(kH)}$ is an isomorphism of the appropriate functors.

The assertion (b) follows from (a), since by $(*)$, we have $s(u) = s(u')$, (c) is a consequence of (b), (d) follows from (b) and (c). □

We apply the lemma to the longest (for a given f) sequence $\bar{s} = \bar{s}(f)$, with all $r-1$ components equal to 1. We denote for simplicity by $B(f)$ the sequence

$$B(f, \bar{s}(f)) : \qquad B_1 \xleftarrow{f} B_2 \leftarrow \cdots \leftarrow B_{n-1} \xleftarrow{f} B_r$$

($B_i = B$ for every $i = 1, \ldots, r$); consequently, $\Phi^{B(f)} = \Phi^{B(f,\bar{s})}$. Then we obtain immediately.

Corollary. (a) *Let V be an object in I_r-spr (kH) with* csupp$(V) = u$, $u = (u_1, \ldots, u_m)$, *V' the corresponding to V sincere object in I_m-spr (kH) such that $V = \varepsilon^u_r(V')$, and s the sequence $s = (u_2 - u_1, \ldots, u_m - u_{m-1})$ $(= \bar{s}(u))$. Then the R/G-modules $\Phi^{B(f,s)}(V')$ and $\Phi^{B(f)}(V)$ are isomorphic. Moreover, the functors*

$$\Phi^{B(f,s)},\ \Phi^{B(f)} \circ \varepsilon^u_r : I_m\text{-spr}\,(kH) \to \mathrm{mod}\,(R/G)$$

are isomorphic, in fact for any arbitrary sequence $u = (u_1, \ldots, u_m)$, $1 \le u_1 < u_2 < \cdots < u_m \le r$, where s is given as above.

(b) *Let $u = (u_1, \ldots, u_m) \in \boldsymbol{N}^m$ be a sequence such that $1 \le u_1 < \cdots < u_m \le r$.*

Then for any i, $r-u_m \le i \le u_1-1$, the functors

$$\Phi^{B(f)} \circ \varepsilon_r^u,\ \Phi^{B(f)} \circ \varepsilon_r^{u[i]} : I_m\text{-spr}\,(kH) \to \text{mod}\,(R/G)$$

are isomorphic; *equivalently, the functors*

$$\Phi^{B(f)}{}_{|\, I_r^{u[i]}\text{-}spr(kH)} \circ \varepsilon_r^{u[i],\,u},\ \Phi^{B(f)}{}_{|\, I_r^u\text{-}spr(kH)} : I_r^u\text{-spr}\,(kH) \to \text{mod}\,(R/G)$$

are isomorphic.

(c) *Let V be an object in I_r-spr (kH) with* csupp$(V) = (u_1,\dots,u_m)$ *and V' be a sincere object in I_m-spr (kH) such that $V = \varepsilon_r^u(V')$. Then for any $u' = u[i]$, where i is as above* (*in particular, for $u' = (1, u_2-u_1+1,\dots,u_m-u_1+1)$*), *the R/G-modules $\Phi^{B(f)}(V)$ and $\Phi^{B(f)}\varepsilon_m^{u'}(V')$ are isomorphic.*

(d) *The full subcategories formed by all modules in the images of the functors $\Phi^{B(f)}$ and $\Phi^{B(f)}{}_{|\, I_r\text{-}spr_1(kH)}$ are equivalent. Moreover, for any V and V' in I_r-spr (kH) with coordinate supports u and u', respectively, we have $\Phi^{B(f)}(V) \simeq \Phi^{B(f)}(V')$ if and only if $\Phi^{B(f)}\varepsilon_r^{\bar u,\,u}(V) \simeq \Phi^{B(f)}\varepsilon_r^{\bar u',\,u'}(V')$, where $\bar u = u[u_1-1]$ and $\bar u' = u'[u_1'-1]$.*

Remark. The above results do not answer the basic questions concerning the functor $\Phi^{B(f,s)}|\, I_n\text{-spr}_{l(s)}(kH)$: when $\Phi^{B(f,s)}(V) \simeq \Phi^{B(f,s)}(V')$ for indecomposables V, V' in I_n-spr$\,_{l(s)}(kH)$, and if $\Phi^{B(f,s)}(V)$ is indecomposable provided V is so.

2.5.

Now we study more precisely "intersections of the images by different functors $\Phi^{B(f,s)}$". We formulate the answer comparing the functors $\Phi^{B(f,s)}$ to the functor $\Phi^{B(f)}$.

Lemma. (a) *Let $s = (s_2,\dots,s_n)$ be a sequence of positive integers such that $s_2+\dots+s_n < r$, $p \in \boldsymbol{N}$ a positive number such that $p \le r-(s_2+\dots+s_n)$, and $u = (u_1,\dots,u_n)$ the sequence given by $u_i = p+s_1+\dots+s_i$, $i = 1,\dots,n$* (*we set $s_1 = 0$*). *Then the functors*

$$\Phi^{B(f,s)},\ \Phi^{B(f)} \circ \varepsilon_r^u : I_n\text{-spr}\,(kH) \to \text{mod}\,(R/G)$$

are isomorphic. In particular, for any object V in I_n-spr (kH) there exists V' in I_r^u-spr (kH) ($V' = \varepsilon_r^u(V)$) *such that the R/G-modules $\Phi^{B(f,s)}(V)$ and $\Phi^{B(f)}(V')$ are isomorphic.*

(b) *Let s be as in* (*a*), *V an object in I_n-spr (kH) with* csupp$(V) = u$, *$u = (u_1,\dots,u_m)$, and V' the corresponding to V sincere object in I_m-spr (kH) such that $V = \varepsilon_n^u(V')$. Then for any pair of sequences, the sequence $u' = (u_1',\dots,u_m')$ of positive integers, $1 \le u_1' < \dots < u_m' \le n$, satisfying* (∗) *together with u, and the sequence $v = (p, p+s_{u_1+1}+\dots+s_{u_2},\dots,p+s_{u_1+1}+\dots+s_{u_m})$ determined by $p \in \boldsymbol{N}$, $1 \le p \le r-(s_{u_1+1}+\dots+s_{u_m})$, we have the R/G-isomorphisms*

$$\Phi^{B(f,s)}(V) \cong \Phi^{B(f,s)}\varepsilon_n^{u'}(V') \cong \Phi^{B(f)}\varepsilon_r^{v}(V')\,;$$

in particular,

$$\Phi^{B(f,s)}(V) \cong \Phi^{B(f,s)}\varepsilon_n^{\bar{u}^s,u}(V) \cong \Phi^{B(f)}\varepsilon_r^v(\varepsilon_n^u)^{-1}(V)$$

where $v = (1, 1 + s_{u_1+1} + \cdots + s_{u_2}, \ldots, 1 + s_{u_1+1} + \cdots + s_{u_m})$.

Proof. Note that $s = \bar{s}(u)$ and (a) follows by applying Lemma 2.4(c) and Corollary 2.4(c). To prove (b) we use similar arguments. □

Combining the lemma and the facts from 2.4 we obtain the following result.

Proposition. (a) *The full subcategories of* $\mathrm{mod}\, R/G$ *formed by all modules in the images of the functors* $\Phi^{B(f,s)}$ *(respectively,* $\Phi^{B(f,s)}|_{I_n\text{-}spr_{l(s)}(kH)}$*), for all sequences* s*, and all modules in the image of the functor* $\Phi^{B(f)}|_{I_r\text{-}spr_1(kH)}$ *are equivalent.*

(b) *Let* $s = (s_2, \ldots, s_n)$ *and* $s' = (s'_2, \ldots, s'_{n'})$ *be a pair of sequences of positive integers such that* $s_2 + \cdots + s_n$, $s'_2 + \cdots + s'_{n'} < r$, *and* $w = (w_1, \ldots, w_n)$, $w' = (w'_1, \ldots, w'_{n'})$ *be a pair of increasing sequences of positive integers such that* $w_n, w'_{n'} \le r$, *satisfying the equalities* $w_i - w_{i-1} = s_i$, *for* $i = 2, \ldots, n$, *and* $w'_{i'} - w_{i'-1} = s_{i'}$, *for* $i' = 2, \ldots, n'$, *respectively. Then for any* V *in* $I_n\text{-spr}\,(kH)$ *and* V' *in* $I_{n'}\text{-spr}\,(kH)$ *with coordinate supports* $u = (u_1, \ldots, u_m)$ *and* $u' = (u'_1, \ldots, u'_{m'})$*, respectively, the following conditions are equivalent*:

(i) $\Phi^{B(f,s)}(V) \cong \Phi^{B(f,s')}(V')$,

(ii) $\Phi^{B(f)}\varepsilon_r^w(V) \cong \Phi^{B(f)}\varepsilon_r^{w'}(V')$,

(iii) $\Phi^{B(f,s)}\varepsilon_n^{\bar{u}^s,u}(V) \cong \Phi^{B(f,s')}\varepsilon_{n'}^{\bar{u'}^{s'},u'}(V')$,

(iv) $\Phi^{B(f)}\varepsilon_r^v(\varepsilon_n^u)^{-1}(V) \cong \Phi^{B(f)}\varepsilon_r^{v'}(\varepsilon_{n'}^{u'})^{-1}(V')$, *where* $v = (1, 1 + s_{u_1+1} + \cdots + s_{u_2}, \ldots, 1 + s_{u_1+1} + \cdots + s_{u_m})$ *and* $v' = (1, 1 + s'_{u'_1+1} + \cdots + s'_{u'_2}, \ldots, 1 + s'_{u'_1+1} + \cdots + s'_{u'_{m'}})$.

As a consequence of the above, the proof of the main assertion of Theorem 2.2, stating that the functor $\Phi^{B(f,s)}|_{I_n\text{-spr}_{l(s)}(kH)}$ is a representation embedding, reduces to the case of the sequence $s = \bar{s}$ and the functor $\Phi^{B(f)}|_{I_r\text{-spr}_1(kH)}$ (cf. also Remark 2.4).

In the next sections we develop the tools we need for the proof of that case (see 6.4).

3. Another description of the category $\mathrm{mod}_B R/H$.

Let B be a periodic G-atom together with a fixed R-action of ν of G_B on itself. The main aim of this section is to describe the category $\mathrm{mod}_B(R/G_B)$ in terms of the module category of the skew group algebra of the stabilizer G_B over the endomorphism algebra $\mathrm{End}_R(B)$, with respect to some natural action induced by ν. We also express in this language the functors used for creating R/G-modules, in particular the generalized tensor functor.

3.1.

Let H be a group, E a k-algebra and σ an action of H on E which can be regarded as a group homomorphism $\sigma : H \to \mathrm{Aut}_{k-\mathrm{alg}}(E)$, where $\mathrm{Aut}_{k-\mathrm{alg}}(E)$ denotes the group of all k-algebra automorphisms of E. Then we denote by $EH = E_\sigma H$ the skew group algebra of H over E under the action σ. By definition, $E_\sigma H$ is the k-vector space

$$EH = \bigoplus_{h\in H} Eh$$

($_E Eh \simeq {}_E E$, for every $h \in H$) equipped with the multiplication given by the formula

$$e_1 h_1 \cdot e_2 h_2 = (e_1(\sigma(h_1)(e_2)))(h_1 h_2)$$

for $h_1, h_2 \in H$ and $e_1, e_2 \in E$. It is a straightforward observation that E is in natural way a subalgebra of EH, H is a subgroup of the unit group of EH and we have the "relations"

$$h \cdot e = \sigma(h)(e)h$$

$e \in E$, $h \in H$, connecting this two embeddings. Therefore any left EH-module M can be regarded as a k vector space equipped with structures of a left E-module and a k-representation of H (that is a left kH-module), related by the equalities

$$(h\cdot) \circ (e\cdot) = (\sigma(h)(e)\cdot) \circ (h\cdot)$$

in $\mathrm{End}_k(M)$, for all $e \in E$, $h \in H$. They can be rephrased in terms of elements in the following form

$$h(em) = (\sigma(h)(e)(hm))$$

for all $m \in M$, $e \in E$ and $h \in H$.

By analogy a right EH-module M is just a k-vector space equipped with the structures of right E-module and right kH-module related by the equalities

$$(mh)e = (m(\sigma(h)(e)))h$$

or equivalently

$$(me)h = (mh)\sigma(h^{-1})(e)$$

for all $m \in M$, $e \in E$ and $h \in H$. The bijection $(-)^{-1} : H \to H$ induces an equivalence between the categories of left and right kH-modules. Consequently, by the last equality a right EH-module M can be regarded as a (left) k-representation of H equipped simultaneously with a structure of a right E-module, such that

$$h(me) = (hm)\sigma(h)(e) \qquad (*)$$

for all $m \in M$, $e \in E$ and $h \in H$.

Another approach to the above interpretation of right EH-modules refers to fact that the action σ induces the action of H on the category $\mathrm{MOD}\, E$ by translations ${}^h(-)$, $h \in H$. Recall that to each M in $\mathrm{MOD}\, E$, the translation ${}^h(-)$ assigns the module ${}^hM = (M, \cdot_h)$, where $\cdot_h : M \times E \to M$ is the multiplication given by the formula $m \cdot_h e = m\sigma(h^{-1})(e)$, for $m \in M$ and $e \in E$. Then any right EH-module M can be treated as a pair (M_E, μ), where

M_E is a right E-module and μ is the right E-action of H an M, that is the family $\mu = (\mu_h : M \to {}^{h^{-1}}M)_{h\in H}$ of E-homomorphisms such that $\mu_1 = \mathrm{id}_M$ and ${}^{h^{-1}}\mu_{h_1} \cdot \mu_h = \mu_{h_1 h}$ for all $h, h_1 \in H$. The category formed by all pairs (M_E, μ) and E-homomorphisms compatible with E-actions of H is traditionally denoted by $\mathrm{MOD}\,{}^H E$. We will identify the categories $\mathrm{MOD}\,(EH)$ with $\mathrm{MOD}\,{}^H E$ via the correspondence $M \mapsto (M, (h\cdot : M \to {}^{h^{-1}}M)_{h\in H})$, where M is an EH-module given by original fashion data (see $(*)$).

It is clear now that the module E_E is equipped with the canonical natural structure of a right EH-module, given by the E-action $\pi = (\sigma(h) : E_E \to {}^{h^{-1}}(E_E))_{h\in H}$ of H. In the paper we will consider some special class of right EH-modules containing the module (E_E, π); namely, consisting of all those M in $\mathrm{MOD}\,(EH)$ that M_E is a finitely generated projective E-module. Following the idea of [**2**] we call these modules the maximal Cohen-Macaulay EH-modules with respect to the algebra embedding $E \subseteq EH$. The full subcategory of $\mathrm{MOD}\,(EH)$ formed by all EH-modules from this class will be denoted by $\mathrm{CM}\,(EH)$ $(= \mathrm{CM}_E(EH))$.

3.2.

Let $B = (B, \nu)$ be a fixed R-module together with an R-action of a subgroup H of the group G on B (clearly, $H \subseteq G_B$). Denote by E the endomorphism algebra $\mathrm{End}_R(B)$. Then the k-linear action

$$\mathrm{Hom}_R(\nu, \nu) : H \times E \to E$$

given by the mapping $(h, e) \mapsto {}^h\nu_h\, {}^h e\, \nu_{h^{-1}}$, $h \in H$, $e \in E$ (see [**6**]), induces the group homomorphism

$$\sigma : H \to \mathrm{Aut}_k(E)$$

defined by the family $\sigma(h) = {}^h\nu_h\, {}^h(-)\, \nu_{h^{-1}} : E \to E$, $h \in H$, of the maps.

Lemma. (a) *The inclusion* $\mathrm{Im}\,\sigma \subseteq \mathrm{Aut}_{k-\mathrm{alg}}(E)$ *holds.*

(b) *If* $E = \mathrm{Hom}_R^H(B, B)$ *then* EH $(= E_\sigma H)$ *is a group algebra of* H *over* E *in the classical sense. In particular, this is always the case when* $E = k$.

Proof. (a) For any $e, e' \in E$, $h \in H$ we have ${}^h\nu_h\, {}^h(ee')\, \nu_{h^{-1}} = {}^h\nu_h\, {}^h e\, {}^h e'\, \nu_{h^{-1}} = ({}^h\nu_h\, {}^h e\, \nu_{h^{-1}})({}^h\nu_h\, {}^h e'\, \nu_{h^{-1}})$ and ${}^h\nu_h\, {}^h\mathrm{id}_B\, \nu_{h^{-1}} = {}^h\nu_h\, \mathrm{id}_{({}^hB)}\, \nu_{h^{-1}} = \mathrm{id}_B$. Consequently, $\sigma(h)$ is an algebra homomorphism, for every $h \in H$, and (a) is proved.

(b) Note that under the assumptions $\mathrm{Im}\,\sigma = \{\mathrm{id}_E\}$, so the first assertion is straightforward. The last assertion follows easily from the equality ${}^h(\alpha\,\mathrm{id}_B) = \alpha\,\mathrm{id}_{({}^hB)}$, $\alpha \in k$, $h \in H$. □

3.3.

Consider some special case of the situation discussed above. Let $B_i = (B_i, \nu_i)$, $i = 1, \ldots, n$, be a family of objects in $\mathrm{Mod}\,{}_{\mathrm{f}}^H R$. We assume that all R-modules B_i are indecomposable (so H-atoms) and are pairwise nonisomorphic. We set $\mathscr{B}_{\mathrm{o}} = \{B_1, \ldots, B_n\}$. Denote by $B = (B, \nu)$ the direct sum of all objects B_i, $i = 1, \ldots, n$, in $\mathrm{Mod}\,{}_{\mathrm{f}}^H R$, that is the pair consisting of the R-module $B = \bigoplus_{i=1}^n B_i$ and the R-action

$\nu = (\bigoplus_{i=1}^{n}(\nu_i)_h : \bigoplus_{i=1}^{n} B_i \to \bigoplus_{i=1}^{n} {}^{h^{-1}}B_i)_{h\in H}$ of H (cf. [**9**]). We have the induced by ν action $\sigma : H \to \mathrm{Aut}_{k-\mathrm{alg}}(E)$ of H on the endomorphism algebra $E = \mathrm{End}_R(B)$.

Now we formulate the most important result of this section.

Theorem. *For $\mathscr{B}_o = \{B_1, \ldots, B_n\}$ as above, there exists an equivalence of categories*

$$\mathrm{mod}_{\mathscr{B}_o}(R/H) \simeq \mathrm{CM}\,(EH)$$

where $EH = E_\sigma H$. If $n = 1$, the equivalence has the form

$$\mathrm{mod}_B(R/H) \simeq \mathrm{CM}\,(EH) \qquad (**)$$

where E is a local algebra.

The proof of the theorem needs some preparation.

3.4.

Let $B = (B, \nu)$, E and $\sigma : H \to \mathrm{Aut}_{k-\mathrm{alg}}(E)$ be as in 3.2. We set $EH = E_\sigma H$. Consider the functors

$$\mathrm{Hom}_R(B, -) : \mathrm{MOD}\,R \to \mathrm{MOD}\,E$$

and

$$\mathscr{H}_B = \mathscr{H}_{(B,\,\nu)} : \mathrm{MOD}^H R \to \mathrm{MOD}\,(kH)^{\mathrm{op}}$$

assigning to any $M = (M, \mu)$ in $\mathrm{MOD}^H R$, the left kH-module defined by the action $\mathrm{Hom}_R(\nu, \mu)$ on the k-vector space $\mathrm{Hom}_R(B, M)$.

Lemma. (a) *The pair $(\mathrm{Hom}_R(B, M), \mathrm{Hom}_R(\nu, \mu))$ is a right EH-module.*

(b) *The mapping $M \mapsto (\mathrm{Hom}_R(B, M)_E, \mathrm{Hom}_R(\nu, \mu))$ defines a functor*

$$\widetilde{\mathscr{H}_B} = \widetilde{\mathscr{H}_{(B,\nu)}} : \mathrm{MOD}^H R \to \mathrm{MOD}\,EH\,.$$

Proof. (a) It is enough to check $(*)$ for an arbitrary module $M = (M, \mu)$ in $\mathrm{MOD}^H R$. Fix any $e \in E$ and $f \in \mathrm{Hom}_R(B, M)$. Then we have the sequence of equalities

$${}^h\mu_h\, {}^h(fe)\,\nu_{h^{-1}} = {}^h\mu_h\, {}^hf\, {}^he\,\nu_{h^{-1}} = ({}^h\mu_h\, {}^hf\,\nu_{h^{-1}})({}^h\mu_h\, {}^he\,\nu_{h^{-1}})$$

and $(*)$ holds for M.

The statement (b) follows immediately from functoriality of $\mathrm{Hom}_R(B, -)$ and $\mathscr{H}_{(B,\,\nu)}$. □

3.5.

Now we return to the context of the last theorem. We assume that $B = (B, \nu)$, E and $\sigma : H \to \mathrm{Aut}_{k-\mathrm{alg}}(E)$ are as in 3.3.

PROPOSITION. *The functor* $\widetilde{\mathscr{H}_B} : \mathrm{MOD}^H R \to \mathrm{MOD}\, EH$ *restricts to an equivalence of categories*

$$\mathrm{Mod}^H_{\mathrm{f},\mathscr{B}_o} R \simeq \mathrm{CM}\,(EH)\,. \qquad (***)$$

PROOF. Denote by P_i, $i = 1, \ldots, n$, the indecomposable projective E-modules $\mathrm{Hom}_R(B, B_i)$. Recall that the functor $\mathrm{Hom}_R(B, -)$ yields the isomorphism of the additive closures of $\mathscr{B}_\mathrm{o}$ in $\mathrm{MOD}\, R$ and $\{P_i\}_{i=1,\ldots,n}$ in $\mathrm{MOD}\, E$. (In fact, the second closure is equivalent to the category of all finitely generated projective E-modules). Therefore $\widetilde{\mathscr{H}_B}$ is faithful and we have $\widetilde{\mathscr{H}_B}(\mathrm{Mod}^H_{\mathrm{f},\mathscr{B}_\mathrm{o}} R) \subset \mathrm{CM}\,(EH)$, so we need only to show that $\widetilde{\mathscr{H}_B}$ is dense and full.

To prove density of $\widetilde{\mathscr{H}_B}$ it is enough, for any object $X = (X, \alpha)$ in $\mathrm{CM}\,(EH)$, $X = \bigoplus_{i=1}^n P_i^{d_i}$, $d_1, \ldots, d_n \in \boldsymbol{N}$, and $\alpha = (\alpha_h : X \to {}^{h^{-1}}X)_{h\in H}$, to construct an R-action $\mu(\alpha) = (\mu_h : \bigoplus_{i=1}^n B_i^{d_i} \to \bigoplus_{i=1}^n {}^{h^{-1}}B_i^{d_i})_{h\in H}$ of H on the R-module $M = \bigoplus_{i=1}^n B_i^{d_i}$, such that the canonical E-isomorphism $\mathrm{Hom}_R(B, M) \simeq \bigoplus_{i=1}^n P_i^{d_i}$ yields the isomorphism $\widetilde{\mathscr{H}_B}(X) \simeq M$ in $\mathrm{CM}\,(EH)$, where $M = (M, \mu(\alpha))$.

Fix X as above. By the projectivity of P_i's each E-isomorphism $\alpha_h : \bigoplus_{i=1}^n P_i^{d_i} \to \bigoplus_{i=1}^n {}^{h^{-1}}P_i^{d_i}$, $h \in H$, is uniquely determined by the R-homomorphism $f(h) = (f_{i,j}^{s,t}(h))_{i,s;\,j,t} : \bigoplus_{j=1}^n \bigoplus_{t=1}^{d_j} B_j^{(t)} \to \bigoplus_{i=1}^n \bigoplus_{s=1}^{d_i} B_i^{(s)}$, $B_j^{(t)} = B_j$, $B_i^{(s)} = B_i$, which is given by $\alpha_h(\pi_j^t) = (f_{i,j}^{s,t}(h))_{i,s} \in \bigoplus_{i=1}^n \bigoplus_{s=1}^{d_i} \mathrm{Hom}_R(B, B_i^{(s)}) = \bigoplus_{i=1}^n {}^{h^{-1}}P_i^{d_i}$, $j = 1, \ldots, n$, $t = 1, \ldots, d_j$, where $\pi_j^t : B \to B_j$ is the canonical jth projection in the tth copy of P_j. Note that in fact, we have $\alpha_h(\pi_j^t) = (f_{i,\,j}^{s,\,t}(h))_{i,s} \in \bigoplus_{i=1}^n \bigoplus_{s=1}^{d_i} \mathrm{Hom}_R(B_j, B_i^{(s)}) \subseteq \bigoplus_{i=1}^n \bigoplus_{s=1}^{d_i} \mathrm{Hom}_R(B, B_i^{(s)}) = \bigoplus_{i=1}^n {}^{h^{-1}}P_i^{d_i}$. This follows from the fact that each standard primitive idempotent in E (the composition of the canonical jth projection and jth embedding, $j = 1, \ldots, n$) is $\sigma(h)$-invariant, for every $h \in H$ (all ν_h's are diagonal!). Let $\mu(\alpha) = (\mu_h : M \to {}^{h^{-1}}M)_{h\in H}$ be the family of R-homomorphisms μ_h defined by the composite maps

$$\bigoplus_{i=1}^n B_i^{d_i} \xrightarrow{\nu_h^M} \bigoplus_{i=1}^n {}^{h^{-1}}B_i^{d_i} \xrightarrow{{}^{h^{-1}}f(h)} \bigoplus_{i=1}^n {}^{h^{-1}}B_i^{d_i}$$

where $\nu_h^M = \bigoplus_{i=1}^n (\nu_i)_h^{d_i}$. We show that $\mu(\alpha)$ is an R-action of H on the R-module M. It is sufficient to prove that the formula

$$f(h_1 h) = f(h_1) \cdot ({}^{h_1}\nu_{h_1}^M \cdot {}^{h_1}f(h) \cdot \nu_{h_1^{-1}}^M) \qquad \text{(i)}$$

holds for all $h, h_1 \in H$, since then we have

$$\begin{aligned}
\mu_{h_1 h} &= {}^{(h_1 h)^{-1}}f(h_1 h) \cdot \nu_{h_1 h}^M \\
&= {}^{h^{-1}h_1^{-1}}f(h_1) \cdot \big({}^{h^{-1}}\nu_{h_1}^M \cdot {}^{h^{-1}}f(h) \cdot {}^{h^{-1}h_1{}^{-1}}\nu_{h_1^{-1}}^M\big) \cdot \big({}^{h^{-1}}\nu_{h_1}^M \cdot \nu_h^M\big) \\
&= {}^{h^{-1}}\big({}^{h_1{}^{-1}}f(h_1) \cdot \nu_{h_1}^M\big) \cdot \big({}^{h^{-1}}f(h) \cdot \nu_h^M\big) = {}^{h^{-1}}\mu_{h_1} \cdot \mu_h
\end{aligned}$$

(clearly, by the construction of $f(1)$, $1 \in H$, we have $\mu_1 = \mathrm{id}_M$). To prove (i) we compare for any (i,s) and (l,u), $1 \le i,l \le n$, $1 \le s \le d_i$, $1 \le u \le d_l$, the (i,s)th components of the equalities

$$\alpha_{h_1 h}(\pi_l^u) = \big({}^{h^{-1}}\alpha_{h_1} \cdot \alpha_h\big)(\pi_l^u) \qquad \text{(ii)}_{(l,u)}$$

$h, h_1 \in H$, regarded as element of $\mathrm{Hom}_R(B_l, B_i^{(s)})$ (see below). For this purpose we compute the image $\alpha_{h_1}((f_{j,l}^{t,u}(h))_{j,t})$ of the element $(f_{j,l}^{t,u}(h))_{j,t} \in \bigoplus_{j=1}^n \bigoplus_{t=1}^{d_j} \mathrm{Hom}_R(B_l, B_j^{(t)}) \subseteq \bigoplus_{j=1}^n \bigoplus_{t=1}^{d_j} \mathrm{Hom}_R(B, B_j^{(t)})$ by α_{h_1}, where $\alpha_h(\pi_l^u) = (f_{j,l}^{t,u}(h))_{j,t}$. Note that each $f_{j,l}^{t,u}(h) \in \mathrm{Hom}_R(B, B_j^{(t)})$ can be viewed in the form $f_{j,l}^{t,u}(h) = \pi_j^t \cdot e$, where $e = e_{j,l}(f_{j,l}^{t,u}(h)) \in \mathrm{End}_R(B)$ is given by the matrix with all but one components equal to zero, and only nonzero, the (j,l)th component, equal to $f_{j,l}^{t,u}(h) \in \mathrm{Hom}_R(B_l, B_j)$. Therefore, for a fixed (j,t), we have

$$\alpha_{h_1}(f_{j,l}^{t,u}(h)) = \alpha_{h_1}(\pi_j^t) \cdot_{h_1^{-1}} e = (f_{j,l}^{t,u}(h))_{i,s} \cdot \sigma(h_1)(e)\,.$$

Hence, the (i,s)th component of $\alpha_{h_1}(f_{j,l}^{t,u}(h))$ belongs to $\mathrm{Hom}_R(B_l, B_i^{(s)}) \subseteq \mathrm{Hom}_R(B, B_i^{(s)})$ and is equal to $f_{i,j}^{s,t}(h_1) \cdot ({}^{h_1}(\nu_j)_{h_1} \cdot {}^{h_1}f_{j,l}^{t,u}(h) \cdot (\nu_l)_{h_1^{-1}})$. Then, by $\text{(ii)}_{(l,u)}$, for any (i,s) and (l,u) we have the equalities

$$f_{i,l}^{s,u}(h_1 h) = \sum_{j=1}^{n} \sum_{t=1}^{d_j} f_{i,j}^{s,t}(h_1) \cdot \big({}^{h_1}(\nu_j)_{h_1} \cdot {}^{h_1}f_{j,l}^{t,u}(h) \cdot (\nu_l)_{h_1^{-1}}\big) \qquad \text{(ii)}_{(l,u)}^{(s,u)}$$

of elements in $\mathrm{Hom}_R(B_l, B_i^{(s)})$, where $\alpha_{h_1 h}(\pi_l^u) = (f_{i,l}^{s,u}(h_1 h))_{i,s} \in \bigoplus_{i=1}^n \bigoplus_{s=1}^{d_i} \mathrm{Hom}_R(B_l, B_i^{(s)}) \subseteq \bigoplus_{i=1}^n \bigoplus_{s=1}^{d_i} \mathrm{Hom}_R(B, B_i^{(s)})$. Consequently, (i) holds for all $h, h_1 \in H$, since by $\text{(ii)}_{(l,u)}^{(s,u)}$, all components $\text{(i)}_{(l,u)}^{(s,u)}$ of the equality (i) hold, for $i,l = 1,\dots,n$, $u = 1,\dots,d_l$, $s = 1,\dots,d_i$. In this way $\mu(\alpha)$ is really an R-action of H on M.

Next we show that $\widetilde{\mathscr{H}_B}(M) \simeq X$ in $\mathrm{CM}\,(EH)$. By definition of $\mu(\alpha)$ and the action $\mathrm{Hom}_R(\mu(\alpha), \nu)$ of H on $\mathrm{Hom}_R(B, M)$, we have

$$h \cdot \gamma = {}^h\mu_h \cdot {}^h\gamma \cdot \nu_{h^{-1}} = f(h) \cdot ({}^h\nu_h^M \cdot {}^h\gamma \cdot \nu_{h^{-1}})$$

for all $h \in H$ and $\gamma \in \mathrm{Hom}_R(B, M)$. In particular, for $\gamma = \pi_j^t$, $j = 1,\dots,n$, $t = 1,\dots,d_j$, we obtain

$$h \cdot \pi_j^t = (f_{i,j}^{s,t}(h))_{i,s} = \alpha_h(\pi_j^t)$$

since ${}^h\nu_h^M \cdot {}^h\pi_j^t \cdot \nu_{h^{-1}} = \pi_j^t$ by the definition (π_j^t is regarded here as an element of $\mathrm{Hom}_R(B, M)$). Consequently, $\widetilde{\mathscr{H}_B}(M, \mu(\alpha)) \simeq (X, \alpha)$ and the functor $\widetilde{\mathscr{H}_B}$ is dense.

To prove that $\widetilde{\mathscr{H}_B}$ is full, it suffices to show that for any morphism $\varphi : X \to X'$ in $\mathrm{CM}\,(EH)$, $X = (X, \alpha)$, $X' = (X', \alpha')$, $X = \bigoplus_{j=1}^n P_j^{d_j}$, $X' = \bigoplus_{i=1}^n P_i^{d_i'}$, the unique

R-homomorphism $\psi : \bigoplus_{j=1}^{n} \bigoplus_{t=1}^{d_j} B_j \to \bigoplus_{i=1}^{n} \bigoplus_{s'=1}^{d'_i} B_i$, with coordinates $\psi_{i,\,j}^{s',\,t}$, $i, j = 1, \ldots, n$, $t = 1, \ldots, d_j$, $s' = 1, \ldots, d'_i$, such that $\mathrm{Hom}_R(B, M) = \varphi$, is compatible with the R-actions $\mu(\alpha) = (\mu_h : M \to {}^{h^{-1}}M)_{h \in H}$ and $\mu(\alpha') = (\mu'_h : M' \to {}^{h^{-1}}M')_{h \in H}$ of H on the R-modules $M = \bigoplus_{j=1}^{n} B_j^{d_j}$ and $M' = \bigoplus_{i=1}^{n} B_i^{d'_i}$, respectively. By definitions of $\mu(\alpha)$ and $\mu(\alpha')$, the required equality

$$\mu'_h \cdot \psi = {}^{h^{-1}}\psi \cdot \mu_h \,, \tag{iii}$$

for $h \in H$, has the form

$${}^{h^{-1}}f'(h) \cdot \nu_h^{M'} \cdot \psi = {}^{h^{-1}}\psi \cdot {}^{h^{-1}}f(h) \cdot \nu_h^{M} \tag{iv}$$

where $f(h)$ and $f'(h)$ are determined by α_h and α'_h as before. Therefore we need only to show that

$$\psi \cdot f(h) = f'(h) \cdot \left({}^{h}\nu_h^{M'} \cdot {}^{h}\psi \cdot \nu_{h^{-1}}^{M}\right) \tag{v}$$

in $\mathrm{Hom}_R(M, M')$, for all $h \in H$ (apply to (iv) autoequivalence ${}^{h^{-1}}(-)$ and then the composition with ν_h^M from the right).

We know that $\varphi = \mathrm{Hom}_R(B, \psi)$, as a morphism in $\mathrm{CM}\,(EH)$, satisfies equalities $\varphi \cdot \alpha_h = \alpha'_h \cdot \varphi$, for all $h \in H$. Consequently, for any $\gamma \in \mathrm{Hom}_R(B, M)$ we have

$$\psi \cdot (\alpha_h(\gamma)) = \alpha'_h(\psi\gamma)\,. \tag{vi}$$

In particular, (vi) holds for $\gamma = \pi_l^u$, for any $l = 1, \ldots, n$, $u = 1, \ldots, d_l$. To prove (v), observe first that passing to components $(l, u), (i, s')$, $i = 1, \ldots, n$, $s' = 1, \ldots, d'_i$, (v) has the form

$$\sum_{j=1}^{n} \sum_{t=1}^{d_j} \psi_{i,\,j}^{s',\,t} \cdot f_{j,\,l}^{t,\,u}(h) = \sum_{j=1}^{n} \sum_{t'=1}^{d'_j} {f'}_{i,\,j}^{s',\,t'}(h) \left({}^{h}(\nu_j)_h \cdot {}^{h}\!\left(\psi_{j,\,l}^{t',\,u}\right) \cdot (\nu_l)_{h^{-1}}\right) \tag*{$(\mathrm{v})_{(i,\,l)}^{(s',\,u)}$}$$

of equality in $\mathrm{Hom}_R(B_l, B_i^{(s')})$, where $B_i^{(s')} = B_i$ for every (i, s'). Next, that for any (l, u), we have the equality

$$\psi \cdot (\alpha_h(\pi_l^u)) = \left(\sum_{j=1}^{n} \sum_{t=1}^{d_j} \psi_{i,\,j}^{s',\,t} \cdot f_{j,\,l}^{t,\,u}(h)\right)_{i,s'}$$

in $\bigoplus_{i=1}^{n} \bigoplus_{s'=1}^{d'_i} \mathrm{Hom}_R(B_l, B_i^{(s')}) \subseteq \bigoplus_{i=1}^{n} \bigoplus_{s'=1}^{d'_i} \mathrm{Hom}_R(B, B_i^{(s')})$, as an immediate consequence of the definition of $f_{j,\,l}^{t,\,u}(h)$'s. Finally, we have also

$$\alpha'_h(\psi\pi_l^u) = \Big(\sum_{j=1}^{n}\sum_{t'=1}^{d'_j} {f'}_{i,j}^{s',t'}(h)\big({}^h(\nu_j)_h \cdot {}^h\psi \cdot (\nu_l)_{h^{-1}}\big)\Big)_{i,s'}.$$

The final equality follows by arguments similar to those from the first part of the proof. Namely, $\psi\pi_l^u$ regarded as an element of $\bigoplus_{j=1}^n \bigoplus_{t'=1}^{d'_j} \mathrm{Hom}_R(B_l, B_j^{(t')}) \subseteq \bigoplus_{j=1}^n \bigoplus_{t'=1}^{d'_j} \mathrm{Hom}_R(B, B_j^{(t')})$ is equal to $(\psi_{j,l}^{t',u})_{j,t'}$, and for any (j,t') we have

$$\alpha'_h(\psi_{j,l}^{t',u}) = \Big({f'}_{i,j}^{s',t'}(h)\big({}^h(\nu_j)_h \cdot {}^h(\psi_{j,l}^{t',u}) \cdot (\nu_l)_{h^{-1}}\big)\Big)_{i,s'}$$

in $\bigoplus_{i=1}^n \bigoplus_{s'=1}^{d'_i} \mathrm{Hom}_R(B_l, B_i^{(s')}) \subseteq \bigoplus_{i=1}^n \bigoplus_{s'=1}^{d'_i} \mathrm{Hom}_R(B, B_i^{(s')})$. We finish the proof by observing that (v) holds if and only if it holds after passing to the components $(\mathrm{v})_{(i,l)}^{(s',u)}$, for all $i,l = 1,\ldots,n$, $u = 1,\ldots,d_l$, $s' = 1,\ldots,d'_i$; but this is the case because the all components $(\mathrm{vi})_{(i,l)}^{(s',u)}$, of (vi), for $\gamma = \pi_l^u$, hold. In this way the proof of the proposition is complete. □

Proof of Theorem 3.3. Denote by $F' : R \to R/H$ the canonical Galois covering functor. As usually the "pull-up" functor $F'_\bullet : \mathrm{MOD}\,(R/H) \to \mathrm{MOD}\,R$ induce the equivalence

$$\mathrm{mod}_{\mathscr{B}_\mathrm{o}}(R/H) \simeq \mathrm{Mod}_{\mathrm{f},\mathscr{B}_\mathrm{o}}^{H} R\,. \qquad (****)$$

Consequently, the functors $\widetilde{\mathscr{H}_B}$ and $F'_\bullet$ induces the equivalence $\mathrm{mod}_{\mathscr{B}_\mathrm{o}}(R/H) \simeq \mathrm{CM}\,(EH)$. □

As consequence, we can give an alternative description of the functors $\Phi^{(B',\nu')} = -\otimes_{kH} F_\lambda(B')$, $H = G_{B'}$, for a G-atoms $B' = (B',\nu')$. Recall that these functors are used for constructing regularly orbicular indecomposable R/G-modules.

Consider the tensor product functor

$$-\otimes_k E : \mathrm{MOD}\,(kH)^{\mathrm{op}} \to \mathrm{CM}\,(EH)$$

defined by the mapping $(V,\mu) \mapsto (V\otimes_k E, \mu\otimes\pi)$, where $\mu\otimes\pi$ is given by the homomorphisms $\mu(h)\otimes\sigma(h) : V\otimes_k E \to V\otimes_k {}^{h^{-1}}E$, $h\in H$ (note that ${}^{h^{-1}}(V\otimes_k E) = V\otimes_k {}^{h^{-1}}E$ since $(V\otimes_k E)_E = V\otimes_k E_E$).

Corollary. *Let $n = 1$. The functors*

$$(\widetilde{\mathscr{H}_B}\circ F'_\bullet)\circ(-\otimes_{kH} F'_\lambda(B_1)),\ -\otimes_k E : \mathrm{MOD}\,(kH)^{op} \longrightarrow \mathrm{CM}\,(EH)$$

are isomorphic.

Proof. Apply the description of (∗∗) as a composition of (∗∗∗) and (∗∗∗∗) (see [**4**, 2.3]), and the fact that canonical k-isomorphism $\mathrm{Hom}_R(B_1, V\otimes_k B_1) \simeq V\otimes_k E$ is a natural, with respect to V in $\mathrm{MOD}\,(kH)^{\mathrm{op}}$, EH-homomorphism. □

3.6.

One can also consider more general situation and form the object $\theta_H^G(B)$ in $\mathrm{Mod}_{\mathrm{f}}^G R$, for $B = (B, \nu)$ is as in 3.3, where $\theta_H^G : \mathrm{Mod}_{\mathrm{f}}^H R \to \mathrm{Mod}_{\mathrm{f}}^G R$ denotes the induction functor (see 1.3). The object $\theta_H^G(B)$ is a pair $(\widetilde{B}, \tilde{\nu})$, where $\widetilde{B} = \bigoplus_{g \in S_H} {}^gB$ $(= \bigoplus_{g \in S_H} \bigoplus_{i=1}^n {}^gB_i)$ is an R-module and $\tilde{\nu} = \nu^G$ is a standard R-action of G on $\tilde{B}$ induced by ν. We denote by $\widetilde{\sigma} : G \to \mathrm{Aut}_{k-\mathrm{alg}}(\widetilde{E})$ the action given by $\mathrm{Hom}_R(\tilde{\nu}, \tilde{\nu})$ of G on $\widetilde{E} = \mathrm{End}_R(\widetilde{B})$ induced by $\tilde{\nu}$ (see 3.2).

One can prove the result analogous to Proposition 3.5 and Theorem 3.3.

Theorem. *Assume that all objects B_i are periodic G-atoms with a common stabilizer $H = G_{B_i}$ for $i = 1, \ldots, n$. If the index $[G : H]$ of H in G is finite, then we have the equivalences*

$$\mathrm{mod}_{\mathscr{B}_o}(R/G) \simeq \mathrm{Mod}_{\mathrm{f},\mathscr{B}_o}^G R \simeq \mathrm{CM}\,(\tilde{E}G)$$

of categories, where $\tilde{E}G = \tilde{E}_{\tilde{\sigma}}G$.

Remark. If $[G : H]$ is infinite then $\mathrm{CM}\,(\tilde{E}_{\tilde{\sigma}}G)$ is not the right object to describe $\mathrm{mod}_{\mathscr{B}_o}(R/G)$ (the algebra $\widetilde{E}$ should be replaced by the category, namely, the full subcategory of $\mathrm{Mod}\,R$ formed by the set $\widetilde{\mathscr{B}} = \{{}^gB_i\}_{i=1,\ldots,n;\, g \in S_H}$).

4. Categories with a trivial action of group on objects.

4.1.

To study the category $\mathrm{mod}_B(R/H)$ we construct certain covering of the category $\mathrm{CM}\,(EH)$. For this purpose we need some generalization of the notion of locally bounded k-category.

Definition. A k-category $\mathscr{E}$ is called *weakly locally bounded* provided $\mathscr{E}$ satisfies the following three conditions:

(a) $x \simeq y$ if and only if $x = y$, for all $x, y \in \mathrm{ob}\,\mathscr{E}$,

(b) $\mathscr{E}(x,x)$ is a local semiprimary k-algebra for every $x \in \mathrm{ob}\,\mathscr{E}$,

(c) for any $x \in \mathrm{ob}\,\mathscr{E}$, $\mathscr{E}(x,y) = 0$ (respectively, $\mathscr{E}(y,x) = 0$) for almost all $y \in \mathrm{ob}\,\mathscr{E}$.

We usually consider weakly locally bounded categories $\mathscr{E}$ satisfying the following extra condition:

(d) $\mathscr{E}(x,x)/J(\mathscr{E}(x,x)) = k$, for every $x \in \mathrm{ob}\,\mathscr{E}$.

Note that any locally bounded k-category $\mathscr{E}$ is weakly locally bounded ; moreover, if k is algebraically closed field then (d) is satisfied for $\mathscr{E}$.

The lemma below presents an example being a motivation of the introduced notion.

Lemma. *The full subcategory $\mathscr{E} = \mathscr{E}(\mathscr{B})$ of $\mathrm{Mod}\,R$, formed by any finite set $\mathscr{B} = \{B_1, \ldots, B_n\}$ of pairwise nonisomorphic G-atoms, is a weakly locally bounded k-category. Moreover, if k is algebraically closed field then $\mathscr{E}$ satisfies the condition (d).*

Proof. The first assertion follows immediately from [**7**, Theorem 2.9]. To prove the second it suffices to observe that since $\mathrm{End}_R(B_i)$ is a local algebra, the factor field $\mathrm{End}(B_i)/J(\mathrm{End}(B_i))$ is always a k-subalgebra of the finite dimensional algebra $\mathrm{End}_k(B_i(x))$, for any $x \in \mathrm{supp}\, B_i$. □

REMARK. Each finitely generated projective $\mathscr{E}$-module has a unique, up to isomorphism, decomposition into a finite direct sum of indecomposable (projective) $\mathscr{E}$-modules of the form $\mathscr{E}(-,x)$, $x \in \mathrm{ob}\,\mathscr{E}$. Note that the discussed problem can be regarded as the analogous one for a semiprimary ring, and then the assertion follows by the uniqueness property for decomposition into a direct sum of indecomposables for finitely generated projective $\mathscr{E}$-modules over semiperfect rings (see [**1**]).

From now on we consider only weakly locally bounded categories $\mathscr{E}$ that satisfy the condition (d)!

4.2.

Suppose we are given a k-liner action of an abstract group H on a weakly locally bounded k-category $\mathscr{E}$, viewed as a group homomorphism $\sigma : H \to \mathrm{Aut}_{k-\mathrm{cat}}(\mathscr{E})$. Then σ induces the action of H on the category $\mathrm{MOD}\,\mathscr{E}$ of all $\mathscr{E}$-modules. We can also consider the category $\mathrm{MOD}^H\mathscr{E}$ of all $\mathscr{E}$-modules with $\mathscr{E}$-action of H whose object as always are pairs (M,μ), where M is an $\mathscr{E}$-module and $\mu = (\mu_h : M \to {}^{h^{-1}}M)_{h\in H}$ is an $\mathscr{E}$ action of H on M (cf. 1.2). Analogously as in the algebra case there exists a construction of a skew group category $\mathscr{E}_\sigma H$ and the module category $\mathrm{MOD}\,\mathscr{E}_\sigma H$ is equivalent to $\mathrm{MOD}^H\mathscr{E}$ (see [**26**]). We do not present it here, since we do not need its precise description but only the fact that we can identify these two module categories.

From now on writing $\mathrm{MOD}\,\mathscr{E}H$ we mean simply $\mathrm{MOD}^H\mathscr{E}$ (whenever this does not lead to any confusion we usually write for simplicity $\mathscr{E}H$ instead of $\mathscr{E}_\sigma H$).

Observe that, if σ is a trivial action then the category $\mathrm{MOD}^H\mathscr{E}$ can be viewed as the category of all kH-$\mathscr{E}$-bimodules, that is the k-functors $M : \mathscr{E}^{\mathrm{op}} \to \mathrm{MOD}\,(kH)^{\mathrm{op}}$. In particular, if H is the trivial group then $\mathrm{MOD}^H\mathscr{E} = \mathrm{MOD}\,\mathscr{E}$.

Analogously as in the algebra case, we denote by $\mathrm{CM}\,(\mathscr{E}H)$ the full subcategory of $\mathrm{MOD}\,\mathscr{E}H$ formed by all the pairs (M,μ) such that M is a finitely generated projective $\mathscr{E}$-module, in fact isomorphic to a finite direct sum of $\mathscr{E}$-modules $P_x = \mathscr{E}(-,x)$, $x \in \mathrm{ob}\,\mathscr{E}$ (see Remark 4.1). Note that $P_x = \mathscr{E}(-,x)$ itself carries the canonical structure of object in $\mathrm{CM}\,(\mathscr{E}H)$; namely, $P_x = (P_x, \pi_x)$, where $\pi_x = ((\pi_x)_h : \mathscr{E}(-,x) \to \mathscr{E}(-,hx))_{h\in H}$ is given by $(\pi_x)_h(z) = \sigma(h)(z,x)$, for $h \in H$ and $z \in \mathrm{ob}\,\mathscr{E}$.

REMARK. The subcategory $\mathrm{CM}\,(\mathscr{E}H)$ of $\mathrm{MOD}\,\mathscr{E}H$ is closed under direct summands.

A fundamental role in studying the category $\mathrm{CM}\,(\mathscr{E}H)$ is played by the following result.

THEOREM. *If $M = (M,\mu)$ is an indecomposable object in $\mathrm{CM}\,(\mathscr{E}H)$ then the endomorphism algebra $\mathrm{End}_{\mathscr{E}H}(M)$ is local.*

The proof of the theorem is based on the following well known fact.

LEMMA. *Let C be a k-category (respectively, k-algebra), $a \in E = \mathrm{End}_C(X)$ an endomorphism of an object X in $\mathrm{MOD}\,C$, $a_| \in \mathrm{End}_C(\mathrm{Im}\,a)$ the restriction of a to $\mathrm{Im}\,a$*

and $\operatorname{Im} a \xrightarrow{a'} \operatorname{Im} a^2 \xrightarrow{a''} \operatorname{Im} a$ *the standard factorization of* $a_|$ *via* $\operatorname{Im}(a_|)$. *Then the following conditions are equivalent*:

(a) a' *is a monomorphism and* a'' *is an epimorphism,*

(b) $a_| \in \operatorname{End}_C(\operatorname{Im} a)$ *is an automorphism,*

(c) $X = \operatorname{Im} a \oplus \operatorname{Ker} a$

(d) *there exists an idempotent* $e \in E$ *and element* $u \in eEe$, *invertible in* eEe, *such that* $a = ue$.

Each $a \in \operatorname{End}_C(X)$ satisfying one of the equivalent conditions above, will be called a *splitting endomorphism* (of X).

Corollary. *Assume that* C *is a* k*-algebra.*

(a) *If* $\dim_k(\operatorname{Im} a)$ *is finite, then* $a \in \operatorname{End}_C(X)$ *is a splitting endomorphism if and only if either* a' *is a monomorphism or* a'' *is an epimorphism. Moreover, for any* $a \in \operatorname{End}_C(X)$, *there exists positive* $m \in \boldsymbol{N}$ *such that* a^m *is a splitting endomorphism.*

(b) *If* C *is the Laurent polynomial algebra* $k[t, t^{-1}]$ *and* $(\operatorname{Im} a)_C$ *is a finitely generated* C*-module, then* a *is a splitting epimorphism if and only if* a'' *is an epimorphism.*

Proof. (a) The first assertion is straightforward, the second follows easily if one consider a decreasing sequence of C-submodules $\{\operatorname{Im} a^m\}_{m \in \boldsymbol{N}}$ of X.

To show (b) assume that $\operatorname{Im} a^2 = \operatorname{Im} a$. We can present $\operatorname{Im} a$ as a direct sum $\operatorname{Im} a = F \oplus T$, where F is a finitely generated free and T a finite-dimensional C-module. Then the epimorphism $a_|$ has the form $a_| = \begin{bmatrix} a_{11} & 0 \\ a_{21} & a_{22} \end{bmatrix}$, where a_{11} is an isomorphism by the uniqueness of the decomposition into indecomposables. Consequently, $a_|$ is an isomorphism since so is the epimorphism $\begin{bmatrix} \mathrm{id}_F & 0 \\ 0 & a_{22} \end{bmatrix} = \begin{bmatrix} a_{11}^{-1} & 0 \\ -a_{21}\, a_{11}{}^{-1} & \mathrm{id}_T \end{bmatrix} \cdot a_|$ (a_{22} is an epimorphic endomorphism of T, so isomorphism). □

4.3.

In the proof of Theorem 4.2 we will also apply the change of base field technique.

Let C be a k-category and K a commutative field containing k. Then using the functor $K \otimes_k - : \operatorname{MOD} k \to \operatorname{MOD} K$, one can form the category $C^{(K)} = K \otimes_k C$, analogously as in algebra case. The functor $K \otimes_k -$ induces also the "scalar extension" functor

$$(-)^{(K)} : \operatorname{MOD} C \to \operatorname{MOD} C^{(K)}$$

which is exact. Note that, $Y = 0$ if and only if $Y^{(K)} = 0$, for any Y in $\operatorname{MOD} C$.

Lemma. (a) *If* Y *is a finitely generated projective* C*-module then* $Y^{(K)}$ *is a finitely generated projective* $C^{(K)}$*-module.*

(b) *Let* a *be as in Lemma* 4.2. *Then* $a \in \operatorname{End}_C(X)$ *is a splitting endomorphism if and only if* $a^{(K)} \in \operatorname{End}_{C^{(K)}}(X^{(K)})$ *is a splitting endomorphism.*

(c) *If* C *is a weakly locally bounded* k*-category then* $C^{(K)}$ *is a weakly locally bounded* K*-category.*

Proof. The assertion (a) is an immediate consequence of the definition of $(-)^{(K)}$; (b) follows easily from Lemma 4.2(a), by basic properties of $(-)^{(K)}$ formulated before the statement of the lemma.

To prove (c), we verify the conditions 4.1(a)–(d). Note first that for any $x \in \operatorname{ob} C^{(K)} = \operatorname{ob} C$, the K-subspace $J^{(K)}(x,x) = K \otimes_k J(C(x,x))$ is a nilpotent ideal in $C^{(K)}(x,x) = K \otimes_k C(x,x)$. Moreover, since $(-)^{(K)}$ is exact, by 4.1(d) we have $C^{(K)}(x,x)/J^{(K)}(x,x) \simeq K \otimes_k k \simeq K$. Consequently, $J^{(K)}(x,x) = J(C^{(K)}(x,x))$ and $C^{(K)}(x,x)$ is a local semiprimary K-algebra with $C^{(K)}(x,x)/J(C^{(K)}(x,x)) \simeq K$.

Since 4.1(c) is trivially satisfied for $C^{(K)}$, it remains to show the condition 4.1(a). Observe that the inclusion $C(x,y) \circ C(y,x) \subseteq J(C(x,x))$ holds for all $x, y \in \operatorname{ob} C$, $x \neq y$ (otherwise, x is a direct summand of y, so $x = y$ by 4.1(a) for C, a contradiction). Consequently, $C^{(K)}(x,y) \circ C^{(K)}(y,x) \subseteq J^{(K)}(x,x) = J(C^{(K)}(x,x))$, and 4.1(a) is satisfied for $C^{(K)}$. □

Proof of Theorem 4.2. We show that any endomorphism $f \in \operatorname{End}_{\mathscr{E}H}(M)$ is either invertible or nilpotent. Then applying standard arguments we infer that $\operatorname{End}_{\mathscr{E}H}(M)$ is local.

To prove our claim it suffices to show the following:

(i) for any finitely generated projective $\mathscr{E}$-module M and endomorphism $f \in \operatorname{End}_{\mathscr{E}}(M)$, f^m is a splitting endomorphism of M, for some positive $m \in \boldsymbol{N}$.

Note that, if (i) is satisfied then we have a decomposition $M = \operatorname{Im} f^m \oplus \operatorname{Ker} f^m$ in $\operatorname{MOD} \mathscr{E}$; in case $f \in \operatorname{End}_{\mathscr{E}H}(M)$, it is also a decomposition in $\operatorname{MOD} \mathscr{E}H$, so in $\operatorname{CM}(\mathscr{E}H)$ (see Remark 4.2). Consequently, by indecomposability of the object $M = (M, \mu)$ in $\operatorname{CM}(\mathscr{E}H)$, we infer that, either $\operatorname{Ker} f^m = M$, so f is nilpotent; or $\operatorname{Ker} f^m = 0$ and $\operatorname{Im} f^m = M$, so f is invertible.

From now on we assume that M and f are as in (i). Observe that by Lemma 4.3, we have to prove the assertion of (i) only in the case $k = \bar{k}$ (k is an algebraically closed field). Moreover, note that f^m is a splitting endomorphism of M if and only if so is f_φ for some $\varphi \in \operatorname{Aut}_{\mathscr{E}}(M)$, where $f_\varphi = \varphi^{-1} \cdot f \cdot \varphi$.

The idea of the proof of (i) is the following. For $f \in \operatorname{End}_{\mathscr{E}}(M)$ (in case $k = \bar{k}$), we construct an automorphism $\varphi \in \operatorname{Aut}_{\mathscr{E}}(M)$ and the subalgebra $\Lambda \subseteq E = \operatorname{End}_{\mathscr{E}}(M)$ such that Λ contains an element $v = (f_\varphi)^{m'}$ for some positive $m' \in \boldsymbol{N}$, and that $v^{m''} \in \Lambda = \operatorname{End}_\Lambda(\Lambda_\Lambda)$ is a splitting endomorphism of Λ_Λ for some positive $m'' \in \boldsymbol{N}$. Then $v^{m''} \in \operatorname{End}_\Lambda(\Lambda_\Lambda)$ satisfies 4.2(d); consequently, so does the endomorphism $(f_\varphi)^{m'm''} = v^{m''} \in \operatorname{End}_{\mathscr{E}}(M)$ of M (apply Lemma 4.2(d), $e\Lambda e$ is a subalgebra of eEe!). Hence, by the second observation, $f^m \in \operatorname{End}_{\mathscr{E}}(M)$ is a splitting endomorphism of M, for $m = m'm''$.

To construct, for a given $f \in \operatorname{End}_{\mathscr{E}}(M)$, the pair (φ, Λ) as above, we need more information on the structure of the algebra $E = \operatorname{End}_{\mathscr{E}}(M)$.

Without loss of generality, we can assume that the $\mathscr{E}$-module M is of the form $M = \bigoplus_{i=1}^n P_i^{d_i}$, where $P_i = \mathscr{E}(-, x_i)$ for some $x_1, \dots, x_n \in \operatorname{ob} \mathscr{E}$ and $d_1, \dots, d_n \in \boldsymbol{N}$ (see Remark 4.1). By the Yoneda Lemma and the general assumption 4.1(d) (4.1(d) implies the isomorphism $\mathscr{E}(x,x) = k \cdot \mathrm{id}_x \oplus J(\mathscr{E}(x,x))$, for any $x \in \operatorname{ob} \mathscr{E}$) the equality defining M yields the standard isomorphism

$$E \simeq \bigoplus_{i=1}^n \mathrm{M}_{d_i}(k) \oplus \bigoplus_{i,j=1}^n \mathrm{M}_{d_j \times d_i}(\mathscr{J}_{\mathscr{E}}(P_i, P_j)) . \qquad \text{(ii)}$$

Observe that under the above identification we have

$$\begin{aligned} J(E) &= \bigoplus_{i,j=1}^{n} \mathrm{M}_{d_j \times d_i}(\mathscr{J}_{\mathscr{E}}(P_i, P_j)) \\ &\simeq \bigoplus_{i=1}^{n} \mathrm{M}_{d_i}(J(\mathscr{E}(x_i, x_i))) \oplus \bigoplus_{i \neq j;\ i,j=1,\ldots,n} \mathrm{M}_{d_j \times d_i}(\mathscr{E}(x_i, x_j))\,. \end{aligned}$$

Note that, since all algebras $\mathrm{End}_{\mathscr{E}}(P_i)$ are semiprimary, the k-space $J = \bigoplus_{i,j=1}^{n} \mathrm{M}_{d_j \times d_i}(\mathscr{J}_{\mathscr{E}}(P_i, P_j))$ is a nilpotent ideal of E. More precisely, we have $J^{ns} = 0$, where s is a common bound of nilpotency degrees for all $J(\mathrm{End}_{\mathscr{E}}(P_i))$, $i = 1, \ldots, n$. Moreover, for any $f \in \mathrm{End}_{\mathscr{E}}(M)$, we denote by $(\bar{f}, f')$, $\bar{f} = (\bar{f}_i)_{i=1,\ldots,n} \in \bigoplus_{i=1}^{n} \mathrm{M}_{d_i}(k)$ and $f' \in \bigoplus_{i,j=1}^{n} \mathrm{M}_{d_j \times d_i}(\mathscr{J}_{\mathscr{E}}(P_i, P_j))$ the pair corresponding to f via (ii). Then the map f has the canonical decomposition

$$f = \bar{f} + f'\,. \tag{iii}$$

(For any $U = (U_j)_{j=1}^{p} \in \bigoplus_{j=1}^{p} \mathrm{M}_{d'_j}(k)$, $d'_1, \ldots, d'_p \in \boldsymbol{N}$, $1 \le i(1), \ldots, i(p) \le n$, we can identify U with the $\mathscr{E}$-homomorphism $\varphi_U = \bigoplus_{j=1}^{p} U_j \cdot \mathrm{id}_{P_{i(j)}} : \bigoplus_{j=1}^{p} P_{i(j)}^{d'_j} \to \bigoplus_{j=1}^{p} P_{i(j)}^{d'_j}$). Since J is an ideal in E, we have $\overline{f^1 f^2} = \bar{f}^1 \bar{f}^2$ in $\prod_{i=1}^{n} \mathrm{M}_{d_i}(k)$, for any $f^1, f^2 \in E$; and therefore, $E/J \simeq \prod_{i=1}^{n} \mathrm{M}_{d_i}(k)$.

In conclusion, $J(E) = J$ and E is a semiprimary k-algebra.

Fix now $f \in \mathrm{End}_{\mathscr{E}}(M)$. We construct first the announced automorphism $\varphi \in \mathrm{Aut}_{\mathscr{E}}(M)$. By the structure theorem for finitely generated modules over principal ideal domains, for every $i = 1, \ldots, n$, there exists a nonsingular matrix $C_i \in \mathrm{M}_d(k)$ such that

$$C_i^{-1} \bar{f}_i\, C_i = \begin{bmatrix} \bar{f}_i^{(1)} & 0 \\ 0 & \bar{f}_i^{(2)} \end{bmatrix}$$

where $\bar{f}_i^{(1)} \in \mathrm{M}_{d_i^{(1)}}(k)$ is invertible (the block diagonal block matrix, consisting of Jordan blocks with nonzero eigenvalues), $\bar{f}_i^{(2)} \in \mathrm{M}_{d_i^{(2)}}(k)$ is nilpotent and $d_i^{(1)} + d_i^{(2)} = d_i$. Let $C \in \mathrm{M}_d(k)$, $d = \sum_{i=1}^{n} d_i$, be the block diagonal nonsingular matrix defined by blocks C_i, $i = 1, \ldots, n$. Modulo some natural slight renumbering of indices, the matrix $C^{-1} \bar{f}\, C$ can be regarded as a matrix

$$\begin{bmatrix} \bar{f}^{(1)} & 0 \\ 0 & \bar{f}^{(2)} \end{bmatrix}$$

where $\bar{f}^{(1)} \in \mathrm{M}_{d^{(1)}}(k)$, $d^{(1)} = \sum_{i=1}^{n} d_i^{(1)}$, is an invertible block diagonal matrix given by the square matrices $\bar{f}_i^{(1)}$, $i = 1, \ldots, n$, and $\bar{f}^{(2)} \in \mathrm{M}_{d_i^{(2)}}(k)$, $d^{(2)} = \sum_{i=1}^{n} d_i^{(2)}$, is a nilpotent block diagonal matrix given by the square matrices $\bar{f}_i^{(2)}$, $i = 1, \ldots, n$. We set

$\varphi = \varphi_{(C_i)_{i=1,\ldots,n}}$. Note that we have $\overline{f_\varphi} = C^{-1}\bar{f}\,C$ since $\overline{\varphi} = C$ and $\overline{\varphi^{-1}} = C^{-1}$.

Let m' be an upper bound of nilpotency degrees of all homomorphisms $\bar{f}_i^{(2)}$, $i = 1, \ldots, n$. Then

$$\overline{(f_\varphi)^{m'}} = (C^{-1}\bar{f}\,C)^{m'} = \begin{bmatrix} (\bar{f}^{(1)})^{m'} & 0 \\ 0 & 0 \end{bmatrix}$$

and consequently the $\mathscr{E}$-homomorphism $v = (f_\varphi)^{m'} : M \to M$ has the form

$$v = \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix} : P^{(1)} \oplus P^{(2)} \to P^{(1)} \oplus P^{(2)}$$

where $P^{(1)} = \bigoplus_{i=1}^n P_i^{d_i^{(1)}}$, $P^{(2)} = \bigoplus_{i=1}^n P_i^{d_i^{(2)}}$, $v_{11} = \bar{v}_{11} + v'_{11}$, $\bar{v}_{11} = (\bar{f}^{(1)})^{m'}$ and $v'_{11}, v_{12}, v_{21}, v_{22} \in J(E)$.

By the construction of homomorphisms $\bar{f}_i^{(1)}$, $i = 1, \ldots, n$, there exist natural numbers $d'_1, \ldots d'_p$, $i(1), \ldots, i(p) \in \boldsymbol{N}$, matrices $v_{11}^{(j)}, N_j \in \mathrm{M}_{d'_j}(k)$, $j = 1 \ldots, p$, and scalars $\lambda_1, \ldots, \lambda_p \in k \setminus \{0\}$, such that $\sum_{j=1}^p d'_{i(j)} = d^{(1)}$, $1 \le i(1), \ldots, i(p) \le n$, $P^{(1)} = \bigoplus_{j=1}^p P_{i(j)}^{d'_j}$ (some refinement of the decomposition $P^{(1)} = \bigoplus_{i=1}^n P_i^{d_i^{(1)}}$), $\bar{v}_{11} = \varphi_{(\bar{v}_{11}^{(j)})_{j=1,\ldots,p}}$ (as elements of $\mathrm{End}_{\mathscr{E}}(P^{(1)})$); finally, $\bar{v}_{11}^{(j)} = \lambda_j \mathrm{I}_{d'_j} + N_j$ and $N_j^{d'_j} = 0$, for $j = 1, \ldots p$. We can view v_{11} (respectively, v'_{11}) in the form $(v_{11}^{(j,\,l)})_{j,\,l=1,\ldots,p}$ (respectively, $(v_{11}'^{(j,\,l)})_{j,\,l=1,\ldots,p}$), as elements of $\bigoplus_{j,\,l=1}^p \mathrm{Hom}_{\mathscr{E}}(P_{i(l)}^{d'_l}, P_{i(j)}^{d'_j})$. Then we have $v_{11}^{(j,j)} = \bar{v}_{11}^{(j)} + v_{11}'^{(j,j)}$ and $v_{11}^{(j,\,l)} = v_{11}'^{(j,\,l)}$ for all $j, l = 1 \ldots, p$, $j \ne l$. Moreover, we have also presentations $v_{12} = (v_{12}^{(j)})_j \in \bigoplus_{j=1}^p \mathrm{Hom}_{\mathscr{E}}(P^2, P_{i(j)}^{d'_j})$ and $v_{2,1} = (v_{21}^{(j)})_j \in \bigoplus_{j=1}^p \mathrm{Hom}_{\mathscr{E}}(P_{i(j)}^{d'_j}, P^{(2)})$. We set $e_1 = \mathrm{id}_{P^{(1)}} = \varphi_{(\mathrm{I}_{d'_j})_{j=1,\ldots,p}}$, $e_2 = \mathrm{id}_{P^{(2)}}$, $u = \bar{v}_{11}$ and $\tilde{u} = (\bar{v}_{11})^{-1}$ (in $\mathrm{End}_{\mathscr{E}}(P^{(1)})$).

Now we define the subalgebra Λ of E. We set $\Lambda = k[\mathscr{X}]$, where $\mathscr{X} = \{e_1, e_2\} \cup \mathscr{V} \cup \mathscr{N} \cup \mathscr{U}$, $\mathscr{V} = \{v_{22}\} \cup \{v_{12}^{(j)}\}_{j=1,\ldots,p} \cup \{v_{21}^{(j)}\}_{j=1,\ldots,p} \cup \{v_{11}'^{(j,\,l)}\}_{j,\,l=1,\ldots,p}$, $\mathscr{N} = \{N_j\}_{j=1,\ldots,p}$ and $\mathscr{U} = \{u, \tilde{u}\}$. Note that the element

$$v = u + \sum_{i,\,l=1}^{p} v_{11}'^{(j,\,l)} + \sum_{i=1}^{p} v_{12}^{(j)} + \sum_{i=1}^{p} v_{21}^{(j)} + v_{22}$$

belongs to Λ. To show that $v^{m''} \in \Lambda = \mathrm{End}_\Lambda(\Lambda_\Lambda)$ is a splitting endomorphism for some positive $m'' \in \boldsymbol{N}$, we need more information on the structure of Λ. For this aim we compare Λ to the subalgebra $\Lambda' = k[\mathscr{X}']$ of E, where $\mathscr{X}' = \{e_1, e_2\} \cup \mathscr{V} \cup \mathscr{N}$.

First we prove that Λ' is a finite-dimensional k-algebra. For this aim it suffices to find $q \in N$ such that ${}_k\Lambda'$, as k-linear vector space, is generated by words in $\mathscr{X}'$ of length bounded by q (that is products of at most q elements from $\mathscr{X}'$). Observe that e_1, e_2 are orthogonal idempotents in Λ' (so in Λ) such that $e_1 + e_2 = 1_E$, and that $e_z x$, $x e_z$ belong to $\{x, 0\}$, for all $x \in \mathscr{X}$, $z = 1, 2$. Moreover, $\mathscr{V}$ is contained in $J(E)$ and we have $xy, yx \in J(E)$ for all $x \in \mathscr{N}$, $y \in \mathscr{V}$. Therefore, since $J(E)^{ns} = 0$, ${}_k\Lambda'$ (over k) is generated by e_1, e_2 and all words w in $\mathscr{V} \cup \mathscr{N}$ that contains at most $ns - 1$ generators from $\mathscr{V}$. Note that $N^{d'} = 0$ and $N_j N_l = 0$ for all $j, l = 1, \dots, p$, $j \neq l$, where d' is an upper bound of $d'_1, \dots, d'_p$. Consequently, ${}_k\Lambda'$ is generated by all words in generators from $\mathscr{X}'$ of length bounded by $q = nsd' - 1$.

Now we prove the fundamental property of Λ for further considerations; namely, that $e_1\Lambda e_2$, $e_2\Lambda e_1$, $e_2\Lambda e_2$ are finite-dimensional k-vector spaces and $e_1\Lambda e_1$ is a finitely generated (left and right) A-module, where A is the subalgebra of $e_1\Lambda e_1$ generated by $\mathscr{U}$ ($\mathscr{U} \subseteq e_1\Lambda e_1$ since $ue_2, e_2u, \tilde{u}e_2, e_2\tilde{u} = 0$). Note that either $A \simeq k[t, t^{-1}]$ or A is a finite-dimensional k-algebra, since A is the image of the canonical k-algebra homomorphism $k[t, t^{-1}] \to e_1\Lambda e_1$ defined by $t \mapsto u$.

In the main step of the proof of the assertion above, we show that any (nonzero!) word w in generators from $\mathscr{V} \cup \mathscr{N} \cup \mathscr{U}$ belongs either to Λ' or to A. We apply an induction on the number $\deg_{\mathscr{U}} w$ of occurring in w (as a formal product) generators from $\mathscr{U}$. If $\deg_{\mathscr{U}} w = 0$ then clearly we have $w \in \Lambda'$. Assume that $w \neq 0$ and $\deg_{\mathscr{U}} w > 0$. Then w contains as a subword one of the words xy or yx, where $x \in \mathscr{U}$, $y \in (\mathscr{V} \setminus \{v_{22}\}) \cup \mathscr{N}$. In the first case we have to consider three possibilities $y = v_{12}^{(j)}$, $y = {v'}_{11}^{(j,l)}$ and $y = N_j$, for suitable j, l. In all these cases we have the equalities $xy = \lambda_j y + N_j y$ and $xy = \sum_{z=1}^{d'} (-1)^{z-1} \lambda_j^{-z} N_j^{z-1} y$, when $x = u$ and $x = \tilde{u}$, respectively. Then, by the inductive assumption, we infer that w belongs to Λ. Analogously, in the second case, we have to consider again three possibilities $y = v_{21}^{(l)}$, $y = {v'}_{11}^{(j,l)}$ and $y = N_l$. Then, applying the equalities $yx = \lambda_l y + yN_l$ and $yx = \sum_{z=1}^{d'} (-1)^{z-1} \lambda_l^{-z} y\, N_l^{z-1}$ for $x = u$ and $x = \tilde{u}$, respectively, we infer similarly that w belongs to Λ.

A straightforward consequence of that just proved above is the equality $\Lambda = \Lambda' + A$. Multiplying this equality from both sides by suitable idempotents, we get the equalities $e_1\Lambda e_2 = e_1\Lambda' e_2$, $e_2\Lambda e_1 = e_2\Lambda' e_1$, $e_2\Lambda e_2 = e_2\Lambda' e_2$ and $e_1\Lambda e_1 = e_1\Lambda' e_1 + A$. Now the required properties of Λ follows from the fact that Λ is finite-dimensional. In particular, we also obtain that $(\Lambda e_2)_{e_2\Lambda e_2}$ is a finite-dimensional $e_2\Lambda e_2$-module and $(\Lambda e_1)_A$ is finitely generated A-module.

Now we complete the proof of the theorem and show that $v^{m''} \in \Lambda = \mathrm{End}_\Lambda(\Lambda_\Lambda)$ is a splitting endomorphism for some positive $m'' \in \boldsymbol{N}$. For any $m \in \boldsymbol{N}$, v^m has the form $v^m = v^m(\cdot)e_1 \oplus v^m(\cdot)e_2 : \Lambda e_1 \oplus \Lambda e_2 \to \Lambda e_1 \oplus \Lambda e_2$. Moreover, we have $v^m(\cdot)e_1 = (v(\cdot)e_1)^m$, $v^m(\cdot)e_2 = (v(\cdot)e_2)^m$ and $\mathrm{Im}\, v^m = (\mathrm{Im}\, v^m)e_1 \oplus (\mathrm{Im}\, v^m)e_2 = \mathrm{Im}\,(v^m(\cdot)e_1) \oplus \mathrm{Im}\,(v^m(\cdot)e_2)$. It suffices to show that if $A \simeq k[t, t^{-1}]$ then $\mathrm{Im}\,(v^{m_1}(\cdot)e_1) = \mathrm{Im}\,(v^{m_1+1}(\cdot)e_1)$, for some positive $m_1 \in \boldsymbol{N}$. Then, since $(\Lambda e_2)_{e_2\Lambda e_2}$ is finite-dimensional and $(\Lambda e_1)_A$ is finitely generated (= finite-dimensional if $A \not\simeq k[t, t^{-1}]$), we infer by Corollary 4.2 that both, $v^{m_1}(\cdot)e_1$ and $v^{m_2}(\cdot)e_2$ (for some positive $m_1, m_2 \in \boldsymbol{N}$) are splitting endomorphisms of $(\Lambda e_1)_{e_1\Lambda e_1}$ and $(\Lambda e_2)_{e_2\Lambda e_2}$, respectively. Consequently, $v^{m''} \in \Lambda = \mathrm{End}_\Lambda(\Lambda_\Lambda)$ is a splitting endomorphism of Λ_Λ, where $m'' = m_1 m_2$.

To prove the final claim, we show that the decreasing sequence

$$v\Lambda e_1 \supseteq v^2\Lambda e_1 \supseteq \dots \supseteq v^m\Lambda e_1 \supseteq v^{m+1}\Lambda e_1 \supseteq \dots \qquad \text{(iv)}$$

of $e_1\Lambda e_1$-submodules of Λe_1 stabilizes (note that, $v^m\Lambda e_1 = \mathrm{Im}\,(v^m(\cdot)e_1)$).

Let

$$v^m = \begin{bmatrix} (v^m)_{11} & (v^m)_{12} \\ (v^m)_{21} & (v^m)_{22} \end{bmatrix}$$

be a matrix presentation of v^m with respect to the decomposition $M = P^{(1)} \oplus P^{(2)}$. By the definition of $(v^m)_{11}$, $(v^m)_{11} - u^m$ belongs to the nilpotent ideal $J(E) \cap e_1 \Lambda e_1$ of $e_1 \Lambda e_1$, so $(v^m)_{11}$ is invertible in $e_1 \Lambda e_1$ ($((v^m)_{11})^{-1} \in e_1 \Lambda e_1$), for every $m \in \boldsymbol{N}$. Therefore the composite Λ-homomorphism

$$v^m \Lambda_\Lambda \hookrightarrow \Lambda_\Lambda \xrightarrow{e_1 \cdot} e_1 \Lambda$$

$m \in \boldsymbol{N}$, are surjective since the equality

$$\begin{bmatrix} e_{11} & 0 \\ 0 & 0 \end{bmatrix} v^m \begin{bmatrix} ((v^m)_{11})^{-1} & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} e_{11} & 0 \\ 0 & 0 \end{bmatrix}$$

in $\mathrm{End}_\Lambda(e_1 \Lambda \oplus e_2 \Lambda)$ holds. Hence, the $e_1 \Lambda e_1$-homomorphisms

$$\pi_m : v^m \Lambda e_1 \hookrightarrow \Lambda e_1 \xrightarrow{\pi} e_1 \Lambda e_1$$

$m \in \boldsymbol{N}$, are also surjective, where each π is a restriction of $e_1 \cdot$ to Λe_1. By (iv), the family $\{\mathrm{Ker}\, \pi_m\}_{m \in \boldsymbol{N}}$ forms a decreasing sequence

$$\mathrm{Ker}\, \pi_1 \supseteq \mathrm{Ker}\, \pi_2 \supseteq \cdots \supseteq \mathrm{Ker}\, \pi_1 \supseteq \mathrm{Ker}\, \pi_{m+1} \supseteq \ldots$$

of submodules of the finite-dimensional right $e_1 \Lambda e_1$-module $\mathrm{Ker}\, \pi = e_2 \Lambda e_1$. Consequently, $\mathrm{Ker}\, \pi_{m_1} = \mathrm{Ker}\, \pi_{m_1+1}$ for some positive $m_1 \in \boldsymbol{N}$. Then we have $v^{m_1} \Lambda e_1 / v^{m_1+1} \Lambda e_1 \simeq (v^{m_1} \Lambda e_1 / \mathrm{Ker}\, \pi_{m_1}) / (v^{m_1+1} \Lambda e_1 / \mathrm{Ker}\, \pi_{m_1+1}) = 0$, and $v^{m_1} \Lambda e_1 = v^{m_1+1} \Lambda e_1$.

In conclusion, (iv) stabilizes, so $v^{m''} \in \mathrm{End}_\Lambda(\Lambda_\Lambda)$, $v^{m''} \in E = \mathrm{End}_{\mathscr{E}}(M))$ and $f^m \in E = \mathrm{End}_{\mathscr{E}}(M)$ are splitting endomorphisms, where $m = m'm''$, m', m'' as above. In this way the proof of the theorem is finished. □

4.4.

The considered here actions of the group H on weakly locally bounded k-categories $\mathscr{E}$ are usually trivial on objects ($H_x = H$ for every $x \in \mathrm{ob}\, \mathscr{E}$). The general aim of this section is to study the category $\mathrm{CM}\,(\mathscr{E}H)$ under this assumption.

The typical example of this situation is in fact that one discussed in 3.3. To interpret it in the context of notions introduced in 4.2 consider the family

$$\mathrm{Hom}_R(\nu_j, \nu_i) : \mathrm{Hom}_R(B_i, B_j) \to \mathrm{Hom}_R(B_i, B_j)$$

$i, j = 1, \ldots, n$, of k-linear actions of the group H. We keep further the notation and assumptions from 3.3 concerning the group H, the set $\mathscr{B}_{\mathrm{o}}$ and the action σ of H on the algebra E.

Lemma. (a) *The above family gives rise to the action* $\sigma' : H \to \mathrm{Aut}_{k-\mathrm{cat}}(\mathscr{E})$ *of the* *group H on the weakly locally bounded k-category $\mathscr{E} = \mathscr{E}(\mathscr{B}_o)$ that is trivial on objects.*

(b) *The standard mapping $M \mapsto \bigoplus_{i=1}^{n} M(B_i)$, M in $\mathrm{MOD}\,\mathscr{E}$, induces an equivalence*

$$\mathrm{MOD}\,(\mathscr{E}_{\sigma'}H) \simeq \mathrm{MOD}\,(E_\sigma H)$$

of categories, that restricts to an equivalence

$$\mathrm{CM}\,(\mathscr{E}_{\sigma'}H) \simeq \mathrm{CM}\,(E_\sigma H)$$

that is induced by the mapping $\mathscr{E}(-, B_i) \mapsto \mathrm{Hom}_E(B, B_i)$, $i = 1, \ldots, n$.

Proof. (a) An easy check on definitions.

(b) Follows by slight adaptation of the classical arguments. □

Corollary. *There are equivalences of categories*

$$\mathrm{mod}_{\mathscr{B}_o}(R/H) \simeq \mathrm{Mod}^{H}_{\mathrm{f},\mathscr{B}_o} R \simeq \mathrm{CM}\,(E_\sigma H) \simeq \mathrm{CM}\,(\mathscr{E}_{\sigma'}H)\,.$$

Note that, under the above equivalence, ${F'_\bullet}^{-1}(\bigoplus_{i=1}^{n} B_i^{d_i}, \mu)$ corresponds to $(\bigoplus_{i=1}^{n} P_i^{d_i}, \mu')$, where μ' is the $\mathscr{E}$-action of H induced by μ.

4.5.

The above result leads to a definition of generalized tensor product in the category $\mathrm{MOD}\,(\mathscr{E}H)$, and in particular in $\mathrm{CM}\,(\mathscr{E}H)$, where $\mathscr{E}$, H satisfy the general assumptions of 4.4 (H acts by $\sigma' : H \to \mathrm{Aut}_{k-\mathrm{cat}}(\mathscr{E})$ trivially on objects of a weakly locally bounded k-category $\mathscr{E}$) and $\mathscr{E}H$ stands for $\mathscr{E}_{\sigma'}H$.

Let

$$P: \qquad P_1 \overset{p_2}{\leftarrow} P_2 \leftarrow \cdots \leftarrow P_{n-1} \overset{p_n}{\leftarrow} P_n \tag{$*$}$$

be a sequence of objects and morphisms in $\mathrm{MOD}\,(\mathscr{E}H)$, where $P_i = (P_i, \nu_i)$ for every $i = 1, \ldots, n$. Then for any

$$V: \qquad V_1 \subseteq V_2 \subseteq \cdots \subseteq V_{n-1} \subseteq V_n$$

in I_n-spr$\,(kH)$, $V_n = (V_n, \mu)$, we denote by $\underline{V} \otimes_k P$ the $\mathscr{E}$-module

$$\underline{V} \otimes_k P = \bigoplus_{i=1}^{n} \underline{V}_i \otimes_k P_i$$

where $\underline{V} = (\underline{V}_i)_{i=1,\ldots,n}$ is a fixed sequence of complementary direct summands for V (see 1.4). The $\mathscr{E}$-module $\underline{V} \otimes_k P$ is equipped with the structure of H-representation. It is given by the family $\underline{\mu} \otimes_k p = ((\underline{\mu} \otimes_k p)(h))_{h \in H}$ of the maps $(\underline{\mu} \otimes_k p)(h) : \underline{V} \otimes_k P \to \underline{V} \otimes_k P$, $h \in H$, with components $\mu(h)_{i,j} \otimes_k p_{i,j}(h) : \underline{V}_j \otimes_k P_j \to \underline{V}_i \otimes_k P_i$, $1 \le i, j \le n$, where for any $h \in H$,

$$\underline{\mu}(h) = [\,\mu(h)_{i,j}\,]_{1 \le i,j \le n}$$

is the matrix presentation of the k-automorphism

$$\mu(h) : \bigoplus_{j=1}^{n} \underline{V}_j \to \bigoplus_{i=1}^{n} \underline{V}_i ,$$

and the k-linear homomorphisms $p_{i,j}(h) : P_j \to P_i$, $1 \le i,j \le n$, are defined as follows

$$p_{i,j}(h) = \begin{cases} \nu_i(h) \cdot p_{i+1} \cdot \dots \cdot p_j & \text{if } i < j , \\ \nu_i(h) & \text{if } i = j , \\ 0 & \text{if } i > j . \end{cases}$$

Note that the maps $p_{i,j}(h)$, $h \in H$, are in fact $\mathscr{E}$-homomorphisms $p_{i,j}(h) : P_j \to {}^{h^{-1}}P_i$. It is easy to check (see [**9**]) that the pair

$$(\underline{V} \otimes_k P, \underline{\mu} \otimes_k p) ,$$

where

$$\underline{\mu} \otimes_k p = ((\underline{\mu} \otimes_k p)(h) : \underline{V} \otimes_k P \to {}^{h^{-1}}(\underline{V} \otimes_k P)_{h \in H})$$

is the family of $\mathscr{E}$-homomorphisms given by the $\mathscr{E}$-homomorphisms $\mu_{i,j}(h) \otimes_k p_{i,j}(h)$, $h \in H$, $1 \le i,j \le n$, defines correctly an $\mathscr{E}H$-module structure on $\underline{V} \otimes_k P$.

Analogously as in the original situation, we fix a selection of sequences $\underline{V}$, for all V in I_n-spr$\,(kH)$. Then the mapping $V \mapsto \underline{V} \otimes_k P$ extends to the (*generalized tensor product*) functor

$$- \otimes_k P : I_n\text{-spr}\,(kH) \to \mathrm{MOD}\,(\mathscr{E}H) .$$

In particular, if all $\mathscr{E}$-modules P_i occurring in the sequence P are finitely generated projective then we obtain the functor

$$- \otimes_k P : I_n\text{-spr}\,(kH) \to \mathrm{CM}\,(\mathscr{E}H) .$$

Applying a precise description of the equivalences from Corollary 4.4, we easily obtain the following result.

Proposition. *Let B be a sequence in $\mathrm{Mod}_{\mathrm{f}}^{H} R$ as 1.4 $(*)$ such that all B_i's are indecomposable and pairwise nonisomorphic, and $\mathscr{E} = \mathscr{E}(\mathscr{B}_o)$ a weakly locally bounded k-category associated to $\mathscr{B}_o = \{B_1, \dots, B_n\}$ equipped with the induced by ν_i's action σ' of H that is trivial on objects. Assume that P is a sequence $(*)$, given by $(P_i, \nu_i) = (\mathrm{Hom}_R(-, B_i), \pi_i)$, $i = 1, \dots, n$; $p_i = \mathrm{Hom}_R(-, \beta_i)$, $i = 2, \dots, n$. Then the functors*

$$- \otimes_k P : I_n\text{-spr}\,(kH) \to \mathrm{CM}\,(\mathscr{E}H)$$

and

$$- \otimes_k B : I_n\text{-spr}\,(kH) \to \mathrm{Mod}\,^{H}_{\mathrm{f},\mathscr{B}_o} R\,,$$

under the identification from Corollary 4.4*, are isomorphic.*

Now we formulate the appropriate version of [**9**, Theorem 3.1] for the Cohen-Macauley modules over $\mathscr{E}H$.

THEOREM. *Let $\mathscr{E}$ be a finite weakly locally bounded k-category,* $\mathrm{ob}\,\mathscr{E} = \{x_1, \dots, x_n\}$*, equipped with an action $\sigma' : H \to \mathrm{Aut}_{k-\mathrm{cat}}(\mathscr{E})$ of a group H on itself, that is trivial on objects. Assume that there exist nonzero H-invariant morphisms $\beta_i \in \mathscr{E}(x_i, x_{i-1})$, $i = 2, \dots, n$, and H-invariant ideal $\mathscr{N}$ in $\mathscr{E}$, satisfying the following conditions*:

(a) $\beta_{1,n} \neq 0$,

(b) $\mathscr{E}(x_j, x_i) = \mathscr{N}(x_j, x_i) \oplus k\beta_{i,j}$ *for all* $1 \leq i, j \leq n$,

where

$$\beta_{i,j} = \begin{cases} \beta_{i+1} \cdot \dots \cdot \beta_j & \text{if } \ i < j\,, \\ \mathrm{id}_{x_i} & \text{if } \ i = j\,, \\ 0 & \text{if } \ i > j\,. \end{cases}$$

Then the functor

$$- \otimes_k P : I_n\text{-spr}\,(kH) \to \mathrm{CM}\,(\mathscr{E}H)$$

where $(P_i, \nu_i) = (\mathscr{E}(-, x_i), \pi_i)$, $i = 1, \dots, n$; $p_i = \mathscr{E}(-, \beta_i)$, $i = 2, \dots, n$, is a representation embedding.

Note that, by (b), $\mathscr{N}$ is contained in the Jacobson radical of $\mathscr{E}$, so is a nilpotent ideal of $\mathscr{E}$.

PROOF. After adaptation and slight modifications follows from that in [**9**], specialized to the case $G = H$. □

5. Galois covering for weakly locally bounded categories.

Now we briefly show how the classical scheme of Galois coverings [**19**] can be adopted for the case of weakly locally bounded categories equipped with an action of a group H that is trivial on objects.

5.1.

Let $\mathscr{E}$ be a weakly locally bounded k-category. Assume we are given a k-linear action $\cdot : \Gamma \times \mathscr{E} \to \mathscr{E}$ of a group Γ on the category $\mathscr{E}$ which is free on $\mathrm{ob}\,\mathscr{E}$ (so $\Gamma \hookrightarrow \mathrm{Aut}_{k-\mathrm{cat}}(\mathscr{E})$). We have also the induced action $\Gamma \times \mathrm{MOD}\,\mathscr{E} \to \mathrm{MOD}\,\mathscr{E}$ of Γ on $\mathrm{MOD}\,\mathscr{E}$ by the shift of structure, $(\gamma, M) \mapsto {}^{\gamma}M$, where ${}^{\gamma}M(x) = M(\gamma^{-1}x)$, for $\mathscr{E}$-module M, $\gamma \in \Gamma$ and

$x \in \operatorname{ob}\mathscr{E}$. Similarly as in the case of locally bounded categories we can construct the orbit category $\bar{\mathscr{E}} = \mathscr{E}/\Gamma$ and the Galois covering functor $\mathscr{F} : \mathscr{E} \to \mathscr{E}/\Gamma$ inducing two functors $\mathscr{F}_\bullet : \operatorname{MOD}(\mathscr{E}/\Gamma) \to \operatorname{MOD}\mathscr{E}$ and $\mathscr{F}_\lambda : \operatorname{MOD}\mathscr{E} \to \operatorname{MOD}(\mathscr{E}/\Gamma)$ with nice properties.

We define $\bar{\mathscr{E}}$ in a little bit different, but in fact equivalent, way as usually (cf. [**19**]). An object class $\operatorname{ob}\bar{\mathscr{E}}$ of $\bar{\mathscr{E}}$ is defined as a fixed set $(\operatorname{ob}\mathscr{E})_{\rm o}$ of representatives of all Γ-orbits in $\operatorname{ob}\mathscr{E}$. For any $x_{\rm o}, y_{\rm o} \in \operatorname{ob}\bar{\mathscr{E}}$ we set

$$\bar{\mathscr{E}}(x_{\rm o}, y_{\rm o}) = \prod_{\gamma\in\Gamma} \mathscr{E}(x_{\rm o}, \gamma\, y_{\rm o})\,.$$

(Note that $\prod_{\gamma\in\Gamma}\mathscr{E}(x_{\rm o},\gamma\, y_{\rm o}) = \bigoplus_{\gamma\in\Gamma}\mathscr{E}(x_{\rm o},\gamma\, y_{\rm o})$ by 4.1(c) for $\mathscr{E}$). If $\alpha' = (\alpha'_\gamma)_{\gamma\in\Gamma} \in \bar{\mathscr{E}}(x_{\rm o}, y_{\rm o})$ and $\alpha'' = (\alpha''_\gamma)_{\gamma\in\Gamma} \in \bar{\mathscr{E}}(y_{\rm o}, z_{\rm o})$ are two morphisms in $\bar{\mathscr{E}}$ then the composition $\alpha''\circ\alpha'$ in $\bar{\mathscr{E}}$ is by definition the collection $\alpha = (\alpha_\gamma)_{\gamma\in\Gamma} \in \bar{\mathscr{E}}(x_{\rm o}, z_{\rm o})$, with the components given by the formula

$$\alpha_\gamma = \sum_{\gamma',\gamma'';\,\gamma'\gamma''=\gamma} \gamma'(\alpha''_{\gamma''})\cdot\alpha'_{\gamma'},$$

where $\gamma'(\alpha''_{\gamma''})\cdot\alpha'_{\gamma'}$ denotes the composition of the respective morphisms in $\mathscr{E}$ (the sum is finite since 4.1(c) holds for $\mathscr{E}$).

For any $x \in \operatorname{ob}\mathscr{E}$, we set $\mathscr{F}(x) = x_{\rm o}$, where $x_{\rm o} \in \operatorname{ob}\bar{\mathscr{E}}$ is such that $x = \gamma_x x_{\rm o}$ for some $\gamma_x \in \Gamma$ (γ_x and $x_{\rm o}$ are uniquely determined for x). For any $\alpha : x \to y$ in $\mathscr{E}$, we denote by $\mathscr{F}(\alpha)$ the morphism $\gamma_x^{-1}(\alpha) : x_{\rm o} \to \gamma_x^{-1} y$ $(= \gamma_x^{-1}\gamma_y\, y_{\rm o})$, regarded as an element of $\mathscr{E}(x_{\rm o}, \gamma_x^{-1}\gamma_y\, y_{\rm o}) \subseteq \bar{\mathscr{E}}(x_{\rm o}, y_{\rm o})$.

Proposition. (a) *The category $\bar{\mathscr{E}}$, equipped with the structure defined above, is a weakly locally bounded k-category.*

(b) *The mapping $\mathscr{F}$ defines a k-linear functor such that $\mathscr{F}\circ\gamma = \mathscr{F}$, for every $\gamma\in\Gamma$. Moreover, for any pair $x \in \operatorname{ob}\mathscr{E}$, $y_o \in \operatorname{ob}\bar{\mathscr{E}}$, $\mathscr{F}$ induces two k-isomorphisms*:

$$\bigoplus_{y,\mathscr{F}(y)=y_o} \mathscr{E}(x,y) \simeq \bar{\mathscr{E}}(\mathscr{F}(x), y_o)$$

and

$$\bigoplus_{y,\mathscr{F}(y)=y_o} \mathscr{E}(y,x) \simeq \bar{\mathscr{E}}(y_o, \mathscr{F}(x))$$

(*cf.* [**19**], *for definition of Galois covering functor*).

Proof. (a) It is easy to check that the data $\bar{\mathscr{E}}$ as above define correctly a k-category structure. Moreover, $\bar{\mathscr{E}}$ satisfies the condition 4.1(c) since so does $\mathscr{E}$. We prove that 4.1(b) holds for $\bar{\mathscr{E}}$, more precisely, that for every $x_{\rm o} \in \operatorname{ob}\bar{\mathscr{E}}$, $\bar{\mathscr{E}}(x_{\rm o}, x_{\rm o})$ is local semiprimary k-algebra with the factor field $\bar{\mathscr{E}}(x_{\rm o}, x_{\rm o})/J(\bar{\mathscr{E}}(x_{\rm o}, x_{\rm o}))$ isomorphic to $\mathscr{E}(x_{\rm o}, x_{\rm o})/J(\mathscr{E}(x_{\rm o}, x_{\rm o}))$. It suffices to show that the k-space $N = J(\mathscr{E}(x_{\rm o}, x_{\rm o})) \oplus \bigoplus_{\epsilon\neq\gamma\in\Gamma}\mathscr{E}(x_{\rm o}, \gamma\, x_{\rm o})$ forms a nilpotent ideal in $\bar{\mathscr{E}}(x_{\rm o}, x_{\rm o})$, where ϵ denotes the neutral element in Γ. Then clearly, $\bar{\mathscr{E}}(x_o, x_o)/N$ is isomorphic to $\mathscr{E}(x_o, x_o)/J(\mathscr{E}(x_o, x_o))$, so it is semisimple; hence, $N = J(\bar{\mathscr{E}}(x_o, x_o))$ and $\bar{\mathscr{E}}(x_o, x_o)$ is a local algebra with the same factor field as $\mathscr{E}(x_o, x_o)$. (Clearly, 4.1(d) holds for $\bar{\mathscr{E}}$ if and only if it holds for $\mathscr{E}$).

We start by observing that N is an ideal since Γ acts freely on $\operatorname{ob}\mathscr{E}$. Let s be a common bound of the nilpotency degree of $J(\mathscr{E}(x_o, x_o))$ and the cardinality of the set $\{\gamma \in \Gamma : \mathscr{E}(x_o, \gamma\, x_o) \neq 0\}$. We claim that $N^{s^2} = 0$. Suppose it is not the case. Then there exist $\alpha_i \in \mathscr{E}(x_o, \gamma_i x_o)$, $i = 1, \ldots, s^2$, such that each partial product $\alpha_i \circ \cdots \circ \alpha_1$ $(= \gamma_1 \ldots \gamma_{i-1}(\alpha_i) \cdot \cdots \cdot \alpha_1)$ in $\bar{\mathscr{E}}$ is nonzero. Therefore, we have $\gamma_1 \ldots \gamma_i \in \{\gamma \in \Gamma : \mathscr{E}(x_o, \gamma\, x_o) \neq 0\}$, for every $i = 1, \ldots, s^2$, and there exists $\gamma \in \Gamma$ ($\mathscr{E}(x_o, \gamma\, x_o) \neq 0\,!$) such that $\gamma_0\gamma_1 \ldots \gamma_i = \gamma$ for at least $s + 1$ indices $i = 0, 1, \ldots, s^2$, where $\gamma_0 = \epsilon$. Observe that, if $\gamma_0\gamma_1 \ldots \gamma_i = \gamma_0\gamma_1 \ldots \gamma_j$ and $j > i$ then $\gamma_{i+1} \ldots \gamma_j = \epsilon$. Consequently, either $\gamma_l = \epsilon$ and α_l belongs to $J(\mathscr{E}(x_o, x_o))$ for at least one l with $i + 1 \le l \le j$, or $j > i + 1$ and all α_l are nonisomorphisms, for $l = i+1, \ldots, j$. Thus always we have $\alpha_j \circ \cdots \circ \alpha_{i+1} \in J(\bar{\mathscr{E}}(x_o, x_o))$. Applying this observation to all elements $\gamma_0\gamma_1 \ldots \gamma_i$ such that $\gamma_0\gamma_1 \ldots \gamma_i = \gamma$, we infer that $\alpha_{s^2} \circ \cdots \circ \alpha_1 = 0$, a contradiction. Hence, we have $N^{s^2} = 0$ and the claim is proved.

To complete the proof of (a) note that 4.1(a) holds for $\bar{\mathscr{E}}$ since it does so for $\mathscr{E}$. This follows easily from the description of the Jacobson radicals $J(\bar{\mathscr{E}}(x_o, x_o))$, $x_o \in \operatorname{ob}\bar{\mathscr{E}}$.

(b) The first isomorphism follows immediately from the fact that γ_x^{-1} yields the k-isomorphism $\bigoplus_{y, \mathscr{F}(y) = y_o} \mathscr{E}(x, y) \simeq \bigoplus_{\gamma \in \Gamma} \mathscr{E}(x_o, \gamma\, y_o)$. To show the second one, we use the k-ismorphisms $\mathscr{E}(y, x) \simeq \mathscr{E}(y_o, \gamma_y^{-1} x)$, $y \in \mathscr{F}^{-1}(y_o)$, induced by γ_y^{-1}, where $y = \gamma_y y_o$. □

Remark. (a) The category $\bar{\mathscr{E}}$ admits a natural structure of Γ-graded category. The decomposition $\bar{\mathscr{E}}(x_o, y_o) = \bigoplus_{\gamma \in \Gamma} \bar{\mathscr{E}}(x_o, y_o)_\gamma$ of the morphism spaces $\bar{\mathscr{E}}(x_o, y_o)$, $x_o, y_o \in \operatorname{ob}\bar{\mathscr{E}}$, defining the Γ-graded category structure on $\bar{\mathscr{E}}$, is given by setting $\bar{\mathscr{E}}(x_o, y_o)_\gamma = \mathscr{E}(x_o, \gamma^{-1} y_o)$, for $\gamma \in \Gamma$.

(b) The different choices of the representative sets of Γ-orbits in $\operatorname{ob}\mathscr{E}$, as objects of $\bar{\mathscr{E}}$, lead to the isomorphic categories. More precisely, if $x'_o = \gamma_{x_o} x_o$, $x_o \in \operatorname{ob}\bar{\mathscr{E}}$, is another choice of representatives of Γ-orbits in $\operatorname{ob}\bar{\mathscr{E}}$ ($\gamma_{x_o} \in \Gamma$ are fixed elements for all $x_o \in \operatorname{ob}\bar{\mathscr{E}}$) then the maps

$$\varphi_{y_o, x_o} : \bigoplus_{\gamma \in \Gamma} \mathscr{E}(x_o, \gamma y_o) \to \bigoplus_{\gamma' \in \Gamma} \mathscr{E}(x'_o, \gamma' y'_o)$$

$x_o, y_o \in \operatorname{ob}\bar{\mathscr{E}}$, given by $\mathscr{E}(x_o, \gamma y_o) \ni \alpha_\gamma \mapsto \gamma_{x_o}(\alpha_\gamma) \in \mathscr{E}(x'_o, \gamma' y'_o)$ with $\gamma' = \gamma_{x_o} \gamma\, \gamma_{y_o}^{-1}$, induce an equivalence of categories $\bar{\mathscr{E}}$ and $\bar{\mathscr{E}}'$ (here $\bar{\mathscr{E}}'$ denotes the orbit category $\mathscr{E}/\Gamma$ with the object class $\{x'_o\}_{x_o \in \operatorname{ob}\bar{\mathscr{E}}}$). The maps φ_{y_o, x_o}, $x_o, y_o \in \operatorname{ob}\bar{\mathscr{E}}$, yield an equivalence of $\bar{\mathscr{E}}$ and $\bar{\mathscr{E}}'$ as graded categories in the sense explained below. Each φ_{y_o, x_o} is given by the k-isomorphisms $\varphi_{y_o, x_o}(\gamma) : \mathscr{E}(x_o, y_o)_\gamma \to \mathscr{E}(x'_o, y'_o)_{\psi_{y_o, x_o}(\gamma)}$, $\gamma \in \Gamma$, where $\psi_{y_o, x_o} : \Gamma \to \Gamma$ is a family of maps compatible with multiplication in Γ ($\psi_{z_o, y_o}(\gamma_2)\, \psi_{y_o, x_o}(\gamma_1) = \psi_{z_o, x_o}(\gamma_2\gamma_1)$, for $\gamma_1, \gamma_2 \in \Gamma$) defined by formula $\psi_{y_o, x_o}(\gamma) = \gamma_{y_o} \gamma^{-1} \gamma_{x_o}^{-1}$ (see also 5.5).

5.2.

Denote by $\mathscr{F}_\bullet : \operatorname{MOD}\bar{\mathscr{E}} \to \operatorname{MOD}\mathscr{E}$ the "pull-back" functor associated to $\mathscr{F}$, given by $\mathscr{F}_\bullet X = X \circ \mathscr{F}^{\mathrm{op}}$, for X in $\operatorname{MOD}\mathscr{E}$. More precisely, for any $x, y \in \operatorname{ob}\mathscr{E}$ and $\alpha \in \mathscr{E}(x, y)$

we have $\mathscr{F}_\bullet X(x) = X(\gamma_x^{-1}x)$; $\mathscr{F}_\bullet X(\alpha)$ is just the map $X(\gamma_x^{-1}\alpha) : X(\gamma_y^{-1}y) \to X(\gamma_x^{-1}x)$, where $x = \gamma_x x_{\rm o}$, $y = \gamma_y y_{\rm o}$ for $x_{\rm o}, y_{\rm o} \in \operatorname{ob}\bar{\mathscr{E}}$, $\gamma_x, \gamma_y \in \Gamma$, and $\gamma_x^{-1}\alpha \in \mathscr{E}(x_{\rm o}, \gamma_x^{-1}\gamma_y y_{\rm o}) \subseteq \bar{\mathscr{E}}(x_{\rm o}, y_{\rm o})$. The functor $\mathscr{F}_\bullet$ admits the left adjoint, the so called "push-down" functor $\mathscr{F}_\lambda : \operatorname{MOD}\mathscr{E} \to \operatorname{MOD}\bar{\mathscr{E}}$. For any M in $\operatorname{MOD}\mathscr{E}$, $\mathscr{F}_\lambda M$ is defined as usually by setting $(\mathscr{F}_\lambda M)(x_{\rm o}) = \bigoplus_{\gamma\in\Gamma} M(\gamma x_{\rm o})$, where $x_{\rm o} \in \operatorname{ob}\bar{\mathscr{E}}$; the maps $(\mathscr{F}_\lambda M)(\alpha) : \bigoplus_{\gamma'\in\Gamma} M(\gamma' y_{\rm o}) \to \bigoplus_{\gamma\in\Gamma} M(\gamma x_{\rm o})$ for $\alpha = (\alpha_{\gamma''})_{\gamma''\in\Gamma} \in \bigoplus_{\gamma''\in\Gamma} \mathscr{E}(x_{\rm o}, \gamma'' y_{\rm o}) = \bar{\mathscr{E}}(x_{\rm o}, y_{\rm o})$ are given by the components $M(\gamma\alpha_{\gamma^{-1}\gamma'}) : M(\gamma' y_{\rm o}) \to M(\gamma x_{\rm o})$, $\gamma, \gamma' \in \Gamma$. It is easy to see that $\mathscr{F}_\lambda(\mathscr{E}(-,x)) = \bar{\mathscr{E}}(-,\mathscr{F}(x)) \simeq \mathscr{F}_\lambda(\mathscr{E}(-,\gamma x))$, for every $x \in \operatorname{ob}\mathscr{E}$ and $\gamma \in \Gamma$. Moreover, we have $\mathscr{F}_\bullet(\bar{\mathscr{E}}(-,x_{\rm o})) = \bigoplus_{x,\mathscr{F}(x)=x_{\rm o}} \mathscr{E}(-,x)$, for every $x_{\rm o} \in \operatorname{ob}\bar{\mathscr{E}}$. Modifying arguments from [**19**], one easily shows classical properties of the functors $\mathscr{F}_\lambda$ and $\mathscr{F}_\bullet$, which are formulated below.

Lemma. *Let M be an $\mathscr{E}$-module. Then*
(a) $\mathscr{F}_\lambda M \simeq \mathscr{F}_\lambda{}^\gamma M$, *for every* $\gamma \in \Gamma$,
(b) $\mathscr{F}_\bullet\mathscr{F}_\lambda M \simeq \bigoplus_{\gamma\in\Gamma} {}^\gamma M$.

5.3.

Assume that, under the general assumptions of 5.1, we have at our disposal also the action $\sigma : H \to \operatorname{Aut}_{k-\rm cat}(\mathscr{E})$ of the group H on $\mathscr{E}$ which *commutes* with the action of Γ on $\mathscr{E}$, that is:

$$\gamma \circ \sigma(h) = \sigma(h) \circ \gamma \qquad (*)$$

in $\operatorname{Aut}_{k-\rm cat}(\mathscr{E})$, for all $\gamma \in \Gamma$, $h \in H$. We assume here that H always acts trivially on $\operatorname{ob}\mathscr{E}$. The action σ induces the family $\bar\sigma(h)(x_{\rm o}, y_{\rm o}) : \bar{\mathscr{E}}(x_{\rm o}, y_{\rm o}) \to \bar{\mathscr{E}}(x_{\rm o}, y_{\rm o})$, $h \in H$, $x_{\rm o}, y_{\rm o} \in \operatorname{ob}\bar{\mathscr{E}}$ of k-linear maps, where $\bar\sigma(h)(x_{\rm o}, y_{\rm o}) : \bigoplus_{\gamma\in\Gamma} \mathscr{E}(x_{\rm o}, \gamma y_{\rm o}) \to \bigoplus_{\gamma\in\Gamma} \mathscr{E}(x_{\rm o}, \gamma y_{\rm o})$ is the diagonal map given by mapping $(\alpha_\gamma)_{\gamma\in\Gamma} \mapsto (\sigma(h)(\alpha_\gamma))_{\gamma\in\Gamma}$.

Lemma. *The family $(\bar\sigma(h)(x_o, y_o))_{h\in H, x_o, y_o \in \operatorname{ob}\bar{\mathscr{E}}}$ defines a group homomorphism $\bar\sigma : H \to \operatorname{Aut}_{k-\rm cat}(\bar{\mathscr{E}})$ and a trivial on objects action of H on $\bar{\mathscr{E}} = \mathscr{E}/\Gamma$ such that $\bar\sigma(h) \circ \mathscr{F} = \mathscr{F} \circ \sigma(h)$, for all $h \in H$.*

Proof. An easy check on definitions. □

From now on we write simply h instead the operators $\sigma(h)$ and $\bar\sigma(h)$. Note that this notation does not lead to any confusion since the actions of H and Γ on $\mathscr{E}$ commute and they can be combined in a natural way into one action of the group $\Gamma \times H$ (on $\mathscr{E}$).

5.4.

Consider now the induced by σ (respectively, $\bar\sigma$) action of H on the category $\operatorname{MOD}\mathscr{E}$ (respectively, $\operatorname{MOD}\bar{\mathscr{E}}$), and the category $\operatorname{MOD}(\mathscr{E}H)$ (respectively, $\operatorname{MOD}(\bar{\mathscr{E}}H)$), where for simplicity $\mathscr{E}H$ stands for $\mathscr{E}_\sigma H$ (respectively, $\bar{\mathscr{E}}H$ for $\bar{\mathscr{E}}_{\bar\sigma}H$). Then the action of the group Γ on $\operatorname{MOD}\mathscr{E}$ induced by the action $\cdot : \Gamma \times \mathscr{E} \to \mathscr{E}$, yields the action of Γ on the category $\operatorname{MOD}(\mathscr{E}H)$, which is given by the mapping $(\gamma, (M,\mu)) \mapsto ({}^\gamma M, {}^\gamma \mu)$, for $\gamma \in \Gamma$ and (M,μ) in $\operatorname{MOD}(\mathscr{E}H)$, $\mu = (\mu_h : M \to {}^{h^{-1}}M)_{h\in H}$, where ${}^\gamma\mu = ({}^\gamma\mu_h : {}^\gamma M \to {}^{\gamma h^{-1}}M)_{h\in H}$. (Note that ${}^\gamma\mu$ is an $\mathscr{E}$-action of H on ${}^\gamma M$, since by $(*)$ we have ${}^{\gamma h^{-1}}M = {}^{h^{-1}\gamma}M$).

Let (X,χ) be an object in $\mathrm{MOD}\,(\bar{\mathscr{E}}H)$, where the $\bar{\mathscr{E}}$-action $\chi = (\chi_h : X \to {}^{h^{-1}}X)_{h\in H}$ of H on X is given by the compatible family $(\chi_h(x_{\mathrm{o}}) : X(x_{\mathrm{o}}) \to X(x_{\mathrm{o}}))_{h\in H,\, x_{\mathrm{o}}\in \mathrm{ob}\bar{\mathscr{E}}}$ of k-linear maps. Then χ induces the family $(\tilde{\chi}_h(x) : (\mathscr{F}_\bullet X)(x) \to (\mathscr{F}_\bullet X)(x))_{h\in H,\, x\in \mathrm{ob}\mathscr{E}}$ of k-linear maps, where $\tilde{\chi}_h(x)$ is the map $\chi_h(\gamma_x^{-1}x) : X(\gamma_x^{-1}x) \to X(\gamma_x^{-1}x)$.

Let (M,μ) be an object in $\mathrm{MOD}\,(\mathscr{E}H)$, where the $\mathscr{E}$-action $\mu = (\mu_h : M \to {}^{h^{-1}}M)_{h\in H}$ of H on M is given by the compatible family $(\mu_h(x) : M(x) \to M(x))_{h\in H,\, x\in \mathrm{ob}\mathscr{E}}$ of k-linear maps. Then μ induces the family $(\bar{\mu}_h(x_{\mathrm{o}}) : (\mathscr{F}_\lambda M)(x_{\mathrm{o}}) \to (\mathscr{F}_\lambda M)(x_{\mathrm{o}}))_{h\in H,\, x_{\mathrm{o}}\in \mathrm{ob}\bar{\mathscr{E}}}$ of k-linear maps, where $\bar{\mu}_h(x_{\mathrm{o}})$ is the map $\bigoplus_{\gamma\in\Gamma} \mu_h(\gamma x_{\mathrm{o}}) : \bigoplus_{\gamma\in\Gamma} M(\gamma x_{\mathrm{o}}) \to \bigoplus_{\gamma\in\Gamma} M(\gamma x_{\mathrm{o}})$.

LEMMA. (a) *The family $(\tilde{\chi}_h(x))_{h\in H,\, x\in \mathrm{ob}\mathscr{E}}$ defines the $\mathscr{E}$-action $\tilde{\chi}$ of H on the $\mathscr{E}$-module $\mathscr{F}_\bullet X$. The mapping $X = (X,\chi) \mapsto (\mathscr{F}_\bullet X, \tilde{\chi})$, for (X,χ) in* $\mathrm{MOD}\,(\bar{\mathscr{E}}H)$, *yields the functor*

$$\mathscr{F}_\bullet : \mathrm{MOD}\,(\bar{\mathscr{E}}H) \to \mathrm{MOD}\,(\mathscr{E}H)\,;$$

moreover, ${}^{\gamma}\mathscr{F}_\bullet X = \mathscr{F}_\bullet X$ in $\mathrm{MOD}\,(\mathscr{E}H)$, *for every $\gamma \in \Gamma$.*

(b) *The family $(\bar{\mu}_h(x_o))_{h\in H,\, x_o\in \mathrm{ob}\bar{\mathscr{E}}}$ defines the $\bar{\mathscr{E}}$-action $\bar{\mu}$ of H on the $\bar{\mathscr{E}}$-module $\mathscr{F}_\lambda M$. The mapping $(M,\mu) \mapsto (\mathscr{F}_\lambda M, \bar{\mu})$, for $M = (M,\mu)$ in* $\mathrm{MOD}\,(\mathscr{E}H)$, *yields the functor*

$$\mathscr{F}_\lambda : \mathrm{MOD}\,(\mathscr{E}H) \to \mathrm{MOD}\,(\bar{\mathscr{E}}H)$$

such that $\mathscr{F}_\lambda(\mathrm{CM}\,(\mathscr{E}H)) \subset \mathrm{CM}\,(\bar{\mathscr{E}}H)$; moreover, $\mathscr{F}_\lambda({}^{\gamma}M) \simeq \mathscr{F}_\lambda M$ in $\mathrm{MOD}\,(\bar{\mathscr{E}}H)$, *for every $\gamma \in \Gamma$.*

(c) *For any $M = (M,\mu)$ in* $\mathrm{MOD}\,(\mathscr{E}H)$, *$\mathscr{F}_\bullet\mathscr{F}_\lambda M \simeq \bigoplus_{\gamma\in\Gamma} {}^{\gamma}M$ (in* $\mathrm{MOD}\,(\mathscr{E}H)$ *).*

PROOF. (a) By definition of $\mathscr{F}_\bullet$ and Lemma 5.3, we have $\mathscr{F}_\bullet({}^{h^{-1}}Y) = {}^{h^{-1}}(\mathscr{F}_\bullet Y)$, for any $h \in H$ and Y in $\mathrm{MOD}\,\bar{\mathscr{E}}$. Moreover, each family $\tilde{\chi}_h = (\tilde{\chi}_h(x))_{x\in \mathrm{ob}\,\mathscr{E}}$ defines in fact the $\mathscr{E}$-homomorphism $\mathscr{F}_\bullet(\chi_h) : \mathscr{F}_\bullet X \to \mathscr{F}_\bullet({}^{h^{-1}}X)$ $(= {}^{h^{-1}}(\mathscr{F}_\bullet X))$, therefore $\tilde{\chi} = (\mathscr{F}_\bullet(\chi_h))_{h\in H}$ is an $\mathscr{E}$-action of H on $\mathscr{F}_\bullet X$ (we denote it by $\mathscr{F}_\bullet(\chi)$). Now a functoriality of the mapping $X = (X,\chi) \mapsto (\mathscr{F}_\bullet X, \mathscr{F}_\bullet(\chi))$ is straightforward. The last assertion follows from the equality of the functors ${}^{\gamma}(-) \circ \mathscr{F}_\bullet$, $\mathscr{F}_\bullet : \mathrm{MOD}\,\bar{\mathscr{E}} \to \mathrm{MOD}\,\mathscr{E}$, $\gamma \in \Gamma$ (see [**19**]), applied to $\mathscr{E}$-homomorphisms $\chi_h : X \to {}^{h^{-1}}X$, $h \in H$.

(b) Using similar arguments as before we have $\mathscr{F}_\lambda({}^{h^{-1}}N) = {}^{h^{-1}}\mathscr{F}_\lambda N$, for any $h \in H$ and N in $\mathrm{MOD}\,\mathscr{E}$. Now each family $\bar{\mu}_h = (\bar{\mu}_h(x_{\mathrm{o}}))_{x_{\mathrm{o}}\in \mathrm{ob}\bar{\mathscr{E}}}$ defines $\bar{\mathscr{E}}$-homomorphisms $\mathscr{F}_\lambda(\mu_h) : \mathscr{F}_\lambda M \to \mathscr{F}_\lambda({}^{h^{-1}}M)$ $(= {}^{h^{-1}}\mathscr{F}_\lambda M)$ and therefore $\bar{\mu} = (\mathscr{F}_\lambda(\mu_h))_{h\in H}$ is an $\mathscr{E}$-action of H on $\mathscr{F}_\lambda M$. The remaining assertions follow now easily.

(c) The classical $\mathscr{E}$-isomorphism $\mathscr{F}_\bullet\mathscr{F}_\lambda N \simeq \bigoplus_{\gamma\in\Gamma} {}^{\gamma}N$, for N in $\mathrm{MOD}\,\mathscr{E}$ (see [**19**]), is natural with respect to N. Now applying this fact to $\mathscr{E}$-isomorphisms $\mu_h : M \to {}^{h^{-1}}M$, $h \in H$, we obtain by $(*)$ the required assertion. □

The main result of this section is the following.

THEOREM. *Let* $\mathrm{M} = (M,\mu)$ *be an indecomposable object in* $\mathrm{CM}\,(\mathscr{E}H)$. *Then*
(a) *$\mathscr{F}_\lambda M$ is an indecomposable object in* $\mathrm{CM}\,(\bar{\mathscr{E}}H)$,

(b) *if* $\mathscr{F}_\lambda M \simeq \mathscr{F}_\lambda N$, *for some* N *in* CM$(\mathscr{E}H)$, *then there exists* $\gamma \in \Gamma$ *such that* $N \simeq {}^\gamma M$.

PROOF. (a) Suppose that $\mathscr{F}_\lambda M \simeq X \oplus Y$ for some X, Y in CM$(\mathscr{E}H)$. Then, by Lemma 5.4, we have $\bigoplus_{\gamma\in\Gamma} {}^\gamma M \simeq \mathscr{F}_\bullet \mathscr{F}_\lambda M \simeq \mathscr{F}_\bullet X \oplus \mathscr{F}_\bullet Y$. Consequently, M is isomorphic to a direct summand of $\mathscr{F}_\bullet X$ or $\mathscr{F}_\bullet Y$ in MOD$(\mathscr{E}H)$. Assume that the first possibility holds. We show that $(\mathscr{F}_\bullet Y)(x) = 0$ for every $x \in \operatorname{ob}\mathscr{E}$; hence $Y = 0$ and M is indecomposable.

Note that ${}^\gamma M$ is isomorphic to a direct summand of $\mathscr{F}_\bullet X$ in MOD$(\mathscr{E}H)$, for every $\gamma \in \Gamma$, since ${}^\gamma\mathscr{F}_\bullet X = \mathscr{F}_\bullet X$. Consequently, $\bigoplus_{s=1}^n {}^{\gamma_s} M$ is a direct summand of $\mathscr{F}_\bullet X$ for any pairwise different $\gamma_1, \ldots, \gamma_n \in \Gamma$. This follows from the fact that the composite maps

$$\bigoplus_{s=1}^n {}^{\gamma_s} M \xrightarrow{\mathbf{i}} \mathscr{F}_\bullet X \xrightarrow{\mathbf{p}} \bigoplus_{s=1}^n {}^{\gamma_s} M$$

where $\mathbf{i}$ and $\mathbf{p}$ are given, respectively, by the components ${}^{\gamma_s} i$ and ${}^{\gamma_s} p$ (here $i : M \to \mathscr{F}_\bullet X$ and $p : \mathscr{F}_\bullet X \to M$ are such that $pi = \mathrm{id}_M$), are isomorphisms in MOD$(\mathscr{E}H)$. Note that we have ${}^{\gamma_s} p\, {}^{\gamma_s} i = \mathrm{id}_{{}^{\gamma_s} M}$ and that ${}^{\gamma_s} p\, {}^{\gamma_{s'}} i$'s are nonisomorphisms, for all s, s', $s \neq s'$ (by Theorem 4.2 all $\mathrm{End}_{\mathscr{E}H}({}^{\gamma_s} M)$'s are local, and ${}^{\gamma_s} M$'s are pairwise nonisomorphic since the $\mathscr{E}$-module M is isomorphic to a finite direct sum of the modules $P_z = \mathscr{E}(-, z)$, $z \in \operatorname{ob}\mathscr{E}$).

Fix $x \in \operatorname{ob}\mathscr{E}$ and establish the notation

$$\begin{aligned} \{x\}^+ &= \{y \in \operatorname{ob}\mathscr{E} : \mathscr{E}(x, y) \neq 0\}, \\ \Gamma_0 &= \{\gamma \in \Gamma : {}^\gamma M(x) \neq 0\}, \\ \Gamma_1 &= \Big\{\gamma \in \Gamma : \operatorname{ob}\Big(\operatorname{supp} {}^\gamma M \cap \bigcup_{\gamma' \in \Gamma_0} \operatorname{supp} {}^{\gamma'} M\Big) \neq \varnothing\Big\}. \end{aligned}$$

By 4.1 all of these three sets are finite. Observe that, for any $\gamma \in \Gamma \setminus \Gamma_0$ and $y \in \{x\}^+$, P_y is not a direct summand of ${}^\gamma M$, therefore the multiplicities of P_y as a direct summand in $\bigoplus_{\gamma\in\Gamma_0} {}^\gamma M$ and $\bigoplus_{\gamma\in\Gamma_1} {}^\gamma M$ are the same. Moreover, there is no homomorphism between $\mathscr{E}$-modules $\bigoplus_{\gamma\in\Gamma_0} {}^\gamma M$ and $\bigoplus_{\gamma\in\Gamma\setminus\Gamma_1} {}^\gamma M$. On the other hand, by the first observation we have the splittable monomorphisms

$$\bigoplus_{\gamma\in\Gamma_0} {}^\gamma M \to \mathscr{F}_\bullet X \to \bigoplus_{\gamma\in\Gamma} {}^\gamma M = \Big(\bigoplus_{\gamma\in\Gamma_1} {}^\gamma M\Big) \oplus \Big(\bigoplus_{\gamma\in\Gamma\setminus\Gamma_1} {}^\gamma M\Big).$$

Consequently, the composite map induces a splittable monomorphism

$$\iota : \bigoplus_{\gamma\in\Gamma_0} {}^\gamma M \to \bigoplus_{\gamma\in\Gamma_1} {}^\gamma M.$$

Therefore we have a decomposition

$$\bigoplus_{\gamma\in\Gamma_1} {}^{\gamma}M = \iota\Big(\bigoplus_{\gamma\in\Gamma_0} {}^{\gamma}M\Big) \oplus M'$$

into a direct sum of $\mathscr{E}$-submodules in the category of finitely generated projective (free) $\mathscr{E}$-modules and M' is isomorphic to a finite direct sum of indecomposable projectives P_z (note that Γ_1 is finite). By the uniqueness of decomposition into a direct sum of indecomposables for finitely generated $\mathscr{E}$-modules (see Remark 4.1) and the considerations above, non of modules P_y, for $y \in \{x\}^+$, can occur in the decomposition of M'. Consequently, $M'(x) = 0$ and $(\mathscr{F}_\bullet Y)(x) = 0$. Since this holds for an arbitrary $x \in \operatorname{ob}\mathscr{E}$, we infer that $Y = 0$, and the proof of (a) is complete.

(b) Assume that $\mathscr{F}_\lambda M \simeq \mathscr{F}_\lambda N$, where M, N satisfy the assumptions. By (a), one can assume that N is also an indecomposable object in $\operatorname{CM}(\mathscr{E}H)$. By Lemma 5.4, we have the isomorphism $\bigoplus_{\gamma\in\Gamma} {}^{\gamma}M \simeq \bigoplus_{\gamma\in\Gamma} {}^{\gamma}N$ in $\operatorname{MOD}(\mathscr{E}H)$. Consequently, M is isomorphic to a direct summand of $\bigoplus_{\gamma\in\Gamma} {}^{\gamma}N$. Denote by Γ_2 a finite set consisting of all $\gamma \in \Gamma$ such that $\operatorname{ob}(\operatorname{supp} M \cap \operatorname{supp} {}^{\gamma}N) \neq \varnothing$. Note that since there is no nonzero $\mathscr{E}$-homomorphism between M and $\bigoplus_{\gamma\in\Gamma\setminus\Gamma_2} {}^{\gamma}N$, M is isomorphic to a direct summand of $\bigoplus_{\gamma\in\Gamma_2} {}^{\gamma}N$ in $\operatorname{CM}(\mathscr{E}H)$. Then M is isomorphic to a direct summand of ${}^{\gamma}N$ in $\operatorname{CM}(\mathscr{E}H)$, for some $\gamma \in \Gamma_2$ ($\operatorname{End}_{\mathscr{E}H}(M)$ is local algebra). Consequently, we have $M \simeq {}^{\gamma}N$ since N is indecomposable. □

5.5.

Now we consider the situation in some sense converse to that from 5.1. Under special assumptions we try to present a given weakly locally bounded k-category $\mathscr{E}$ in the form $\tilde{\mathscr{E}}/\Gamma$, for certain group $\Gamma \subseteq \operatorname{Aut}_{k-\mathrm{cat}}(\tilde{\mathscr{E}})$ acting freely on objects of some weakly locally bounded k-category $\tilde{\mathscr{E}}$.

Assume, we are given a k-category $\mathscr{E}$ admitting a Γ-grading defined by the decompositions

$$\mathscr{E}(a,b) \simeq \bigoplus_{\gamma\in\Gamma} \mathscr{E}(a,b)_\gamma \qquad (**)$$

$a, b \in \operatorname{ob}\mathscr{E}$. Then we define a new category $\tilde{\mathscr{E}}$. We set

$$\operatorname{ob}\tilde{\mathscr{E}} = \Gamma \times \operatorname{ob}\mathscr{E},$$

the morphisms in $\tilde{\mathscr{E}}$ are defined by the formula

$$\tilde{\mathscr{E}}((\gamma_1, a), (\gamma_2, b)) = \mathscr{E}(a,b)_{\gamma_2{}^{-1}\gamma_1}$$

for $\gamma_1, \gamma_2 \in \Gamma$ and $a, b \in \operatorname{ob}\mathscr{E}$, the composition in $\tilde{\mathscr{E}}$ is given by the composition in $\mathscr{E}$. Moreover, we have $\tilde{\mathscr{E}}((\gamma\gamma_1, a), (\gamma\gamma_2, b)) = \mathscr{E}(a,b)_{(\gamma\gamma_2)^{-1}(\gamma\gamma_2)} = \mathscr{E}(a,b)_{\gamma_2{}^{-1}}\gamma_1$, for $\gamma \in \Gamma$. Consequently, the mapping $(\gamma, (\gamma_1, a)) \mapsto (\gamma\gamma_1, a)$ induces, in an obvious way, an action

$$\cdot : \Gamma \times \tilde{\mathscr{E}} \to \tilde{\mathscr{E}}$$

of the group Γ by k-linear automorphisms on $\tilde{\mathscr{E}}$, that is free on objects (then clearly, Γ can be embedded into $\mathrm{Aut}_{k-\mathrm{cat}}(\tilde{\mathscr{E}})$). We fix the set $\{(\epsilon, a)\}_{a\in \mathrm{ob}\,\mathscr{E}}$ as a set representatives of $(\mathrm{ob}\,\tilde{\mathscr{E}})/\Gamma$. Then analogously as in 5.1, we can form the orbit category $\tilde{\mathscr{E}}/\Gamma$ and the respective Galois covering functor $\mathscr{F} : \tilde{\mathscr{E}} \to \tilde{\mathscr{E}}/\Gamma$ provided, for any pair $a, b \in \mathrm{ob}\,\mathscr{E}$, $\mathscr{E}(a,b)_\gamma = 0$ for almost all $\gamma \in \Gamma$.

Proposition. *Assume that $\mathscr{E}$ is a weakly locally bounded k-category equipped with the Γ-grading* (∗∗) *satisfying the following two conditions*:

(a) *for every pair $a, b \in \mathrm{ob}\,\mathscr{E}$, $\mathscr{E}(a,b)_\gamma = 0$ for almost all $\gamma \in \Gamma$,*

(b) $\bigoplus_{\epsilon\neq\gamma\in\Gamma} \mathscr{E}(a,a)_\gamma$ *is contained in $J(\mathscr{E}(a,a))$ for every $a \in \mathrm{ob}\,\mathscr{E}$ (follows from* (a) *if Γ is torsion-free).*

Then $\tilde{\mathscr{E}}$ is a weakly locally bounded category (the group $\Gamma \subseteq \mathrm{Aut}_{k-\mathrm{cat}}(\tilde{\mathscr{E}})$ acts freely on $\mathrm{ob}\,\tilde{\mathscr{E}}$) and the mapping $\mathrm{ob}\,\mathscr{E} \ni a \mapsto (\epsilon, a) \in \mathrm{ob}\,(\tilde{\mathscr{E}}/\Gamma)$ induces the isomorphism

$$\mathscr{E} \simeq \tilde{\mathscr{E}}/\Gamma$$

of Γ-graded weakly locally bounded k-categories (see Remark 5.1 (*i*)) *for definition of natural Γ-grading on $\tilde{\mathscr{E}}/\Gamma$).*

Proof. We show first that (b) follows from (a) if Γ is torsion-free. Suppose that $\alpha = \sum_{\epsilon\neq\gamma\in\Gamma} \alpha_\gamma \in \bigoplus_{\epsilon\neq\gamma\in\Gamma} \mathscr{E}(a,a)_\gamma \setminus J(\mathscr{E}(a,a))$. Then at least one homogeneous component α_γ, $\epsilon \neq \gamma \in \Gamma$, does not belong to $J(\mathscr{E}(a,a))$, so is invertible. Hence, $\mathscr{E}(a,a)_{\gamma^n} \neq 0$ for all $n \geq 0$, so γ is torsion since from (a) the set $\{\gamma^n : n \in \boldsymbol{N}\}$ is finite, a contradiction. Consequently, (b) holds.

To prove that $\tilde{\mathscr{E}}$ is a weakly locally bounded category note first that 4.1(c) holds for $\tilde{\mathscr{E}}$ by (a). To show the property 4.1(b) for $\tilde{\mathscr{E}}$ observe that, the ideal $J(\mathscr{E}(a,a)) \cap \mathscr{E}(a,a)_\epsilon$ of the algebra $\mathscr{E}(a,a)_\epsilon$ is nilpotent (note that $\mathrm{id}_a \in \mathscr{E}(a,a)_\epsilon$) and by (b) we have the isomorphisms

$$\begin{aligned}\mathscr{E}(a,a)_\epsilon/(J(\mathscr{E}(a,a)) \cap \mathscr{E}(a,a)_\epsilon) &\simeq (\mathscr{E}(a,a)_\epsilon + J(\mathscr{E}(a,a)))/J(\mathscr{E}(a,a))\\ &= \mathscr{E}(a,a)/J(\mathscr{E}(a,a))\,.\end{aligned}$$

Consequently, $J(\mathscr{E}(a,a)_\epsilon) = J(\mathscr{E}(a,a)) \cap \mathscr{E}(a,a)_\epsilon$ and $\mathscr{E}(a,a)$ is a local, semiprimary k-algebra, for every $a \in \mathrm{ob}\,\mathscr{E}$. Therefore, $\tilde{\mathscr{E}}$ satisfies 4.1(a), since so does $\mathscr{E}$ (we apply (b) and the last equality). It is clear that Γ acts freely on $\mathrm{ob}\,\tilde{\mathscr{E}}$, so one can form the quotient $\tilde{\mathscr{E}}/\Gamma$ and start to prove the last assertion. By the definition the equality

$$(\tilde{\mathscr{E}}/\Gamma)((\epsilon,a),(\epsilon,b)) = \bigoplus_{\gamma\in\Gamma} \tilde{\mathscr{E}}((\epsilon,a),(\gamma,b)) = \bigoplus_{\gamma\in\Gamma} \mathscr{E}(a,b)_{\gamma^{-1}}$$

holds for every pair $a, b \in \mathrm{ob}\,\mathscr{E}$. Then for any $\alpha^1_{\gamma_1} \in \mathscr{E}(a,b)_{\gamma_1^{-1}} \subseteq (\tilde{\mathscr{E}}/\Gamma)((\epsilon,a),(\epsilon,b))$, $\alpha^2_{\gamma_2} \in \mathscr{E}(b,c)_{\gamma_2^{-1}} \subseteq (\tilde{\mathscr{E}}/\Gamma)((\epsilon,b),(\epsilon,c))$ we have $\alpha^2_{\gamma_2} \circ \alpha^1_{\gamma_1} = \gamma_1(\alpha^2_{\gamma_2}) \cdot \alpha^1_{\gamma_1} = \alpha^2_{\gamma_2}\alpha^1_{\gamma_1}$ ($\in \mathscr{E}(a,c)_{\gamma_2^{-1}\gamma_1^{-1}} = \mathscr{E}(a,c)_{(\gamma_1\gamma_2)^{-1}}$), where the first composition refers to $(\tilde{\mathscr{E}}/\Gamma)$, the second to $\tilde{\mathscr{E}}$, the third to $\mathscr{E}$. Finally, note that $(\tilde{\mathscr{E}}/\Gamma)((\epsilon,a),(\epsilon,b))_\gamma = \tilde{\mathscr{E}}((\epsilon,a),\gamma^{-1}(\epsilon,b)) = \mathscr{E}(a,b)_\gamma$, for all $a, b \in \mathrm{ob}\,\mathscr{E}$ and $\gamma \in \Gamma$. □

Note that in the case Γ is not torsion-free (a) not always implies (b).

Example. Let $\Gamma = \boldsymbol{Z}_2 (= \{0,1\})$ and $\mathscr{E}$ consists of one object a such that $\mathscr{E}(a,a) = k[t]/(t^2)$. If $\operatorname{char} k = 2$ then $\mathscr{E}$ admits $\boldsymbol{Z}_2$-grading given by $k[t]/(t^2) = k1 \oplus k(1+\bar{t})$; $\deg 1 = 0$, $\deg(1+\bar{t}) = 1$, where $\bar{t} = t + (t^2)$. Then $\mathscr{E}(a,a)_1$ is not contained in $J(\mathscr{E}(a,a))$ and $\tilde{\mathscr{E}}$ does not satisfy 4.1(a).

Remark. (a) The functor $\zeta : \tilde{\mathscr{E}} \to \mathscr{E}$, corresponding under the identification $\mathscr{E} \simeq \tilde{\mathscr{E}}/\Gamma$ above to the Galois covering functor $\mathscr{F} : \tilde{\mathscr{E}} \to \tilde{\mathscr{E}}/\Gamma$, is given by the projection $\pi_2 : \Gamma \times \operatorname{ob}\mathscr{E} \to \operatorname{ob}\mathscr{E}$ on the second component (on objects) and by the embeddings $\mathscr{E}(a,b)_{\gamma_2^{-1}\gamma_1} \subseteq \mathscr{E}(a,b)$, $\gamma_1, \gamma_2 \in \Gamma$, $a, b \in \operatorname{ob}\mathscr{E}$ (on morphisms).

(b) The category $\operatorname{MOD}\tilde{\mathscr{E}}$ is equivalent to the category formed by all Γ-graded $\mathscr{E}$-modules and $\mathscr{E}$-homomorphisms of "zero degree".

5.6.

Assume that $\mathscr{E}$ is weakly locally bounded k-category with a fixed Γ-grading $(**)$. Suppose we are given an action $\sigma : H \to \operatorname{Aut}_{k-\mathrm{cat}}(\mathscr{E})$ of the group H on $\mathscr{E}$ which is *compatible* with the Γ-grading, that is each $\sigma(h)$, $h \in H$, induces the k-isomorphism

$$\mathscr{E}(a,b)_\gamma \simeq \mathscr{E}(\sigma(h)(a), \sigma(h)(b))_\gamma$$

for all $a, b \in \operatorname{ob}\mathscr{E}$ and $\gamma \in \Gamma$. In particular, if H acts trivially on $\operatorname{ob}\mathscr{E}$ this simply means that for any $a, b \in \operatorname{ob}\mathscr{E}$ all subspaces $\mathscr{E}(a,b)_\gamma \subseteq \mathscr{E}(a,b)$, $\gamma \in \Gamma$, are H-invariant.

Consider the family $\tilde{\sigma}(h)((\gamma_1,a),(\gamma_2,b)) : \tilde{\mathscr{E}}((\gamma_1,a),(\gamma_2,b)) \to \tilde{\mathscr{E}}((\gamma_1,a),(\gamma_2,b))$, $h \in H$, $a, b \in \operatorname{ob}\mathscr{E}$, $\gamma_1, \gamma_2 \in \Gamma$, of k-linear maps given by the restrictions of $\sigma(h) : \mathscr{E}(a,b) \to \mathscr{E}(a,b)$ to $\mathscr{E}(a,b)_{\gamma_2^{-1}\gamma_1}$.

Lemma. (a) *The family* $(\tilde{\sigma}(h)((\gamma_1,a),(\gamma_2,b)))_{h\in H;\,(\gamma_1,a),(\gamma_2,b)\in \operatorname{ob}\tilde{\mathscr{E}}}$ *yields a group homomorphism* $\tilde{\sigma} : \Gamma \to \operatorname{Aut}_{k-\mathrm{cat}}(\tilde{\mathscr{E}})$, *in fact an action of* H *on the* k*-category* $\tilde{\mathscr{E}}$, *that is trivial on objects and commutes with the action of* Γ *on* $\tilde{\mathscr{E}}$.

(b) *Assume that* 5.5(*a*) *and* 5.5(*b*) *hold. The action* $\bar{\tilde{\sigma}} : H \to \operatorname{Aut}_{k-\mathrm{cat}}(\tilde{\mathscr{E}}/\Gamma)$, *induced by* $\tilde{\sigma}$, *is trivial on objects and coincides with* σ *under the identification* $\mathscr{E} \simeq \tilde{\mathscr{E}}/\Gamma$, *given by the mapping* $a \mapsto (\epsilon, a)$ (*see Lemma* 5.3 *and Proposition* 5.5).

Proof. Follows easily from definitions. □

From now on we will identify $\mathscr{E}$ with $\tilde{\mathscr{E}}/\Gamma$, $\operatorname{MOD}(\mathscr{E}_\sigma H)$ with $\operatorname{MOD}((\tilde{\mathscr{E}}/\Gamma)_{\bar{\tilde{\sigma}}} H)$, and $\operatorname{CM}(\mathscr{E}_\sigma H)$ with $\operatorname{CM}((\tilde{\mathscr{E}}/\Gamma)_{\bar{\tilde{\sigma}}} H)$. Then, as an immediate consequence of Theorem 5.4, Proposition 5.5 and Lemma 5.6, we obtain the following.

Corollary. *Given a Galois covering* $\mathscr{F} : \tilde{\mathscr{E}} \to \mathscr{E}$ $(= \tilde{\mathscr{E}}/\Gamma)$, *the "push-down" functor* $\mathscr{F}_{\lambda \mid \operatorname{CM}(\tilde{\mathscr{E}}_{\tilde{\sigma}} H)} : \operatorname{CM}(\tilde{\mathscr{E}}_{\tilde{\sigma}} H) \to \operatorname{CM}(\mathscr{E}_\sigma H)$ *induces an injection of the* G*-orbits of the isoclasses of indecomposables in* $\operatorname{CM}(\tilde{\mathscr{E}}_{\tilde{\sigma}} H)$ *into the set of the isoclasses of indecomposables in* $\operatorname{CM}(\mathscr{E}_\sigma H)$.

6. The proof of the main result.

We start with three preparatory facts used in the proof.

6.1.
Let B be an arbitrary G-atom in $\operatorname{Mod} R$ and

$$\theta : \operatorname{Mod}^{H}_{\mathrm{f},B} R \to \operatorname{Mod}^{G}_{\mathrm{f},B} R\,,$$

the restriction of the induction functor

$$\theta^G_H : \operatorname{Mod}^H_{\mathrm{f}} R \to \operatorname{Mod}^G_{\mathrm{f}} R$$

to $\operatorname{Mod}^{H}_{\mathrm{f},B} R$, where $H = G_B$. The following property of θ is used in the proof.

Lemma. (a) *If M is an indecomposable object in $\operatorname{Mod}^{H}_{\mathrm{f},B} R$ then $\theta(M)$ is an indecomposable object in $\operatorname{Mod}^{G}_{\mathrm{f},B} R$.*

(b) *Let M, M' be a pair of indecomposable objects in $\operatorname{Mod}^{H}_{\mathrm{f},B} R$. Then $M \simeq M'$ in $\operatorname{Mod}^{H}_{\mathrm{f},B} R$ if and only if $\theta(M) \simeq \theta(M')$ in $\operatorname{Mod}^{G}_{\mathrm{f},B} R$.*

Proof. (a) Consider the restriction $\mathscr{R} : \operatorname{Mod}^{G}_{\mathrm{f},B} R \to \operatorname{Mod}^{H} R$ of the functor $\mathscr{R}^G_H : \mathrm{MOD}^G R \to \mathrm{MOD}^H R$. Recall that, $\mathscr{R}^G_H$ attaches to any (N, ν), $\nu = (\nu_g)_{g\in G}$, the pair $(N, \nu_{|H})$, where $\nu_{|H} = (\nu_h)_{h\in H}$. (It is clear that θ^G_H is the left adjoint functor to $\mathscr{R}^G_H$.) By the definition of θ and $\mathscr{R}$, for any $M = (B^n, \mu)$ in $\operatorname{Mod}^H_{\mathrm{f},B} R$, $n \in \boldsymbol{N}$, the decomposition

$$\mathscr{R}\theta(M) = B^n \oplus \Bigg(\bigoplus_{e\neq g\in S_B} {}^{g}B^n\Bigg) \tag{i}$$

($e = \mathrm{id}_R$) in $\operatorname{Mod} R$ yields a decomposition of the object $\mathscr{R}\theta(M)$ in $\operatorname{Mod}^H R$ into a direct sum of $M = (B^n, \mu)$ and the complementary direct summand $(\bigoplus_{e\neq g\in S_B} {}^{g}B^n)$ in $\operatorname{Mod}^H R$. It is clear (see 1.2) that in the proof we can restrict our attention only to the objects M in $\operatorname{Mod}^H_{\mathrm{f},B} R$ of the form as above.

Fix an indecomposable object $M = (B^n, \mu)$ in $\operatorname{Mod}^H_{\mathrm{f},B} R$. Suppose that we have $\theta(M) \simeq M_1 \oplus M_2$ in $\operatorname{Mod}^G_{\mathrm{f},B} R$, for some M_1, M_2. We can assume that as R-modules they have the form $M_1 = (\bigoplus_{g\in S_B} {}^{g}B^{n_1})$ and $M_2 = (\bigoplus_{g\in S_B} {}^{g}B^{n_2})$, respectively. By the uniqueness of the decomposition into a direct sum of indecomposables in $\operatorname{Mod} R$ (see [**7**]) we have $n_1 + n_2 = n$, since $\bigoplus_{g\in S_B} {}^{g}B^n \simeq (\bigoplus_{g\in S_B} {}^{g}B^{n_1}) \oplus (\bigoplus_{g\in S_B} {}^{g}B^{n_2})$. On the other hand $\mathscr{R}\theta(M) \simeq \mathscr{R}(M_1) \oplus \mathscr{R}(M_2)$ in $\operatorname{Mod}^H R$; consequently, by (i), M is a direct summand of $\mathscr{R}(M_1)$ or $\mathscr{R}(M_2)$ in $\operatorname{Mod}^H R$. Assume that M is a direct summand of $\mathscr{R}(M_1)$. Then $\bigoplus_{g\in S_B} {}^{g}B^n$ is a direct summand of $\bigoplus_{g\in S_B} {}^{g}B^{n_2}$ in $\operatorname{Mod} R$ and by arguments as before we have $n \le n_2$, hence $n_2 = 0$ and $M_2 = 0$. This immediately implies that M is an indecomposable object in $\operatorname{Mod}^G_{\mathrm{f},B} R$ and the proof of (a) is finished.

(b) Suppose now that $M = (B^n, \mu)$, $M' = (B^{n'}, \mu)$ are objects in $\operatorname{Mod}^H_{\mathrm{f},B} R$ such that $\theta(M) \simeq \theta(M')$. Then we obtain the isomorphism

$$u :\ \mathscr{R}\theta(M) \to \mathscr{R}\theta(M')$$

in $\operatorname{Mod}^H R$. The isomorphism u, regarded as a map

$$u:\ B^n \oplus \Big(\bigoplus_{e\neq g\in S_B} {}^gB^n\Big) \longrightarrow B^{n'} \oplus \Big(\bigoplus_{e\neq g\in S_B} {}^gB^{n'}\Big),$$

is given by the matrix

$$\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix},$$

where the components $u_{11} : M \to M'$, $u_{12} : \bigoplus_{e\neq g\in S_B} {}^gB^n \to M'$, $u_{21} : M \to \bigoplus_{e\neq g\in S_B} {}^gB^{n'}$ and $u_{22} : \bigoplus_{e\neq g\in S_B} {}^gB^n \to \bigoplus_{e\neq g\in S_B} {}^gB^{n'}$ are morphisms in $\mathrm{Mod}^H R$ (see (i)). Then by [**7**, Proposition 2.2] the R-homomorphisms u_{11}, u_{22} are R-isomorphisms and we have $M \simeq M'$ in $\mathrm{Mod}^H_{\mathrm{f},B} R$. Consequently, the proof of (b) is complete. □

Remark. Applying the arguments from the proof one can show that (a) and (b) hold also for the functor $\theta^G_H : \mathrm{Mod}_{\mathrm{f}}^H R \to \mathrm{Mod}_{\mathrm{f}}^G R$.

The crucial role in the proof of the main theorem is played by the following consequence of the lemma.

Corollary. *For any $f \in \mathrm{End}^H_R(B) \cap J(\mathrm{End}_R(B))$ and a sequence $s = (s_2, \ldots, s_n)$ of positive integers, $n \geq 2$, the functor*

$$\Phi^{B(f,s)}{}_{|\, I_n\text{-}spr_{l(s)}(kH)} : I_n\text{-spr}_{\,l(s)}(kH) \to \mathrm{mod}_B(R/G)$$

is a representation embedding if and only if so is

$$-\otimes_k B(f,s)_{|\, I_n\text{-}spr_{l(s)}(kH)} : I_n\text{-spr}_{\,l(s)}(kH) \to \mathrm{Mod}^{\,H}_{\mathrm{f},B} R\,.$$

6.2.

Let $\mathscr{E}_0$ be a full subcategory of a weakly locally bounded k-category $\mathscr{E}$. Assume, as usually, that $\mathscr{E}$ is equipped with an action $\sigma : H \to \mathrm{Aut}_{k-\mathrm{cat}}(\mathscr{E})$ that is trivial on objects. We denote by $\sigma_| : H \to \mathrm{Aut}_{k-\mathrm{cat}}(\mathscr{E}_0)$ the action given by the family $(\sigma(h)_{|\mathscr{E}_0})_{h\in H}$ of k-equivalences. Let $e_\bullet : \mathrm{MOD}\,\mathscr{E} \to \mathrm{MOD}\,\mathscr{E}_0$ be the restriction functor induced by the embedding $\mathscr{E}_0 \hookrightarrow \mathscr{E}$, and $e_\lambda : \mathrm{MOD}\,\mathscr{E}_0 \to \mathrm{MOD}\,\mathscr{E}$ the associated left Kan extension functor, that is the left adjoint to $e_\bullet$ (see [**23**] for the precise definition). It is easily seen that $e_\bullet$ induces the functor

$$e_\bullet : \mathrm{MOD}\,(\mathscr{E}H) \to \mathrm{MOD}\,(\mathscr{E}_0 H)$$

where $\mathscr{E}H = \mathscr{E}_\sigma H$ and $\mathscr{E}_0 H = (\mathscr{E}_0)_{\sigma_1} H$ (note that ${}^he_\bullet(N) = e_\bullet({}^hN)$, for any N in $\mathrm{MOD}\,\mathscr{E}$ and $h \in H$). We discuss briefly the analogous problem for e_λ.

Lemma. (a) *Let M be an object in $\mathrm{MOD}\,\mathscr{E}_0$. For any $h \in H$, the maps ${}^hM \otimes_{\mathscr{E}_0} \mathscr{E}(x,-) \to M \otimes_{\mathscr{E}_0} \mathscr{E}(x,-)$, $x \in \mathrm{ob}\,\mathscr{E}$, given by $m \otimes \psi \mapsto m \otimes \sigma(h^{-1})(\psi)$, $m \in M(z)$, $\psi \in \mathscr{E}(x,z)$, $z \in \mathrm{ob}\,\mathscr{E}_0$, yield the isomorphisms*

$$\eta_h = \eta_h(M) : e_\lambda({}^hM) \to {}^he_\lambda(M)$$

in MOD $\mathscr{E}$. *Moreover, the family* $(\eta_h(M))_{M\in \mathrm{ob}\,\mathrm{MOD}\,\mathscr{E}_0}$ *is a natural family of* $\mathscr{E}$*-homomorphisms, for every* $h \in H$.

(b) *Let* $M = (M,\mu)$ *be an object in* MOD $(\mathscr{E}_0H)$. *The family* $e_\lambda(\mu) = (\eta_{h^{-1}} \cdot e_\lambda(\mu_h))_{h\in H}$ *forms an* $\mathscr{E}$*-action of* H *on* $e_\lambda(M)$.

PROOF. (a) Note first that the maps above are well defined k-isomorphisms since ${}^hM(\alpha)(m)\otimes\psi - m\otimes\alpha\psi$ is mapped to the element $M(\sigma(h^{-1})(\alpha))(m)\otimes\sigma(h^{-1})(\psi) - m\otimes\sigma(h^{-1})(\alpha\psi) = 0$, where $m\in M(z)$, $\alpha\in\mathscr{E}(z',z)$, $z',z\in \mathrm{ob}\,\mathscr{E}_0$, $x\in\mathrm{ob}\,\mathscr{E}$. Moreover, for any $m\in M(z)$, $\psi\in\mathscr{E}(x,z)$, $\alpha\in\mathscr{E}(y,x)$, $z\in\mathrm{ob}\,\mathscr{E}_0$, $x,y\in\mathrm{ob}\,\mathscr{E}$, we have $m\otimes\alpha\cdot_h \sigma(h^{-1})(\psi) = m\otimes\sigma h^{-1}(\alpha\psi)$, so they yield an $\mathscr{E}$-homomorphism. The final assertion follows by an easy check.

(b) Follows by the equality of $\mathscr{E}$-homomorphisms

$$^{h^{-1}}\eta_{h_1^{-1}}\, {}^{h^{-1}}\big(e_\lambda(\mu_{h_1})\big)\,\eta_{h^{-1}} = \eta_{(h_1h)^{-1}}\, e_\lambda\big({}^{h^{-1}}\mu_{h_1}\big) : e_\lambda\big({}^{h^{-1}}M\big) \to {}^{(h_1h)^{-1}}(e_\lambda(M))$$

which for any $m\in {}^{h^{-1}}M(z)$, $\psi\in\mathscr{E}(x,z)$, $z\in\mathrm{ob}\,\mathscr{E}_0$, $x\in\mathrm{ob}\,\mathscr{E}$, maps the element $m\otimes\psi$ to $\mu_{h_1}(m)\otimes {}^{h^{-1}h_1^{-1}}\psi$. □

PROPOSITION. *The mapping* $(M,\mu)\mapsto(e_\lambda(M), e_\lambda(\mu))$ *yields a full and faithful functor*

$$e_\lambda : \mathrm{MOD}\,(\mathscr{E}_0H) \to \mathrm{MOD}\,(\mathscr{E}H)$$

such that $e_\lambda(\mathrm{CM}\,(\mathscr{E}_0H)) \subset \mathrm{CM}\,(\mathscr{E}H)$.

PROOF. Since the functor $e_\lambda : \mathrm{MOD}\,\mathscr{E}_0 \to \mathrm{MOD}\,\mathscr{E}$ is full and faithful, it suffices to show that, for any $M = (M,\mu)$, $M' = (M',\mu')$ in MOD $(\mathscr{E}_0H)$ and $\varphi\in\mathrm{Hom}_{\mathscr{E}_0}(M,M')$, φ is a morphism from M to M' in MOD $(\mathscr{E}_0H)$ if and only if $e_\lambda(\varphi)$ is a morphism from $e_\lambda(M)$ to $e_\lambda(M')$ in MOD $(\mathscr{E}H)$. But this follows (by the definition of $e_\lambda(\mu)$, $e_\lambda(\mu')$ and properties of e_λ) from the fact that $(\eta_h(N))_{N\in\mathrm{ob}\,\mathrm{MOD}\,\mathscr{E}_0}$ is a natural family of $\mathscr{E}$-homomorphisms, for every $h\in H$.

The final assertion is an immediate consequence of the $\mathscr{E}$-isomorphisms $e_\lambda(\mathscr{E}_0(x,-)) \simeq \mathscr{E}(x,-)$, $x\in\mathrm{ob}\,\mathscr{E}_0$. □

6.3.

The following fact allows us to describe the structure of the k-algebra $\mathrm{End}_R(B)$ under the assumption of Theorem 2.2.

LEMMA. *Let* E *be a local* k*-algebra equipped with a grading* $E = \bigoplus_{\gamma\in\Gamma} E_\gamma$, *by an infinite cyclic group* $\Gamma = \mathbf{Z}$, *and with an action* $\sigma : H \to \mathrm{Aut}_{k-\mathrm{alg}}(E)$ *of a group* H, *compatible with the grading. Assume that there exists a homogeneous nilpotent* H*-invariant* $f\in E$ *which admits surjective* H*-invariant algebra homomorphism* $\pi : \mathrm{End}_R(B) \to A$, *where* $A = k[f] = \bigoplus_{i=0}^r kf^i$, $r = r(f)$, *is the subalgebra of* $\mathrm{End}_R(B)$ *generated by* f, *such that* $\pi_{|A} = \mathrm{id}_A$ *and*

$$\pi(E_\gamma) = \begin{cases} kf^\gamma & \text{if } \gamma \geq 0 \\ 0 & \text{if } \gamma < 0 \ . \end{cases} \tag{$*$}$$

Then the algebra E has the following structure:

(a) $A \subseteq E^H$, $A \simeq k[t]/(t^r)$ *as Γ-graded algebras and* $N = \operatorname{Ker}\pi \subseteq E$ *is a homogeneous H-invariant ideal,*

(b) $E = A \oplus N$ *as "Γ-graded H-representations"; more precisely,* $E_\gamma = A_\gamma \oplus N_\gamma$ *as kH-modules, for every* $\gamma \in \Gamma$, *where* $N_\gamma = N \cap E_\gamma$ *and* $A_\gamma = kf^\gamma \, (\neq 0)$ *for* $0 \leq \gamma < r$, *and* $A_\gamma = 0$, *otherwise.*

(c) $J(E) = N_0 \oplus \bigoplus_{0 \neq \gamma \in \Gamma} E_\gamma$ *and* $J(E) = J(A) \oplus N$ *as "Γ-graded H-representations",*

(d) E_0 *is a local algebra with* $J(E_0) = N_0$,

(e) $E^H = A \oplus (E^H \cap N)$ *and* $J(E^H) \, (= E^H \cap J(A)) = J(A) \oplus (E^H \cap N)$ *as "Γ-graded H-representations".*

PROOF. Assume that $0 \neq f \in E_\gamma$, then $\gamma \geq 0$, $r = r(f) \geq 2$, $f^i \in E_{i\gamma}$ are nonzero H-invariants for $i = 0, \ldots, r-1$, and $f_i = 0$ for all $i \geq r$. In fact $\gamma = 1$ since $f = \pi(f) \in \pi(E_\gamma) \subseteq kf^\gamma \subseteq E_{\gamma^2}$ so $E_\gamma \cap E_{\gamma^2} \neq 0$, hence $\gamma^2 = \gamma$ and $\gamma = 1$ ($\gamma = 0$ implies $f \in k1_E$, a contradiction). Consequently, the algebra homomorphism $k[t] \to A$, $t \mapsto f$, induces Γ-graded algebra isomorphism $k[t]/(t^r) \simeq A$, where $k[t]/(t^r)$ is equipped with the standard $\boldsymbol{Z}$-grading and $A = \bigoplus_{\gamma \in \Gamma} A \cap E_\gamma = \bigoplus_{i=0}^{r} kf^\gamma$. Since π preserves Γ-grading and H-action, we have $E = A \oplus N$, where $N = \operatorname{Ker} \pi$ is a homogeneous, H-invariant ideal contained in $J(E)$. Moreover, all summands $N_\gamma = N \cap E_\gamma$ in the decomposition $N = \bigoplus_{\gamma \in \Gamma} N_\gamma$, are H-invariant and for every $\gamma \in \Gamma$ we have a decomposition $E_\gamma = A_\gamma \oplus N_\gamma$ of H-representations. Finally, $J(E) = \operatorname{Ker} p\pi = \pi^{-1}((f)) \supseteq \operatorname{Ker} \pi$, where $p : K[t]/(t^r) \to k$ is the canonical projection, so $E/J(E) \simeq k$ and $J(E) \supseteq \bigoplus_{0 \neq \gamma \in \Gamma} E_\gamma$.
The remaining assertions follow immediately from the previous and the proof. □

REMARK. Let E be as above. Suppose we are given a homogeneous invariant $f \in E$ and a homogeneous ideal $N \subseteq E$. If a decomposition $E = A \oplus N$ of the k-linear space E holds, where $A = k[f]$, then $E = A \oplus N$ as "Γ-graded H-representations" (A is equipped with the standard Γ-grading determined by the degree of f). More precisely, this means that $E_\gamma = A_\gamma \oplus N_\gamma$ as kH-modules, for every $\gamma \in \Gamma$, where $N_\gamma = N \cap E_\gamma$ and $A_\gamma = A \cap E_\gamma$. In this situation, the canonical projection $\pi : E \to A$ is not only an H-invariant surjective algebra homomorphism such that $\pi_{|A} = \mathrm{id}_A$, but also a homomorphism of Γ-graded algebras. In particular, π satisfies $(*)$ if a degree of f is 1.

6.4.

PROOF OF THEOREM 2.2. We have to prove that for a given sequence $B(f,s)$ as in 2.2, the functor $\Phi^{B(f,s)}{}_{| I_n\text{-spr}_{l(s)}(kH)}$ is a representation embedding, under the assumptions of Theorem 2.2. Due to considerations in 2.4 and 2.5 (see the final conclusion), we can assume without loss of generality that $s = \bar{s}$ (then $n = r$ and $\Phi^{B(f,s)}{}_{| I_n\text{-spr}_{l(s)}(kH)} = \Phi^{B(f)}{}_{| I_r\text{-spr}_1(kH)}$). By Corollary 6.1 and Proposition 3.5, it suffices to show that the functor $\widetilde{\mathscr{H}_B} \circ (- \otimes_k B(f))_{| I_r\text{-spr}_1(kH)} : I_r\text{-spr}_1(kH) \to \mathrm{CM}\,(EH)$ is a representation embedding, where $E = \operatorname{End}_R(B)$, $EH = E_\sigma H$, σ is the action of H on

E given by $\mathrm{Hom}_R(\nu,\nu)$ (see 3.2) and $\mathscr{B}_o = \{B\}$. This is equivalent to the fact that, the functor

$$(-\otimes_k P)_{|\, I_r\text{-spr}_1(kH)} : I_r\text{-spr}_1(kH) \longrightarrow \mathrm{CM}\,(\mathscr{E}H)$$

is a representation embedding, where $\mathscr{E} = \mathscr{E}(\mathscr{B}_o)$, $\mathscr{E}H = \mathscr{E}_{\sigma'}H$, σ' is the induced by σ action of H on $\mathscr{E}$ and

$$P: \qquad P_1 \overset{p_2}{\leftarrow} P_2 \leftarrow \cdots \leftarrow P_{r-1} \overset{p_r}{\leftarrow} P_r$$

is the sequence in $\mathrm{CM}\,(\mathscr{E}H)$ such that $P_i = (\mathrm{Hom}_R(-,B),\pi_B)$, for $i = 1,\dots,r$; $p_i = \mathrm{Hom}_R(-,f)$, for $i = 2,\dots,r$ (see Proposition 4.5).

On the other hand, by the assumptions, the category $\mathscr{E}$ is equipped with a grading by an infinite cyclic group $\varGamma = \boldsymbol{Z}$, compatible with the action σ' of H on $\mathscr{E}$ that is trivial on objects (cf. 5.5(∗∗)). Consequently, $\mathscr{E}$ admits the Galois covering $\mathscr{F} : \tilde{\mathscr{E}} \to \mathscr{E}$ by the associated with the $\varGamma$-grading of $\mathscr{E}$ weakly locally bounded k-category $\tilde{\mathscr{E}}$ (see Proposition 5.5). The functor $\mathscr{F}$ is compatible with the actions $\tilde{\sigma}'$ and σ' of H on $\tilde{\mathscr{E}}$ and $\mathscr{E}$, respectively (see Lemma 5.6 and Lemma 5.3). Therefore we have at our disposal the nicely behaved "push-down" functor $\mathscr{F}_{\lambda\,|\,\mathrm{CM}(\tilde{\mathscr{E}}H)} : \mathrm{CM}\,(\tilde{\mathscr{E}}H) \to \mathrm{CM}\,(\mathscr{E}H)$, where $\tilde{\mathscr{E}}H = \tilde{\mathscr{E}}_{\tilde{\sigma}'}H$ (see Corollary 5.6).

For any $i \in \varGamma\,(= \mathrm{ob}\,\tilde{\mathscr{E}})$, we set $\tilde{P}_i = (\tilde{\mathscr{E}}(-,i),\tilde{\pi}_i)$, where $\tilde{\pi}_i$ is the canonical $\tilde{\mathscr{E}}$-action of H on $\tilde{\mathscr{E}}(-,i)$ induced by $\tilde{\sigma}'$ (see 4.2). Let

$$\tilde{P}: \qquad \tilde{P}_1 \overset{\tilde{p}_2}{\leftarrow} \tilde{P}_2 \leftarrow \cdots \leftarrow \tilde{P}_{r-1} \overset{\tilde{p}_r}{\leftarrow} \tilde{P}_r$$

be the sequence in $\mathrm{CM}\,(\tilde{\mathscr{E}}H)$, where $\tilde{p}_i = \tilde{\mathscr{E}}(-,f)$, $f \in E_1^H \subseteq \tilde{\mathscr{E}}(i,i-1)$ (see Lemma 6.3(b)). Observe first that by the definition the functors

$$\mathscr{F}_\lambda \circ (-\otimes_k \tilde{P}),\ -\otimes_k P :\ I_r\text{-spr}(kH) \longrightarrow \mathrm{CM}\,(\mathscr{E}H)$$

are isomorphic, since for $\mathscr{F}_\lambda(\tilde{P}_i) = \mathrm{Hom}_R(-,B)$ $(=: P_0)$ as $\mathscr{E}$-modules and for any V in I_r-spr(kH) the values of the both functors are canonically isomorphic to $\bigoplus_{i=1}^r \underline{V}_i \otimes_k P_0$, where $\underline{V} = (\underline{V}_i)_{i=1,\dots,r}$ is a fixed sequence of complementary direct summands for V (it is easy to check that the respective actions of H correspond each to other).

Next we show that the functor $-\otimes_k \tilde{P}$ is a representation embedding. For this purpose, we present the functor $-\otimes_k \tilde{P}$ as a composition of the functor

$$-\otimes_k P' :\ I_r\text{-spr}_1(kH) \to \mathrm{CM}\,(\mathscr{E}'H)$$

and the left Kan extension functor

$$e_\lambda : \mathrm{CM}\,(\mathscr{E}'H) \to \mathrm{CM}\,(\tilde{\mathscr{E}}H)\,,$$

where $\mathscr{E}'$ is the full subcategory of $\tilde{\mathscr{E}}$ formed by the set $\{1,\dots,r\}$, $\mathscr{E}'H = \mathscr{E}'_{\tilde{\sigma}'_|}H$, $\tilde{\sigma}'_|$ is the restricted action of H on $\mathscr{E}'$ induced by $\tilde{\sigma}'$ (see 6.2) and

$$P' : \qquad P_1 \overset{p'_2}{\leftarrow} P'_2 \leftarrow \cdots \leftarrow P'_{r-1} \overset{p'_r}{\leftarrow} P_r$$

is the sequence in $\mathrm{CM}(\mathscr{E}'H)$ such that $P'_i = (\mathscr{E}'(-,i), \pi'_i)$, for $i = 1, \ldots, r$, and $p'_i = \mathscr{E}'(-, f)$, for $i = 2, \ldots, r$. Since E has the structure as described in Lemma 6.3, the family $\mathscr{N}' = (\mathscr{N}'(i,j))_{1\le i,j\le r}$, $\mathscr{N}'(i,j) = N_{i-j} \subseteq E_{i-j} = \mathscr{E}'(i,j)$, forms an H-invariant ideal of $\mathscr{E}'$, and together with H-invariant morphisms $\beta_i = f \in E_1^H \subseteq \mathscr{E}'(i, i-1)$, $i = 1, \ldots r$, satisfy the assumptions of Theorem 4.5 for the category $\mathscr{E}'$. Consequently, the functor $-\otimes_k P'$ is a representation embedding and by Proposition 6.2, so is $-\otimes_k \tilde{P}$.

Now, we immediately obtain from Theorem 5.4(a) that the functor $-\otimes_k P$ preserves indecomposability. To complete our proof it suffices to show that for any two nonisomorphic indecomposables V, V' in $I_r\text{-spr}_1(kH)$, isoclasses of the modules $M = V \otimes_k \tilde{P}$ and $M' = V' \otimes_k \tilde{P}$ from $\mathrm{CM}(\tilde{\mathscr{E}}H)$ do not lie in the same Γ-orbit (see Theorem 5.4(b)). Note that M and M' are nonisomorphic in $\mathrm{CM}(\tilde{\mathscr{E}}H)$ and they both, as $\tilde{\mathscr{E}}$-modules, have a direct summand isomorphic to $\tilde{P}_1$. Moreover, for any $0 \ne \gamma \in \Gamma$, either ${}^{\gamma}M$ or ${}^{\gamma^{-1}}M'$ do not contain a direct summand isomorphic to $\tilde{P}_1$. Consequently, $M' \not\simeq {}^{\gamma}M$ for all $\gamma \in \Gamma$.

As conclusion, the functor $(-\otimes_k P)_{|\, I_r\text{-spr}_1(kH)}$ is a representation embedding and so is $\Phi^{B(f)}{}_{|\, I_r\text{-spr}_1(kH)}$ (respectively, $\Phi^{B(f,s)}{}_{|\, I_n\text{-spr}_{l(s)}(kH)}$, for any sequence s). Moreover, by Corollary 2.3, we immediately infer that all indecomposables in the images of the functors $\Phi^{B(f)}{}_{|\, I_r\text{-spr}'_1(kH)}$ (respectively, $\Phi^{B(f,s)}{}_{|\, I_n\text{-spr}'_{l(s)}(kH)}$) are non-regularly orbicular.

The final assertion of the theorem follows from [**9**, Lemma 3.7], since $I_2\text{-spr}'(kH)$ can be fully imbedded into $I_n\text{-spr}'_{l(s)}(kH)$, for any $n \ge 2$ and any sequence s. □

Remark. If char $k = p > 0$, then, taking [**33**, Theorem 1.4] into consideration, we can slightly modify the final assertion of Theorem 2.2.

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