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A generalization of the $\varDelta$-genus of quasi-polarized varieties

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Abstract. Let (X, L) be a quasi-polarized variety defined over the complex number field. Then there are several invariants of (X, L), for example, the sectional genus and the $\varDelta$-genus. In this paper we introduce the i-th $\varDelta$-genus $\varDelta_i(X, L)$ for every integer i with $0 \leq i \leq n = \dim X$. This is a generalization of the $\varDelta$-genus. Furthermore we study some properties of $\varDelta_i(X, L)$ and we will propose some problems.

Introduction.

Let X be a projective variety of dimension n defined over the complex number field and let L be a line bundle on X. If L is ample (resp. nef and big), then (X, L) is called a *polarized* (*resp. quasi-polarized*) *variety*. Furthermore if X is smooth and L is ample (resp. nef and big), we say that (X, L) is a polarized (resp. quasi-polarized) *manifold*. For this (X, L), there are some invariants, for example, the sectional genus $g(L)$ and the $\varDelta$-genus $\varDelta(L)$ (see [**Fj1**]). Fujita studied polarized varieties by using these invariants, and he gave a beautiful theory (see [**Fj3**] in detail). But there is a limit to studying polarized varieties by using these invariants. So in order to study polarized varieties more deeply, the author thought that he wants to give a new invariant of (X, L) which is a generalization of these invariants.

In [**Fk**], we defined the i-th sectional geometric genus $g_i(X, L)$ of (X, L) for every integer i with $0 \leq i \leq n$, which is a generalization of the degree L^n and the sectional genus $g(L)$ of (X, L). (We remark that $g_0(X, L) = L^n$, $g_1(X, L) = g(L)$, and $g_n(X, L) = h^n(\mathscr{O}_X)$.) Some properties of the i-th sectional geometric genus which are obtained in [**Fk**] also show that the i-th sectional geometric genus is a natural generalization of the sectional genus. For example, in [**Fk**] we proved the following theorem which is analogous to a theorem of Sommese ([**So**, Theorem 4.1]).

Theorem (See [**Fk**, Corollary 3.5]). *Let (X, L) be a polarized manifold of dimension $n \geq 3$. Assume that L is spanned. Then the following are equivalent*:

(1) $g_2(X, L) = h^2(\mathscr{O}_X)$.
(2) $h^0(K_X + (n-2)L) = 0$.
(3) $\kappa(K_X + (n-2)L) = -\infty$.
(4) $K_{X'} + (n-2)L'$ *is not nef, where* (X', L') *is a reduction of* (X, L). (See Definition 1.4(2) below.)

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(5) (X, L) *is one of the types from* (1) *to* (7.4) *in Theorem* 1.7 *below.*

As the next step, we want to give a generalization of the Δ-genus.

In this paper, we will give a definition of the i-th Δ-genus $\Delta_i(X, L)$ of (X, L) for $0 \leq i \leq n$. If $i = 1$, then $\Delta_1(X, L)$ is the Δ-genus $\Delta(L)$ of (X, L). (When we define the i-th Δ-genus of (X, L), we need the sectional geometric genus of (X, L).)

Furthermore we will study some properties of $\Delta_i(X, L)$. If $\mathrm{Bs}|L| = \emptyset$, then some properties of $\Delta_i(X, L)$ is similar to that of the Δ-genus $\Delta(L)$ of (X, L) (see Section 3), and the i-th Δ-genus is useful in order to study polarized manifolds (X, L) with $\mathrm{Bs}|L| = \emptyset$.

So we expect that the i-th Δ-genus has good properties for general polarized varieties. For example, we expect that $\Delta_i(X, L) \geq 0$ for $2 \leq i \leq n$. But unfortunately there exists an example of (X, L) with $\Delta_i(X, L) < 0$ (see Section 4). Hence it is important to consider when the i-th Δ-genus is nonnegative. We treat this problem in a forthcoming paper.

The contents of this paper are the following.

In Section 1, we propose some results which are used later.

In Section 2, we will give a definition of the i-th Δ-genus $\Delta_i(X, L)$ of (X, L) (see Definition 2.1), and we will prove some results under the condition that L has a k-ladder. (For the definition of a k-ladder, see Definition 2.7.)

In Section 3, we consider the case where (X, L) is a (quasi-)polarized manifold with $\mathrm{Bs}|L| = \emptyset$, and we will get results similar to that of the Δ-genus $\Delta(L)$ of (X, L). In particular we will prove $\Delta_i(X, L) \geq 0$ for $1 \leq i \leq n$ (see Corollary 3.3) and we give a classification of (X, L) such that L is base point free (resp. very ample) and $\Delta_2(X, L) = 0$ (resp. 1) (see Theorem 3.13 and Remark 3.13.1 (resp. Theorem 3.17)). (We will study the i-th Δ-genus of (X, L) with $\dim \mathrm{Bs}|L| \geq 0$ in a forthcoming paper.)

In Section 4, we propose some problems and we will give some examples of (X, L) such that $\Delta_i(X, L) < 0$.

Our dream is to construct a classification theory of polarized manifolds by using the i-th sectional geometric genus and the i-th Δ-genus. If $i = 1$, then this case has been studied by Fujita, and a series of his studies is called Fujita's Δ-genus theory (see [**Fj3**]). So, as the next step, we want to study the case where $i = 2$ in detail. As the first step, in a future paper, we will study a classification of (X, L) with $2 \leq g_2(X, L) - h^2(\mathscr{O}_X) \leq 5$ and $2 \leq \Delta_2(X, L) \leq 5$ when L is very ample.

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Notation and Conventions.

In this paper, we work throughout over the complex number $\boldsymbol{C}$. The words "line bundles" and "Cartier divisors" are used interchangeably. The tensor products of line bundles are denoted additively.

$\mathscr{O}(D)$: invertible sheaf associated with a Cartier divisor D on X.
$\mathscr{O}_X$: the structure sheaf of X.
$\chi(\mathscr{F})$: the Euler-Poincaré characteristic of a coherent sheaf $\mathscr{F}$.
$\chi(X) = \chi(\mathscr{O}_X)$.

$h^i(\mathscr{F}) = \dim H^i(X, \mathscr{F})$ for a coherent sheaf $\mathscr{F}$ on X.
$h^i(D) = h^i(\mathscr{O}(D))$ for a divisor D.
$D|_C$: the restriction of D to C.
$|D|$: the complete linear system associated with a divisor D.
K_X: the canonical divisor of X.
$q(X)$ (or q): the irregularity $h^1(\mathscr{O}_X)$ of a smooth projective variety X.
$\kappa(D)$: the Iitaka dimension of a Cartier divisor D on X.
$\kappa(X)$: the Kodaira dimension of X.
$\boldsymbol{P}^n$: the projective space of dimension n.
$\boldsymbol{Q}^n$: a hyperquadric surface in $\boldsymbol{P}^{n+1}$.
$\boldsymbol{P}_Y(\mathscr{E})$: the $\boldsymbol{P}^{r-1}$-bundle associated with a locally free sheaf $\mathscr{E}$ of rank r over Y.
$H(\mathscr{E})$: the tautological invertible sheaf of $\boldsymbol{P}_Y(\mathscr{E})$.
$\sim$ (or $=$): linear equivalence.
$\equiv$: numerical equivalence.

1. Preliminaries.

NOTATION 1.1. Let (X, L) be a quasi-polarized variety of dimension n and let $\chi(tL)$ be the Euler-Poincaré characteristic of tL. Then we put

$$\chi(tL) = \sum_{j=0}^{n} \chi_j(X, L) \frac{t^{[j]}}{j!},$$

where $t^{[j]} = t(t+1)\cdots(t+j-1)$ for $j \geq 1$ and $t^{[0]} = 1$.

DEFINITION 1.2 ([**Fk**, Definition 2.1]). Let (X, L) be a quasi-polarized variety of dimension n. Then, for every integer i with $0 \leq i \leq n$, the *i-th sectional geometric genus $g_i(X, L)$ of* (X, L) is defined by the following formula:

$$g_i(X, L) = (-1)^i(\chi_{n-i}(X, L) - \chi(\mathscr{O}_X)) + \sum_{j=0}^{n-i}(-1)^{n-i-j} h^{n-j}(\mathscr{O}_X).$$

REMARK 1.2.1.

(1) If $i = 0$ (resp. $i = 1$), then $g_i(X, L)$ is equal to the degree (resp. the sectional genus) of (X, L).
(2) If $i = n$, then $g_n(X, L) = h^n(\mathscr{O}_X)$ and $g_n(X, L)$ is independent of L.

THEOREM 1.3. (1) *Let (X, L) be a quasi-polarized variety of dimension n. Let i be an integer with $0 \leq i \leq n-1$. Then*

$$g_i(X, L) = \sum_{j=0}^{n-i-1} (-1)^{n-j} \binom{n-i}{j} \chi(-(n-i-j)L) + \sum_{k=0}^{n-i}(-1)^{n-i-k} h^{n-k}(\mathscr{O}_X).$$

(2) *If (X, L) is a quasi-polarized manifold of dimension n, then for every integer i with*

$0 \leq i \leq n-1$

$$g_i(X,L) = \sum_{j=0}^{n-i-1}(-1)^j\binom{n-i}{j}h^0(K_X+(n-i-j)L) + \sum_{k=0}^{n-i}(-1)^{n-i-k}h^{n-k}(\mathcal{O}_X).$$

Proof. (1) By [**Fk**, Theorem 2.2], we obtain

$$\begin{aligned}\chi_{n-i}(X,L) &= \sum_{j=0}^{n-i}(-1)^{n-i-j}\binom{n-i}{j}\chi(-(n-i-j)L) \\ &= \sum_{j=0}^{n-i-1}(-1)^{n-i-j}\binom{n-i}{j}\chi(-(n-i-j)L) + \chi(\mathcal{O}_X).\end{aligned}$$

Hence by Definition 1.2, we get the assertion.
(2) By the Serre duality and the Kawamata-Viehweg vanishing theorem, we get the assertion (See also [**Fk**, Theorem 2.3]). □

Remark 1.3.1. Let (X,L) be a quasi-polarized manifold of dimension n. Then by Theorem 1.3(2) and the Serre duality, we get

$$g_{n-1}(X,L) = h^0(K_X+L) - h^0(K_X) + h^{n-1}(\mathcal{O}_X).$$

Definition 1.4. (1) Let X (resp. Y) be an n-dimensional projective manifold, and let L (resp. A) be an ample line bundle on X (resp. Y). Then (X,L) is called a *simple blowing up of* (Y,A) if there exists a birational morphism $\pi : X \to Y$ such that π is a blowing up at a point of Y and $L = \pi^*(A) - E$, where E is the π-exceptional effective reduced divisor.
(2) Let X (resp. Y) be an n-dimensional projective manifold, and let L (resp. A) be an ample line bundle on X (resp. Y). Then we say that (Y,A) is a *reduction of* (X,L) if there exists a birational morphism $\mu : X \to Y$ such that μ is a composite of simple blowing ups and (Y,A) is not obtained by a simple blowing up of any polarized manifold. In this case the morphism μ is called the *reduction map*.

Remark 1.4.1. Let (X,L) be a polarized manifold and let (Y,A) be a reduction of (X,L). Let $\mu : X \to Y$ be the reduction map.

(1) We obtain $g_i(X,L) = g_i(Y,A)$ for every integer i with $1 \leq i \leq n$ (see [**Fk**, Proposition 2.6]).
(2) Assume that $\mathrm{Bs}|L| = \varnothing$. Then for a general member D of $|L|$, D and $\mu(D) \in |A|$ are smooth.
(3) If (X,L) is not obtained by a simple blowing up of another polarized manifold, then (X,L) is a reduction of itself.
(4) A reduction of (X,L) always exists (see [**Fj3**, Chapter II, (11.11)]).

Definition 1.5. Let (X,L) be a polarized manifold of dimension n. We say that

(X,L) is a *scroll* (*resp. quadric fibration, Del Pezzo fibration*) *over a normal variety* Y *of dimension* m if there exists a surjective morphism with connected fibers $f: X \to Y$ such that $K_X + (n-m+1)L = f^*A$ (resp. $K_X + (n-m)L = f^*A$, $K_X + (n-m-1)L = f^*A$) for some ample line bundle A on Y.

LEMMA 1.6. *Let X (resp. Y) be a smooth projective variety (resp. normal projective variety) of dimension n (resp. m) with $n > m \geq 1$ such that there exists a surjective morphism $f: X \to Y$ with connected fibers. Let L be a nef and big line bundle on X such that $\mathcal{O}(K_X + tL) = f^*(A)$ for a line bundle A on Y, where t is a positive integer. Then $h^i(L) = 0$ and $h^i(\mathcal{O}_X) = 0$ for $i > m$.*

PROOF. By assumption, we get $\mathcal{O}(K_X + (t+1)L) = L \otimes f^*(A)$. By the Kawamata-Viehweg vanishing theorem ([**KMM**, Theorem 1-2-5]), we get $R^i f_*(L \otimes f^*(A)) = 0$ for every integer i with $i > 0$. Since $R^i f_*(L \otimes f^*(A)) = R^i f_*(L) \otimes A$, we get $R^i f_*(L) \otimes A = 0$. Hence $R^i f_*(L) = 0$ for every $i > 0$. Therefore $h^i(L) = h^i(f_*(L))$. By [**Ha**, Theorem 2.7, Chapter III], we obtain $h^i(f_*(L)) = 0$ for every $i > m$. Hence $h^i(L) = 0$ for every integer i with $i > m$. Next we prove the second statement. Since $\mathcal{O}(K_X + tL) = f^*(A)$, by the Kawamata-Viehweg vanishing theorem ([**KMM**, Theorem 1-2-5]), we get $R^i f_*(f^*(A)) = 0$ for every $i > 0$. Since $R^i f_*(f^*(A)) = R^i f_*(\mathcal{O}_X) \otimes A$, we get $R^i f_*(\mathcal{O}_X) \otimes A = 0$, and $R^i f_*(\mathcal{O}_X) = 0$ for every $i > 0$. Therefore $h^i(\mathcal{O}_X) = h^i(f_*(\mathcal{O}_X)) = h^i(\mathcal{O}_Y)$. By [**Ha**, Theorem 2.7, Chapter III], we obtain $h^i(\mathcal{O}_Y) = 0$ for every $i > m$. Hence $h^i(\mathcal{O}_X) = 0$ for every integer i with $i > m$. □

THEOREM 1.7. *Let (X,L) be a polarized manifold of dimension $n \geq 3$. Then (X,L) is one of the following types.*

(1) $(\boldsymbol{P}^n, \mathcal{O}_{\boldsymbol{P}^n}(1))$.
(2) $(\boldsymbol{Q}^n, \mathcal{O}_{\boldsymbol{Q}^n}(1))$.
(3) *A scroll over a smooth curve.*
(4) $K_X \sim -(n-1)L$, *that is, (X,L) is a Del Pezzo manifold.*
(5) *A quadric fibration over a smooth curve.*
(6) *A scroll over a smooth surface.*
(7) *Let (X', L') be a reduction of (X,L).*
  (7.1) $n = 4$, $(X', L') = (\boldsymbol{P}^4, \mathcal{O}_{\boldsymbol{P}^4}(2))$.
  (7.2) $n = 3$, $(X', L') = (\boldsymbol{Q}^3, \mathcal{O}_{\boldsymbol{Q}^3}(2))$.
  (7.3) $n = 3$, $(X', L') = (\boldsymbol{P}^3, \mathcal{O}_{\boldsymbol{P}^3}(3))$.
  (7.4) $n = 3$, *X' is a $\boldsymbol{P}^2$-bundle over a smooth curve C with* $(F', L'|_{F'}) = (\boldsymbol{P}^2, \mathcal{O}_{\boldsymbol{P}^2}(2))$ *for every fiber F' of it.*
  (7.5) $K_{X'} + (n-2)L'$ *is nef.*

PROOF. See [**BeSo**, Proposition 7.2.2, Theorem 7.2.4, Theorem 7.3.2, and Theorem 7.3.4]. □

LEMMA 1.8. *Let X be a complete normal variety of dimension n defined over the complex number field, and let D_1 and D_2 be effective Weil divisors on X. Then $h^0(D_1 + D_2) \geq h^0(D_1) + h^0(D_2) - 1$.*

PROOF (See also [**I**, Chapter 6, §6.2, b]). We put $D_1 = \sum_{j=1}^{s} n_j \Gamma_j$ and $D_2 =$

$\sum_{j=1}^{s} m_j \Gamma_j$, where Γ_j is a prime divisor on X for any integer j with $1 \leq j \leq s$ such that $\Gamma_k \neq \Gamma_l$ for $k \neq l$, and n_j and m_j are non-negative integers.

For a divisor B on X we put

$$L(B) := \{\phi \in R(X) \mid \phi = 0 \text{ or } B + \operatorname{div}(\phi) \geq 0\},$$

where $R(X)$ is the rational function field of X. Then $L(B)$ is a vector space, and we put $l(B) := \dim L(B)$.

Let

$$D_1 \wedge D_2 := \sum_{j=1}^{s} \min\{n_j, m_j\} \Gamma_j,$$

$$D_1 \vee D_2 := \sum_{j=1}^{s} \max\{n_j, m_j\} \Gamma_j.$$

Then there are the following relations:

$$L(D_1) \cap L(D_2) = L(D_1 \wedge D_2)$$

and

$$L(D_1) \cup L(D_2) \subset L(D_1 \vee D_2).$$

Here we note that by a theorem on vector spaces we get

$$\begin{aligned} l(B_1) + l(B_2) &= \dim(L(B_1) \cap L(B_2)) + \dim(L(B_1) + L(B_2)) \\ &\leq l(B_1 \wedge B_2) + l(B_1 \vee B_2) \end{aligned} \tag{1.8.1}$$

for any effective divisors B_1 and B_2 on X.

Let Z be the fixed part of $|D_1|$, and we put $D_1' = D_1 - Z$. Then $l(D_1) = l(D_1')$ and by taking a general member of $|D_1'|$, we may assume that $D_1' \wedge D_2 = 0$ and $D_1' \vee D_2 = D_1' + D_2$. By (1.8.1), we get

$$\begin{aligned} l(D_1) + l(D_2) &= l(D_1') + l(D_2) \\ &\leq l(0) + l(D_1' + D_2) \\ &\leq 1 + l(D_1 + D_2 - Z) \\ &\leq 1 + l(D_1 + D_2). \end{aligned}$$

Since $h^0(D_1 + D_2) = l(D_1 + D_2)$ and $h^0(D_i) = l(D_i)$ for $i = 1, 2$, we get the assertion. □

LEMMA 1.9. *Let X be a smooth projective variety of dimension $n \geq 2$ and let L be a divisor on X such that* $\mathrm{Bs}|L| = \varnothing$. *Let D be an effective divisor on X. Then*

$h^0(D|_{X_1}) > 0$ for a general $X_1 \in |L|$.

Proof. If $\mathcal{O}(D) = \mathcal{O}_X$, then this is true.
So we may assume that D is a nonzero effective divisor.
We use the following exact sequence:

$$0 \to \mathcal{O}(D - X_1) \to \mathcal{O}(D) \to \mathcal{O}(D_{X_1}) \to 0.$$

By this exact sequence, we get

$$0 \to H^0(D - X_1) \to H^0(D) \to H^0(D|_{X_1}).$$

Assume that $h^0(D|_{X_1}) = 0$. Then $h^0(D - X_1) = h^0(D) > 0$. Since $h^0(X_1) = h^0(L) \geq n + 1$, by Lemma 1.8 we get

$$\begin{aligned} h^0(D) &\geq h^0(D - X_1) + h^0(X_1) - 1 \\ &\geq h^0(D - X_1) + n \\ &> h^0(D - X_1) \end{aligned}$$

and this is a contradiction. Hence $h^0(D|_{X_1}) \neq 0$. □

Proposition 1.10. *Let Y be a smooth projective variety of dimension 3 and let $\mathcal{E}$ be an ample vector bundle of rank $r \geq 3$ on Y. Assume that $(Y, c_1(\mathcal{E}))$ is a Del Pezzo fibration over a smooth curve C. Let $\pi : Y \to C$ be its morphism. Then there exist vector bundles $\mathcal{F}$ and $\mathcal{G}$ on C with $\operatorname{rank}\mathcal{F} = 3$ and $\operatorname{rank}\mathcal{G} = 3$ such that $Y = \boldsymbol{P}_C(\mathcal{F})$ and $\mathcal{E} \cong H(\mathcal{F}) \otimes \pi^*(\mathcal{G})$.*

Proof. Since $\operatorname{rank}(\mathcal{E}) = r \geq 3$ and $\mathcal{E}$ is ample, we have

$$c_1(\mathcal{E})Z \geq 3 \tag{1.10.a}$$

for any rational curve Z on Y. Hence $(F, c_1(\mathcal{E})|_F) \cong (\boldsymbol{P}^2, \mathcal{O}_{\boldsymbol{P}^2}(3))$ for any general fiber F of π because any general fiber of π is a Del Pezzo surface.

On the other hand, if π has a singular fiber F', then by [**Fj4**, (2.9), (2.12), (2.19) and (2.20)] there exists a rational curve Z' on F' such that $c_1(\mathcal{E})Z' \leq 2$.

Therefore, by (1.10.a), π has no singular fibers, that is, any fiber of π is $\boldsymbol{P}^2$. Hence Y is a $\boldsymbol{P}^2$-bundle on C and there exists a vector bundle $\mathcal{F}$ of rank 3 on C such that $Y \cong \boldsymbol{P}_C(\mathcal{F})$. Since $\operatorname{rank}(\mathcal{E}) \geq 3$ and $c_1(\mathcal{E})|_F = \mathcal{O}_{\boldsymbol{P}^2}(3)$, we get $\mathcal{E}|_F \cong \mathcal{O}_{\boldsymbol{P}^2}(1)^{\oplus 3}$ for any fiber F of π.

Therefore there exists a vector bundle $\mathcal{G}$ of rank 3 on C such that $\mathcal{E} \cong H(\mathcal{F}) \otimes \pi^*(\mathcal{G})$. This completes the proof. □

Remark 1.10.1. Let (X, L) be a polarized manifold. Assume that (X, L) is of the type (4.2) in [**Fk**, Theorem 3.6], that is, (X, L) is a scroll over a smooth projective 3-fold Y and $\mathcal{E}$ is an ample vector bundle of rank 3 on Y such that $X = \boldsymbol{P}_Y(\mathcal{E})$, $L = H(\mathcal{E})$, and $(Y, c_1(\mathscr{E}))$ is a Del Pezzo fibration over a smooth curve C. Let $\pi : Y \to C$ be its morphism. Then by Proposition 1.10, there exist vector bundles $\mathscr{F}$ and $\mathscr{G}$ on C with $\operatorname{rank} \mathscr{F} = 3$ and $\operatorname{rank} \mathscr{G} = 3$ such that $Y = \boldsymbol{P}_C(\mathscr{F})$ and $\mathscr{E} \cong H(\mathscr{F}) \otimes \pi^*(\mathscr{G})$.

2. Definition and some general results.

In this section, first we give the definition of the i-th Δ-genus of quasi-polarized varieties, which is a generalization of the Δ-genus of quasi-polarized varieties.

Definition 2.1. Let (X, L) be a quasi-polarized variety of dimension n. For every integer i with $0 \le i \le n$, the *i-th Δ-genus* $\Delta_i(X, L)$ *of* (X, L) is defined by the following formula:

$$\Delta_i(X, L) = \begin{cases} 0 & \text{if } i = 0, \\ g_{i-1}(X, L) - \Delta_{i-1}(X, L) & \\ \quad +(n-i+1)h^{i-1}(\mathscr{O}_X) - h^{i-1}(L) & \text{if } 1 \le i \le n, \end{cases}$$

where $g_{i-1}(X, L)$ is the $(i-1)$-th sectional geometric genus of (X, L).

Remark 2.2.

(1) If $i = 1$, then $\Delta_1(X, L)$ is equal to the Δ-genus of (X, L) (See [**Fj1**]).
(2) In this section, we will give another reason why this invariant is a generalization of the Δ-genus of quasi-polarized varieties (See Theorem 2.8).

Proposition 2.3. *Let (X, L) be a quasi-polarized variety of dimension n. Then for every integer i with $1 \le i \le n$*

$$\begin{aligned} \Delta_i(X, L) = (-1)^{i-1} \sum_{j=0}^{i-1} \chi_{n-j}(X, L) + (n-i+1)(-1)^{i-1} \left(\sum_{k=0}^{i-1} (-1)^k h^k(\mathscr{O}_X) \right) \\ + (-1)^i \left(\sum_{k=0}^{i-1} (-1)^k h^k(L) \right). \end{aligned}$$

Proof. We prove this proposition by induction.

If $i = 1$, then

$$\begin{aligned} \Delta_1(X, L) &= n + L^n - h^0(L) \\ &= \chi_n(X, L) + nh^0(\mathscr{O}_X) - h^0(L). \end{aligned}$$

This is true.

Assume that the assertion is true for $i = t \ge 1$. We consider the case where $i = t+1$. Then

$$\begin{aligned}\Delta_{t+1}(X,L) &= g_t(X,L) - \Delta_t(X,L) + (n-t)h^t(\mathcal{O}_X) - h^t(L)\\ &= g_t(X,L) - (-1)^{t-1}\Biggl\{\sum_{j=0}^{t-1}\chi_{n-j}(X,L) + (n-t+1)\Biggl(\sum_{k=0}^{t-1}(-1)^k h^k(\mathcal{O}_X)\Biggr)\\ &\qquad - \Biggl(\sum_{k=0}^{t-1}(-1)^k h^k(L)\Biggr)\Biggr\} + (n-t)h^t(\mathcal{O}_X) - h^t(L).\end{aligned}$$

By the definition of the t-th sectional geometric genus of (X,L), we get

$$g_t(X,L) = (-1)^t(\chi_{n-t}(X,L) - \chi(\mathcal{O}_X)) + \sum_{j=0}^{n-t}(-1)^{n-t-j}h^{n-j}(\mathcal{O}_X).$$

Hence

$$\begin{aligned}\Delta_{t+1}(X,L) &= (-1)^t(\chi_{n-t}(X,L) - \chi(\mathcal{O}_X)) + \sum_{j=0}^{n-t}(-1)^{n-t-j}h^{n-j}(\mathcal{O}_X)\\ &\quad + (-1)^t\Biggl\{\sum_{j=0}^{t-1}\chi_{n-j}(X,L) + (n-t+1)\Biggl(\sum_{k=0}^{t-1}(-1)^k h^k(\mathcal{O}_X)\Biggr)\\ &\qquad - \Biggl(\sum_{k=0}^{t-1}(-1)^k h^k(L)\Biggr)\Biggr\} + (n-t)h^t(\mathcal{O}_X) - h^t(L)\\ &= (-1)^t\sum_{j=0}^{t}\chi_{n-j}(X,L) - (-1)^t\sum_{k=0}^{t}(-1)^k h^k(L)\\ &\quad + (-1)^{t+1}\chi(\mathcal{O}_X) + \sum_{j=0}^{n-t}(-1)^{n-t-j}h^{n-j}(\mathcal{O}_X)\\ &\quad + (-1)^t(n-t+1)\Biggl(\sum_{k=0}^{t-1}(-1)^k h^k(\mathcal{O}_X)\Biggr) + (n-t)h^t(\mathcal{O}_X)\\ &= (-1)^t\sum_{j=0}^{t}\chi_{n-j}(X,L) + (-1)^{t+1}\sum_{k=0}^{t}(-1)^k h^k(L)\\ &\quad + (-1)^{t+1}\chi(\mathcal{O}_X) - (-1)^{t+1}\sum_{j=0}^{n-t}(-1)^{n-j}h^{n-j}(\mathcal{O}_X)\\ &\quad + (-1)^t(n-t+1)\Biggl(\sum_{k=0}^{t-1}(-1)^k h^k(\mathcal{O}_X)\Biggr) + (n-t)h^t(\mathcal{O}_X).\end{aligned}$$

On the other hand

$$
\begin{aligned}
&(-1)^{t+1}\chi(\mathcal{O}_X)-(-1)^{t+1}\sum_{j=0}^{n-t}(-1)^{n-j}h^{n-j}(\mathcal{O}_X)\\
&\quad+(-1)^t(n-t+1)\left(\sum_{k=0}^{t-1}(-1)^k h^k(\mathcal{O}_X)\right)+(n-t)h^t(\mathcal{O}_X)\\
&=(-1)^{t+1}\left(\sum_{k=0}^{t-1}(-1)^k h^k(\mathcal{O}_X)\right)+(-1)^t(n-t+1)\sum_{k=0}^{t-1}(-1)^k h^k(\mathcal{O}_X)+(n-t)h^t(\mathcal{O}_X)\\
&=(-1)^t(n-t)\sum_{k=0}^{t-1}(-1)^k h^k(\mathcal{O}_X)+(n-t)h^t(\mathcal{O}_X)\\
&=(-1)^t(n-t)\sum_{k=0}^{t}(-1)^k h^k(\mathcal{O}_X).
\end{aligned}
$$

Therefore we get the assertion. □

Next we consider the case where $i=n$. This result is very useful to calculate the i-th Δ-genus (see Example 2.12 below).

Proposition 2.4. *Let (X,L) be a quasi-polarized variety of dimension n. Then*

$$\Delta_n(X,L)=h^n(\mathcal{O}_X)-h^n(L).$$

Proof. By definition of the n-th Δ-genus of (X,L), we get

$$
\begin{aligned}
&\Delta_n(X,L)\\
&=g_{n-1}(X,L)-\Delta_{n-1}(X,L)+h^{n-1}(\mathcal{O}_X)-h^{n-1}(L)\\
&=g_{n-1}(X,L)-g_{n-2}(X,L)+\Delta_{n-2}(X,L)+\big(h^{n-1}(\mathcal{O}_X)-2h^{n-2}(\mathcal{O}_X)\big)\\
&\quad-\big(h^{n-1}(L)-h^{n-2}(L)\big)\\
&=\cdots\\
&=\sum_{i=0}^{n-1}(-1)^{n-1-i}g_i(X,L)+\sum_{i=0}^{n-1}(-1)^{n-1-i}(n-i)h^i(\mathcal{O}_X)-\sum_{i=0}^{n-1}(-1)^{n-1-i}h^i(L)\\
&=(-1)^{n-1}(\chi_1(X,L)+\chi_2(X,L)+\cdots+\chi_n(X,L))+(-1)^n n\chi(\mathcal{O}_X)\\
&\quad+\sum_{i=0}^{n-1}\sum_{j=0}^{n-i}(-1)^{-1-j}h^{n-j}(\mathcal{O}_X)+\sum_{i=0}^{n-1}(-1)^{n-1-i}(n-i)h^i(\mathcal{O}_X)-\sum_{i=0}^{n-1}(-1)^{n-1-i}h^i(L)\\
&=(-1)^{n-1}(\chi(L))+(-1)^n\chi(\mathcal{O}_X)+(-1)^n n\chi(\mathcal{O}_X)\\
&\quad+\sum_{i=0}^{n-1}\sum_{j=0}^{n-i}(-1)^{-1-j}h^{n-j}(\mathcal{O}_X)+\sum_{i=0}^{n-1}(-1)^{n-1-i}(n-i)h^i(\mathcal{O}_X)-\sum_{i=0}^{n-1}(-1)^{n-1-i}h^i(L).
\end{aligned}
$$

Since

$$\begin{aligned}
&\sum_{i=0}^{n-1}\sum_{j=0}^{n-i}(-1)^{-1-j}h^{n-j}(\mathscr{O}_X)\\
&\quad= (-h^n(\mathscr{O}_X)+\cdots+(-1)^{n-1}h^0(\mathscr{O}_X)) + (-h^n(\mathscr{O}_X)+\cdots+(-1)^{n-2}h^1(\mathscr{O}_X))\\
&\qquad+\cdots+(-h^n(\mathscr{O}_X)+h^{n-1}(\mathscr{O}_X))\\
&\quad= -nh^n(\mathscr{O}_X)+nh^{n-1}(\mathscr{O}_X)-(n-1)h^{n-2}(\mathscr{O}_X)+\cdots+(-1)^{n-1}h^0(\mathscr{O}_X),
\end{aligned}$$

we get

$$\begin{aligned}
&\sum_{i=0}^{n-1}\sum_{j=0}^{n-i}(-1)^{-1-j}h^{n-j}(\mathscr{O}_X)+\sum_{i=0}^{n-1}(-1)^{n-1-i}(n-i)h^i(\mathscr{O}_X)\\
&\quad= h^n(\mathscr{O}_X)-(-1)^n(n+1)\chi(\mathscr{O}_X).
\end{aligned}$$

Therefore we obtain

$$\begin{aligned}
\Delta_n(X,L) &= (-1)^{n-1}(\chi(L))+(-1)^n\chi(\mathscr{O}_X)+(-1)^n n\chi(\mathscr{O}_X)+h^n(\mathscr{O}_X)\\
&\qquad-(-1)^n(n+1)\chi(\mathscr{O}_X)-\sum_{i=0}^{n-1}(-1)^{n-1-i}h^i(L)\\
&= h^n(\mathscr{O}_X)-h^n(L).
\end{aligned}$$

This completes the proof of Proposition 2.4. □

COROLLARY 2.5. *Let (X,L) be a quasi-polarized manifold of dimension n. Assume that $\kappa(X)\neq \dim X$. Then $\Delta_n(X,L)\geq 0$.*

PROOF. By the Serre duality, we get $h^n(L)=h^0(K_X-L)$. If $h^n(L)\neq 0$, then there exists an effective divisor D on X such that $K_X\sim L+D$. Since L is big, we obtain that K_X is big. But this is impossible. Hence $h^n(L)=0$. Therefore by Proposition 2.4, $\Delta_n(X,L)=h^n(\mathscr{O}_X)-h^n(L)=h^n(\mathscr{O}_X)\geq 0$. This completes the proof. □

COROLLARY 2.6. *Let (X,L) be a quasi-polarized manifold of dimension n. Assume that $h^0(L)>0$. Then $\Delta_n(X,L)\geq 0$.*

PROOF. By Proposition 2.4, we have

$$\Delta_n(X,L)=h^n(\mathscr{O}_X)-h^n(L).$$

By the Serre duality, we have

$$\Delta_n(X,L)=h^0(K_X)-h^0(K_X-L).$$

If $h^0(K_X - L) = 0$, then $\Delta_n(X,L) = h^0(K_X) \geq 0$.
If $h^0(K_X - L) \neq 0$, then by Lemma 1.8 we get

$$\begin{aligned}\Delta_n(X,L) &= h^0(K_X) - h^0(K_X - L)\\ &\geq h^0(L) - 1\\ &\geq 0.\end{aligned}$$

This completes the proof. □

Definition 2.7. Let (X,L) be a quasi-polarized variety of dimension n. Then L has a *k-ladder* if there exists an irreducible and reduced subvariety X_i of X_{i-1} such that $X_i \in |L_{i-1}|$ for every integer i with $1 \leq i \leq k$, where $X_0 := X$, $L_0 := L$, and $L_i := L_{i-1}|_{X_i}$.

Notation 2.7.1. Let (X,L) be a quasi-polarized variety of dimension n, and let k be an integer with $1 \leq k \leq n-1$. Assume that L has a k-ladder. We put $X_0 := X$ and $L_0 := L$. Let $X_i \in |L_{i-1}|$ be an irreducible and reduced member, and $L_i := L_{i-1}|_{X_i}$ for every integer i with $1 \leq i \leq k$. Let $r_{p,q} : H^p(X_q, L_q) \to H^p(X_{q+1}, L_{q+1})$ be the natural map. If $h^0(L_k) > 0$, then we take an element $X_{k+1} \in |L_k|$ and we put $L_{k+1} = L_k|_{X_{k+1}}$.

The following conditions are used in Theorem 2.8 and Corollary 2.9.

2.7.2. Let (X,L) be a quasi-polarized variety of dimension n. Let i and j be integers with $1 \leq i \leq n$ and $1 \leq j \leq i$. (We use notation in Notation 2.7.1.)
Condition $A_1(i)$: L has an $(n-i)$-ladder.
Condition $A_2(i)$: $h^0(L_{n-i}) > 0$.
Condition $B(i,j)$: $\sum_{k=0}^{j-1}(-1)^k h^k(\mathcal{O}_X) = \cdots = \sum_{k=0}^{j-1}(-1)^k h^k(\mathcal{O}_{X_{n-i}})$.

In Theorem 2.8 and Corollary 2.9, we use Notation 2.7.1.

Theorem 2.8. *Let (X,L) be a quasi-polarized variety of dimension n.*
(1) *Let i and j be integers with $1 \leq i \leq n-1$ and $1 \leq j \leq i$. Assume that Condition $A_1(i)$ and Condition $B(i,j)$ in 2.7.2 are satisfied. Then for every integer s with $1 \leq s \leq n-i$*

$$\Delta_j(X,L) = \Delta_j(X_s, L_s) + \sum_{k=0}^{s-1} \dim \operatorname{Coker}(r_{j-1,k}).$$

(2) *Let i be an integer with $1 \leq i \leq n$. Assume that Condition $A_1(i)$, Condition $A_2(i)$, and Condition $B(i,i)$ in 2.7.2 are satisfied. Then*

$$\Delta_i(X,L) = \sum_{k=0}^{n-i} \dim \operatorname{Coker}(r_{i-1,k}).$$

Proof. (1) Assume that $1 \leq i \leq n-1$. By Proposition 2.3 we have

$$\Delta_j(X,L) = (-1)^{j-1}\sum_{k=0}^{j-1}\chi_{n-k}(X,L) + (n-j+1)(-1)^{j-1}\left(\sum_{k=0}^{j-1}(-1)^k h^k(\mathscr{O}_X)\right) + (-1)^j\left(\sum_{k=0}^{j-1}(-1)^k h^k(L)\right).$$

By the exact sequence

$$0 \to \mathscr{O}_{X_t} \to L_t \to L_{t+1} \to 0,$$

we get the following exact sequence

$$\begin{aligned} 0 &\to H^0(\mathscr{O}_{X_t}) \to H^0(L_t) \to H^0(L_{t+1}) \\ &\to H^1(\mathscr{O}_{X_t}) \to H^1(L_t) \to H^1(L_{t+1}) \\ &\to \cdots \\ &\to H^{j-1}(\mathscr{O}_{X_t}) \to H^{j-1}(L_t) \to H^{j-1}(L_{t+1}) \\ &\to \cdots . \end{aligned}$$

By this exact sequence, we have

$$\begin{aligned} &(-1)^{j-1}\sum_{k=0}^{j-1}(-1)^k h^k(\mathscr{O}_{X_t}) - (-1)^{j-1}\sum_{k=0}^{j-1}(-1)^k h^k(L_t) \\ &= (-1)^j\sum_{k=0}^{j-1}(-1)^k h^k(L_{t+1}) + \dim \operatorname{Coker}(r_{j-1,t}) \end{aligned}$$

for every integer t with $0 \leq t \leq n-i-1$. Furthermore we have $\chi_s(X_t, L_t) = \chi_{s-1}(X_{t+1}, L_{t+1})$.

By Condition $B(i,j)$ in 2.7.2, we have

$$\sum_{k=0}^{j-1}(-1)^k h^k(\mathscr{O}_X) = \sum_{k=0}^{j-1}(-1)^k h^k(\mathscr{O}_{X_1}) = \cdots = \sum_{k=0}^{j-1}(-1)^k h^k(\mathscr{O}_{X_{n-i}}).$$

Hence

$$\Delta_j(X,L) = (-1)^{j-1}\sum_{k=0}^{j-1}\chi_{n-k}(X,L) + (n-j+1)(-1)^{j-1}\left(\sum_{k=0}^{j-1}(-1)^k h^k(\mathscr{O}_X)\right) + (-1)^j\left(\sum_{k=0}^{j-1}(-1)^k h^k(L)\right)$$

$$= (-1)^{j-1}\sum_{k=0}^{j-1}\chi_{n-k-1}(X_1,L_1) + (n-j)(-1)^{j-1}\left(\sum_{k=0}^{j-1}(-1)^k h^k(\mathcal{O}_{X_1})\right)$$
$$+ (-1)^j\left(\sum_{k=0}^{j-1}(-1)^k h^k(L_1)\right) + \dim \operatorname{Coker}(r_{j-1,0})$$
$$\vdots$$
$$= (-1)^{j-1}\sum_{k=0}^{j-1}\chi_{i-k}(X_{n-i},L_{n-i}) + (i-j+1)(-1)^{j-1}\left(\sum_{k=0}^{j-1}(-1)^k h^k(\mathcal{O}_{X_{n-i}})\right)$$
$$+ (-1)^j\left(\sum_{k=0}^{j-1}(-1)^k h^k(L_{n-i})\right) + \sum_{k=0}^{n-i-1}\dim \operatorname{Coker}(r_{j-1,k}).$$

Namely

$$\begin{aligned}\varDelta_j(X,L) &= \varDelta_j(X_1,L_1) + \dim \operatorname{Coker}(r_{j-1,0})\\ &\ \vdots\\ &= \varDelta_j(X_{n-i},L_{n-i}) + \sum_{k=0}^{n-i-1}\dim \operatorname{Coker}(r_{j-1,k}).\end{aligned}$$

(2) If $i = n$, then by Proposition 2.4 we have

$$\varDelta_n(X,L) = h^n(\mathcal{O}_X) - h^n(L).$$

By Condition $A_2(n)$ in 2.7.2, there exists the following exact sequence.

$$0 \to \mathcal{O}_X \to L \to L_1 \to 0.$$

Hence we get the exact sequence

$$H^{n-1}(L) \to H^{n-1}(L_1) \to H^n(\mathcal{O}_X) \to H^n(L) \to 0,$$

and we have $h^n(\mathcal{O}_X) - h^n(L) = \dim \operatorname{Coker}(r_{n-1,0})$. Hence we get the assertion for $i = n$.

Assume that $1 \le i \le n-1$. Then by (1) above and Proposition 2.4, we get

$$\begin{aligned}\varDelta_i(X,L) &= \varDelta_i(X_{n-i},L_{n-i}) + \sum_{j=0}^{n-i-1}\dim \operatorname{Coker}(r_{i-1,j})\\ &= h^i(\mathcal{O}_{X_{n-i}}) - h^i(L_{n-i}) + \sum_{j=0}^{n-i-1}\dim \operatorname{Coker}(r_{i-1,j}).\end{aligned}$$

Here we use Condition $A_2(i)$ in 2.7.2. Then there is the following exact sequence:

$$0 \to \mathscr{O}_{X_{n-i}} \to L_{n-i} \to L_{n-i+1} \to 0.$$

Since $H^{i-1}(L_{n-i}) \to H^{i-1}(L_{n-i+1}) \to H^i(\mathscr{O}_{X_{n-i}}) \to H^i(L_{n-i}) \to 0$ is exact, we get $h^i(\mathscr{O}_{X_{n-i}}) - h^i(L_{n-i}) = \dim \operatorname{Coker}(r_{i-1,n-i})$. Hence

$$\Delta_i(X, L) = \sum_{j=0}^{n-i} \dim \operatorname{Coker}(r_{i-1,j}).$$

This completes the proof. □

Remark 2.8.1. Let (X, L) be a quasi-polarized variety of dimension n.
(1) Let i be an integer with $1 \leq i \leq n-1$. Assume that L has an $(n-i)$-ladder. We use notation in Notation 2.7.1. If $h^r(-L_s) = 0$ for every integers s and r with $0 \leq s \leq n-i-1$ and $0 \leq r \leq i$, we have $h^r(\mathscr{O}_X) = h^r(\mathscr{O}_{X_1}) = \cdots = h^r(\mathscr{O}_{X_{n-i}})$ for every integer r with $0 \leq r \leq i-1$. In particular, we get Condition $B(i,j)$ in 2.7.2 for every integer j with $1 \leq j \leq i$.

Hence, for example, if X is smooth and $\mathrm{Bs}|L| = \emptyset$, then, by the Kawamata-Viehweg vanishing theorem, Condition $B(i,j)$ in 2.7.2 holds for every integers i and j with $1 \leq i \leq n-1$ and $1 \leq j \leq i$.
(2) If L has an $(n-1)$-ladder, then Condition $B(1,1)$ in 2.7.2 always holds.

Corollary 2.9. *Let (X, L) be a quasi-polarized variety of dimension n.*
(1) *Let i and j be integers with $1 \leq i \leq n-1$ and $1 \leq j \leq i$. Assume that Condition $A_1(i)$ and Condition $B(i,j)$ in 2.7.2 are satisfied. Then*

$$\Delta_j(X, L) \geq \Delta_j(X_1, L_1) \geq \cdots \geq \Delta_j(X_{n-i}, L_{n-i}).$$

(2) *Let i be an integer with $1 \leq i \leq n$. Assume that Condition $A_1(i)$, Condition $A_2(i)$, and Condition $B(i,i)$ in 2.7.2 are satisfied. Then*

$$\Delta_i(X, L) \geq \Delta_i(X_1, L_1) \geq \cdots \geq \Delta_i(X_{n-i}, L_{n-i}) \geq 0.$$

Proposition 2.10. *Let (X, L) be a polarized manifold of dimension $n \geq 3$. Assume that there exists a polarized manifold (Y, A) such that $\pi : X \to Y$ is a one point blowing up and $L = \pi^*(A) - E$, where E is the reduced exceptional divisor of π. Then*

$$\Delta_1(X, L) \leq \Delta_1(Y, A)$$

and

$$\Delta_j(X, L) = \Delta_j(Y, A)$$

for every integer j with $2 \leq j \leq n$.

Proof. We consider the following exact sequence:

$$0 \to L \to \pi^*(A) \to \mathscr{O}_E \to 0.$$

Here we remark that $E \cong \boldsymbol{P}^{n-1}$. Then we get the following exact sequence:

$$\begin{aligned} 0 &\to H^0(L) \to H^0(\pi^*(A)) \to H^0(\mathscr{O}_E) \\ &\to H^1(L) \to H^1(\pi^*(A)) \to 0 \end{aligned} \tag{$\clubsuit$}$$

because $h^1(\mathscr{O}_E) = 0$.

(A) The case of $\varDelta_1(X, L)$.

Then since $h^0(A) = h^0(\pi^*(A)) \leq h^0(L) + h^0(\mathscr{O}_E) = h^0(L) + 1$ and $A^n = L^n + 1$, we get

$$\begin{aligned} \varDelta_1(X, L) &= n + L^n - h^0(L) \\ &\leq n + A^n - 1 - h^0(A) + 1 \\ &= n + A^n - h^0(A) \\ &= \varDelta_1(Y, A). \end{aligned}$$

(B) The case of $\varDelta_2(X, L)$.

Then by definition

$$\varDelta_2(X, L) = g_1(X, L) - \varDelta_1(X, L) + (n-1)h^1(\mathscr{O}_X) - h^1(L).$$

Here we remark that $g_1(X, L) = g_1(Y, A)$ by Remark 1.4.1(1) and $h^1(\mathscr{O}_X) = h^1(\mathscr{O}_Y)$. By the exact sequence ($\clubsuit$), we get

$$h^0(L) - h^0(A) + h^0(\mathscr{O}_E) - h^1(L) + h^1(\pi^*(A)) = 0.$$

Hence $h^0(L) - h^1(L) = h^0(A) - h^1(\pi^*(A)) - 1$. Therefore

$$\begin{aligned} \varDelta_1(X, L) + h^1(L) &= n + L^n - h^0(L) + h^1(L) \\ &= n + A^n - h^0(A) + h^1(\pi^*(A)) \\ &= \varDelta_1(Y, A) + h^1(\pi^*(A)). \end{aligned}$$

Since π is a one point blowing up, $R^i\pi_*\mathscr{O}_X = 0$ for every integer i with $i \geq 1$. Hence $h^1(A) = h^1(\pi^*(A))$. Therefore $\varDelta_1(X, L) + h^1(L) = \varDelta_1(Y, A) + h^1(A)$ and

$$\begin{aligned} \varDelta_2(X, L) &= g_1(X, L) - \varDelta_1(X, L) + (n-1)h^1(\mathscr{O}_X) - h^1(L) \\ &= g_1(Y, A) - \varDelta_1(Y, A) + (n-1)h^1(\mathscr{O}_Y) - h^1(A) \\ &= \varDelta_2(Y, A). \end{aligned}$$

(C) The case of $\Delta_j(X,L)$ for $j \geq 3$.

We remark that $g_i(X,L) = g_i(Y,A)$ by Remark 1.4.1(1) and $h^i(\mathscr{O}_X) = h^i(\mathscr{O}_Y)$ for every integer i with $i \geq 1$. Since $R^i\pi_*(\mathscr{O}_X) = 0$ and $h^i(\mathscr{O}_E) = 0$ for every integer i with $i \geq 1$, we get $h^i(L) = h^i(\pi^*(A)) = h^i(A)$ for every integer i with $i \geq 1$. Hence we get the assertion by using induction. □

By using this we can prove the following:

COROLLARY 2.11. *Let (X,L) be a polarized manifold of dimension $n \geq 3$, and let (X',L') be a reduction of (X,L). Then*

$$\Delta_1(X,L) \leq \Delta_1(X',L')$$

and

$$\Delta_j(X,L) = \Delta_j(X',L')$$

for every integer j with $2 \leq j \leq n$.

Next we calculate the i-th Δ-genus of some examples of polarized manifolds for an integer i with $i \geq 2$.

EXAMPLE 2.12.

(1) If (X,L) is $(\boldsymbol{P}^n, \mathscr{O}_{\boldsymbol{P}^n}(1))$ or $(\boldsymbol{Q}^n, \mathscr{O}_{\boldsymbol{Q}^n}(1))$, then L is very ample, $h^i(\mathscr{O}_X) = 0$ and $h^i(L) = 0$ for $1 \leq i$, and $g_1(X,L) = 0$ and $\Delta_1(X,L) = 0$. By Theorem 1.3(2), we have $g_i(X,L) = 0$ for every integer i with $i \geq 2$ (see also [**Fk**, Example 2.10(1), (2)]). Hence $\Delta_i(X,L) = 0$ for $i \geq 2$.

(2) Assume that (X,L) is a Del Pezzo manifold, that is, $K_X + (n-1)L \sim \mathscr{O}_X$. Then $h^i(L) = 0$ and $h^i(\mathscr{O}_X) = h^{n-i}(K_X) = 0$ for $i \geq 1$. In this case, $\Delta_1(X,L) = 1$ and $g_1(X,L) = 1$. By Theorem 1.3(2), we have $g_i(X,L) = 0$ for every integer i with $i \geq 2$. By the definition of the i-th Δ-genus, we have $\Delta_i(X,L) = 0$ for $i \geq 2$.

(3.1) Assume that (X,L) is $(\boldsymbol{P}^4, \mathscr{O}_{\boldsymbol{P}^4}(2))$ (resp. $(\boldsymbol{P}^3, \mathscr{O}_{\boldsymbol{P}^3}(3))$ and $(\boldsymbol{Q}^3, \mathscr{O}_{\boldsymbol{Q}^3}(2))$). Here we note that $h^i(\mathscr{O}_X) = 0$ and $h^i(L) = 0$ for every integer i with $i \geq 1$. Since $g_1(X,L) = 5$ (resp. 10, 5) and $\Delta_1(X,L) = 5$ (resp. 10, 5), we get $\Delta_2(X,L) = 0$. By the definition of the i-th Δ-genus, $\Delta_i(X,L) = 0$ for every integer i with $i \geq 3$ because $g_i(X,L) = 0$ for every integer i with $i \geq 2$ by Theorem 1.3(2) (see also [**Fk**, Example 2.10, (4), (5), (6)]).

(3.2) Assume that (X,L) is a $\boldsymbol{P}^2$-bundle over a smooth curve C with $L|_F \cong \mathscr{O}_{\boldsymbol{P}^2}(2)$ for every fiber F. Let $f: X \to C$ be its fibration. Then $R^if_*(L) = 0$ for any $i > 0$ because $L|_F \cong \mathscr{O}_{\boldsymbol{P}^2}(2)$ and $F = \boldsymbol{P}^2$. Therefore $h^i(L) = h^i(f_*(L))$. In particular $h^i(L) = 0$ for every integer i with $i \geq 2 > \dim C$. By the Hirzebruch-Riemann-Roch theorem ([**Hi**, Chapter IV]),

$$\mathscr{X}(L) = \frac{1}{6}(L)^3 - \frac{1}{4}K_X(L)^2 + \frac{1}{12}\big((K_X)^2 + c_2(X)\big)L + \chi(\mathscr{O}_X).$$

Since $\mathscr{X}(L) = h^0(L) - h^1(L)$ and $\chi(\mathscr{O}_X) = h^0(\mathscr{O}_X) - h^1(\mathscr{O}_X)$, we have

$$h^0(L) - h^1(L) = \frac{1}{6}(L)^3 - \frac{1}{4}K_X(L)^2 + \frac{1}{12}\big((K_X)^2 + c_2(X)\big)L + 1 - h^1(\mathscr{O}_X). \qquad (\dagger)$$

By the definition of the second Δ-genus and ($\dagger$),

$$\begin{aligned}
\Delta_2(X,L) &= g_1(X,L) - \Delta_1(X,L) + 2h^1(\mathscr{O}_X) - h^1(L)\\
&= 1 + \frac{1}{2}(K_X + 2L)(L)^2 - \big(3 + (L)^3 - h^0(L)\big) + 2h^1(\mathscr{O}_X) - h^1(L)\\
&= -2 + \frac{1}{2}K_X(L)^2 + 2h^1(\mathscr{O}_X) + h^0(L) - h^1(L)\\
&= -1 + \frac{1}{6}(L)^3 + \frac{1}{4}K_X(L)^2 + \frac{1}{12}\big((K_X)^2 + c_2(X)\big)L + h^1(\mathscr{O}_X)\\
&= -1 + h^1(\mathscr{O}_X) + \frac{1}{12}\big((K_X + 2L)(K_X + L) + c_2(X)\big)L\\
&= g_2(X,L).
\end{aligned}$$

On the other hand $g_2(X,L) = 0$ by [**Fk**, Example 2.10(11)]. Hence $\Delta_2(X,L) = 0$. By the definition of the i-th Δ-genus, we get $\Delta_3(X,L) = 0$ because $h^2(\mathscr{O}_X) = 0$ and $h^2(L) = 0$.
(4) Let (X,L) be a Mukai manifold of dimension n, that is, $K_X + (n-2)L = \mathscr{O}_X$. Then $h^0(K_X + (n-1)L) = h^0(L)$, $h^0(K_X + (n-2)L) = 1$, and $h^0(K_X + mL) = 0$ for every integer m with $1 \le m \le n-3$. Furthermore $h^i(\mathscr{O}_X) = 0$ and $h^i(L) = 0$ for $i \ge 1$. We note that by [**Fk**, Example 2.10(7)]

$$\begin{aligned}
g_1(X,L) &= 1 + \frac{1}{2}L^n,\\
g_2(X,L) &= h^0(K_X + (n-2)L) = 1,
\end{aligned}$$

and

$$g_i(X,L) = 0 \text{ for } i \ge 3.$$

By the definition of the i-th Δ-genus, we get

$$\begin{aligned}
\Delta_2(X,L) &= g_1(X,L) - \Delta_1(X,L) + (n-1)h^1(\mathscr{O}_X) - h^1(L)\\
&= 1 - n - \frac{1}{2}L^n + h^0(L),\\
\Delta_3(X,L) &= g_2(X,L) - \Delta_2(X,L)\\
&= n + \frac{1}{2}L^n - h^0(L),
\end{aligned}$$

and

$$\Delta_j(X,L) = g_{j-1}(X,L) - \Delta_{j-1}(X,L) \qquad (\sharp)$$

for every integer j with $j \geq 4$. On the other hand, $h^0(L) = n + \frac{1}{2}L^n$ (for example, see [**AGV**, Corollary 2.1.14(ii)]). So we obtain $\Delta_2(X,L) = 1$ and $\Delta_3(X,L) = 0$. Since $g_i(X,L) = 0$ for every integer i with $i \geq 3$, by ($\sharp$) we get $\Delta_i(X,L) = 0$ for every integer i with $i \geq 4$.

Next we prove the following.

LEMMA 2.12.1. *Let (X,L) be a scroll (resp. a quadric fibration, a Del Pezzo fibration) over a normal variety Y. Let $n := \dim X$ and $m := \dim Y$ with $n \geq 3$ and $n > m \geq 1$. Then $\Delta_i(X,L) = 0$ for every integer i with $i \geq m+1$ (resp. $m+1$, $m+2$).*

PROOF. Let $\pi : X \to Y$ be its morphism. In this case by Lemma 1.6 we get

$$h^i(\mathscr{O}_X) = 0 \text{ and } h^i(L) = 0 \text{ for } i \geq m+1. \tag{2.12.1.1}$$

By [**Fk**, Example 2.10], we get

$$g_i(X,L) = 0 \text{ for } i \geq m+1 \text{ (resp. } m+1,\ m+2). \tag{2.12.1.2}$$

By the definition of the i-th Δ-genus, we have

$$\Delta_i(X,L) = g_i(X,L) - \Delta_{i+1}(X,L) + (n-i)h^i(\mathscr{O}_X) - h^i(L) \tag{2.12.1.3}$$

for $1 \leq i \leq n-1$. Since by Proposition 2.4, we have

$$\Delta_n(X,L) = h^n(\mathscr{O}_X) - h^n(L) = 0. \tag{2.12.1.4}$$

By (2.12.1.1), (2.12.1.2), (2.12.1.3), and (2.12.1.4), we have $\Delta_i(X,L) = 0$ for every integer i with $i \geq m+1$ (resp. $m+1$, $m+2$). This completes the proof of Lemma 2.12.1. $\square$

(5) Let (X,L) be a scroll over a smooth curve C, that is, there exists a surjective morphism $f : X \to C$ such that $K_X + nL = f^*(A)$ for an ample line bundle A on C. If $i \geq 2$, then $\Delta_i(X,L) = 0$ by Lemma 2.12.1.

(6) Let (X,L) be a scroll over a normal surface S, that is, there exists a surjective morphism $f : X \to S$ such that $K_X + (n-1)L = f^*(A)$ for an ample line bundle A on S.

If $i \geq 3$, then $\Delta_i(X,L) = 0$ by Lemma 2.12.1.

Next we calculate $\Delta_2(X,L)$. Here we note that $g_2(X,L) = h^2(\mathscr{O}_X)$ by [**Fk**, Example 2.10(8)]. Since

$$\Delta_2(X,L) = g_2(X,L) - \Delta_3(X,L) + (n-2)h^2(\mathscr{O}_X) - h^2(L),$$

we get

$$\Delta_2(X,L) = (n-1)h^2(\mathscr{O}_X) - h^2(L).$$

(7) Let (X, L) be a scroll over a normal projective variety Y of dimension 3, that is, there exists a surjective morphism $f : X \to Y$ such that $K_X + (n-2)L = f^*(A)$ for an ample line bundle A on Y.

If $i \geq 4$, then $\Delta_i(X, L) = 0$ by Lemma 2.12.1.

Next we calculate $\Delta_2(X, L)$ and $\Delta_3(X, L)$. Here we note that by [**Fk**, Example 2.10(8)]

(A) $g_3(X, L) = h^3(\mathcal{O}_X)$,
(B) $g_2(X, L) = h^0(K_X + (n-2)L) + h^2(\mathcal{O}_X) - h^3(\mathcal{O}_X)$.

Since

$$\Delta_3(X, L) = g_3(X, L) - \Delta_4(X, L) + (n-3)h^3(\mathcal{O}_X) - h^3(L),$$

we get

$$\Delta_3(X, L) = (n-2)h^3(\mathcal{O}_X) - h^3(L).$$

Since

$$\Delta_2(X, L) = g_2(X, L) - \Delta_3(X, L) + (n-2)h^2(\mathcal{O}_X) - h^2(L),$$

we get

$$\Delta_2(X, L) = h^0\big(K_X + (n-2)L\big) - h^2(L) + h^3(L) + (n-1)\big(h^2(\mathcal{O}_X) - h^3(\mathcal{O}_X)\big).$$

(8) Let (X, L) be a quadric fibration over a smooth curve Y, that is, there exists a surjective morphism $f : X \to Y$ such that $K_X + (n-1)L = f^*(A)$ for an ample line bundle A on Y.

By Lemma 2.12.1 we get $\Delta_i(X, L) = 0$ for every integer i with $i \geq 2$.

(9) Let (X, L) be a quadric fibration over a normal surface Y, that is, there exists a surjective morphism $f : X \to Y$ such that $K_X + (n-2)L = f^*(A)$ for an ample line bundle A on Y.

If $i \geq 3$, then $\Delta_i(X, L) = 0$ by Lemma 2.12.1.

Next we calculate $\Delta_2(X, L)$. Here we note that by [**Fk**, Example 2.10(9)] $g_2(X, L) = h^0(K_X + (n-2)L) + h^2(\mathcal{O}_X)$. Since

$$\Delta_2(X, L) = g_2(X, L) - \Delta_3(X, L) + (n-2)h^2(\mathcal{O}_X) - h^2(L),$$

we get

$$\Delta_2(X, L) = h^0\big(K_X + (n-2)L\big) + (n-1)h^2(\mathcal{O}_X) - h^2(L).$$

(10) Let (X, L) be a Del Pezzo fibration over a smooth curve C, that is, there exists a surjective morphism $f : X \to C$ such that $K_X + (n-2)L = f^*(A)$ for an ample line

bundle A on C.

If $i \geq 3$, then $\Delta_i(X, L) = 0$ by Lemma 2.12.1.

Next we calculate $\Delta_2(X, L)$. Here we note that by [**Fk**, Example 2.10(10)] $g_2(X, L) = h^0(K_X + (n-2)L)$. Hence

$$\begin{aligned}\Delta_2(X, L) &= g_2(X, L) - \Delta_3(X, L) + (n-2)h^2(\mathcal{O}_X) - h^2(L) \\ &= h^0\big(K_X + (n-2)L\big) + (n-2)h^2(\mathcal{O}_X) - h^2(L).\end{aligned}$$

Since $h^i(L) = 0$ and $h^i(\mathcal{O}_X) = 0$ for every integer i with $i \geq 2$ by Lemma 1.6, we get

$$\begin{aligned}\Delta_2(X, L) &= h^0\big(K_X + (n-2)L\big) + (n-2)h^2(\mathcal{O}_X) - h^2(L) \\ &= h^0\big(K_X + (n-2)L\big).\end{aligned}$$

3. The case where X is smooth and $\mathrm{Bs}|L| = \varnothing$.

In this section we mainly consider the case where X is smooth and $\mathrm{Bs}|L| = \varnothing$. First we fix the notation.

Notation 3.0. Let (X, L) be a quasi-polarized manifold of dimension $n \geq 3$ and $\mathrm{Bs}|L| = \varnothing$.

(1) We put $X_0 := X$ and $L_0 := L$. Let $X_j \in |L_{j-1}|$ be a smooth member of $|L_{j-1}|$ and $L_j = L_{j-1}|_{X_j}$ for every integer j with $1 \leq j \leq n-1$.
(2) Let $r_{j,k} : H^j(X_k, L_k) \to H^j(X_{k+1}, L_{k+1})$ be the natural map for every integers j and k with $0 \leq j \leq n-k-1$ and $0 \leq k \leq n-2$.

First we state some results about the i-th sectional geometric genus which are used in this section.

Theorem 3.1. *Let (X, L) be a quasi-polarized manifold of dimension n and let i be an integer with $0 \leq i \leq n$. Assume that L is base point free. Then the following hold.*
(1) *Here we use Notation* 3.0. *For every integer k with $0 \leq k \leq n-i-1$,*

$$g_i(X_k, L_k) = g_i(X_{k+1}, L_{k+1}).$$

In particular, by Remark 1.2.1(2) *we get*

$$g_i(X, L) = g_i(X_1, L_1) = \cdots = g_i(X_{n-i}, L_{n-i}) = h^i(\mathcal{O}_{X_{n-i}}).$$

(2) *$g_i(X, L) \geq h^i(\mathcal{O}_X)$. (In particular $g_i(X, L) \geq 0$.) Furthermore if $i = 2$, then the following are equivalent*:

(a) *$g_2(X, L) = h^2(\mathcal{O}_X)$.*
(b) *$h^0(K_X + (n-2)L) = 0$.*
(c) *$\kappa(K_X + (n-2)L) = -\infty$.*

(d) $K_{X'}+(n-2)L'$ *is not nef, where* (X', L') *is a reduction of* (X, L).
(e) (X, L) *is one of the types from* (1) *to* (7.4) *in Theorem* 1.7.

Proof. (1) See in [**Fk**, Theorem 2.4].
(2) See in [**Fk**, Theorem 3.1 and Corollary 3.5]. □

(3.A) Some basic results.

Here we study some basic properties of the i-th Δ-genus. First we consider a lower bound for $\Delta_i(X, L)$. By Theorem 2.8(2), Corollary 2.9(2), and Remark 2.8.1, we get the following two corollaries.

Corollary 3.2. *Let* (X, L) *be a quasi-polarized manifold of dimension* n. *Assume that* $\mathrm{Bs}|L| = \varnothing$. *Then*

$$\Delta_i(X, L) = \sum_{k=0}^{n-i} \dim \mathrm{Coker}(r_{i-1,k})$$

for every integer i *with* $1 \leq i \leq n$.

Corollary 3.3. *Let* (X, L) *be a quasi-polarized manifold of dimension* n. *Assume that* $\mathrm{Bs}|L| = \varnothing$. *Then*

$$\Delta_i(X, L) \geq \Delta_i(X_1, L_1) \geq \cdots \geq \Delta_i(X_{n-i}, L_{n-i}) \geq 0$$

for every integer i *with* $1 \leq i \leq n$.

Next result is useful when we classify (X, L) by the value of the i-th Δ-genus.

Theorem 3.4. *Let* (X, L) *be a quasi-polarized manifold of dimension* n, *and let* i *be an integer with* $1 \leq i \leq n$. *Assume that* $\mathrm{Bs}|L| = \varnothing$ *and* $h^0(K_{X_{n-i}} - L_{n-i}) > 0$. *Then*

$$\Delta_i(X, L) \geq h^0(L) - (n - i + 1).$$

Proof. By Corollary 3.3, we get

$$\Delta_i(X, L) \geq \Delta_i(X_1, L_1) \geq \cdots \geq \Delta_i(X_{n-i}, L_{n-i}) \geq 0.$$

By Proposition 2.4, we have

$$\begin{aligned} \Delta_i(X_{n-i}, L_{n-i}) &= h^i(\mathscr{O}_{X_{n-i}}) - h^i(L_{n-i}) \\ &= h^0(K_{X_{n-i}}) - h^0(K_{X_{n-i}} - L_{n-i}). \end{aligned}$$

Since $h^0(K_{X_{n-i}} - L_{n-i}) > 0$, we have $h^0(K_{X_{n-i}}) \geq h^0(K_{X_{n-i}} - L_{n-i}) + h^0(L_{n-i}) - 1$ by Lemma 1.8. Hence

$$
\begin{aligned}
\Delta_i(X_{n-i}, L_{n-i}) &\geq h^0(L_{n-i}) - 1 \\
&\geq h^0(L_{n-i-1}) - 2 \\
&\ \ \vdots \\
&\geq h^0(L) - (n-i+1).
\end{aligned}
$$

This completes the proof of Theorem 3.4. □

COROLLARY 3.5. *Let (X, L) be a quasi-polarized manifold of dimension n, and let i be an integer with $1 \leq i \leq n$. Assume that* $\mathrm{Bs}|L| = \varnothing$ *and $h^0(K_X + (n-i-1)L) > 0$. Then*

$$\Delta_i(X, L) \geq h^0(L) - (n-i+1).$$

PROOF. Since $h^0(K_X+(n-i-1)L) > 0$, by using Lemma 1.9 we can get $h^0(K_{X_{n-i}} - L_{n-i}) > 0$. Hence by Theorem 3.4 we get the assertion. □

COROLLARY 3.6. *Let (X, L) be a quasi-polarized manifold of dimension n, and let i be an integer with $1 \leq i \leq n$. Assume that* $\mathrm{Bs}|L| = \varnothing$ *and $g_i(X, L) > \Delta_i(X, L)$. Then*

$$\Delta_i(X, L) \geq h^0(L) - (n-i+1).$$

PROOF. If $h^0(K_{X_{n-i}} - L_{n-i}) = 0$, then by Proposition 2.4 and Corollary 3.3, we get

$$
\begin{aligned}
\Delta_i(X, L) &\geq \Delta_i(X_{n-i}, L_{n-i}) \\
&= h^i(\mathcal{O}_{X_{n-i}}) - h^i(L_{n-i}) \\
&= h^i(\mathcal{O}_{X_{n-i}}) \\
&= g_i(X, L),
\end{aligned}
$$

and this contradicts the assumption. Therefore we get $h^0(K_{X_{n-i}} - L_{n-i}) > 0$, and by Theorem 3.4 we get the assertion. □

Next we consider some relations between the i-th sectional geometric genus and the i-th Δ-genus.

PROPOSITION 3.7. *Let (X, L) be a quasi-polarized manifold of dimension n, and let i be an integer with $i \geq 1$. Assume that* $\mathrm{Bs}|L| = \varnothing$. *If $\Delta_i(X, L) \leq i - 1$, then $g_i(X, L) \leq \Delta_i(X, L)$.*

PROOF. If $h^0(K_{X_{n-i}} - L_{n-i}) \neq 0$, then by Theorem 3.4 we get

$$
\begin{aligned}
\Delta_i(X, L) &\geq h^0(L) - (n-i+1) \\
&\geq i.
\end{aligned}
$$

But this contradicts the assumption. Hence $h^0(K_{X_{n-i}} - L_{n-i}) = 0$ and

$$\begin{aligned}\Delta_i(X,L) &\geq \Delta_i(X_{n-i}, L_{n-i}) \\ &= h^i(\mathscr{O}_{X_{n-i}}) - h^i(L_{n-i}) \\ &= h^i(\mathscr{O}_{X_{n-i}}) \\ &= g_i(X,L).\end{aligned}$$

This completes the proof. □

Corollary 3.8. *Let (X,L) be a quasi-polarized manifold of dimension n, and let i be an integer with $i \geq 1$. Assume that* $\mathrm{Bs}|L| = \varnothing$. *If $\Delta_i(X,L) \leq i-1$ and $g_i(X,L) \geq \Delta_i(X,L)$, then $g_i(X,L) = \Delta_i(X,L)$.*

Remark 3.8.1. By Proposition 3.7, we find that a classification of (X,L) with $\Delta_i(X,L) = k$ for $k \leq i-1$ can be obtained by a classification of (X,L) with $g_i(X,L) \leq k$.

Proposition 3.9. *Let (X,L) be a quasi-polarized manifold of dimension n, and let i be an integer with $1 \leq i \leq n-1$. Assume that* $\mathrm{Bs}|L| = \varnothing$. *If $\Delta_i(X,L) \leq i-1$, then $h^0(K_X + (n-i)L) \leq \Delta_i(X,L)$ and $g_{i+1}(X,L) = \Delta_{i+1}(X,L) = 0$.*

Proof. By assumption, we get $g_i(X,L) \leq \Delta_i(X,L)$ by Proposition 3.7. So by Theorem 3.1 (1) and Remark 1.3.1, we have

$$\begin{aligned}\Delta_i(X,L) \geq g_i(X,L) &= g_i(X_{n-i-1}, L_{n-i-1}) \\ &= h^0(K_{X_{n-i-1}} + L_{n-i-1}) - h^0(K_{X_{n-i-1}}) + h^i(\mathscr{O}_{X_{n-i-1}}) \\ &\geq h^0(K_{X_{n-i-1}} + L_{n-i-1}) - h^0(K_{X_{n-i-1}}).\end{aligned}$$

If $h^0(K_{X_{n-i-1}}) \neq 0$, then by Lemma 1.8

$$\begin{aligned}\Delta_i(X,L) &\geq h^0(K_{X_{n-i-1}} + L_{n-i-1}) - h^0(K_{X_{n-i-1}}) \\ &\geq h^0(L_{n-i-1}) - 1 \\ &\geq i+1 \geq \Delta_i(X,L) + 2,\end{aligned}$$

and this is impossible. Therefore $h^0(K_{X_{n-i-1}}) = 0$ and $h^0(K_{X_{n-i-1}} + L_{n-i-1}) \leq \Delta_i(X,L)$. By using Lemma 1.9 we can get $h^0(K_{X_k} + (n-i-1-k)L_k) = 0$ for every integer k with $0 \leq k \leq n-i-2$.

By using the following exact sequence

$$\begin{aligned}0 &\to H^0\big(K_{X_j} + (n-i-1-j)L_j\big) \to H^0\big(K_{X_j} + (n-i-j)L_j\big) \\ &\to H^0\big(K_{X_{j+1}} + (n-i-1-j)L_{j+1}\big) \to 0\end{aligned}$$

for every integer j with $0 \leq j \leq n-i-2$, we get $H^0(K_{X_j} + (n-i-j)L_j) = H^0(K_{X_{j+1}} +$

$(n-i-1-j)L_{j+1})$. Hence

$$\begin{aligned} h^0\bigl(K_X+(n-i)L\bigr) &= h^0\bigl(K_{X_1}+(n-i-1)L_1\bigr) \\ &= \cdots \\ &= h^0(K_{X_{n-i-1}}+L_{n-i-1}) \\ &\leq \Delta_i(X,L). \end{aligned}$$

Since $h^0(K_{X_{n-i-1}})=0$, by the Serre duality we get $h^{i+1}(\mathcal{O}_{X_{n-i-1}})=0$. Therefore

$$h^{i+1}(\mathcal{O}_X)=h^{i+1}(\mathcal{O}_{X_1})=\cdots=h^{i+1}(\mathcal{O}_{X_{n-i-2}})\leq h^{i+1}(\mathcal{O}_{X_{n-i-1}})=0.$$

Hence $\dim \operatorname{Coker}(r_{i,k})=0$ for every integer k with $0\leq k\leq n-i-1$. By Corollary 3.2, we get

$$\Delta_{i+1}(X,L)=\Delta_{i+1}(X_1,L_1)=\cdots=\Delta_{i+1}(X_{n-i-1},L_{n-i-1})=0.$$

Furthermore $g_{i+1}(X,L)=h^{i+1}(\mathcal{O}_{X_{n-i-1}})=0$ by Theorem 3.1(1). This completes the proof. $\square$

As a corollary of Proposition 3.9, we get a relation between $\Delta_i(X,L)$ and $\Delta_{i+1}(X,L)$.

Corollary 3.10. *Let (X,L) be a quasi-polarized manifold of dimension n, and let i be an integer with $1\leq i\leq n$. Assume that* $\mathrm{Bs}|L|=\varnothing$. *If $\Delta_i(X,L)=0$, then $\Delta_{i+1}(X,L)=0$.*

By using Corollary 3.10, we obtain the following theorem.

Theorem 3.11. *Let (X,L) be a quasi-polarized manifold of dimension n, and let i be an integer with $1\leq i\leq n-1$. Assume that* $\mathrm{Bs}|L|=\varnothing$. *If $g_i(X,L)-h^i(\mathcal{O}_X)\leq i$, then $\Delta_k(X,L)=0$ for every integer k with $k\geq i+1$.*

Proof. By assumption, the Lefschetz theorem, Remark 1.3.1, and Theorem 3.1 (1), we have

$$\begin{aligned} i &\geq g_i(X,L)-h^i(\mathcal{O}_X) \\ &= g_i(X_{n-i-1},L_{n-i-1})-h^i(\mathcal{O}_{X_{n-i-1}}) \\ &= h^0(K_{X_{n-i-1}}+L_{n-i-1})-h^0(K_{X_{n-i-1}}). \end{aligned}$$

If $h^0(K_{X_{n-i-1}})\neq 0$, then by Lemma 1.8

$$\begin{aligned} &h^0(K_{X_{n-i-1}}+L_{n-i-1})-h^0(K_{X_{n-i-1}}) \\ &\geq h^0(L_{n-i-1})-1 \\ &\geq i+1. \end{aligned}$$

But this is impossible. Hence $h^0(K_{X_{n-i-1}}) = 0$. By the same argument as in the proof of Proposition 3.9, we get $\Delta_{i+1}(X, L) = 0$. By Corollary 3.10 we have $\Delta_k(X, L) = 0$ for every integer k with $k \geq i + 1$. This completes the proof. □

Next we assume that (X, L) is a polarized manifold. Next result is useful in order to classify polarized manifolds by using the i-th Δ-genus.

Proposition 3.12. *Let (X, L) be a polarized manifold of dimension n, and let i be an integer with $1 \leq i \leq n$. Assume that* $\mathrm{Bs}|L| = \emptyset$ *and $\Delta_i(X, L) = i$. Then either $g_i(X, L) \leq i$ or there exists a covering $\pi : X \to \boldsymbol{P}^n$ of degree L^n such that $h^0(L) = n+1$ and $\Delta_i(X, L) = \cdots = \Delta_i(X_{n-i}, L_{n-i})$.*

Proof. In this case by Proposition 2.4, Corollary 3.3, and the Serre duality, we have

$$
\begin{aligned}
i = \Delta_i(X, L) &\geq \Delta_i(X_1, L_1) \\
&\ \ \vdots \\
&\geq \Delta_i(X_{n-i}, L_{n-i}) \\
&= h^i(\mathcal{O}_{X_{n-i}}) - h^i(L_{n-i}) \\
&= h^0(K_{X_{n-i}}) - h^0(K_{X_{n-i}} - L_{n-i}).
\end{aligned}
$$

If $h^0(K_{X_{n-i}} - L_{n-i}) = 0$, then $i = \Delta_i(X, L) \geq g_i(X, L)$ by the same argument as in the proof of Corollary 3.6.

If $h^0(K_{X_{n-i}} - L_{n-i}) \neq 0$, then by Lemma 1.8

$$
\begin{aligned}
h^0(K_{X_{n-i}}) - h^0(K_{X_{n-i}} - L_{n-i}) &\geq h^0(L_{n-i}) - 1 && (\spadesuit) \\
&\geq h^0(L_{n-i-1}) - 2 \\
&\ \ \vdots \\
&\geq h^0(L) - (n - i + 1) \\
&\geq n + 1 - n + i - 1 \\
&= i.
\end{aligned}
$$

Hence $\Delta_i(X_j, L_j) = \Delta_i(X_{j+1}, L_{j+1}) = i$ and $h^0(L_j) = h^0(L_{j+1}) + 1$ for $j = 0, \cdots, n - i - 1$. Furthermore $h^0(L) = n + 1$ by ($\spadesuit$). Since $\mathrm{Bs}|L| = \emptyset$, there exists a morphism $\Phi_{|L|} : X \to \boldsymbol{P}^n$ such that $\Phi_{|L|}$ is finite of degree L^n. This completes the proof. □

(3.B) The case where $\Delta_i(X, L) = 0$.
Here we study (X, L) with $\Delta_i(X, L) = 0$.

Theorem 3.13. *Let (X, L) be a quasi-polarized manifold of dimension n, and let i be an integer with $1 \leq i \leq n$. Assume that* $\mathrm{Bs}|L| = \emptyset$*. Then $\Delta_i(X, L) = 0$ if and only if $g_i(X, L) = 0$.*

PROOF. Assume that $g_i(X,L)=0$. Then $h^i(\mathscr{O}_{X_{n-i}})=0$. Therefore $h^i(\mathscr{O}_X)=h^i(\mathscr{O}_{X_1})=\cdots=h^i(\mathscr{O}_{X_{n-i-1}})\le h^i(\mathscr{O}_{X_{n-i}})=0$. Hence $H^{i-1}(L_j)\to H^{i-1}(L_{j+1})$ is surjective for every integer j with $0\le j\le n-i$. Namely $\dim\operatorname{Coker}(r_{i-1,j})=0$ for every integer j with $0\le j\le n-i$. Therefore by Corollary 3.2,

$$\Delta_i(X,L)=\sum_{k=0}^{n-i}\dim\operatorname{Coker}(r_{i-1,k})=0.$$

Assume that $\Delta_i(X,L)=0$. Then $\dim\operatorname{Coker}(r_{i-1,k})=0$ for every integer k with $0\le k\le n-i$, and $\Delta_i(X,L)=\Delta_i(X_1,L_1)=\cdots=\Delta_i(X_{n-i},L_{n-i})$. We consider the following exact sequence

$$H^{i-1}(L_{n-i})\to H^{i-1}(L_{n-i+1})\to H^i(\mathscr{O}_{X_{n-i}})\to H^i(L_{n-i})\to 0.$$

Since $H^{i-1}(L_{n-i})\to H^{i-1}(L_{n-i+1})$ is surjective, we obtain $h^i(\mathscr{O}_{X_{n-i}})=h^i(L_{n-i})$.

If $h^i(\mathscr{O}_{X_{n-i}})\neq 0$, then $h^i(L_{n-i})\neq 0$ and by Lemma 1.8 and the Serre duality, we get

$$\begin{aligned}h^i(\mathscr{O}_{X_{n-i}})&=h^0(K_{X_{n-i}})\\&\ge h^0(K_{X_{n-i}}-L_{n-i})+h^0(L_{n-i})-1\\&=h^i(L_{n-i})+h^0(L_{n-i})-1\\&\ge h^i(L_{n-i})+i\\&>h^i(L_{n-i}).\end{aligned}$$

But this is a contradiction. Hence $h^i(\mathscr{O}_{X_{n-i}})=0$ and by Theorem 3.1(1) we get

$$g_i(X,L)=g_i(X_{n-i},L_{n-i})=h^i(\mathscr{O}_{X_{n-i}})=0.$$

This completes the proof of Theorem 3.13. □

REMARK 3.13.1. If $n\ge 3$, then by Theorem 3.1(2) and Theorem 3.13, we get a classification of polarized manifolds (X,L) with $\Delta_2(X,L)=0$ and $\operatorname{Bs}|L|=\varnothing$. In particular, if $\Delta_2(X,L)=0$ and $\operatorname{Bs}|L|=\varnothing$, then (X,L) is one of the types from (1) to (7.4) in Theorem 1.7. (Here we remark that if (X,L) is a scroll over a smooth surface, then $h^2(\mathscr{O}_X)=0$.)

COROLLARY 3.14. *Let (X,L) be a quasi-polarized manifold of dimension n, and let i be an integer with $1\le i\le n-1$. Assume that $\operatorname{Bs}|L|=\varnothing$. If $g_i(X,L)-h^i(\mathscr{O}_X)\le i$, then $g_k(X,L)=0$ for every integer k with $k\ge i+1$.*

PROOF. By Theorem 3.11 and Theorem 3.13, we get the assertion. □

Next result is a vanishing theorem of cohomology of tL. This result is analogous to [**Fj3**, (3.5) Theorem 3].

Theorem 3.15. *Let (X, L) be a quasi-polarized manifold of dimension n, and let i be an integer with $1 \leq i \leq n-1$. Assume that* $\mathrm{Bs}|L| = \varnothing$ *and $\Delta_i(X, L) = 0$. Then $h^k(tL) = 0$ for every integers t and k with $t \geq 0$ and $i \leq k \leq n$.*

Proof. (A) Assume that $t = 0$. By $\Delta_i(X, L) = 0$, we have $g_i(X, L) = 0$ and $h^i(\mathcal{O}_X) = 0$ by Theorem 3.1(2) and Theorem 3.13. Furthermore by Theorem 3.11 we have $\Delta_k(X, L) = 0$ for every integer k with $k \geq i+1$. Hence by Theorem 3.1(2) and Theorem 3.13, $g_k(X, L) = 0$ and $h^k(\mathcal{O}_X) = 0$ for every integer k with $k \geq i+1$.

Hence $h^k(\mathcal{O}_X) = 0$ for every integer k with $k \geq i \geq 1$.

(B) Assume that $t > 0$. Since $\Delta_i(X, L) = 0$, we have $0 = \Delta_i(X_{n-i}, L_{n-i})$. In particular $h^i(\mathcal{O}_{X_{n-i}}) - h^i(L_{n-i}) = 0$ by Proposition 2.4. By the same argument as the proof of Theorem 3.13, we have $h^i(L_{n-i}) = 0$. Since $h^i(tL_{n-i}) = h^0(K_{X_{n-i}} - tL_{n-i}) \leq h^0(K_{X_{n-i}} - L_{n-i}) = h^i(L_{n-i})$, we have $h^i(tL_{n-i}) = 0$ for every integer t with $t \geq 1$.

Assume that $h^k(tL_m) = 0$ for every integers t and k with $t \geq 1$ and $i \leq k \leq n-m$. We study the value of $h^k(tL_{m-1})$. Then

$$H^k\big((s-1)L_{m-1}\big) \to H^k(sL_{m-1})$$

is surjective for every integers s and k with $s \geq 1$ and $i \leq k \leq n-m+1$ because $h^k(tL_m) = 0$ for every integer t with $t \geq 1$. Therefore

$$h^k(\mathcal{O}_{X_{m-1}}) \geq h^k(L_{m-1}) \geq \cdots \geq h^k(sL_{m-1}) \geq \cdots$$

for every integer k with $i \leq k \leq n-m+1$. We remark that

$$h^k(\mathcal{O}_X) = h^k(\mathcal{O}_{X_1}) = \cdots = h^k(\mathcal{O}_{X_{m-1}})$$

for every integer k with $i \leq k \leq n-m$. By assumption, Corollary 3.10, and Theorem 3.13, we get $g_k(X, L) = 0$ for every integer k with $k \geq i$. Hence by Theorem 3.1(2) we get $0 = g_k(X, L) \geq h^k(\mathcal{O}_X)$, and $h^k(\mathcal{O}_{X_{m-1}}) = 0$ for every integer k with $i \leq k \leq n-m$.

If $k = n-m+1$, then by Theorem 3.1(1) we get

$$0 = g_k(X, L) = g_k(X_{m-1}, L_{m-1}) = h^k(\mathcal{O}_{X_{m-1}}).$$

Hence $h^k(\mathcal{O}_{X_{m-1}}) = 0$. Therefore $h^k(tL_{m-1}) = 0$ for all integers t and k with $t \geq 1$ and $i \leq k \leq n-m+1$. By induction $h^k(tL) = 0$ for all integers t and k with $t \geq 1$ and $i \leq k \leq n$. This completes the proof. $\square$

(3.C) The case where $\Delta_i(X, L) = 1$ with $2 \leq i \leq n$.

Let i be an integer with $2 \leq i \leq n$. Here we study (X, L) with $\Delta_i(X, L) = 1$. The following result can be proved as a corollary of Corollary 3.8, Proposition 3.9, and Theorem 3.13.

THEOREM 3.16. *Let (X, L) be a quasi-polarized manifold of dimension n, and let i be an integer with $2 \leq i \leq n$. Assume that* $\mathrm{Bs}|L| = \emptyset$. *If $\Delta_i(X, L) = 1$, then $g_i(X, L) = 1$. Furthermore if $\Delta_i(X, L) = 1$ for an integer i with $2 \leq i \leq n-1$, then $g_{i+1}(X, L) = \Delta_{i+1}(X, L) = 0$.*

REMARK 3.16.1. Let (X, L) be a polarized manifold of dimension n. If $g_1(X, L) = \Delta_1(X, L) = 1$, then (X, L) is a Del Pezzo manifold. (See [**Fj3**, (6.5) Corollary].)

If $n \geq 3$, $i = 2$, and L is very ample, then we get a classification of (X, L) with $\Delta_2(X, L) = 1$ as follows.

THEOREM 3.17. *Let (X, L) be a polarized manifold of dimension $n \geq 3$ and let (M, A) be a reduction of (X, L). Assume that L is very ample. If $\Delta_2(X, L) = 1$, then (X, L) is one of the following.*

(1) *(M, A) is a Mukai manifold.*
(2) *(M, A) is a Del Pezzo fibration over a smooth elliptic curve C. Let $f : M \to C$ be its fibration. Then $K_M + (n-2)A = f^*(H)$ for some ample line bundle H on C with* $\deg H = 1$.
(3) *(M, A) is a quadric fibration over a smooth surface S. Let $f : M \to S$ be its fibration. Then $K_M + (n-2)A = f^*(K_S + H)$ for some ample line bundle H on S.*
(3.1) S is a $\boldsymbol{P}^1$-bundle, $p : S \to B$, over an elliptic curve B and $H = 3C_0 - F$, where C_0 (resp. F) denotes the minimal section of S with $C_0^2 = 1$ (resp. a fiber of p).
(3.2) S is a hyperelliptic surface, $H^2 = 2$, and $h^0(H) = 1$.
(4) *$(X, L) = (M, A)$, $n = \dim X \geq 4$, and (X, L) is a scroll over a normal 3-fold Y with $h^2(\mathcal{O}_Y) = 0$. If $\dim X \geq 5$, then Y is smooth and there exists an ample vector bundle $\mathcal{E}$ of rank $n-2$ on Y such that $X = \boldsymbol{P}_Y(\mathcal{E})$ and $L = H(\mathcal{E})$, where $H(\mathcal{E})$ is the tautological line bundle on X. In this case $(Y, c_1(\mathcal{E}))$ is one of the following.*
(4.1) $(Y, c_1(\mathcal{E}))$ is a Mukai manifold. In this case, $(Y, \mathcal{E})$ is one of the following.
(4.1.1) $(Y, \mathcal{E}) \cong (\boldsymbol{P}^3, \mathcal{O}_{\boldsymbol{P}^3}(1)^{\oplus 4})$.
(4.1.2) $(Y, \mathcal{E}) \cong (\boldsymbol{P}^3, \mathcal{O}_{\boldsymbol{P}^3}(2) \oplus \mathcal{O}_{\boldsymbol{P}^3}(1)^{\oplus 2})$.
(4.1.3) $(Y, \mathcal{E}) \cong (\boldsymbol{P}^3, T_{\boldsymbol{P}^3})$, where $T_{\boldsymbol{P}^3}$ is the tangent bundle of $\boldsymbol{P}^3$.
(4.1.4) $(Y, \mathcal{E}) \cong (\boldsymbol{Q}^3, \mathcal{O}_{\boldsymbol{Q}^3}(1)^{\oplus 3})$.
(4.2) $(Y, c_1(\mathcal{E}))$ is a Del Pezzo fibration over a smooth curve C such that $(Y, c_1(\mathcal{E}))$ is of the type (2) above. In this case $\dim X = 5$ and there exist vector bundles $\mathcal{F}$ and $\mathcal{G}$ on C with $\mathrm{rank}\mathcal{F} = 3$ and $\mathrm{rank}\mathcal{G} = 3$ such that $Y = \boldsymbol{P}_C(\mathcal{F})$ and $\mathcal{E} \cong H(\mathcal{F}) \otimes \pi^*(\mathcal{G})$.

Furthermore if (X, L) is one of the types from (1) to (4) above unless (X, L) is a 4-dimensional scroll over a normal 3-fold Y with $h^2(\mathcal{O}_Y) = 0$, then $\Delta_2(X, L) = 1$.

PROOF. By Theorem 3.16 we obtain $g_2(X, L) = 1$. In particular, we get $g_2(X, L) \leq h^2(\mathcal{O}_X) + 1$. Hence one of the following holds.

(A) $g_2(X, L) = 1 = h^2(\mathcal{O}_X) + 1$, that is, $h^2(\mathcal{O}_X) = 0$.
(B) $g_2(X, L) = 1 = h^2(\mathcal{O}_X)$.

Here we note that by Corollary 2.11 we get $\Delta_2(X,L)=\Delta_2(M,A)$.
(I) First we consider the case (A).

Then by [**Fk**, Theorem 3.6], one of the following holds. (Here we use the assumption that L is very ample.)

(A.1) (M,A) is a Mukai manifold.

(A.2) (M,A) is a Del Pezzo fibration over a smooth curve C. Let $f: M\to C$ be its morphism. Then there exists an ample line bundle H on C such that $K_M+(n-2)A=f^*(H)$. In this case $(g(C),\deg H)=(1,1)$.

(A.3) (M,A) is a quadric fibration over a smooth surface S. Let $f: M\to S$ be its morphism. Then there exists an ample line bundle H on S such that $K_M+(n-2)A=f^*(K_S+H)$. In this case (S,H) is one of the following types:

(A.3.1) S is a $\boldsymbol{P}^1$-bundle, $p: S\to B$, over a smooth elliptic curve B, and $H=3C_0-F$, where C_0 (resp. F) denotes the minimal section of S with $C_0^2=1$ (resp. a fiber of p).

(A.3.2) S is an abelian surface, $H^2=2$, and $h^0(H)=1$.

(A.3.3) S is a hyperelliptic surface, $H^2=2$, and $h^0(H)=1$.

(A.4) $(M,A)=(X,L)$, $n=\dim X\geq 4$, and (X,L) is a scroll over a normal projective variety Y of dimension 3. If $\dim X\geq 5$, then Y is smooth and there exists an ample vector bundle $\mathscr{E}$ of rank $n-2$ on Y such that $X=\boldsymbol{P}_Y(\mathscr{E})$ and $L=H(\mathscr{E})$, where $H(\mathscr{E})$ is the tautological line bundle on X. In this case $(Y,c_1(\mathscr{E}))$ is one of the following.

(A.4.1) $(Y,c_1(\mathscr{E}))$ is a Mukai manifold. In this case, $(Y,\mathscr{E})$ is one of the following:

(A.4.1.1) $(Y,\mathscr{E})\cong(\boldsymbol{P}^3,\mathscr{O}_{\boldsymbol{P}^3}(1)^{\oplus 4})$.

(A.4.1.2) $(Y,\mathscr{E})\cong(\boldsymbol{P}^3,\mathscr{O}_{\boldsymbol{P}^3}(2)\oplus\mathscr{O}_{\boldsymbol{P}^3}(1)^{\oplus 2})$.

(A.4.1.3) $(Y,\mathscr{E})\cong(\boldsymbol{P}^3,T_{\boldsymbol{P}^3})$, where $T_{\boldsymbol{P}^3}$ is the tangent bundle of $\boldsymbol{P}^3$.

(A.4.1.4) $(Y,\mathscr{E})\cong(\boldsymbol{Q}^3,\mathscr{O}_{\boldsymbol{Q}^3}(1)^{\oplus 3})$.

(A.4.2) $(Y,c_1(\mathscr{E}))$ is a Del Pezzo fibration over a smooth curve such that $(Y,c_1(\mathscr{E}))$ is of the type (A.2) above. In this case $\dim X=5$.

(I.1) If (M,A) is as in the case (A.1), then by Example 2.12(4) we have $\Delta_2(X,L)=\Delta_2(M,A)=1$.

(I.2) If (M,A) is as in the case (A.2), then we obtain

$$h^0(K_M+(n-2)A)=h^0(f^*(H))=h^0(H)=1.$$

Hence by Example 2.12(10), we obtain

$$\begin{aligned}\Delta_2(M,A)&=g_2(M,A)-\Delta_3(M,A)+(n-2)h^2(\mathscr{O}_M)-h^2(A)\\&=h^0(K_M+(n-2)A)\\&=1.\end{aligned}$$

(I.3) If (M,A) is as in the case (A.3), then $K_M+(n-2)A=f^*(K_S+H)$.

(I.3.1) The case (A.3.2) is impossible because $h^2(\mathscr{O}_S)=0$ under this situation.

(I.3.2) Next we consider the cases (A.3.1) and (A.3.3). Then $h^2(\mathscr{O}_M)=h^2(\mathscr{O}_S)=0$. Hence by Example 2.12 (9) we get

$$\begin{aligned}\Delta_2(X,L) &= \Delta_2(M,A)\\ &= h^0(K_M+(n-2)A)-h^2(A)\\ &= h^0(K_S+H)-h^2(A).\end{aligned}$$

Next we calculate $h^0(K_S+H)$.

If (M,A) is as in the case (A.3.1), then $K_S+H=-2C_0+F+(3C_0-F)=C_0$. By the Riemann-Roch theorem and the vanishing theorem, we get

$$\begin{aligned}h^0(K_S+H) &= g(H)-q(S)+h^2(\mathscr{O}_S)\\ &= 2-1=1,\end{aligned}$$

where $g(H)$ is the sectional genus of (S,H).

If (M,A) is as in the case (A.3.3), then by the Riemann-Roch theorem and the vanishing theorem

$$\begin{aligned}h^0(K_S+H) &= g(H)-q(S)+h^2(\mathscr{O}_S)\\ &= 2-1=1.\end{aligned}$$

In each case, we get $h^0(K_S+H)=1$. Therefore $\Delta_2(X,L)=\Delta_2(M,A)=1-h^2(A)$.

If $\Delta_2(X,L)=0$, then $g_2(X,L)=0$ by Theorem 3.13. Hence $g_2(X,L)=h^2(\mathscr{O}_X)$ and this is a contradiction. Therefore $\Delta_2(X,L)>0$. So we obtain $h^2(A)=0$ and $\Delta_2(X,L)=1$.

(I.4) We consider the case (A.4). In this case, by Example 2.12 (7), we get

$$\begin{aligned}\Delta_2(X,L) = h^0(K_X+(n-2)L)-h^2(L)+h^3(L)\\ +(n-1)(h^2(\mathscr{O}_X)-h^3(\mathscr{O}_X)).\end{aligned} \tag{$\heartsuit$}$$

Here we assume that $\dim X\geq 5$. Then Y is smooth and there exists an ample vector bundle $\mathscr{E}$ of rank $n-2$ on Y such that $X=\boldsymbol{P}_Y(\mathscr{E})$ and $L=H(\mathscr{E})$, where $H(\mathscr{E})$ is the tautological line bundle of $\boldsymbol{P}_Y(\mathscr{E})$. Let $f: X\to Y$ be its morphism. Here we note that

$$\begin{aligned}&K_X+(n-2)L\\ &\quad= -(n-2)H(\mathscr{E})+f^*\big(K_Y+c_1(\mathscr{E})\big)+(n-2)H(\mathscr{E})\\ &\quad= f^*(K_Y+c_1(\mathscr{E})).\end{aligned}$$

(I.4.1) We consider the case (A.4.1).

Then $(Y,\mathscr{E})$ is one of the cases (A.4.1.1), (A.4.1.2), (A.4.1.3), and (A.4.1.4). In these cases, we get $h^2(\mathscr{O}_X)=0$ and $h^3(\mathscr{O}_X)=0$.

On the other hand $K_X+(n-2)L=f^*(K_Y+c_1(\mathscr{E}))=\mathscr{O}_X$ because $(Y,c_1(\mathscr{E}))$ is a Mukai manifold. Hence $h^0(K_X+(n-2)L)=1$. Next we calculate $h^2(L)$ and $h^3(L)$.

$$\begin{aligned} h^2(L) &= h^2(H(\mathscr{E})) \\ &= h^{n-2}(K_X - H(\mathscr{E})) \\ &= h^{n-2}(-(n-1)H(\mathscr{E})) \\ &= 0, \end{aligned}$$

and

$$\begin{aligned} h^3(L) &= h^3(H(\mathscr{E})) \\ &= h^{n-3}(K_X - H(\mathscr{E})) \\ &= h^{n-3}(-(n-1)H(\mathscr{E})) \\ &= 0. \end{aligned}$$

Hence by (♡) we have $\Delta_2(X, L) = 1$.
(I.4.2) We consider the case (A.4.2).
Then $(Y, c_1(\mathscr{E}))$ is a Del Pezzo fibration over a smooth elliptic curve. Let $\pi : Y \to C$ be its morphism. Then by Proposition 1.10, there exist vector bundles $\mathscr{F}$ and $\mathscr{G}$ on C with $\mathrm{rank}\mathscr{F} = 3$ and $\mathrm{rank}\mathscr{G} = 3$ such that $Y = \boldsymbol{P}_C(\mathscr{F})$ and $\mathscr{E} \cong H(\mathscr{F}) \otimes \pi^*(\mathscr{G})$.

Next we calculate $\Delta_2(X, L)$ in this case. Since $K_Y + c_1(\mathscr{E}) = \pi^*(H)$ for some ample line bundle H on C, we get

$$\begin{aligned} h^0\bigl(K_X + (n-2)L\bigr) &= h^0\bigl(f^*(K_Y + c_1(\mathscr{E}))\bigr) \\ &= h^0\bigl(f^* \circ \pi^*(H)\bigr) \\ &= h^0(H) = 1 \end{aligned}$$

because $g(C) = 1$ and $\deg H = 1$.

Next we calculate $h^j(L)$ for $j = 2, 3$. Here we note that by the Serre duality

$$\begin{aligned} h^j(L) &= h^j(H(\mathscr{E})) \\ &= h^{n-j}\bigl(K_X - H(\mathscr{E})\bigr) \\ &= h^{n-j}\bigl(-(n-1)H(\mathscr{E}) + f^* \circ \pi^*(H)\bigr). \end{aligned}$$

Claim 3.17.1. *$h^{n-j}(-tH(\mathscr{E})|_F) = 0$ for any fiber F of $\pi \circ f$ if $j \geq 2$ and $t \geq 0$.*

Proof. By the following exact sequence

$$0 \to -tH(\mathscr{E}) - F \to -tH(\mathscr{E}) \to -tH(\mathscr{E})|_F \to 0,$$

we get the following exact sequence

$$\begin{aligned} H^{n-j}(-tH(\mathscr{E})-F) &\to H^{n-j}(-tH(\mathscr{E})) \\ &\to H^{n-j}(-tH(\mathscr{E})|_F) \\ &\to H^{n-j+1}(-tH(\mathscr{E})-F). \end{aligned}$$

Since $tH(\mathscr{E})$ and $tH(\mathscr{E})+F$ is ample for $t>0$, we obtain $h^{n-j}(-tH(\mathscr{E})-F)=0$, $h^{n-j+1}(-tH(\mathscr{E})-F)=0$, and $h^{n-j}(-tH(\mathscr{E}))=0$ for $j\geq 2$.

Hence $h^{n-j}(-tH(\mathscr{E})|_F)=0$. This completes the proof of Claim 3.17.1. □

Claim 3.17.2. *$h^j(L)=0$ for $j=2,3$.*

Proof. We consider the following exact sequence.

$$\begin{aligned} &0\to -(n-1)H(\mathscr{E})\to -(n-1)H(\mathscr{E})+f^*\circ\pi^*(H) \\ &\quad\to -(n-1)H(\mathscr{E})|_F\to 0 \end{aligned}$$

because $\deg(H)=1$ and $h^0(H)=1$. On the other hand, $h^{n-j}(-(n-1)H(\mathscr{E}))=0$, and by Claim 3.17.1, we get $h^{n-j}(-(n-1)H(\mathscr{E})|_F)=0$. Hence

$$h^j(L)=h^{n-j}(-(n-1)H(\mathscr{E})+f^*\circ\pi^*(H))=0.$$

This completes the proof of Claim 3.17.2. □

Since $h^j(\mathscr{O}_X)=h^j(\mathscr{O}_Y)=0$ for $j=2,3$, we get

$$\begin{aligned} \Delta_2(X,L) &= h^0\big(K_X+(n-2)L\big)-h^2(L)+h^3(L)+(n-1)\big(h^2(\mathscr{O}_X)-h^3(\mathscr{O}_X)\big) \\ &= 1. \end{aligned}$$

(II) Next we consider the case (B). By Theorem 3.1(2), (X,L) is one of the types from (1) to (7.4) in Theorem 1.7 because L is very ample. Since $h^2(\mathscr{O}_X)=1$ in this case, (X,L) is a scroll over a smooth surface S with $h^2(\mathscr{O}_S)=1$.

Claim 3.17.3. *In this case, $\Delta_2(X,L)\geq 2$.*

Proof. There exists an ample and spanned vector bundle $\mathscr{E}$ of rank $n-1$ on S such that $X=\boldsymbol{P}_S(\mathscr{E})$ and $L=H(\mathscr{E})$, where $H(\mathscr{E})$ is the tautological line bundle of $\boldsymbol{P}_S(\mathscr{E})$. Let $f:X\to S$ be its morphism.

(a) The case where $\dim X=3$.

First we prove the following claim.

Claim 3.17.3.1. *$h^2(L)=0$.*

Proof. (i) First we consider the case where $K_S\neq\mathscr{O}_S$.

Assume that $h^2(L)>0$. Here we remark that $h^2(L)=h^2(f_*(L))$ by the proof of Lemma 1.6. Since

$$\begin{aligned} h^2(L) &= h^2(H(\mathscr{E})) \\ &= h^2\big(f_*(H(\mathscr{E}))\big) \\ &= h^2(\mathscr{E}) \\ &= h^0(K_S \otimes \mathscr{E}^\vee) \\ &= \dim \mathrm{Hom}(\mathscr{E}, K_S), \end{aligned}$$

we get a nontrivial map $\mu : \mathscr{E} \to K_S$. Then there exists an exact sequence

$$0 \to \mathrm{Ker}\mu \to \mathscr{E} \to \mathrm{Im}\mu \to 0.$$

Here we calculate rank(Imμ). If rank(Imμ) $= 0$, then $\dim \mathrm{Supp}(\mathrm{Im}\mu) < \dim S$ and Imμ is a torsion sheaf. On the other hand since Imμ is a subsheaf of K_S, Imμ is a torsion free sheaf. Hence Im$\mu = 0$ and this is a contradiction because $\mu : \mathscr{E} \to K_S$ is a nontrivial map. Hence rank(Imμ) > 0 and rank(Imμ) $= 1$ because Imμ is a subsheaf of K_S.

Since Imμ is a torsion free sheaf, by [**OSS**, p. 148 Corollary] there exists an open set U of S such that $\dim(S \backslash U) \leq 0$ and $(\mathrm{Im}\mu)|_U$ is a locally free sheaf of rank 1.

Since $\dim(S\backslash U) \leq 0$, $h^0(K_S) = h^2(\mathscr{O}_S) = 1$, and $K_S \neq \mathscr{O}_S$, there exists a point $x \in U$ such that $t(x) = 0$ for every $t \in H^0(S, K_S)$. On the other hand, since Imμ is a subsheaf of $\mathscr{O}(K_S)$, we get $u(x) = 0$ for every $u \in H^0(S, \mathrm{Im}\mu)$.

Because

$$\mathscr{E} \to \mathrm{Im}\mu \to 0$$

is exact and $\mathscr{E}$ is generated by its global sections, Imμ is generated by its global sections. But this is a contradiction because $(\mathrm{Im}\mu)|_U$ is an invertible sheaf and there exists a point $x \in U$ such that $u(x) = 0$ for every $u \in H^0(S, \mathrm{Im}\mu)$. Therefore we get $h^2(L) = 0$.
(ii) Next we consider the case where $K_S = \mathscr{O}_S$.

Since rank$\mathscr{E} = 2 = \dim S$, by a Le Potier's theorem [**ShSo**, p. 96 (5.17) Corollary], we obtain

$$\begin{aligned} h^2(L) &= h^2(\mathscr{E}) \\ &= h^2(K_S \otimes \mathscr{E}) \\ &= 0. \end{aligned}$$

These complete the proof of Claim 3.17.3.1. □

Therefore by Example 2.12(6) we have

$$\begin{aligned} \varDelta_2(X, L) &= 2h^2(\mathscr{O}_X) - h^2(L) \\ &= 2. \end{aligned}$$

(b) The case where $\dim X \geq 4$.

Since $\mathrm{Bs}|L| = \emptyset$, there exists a member $X_1 \in |L|$ such that X_1 is a smooth projective variety of dimension $n-1$. On the other hand, since $K_X + (n-1)L = f^*(B)$ for some ample line bundle $B \in \mathrm{Pic}(S)$ by hypothesis, we get $K_{X_1} + (n-2)L_1 = (f_1)^*(B)$, where $f_1 := f|_{X_1} : X_1 \to S$. Because X_1 is an ample divisor on X, f_1 is a surjective morphism with connected fibers. Therefore (X_1, L_1) is a scroll over a smooth surface S with $h^2(\mathscr{O}_{X_1}) = 1$ and $\mathrm{Bs}|L_1| = \emptyset$. Hence by [**BeSo**, Theorem 11.1.1], $\mathscr{E}_1 := (f_1)_*(L_1)$ is a locally free sheaf, $X_1 = \boldsymbol{P}_S(\mathscr{E}_1)$, and $L_1 = H(\mathscr{E}_1)$. (Here we note that $\mathscr{E}_1$ is ample.)

By the same argument as above, there exists an $(n-3)$-ladder $X_{n-3} \subset \cdots \subset X_1 \subset X_0 = X$ such that for every integer j with $0 \leq j \leq n-3$, we put $L_j = L_{j-1}|_{X_j}$, and (X_j, L_j) is a scroll over a smooth surface S with $h^2(\mathscr{O}_{X_j}) = 1$ and $\mathrm{Bs}|L_j| = \emptyset$. Let $f_j : X_j \to S$ be its morphism. By putting $\mathscr{E}_j := (f_j)_*(L_j)$, $\mathscr{E}_j$ is a locally free sheaf, $X_j = \boldsymbol{P}_S(\mathscr{E}_j)$, and $L_j = H(\mathscr{E}_j)$. (Here we note that $\mathscr{E}_j$ is ample.)

By Corollary 3.3, we get

$$\Delta_2(X, L) \geq \cdots \geq \Delta_2(X_{n-3}, L_{n-3}).$$

By the case (a) above, we obtain $\Delta_2(X_{n-3}, L_{n-3}) \geq 2$ and $\Delta_2(X, L) \geq 2$. These complete the proof of Claim 3.17.3. $\square$

Therefore we get the assertion of Theorem 3.17. $\square$

Remark 3.17.4. Let X be a $\boldsymbol{P}^{n-m}$-bundle over a smooth projective variety Y of dimension m with $h^m(\mathscr{O}_Y) \geq 1$ and let L be an ample and spanned line bundle on X such that $L|_F = \mathscr{O}_{\boldsymbol{P}^{n-m}}(1)$ for every fiber F. Then by the same argument as in the proof of Claim 3.17.3, we can prove that $\Delta_m(X, L) \geq 2$. A proof is the following.

Proof. First we consider the case where $\dim X = m+1$. We can prove $h^m(L) = 0$ by the same argument as Claim 3.17.3.1.

By Lemma 2.12.1, we obtain $\Delta_{m+1}(X, L) = 0$. By [**Fk**, Example 2.10(8)] we get $g_m(X, L) = h^m(\mathscr{O}_X)$. By the definition of the i-th Δ-genus, we get

$$\begin{aligned}\Delta_m(X, L) &= g_m(X, L) - \Delta_{m+1}(X, L) + h^m(\mathscr{O}_X) - h^m(L) \\ &= 2h^m(\mathscr{O}_X) \\ &\geq 2.\end{aligned}$$

Next we consider the case where $\dim X = n \geq m+2$. Then there exists an $(n-m-1)$-ladder $X_{n-m-1} \subset \cdots \subset X_1 \subset X_0 = X$ such that for every integer j with $0 \leq j \leq n-m-1$, we put $L_j = L_{j-1}|_{X_j}$, and (X_j, L_j) is a scroll over Y with $h^m(\mathscr{O}_{X_j}) = 1$ and $\mathrm{Bs}|L_j| = \emptyset$. Let $f_j : X_j \to Y$ be its morphism. By putting $\mathscr{E}_j := (f_j)_*(L_j)$, $\mathscr{E}_j$ is a locally free sheaf, $X_j = \boldsymbol{P}_Y(\mathscr{E}_j)$, and $L_j = H(\mathscr{E}_j)$. (Here we note that $\mathscr{E}_j$ is ample.)

By Corollary 3.3, we get

$$\Delta_m(X, L) \geq \cdots \geq \Delta_m(X_{n-m-1}, L_{n-m-1}).$$

Since $\dim X_{n-m-1} = m+1$, by above we get $\Delta_m(X_{n-m-1}, L_{n-m-1}) \geq 2$. Hence we get the assertion. $\square$

Here we study a polarized manifold (X, L) with $g_2(X, L) = 1$ by using the second Δ-genus.

Proposition 3.18. *Let (X, L) be a polarized manifold of dimension $n \geq 3$. Assume that $\mathrm{Bs}|L| = \varnothing$. If $\Delta_2(X, L) > g_2(X, L) = 1$, then (X, L) is a scroll over a smooth surface S with $h^2(\mathcal{O}_S) = 1$.*

Proof. We use Notation 3.0. By Corollary 3.2, we get

$$\Delta_2(X, L) = \sum_{k=0}^{n-2} \dim \mathrm{Coker}(r_{1,k}).$$

By the Lefschetz theorem, we have

$$0 \leq h^2(\mathcal{O}_X) = h^2(\mathcal{O}_{X_1}) = \cdots = h^2(\mathcal{O}_{X_{n-3}}) \leq h^2(\mathcal{O}_{X_{n-2}}).$$

By Theorem 3.1(1) we obtain $1 = g_2(X, L) = h^2(\mathcal{O}_{X_{n-2}})$. Hence

$$0 \leq h^2(\mathcal{O}_X) = h^2(\mathcal{O}_{X_1}) = \cdots = h^2(\mathcal{O}_{X_{n-3}}) \leq h^2(\mathcal{O}_{X_{n-2}}) = 1.$$

If $h^2(\mathcal{O}_{X_{n-3}}) = 0$, then $\dim \mathrm{Coker}(r_{1,i}) = 0$ for $i = 0, \cdots, n-3$. Hence $\Delta_2(X, L) = \dim \mathrm{Coker}(r_{1,n-2}) \leq h^2(\mathcal{O}_{X_{n-2}}) = 1 = g_2(X, L)$ and this is impossible. Therefore $h^2(\mathcal{O}_{X_{n-3}}) = 1 = h^2(\mathcal{O}_{X_{n-2}})$. In particular $h^2(\mathcal{O}_{X_{n-2}}) = h^2(\mathcal{O}_X) = 1$.

Therefore, by Theorem 3.1(1), we obtain $g_2(X, L) = h^2(\mathcal{O}_{X_{n-2}}) = h^2(\mathcal{O}_X) = 1$. By Theorem 3.1(2) and $h^2(\mathcal{O}_X) = 1$, we get the assertion. $\square$

Lemma 3.19. *Let (X, L) be a quasi-polarized manifold of dimension n. Assume that $\mathrm{Bs}|L| = \varnothing$. If $\Delta_2(X, L) \leq g_2(X, L) = 1$, then $\Delta_2(X, L) = 1$.*

Proof. Since $\Delta_2(X, L) \geq 0$, we get $\Delta_2(X, L) = 0$ or 1. If $\Delta_2(X, L) = 0$, then $g_2(X, L) = 0$ by Theorem 3.13. Hence we get the assertion. $\square$

By using Proposition 3.18 and Lemma 3.19 we get the following.

Theorem 3.20. *Let (X, L) be a polarized manifold of dimension $n \geq 3$. Assume that $\mathrm{Bs}|L| = \varnothing$. If $g_2(X, L) = 1$, then (X, L) is one of the following.*

(1) $\Delta_2(X, L) = 1$ *and* $h^2(\mathcal{O}_X) = 0$.
(2) (X, L) *is a scroll over a smooth surface with* $h^2(\mathcal{O}_X) = 1$.

Proof. (A) If $\Delta_2(X, L) > g_2(X, L) = 1$, then (X, L) is of the type (2) by Proposition 3.18.
(B) If $\Delta_2(X, L) \leq g_2(X, L) = 1$, then $\Delta_2(X, L) = 1$ by Lemma 3.19. By Theorem 3.1(2), $h^2(\mathcal{O}_X) \leq g_2(X, L) = 1$.
(B-1) If $h^2(\mathcal{O}_X) = 0$, then (X, L) is of the type (1).

(B-2) If $h^2(\mathcal{O}_X)=1$, then $g_2(X,L)=h^2(\mathcal{O}_X)=1$. By Theorem 1.7 and Theorem 3.1 (2), (X,L) is a scroll over a smooth surface with $h^2(\mathcal{O}_X)=1$. This is of the type (2). This completes the proof. $\square$

(3.D) The case where $\Delta_i(X,L)=2$ with $2\le i\le n$.

Let (X,L) be a quasi-polarized manifold of dimension n with $\mathrm{Bs}|L|=\varnothing$. Assume that i is an integer with $n-1\ge i\ge 3$. Then by Proposition 3.7 and Proposition 3.9, we get $g_i(X,L)\le 2$, and $g_{i+1}(X,L)=\Delta_{i+1}(X,L)=0$.

Assume that $i=2$. Then by Proposition 3.12, one of the following holds.

(3.D.1) $g_2(X,L)\le 2$.

(3.D.2) There exists a covering $\pi: X\to \boldsymbol{P}^n$ of degree L^n such that $\Delta_2(X,L)=\cdots=\Delta_2(X_{n-2},L_{n-2})$.

In particular, if L is very ample, then $g_2(X,L)\le 2$. We will study a polarized manifold (X,L) such that $\dim X=n\ge 4$, L is very ample, and $\Delta_2(X,L)=2$ in a future paper.

4. Remark.

In this section, we propose some problems about the i-th Δ-genus. First we propose the following problem.

PROBLEM 4.1. Let (X,L) be a quasi-polarized variety of dimension n. Is it true that $\Delta_i(X,L)\ge 0$ for every integer i with $1\le i\le n$?

If $i=1$, then this is true by Fujita's result ([**Fj1**], [**Fj2**]). If X is smooth and $\mathrm{Bs}|L|=\varnothing$, then this is true by Corollary 3.3. But this problem is not true in general. Here we give some examples of (X,L) such that $\Delta_i(X,L)<0$.

EXAMPLE 4.1.1. Let $\boldsymbol{P}^{n+1}$ be the projective space of dimension $n+1$ with $n\ge 4$. Let $(\xi_0:\xi_1:\cdots:\xi_{n+1})$ be the homogeneous coordinate of it. Let $k=n+3$ be a prime number. Let $G=\boldsymbol{Z}/k\boldsymbol{Z}$ be a cyclic group of order k generated by the primitive k-th root of unity. Then $\rho\in G$ acts on $\boldsymbol{P}^{n+1}$ as the following.

$$(\rho)\cdot(\xi_0:\xi_1:\cdots:\xi_{n+1})=(\xi_0:\rho\xi_1:\cdots:\rho^{n+1}\xi_{n+1}),$$

where $\rho=\exp(2\pi i/k)$. The fixed points of this action are the following.

$$(1:0:\cdots:0),(0:1:\cdots:0),\cdots,(0:0:\cdots:1). \tag{4.1.1.1}$$

Let Y be a hypersurface in $\boldsymbol{P}^{n+1}$ which is defined by $\sum_{i=0}^{n+1}{\xi_i}^k=0$. We note that the above action of G on $\boldsymbol{P}^{n+1}$ induces the action of G on Y. All points in (4.1.1.1) are not on Y. Hence $X:=Y/G$ is smooth and $\pi: Y\to X$ is an etale covering of degree $k=n+3$. Since $K_Y=(\mathcal{O}(-n-2)+\mathcal{O}(n+3))|_Y=\mathcal{O}_Y(1)$, we get $n+3=K_Y^n=(\pi^*K_X)^n=(\deg\pi)(K_X)^n=(n+3)(K_X)^n$. Namely $(K_X)^n=1$. Here we remark that $\pi_*\mathcal{O}_Y=\mathcal{O}_X\oplus\mathcal{E}$, where $\mathcal{E}$ is a locally free sheaf of rank $n+2$ on X. Since

$$H^i(\mathcal{O}_Y)=H^i(\pi_*\mathcal{O}_Y)=H^i(\mathcal{O}_X)\oplus H^i(\mathcal{E})$$

and $h^i(\mathscr{O}_Y)=0$ for every integer i with $1\le i\le n-1$, we get $h^i(\mathscr{O}_X)=0$ for $1\le i\le n-1$. In particular, $h^1(K_X)=h^{n-1}(\mathscr{O}_X)=0$.

Next we calculate $h^0(K_X)$. Since $n+3$ is prime, n is even. Hence

$$\chi(\mathscr{O}_Y)=1+h^n(\mathscr{O}_Y)=1+h^0(K_Y)=n+3.$$

Since π is etale,

$$\chi(\mathscr{O}_X)=\frac{1}{\deg\pi}\chi(\mathscr{O}_Y)=1.$$

Hence $h^n(\mathscr{O}_X)=0$. By the Serre duality, we have $h^0(K_X)=0$.

Here we remark that K_X is ample. We calculate $\Delta_2(X,K_X)$. By definition

$$\begin{aligned}\Delta_2(X,K_X)&=g_1(X,K_X)-\Delta_1(X,K_X)+(n-1)h^1(\mathscr{O}_X)-h^1(K_X)\\&=1+\frac{1}{2}\bigl(K_X+(n-1)K_X\bigr){K_X}^{n-1}-\bigl(n+{K_X}^n-h^0(K_X)\bigr)\\&=1+\frac{n}{2}-n-1\\&=-\frac{n}{2}<0.\end{aligned}$$

Here we remark that since $k=n+3$ is a prime number, $n=2,4,8,\cdots$.

Example 4.1.2. Let $\boldsymbol{P}^{n+1}$ be the projective space of dimension $n+1$ with $n\ge 4$. Let $(\xi_0:\xi_1:\cdots:\xi_{n+1})$ be the homogeneous coordinate of it. Let $G=\boldsymbol{Z}/k\boldsymbol{Z}$ for a prime number $k=n+3$. We assume that the action of G on $\boldsymbol{P}^{n+1}$ is the same action as in Example 4.1.1. Let H_j be a hyperplane $\xi_j=0$. Let Y be a hypersurface of $\boldsymbol{P}^{n+1}$ which is defined by $\sum_{i=0}^{n+1}{\xi_i}^k=0$, $X:=Y/G$, and $\pi:Y\to X$ be as in Example 4.1.1. Then

$$Y_j:=Y\cap H_j$$

is smooth for any j. The action of G on $\boldsymbol{P}^{n+1}$ induces the action of G on Y_j, and Y_j has no fixed point. Here we consider $X_j:=\pi(Y\cap H_j)$. Then X_j is smooth, $Y_j=\pi^*(X_j)$, $\dim X_j=n-1$, and $K_X|_{X_j}$ is ample. Here we remark that

$$(K_Y)^{n-i}(Y_j)^i=\mathscr{O}_Y(1)^n=n+3$$

for every integer i with $0\le i\le n$. On the other hand

$$\begin{aligned}(K_Y)^{n-i}(Y_j)^i&=\bigl(\pi^*(K_X)\bigr)^{n-i}\bigl(\pi^*(X_j)\bigr)^i\\&=(\deg\pi)\bigl((K_X)^{n-i}(X_j)^i\bigr)\\&=(n+3)\bigl((K_X)^{n-i}(X_j)^i\bigr).\end{aligned}$$

Hence $(K_X)^{n-i}(X_j)^i = 1$ for every integer i with $0 \le i \le n$.

CLAIM 4.1.2.1. *$h^i(K_X|_{X_j}) = 0$ for every integer i with $0 \le i \le n-2$.*

PROOF. We consider the following exact sequence.

$$0 \to K_X - X_j \to K_X \to K_X|_{X_j} \to 0.$$

Then

$$H^i(K_X) \to H^i(K_X|_{X_j}) \to H^{i+1}(K_X - X_j)$$

is exact. By Example 4.1.1, we get $h^i(K_X) = 0$ for every integer i with $0 \le i \le n-1$. By the Serre duality we have $h^{i+1}(K_X - X_j) = h^{n-i-1}(X_j)$. Here we remark that

$$\begin{aligned} \pi_*\bigl(\mathscr{O}(Y_j)\bigr) &= \pi_*\pi^*\bigl(\mathscr{O}(X_j)\bigr) \\ &= \mathscr{O}(X_j) \oplus \bigl(\mathscr{E} \otimes \mathscr{O}(X_j)\bigr) \end{aligned}$$

because $\pi_*\mathscr{O}_Y = \mathscr{O}_X \oplus \mathscr{E}$, where $\mathscr{E}$ is a locally free sheaf of rank $n+2$ on X. Since

$$\begin{aligned} H^{n-i-1}\bigl(\mathscr{O}(Y_j)\bigr) &= H^{n-i-1}\bigl(\pi_*(\mathscr{O}(Y_j)\bigr) \\ &= H^{n-i-1}\bigl(\mathscr{O}(X_j)\bigr) \oplus H^{n-i-1}\bigl(\mathscr{E} \otimes \mathscr{O}(X_j)\bigr), \end{aligned}$$

and $h^{n-i-1}(\mathscr{O}(Y_j)) = 0$ for $0 \le i \le n-2$, we have $h^{n-i-1}(\mathscr{O}(X_j)) = 0$ for $0 \le i \le n-2$. Hence $h^i(K_X|_{X_j}) = 0$ for every integer i with $0 \le i \le n-2$. □

Here we remark that $h^1(\mathscr{O}_{X_j}) = 0$. Actually, since Y_j is ample and $Y_j = \pi^*(X_j)$, X_j is ample on X. Since $\dim X = n \ge 4$, we get $h^1(-X_j) = h^2(-X_j) = 0$ by the Kodaira vanishing theorem. By Example 4.1.1 we also get $h^1(\mathscr{O}_X) = 0$. Hence $h^1(\mathscr{O}_{X_j}) = 0$.

Here we calculate the second Δ-genus of $(X_j, K_X|_{X_j})$. By Claim 4.1.2.1 we get $h^0(K_X|_{X_j}) = 0$ and $h^1(K_X|_{X_j}) = 0$. Hence

$$\begin{aligned} \Delta_2(X_j, K_X|_{X_j}) &= g_1(X_j, K_X|_{X_j}) - \Delta_1(X_j, K_X|_{X_j}) + (n-2)h^1(\mathscr{O}_{X_j}) - h^1(K_X|_{X_j}) \\ &= 1 + \frac{1}{2}\bigl(K_{X_j} + (n-2)(K_X|_{X_j})\bigr)(K_X|_{X_j})^{n-2} \\ &\quad - \bigl(n - 1 + (K_X|_{X_j})^{n-1} - h^0(K_X|_{X_j})\bigr) \\ &= 1 + \frac{1}{2}\bigl((n-1)K_X + X_j\bigr)(K_X)^{n-2}X_j - n \\ &= -\frac{n}{2} + 1. \end{aligned}$$

If $n \ge 4$, then $\Delta_2(X_j, K_X|_{X_j}) < 0$.

EXAMPLE 4.1.3.

(1) Let X be a smooth projective variety of dimension $n \geq 2$. Assume that K_X is ample with $h^0(K_X) = 0$. (Here we remark that there exists an example of this type. For example, there exists a minimal surface of general type S such that K_S is ample and $h^0(K_S) = 0$ (see [**BaPeVa**, Chapter V, 15]). Let Y' be a smooth projective manifold of dimension $n-2$ such that $K_{Y'}$ is ample. We put $Y = Y' \times S$. Then K_Y is ample and $h^0(K_Y) = h^0(K_{Y'})h^0(K_S) = 0$.)

Then by Proposition 2.4

$$\begin{aligned} \Delta_n(X, K_X) &= h^n(\mathscr{O}_X) - h^n(K_X) \\ &= h^0(K_X) - h^0(\mathscr{O}_X) \\ &= -1 < 0. \end{aligned}$$

(2) We fix a natural number n with $n \geq 3$. For every natural number m, there exists an example of (X, L) with $\Delta_n(X, L) = -m$ and $\dim X = n$. Let Y be a smooth projective variety of dimension $n-1 \geq 2$ such that K_Y is ample with $h^0(K_Y) = 0$. Let C be a smooth projective curve of genus $m+1 \geq 2$, where m is a natural number. Let A be a divisor on C with $\deg A = 1$ and $h^0(A) = 1$. Here we remark that $\mathrm{Bs}|K_C| = \varnothing$. Hence $h^0(K_C - A) = g(C) - 1$. We put $X := Y \times C$ and $L := p_1^*(K_Y) + p_2^*(A)$, where p_i is the i-th projection for $i = 1, 2$. Then L is ample. Moreover we get

$$h^n(\mathscr{O}_X) = h^0(K_X) = h^0(K_Y)h^0(K_C) = 0,$$

and

$$\begin{aligned} h^n(L) &= h^n(p_1^*(K_Y) + p_2^*(A)) \\ &= h^{n-1}(K_Y)h^1(A) \\ &= h^0(\mathscr{O}_Y)h^0(K_C - A) \\ &= g(C) - 1. \end{aligned}$$

Hence

$$\begin{aligned} \Delta_n(X, L) &= h^n(\mathscr{O}_X) - h^n(L) \\ &= -(g(C) - 1) \\ &= -m. \end{aligned}$$

Example 4.1.4. (1) Let Y be a smooth projective variety of dimension $m \geq 2$ such that K_Y is ample with $h^0(K_Y) = 0$. We put $\mathscr{E} = \mathscr{O}(K_Y)^{\oplus n-m+1}$, where n is a natural number with $n > m$. Let $X = \boldsymbol{P}_Y(\mathscr{E})$ and $L = H(\mathscr{E})$, where $H(\mathscr{E})$ is the tautological line bundle on $\boldsymbol{P}_Y(\mathscr{E})$. Then L is ample. Since $g_m(X, L) = h^m(\mathscr{O}_X)$ and by Lemma 2.12.1 $\Delta_{m+1}(X, L) = 0$ holds, we get

$$\begin{aligned}\Delta_m(X,L) &= g_m(X,L) - \Delta_{m+1}(X,L) + (n-m)h^m(\mathscr{O}_X) - h^m(L)\\ &= (n-m+1)h^m(\mathscr{O}_X) - h^m(L).\end{aligned}$$

Since $h^m(\mathscr{O}_X) = h^m(\mathscr{O}_Y) = h^0(K_Y) = 0$ and

$$\begin{aligned}h^m(L) &= h^m(\pi_*(L))\\ &= h^m(\mathscr{E})\\ &= h^m\big(\mathscr{O}(K_Y)^{\oplus n-m+1}\big)\\ &= (n-m+1)h^m\big(\mathscr{O}(K_Y)\big)\\ &= n-m+1,\end{aligned}$$

we get

$$\begin{aligned}\Delta_m(X,L) &= (n-m+1)h^m(\mathscr{O}_X) - h^m(L)\\ &= -(n-m+1) < 0.\end{aligned}$$

(2) We fix a natural number n with $n \geq 3$. For every natural number d, there exists a polarized manifold (X,L) such that $\dim X = n$, $h^0(L) \geq d$ and $\Delta_i(X,L) < 0$ for every integer i with $2 \leq i \leq n-1$ as follows.

Let (Y, K_Y) be a polarized manifold of dimension $m \geq 2$ such that $h^0(K_Y) = 0$. Let A be an ample line bundle on Y such that $h^0(A) \geq d$ and $h^m(A) = 0$. (Here we remark that this A does exist. Let L be an ample line bundle on Y. If t is sufficiently large, $h^0(L^{\otimes t}) \geq d$ holds. Furthermore by the Serre vanishing theorem, we get $h^m(L^{\otimes t}) = 0$ for sufficiently large t. Here we put $A = L^{\otimes t}$.) We put $\mathscr{E} = \mathscr{O}(K_Y)^{\oplus n-m} \oplus A$, where n is a natural number with $n > m$. Let $X = \boldsymbol{P}_Y(\mathscr{E})$ and $L = H(\mathscr{E})$, where $H(\mathscr{E})$ is the tautological line bundle on $\boldsymbol{P}_Y(\mathscr{E})$. Then L is ample with $h^0(L) = h^0(\mathscr{E}) = h^0(A) \geq d$. By using Lemma 2.12.1, we get

$$\Delta_m(X,L) = (n-m+1)h^m(\mathscr{O}_X) - h^m(L).$$

Since $h^m(\mathscr{O}_X) = h^m(\mathscr{O}_Y) = h^0(K_Y) = 0$ and

$$\begin{aligned}h^m(L) &= h^m\big(\pi_*(L)\big)\\ &= h^m(\mathscr{E})\\ &= h^m\big(\mathscr{O}(K_Y)^{\oplus n-m} \oplus A\big)\\ &= (n-m)h^m\big(\mathscr{O}(K_Y)\big) + h^m(A)\\ &= n-m,\end{aligned}$$

we get

$$\begin{aligned}\Delta_m(X,L) &= (n-m+1)h^m(\mathcal{O}_X) - h^m(L) \\ &= -(n-m) < 0.\end{aligned}$$

By considering these examples, we can propose the following problem.

PROBLEM 4.2. List up types of quasi-polarized variety (X, L) with $\Delta_i(X, L) < 0$ for $2 \leq i \leq n = \dim X$.

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