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Large time behavior of solutions to the Klein-Gordon equation with nonlinear dissipative terms

Dedicated to Professor Yujiro Ohya on the occasion of his seventieth birthday.

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Abstract. We consider the Cauchy problem for $\partial_t^2 u - \partial_x^2 u + u = -g(\partial_t u)^3$ on the real line. It is shown that if $g > 0$, the solution has an additional logarithmic time decay in comparison with the free evolution in the sense of L^p, $2 \leq p \leq \infty$. Moreover, the asymptotic profile of $u(t,x)$ as $t \to +\infty$ is obtained. We also discuss a generalization. Consequently we see that the "null condition" in the sense of J. -M. Delort (Ann. Sci. École Norm. Sup., **34** (2001), 1–61) is *not* optimal for small data global existence for nonlinear Klein-Gordon equations.

1. Introduction.

We consider the Cauchy problem for

$$\Box u + u = -g(\partial_t u)^3 \tag{1.1}$$

in $(t,x) \in (0,\infty) \times \boldsymbol{R}$, where $\Box = \partial_t^2 - \partial_x^2$ and $g \in \boldsymbol{R}$. From the heuristic point of view, the nonlinearity in (1.1) plays a role of a dissipative term if $g > 0$. So it is natural to expect that the energy decays in the large time. According to the earlier results (M. Nakao [**11**], K. Mochizuki and T. Motai [**9**], etc.), $\|u(t)\|_E$ behaves like $O((\log t)^{-1/2})$ as $t \to +\infty$ if the initial data belongs to suitable function space, where

$$\|u(t)\|_E^2 = \frac{1}{2} \int_{\boldsymbol{R}} |\partial_t u(t,x)|^2 + |\partial_x u(t,x)|^2 + |u(t,x)|^2 dx.$$

We remark that their proof relies heavily on the conservation law

$$\|u(t)\|_E^2 + \int_0^t \int_{\boldsymbol{R}} g |\partial_t u(\tau,x)|^4 dx d\tau = \|u(0)\|_E^2 \tag{1.2}$$

and thus it seems difficult to obtain any other information from their approach. Here, we raise the following question: *What can we say about the large time behavior of the quantities other than the energy? In particular, what about the pointwise asymptotics of $u(t,x)$ as $t \to +\infty$?* To the author's knowledge, there are no previous results. The first

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aim of the present paper is to give an answer to this question in the case where the Cauchy data is sufficiently small, smooth and compactly-supported. In what follows, we assume that the initial data is given by

$$u(0,x)=\varepsilon u_0(x),\quad \partial_t u(0,x)=\varepsilon u_1(x),\qquad x\in \boldsymbol{R}, \tag{1.3}$$

where u_0, u_1 are real-valued functions which belong to $C_0^\infty(\boldsymbol{R})$ and $\varepsilon>0$ denotes a small parameter. The first result is the following one.

Theorem 1.1. *Suppose $g>0$. There exists $\varepsilon_0>0$ such that if $\varepsilon\in]0,\varepsilon_0]$, the Cauchy problem* (1.1)–(1.3) *admits a unique global classical solution which satisfies*

$$\sum_{|\alpha|\le 1}\|\partial_{t,x}^\alpha u(t,\cdot)\|_{L^p}\le \frac{C(1+t)^{-\left(\frac{1}{2}-\frac{1}{p}\right)}}{\sqrt{\log(2+t)}} \tag{1.4}$$

for any $p\in[1,\infty]$. Here C denotes a positive constant depending only on u_0, u_1, g and p. Moreover, the following asymptotic expression is valid as $t\to+\infty$, uniformly with respect to $x\in\boldsymbol{R}$:

$$u(t,x)=\frac{\frac{1}{\sqrt{t}}\operatorname{Re}\left[a(x/t)e^{i(t^2-|x|^2)_+^{1/2}}\right]}{\sqrt{1+\frac{3g}{4}|a(x/t)|^2(1-|x/t|^2)_+^{-1}\log t}}+O\big(t^{-1/2}(\log t)^{-3/2}\big), \tag{1.5}$$

where $(\,\cdot\,)_+=\max\{\,\cdot\,,0\}$, $i=\sqrt{-1}$ and $a(y)$ is a complex-valued smooth function which satisfies

$$\left|\partial_y^j a(y)\right|\le C_j(1-|y|)_+^{N-j}\quad \textit{for}\quad j=0,1,2,\cdots,N$$

with sufficiently large $N\in\boldsymbol{N}$ and positive constants C_j.

Remark 1.1. The estimate (1.4) implies the solution of (1.1)–(1.3) has an additional logarithmic time decay in comparison with the free evolution in L^p for any $p\in[2,\infty]$. In particular, it covers the energy decay results mentioned before though the assumptions on the initial data are stronger than that of [**9**], [**11**].

The second objective of this work is to generalize the above result. As we shall see below, our approach is available for much wider class of nonlinear Klein-Gordon equations because it does not require the conservation law like (1.2) at all. For instance, let us consider

$$\Box u+u=F(u,\partial_t u,\partial_x u) \tag{1.6}$$

in $(t,x)\in(0,\infty)\times\boldsymbol{R}$, where F is a cubic homogeneous polynomial in $(u,\partial_t u,\partial_x u)$. When we put

$$K_F(z)=\frac{i}{2\pi}\int_0^{2\pi}F(\cos\theta,\ -\cosh z\sin\theta,\ \sinh z\sin\theta)e^{-i\theta}d\theta$$

for $z \in \boldsymbol{R}$, we have the following:

THEOREM 1.2. *Suppose that F satisfies*

$$\inf_{z\in\boldsymbol{R}} \operatorname{Re} K_F(z) \geq 0. \tag{1.7}$$

Then, for sufficiently small ε, the Cauchy problem (1.6)–(1.3) *admits a unique global classical solution $u(t,x)$ which behaves like*

$$u(t,x) = \frac{\frac{1}{\sqrt{t}} \operatorname{Re}\left[a(x/t)e^{\{i(t^2-|x|^2)_+^{1/2}+i\Psi_F(x/t)|a(x/t)|^2\mathscr{L}(t,\,|a(x/t)|^2\Phi_F(x/t))\}}\right]}{\sqrt{1+2\Phi_F(x/t)|a(x/t)|^2\log t}} + O\big(t^{-1/2}(\log t)^{-3/2}\big) \tag{1.8}$$

as $t \to +\infty$, uniformly in $x \in \boldsymbol{R}$. Here $a(y)$ is as before, $\Phi_F(y)$, $\Psi_F(y)$ are given by

$$\Phi_F(y) = (1-y^2)^{1/2}\operatorname{Re}\big[K_F(\tanh^{-1} y)\big],$$
$$\Psi_F(y) = (1-y^2)^{1/2}\operatorname{Im}\big[K_F(\tanh^{-1} y)\big]$$

for $|y|<1$, and $\mathscr{L}(\tau,\varphi)$ is defined by

$$\mathscr{L}(\tau,\varphi) = -\int_1^\tau \frac{d\sigma}{\sigma(1+2\varphi\log\sigma)} = \begin{cases} -\log\tau & \text{if } \varphi = 0, \\ -\frac{1}{2\varphi}\log(1+2\varphi\log\tau) & \text{if } \varphi\neq 0,\ 1+2\varphi\log\tau > 0. \end{cases} \tag{1.9}$$

REMARK 1.2. The above assertion is still valid for cubic *quasilinear* equation

$$\Box u + u = F(u, \partial_t u, \partial_x u, \partial_t\partial_x u, \partial_x^2 u), \qquad (t,x)\in(0,\infty)\times\boldsymbol{R},$$

if the definition of $K_F(z)$ is replaced by

$$\frac{i}{2\pi}\int_0^{2\pi} F(\cos\theta,\ -\cosh z\sin\theta,\ \sinh z\sin\theta,\ \cosh z\sinh z\cos\theta,\ -\sinh^2 z\cos\theta)e^{-i\theta}d\theta.$$

REMARK 1.3. From Theorem 1.2, we see that J.-M. Delort's "null condition" studied in [**2**], [**3**] is *not* optimal for small data global existence for nonlinear Klein-Gordon equations when we consider the *forward* Cauchy problem (i.e., for $t>0$). Indeed, his condition is equivalent to $\operatorname{Re} K_F(z) \equiv 0$ in the cubic nonlinear case. Note that it is nothing but $\Phi_F(y) \equiv 0$, which guarantees the solution has just the logarithmic phase correction as is proved in [**3**]. On the other hand, our condition (1.7) is equivalent to $\Phi_F(y) \geq 0$.

Remark 1.4. In view of the denominator of the leading term of (1.8), it would be very reasonable to conjecture that the solution blows up in finite time if (1.7) is *not* satisfied. (This is a modification of the Delort conjecture. See [**2**], [**3**].) However, the author has no rigorous proof yet.

Remark 1.5. Without loss of generality, cubic homogeneous polynomials of (u, u_t, u_x) are written in the form

$$\begin{aligned} F(u,u_t,u_x) &= \big(\gamma_1 u^2 + \gamma_2 u_t^2 + \gamma_3 u_x^2 + \gamma_4 u_t u_x\big)u \\ &\quad + \big(\gamma_5 u^2 + \gamma_6 u_t^2 + \gamma_7 u_x^2\big)u_t + \big(\gamma_8 u^2 + \gamma_9 u_t^2 + \gamma_{10} u_x^2\big)u_x \end{aligned}$$

with real coefficients $\gamma_1, \cdots, \gamma_{10}$. For this F, it holds that

$$\begin{aligned} K_F(z) &= \frac{i}{8}\big(3\gamma_1 + \gamma_2\cosh^2 z + \gamma_3 \sinh^2 z - \gamma_4 \cosh z \sinh z\big) \\ &\quad - \frac{\cosh z}{8}\big(\gamma_5 + 3\gamma_6 \cosh^2 z + 3\gamma_7 \sinh^2 z\big) \\ &\quad + \frac{\sinh z}{8}\big(\gamma_8 + 3\gamma_9 \cosh^2 z + 3\gamma_{10} \sinh^2 z\big), \end{aligned}$$

whence

$$\varPhi_F(y) = \frac{-(\gamma_5 + 3\gamma_6) + (\gamma_8 + 3\gamma_9)y + (\gamma_5 - 3\gamma_7)y^2 + (3\gamma_{10} - \gamma_8)y^3}{8(1-y^2)}$$

and

$$\varPsi_F(y) = \frac{(3\gamma_1 + \gamma_2) - \gamma_4 y + (\gamma_3 - 3\gamma_1)y^2}{8(1-y^2)^{1/2}}.$$

We see from this expression that both $\varPhi_F(y)$ and $\varPsi_F(y)$ vanish identically if and only if F is written as a linear combination of

$$\big(-u^2 + 3u_t^2 - 3u_x^2\big)u, \quad \big(-3u^2 + u_t^2 - u_x^2\big)u_t, \quad \big(-3u^2 + u_t^2 - u_x^2\big)u_x$$

(cf. [**10**], [**8**]). Also we can check that $\inf_{z\in\boldsymbol{R}} \operatorname{Re} K_F(z) = 0, > 0, < 0$, when $F = u^3$, $-u_t^3$, $u_t^2 u_x$, respectively.

The rest of this paper is organized as follows. In Section 2, we prepare a preliminary lemma. Section 3 is devoted to the reduction of the problem and to getting some a priori estimate. After that, we prove Theorem 1.1 in Sections 4 and finally we give a sketch of the proof of Theorem 1.2. In what follows, all non-negative constants will be denoted by C unless otherwise specified.

2. A lemma on ODE.

This section is devoted to the proof of the following lemma. Similar argument may be found (less explicitly) in the works of N. Hayashi and P. I. Naumkin concerning the large time asymptotics for nonlinear Schrödinger equations (see e.g., [**5**], [**6**]).

LEMMA 2.1. (1) *Let* $\beta_0 \in \boldsymbol{C}$ *and* $\kappa \in \boldsymbol{C}$ *with* $\operatorname{Re}\kappa \geq 0$. *Let* $\varrho(\tau)$ *be a complex-valued function of* $\tau \in \boldsymbol{R}$ *which satisfies*

$$|\varrho(\tau)| \leq C\tau^{-1-\lambda}, \qquad \tau \geq \tau_0$$

with some constants $\lambda > 0$ *and* $\tau_0 > 1$. *If* $\beta(\tau)$ *solves the ODE*

$$\frac{d\beta}{d\tau} = -\frac{\kappa}{\tau}|\beta|^2\beta + \varepsilon\varrho(\tau), \quad \beta(\tau_0) = \varepsilon\beta_0 \tag{2.1}$$

for sufficiently small ε, *then there exists* $\beta_\infty \in \boldsymbol{C}$ *such that*

$$\beta(\tau) = \frac{\beta_\infty e^{i \operatorname{Im}\kappa|\beta_\infty|^2 \mathscr{L}(\tau, \operatorname{Re}\kappa|\beta_\infty|^2)}}{\sqrt{1 + 2\operatorname{Re}\kappa|\beta_\infty|^2 \log\tau}} + O\big((\log\tau)^{-3/2}\big) \tag{2.2}$$

as $\tau \to +\infty$, *where* $\mathscr{L}$ *is given by* (1.9).
(2) *If* $\beta(\tau)$, β_0, κ, $\varrho(\tau)$ *depend smoothly on some parameter* $z \in \boldsymbol{R}$ *and satisfy*

$$\big|\partial_z^j\beta(\tau,z)\big| \leq C_j\tau^\delta, \quad \big|\partial_z^j\kappa(z)\big| \leq C_j, \quad \big|\partial_z^j\varrho(\tau,z)\big| \leq C_j\tau^{-1-\lambda_j} \quad (j = 0, 1, 2, \cdots)$$

with some constants $C_j > 0$, $\lambda_j > 0$ *and sufficiently small* $\delta > 0$, *then* β_∞ *is a bounded smooth function of* z *and* (2.2) *is valid uniformly with respect to* z.
(3) *If, in addition,* $\kappa(z)$ *decays like* $O(1/z)$ *as* $|z| \to \infty$, *then* (2.2) *can be replaced by*

$$\beta(\tau,z) = \frac{\beta_\infty(z) e^{i \operatorname{Im}\kappa(z)|\beta_\infty(z)|^2 \mathscr{L}(\tau\cosh z, \operatorname{Re}\kappa(z)|\beta_\infty(z)|^2)}}{\sqrt{1 + 2\operatorname{Re}\kappa(z)|\beta_\infty(z)|^2 \log(\tau\cosh z)}} + O\bigg(\frac{e^{h|z|}}{\{\log(\tau\cosh z)\}^{3/2}}\bigg)$$

with arbitrary small positive number h.

REMARK 2.1. The assumption $\operatorname{Re}\kappa \geq 0$ is essential in the above lemma. Indeed, in the case of $\operatorname{Re}\kappa < 0$, the leading term in the right hand side of (2.2) blows up as τ tends to $\exp\big(\frac{1}{-2\operatorname{Re}\kappa|\beta_\infty|^2}\big)$ (cf. p. 72 of [**1**], Lemma 1.3.3 of [**7**], etc.).

PROOF OF LEMMA 2.1(1). We first remark that the solution of (2.1) is unique. Accordingly, $\beta(\tau)$ admits the following decomposition:

$$\beta(\tau) = \frac{\varepsilon p(\tau)}{\sqrt{q(\tau)}},$$

where $p(\tau)$ and $q(\tau)$ are the solutions of

$$
\begin{cases}
\dfrac{dp}{d\tau}(\tau) = -i\varepsilon^2 \dfrac{\operatorname{Im}\kappa}{\tau} \dfrac{|p(\tau)|^2}{q(\tau)} p(\tau) + \sqrt{q(\tau)}\varrho(\tau), \\
\dfrac{dq}{d\tau}(\tau) = 2\varepsilon^2 \dfrac{\operatorname{Re}\kappa}{\tau} |p(\tau)|^2, \\
p(\tau_0) = \beta_0, \quad q(\tau_0) = 1.
\end{cases}
\tag{2.3}
$$

Note that $p(\tau)$ is complex-valued, while $q(\tau)$ is real and strictly positive. In order to obtain the desired conclusion, it is sufficent to get the asymptotics of $p(\tau)$ and $q(\tau)$ as $\tau \to +\infty$.

Next we show that there exists a positive constant A such that

$$
|p(\tau)| < A \tag{2.4}
$$

for any $\tau \geq \tau_0$. We prove it by the contradiction argument. Suppose that for any $A > |\beta_0|$, there exists a finite time $T_A \in]\tau_0, \infty[$ such that

$$
\sup_{\tau \in [\tau_0, T_A[} |p(\tau)| \leq A \quad \text{and} \quad |p(T_A)| = A.
$$

Then, from the second equation of (2.3), we have

$$
1 \leq q(\tau) \leq 1 + 2\varepsilon^2 \operatorname{Re}\kappa A^2 \log \tau
$$

for $\tau \in [\tau_0, T_A]$. On the other hand, it follows from the first equation of (2.3) that

$$
\begin{aligned}
\frac{d}{d\tau}\left(|p(\tau)|^2\right) &= 2\operatorname{Re}\left[\overline{p(\tau)}\frac{dp}{d\tau}(\tau)\right] \\
&= 2\operatorname{Re}\left[\overline{p(\tau)}\sqrt{q(\tau)}\varrho(\tau)\right] \\
&\leq \frac{|p|^2}{\tau^{1+\lambda}} + q(\tau)|\varrho(\tau)|^2 \tau^{1+\lambda} \\
&\leq \frac{|p|^2}{\tau^{1+\lambda}} + \frac{C(1 + 2\varepsilon^2 \operatorname{Re}\kappa A^2 \log \tau)}{\tau^{1+\lambda}}.
\end{aligned}
$$

Then the Gronwall lemma gives us

$$
|p(T_A)|^2 \leq C|\beta_0|^2 + C(1 + \varepsilon^2 A^2) \leq C(1 + A^2 \varepsilon^2).
$$

When we choose $A \geq \sqrt{8C}$, we have

$$
|p(T_A)| \leq \frac{A}{2} < A
$$

for $\varepsilon \in]0, 1/\sqrt{8C}]$, which is the desired contradiction. Hence (2.4) must hold for some A.

Also we have

$$1 \le q(\tau) \le 1 + 2\varepsilon^2 \operatorname{Re}\kappa A^2 \log \tau$$

for any $\tau \ge \tau_0$.

Now, let us introduce

$$\theta(\tau) = -\operatorname{Im}\kappa \int_{\tau_0}^{\tau} \frac{|p(\sigma)|^2}{q(\sigma)\sigma} d\sigma$$

so that

$$\frac{d}{d\tau}\big(e^{-i\varepsilon^2\theta(\tau)}p(\tau)\big) = \sqrt{q(\tau)}\varrho(\tau)e^{-i\varepsilon^2\theta(\tau)}.$$

Then we have

$$\left|\frac{d}{d\tau}\big(e^{-i\varepsilon^2\theta(\tau)}p(\tau)\big)\right| \le \frac{C(1+C\varepsilon^2\log\tau)^{1/2}}{\tau^{1+\lambda}} \le \frac{C}{\tau^{1+\lambda/2}},$$

which implies the existence of $p_\infty \in \boldsymbol{C}$ such that

$$\big|p(\tau) - e^{i\varepsilon^2\theta(\tau)}p_\infty\big| \le \frac{C}{\tau^{\lambda/2}}.$$

From this it follows that

$$\big||p(\tau)|^2 - |p_\infty|^2\big| \le \big|p(\tau) - e^{i\varepsilon^2\theta(\tau)}p_\infty\big|(|p(\tau)| + |p_\infty|) \le \frac{C}{\tau^{\lambda/2}},$$

whence

$$\begin{aligned}
&\big|q(\tau) - 1 - 2\varepsilon^2\operatorname{Re}\kappa|p_\infty|^2\log\tau\big| \\
&\le \int_{\tau_0}^{\tau}\left|\frac{dq}{d\tau}(\sigma) - 2\varepsilon^2\frac{\operatorname{Re}\kappa}{\sigma}|p_\infty|^2\right|d\sigma + 2\varepsilon^2|\operatorname{Re}\kappa||p_\infty|^2\log\tau_0 \\
&\le 2\varepsilon^2\operatorname{Re}\kappa\int_{\tau_0}^{\infty}\big||p(\sigma)|^2 - |p_\infty|^2\big|\frac{d\sigma}{\sigma} + C\varepsilon^2 \\
&\le C\varepsilon^2.
\end{aligned}$$

Let us also introduce

$$s(\tau) = -\frac{\operatorname{Im}\kappa}{\tau}\left(\frac{|p(\tau)|^2}{q(\tau)} - \frac{|p_\infty|^2}{1 + 2\varepsilon^2\operatorname{Re}\kappa|p_\infty|^2\log\tau}\right).$$

Then we see that

$$\begin{aligned}|s(\tau)| &\le \frac{C}{\tau}\left(\frac{\big||p(\tau)|^2-|p_\infty|^2\big|}{q(\tau)}+\frac{|p_\infty|^2\big|q(\tau)-1-2\varepsilon^2\operatorname{Re}\kappa|p_\infty|^2\log\tau\big|}{q(\tau)\big\{1+2\varepsilon^2\operatorname{Re}\kappa|p_\infty|^2\log\tau\big\}}\right)\\ &\le \frac{C}{\tau^{1+\lambda/2}}+\frac{C\varepsilon^2}{\tau(1+C\varepsilon^2\log\tau)^2}\\ &\le \frac{C}{\tau(\log\tau)^2}\end{aligned}$$

and that

$$\begin{aligned}\theta(\tau) &= -\operatorname{Im}\kappa|p_\infty|^2\int_{\tau_0}^{\tau}\frac{d\sigma}{\big(1+2\varepsilon^2\operatorname{Re}\kappa|p_\infty|^2\log\sigma\big)\sigma}+\int_{\tau_0}^{\tau}s(\sigma)d\sigma\\ &= -\operatorname{Im}\kappa|p_\infty|^2\int_{1}^{\tau}\frac{d\sigma}{\big(1+2\varepsilon^2\operatorname{Re}\kappa|p_\infty|^2\log\sigma\big)\sigma}+\theta_0-\int_{\tau}^{\infty}s(\sigma)d\sigma\\ &= \operatorname{Im}\kappa|p_\infty|^2\mathscr{L}\big(\tau,\varepsilon^2\operatorname{Re}\kappa|p_\infty|^2\big)+\theta_0+O((\log\tau)^{-1}),\end{aligned}\tag{2.5}$$

where

$$\theta_0 = \operatorname{Im}\kappa|p_\infty|^2\int_{1}^{\tau_0}\frac{d\sigma}{\big(1+2\varepsilon^2\operatorname{Re}\kappa|p_\infty|^2\log\sigma\big)\sigma}+\int_{\tau_0}^{\infty}s(\sigma)d\sigma.\tag{2.6}$$

Thus, putting $\beta_\infty=\varepsilon p_\infty e^{i\varepsilon^2\theta_0}$, we deduce from (2.5) that

$$\big|\varepsilon p_\infty e^{i\varepsilon^2\theta(\tau)}-\beta_\infty e^{i\operatorname{Im}\kappa|\beta_\infty|^2\mathscr{L}(\tau,\operatorname{Re}\kappa|\beta_\infty|^2)}\big|\le\frac{C\varepsilon^3}{\log\tau}.$$

Summing up, we obtain

$$\beta(\tau)=\frac{\varepsilon p(\tau)}{\sqrt{q(\tau)}},$$

$$\big|\varepsilon p(\tau)-\beta_\infty e^{i\operatorname{Im}\kappa|\beta_\infty|^2\mathscr{L}(\tau,\operatorname{Re}\kappa|\beta_\infty|^2)}\big|\le\frac{C\varepsilon}{\log\tau},$$

$$\left|\frac{1}{\sqrt{q(\tau)}}-\frac{1}{\sqrt{1+2\operatorname{Re}\kappa|\beta_\infty|^2\log\tau}}\right|\le\frac{C}{\varepsilon(\log\tau)^{3/2}},$$

which leads to the desired asymptotic expression (2.2).

PROOF OF LEMMA 2.1(2). For any $m\in\boldsymbol{N}_0:=\boldsymbol{N}\cup\{0\}$ and $\tau_2>\tau_1\ge\tau_0$, we have

$$\begin{aligned}&\big\|\partial_z^m\big\{e^{-i\varepsilon^2\theta(\tau_2,\cdot)}p(\tau_2,\cdot)-e^{-i\varepsilon^2\theta(\tau_1,\cdot)}p(\tau_1,\cdot)\big\}\big\|_{L_z^\infty}\\ &\quad\le\int_{\tau_1}^{\tau_2}\big\|\partial_\tau\partial_z^m\big\{e^{-i\varepsilon^2\theta(\tau,\cdot)}p(\tau,\cdot)\big\}\big\|_{L_z^\infty}d\tau.\end{aligned}\tag{2.7}$$

On the other hand, the second equation of (2.3) yields

$$\partial_\tau \sqrt{q(\tau,z)} = \frac{\operatorname{Re}\kappa(z)|\beta(\tau,z)|^2}{\tau}\sqrt{q(\tau,z)},$$

which leads to

$$\sqrt{q(\tau,z)} = e^{\operatorname{Re}\kappa(z)\int_{\tau_0}^{\tau}|\beta(\sigma,z)|^2\frac{d\sigma}{\sigma}} = e^{i\varepsilon^2\theta(\tau,z)}e^{\kappa(z)\int_{\tau_0}^{\tau}|\beta(\sigma,z)|^2\frac{d\sigma}{\sigma}}.$$

Therefore we have

$$\partial_\tau\big(e^{-i\varepsilon^2\theta(\tau,z)}p(\tau,z)\big) = e^{-i\varepsilon^2\theta(\tau,z)}\sqrt{q(\tau,z)}\varrho(\tau,z) = e^{\kappa(z)\int_{\tau_0}^{\tau}|\beta(\sigma,z)|^2\frac{d\sigma}{\sigma}}\varrho(\tau,z). \tag{2.8}$$

From (2.8) and the Leibniz rule, we see that $\partial_\tau\partial_z^m(e^{-i\varepsilon^2\theta(\tau,z)}p(\tau,z))$ consists of the terms like

$$e^{-i\varepsilon^2\theta(\tau,z)}\sqrt{q(\tau,z)}\partial_z^k\varrho(\tau,z)\prod_{\sum_j l_j=l}\partial_z^{l_j}\left\{\kappa(z)\int_{\tau_0}^{\tau}|\beta(\sigma,z)|^2\frac{d\sigma}{\sigma}\right\}\qquad(k+l\le m),$$

which yields

$$\big\|\partial_\tau\partial_z^m\{e^{-i\varepsilon^2\theta(\tau,\cdot)}p(\tau,\cdot)\}\big\|_{L_z^\infty} \le C(1+\varepsilon^2\log\tau)^{1/2}\tau^{-(1+\min_{k\le m}\lambda_k)+C\delta}\in L^1(\tau_0,\infty). \tag{2.9}$$

By (2.7) and (2.9), the limit $\lim_{\tau\to+\infty}e^{-i\varepsilon^2\theta(\tau,z)}p(\tau,z) =: p_\infty(z)$ exists and belongs to $W^{m,\infty}(\boldsymbol{R})$ for any m. Since $\beta_\infty(z)=\varepsilon p_\infty(z)e^{i\varepsilon^2\theta_0(z)}$ with $\theta_0(z)$ given by (2.6), $\beta_\infty(z)$ is a bounded smooth function of z.

Proof of Lemma 2.1(3). Since $\sup_{z\in\boldsymbol{R}}|\kappa(z)\log\cosh z|<\infty$ and $|\beta_\infty(\tau,z)|\le C\varepsilon$, we have

$$\begin{aligned}&\big|q(\tau,z)-1-2\operatorname{Re}\kappa(z)|\beta_\infty(z)|^2\log(\tau\cosh z)\big|\\ &\quad\le\big|q(\tau,z)-1-2\operatorname{Re}\kappa(z)|\beta_\infty(z)|^2\log\tau\big|+2|\beta_\infty(z)|^2\big|\kappa(z)\log\cosh z\big|\\ &\quad\le C\varepsilon^2.\end{aligned}$$

Using this estimate, we can modify the previous argument to obtain

$$\begin{aligned}&\left|\beta(\tau,z)-\frac{\beta_\infty(z)e^{i\operatorname{Im}\kappa(z)|\beta_\infty(z)|^2\mathscr{L}(\tau\cosh z,\,\operatorname{Re}\kappa(z)|\beta_\infty(z)|^2)}}{\sqrt{1+2\operatorname{Re}\kappa(z)|\beta_\infty(z)|^2\log(\tau\cosh z)}}\right|\\ &\quad\le\frac{C}{\{\log(\tau\cosh z)\}^{3/2}}+\frac{C}{\tau^h\log(\tau\cosh z)}\le\frac{Ce^{h|z|}}{\{\log(\tau\cosh z)\}^{3/2}}\end{aligned}$$

with arbitrary small $h > 0$. □

3. Reduced equation and a priori estimate.

In this section, we make some reduction of the problem and derive an a priori estimate for the reduced equation. The argument of this section is almost same as that of the previous work [**13**], in which the $\boldsymbol{C}$ or $\boldsymbol{R}^2$ valued NLKG has been studied (see also [**3**], [**4**], [**12**]).

Let B be a positive constant which satisfies

$$\text{supp } u_0 \cup \text{supp } u_1 \subset \{x \in \boldsymbol{R} \ : \ |x| \le B\}$$

and let τ_0 be a fixed positive number strictly greater than $1 + 2B$. We start with the fact that we may treat the problem as if the Cauchy data is given on the upper branch of the hyperbola

$$\{(t,x) \in \boldsymbol{R}^{1+1} \ : \ (t+2B)^2 - |x|^2 = \tau_0^2, \ t > 0\}$$

and it is sufficiently smooth, small, compactly-supported. This is a consequence of the classical local existence theorem and the finite speed of propagation (see e.g., Proposition 1.4 of [**3**] for detail). Next, let us introduce the hyperbolic coordinate $(\tau, z) \in [\tau_0, \infty[\times \boldsymbol{R}$ in the interior of the light cone, i.e.,

$$t + 2B = \tau \cosh z, \ x = \tau \sinh z \quad \text{for} \quad |x| < t + 2B.$$

Then it is a routine to check that

$$\begin{aligned} \partial_t &= (\cosh z)\partial_\tau - \frac{\sinh z}{\tau}\partial_z, \\ \partial_x &= -(\sinh z)\partial_\tau + \frac{\cosh z}{\tau}\partial_z, \\ \Box &= \frac{\partial^2}{\partial \tau^2} + \frac{1}{\tau}\frac{\partial}{\partial \tau} - \frac{1}{\tau^2}\frac{\partial^2}{\partial z^2} \end{aligned}$$

and

$$\tau = \sqrt{(t+2B)^2 - |x|^2}, \quad z = \tanh^{-1}\left(\frac{x}{t+2B}\right).$$

We also take a weight function $\chi \in C^\infty(\boldsymbol{R})$ satisfying

$$0 < \chi(z) \le C_0 e^{-\eta|z|} \ \text{ and } \ |\chi^{(j)}(z)| \le C_j \chi(z) \ \ (j = 1, 2, \cdots)$$

with a large parameter $\eta \gg 1$ and positive constants C_0, C_j. With this weight function, let us define the new unknown function $v(\tau, z)$ by

$$u(t,x) = \frac{\chi(z)}{\tau^{1/2}} v(\tau, z).$$

Then we see that v satisfies

$$Pv = G(\tau, z, v, \partial_\tau v, \partial_z v),$$

if u solves (1.1), where

$$P = \frac{\partial^2}{\partial \tau^2} - \frac{1}{\tau^2}\left(\frac{\partial^2}{\partial z^2} + 2\frac{\chi'(z)}{\chi(z)}\frac{\partial}{\partial z} + \frac{\chi''(z)}{\chi(z)} - \frac{1}{4}\right) + 1,$$

$$G(\tau, z, v, \partial_\tau v, \partial_z v) = -\frac{g\chi(z)^2}{\tau}\left\{(\cosh z)\partial_\tau v - \frac{\sinh z}{\tau}\partial_z v - \frac{1}{\tau}\left((\sinh z)\frac{\chi'(z)}{\chi(z)} + \frac{\cosh z}{2}\right)v\right\}^3.$$

Therefore, the original problem (1.1)–(1.3) is reduced to

$$\begin{cases} Pv = G(\tau, z, v, \partial_\tau v, \partial_z v), & \tau > \tau_0,\ z \in \boldsymbol{R}, \\ (v, \partial_\tau v)|_{\tau=\tau_0} = (\varepsilon v_0, \varepsilon v_1) & z \in \boldsymbol{R}, \end{cases} \tag{3.1}$$

where v_0 and v_1 are sufficiently smooth functions of z with compact support.

The rest part of this section is devoted to getting an a priori estimate of the solution v to the reduced problem (3.1). Let us define

$$E_s(\tau) = \sum_{k=0}^{s} \frac{1}{2}\int_{\boldsymbol{R}} \left|\partial_\tau \partial_z^k v(\tau, z)\right|^2 + \left|\frac{\partial_z}{\tau}\partial_z^k v(\tau, z)\right|^2 + \left|\partial_z^k v(\tau, z)\right|^2 dz$$

for $s \in \boldsymbol{N}_0$. What we are going to prove is the following lemma.

LEMMA 3.1. *Let $\delta \in]0, \frac{1}{3}]$, $s \geq 3$ and $T > \tau_0$. Suppose that $v(\tau, z)$ is a smooth solution of* (3.1) *for $\tau \in [\tau_0, T[$. Then there exists $\varepsilon_1 > 0$ such that*

$$E_s(\tau) \leq C_1 \varepsilon^2 \tau^\delta \tag{3.2}$$

and

$$|v(\tau, z)| + |\partial_\tau v(\tau, z)| + \frac{1}{\tau}|\partial_z v(\tau, z)| \leq C_2 \varepsilon \tag{3.3}$$

for $\tau \in [\tau_0, T[$, provided $\varepsilon \in]0, \varepsilon_1]$. Here C_1, C_2 are positive constants independent of δ, ε, T, τ, z, but may depend on s.

REMARK 3.1. Using (3.2), (3.3) and the relation

$$\partial_\tau^2 v = -v + \frac{1}{\tau^2}\left(\frac{\partial^2}{\partial z^2} + 2\frac{\chi'(z)}{\chi(z)}\frac{\partial}{\partial z} + \frac{\chi''(z)}{\chi(z)} - \frac{1}{4}\right)v + G(\tau, z, v, \partial_\tau v, \partial_z v),$$

we have

$$\sup_{\tau\in[\tau_0,T^*[}\sum_{j+k\le 2}\left\|\partial_\tau^j\left(\frac{\partial_z}{\tau}\right)^k v(\tau,\cdot)\right\|_{L^\infty} \le C\varepsilon,$$

where T^* denotes the lifespan of the classical solution. Thus the classical blow-up criterion yields $T^* = +\infty$.

Proof of Lemma 3.1. Set

$$M(\tau) = \sup_{(\sigma,z)\in[\tau_0,\tau[\times\boldsymbol{R}}\left(|v(\sigma,z)| + |\partial_\tau v(\sigma,z)| + \frac{1}{\sigma}|\partial_z v(\sigma,z)|\right).$$

To prove Lemma 3.1, it suffices to show (3.2) and (3.3) under the assumption $M(T) \le \varepsilon^{1/2}$. Indeed, when we choose $\varepsilon_2 \in]0, \varepsilon_1]$ so that

$$C_2\varepsilon_2^{1/2} \le \frac{1}{2}$$

and

$$\varepsilon_2^{1/2}\sup_{z\in\boldsymbol{R}}\left(|v_0(z)| + |v_1(z)| + \frac{1}{\tau_0}|\partial_z v_0(z)|\right) \le \frac{1}{2},$$

then, for any $\varepsilon \in]0, \varepsilon_2]$, we see from (3.3) that

$$M(T) \le \frac{\varepsilon^{1/2}}{2}$$

as well as

$$M(\tau_0 + \varDelta) \le \varepsilon^{1/2}$$

with some $\varDelta > 0$. This implies the induction hypothesis $M(T) \le \varepsilon^{1/2}$ is harmless.

Now, we prove (3.2) under the assumption $M(T) \le \varepsilon^{1/2}$. Let us introduce

$$E_s(\tau; w) = \sum_{k=0}^{s}\frac{1}{2}\int_{\boldsymbol{R}}\left|\partial_\tau\partial_z^k w(\tau,z)\right|^2 + \left|\frac{\partial_z}{\tau}\partial_z^k w(\tau,z)\right|^2 + \left|\partial_z^k w(\tau,z)\right|^2 dz$$

for $s \in \boldsymbol{N}_0$ and for smooth function w of $(\tau, z) \in [\tau_0, T[\times\boldsymbol{R}$. We start with the following energy inequality, whose proof is found in Appendix of [**13**] (see also §3 of [**12**] or §2 of [**4**]).

LEMMA 3.2. *For $s \in \boldsymbol{N}_0$ and $l = 0, 1$, we have*

$$\frac{d}{d\tau}E_s(\tau; w) \leq \frac{C^*}{\tau^{1+l}}E_{s+l}(\tau; w) + CE_s(\tau; w)^{1/2}\|Pw(\tau, \cdot)\|_{H^s} + \frac{C}{\tau^2}E_s(\tau; w).$$

Here C^* is given by

$$C^* = 2 \sup_{z \in \boldsymbol{R}} \frac{|\chi'(z)|}{\chi(z)}$$

and $\|\cdot\|_{H^s}$ denotes the standard norm of the Sobolev space H^s.

We shall apply the above lemma with $l = 0$, $s = s_0 + s_1 + 1$ and $w = v$, where s_0 is an integer greater than C^*, and s_1 is a fixed arbitrary non-negative integer. Since the Gagliardo-Nirenberg inequality yields

$$\big\|G\big(\tau, \cdot, v(\tau, \cdot), \partial_\tau v(\tau, \cdot), \partial_z v(\tau, \cdot)\big)\big\|_{H^s} \leq \frac{C}{\tau}M(T)^2 E_s(\tau)^{1/2} \leq \frac{C}{\tau}\varepsilon E_s(\tau)^{1/2},$$

we have

$$\frac{d}{d\tau}E_{s_0+s_1+1}(\tau) \leq \left(\frac{C^* + C\varepsilon}{\tau} + \frac{C}{\tau^2}\right)E_{s_0+s_1+1}(\tau) \leq \left(\frac{s_0 + \frac{1}{2}}{\tau} + \frac{C}{\tau^2}\right)E_{s_0+s_1+1}(\tau).$$

Thus we obtain

$$E_{s_0+s_1+1}(\tau) \leq E_{s_0+s_1+1}(\tau_0) \exp\left(\int_{\tau_0}^{\tau} \frac{s_0 + \frac{1}{2}}{\sigma} + \frac{C}{\sigma^2} d\sigma\right) \leq C\varepsilon^2 \tau^{s_0 + \frac{1}{2}}.$$

Next, we apply Lemma 3.2 with $l = 1$, $s = s_0 + s_1$. Then we have

$$\begin{aligned}\frac{d}{d\tau}E_{s_0+s_1}(\tau) &\leq \frac{C^*}{\tau^2}E_{s_0+s_1+1}(\tau) + \frac{C\varepsilon}{\tau}E_{s_0+s_1}(\tau) + \frac{C}{\tau^2}E_{s_0+s_1}(\tau) \\ &\leq C\varepsilon^2\tau^{s_0 - \frac{3}{2}} + \left(\frac{C\varepsilon}{\tau} + \frac{C}{\tau^2}\right)E_{s_0+s_1}(\tau).\end{aligned}$$

Therefore it follows from the Gronwall lemma that

$$E_{s_0+s_1}(\tau) \leq C\varepsilon^2\tau^{s_0 - \frac{1}{2}}.$$

Repeating the same procedure n times, we have

$$E_{s_0+s_1+1-n}(\tau) \leq C\varepsilon^2\tau^{s_0 - n + \frac{1}{2}}$$

for $n = 1, 2, \cdots, s_0$. In particular we have

$$E_{s_1+1}(\tau) \le C\varepsilon^2\tau^{1/2}.$$

Finally, we again use Lemma 3.2 with $l = 1$, $s = s_1$ to obtain

$$\begin{aligned}\frac{d}{d\tau}E_{s_1}(\tau) &\le \frac{C^*}{\tau^2}E_{s_1+1}(\tau) + \frac{C\varepsilon}{\tau}E_{s_1}(\tau) + \frac{C}{\tau^2}E_{s_1}(\tau)\\ &\le \frac{C\varepsilon^2}{\tau^{3/2}} + \left(\frac{C\varepsilon}{\tau} + \frac{C}{\tau^2}\right)E_{s_1}(\tau),\end{aligned}$$

whence we deduce

$$E_{s_1}(\tau) \le C\varepsilon^2\tau^{C\varepsilon}$$

for $\tau \in [\tau_0, T[$. Replacing s_1 by s and choosing ε so small that $C\varepsilon \le \delta$, we arrive at (3.2).

We next prove (3.3). As in [**12**], [**13**], let us introduce the $\boldsymbol{C}$-valued function $\alpha(\tau, z)$ by

$$\alpha(\tau, z) = e^{-i\tau}\left(1 + \frac{1}{i}\frac{\partial}{\partial\tau}\right)v(\tau, z), \qquad (\tau, z) \in [\tau_0, T[\times\boldsymbol{R},$$

for the solution $v(\tau, z)$ to (3.1). In view of the relations

$$|\alpha(\tau, z)| = \left(|v(\tau, z)|^2 + |\partial_\tau v(\tau, z)|^2\right)^{1/2}$$

and

$$\frac{1}{\tau}|\partial_z v(\tau, z)| \le \frac{CE_2(\tau)^{1/2}}{\tau} \le C\varepsilon\tau^{-(1-\delta)} \le C\varepsilon,$$

it suffices to show that

$$\sup_{(\tau,z)\in[\tau_0,T[\times\boldsymbol{R}} |\alpha(\tau, z)| \le C\varepsilon \tag{3.4}$$

holds true under the assumptions $M(T) \le \varepsilon^{1/2}$ and (3.2).

First we note that

$$\begin{aligned}\frac{\partial\alpha}{\partial\tau} &= -ie^{-i\tau}\left(\partial_\tau^2 + 1\right)v\\ &= -ie^{-i\tau}\left\{G(\tau, z, v, \partial_\tau v, \partial_z v) + \frac{1}{\tau^2}\left(\partial_z^2 + 2\frac{\chi'(z)}{\chi(z)}\partial_z + \frac{\chi''(z)}{\chi(z)} - \frac{1}{4}\right)v\right\}\\ &= ie^{-i\tau}\frac{g\chi(z)^2}{\tau}(\cosh z\partial_\tau v)^3 + \frac{R}{\tau^2},\end{aligned}$$

where

$$
\begin{aligned}
R(\tau,z) = &-i\tau^2 e^{-i\tau}\left\{G(\tau,z,v,\partial_\tau v,\partial_z v) + \frac{g\chi(z)^2}{\tau}(\cosh z\partial_\tau v)^3\right\}\\
&-ie^{-i\tau}\left(\frac{\partial^2}{\partial z^2} + 2\frac{\chi'(z)}{\chi(z)}\frac{\partial}{\partial z} + \frac{\chi''(z)}{\chi(z)} - \frac{1}{4}\right)v(\tau,z).
\end{aligned}
$$

Using the assumption $M(T) \le \varepsilon^{1/2}$, we have

$$
\left|\frac{g\chi(z)^2}{\tau}(\cosh z\partial_\tau v)^3\right| \le \frac{CM(T)^3}{\tau} \le \frac{C\varepsilon^{3/2}}{\tau}.
$$

Also we have

$$
|R(\tau,z)| \le C\varepsilon\tau^{3\delta/2} \tag{3.5}
$$

because

$$
\begin{aligned}
\tau^2\left|G(\tau,z,v,\partial_\tau v,\partial_z v) + \frac{g\chi(z)^2}{\tau}(\cosh z\partial_\tau v)^3\right| &\le Ce^{-(2\eta-3)|z|}E_2(\tau)^{3/2}\\
&\le C\varepsilon^2\tau^{3\delta/2}
\end{aligned}
$$

and

$$
\begin{aligned}
\left|\left(\frac{\partial^2}{\partial z^2} + 2\frac{\chi'(z)}{\chi(z)}\frac{\partial}{\partial z} + \frac{\chi''(z)}{\chi(z)} - \frac{1}{4}\right)v(\tau,z)\right| &\le C\sum_{j=0}^{2}\left|\partial_z^j v(\tau,z)\right|\\
&\le CE_3(\tau)^{1/2}\\
&\le C\varepsilon\tau^{\delta/2}.
\end{aligned}
$$

From them it follows that

$$
\left|\frac{\partial\alpha}{\partial\tau}(\tau,z)\right| \le \frac{C\varepsilon^{3/2}}{\tau} + \frac{C\varepsilon}{\tau^{2-3\delta/2}} \le \frac{C\varepsilon}{\tau}. \tag{3.6}
$$

Moreover, since

$$
\begin{aligned}
ie^{-i\tau}\frac{g\chi(z)^2}{\tau}(\cosh z\partial_\tau v)^3 &= ie^{-i\tau}\frac{g\chi(z)^2\cosh^3 z}{\tau}\left\{\frac{i(\alpha e^{i\tau} - \overline{\alpha}e^{-i\tau})}{2}\right\}^3\\
&= \frac{g\chi(z)^2\cosh^3 z}{8\tau}\left\{\alpha^3 e^{i2\tau} - 3\alpha^2\overline{\alpha} + 3\alpha\overline{\alpha}^2 e^{-i2\tau} - \overline{\alpha}^3 e^{-i4\tau}\right\},
\end{aligned}
$$

we see that α satisfies

$$
\frac{\partial\alpha}{\partial\tau} = -\frac{\kappa(z)}{\tau}|\alpha|^2\alpha + S + \frac{R}{\tau^2},
$$

where

$$\kappa(z) = \frac{3g\chi(z)^2 \cosh^3 z}{8}$$

and

$$S = \frac{g\chi(z)^2 \cosh^3 z}{8}\left[\alpha^3 \frac{e^{i2\tau}}{\tau} + 3\alpha\overline{\alpha}^2 \frac{e^{-i2\tau}}{\tau} - \overline{\alpha}^3 \frac{e^{-i4\tau}}{\tau}\right].$$

Noting that $\operatorname{Re}\kappa(z) > 0$, we have

$$\begin{aligned}
\frac{\partial}{\partial\tau}|\alpha(\tau,z)|^2 &= 2\operatorname{Re}[\overline{\alpha}\partial_\tau\alpha] \\
&= -\frac{2\operatorname{Re}\kappa(z)}{\tau}|\alpha|^4 + 2\operatorname{Re}[\overline{\alpha}S] + 2\operatorname{Re}\left[\frac{\overline{\alpha}R}{\tau^2}\right] \\
&\le 2\operatorname{Re}[\overline{\alpha}S] + \frac{|\alpha|^2}{\tau^2} + \frac{|R|^2}{\tau^2} \\
&\le 2\operatorname{Re}[\overline{\alpha}S] + \frac{|\alpha|^2}{\tau^2} + \frac{C\varepsilon^2}{\tau^{2-3\delta}},
\end{aligned}$$

which yields

$$\begin{aligned}
|\alpha(\tau,z)|^2 &\le C\varepsilon^2 + 2\left|\int_{\tau_0}^{\tau} \overline{\alpha}S\,d\sigma\right| + \int_{\tau_0}^{\tau} |\alpha(\sigma,z)|^2 \frac{d\sigma}{\sigma^2} \\
&\le C\varepsilon^2 + \int_{\tau_0}^{\tau} |\alpha(\sigma,z)|^2 \frac{d\sigma}{\sigma^2} \qquad (3.7)
\end{aligned}$$

for $\tau \in [\tau_0, T[$, provided that

$$\sup_{\tau\in[\tau_0,T[}\left|\int_{\tau_0}^{\tau} \overline{\alpha}S\,d\sigma\right| \le C\varepsilon^2. \qquad (3.8)$$

Once we get (3.7), we can apply the Gronwall lemma to obtain (3.4).

It remains to prove (3.8). To this end, we observe that

$$\begin{aligned}
\int_{\tau_0}^{\tau} \alpha^3\overline{\alpha}\frac{e^{i2\sigma}}{\sigma}d\sigma &= \int_{\tau_0}^{\tau} (\partial_\tau A_1)(\sigma,z) + A_2(\sigma,z)d\sigma \\
&= A_1(\tau,z) - A_1(\tau_0,z) + \int_{\tau_0}^{\tau} A_2(\sigma,z)d\sigma,
\end{aligned}$$

where

$$A_1(\tau,z) = \alpha^3\overline{\alpha}\frac{e^{i2\tau}}{i2\tau}, \quad A_2(\tau,z) = \alpha^3\overline{\alpha}\frac{e^{i2\tau}}{i2\tau^2} - \left(3\alpha_1^2\overline{\alpha}\partial_\tau\alpha + \alpha_1^3\overline{\partial_\tau\alpha}\right)\frac{e^{i2\tau}}{i2\tau}.$$

Using (3.6) and $M(T) \leq \varepsilon^{1/2}$, we have

$$|A_1(\tau,z)| \leq \frac{C\varepsilon^2}{\tau}, \quad |A_2(\tau,z)| \leq \frac{C}{\tau} \cdot \varepsilon^{3/2} \cdot \frac{C\varepsilon}{\tau} \leq \frac{C\varepsilon^2}{\tau^2}.$$

From them we deduce that

$$\sup_{\tau\in[\tau_0,T[} \left| \frac{g\chi(z)^2 \cosh^3 z}{8} \int_{\tau_0}^{\tau} \alpha^3 \overline{\alpha} \frac{e^{i2\sigma}}{\sigma} d\sigma \right| \leq C\varepsilon^2 e^{-(2\eta-3)|z|} \left(1 + \int_{\tau_0}^{\infty} \frac{d\sigma}{\sigma^2}\right) \leq C\varepsilon^2.$$

The other two terms of $\overline{\alpha}S$ can be treated in the same manner because they are of the form

$$(\text{bounded functions of } z) \times (\text{quartic terms of } (\alpha,\overline{\alpha})) \times \frac{e^{i\omega\tau}}{\tau}$$

with $\omega \neq 0$. The proof of Lemma 3.1 is completed. □

4. Proof of Theorem 1.1.

We are in position to prove Theorem 1.1. From the argument of the previous section, we see that

$$\frac{\partial\alpha}{\partial\tau} = -\frac{3g\chi(z)^2 \cosh^3 z}{8\tau} |\alpha|^2 \alpha + S + \frac{R}{\tau^2}$$

and that $S(\tau,z)$ can be written as

$$S = \partial_\tau S_1 + S_2$$

with

$$|S_1(\tau,z)| \leq \frac{C\varepsilon^3 e^{-(2\eta-3)|z|}}{\tau}, \qquad |S_2(\tau,z)| \leq \frac{C\varepsilon^3 e^{-(2\eta-3)|z|}}{\tau^2}.$$

Putting $\beta = \alpha - S_1$, we have

$$\frac{\partial\beta}{\partial\tau} = -\frac{3g\chi(z)^2 \cosh^3 z}{8\tau} |\beta|^2 \beta + \varepsilon\varrho(\tau,z),$$

where

$$\varrho(\tau,z) = \frac{1}{\varepsilon}\left[S_2 - \frac{3g\chi(z)^2 \cosh^3 z}{8\tau} \left\{|\alpha|^2\alpha - |\alpha+S_1|^2(\alpha+S_1)\right\} + \frac{R}{\tau^2}\right].$$

Note that

$$|\varrho(\tau, z)| \le \frac{C}{\varepsilon}\left(\frac{\varepsilon^3}{\tau^2} + \frac{\varepsilon^5}{\tau^2} + \frac{C\varepsilon\tau^{3\delta/2}}{\tau^2}\right) \le C\tau^{-1-\lambda}$$

with $\lambda = 1 - \frac{3\delta}{2} > 0$. Therefore it follows from Lemma 2.1 that

$$\begin{aligned} \alpha(\tau, z) &= \beta(\tau, z) + S_1(\tau, z) \\ &= \frac{\beta_\infty(z)}{\sqrt{1 + \frac{3g}{4}\chi(z)^2 \cosh^3 z |\beta_\infty(z)|^2 \log(\tau \cosh z)}} + O\left(\frac{e^{h|z|}}{\{\log(\tau \cosh z)\}^{3/2}}\right) \end{aligned} \tag{4.1}$$

as $\tau \to +\infty$, where $\beta_\infty(z)$ is a bounded smooth function of z, and h denotes an arbitrary small positive number. In particular we have

$$|\alpha(\tau, z)|\chi(z) \cosh^{3/2} z \le \frac{C}{\sqrt{\log(\tau \cosh z)}}. \tag{4.2}$$

Now, we are going back to the original variables. Remember that our change of variable is

$$u(t, x) = \frac{\chi(z)}{\sqrt{\tau}} \operatorname{Re}\left[\alpha(\tau, z)e^{i\tau}\right] \tag{4.3}$$

with $\tau = \sqrt{(t + 2B)^2 - x^2}$, $z = \tanh^{-1}(x/t)$ and $t \gg 1$, $|x| < t + 2B$. It follows from (4.2) and (4.3) that

$$|u(t, x)| \le \frac{\chi(z) \cosh^{1/2} z}{\sqrt{t + 2B}} |\alpha(\tau, z)| \le \frac{C(1 + t)^{-1/2}}{\sqrt{\log(2 + t)}}. \tag{4.4}$$

Similarly, we have

$$\sum_{|\alpha|=1} |\partial_{t,x}^\alpha u(t, x)| \le \frac{C(1 + t)^{-1/2}}{\sqrt{\log(2 + t)}} \tag{4.5}$$

because $\partial_t u$ and $\partial_x u$ are written as

$$\begin{aligned} \partial_t u(t, x) &= \left((\cosh z)\partial_\tau - \frac{\sinh z}{\tau}\partial_z\right)\left\{\frac{\chi(z)}{\tau^{1/2}} v(\tau, z)\right\} \\ &= -\frac{\chi(z) \cosh^{3/2} z}{(\tau \cosh z)^{1/2}} \operatorname{Im}\left[\alpha(\tau, z)e^{i\tau}\right] \\ &\quad - \frac{\chi(z) \cosh^{3/2} z}{(\tau \cosh z)^{3/2}} \left((\tanh z)\partial_z + (\tanh z)\frac{\chi'(z)}{\chi(z)} + \frac{1}{2}\right) v(\tau, z) \end{aligned}$$

and

$$
\begin{aligned}
\partial_x u(t,x) &= \left(-(\sinh z)\partial_\tau + \frac{\cosh z}{\tau}\partial_z\right)\left\{\frac{\chi(z)}{\tau^{1/2}} v(\tau,z)\right\} \\
&= \frac{\chi(z)\tanh z\cosh^{3/2} z}{(\tau\cosh z)^{1/2}} \operatorname{Im}\left[\alpha(\tau,z)e^{i\tau}\right] \\
&\quad + \frac{\chi(z)\cosh^{3/2} z}{(\tau\cosh z)^{3/2}}\left(\partial_z + \frac{\chi'(z)}{\chi(z)} + \frac{\tanh z}{2}\right) v(\tau,z),
\end{aligned}
$$

respectively. Using (4.4), (4.5) and the finite propagation speed, we have

$$
\begin{aligned}
\sum_{|\alpha|\le 1} \|\partial_{t,x}^\alpha u(t,\cdot)\|_{L^p(\boldsymbol{R})} &= \sum_{|\alpha|\le 1} \|\partial_{t,x}^\alpha u(t,\cdot)\|_{L^p(\{|x|\le t+B\})} \\
&\le \frac{C(1+t)^{-1/2}}{\sqrt{\log(2+t)}}\left(\int_{|x|\le t+B} 1 \ dx\right)^{1/p} \\
&\le \frac{C(1+t)^{-(1/2-1/p)}}{\sqrt{\log(2+t)}},
\end{aligned}
$$

which yields (1.4). As for (1.5), it follows from (4.1) and (4.3) that

$$
\begin{aligned}
u(t,x) &= \frac{\frac{1}{\sqrt{t+2B}} \operatorname{Re}\left[\chi(z)\beta_\infty(z)\cosh^{1/2} z \ e^{i\tau}\right]}{\sqrt{1 + \frac{3g}{4}\left|\chi(z)\beta_\infty(z)\cosh^{1/2} z\right|^2 \cosh^2 z \log(\tau\cosh z)}} + O\left(\frac{e^{-(\eta-\frac{1}{2}-h)|z|}}{t^{1/2}(\log t)^{3/2}}\right) \\
&= \frac{\frac{1}{\sqrt{t+2B}} \operatorname{Re}\left[b(\frac{x}{t+2B})e^{i((t+2B)^2-x^2)^{1/2}}\right]}{\sqrt{1 + \frac{3g}{4}\left|b(\frac{x}{t+2B})\right|^2 (1-|\frac{x}{t+2B}|^2)^{-1}\log(t+2B)}} + O\big(t^{-1/2}(\log t)^{-3/2}\big) \\
&= \frac{\frac{1}{\sqrt{t}} \operatorname{Re}\left[a(\frac{x}{t})e^{i(t^2-x^2)^{1/2}}\right]}{\sqrt{1 + \frac{3g}{4}\left|a(\frac{x}{t})\right|^2 (1-|\frac{x}{t}|^2)^{-1}\log t}} + O\big(t^{-1/2}(\log t)^{-3/2}\big),
\end{aligned}
$$

where

$$
a(y) = \begin{cases} b(y)e^{i2B\sqrt{1-y^2}} & \text{if} \quad |y|<1, \\ 0 & \text{if} \quad |y|\ge 1, \end{cases}
$$

$$
b(y) = \chi(z)\beta_\infty(z)\cosh^{1/2} z\Big|_{z=\tanh^{-1} y}
$$

(cf. p. 58–59 of [**3**]). □

5. Proof of Theorem 1.2.

In this section, we give a sketch of the proof of Theorem 1.2. Since the essential idea is same as that of Theorem 1.1, we omit the detail here.

As before, we look for the solution of (1.6)–(1.3) in the form

$$u(t,x) = \frac{\chi(z)}{\tau^{1/2}} v(\tau,z).$$

Then we see that $v(\tau,z)$ satisfies

$$Pv = G(\tau, z, v, \partial_\tau v, \partial_z v),$$

where

$$P = (\partial_\tau^2 + 1) - \frac{1}{\tau^2}\left(\partial_z^2 + 2\frac{\chi'(z)}{\chi(z)}\partial_z + \frac{\chi''(z)}{\chi(z)} - \frac{1}{4}\right),$$

$$G(\tau, z, v, \partial_\tau v, \partial_z v) = \frac{\chi(z)^2}{\tau} F(v, \cosh z \partial_\tau v, -\sinh z \partial_\tau v) + (\text{remainder terms}).$$

Also, when we put

$$\alpha(\tau,z) = e^{-i\tau}\left(1 + \frac{1}{i}\frac{\partial}{\partial\tau}\right) v(\tau,z),$$

we have

$$\begin{aligned}
\frac{\partial\alpha}{\partial\tau} &= -ie^{-i\tau}(\partial_\tau^2 + 1)v \\
&= -ie^{-i\tau}\frac{\chi(z)^2}{\tau} F(v, \cosh z\partial_\tau v, -\sinh z \partial_\tau v) + \frac{R(\tau,z)}{\tau^2} \\
&= -\frac{i\chi(z)^2}{\tau} e^{-i\tau} F\big(\operatorname{Re}[\alpha e^{i\tau}],\ -\cosh z \operatorname{Im}[\alpha e^{i\tau}],\ \sinh z \operatorname{Im}[\alpha e^{i\tau}]\big) + \frac{R(\tau,z)}{\tau^2} \\
&= -\frac{\kappa(z)}{\tau}|\alpha|^2\alpha + S(\tau,z) + \frac{R(\tau,z)}{\tau^2},
\end{aligned} \tag{5.1}$$

where

$$\kappa(z) = \frac{i\chi(z)^2}{2\pi}\int_0^{2\pi} e^{-i\theta} F(\cos\theta,\ -\cosh z\sin\theta,\ \sinh z \sin\theta)d\theta = \chi(z)^2 K_F(z),$$

$$\begin{aligned}
S(\tau,z) = -\frac{i\chi(z)^2}{2\pi}\bigg\{ &\alpha^3 \frac{e^{i2\tau}}{\tau}\int_0^{2\pi} e^{-i3\theta} F(\cos\theta,\ -\cosh z\sin\theta,\ \sinh z\sin\theta)d\theta \\
&+ \alpha\overline{\alpha}^2 \frac{e^{-i2\tau}}{\tau}\int_0^{2\pi} e^{i\theta} F(\cos\theta,\ -\cosh z\sin\theta,\ \sinh z\sin\theta)d\theta \\
&+ \overline{\alpha}^3 \frac{e^{-i4\tau}}{\tau}\int_0^{2\pi} e^{i3\theta} F(\cos\theta,\ -\cosh z\sin\theta,\ \sinh z\sin\theta)d\theta \bigg\},
\end{aligned}$$

and

$$R(\tau,z) = i\tau^2 e^{-i\tau}\left\{\frac{\chi(z)^2}{\tau}F(v,\cosh z\partial_\tau v, -\sinh z\partial_\tau v) - G(\tau,z,v,\partial_\tau v,\partial_z v)\right\}$$
$$- ie^{-i\tau}\left(\partial_z^2 + 2\frac{\chi'(z)}{\chi(z)}\partial_z + \frac{\chi''(z)}{\chi(z)} - \frac{1}{4}\right)v(\tau,z).$$

(For a detail of the derivation of (5.1), see Appendix below.) Therefore, as in the previous section, we can apply Lemma 2.1 to obtain the asymptotics of $\alpha(\tau,z)$ as $\tau \to +\infty$. Substituting this asymptotics into (4.3), we can obtain the desired conclusion. □

Appendix.

We give a detail of the derivation of (5.1) here. By the cubic homogeneity of F, we have

$$\begin{aligned} &e^{-i\tau}F\big(\operatorname{Re}[\alpha e^{i\tau}],\ -\cosh z\operatorname{Im}[\alpha e^{i\tau}],\ \sinh z\operatorname{Im}[\alpha e^{i\tau}]\big)\\ &= |\alpha|^3 e^{-i\tau}F\left(\operatorname{Re}\left[\frac{\alpha}{|\alpha|}e^{i\tau}\right],\ -\cosh z\operatorname{Im}\left[\frac{\alpha}{|\alpha|}e^{i\tau}\right],\ \sinh z\operatorname{Im}\left[\frac{\alpha}{|\alpha|}e^{i\tau}\right]\right)\\ &= |\alpha|^2\alpha\ H(z,\theta_0(\tau,z)),\end{aligned}$$

where

$$H(z,\theta) = e^{-i\theta}F(\cos\theta, -\cosh z\sin\theta, \sinh z\sin\theta)$$

and

$$\theta_0(\tau,z) = \tau + \arg\alpha(\tau,z).$$

Since $H(z,\theta)$ is 2π-periodic with respect to θ, we have

$$H(z,\theta) = \sum_{n\in\boldsymbol{Z}}\hat{H}_n(z)e^{in\theta},$$

where $\hat{H}_n(z)$ denotes the n-th Fourier coefficient, i.e.,

$$\hat{H}_n(z) = \frac{1}{2\pi}\int_0^{2\pi} H(z,\theta)e^{-in\theta}d\theta.$$

Noting that

$$\int_0^{2\pi} F(\cos\theta, -\cosh z\sin\theta, \sinh z\sin\theta)e^{-i(n+1)\theta}d\theta = 0$$

when $n+1 \notin \{1,\ 3,\ -1,\ -3\}$, we see that

$$H(z,\theta) = \hat{H}_0(z) + \hat{H}_2(z)e^{i2\theta} + \hat{H}_{-2}(z)e^{-i2\theta} + \hat{H}_{-4}(z)e^{-i4\theta}.$$

Summing up, we obtain

$$
\begin{aligned}
&-\frac{i\chi(z)^2}{\tau}e^{-i\tau}F\big(\operatorname{Re}[\alpha e^{i\tau}],\ -\cosh z\operatorname{Im}[\alpha e^{i\tau}],\ \sinh z\operatorname{Im}[\alpha e^{i\tau}]\big)\\
&=-\frac{i\chi(z)^2}{\tau}|\alpha|^2\alpha\, H(z,\theta_0(\tau,z))\\
&=-\frac{i\chi(z)^2}{\tau}|\alpha|^2\alpha\left\{\hat{H}_0(z)+\hat{H}_2(z)\frac{\alpha^2}{|\alpha|^2}e^{i2\tau}+\hat{H}_{-2}(z)\frac{\overline{\alpha}^2}{|\alpha|^2}e^{-i2\tau}+\hat{H}_{-4}(z)\frac{\overline{\alpha}^4}{|\alpha|^4}e^{-i4\tau}\right\}\\
&=-\frac{i\chi(z)^2\hat{H}_0(z)}{\tau}|\alpha|^2\alpha-i\chi(z)^2\left\{\alpha^3\frac{e^{i2\tau}}{\tau}\hat{H}_2(z)+\alpha\overline{\alpha}^2\frac{e^{-i2\tau}}{\tau}\hat{H}_{-2}(z)+\overline{\alpha}^3\frac{e^{-i4\tau}}{\tau}\hat{H}_{-4}(z)\right\}\\
&=-\frac{\kappa(z)}{\tau}|\alpha|^2\alpha+S(\tau,z).
\end{aligned}
$$

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