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Time periodic problem for the compressible Navier–Stokes equation on $\mathbb{R}^2$ with antisymmetry

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Abstract. The compressible Navier–Stokes equation is considered on the two dimensional whole space when the external force is periodic in the time variable. The existence of a time periodic solution is proved for sufficiently small time periodic external force with antisymmetry condition. The proof is based on using the time-T-map associated with the linearized problem around the motionless state with constant density. In some weighted L^∞ and Sobolev spaces the spectral properties of the time-T-map are investigated by a potential theoretic method and an energy method. The existence of a stationary solution to the stationary problem is also shown for sufficiently small time-independent external force with antisymmetry condition on $\mathbb{R}^2$.

1. Introduction.

We consider time periodic problem of the following compressible Navier–Stokes equation for barotropic flow in $\mathbb{R}^2$:

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho v) = 0, \\ \rho(\partial_t v + (v\cdot\nabla)v) - \mu\Delta v - (\mu+\mu')\nabla \operatorname{div} v + \nabla p(\rho) = \rho g. \end{cases} \tag{1.1}$$

Here $\rho = \rho(x,t)$ and $v = (v_1(x,t), v_2(x,t))$ denote the unknown density and the unknown velocity field, respectively, at time $t \geq 0$ and position $x \in \mathbb{R}^2$; $p = p(\rho)$ is the pressure that is assumed to be a smooth function of ρ satisfying

$$p'(\rho_*) > 0,$$

for a given positive constant ρ_*; μ and μ' are the viscosity coefficients that are assumed to be constants satisfying

$$\mu > 0, \quad \mu + \mu' \geq 0;$$

and $g = g(x,t)$ is a given external force periodic in t. We assume that $g = g(x,t)$ satisfies the condition

$$g(x, t+T) = g(x,t) \quad (x \in \mathbb{R}^2,\ t \in \mathbb{R}), \tag{1.2}$$

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for some constant $T > 0$. We also suppose that g has the form $g = \nabla^{\perp} G := ((\partial/\partial_{x_2})G, -(\partial/\partial_{x_1})G)$, where $G(x,t)$ is a scalar function satisfying the following antisymmetry condition for $x \in \mathbb{R}^2$;

$$\begin{cases} G(-x_1, x_2, t) = -G(x_1, x_2, t), \\ G(x_1, -x_2, t) = -G(x_1, x_2, t), \\ G(x_2, x_1, t) = -G(x_1, x_2, t). \end{cases} \tag{1.3}$$

The antisymmetry condition (1.3) was used in the stationary problem for imcompressible Navier–Stokes equation on $\mathbb{R}^2$ ([**12**]).

In this paper time periodic problem and stationary problem are considered for the compressible Navier–Stokes equation (1.1) on $\mathbb{R}^2$. Concerning the time periodic problem for (1.1) on the whole space, Ma, Ukai, and Yang [**11**] showed the existence and stability of a time periodic solution on $\mathbb{R}^n$ with the space dimension $n \geq 5$. In [**11**] it was shown that if $g \in C^0(\mathbb{R}; H^{N-1} \cap L^1)$ with $g(x, t+T) = g(x,t)$ and g is sufficiently small, then there exists a time periodic solution (ρ_{per}, v_{per}) around $(\rho_*, 0)$, where $N \in \mathbb{Z}$ satisfying $N \geq n+2$. It was also shown that for sufficiently small perturbations the time periodic solution is stable and it holds that

$$\begin{aligned} &\|(\rho(t), v(t)) - (\rho_{per}(t), v_{per}(t))\|_{H^{N-1}} \\ &\quad \leq C(1+t)^{-n/4} \|(\rho_0, v_0) - (\rho_{per}(t_0), v_{per}(t_0))\|_{H^{N-1} \cap L^1}, \end{aligned}$$

where t_0 is a certain initial time and $(\rho, v)|_{t=t_0} = (\rho_0, v_0)$. Here the symbol H^k stands for the L^2-Sobolev space on $\mathbb{R}^n$ of order k.

In [**4**] the time periodic problem on $\mathbb{R}^n$ was investigated for $n \geq 3$. It was proved that if g satisfies the following condition for the space variable;

$$g(-x, t) = -g(x,t) \quad (x \in \mathbb{R}^n,\ t \in \mathbb{R}), \tag{1.4}$$

and g is sufficiently small in some weighted L^2-Sobolev space, then there exists a time periodic solution (ρ_{per}, v_{per}) for (1.1) around $(\rho_*, 0)$ and $u_{per}(t) = (\rho_{per}(t) - \rho_*, v_{per}(t))$ satisfies

$$\begin{aligned} &\sup_{t \in [0,T]} (\|u_{per}(t)\|_{L^2} + \|x \nabla u_{per}(t)\|_{L^2}) \\ &\quad \leq C\{\|(1+|x|)g\|_{C([0,T]; L^1 \cap L^2)} + \|(1+|x|)g\|_{L^2(0,T; H^{s-1})}\}, \end{aligned}$$

where s is an integer satisfying $s \geq [n/2] + 1$. Moreover, (ρ_{per}, v_{per}) is asymptotically stable and it holds that

$$\|(\rho(t), v(t)) - (\rho_{per}(t), v_{per}(t))\|_{L^2} = O(t^{-n/4}) \text{ as } t \to \infty \tag{1.5}$$

for sufficiently small initial perturbations. In [**10**], the existence and stability of time periodic solution were proved for $n \geq 3$, without assuming the condition (1.4); it was shown that if g is small enough in some weighted L^∞ and L^2 Sobolev spaces then there exists a time periodic solution (ρ_{per}, v_{per}) around $(\rho_*, 0)$; and the time periodic solution is stable under sufficiently small initial perturbation and the perturbation $(\rho-\rho_{per}, v-v_{per})$ satisfies

$$\|(\rho(t)-\rho_{per}(t), v(t)-v_{per}(t))\|_{L^\infty} \to 0 \quad (t\to\infty).$$

Concerning the stationary problem of (1.1), Shibata and Tanaka [**8**] showed the existence and stability of a stationary solution on $\mathbb{R}^3$. They showed that if $g=g(x)$ is small enough in some weighted L^∞ and L^2 Sobolev spaces then there exists a stationary solution (ρ^*, v^*) around the motionless state $(\rho_*, 0)$. Moreover, it was shown that for sufficiently small initial perturbations the stationary solution is stable and the perturbation $(\rho-\rho^*, v-v^*)$ satisfies

$$\|(\rho(t)-\rho^*, v(t)-v^*)\|_{L^\infty} \to 0 \quad (t\to\infty). \tag{1.6}$$

In [**9**], the convergence rate for (1.6) was studied and it was shown if the initial perturbation $(\rho(0)-\rho^*, v(0)-v^*)$ satisfies the estimate $\|(\rho(0)-\rho^*, v(0)-v^*)\|_{H^3} \ll 1$ and $(\rho(0)-\rho^*, v(0)-v^*) \in L^{6/5}$ then

$$\|(\rho(t)-\rho^*(t), v(t)-v^*(t))\|_{L^\infty} \le Ct^{-(1-\delta)/2} \quad (t\to\infty),$$

where δ is any small positive number.

To our knowledge there seems no existence result on time periodic (and stationary) problem for (1.1) on $\mathbb{R}^2$.

In this paper we consider the existence of a time periodic solution for (1.1) on $\mathbb{R}^2$ under (1.3). It will be proved that if $g=\nabla^\perp G$ satisfies (1.2), (1.3) and the estimate

$$\begin{aligned}&\|(1+|x|)g\|_{C([0,T];L^1)} + \|(1+|x|^3)g\|_{C([0,T];L^\infty)}\\ &\quad +\|(1+|x|^2)G\|_{C([0,T];L^\infty)} + \|(1+|x|^2)G\|_{L^2(0,T;H^s)} \ll 1\end{aligned}$$

for an integer $s\ge 3$, then there exists a time periodic solution $u_{per}=(\rho_{per}-\rho_*, v_{per})\in C(\mathbb{R};L^\infty)$ for (1.1), with $\nabla u_{per}\in C(\mathbb{R};H^{s-1})$ having time period T and u_{per} satisfies the estimate

$$\begin{aligned}&\sup_{t\in[0,T]}\Big\{\sum_{j=0}^{1}\|(1+|x|^{1+j})\partial_x^j(\rho_{per}-\rho_*)(t)\|_{L^\infty} + \sum_{j=0}^{1}\|(1+|x|^{1+j})\partial_x^j v_{per}(t)\|_{L^\infty}\Big\}\\ &\quad \le C\{\|(1+|x|)g\|_{C([0,T];L^1)} + \|(1+|x|^3)g\|_{C([0,T];L^\infty)}\\ &\qquad + \|(1+|x|^2)G\|_{C([0,T];L^\infty)} + \|(1+|x|^2)G\|_{L^2(0,T;H^s)}\}.\end{aligned}$$

Furthermore, we obtain the existence of a stationary solution for the stationary problem of (1.1). It will be proved that if $g=\nabla^\perp G$ is time-independent and satisfies (1.3) and the estimate

$$\begin{aligned}&\|(1+|x|)g\|_{L^1} + \|(1+|x|^3)g\|_{L^\infty}\\ &\quad +\|(1+|x|^2)G\|_{L^\infty} + \|(1+|x|^2)G\|_{H^s} \ll 1\end{aligned}$$

for an integer $s \geq 3$, then there exists a stationary solution $u^* = (\rho^* - \rho_*, v^*) \in L^\infty$ with $\nabla u^* \in H^{s-1}$ for the stationary problem for (1.1), and u^* satisfies the estimate

$$\begin{aligned}\sum_{j=0}^{1} \|(1+|x|^{1+j})\partial_x^j(\rho^* - \rho_*)\|_{L^\infty} + \sum_{j=0}^{1} \|(1+|x|^{1+j})\partial_x^j v^*\|_{L^\infty} \\ \leq C\{\|(1+|x|)g\|_{L^1} + \|(1+|x|^3)g\|_{L^\infty} + \|(1+|x|^2)G\|_{L^\infty} + \|(1+|x|^2)G\|_{H^s}\}.\end{aligned}$$

The existence of a time periodic solution is shown by using time-T-map concerned with the linearized problem around the constant state. We use a coupled system of equations for a low frequency part and high frequency part of solution as in [**4**]. Concerning the low frequency part, we apply the potential theoretic method similar to that in the study of the stationary problem [**8**] which controls spatial decay properties for a solution. The same method was used to study the time periodic problem in [**10**] for the space dimension $n \geq 3$. The main difference between the analysis in this paper and that in [**10**] is stated as follows. We denote by A_1 the linearized operator around $(\rho_*, 0)$ on the low frequency part. Then we estimate $(I - S_1(T))^{-1}$ in some weighted L^∞ space, where S_1 denotes the semigroup generated by A_1. In contrast to [**10**], since we consider the problem on $\mathbb{R}^2$, the integral kernel $(I - S_1(T))^{-1}$ behaves like $O(\log|x|)$ as $x \to \infty$, which is the same as the fundamental solution of the Laplace equation. More precisely, it follows from the spectral resolution that

$$\mathcal{F}(I - S_1(T))^{-1} \sim -\frac{1}{T}\begin{pmatrix} \dfrac{\nu+\tilde{\nu}}{\gamma^2} & -\dfrac{i^\top \xi}{\gamma|\xi|^2} \\ -\dfrac{i\xi}{\gamma|\xi|^2} & \dfrac{1}{\nu|\xi|^2}\left(I_2 - \dfrac{\xi^\top \xi}{|\xi|^2}\right)\end{pmatrix} \quad \text{as} \quad \xi \to 0, \tag{1.7}$$

where the superscript ${}^\top\cdot$ denotes the transposition, I_2 denotes the 2×2 identity matrix and $\mathcal{F}$ denotes the Fourier transform. Then the order $\log|x|$ appears from the Stokes inverse in the right hand side of (1.7). This prevents us from controlling spatial decay properties for the convection term and the external force. To overcome this difficulty, since the slowly decaying order appears from the Stokes inverse, we introduce the antisymmetry condition which was used in the stationary problem for incompressible flow on $\mathbb{R}^2$ ([**12**]). Moreover, we use the following two key observations to estimate the convection term $v \cdot \nabla v$.

The one is concerned with the formulation for the low frequency part. Due to the slow decay of v at spatial infinity, for the low frequency part we formulate the equation not only using the conservation form with the momentum as in [**10**] but also rewriting the convection term into a sum of the incompressible flow part and the potential flow part. More precisely, we rewrite the convection term as

$$\partial_{x_2}\begin{pmatrix} v_1 v_2 \\ (v_2)^2 - (v_1)^2 \end{pmatrix} + \partial_{x_1}\begin{pmatrix} 0 \\ v_2 v_1 \end{pmatrix} + \nabla(v_1)^2. \tag{1.8}$$

This enables us to use of the antisymmetry condition effectively for the low frequency part. (Cf., Remark 4.7 bellow.) Note that in [**12**], since the incompressible flow was considered, the vorticity formlation was used effectively to estimate the convection term under the antisymmetry condition (1.3). On the other hand, since we consider the compressible flow, we use a coupled system of the conservation form of the momentum and the velocity formulation with (1.8) instead of the vorticity formulation.

Another key observation is concerned with the potential theoretic method on $\mathbb{R}^2$. By making use of the antisymmetry condition (1.3), an estimate for convolution is established in a weighted L^∞ space on $\mathbb{R}^2$. (See Lemma 4.11 bellow.) Using this estimate, we obtain the estimate for a convolution with the convection term in the weighted L^∞ space.

As for the high frequency part, we use the velocity formulation to avoid some derivative loss by using the energy method as in [**4**], [**10**].

The existence of the stationary solution is proved similarly. Since the fundamental solution for the linearized stationary problem for the low frequency part is the same as the leading part of $(I - S_1(T))^{-1}$, one can prove the existence of the stationary solution by similar estimates to those used in the proof of the existence of a time periodic solution.

This paper is organized as follows. In section 2, notations and auxiliary lemmas are introduced, which are used in this paper. In section 3, main results of this paper are stated. In section 4, we reformulate the problem. A coupled system with the conservation of momentum for the low frequency part and the equation of motion for the high frequency part is introduced; and we will then rewrite by a system of integral equations in terms of the time-T-map. We also establish some estimates for a convolution which will appear in the low frequency part. In section 5, we derive estimates for a solution related to the time-T-map for the low frequency part. In section 6, some spectral properties of the time-T-map are stated for the high frequency part. In section 7, nonlinear terms are estimated and we then prove the existence of a time periodic solution by the iteration argument.

2. Preliminaries.

In this section we introduce notations which will be used throughout this paper. Furthermore, we introduce some lemmas which will be useful in the proof of the main results.

We denote the norm on X by $\|\cdot\|_X$ for a given Banach space X.

Let $1 \leqq p \leqq \infty$. L^p stands for the usual L^p space on $\mathbb{R}^2$. We denote the inner product of L^2 by $(\cdot,\cdot)$. Let k be a nonnegative integer. H^k denotes the usual L^2-Sobolev space of order k. (As usual, we define that $H^0 := L^2$.)

For simplicity, L^p stands for the set of all vector fields $w = {}^\top(w_1, w_2)$ on $\mathbb{R}^2$ with $w_j \in L^p$ $(j = 1, 2)$ and we denote by $\|\cdot\|_{L^p}$ the norm $\|\cdot\|_{(L^p)^2}$ if no confusion will occur. Similarly, we denote by a function space X the set of all vector fields $w = {}^\top(w_1, w_2)$ on $\mathbb{R}^2$ with $w_j \in X$ $(j = 1, 2)$; and we denote the norm $\|\cdot\|_{X^2}$ on it by $\|\cdot\|_X$ if no confusion will occur.

We take $u = {}^\top(\phi, w)$ with $\phi \in H^k$ and $w = {}^\top(w_1, w_2) \in H^m$. Then the norm of u on $H^k \times H^m$ is denoted by $\|u\|_{H^k \times H^m}$, that is, we define

$$\|u\|_{H^k \times H^m} := \left(\|\phi\|_{H^k}^2 + \|w\|_{H^m}^2\right)^{1/2}.$$

When $m = k$, we simply denote $H^k \times (H^k)^2$ by H^k. We also denote the norm $\|u\|_{H^k \times (H^k)^2}$ by $\|u\|_{H^k}$, i.e., we define that

$$H^k := H^k \times (H^k)^2, \quad \|u\|_{H^k} := \|u\|_{H^k \times (H^k)^2} \quad (u = {}^\top(\phi, w)).$$

Similarly, for $u = {}^\top(\phi, w) \in X \times Y$ with $w = {}^\top(w_1, w_2)$, the norm $\|u\|_{X \times Y}$ stands for

$$\|u\|_{X \times Y} := \left(\|\phi\|_X^2 + \|w\|_Y^2\right)^{1/2} \quad (u = {}^\top(\phi, w)).$$

If $Y = X^2$, the symbol X stands for $X \times X^2$ for simplicity, and we define its norm $\|u\|_{X \times X^2}$ by $\|u\|_X$;

$$X := X \times X^2, \quad \|u\|_X := \|u\|_{X \times X^2} \quad (u = {}^\top(\phi, w)).$$

A function space with spatial weight is defined as follows. For a nonnegative integer ℓ and $1 \le p \le \infty$, the symbol L^p_ℓ denotes the weighted L^p space which is defined by

$$L^p_\ell := \{u \in L^p; \|u\|_{L^p_\ell} := \|(1 + |x|)^\ell u\|_{L^p} < \infty\}.$$

The notations $\hat{f}$ and $\mathcal{F}[f]$ denote the Fourier transform of f:

$$\hat{f}(\xi) = \mathcal{F}[f](\xi) := \int_{\mathbb{R}^2} f(x) e^{-ix\cdot\xi} dx \quad (\xi \in \mathbb{R}^2).$$

In addition, we denote the inverse Fourier transform of f by $\mathcal{F}^{-1}[f]$:

$$\mathcal{F}^{-1}[f](x) := (2\pi)^{-2} \int_{\mathbb{R}^2} f(\xi) e^{i\xi\cdot x} d\xi \quad (x \in \mathbb{R}^2).$$

Let k be a nonnegative integer and let r_1 and r_∞ be positive constants satisfying $r_1 < r_\infty$. The symbol $H^k_{(\infty)}$ stands for the set of all $u \in H^k$ satisfying supp $\hat{u} \subset \{|\xi| \ge r_1\}$, and the symbol $L^2_{(1)}$ stands for the set of all $u \in L^2$ satisfying supp $\hat{u} \subset \{|\xi| \le r_\infty\}$. It follows from Lemma 4.3 (ii) bellow that $H^k \cap L^2_{(1)} = L^2_{(1)}$ for any nonnegative integer k.

Let k and ℓ be nonnegative integers. The weighted L^2-Sobolev space H^k_ℓ is defined by

$$H^k_\ell := \{u \in H^k; \|u\|_{H^k_\ell} < +\infty\},$$

where

$$\|u\|_{H^k_\ell} := \left(\sum_{j=0}^{\ell} |u|^2_{H^k_j}\right)^{1/2},$$

$$|u|_{H^k_\ell} := \left(\sum_{|\alpha| \le k} \| \, |x|^\ell \partial_x^\alpha u \, \|^2_{L^2}\right)^{1/2}.$$

Moreover, $H^k_{(\infty),\ell}$ denotes the weighted L^2-Sobolev space for the high frequency part defined by

$$H^k_{(\infty),\ell} := \{u \in H^k_{(\infty)}; \|u\|_{H^k_\ell} < +\infty\}.$$

Let ℓ be a nonnegative integer. The symbol $L^2_{(1),\ell}$ stands for the weighted L^2 space for the low frequency part defined by

$$L^2_{(1),\ell} := \{f \in L^2_\ell; f \in L^2_{(1)}\}.$$

For $-\infty \le a < b \le \infty$, the symbol $C^k([a,b];X)$ denotes the set of all C^k functions on $[a,b]$ with values in X. Similarly, $L^p(a,b;X)$ and $H^k(a,b;X)$ denote the L^p-Bochner space on (a,b) and the L^2-Bochner–Sobolev space of order k respectively.

The time periodic problem is considered in function spaces with the following anti-symmetry. Γ_j $(j=1,2,3)$ are defined by

$$\begin{aligned}(\Gamma_1 u)(x) &:= {}^\top(\phi(-x_1,x_2), -w_1(-x_1,x_2), w_2(-x_1,x_2)),\\ (\Gamma_2 u)(x) &:= {}^\top(\phi(x_1,-x_2), w_1(x_1,-x_2), -w_2(x_1,-x_2)),\\ (\Gamma_3 u)(x_1,x_2) &:= {}^\top(\phi(x_2,x_1), w_2(x_2,x_1), w_1(x_2,x_1))\end{aligned}$$

for $u(x) = {}^\top(\phi(x), w_1(x), w_2(x)),\ x \in \mathbb{R}^2$. For a function space X on $\mathbb{R}^2$, the space X_{sym} denotes the set of all $u = {}^\top(\phi, w_1, w_2) \in X$ satisfying $\Gamma_j u = u$ $(j=1,2,3)$.

Let X be a function space on $\mathbb{R}^2$. $X_\diamond$ denotes the set of all $f \in X$ satisfying

$$f(-x_1,x_2) = f(x_1,x_2), \quad f(x_1,-x_2) = f(x_1,x_2),$$
$$f(x_2,x_1) = f(x_1,x_2).$$

$X_\#$ denotes the set of all $f = {}^\top(f_1,f_2) \in X$ satisfying

$$\begin{cases} f_1(-x_1,x_2) = -f_1(x_1,x_2), \quad f_1(x_1,-x_2) = f_1(x_1,x_2),\\ f_2(-x_1,x_2) = f_2(x_1,x_2), \quad f_2(x_1,-x_2) = -f_2(x_1,x_2),\\ f_1(x_2,x_1) = f_2(x_1,x_2), \quad f_2(x_2,x_1) = f_1(x_1,x_2).\end{cases}$$

Note that if f in X has the form $f = \nabla^\perp F = {}^\top((\partial/\partial_{x_2})F, -(\partial/\partial_{x_1})F)$, where F satisfies the condition

$$F(-x_1,x_2) = -F(x_1,x_2), \quad F(x_1,-x_2) = -F(x_1,x_2),$$
$$F(x_2,x_1) = -F(x_1,x_2)$$

for $\mathbb{R}^2$, then $f \in X_\#$.

The space $\mathscr{X}_{(1)}$ is defined by

$$\mathscr{X}_{(1)} := \{\phi \in L^\infty_1 \cap L^2; \mathrm{supp}\ \hat{\phi} \subset \{|\xi| \le r_\infty\}, \|\phi\|_{\mathscr{X}_{(1)}} < +\infty\},$$

where the norm is defined by

$$\|\phi\|_{\mathscr{X}_{(1)}} := \|\phi\|_{\mathscr{X}_{(1),L^\infty}} + \|\phi\|_{\mathscr{X}_{(1),L^2}},$$
$$\|\phi\|_{\mathscr{X}_{(1),L^\infty}} := \sum_{k=0}^{1} \|\nabla^k \phi\|_{L^\infty_{k+1}},$$
$$\|\phi\|_{\mathscr{X}_{(1),L^2}} := \sum_{k=0}^{1} \|\nabla^k \phi\|_{L^2_k}.$$

On the other hand, $\mathscr{Y}_{(1)}$ is defined by

$$\mathscr{Y}_{(1)} := \{w \in L^\infty_1, \nabla w \in H^1; \mathrm{supp}\ \hat{w} \subset \{|\xi| \le r_\infty\}, \|w\|_{\mathscr{Y}_{(1)}} < +\infty\},$$

where

$$\|w\|_{\mathscr{Y}_{(1)}} := \|w\|_{\mathscr{X}_{(1),L^\infty}} + \|w\|_{\mathscr{Y}_{(1),L^2}},$$
$$\|w\|_{\mathscr{Y}_{(1),L^2}} := \sum_{j=1}^{2} \|(1+|x|)^{j-1}\nabla^j w\|_{L^2}.$$

We define a weighted space for the low frequency part $\mathscr{Z}_{(1)}(a,b)$ by

$$\mathscr{Z}_{(1)}(a,b) := C^1([a,b];\mathscr{X}_{(1)}) \times \Big[C([a,b];\mathscr{Y}_{(1)}) \cap H^1(a,b;\mathscr{Y}_{(1)})\Big].$$

Let s be a nonnegative integer satisfying $s \ge 3$. We denote by the space $\mathscr{Z}^k_{(\infty),1}(a,b)$ $(k = s-1, s)$ the weighted space for the high frequency part defined by

$$\begin{aligned}\mathscr{Z}^k_{(\infty),1}(a,b) := &\big[C([a,b];H^k_{(\infty),2}) \cap C^1([a,b];L^2_2)\big]\\ &\times \big[L^2(a,b;H^{k+1}_{(\infty),2}) \cap C([a,b];H^k_{(\infty),2}) \cap H^1(a,b;H^{k-1}_{(\infty),2})\big].\end{aligned}$$

Let s be a nonnegative integer satisfying $s \ge 3$ and let $k = s-1, s$. We define a space $X^k(a,b)$ by

$$\begin{aligned}X^k(a,b) := \big\{&\{u_{(1)}, u_{(\infty)}\}; u_{(1)} \in \mathscr{Z}_{(1)}(a,b), u_{(\infty)} \in \mathscr{Z}^k_{(\infty),2}(a,b),\\ &\partial_t \phi_{(\infty)} \in C([a,b];L^2_1), u_{(j)} = {}^\top(\phi_{(j)}, w_{(j)})\,(j=1,\infty)\big\},\end{aligned}$$

and we define the norm by

$$\begin{aligned}\|\{u_{(1)}, u_{(\infty)}\}\|_{X^k(a,b)} :=&\|u_{(1)}\|_{\mathscr{Z}_{(1)(a,b)}} + \|u_{(\infty)}\|_{\mathscr{Z}^k_{(\infty),2}(a,b)}\\ &+ \|\partial_t \phi_{(\infty)}\|_{C([a,b];L^2_1)} + \|\partial_t u_{(1)}\|_{C([a,b];L^2)} + \|\partial_t \nabla u_{(1)}\|_{C([a,b];L^2_1)}.\end{aligned}$$

Let s be a nonnegative integer satisfying $s \ge 3$ and let $k = s-1, s$. We define a space Y^k by

$$\begin{aligned}Y^k := \big\{&\{u_{(1)}, u_{(\infty)}\}; u_{(1)} \in \mathscr{X}_{(1)} \times \mathscr{Y}_{(1)}, u_{(\infty)} \in H^k_{(\infty),2} \times H^{k+1}_{(\infty),2},\\ &u_{(j)} = {}^\top(\phi_{(j)}, w_{(j)})\,(j=1,\infty)\big\}\end{aligned}$$

and we define the norm by

$$\|\{u_{(1)}, u_{(\infty)}\}\|_{Y^k} := \|u_{(1)}\|_{\mathscr{X}_{(1)} \times \mathscr{Y}_{(1)}} + \|u_{(\infty)}\|_{H^k_{(\infty),2} \times H^{k+1}_{(\infty),2}}.$$

Function spaces of time periodic functions with period T are introduced as follows. $C_{per}(\mathbb{R}; X)$ stands for the set of all time periodic continuous functions with values in X and period T whose norm is defined by $\|\cdot\|_{C([0,T];X)}$; Similarly, $L^2_{per}(\mathbb{R}; X)$ denotes the set of all time periodic locally square integrable functions with values in X and period T whose the norm is defined by $\|\cdot\|_{L^2(0,T;X)}$. Similarly, $H^1_{per}(\mathbb{R}; X)$ and $X^k_{per}(\mathbb{R})$, and so on, are defined.

For operators L_1 and L_2, we denote by $[L_1, L_2]$ the commutator of L_1 and L_2, i.e.,

$$[L_1, L_2]f := L_1(L_2 f) - L_2(L_1 f).$$

We next state some lemmas which will be used in the proof of the main results.

The following lemma is the well-known Sobolev type inequality.

Lemma 2.1. *Let s be an integer satisfying $s \geq 2$. Then there holds the inequality*

$$\|f\|_{L^\infty} \leq C\|\nabla f\|_{H^{s-1}},$$

for $f \in H^s$.

The following Hardy's inequality is known for a function satisfying the oddness conditions in (1.3) on $\mathbb{R}^2$.

Lemma 2.2. *Let $u \in H^1$ and we assume that u satisfies*

$$u(-x_1, x_2) = -u(x_1, x_2) \text{ or } u(x_1, -x_2) = -u(x_1, x_2) \tag{2.1}$$

for $x = {}^\top(x_1, x_2)$. Then there holds the inequality

$$\left\|\frac{u}{|x|}\right\|_{L^2} \leq C\|\nabla u\|_{L^2}.$$

See, e.g., [**1**] for the proof of Lemma 2.2.

We state the following inequalities which are concerned with composite functions.

Lemma 2.3. *Let s be an integer satisfying $s \geq 2$. Let s_j and $\mu_{(j)}$ $(j = 1, \cdots, \ell)$ be nonnegative integers and multiindices satisfying $0 \leq |\mu_{(j)}| \leq s_j \leq s + |\mu_{(j)}|$, $\mu = \mu_{(1)} + \cdots + \mu_\ell$, $s = s_1 + \cdots + s_\ell \geq (\ell - 1)s + |\mu|$, respectively. Then there holds*

$$\| \partial_x^{\mu_{(1)}} f_1 \cdots \partial_x^{\mu_\ell} f_\ell \|_{L^2} \leq C \prod_{1 \leq j \leq \ell} \| f_j \|_{H^{s_j}} \quad (f_j \in H^{s_j}).$$

See, e.g., [**3**] for the proof of Lemma 2.3.

Lemma 2.4. *Let s be an integer satisfying $s \geq 2$. Suppose that F is a smooth function on I, where I is a compact interval of $\mathbb{R}$. Then for a multi-index α with $1 \leq |\alpha| \leq s$, there hold the estimates*

$$\|[\partial_x^\alpha, F(f_1)]f_2\|_{L^2} \leq C\|F\|_{C^{|\alpha|}(I)} \left\{1 + \|\nabla f_1\|_{s-1}^{|\alpha|-1}\right\} \|\nabla f_1\|_{H^{s-1}} \|f_2\|_{H^{|\alpha|}},$$

for $f_1 \in H^s$ with $f_1(x) \in I$ for all $x \in \mathbb{R}^2$ and $f_2 \in H^{|\alpha|}$; and

$$\|[\partial_x^\alpha, F(f_1)]f_2\|_{L^2} \leq C\|F\|_{C^{|\alpha|}(I)} \left\{1 + \|\nabla f_1\|_{s-1}^{|\alpha|-1}\right\} \|\nabla f_1\|_{H^{s}} \|f_2\|_{H^{|\alpha|-1}},$$

for $f_1 \in H^{s+1}$ with $f_1(x) \in I$ for all $x \in \mathbb{R}^2$ and $f_2 \in H^{|\alpha|-1}$.

See, e.g., [**2**] for the proof of Lemma 2.4.

3. Main results.

In this section, we state our main result on the existence of a time periodic solution for (1.1). We also state our result on the existence of a stationary solution of (1.1) when g is independent of t. To state our results, the following operators are introduced, which decompose a function into its low and high frequency parts respectively. We define operators P_1 and P_∞ on L^2 by

$$P_j f := \mathcal{F}^{-1}(\hat{\chi}_j \mathcal{F}[f]) \quad (f \in L^2, j = 1, \infty),$$

where

$$\begin{aligned}
&\hat{\chi}_j(\xi) \in C^\infty(\mathbb{R}^2) \quad (j = 1, \infty), \quad 0 \leq \hat{\chi}_j \leq 1 \quad (j = 1, \infty),\\
&\hat{\chi}_1(\xi) := \begin{cases} 1 & (|\xi| \leq r_1),\\ 0 & (|\xi| \geq r_\infty),\end{cases}\\
&\hat{\chi}_\infty(\xi) := 1 - \hat{\chi}_1(\xi),\\
&0 < r_1 < r_\infty.
\end{aligned}$$

r_1 and r_∞ are positive constants satisfying $0 < r_1 < r_\infty < 2\gamma/(\nu + \tilde{\nu})$ in such a way that the estimate (5.6) in Lemma 5.3 below holds for $|\xi| \leq r_\infty$.

Substituting $\phi = (\rho - \rho_*)/\rho_*$ and $w = v/\gamma$ with $\gamma := \sqrt{p'(\rho_*)}$ into (1.1), time periodic problem (1.1) is formulated as

$$\partial_t u + Au = -B[u]u + G(u, g), \tag{3.1}$$

where

$$A := \begin{pmatrix} 0 & \gamma \mathrm{div} \\ \gamma \nabla & -\nu\Delta - \tilde{\nu}\nabla \mathrm{div} \end{pmatrix}, \quad \nu := \frac{\mu}{\rho_*}, \quad \tilde{\nu} := \frac{\mu + \mu'}{\rho_*}, \tag{3.2}$$

$$B[\tilde{u}]u := \gamma \begin{pmatrix} \tilde{w} \cdot \nabla \phi \\ 0 \end{pmatrix} \text{ for } u = {}^\top(\phi, w),\ \tilde{u} = {}^\top(\tilde{\phi}, \tilde{w}), \tag{3.3}$$

and

$$G(u,g) := \begin{pmatrix} F^0(u) \\ \tilde{F}(u,g) \end{pmatrix}, \tag{3.4}$$

$$F^0(u) := -\gamma\phi \mathrm{div} w, \tag{3.5}$$

$$\tilde{F}(u,g) := -\gamma(1+\phi)(w\cdot\nabla w) - \phi\partial_t w - \nabla(\tilde{p}(\phi)\phi^2) + \frac{1+\phi}{\gamma}g, \tag{3.6}$$

$$\tilde{p}(\phi) := \frac{\rho_*}{\gamma}\int_0^1 (1-\theta)p''(\rho_*(1+\theta\phi))d\theta.$$

We now state our result on the existence of a time periodic solution.

THEOREM 3.1. *Let s be an integer satisfying $s \geq 3$. Let $g = \nabla^{\perp} G$, where G is a scaler function. Assume that g and G satisfies* (1.2), (1.3) *and $g \in C_{per}(\mathbb{R}; L^1_1 \cap L^\infty_3)$ with $G \in C_{per}(\mathbb{R}; L^\infty_2) \cap L^2_{per}(\mathbb{R}; H^s_2)$. We define the norm of g by*

$$[g]_s := \|g\|_{C([0,T];L^1_1\cap L^\infty_3)} + \|G\|_{C([0,T];L^\infty_2)\cap L^2(0,T;H^s_2)}.$$

Then there exist constants $\delta_1 > 0$ and $C > 0$ such that if $[g]_s \leq \delta_1$, the problem (3.1) *has a time periodic solution $u = u_{(1)} + u_{(\infty)}$ satisfying $\{u_{(1)}, u_{(\infty)}\} \in X^s_{sym,per}(\mathbb{R})$ with $\|\{u_{(1)}, u_{(\infty)}\}\|_{X^s(0,T)} \leq C[g]_s$. Furthermore, the uniqueness of time periodic solutions of* (3.1) *holds in the class*

$$\{u = {}^\top(\phi, w); u = u_{(1)} + u_{(\infty)}, \{u_{(1)}, u_{(\infty)}\} \in X^s_{sym,per}(\mathbb{R}), \|\{u_{(1)}, u_{(\infty)}\}\|_{X^s(0,T)} \leq C\delta_1\}.$$

We next consider the stationary problem for (1.1). We consider the following stationary problem on $\mathbb{R}^2$:

$$\begin{cases} \mathrm{div}\,(\rho v) = 0, \\ \rho(v\cdot\nabla)v - \mu\Delta v - (\mu+\mu')\nabla \mathrm{div}\, v + \nabla p(\rho) = \rho g, \end{cases} \tag{3.7}$$

where $g = g(x)$ is a given external force satisfying (1.3). Substituting $\phi = (\rho - \rho_*)/\rho_*$ and $w = v/\gamma$ with $\gamma = \sqrt{p'(\rho_*)}$ into (3.7), we rewrite (3.7) to

$$Au = -B[u]u + G(u,g). \tag{3.8}$$

The existence of the stationary solution is stated as follows.

THEOREM 3.2. *Let s be an integer satisfying $s \geq 3$. Let $g = \nabla^{\perp} G$, where G is a scaler function. Assume that G satisfies* (1.3) *and $g \in L^1_1 \cap L^\infty_3$ with $G \in L^\infty_2 \cap H^s_2$. We define the norm of g by*

$$|||g|||_s := \|g\|_{L^1_1\cap L^\infty_3} + \|G\|_{L^\infty_2\cap H^s_2}.$$

Then there exist constants $\delta_2 > 0$ and $C > 0$ such that if $|||g|||_s \leq \delta_2$, the problem (3.8) *has a stationary solution $u = u_{(1)} + u_{(\infty)}$ satisfying $\{u_{(1)}, u_{(\infty)}\} \in Y^s_{sym}$ with $\|\{u_{(1)}, u_{(\infty)}\}\|_{Y^s} \leq C|||g|||_s$. Furthermore, the uniqueness of stationary solutions of* (3.8) *holds in the class* $\{u = {}^\top(\phi, w); u = u_{(1)} + u_{(\infty)}, \{u_{(1)}, u_{(\infty)}\} \in Y^s_{sym}, \|\{u_{(1)}, u_{(\infty)}\}\|_{Y^s} \leq C\delta_2\}$.

In this paper we will give a proof of Theorem 3.1 only, since Theorem 3.2 can be proved in a similar manner to the proof of Theorem 3.1. The only difference appears in the analysis of the high frequency part. In fact, Theorem 3.2 can be proved in the following way. As in [**10**], direct computations show that the low frequency part of the solution operator for the linearized problem for (3.8) coincides with the leading part of $(I - S_1(T))^{-1}$ which provides the key estimates in the proof of Theorem 3.1. Here $S_1(T) = e^{-TA}$ is the low frequency part of the semigroup generated by A. (See Proposition 5.1 below.) More precisely, it holds that

$$\mathcal{F}\{(I - S_1(T))^{-1}\} \sim -\frac{1}{T}\begin{pmatrix} \dfrac{\nu+\tilde{\nu}}{\gamma^2} & -\dfrac{i^\top\xi}{\gamma|\xi|^2} \\ -\dfrac{i\xi}{\gamma|\xi|^2} & \dfrac{1}{\nu|\xi|^2}\left(I_2 - \dfrac{\xi^\top\xi}{|\xi|^2}\right) \end{pmatrix} \quad \text{as } \xi \to 0$$

and the the right-hand side corresponds to the fundamental solution for the linearized problem of (3.8) in the Fourier space for the low frequency part. Therefore, one can obtain the estimates for the low frequency part of the solution operator in $(\mathscr{X}_{(1)} \times \mathscr{Y}_{(1)})_{sym}$ as in Section 5. The high frequency part is analyzed in a similar manner to the case of time periodic problem as in Section 6. The desired estimates for the high frequency part can be obtained by the weighted L^2 energy method. The only difference from the case of the time periodic problem appears in proving the existence of the solution operator for the high frequency part of the linearized problem. In the case of the stationary problem, one can show the existence of the solution operator by the elliptic regularization method as in [**6**], [**8**]. Although we consider the two dimensional problem, the existence of the solution operator can be shown more easily than in [**6**], [**8**], since 0 belongs to the resolvent sets of the elliptic operators $-\epsilon\Delta$ $(\epsilon > 0)$ and $-\nu\Delta - \tilde{\nu}\mathrm{div}$ restricted to the high frequency part.

In the remaining of this paper we will give a proof of Theorem 3.1.

4. Reformulation of the problem.

In this section, we reformulate (3.1). We begin with to decompose u into a low frequency part $u_{(1)}$ and a high frequency part $u_{(\infty)}$, and then, we rewrite (3.1) to equations for $u_{(1)}$ and $u_{(\infty)}$ as in [**4**].

Similarly to [**4**], we define

$$u_{(1)} := P_1 u, \quad u_{(\infty)} := P_\infty u.$$

Applying the operators P_1 and P_∞ to (3.1), we see that

$$\partial_t u_{(1)} + Au_{(1)} = F_{low}(u_{(1)} + u_{(\infty)}, g), \tag{4.1}$$

$$\partial_t u_{(\infty)} + Au_{(\infty)} + P_\infty(B[u_{(1)} + u_{(\infty)}]u_{(\infty)}) = F_{high}(u_{(1)} + u_{(\infty)}, g). \tag{4.2}$$

Here

$$\begin{aligned}F_{low}(u_{(1)}+u_{(\infty)},g)&:=P_1[-B[u_{(1)}+u_{(\infty)}](u_{(1)}+u_{(\infty)})+G(u_{(1)}+u_{(\infty)},g)],\\F_{high}(u_{(1)}+u_{(\infty)},g)&:=P_\infty[-B[u_{(1)}+u_{(\infty)}]u_{(1)}+G(u_{(1)}+u_{(\infty)},g)].\end{aligned}$$

On the other hand, if some functions $u_{(1)}$ and $u_{(\infty)}$ satisfy (4.1) and (4.2), then adding (4.1) to (4.2), we derive that

$$\begin{aligned}&\partial_t(u_{(1)}+u_{(\infty)})+A(u_{(1)}+u_{(\infty)})\\&\quad=-P_\infty(B[u_{(1)}+u_{(\infty)}]u_{(\infty)})+(F_{low}+F_{high})(u_{(1)}+u_{(\infty)},g)\\&\quad=-B[u_{(1)}+u_{(\infty)}](u_{(1)}+u_{(\infty)})+G(u_{(1)}+u_{(\infty)},g).\end{aligned}$$

Defining $u:=u_{(1)}+u_{(\infty)}$, we get

$$\partial_t u+Au+B[u]u=G(u,g).$$

Therefore, in order to obtain a solution u of (3.1), we look for a solution $\{u_{(1)},u_{(\infty)}\}$ satisfying (4.1)–(4.2).

Concerning antisymmetry of (3.1) and (4.1)–(4.2), We state the following lemmas. Recall that Γ_j $(j=1,2,3)$ is defined by

$$\begin{gathered}(\Gamma_1u)(x):={}^\top(\phi(-x),-w_1(-x),w_2(-x)),\quad(\Gamma_2u)(x):={}^\top(\phi(-x),w_1(-x),-w_2(-x)),\\(\Gamma_3u)(x_1,x_2):={}^\top(\phi(x_2,x_1),w_2(x_2,x_1),w_1(x_2,x_1))\end{gathered}$$

for $u(x)={}^\top(\phi(x),w_1(x),w_2(x))$, $x\in\mathbb{R}^2$.

LEMMA 4.1. *We define* $\boldsymbol{g}(x,t):={}^\top(0,g(x,t))$ *and let* g *satisfy* $(\Gamma_j\boldsymbol{g})(x,t)=\boldsymbol{g}(x,t)\,(x\in\mathbb{R}^2,\ t\in\mathbb{R},\ j=1,2,3)$.

(i) $\Gamma_j u$ $(j=1,2,3)$ *is a solution of* (3.1) *if* $u={}^\top(\phi,w)$ *is a solution of* (3.1).

(ii) $\{\Gamma_j u_{(1)},\Gamma_j u_{(\infty)}\}$ $(j=1,2,3)$ *is a solution of* (4.1)–(4.2) *if* $\{u_{(1)},u_{(\infty)}\}$ *is a solution of* (4.1)–(4.2).

LEMMA 4.2. *Let* g *satisfy* $(\Gamma_j\boldsymbol{g})(x,t)=\boldsymbol{g}(x,t)$ $(x\in\mathbb{R}^2,\ t\in\mathbb{R},\ j=1,2,3)$.

(i) *There holds*

$$[\Gamma_j(\partial_t u+Au+B[u]u-G(u,g))](x,t)=[\partial_t u+Au+B[u]u-G(u,g)](x,t)$$

for $x\in\mathbb{R}^2, t\in\mathbb{R}$, $j=1,2,3$ *if* $(\Gamma_j u)(x,t)=u(x,t)$ $(x\in\mathbb{R}^2,\ t\in\mathbb{R},\ j=1,2,3)$.

(ii) *There hold*

$$\begin{aligned}&[\Gamma_j(\partial_t u_{(1)}+Au_{(1)}-F_{low}(u_{(1)}+u_{(\infty)},g))](x,t)\\&\quad=[\partial_t u_{(1)}+Au_{(1)}-F_{low}(u_{(1)}+u_{(\infty)},g)](x,t)\end{aligned}$$

and

$$[\Gamma_j(\partial_t u_{(\infty)}+Au_{(\infty)}+P_\infty(B[u_{(1)}+u_{(\infty)}]u_{(\infty)})-F_{high}(u_{(1)}+u_{(\infty)},g))](x,t)$$

$$= [\partial_t u_{(\infty)} + Au_{(\infty)} + P_\infty(B[u_{(1)} + u_{(\infty)}]u_{(\infty)}) - F_{high}(u_{(1)} + u_{(\infty)}, g)](x,t)$$

for $x \in \mathbb{R}^2, t \in \mathbb{R}, j = 1,2,3$ *if* $\{\Gamma_j u_{(1)}(x,t), \Gamma_j u_{(\infty)}(x,t)\} = \{u_{(1)}(x,t), u_{(\infty)}(x,t)\}$ $(x \in \mathbb{R}^2,\ t \in \mathbb{R},\ j = 1,2,3)$.

Direct computations verify Lemma 4.1 (i) and Lemma 4.2 (i). As for Lemma 4.1 (ii) and Lemma 4.2 (ii), since it holds that $\mathcal{F}\Gamma_j = -\Gamma_j\mathcal{F}$ $(j = 1,2)$, $\mathcal{F}\Gamma_3 = \Gamma_3\mathcal{F}$, $\chi_j(-\xi_1,\xi_2) = \chi_j(\xi_1,-\xi_2) = \chi_j(\xi_2,\xi_1) = \chi_j(\xi_1,\xi_2)$ $(j = 1,\infty)$, we find that $\Gamma_k P_j = P_j \Gamma_k$ $(k = 1,2,3,\ j = 1,\infty)$. Hence Lemma 4.1 (ii) and Lemma 4.2 (ii) follow from the above relation by a direct computation.

Therefore, we consider (4.1)–(4.2) in space of functions satisfying $\{\Gamma_j u_{(1)}, \Gamma_j u_{(\infty)}\} = \{u_{(1)}, u_{(\infty)}\}$ $(j = 1,2,3)$ by Lemma 4.1 and Lemma 4.2.

To prove the existence of time periodic solution on $\mathbb{R}^2$, we use the momentum formulation for the low frequency part due to the slow decay of the low frequency part $u_{(1)}$ in a weighted L^∞ space as in [**10**].

Some inequalities are prepared for the low frequency part to state the momentum formulation. The following lemma is concerned with properties of P_1.

Lemma 4.3. [**4**, Lemma 4.3] (i) *Let k be a nonnegative integer. Then P_1 is a bounded linear operator from L^2 to H^k. In fact, it holds that*

$$\|\nabla^k P_1 f\|_{L^2} \le C\|f\|_{L^2} \qquad (f \in L^2).$$

As a result, for any $2 \le p \le \infty$, P_1 is bounded from L^2 to L^p.

(ii) *Let k be a nonnegative integer. Then there hold the estimates*

$$\|\nabla^k f_{(1)}\|_{L^2} + \|f_{(1)}\|_{L^p} \le C\|f_{(1)}\|_{L^2} \quad (f_{(1)} \in L^2_{(1)}),$$

where $2 \le p \le \infty$.

We state the following inequality for the weighted L^p norm of the low frequency part.

Lemma 4.4. [**10**, Lemma 4.3] *Let k and ℓ be nonnegative integers and let $1 \le p \le \infty$. Then there holds the estimate*

$$\||x|^\ell \nabla^k f_{(1)}\|_{L^p} \le C\||x|^\ell f_{(1)}\|_{L^p} \quad (f_{(1)} \in L^2_{(1)} \cap L^p_\ell).$$

The following inequality holds for the weighted L^2 norm of the low frequency part.

Lemma 4.5. *Let $\phi \in L^\infty_1$ with $\nabla\phi \in L^2_1$ and $w_{(1)} \in \mathscr{Y}_{(1)}$. Then, it holds that*

$$\|P_1(\phi w_{(1)})\|_{\mathscr{Y}_{(1),L^2}} \le C(\|\phi\|_{L^\infty_1} + \|\nabla\phi\|_{L^2_1})(\|w_{(1)}\|_{L^\infty_1} + \|\nabla w_{(1)}\|_{L^2})$$

uniformly for ϕ and $w_{(1)}$.

Lemma 4.5 follows directly from Lemma 4.4.

We introduce $m_{(1)}$ and $u_{(1),m}$ by

$$m_{(1)} := w_{(1)} + P_1(\phi w), \quad u_{(1),m} := {}^{\top}(\phi_{(1)}, m_{(1)}), \tag{4.3}$$

where $\phi = \phi_{(1)} + \phi_{(\infty)}$ and $w = w_{(1)} + w_{(\infty)}$. The following Lemma is related to reformulation to the momentum formulation for the low frequency part.

LEMMA 4.6. [**10**, Lemma 4.5] *Assume that* $\{u_{(1)}, u_{(\infty)}\}$ *satisfies the system* (4.1)–(4.2). *Then* $\{u_{(1),m}, u_{(\infty)}\}$ *satisfies the following system*:

$$\begin{aligned} \partial_t u_{(1),m} + Au_{(1),m} &= F_{low,m}(u_{(1)} + u_{(\infty)}, g), \qquad (4.4)\\ \partial_t u_{(\infty)} + Au_{(\infty)} + P_\infty(B[u_{(1)} + u_{(\infty)}]u_{(\infty)}) &= F_{high}(u_{(1)} + u_{(\infty)}, g). \end{aligned}$$

Here

$$\begin{aligned} F_{low,m}(u_{(1)} + u_{(\infty)}, g) &:= {}^{\top}(0, \tilde{F}_{low,m}(u_{(1)} + u_{(\infty)}, g)),\\ \tilde{F}_{low,m}(u_{(1)} + u_{(\infty)}, g) &:= -P_1\{\mu\Delta(\phi w) + \tilde{\mu}\nabla \mathrm{div}\,(\phi w) + \frac{\rho_*}{\gamma}\nabla(p^{(1)}(\phi)\phi^2)\\ &\quad + \gamma \mathrm{div}\,(\phi w \otimes w) - \frac{1}{\gamma}((1+\phi)g)\\ &\quad + \gamma\partial_{x_2}\begin{pmatrix} w_1 w_2\\ (w_2)^2 - (w_1)^2\end{pmatrix} + \gamma\partial_{x_1}\begin{pmatrix} 0\\ w_2 w_1\end{pmatrix} + \gamma\nabla(w_1)^2\}. \end{aligned} \tag{4.5}$$

REMARK 4.7. Here we rewrite the convection term $\mathrm{div}\,(w \otimes w)$ by

$$\mathrm{div}\,(w\otimes w) = \partial_{x_2}\begin{pmatrix} w_1 w_2\\ (w_2)^2 - (w_1)^2\end{pmatrix} + \partial_{x_1}\begin{pmatrix} 0\\ w_2 w_1\end{pmatrix} + \nabla(w_1)^2$$

to use the antisymmetry effectively. See Proposition 7.1.

Similarly to Lemma 4.2, the following lemma follows from direct computations which implies that the antisymmetry of (4.4) holds.

LEMMA 4.8. (i) $\Gamma_j u_{(1),m}$ $(j = 1, 2, 3)$ *is a solution of* (4.4) *if* $u_{(1),m} = {}^{\top}(\phi_{(1)}, m_{(1)})$ *is a solution of* (4.4).

(ii) *Let* g *satisfy* $(\Gamma_j \boldsymbol{g})(x, t) = \boldsymbol{g}(x, t)$ $(x \in \mathbb{R}^2,\ t \in \mathbb{R},\ j = 1, 2, 3)$. *Then there hold*

$$\begin{aligned} &[\Gamma_j(\partial_t u_{(1),m} + Au_{(1),m} - F_{low,m}(u_{(1),m} + u_{(\infty)}, g))](x, t)\\ &\quad = [\partial_t u_{(1),m} + Au_{(1),m} - F_{low,m}(u_{(1),m} + u_{(\infty)}, g)](x, t) \end{aligned}$$

for $x \in \mathbb{R}^2, t \in \mathbb{R}$, $j = 1, 2, 3$ *if* $\{\Gamma_j u_{(1),m}(x, t), \Gamma_j u_{(\infty)}(x, t)\} = \{u_{(1),m}(x, t), u_{(\infty)}(x, t)\}$ $(x \in \mathbb{R}^2,\ t \in \mathbb{R},\ j = 1, 2, 3)$.

If $\phi = \phi_{(1)} + \phi_{(\infty)}$ is sufficiently small, we obtain the solution $\{u_{(1)}, u_{(\infty)}\}$ of (4.1)–(4.2) from the solution of (4.2), (4.3) and (4.4), i.e., we have the following.

LEMMA 4.9. (i) *Let* s *be an integer satisfying* $s \geq 3$ *and* $u_{(1),m} = {}^{\top}(\phi_{(1)}, m_{(1)})$ *and* $u_{(\infty)} = {}^{\top}(\phi_{(\infty)}, w_{(\infty)})$ *satisfy* $\{u_{(1),m}, u_{(\infty)}\} \in X^s_{sym}(a, b)$. *Then there exists a positive constant* δ_0 *such that if* $\phi = \phi_{(1)} + \phi_{(\infty)}$ *satisfies* $\sup_{t\in[a,b]}(\|\phi\|_{L^\infty_1} + \|\nabla\phi\|_{L^2_1}) \leq \delta_0$, *then* *there uniquely exists* $w_{(1)} \in C([a,b];\mathscr{Y}_{(1),\#}) \cap H^1(a,b;\mathscr{Y}_{(1),\#})$ *satisfying the following equation*

$$w_{(1)} = m_{(1)} - P_1(\phi(w_{(1)} + w_{(\infty)})), \tag{4.6}$$

where $\phi = \phi_{(1)} + \phi_{(\infty)}$. *Furthermore, we have the estimates*

$$\|w_{(1)}\|_{C([a,b];\mathscr{Y}_{(1)})} \le C(\|m_{(1)}\|_{C([a,b];\mathscr{Y}_{(1)})} + \|w_{(\infty)}\|_{C([a,b];L^2_1)}), \tag{4.7}$$

$$\begin{aligned}\int_a^b \|\partial_t w_{(1)}(\tau)\|^2_{\mathscr{Y}_{(1)}} d\tau \le{}& C((\|\partial_t \nabla\phi_{(1)}\|^2_{C([a,b];L^2_1)} + \|\partial_t\phi_{(\infty)}\|^2_{C([a,b];L^2_1)})\|w_{(1)}\|^2_{C([a,b];L^\infty_1)} \\ &+\|\partial_t\phi\|^2_{C([a,b];L^2)}\|w_{(1)}\|^2_{C([a,b];\mathscr{X}_{(1),L^\infty})}) \\ &+\int_a^b C\Big(\|\partial_t m_{(1)}(\tau)\|^2_{\mathscr{Y}_{(1)}} + \|\partial_t\phi\|^2_{C([a,b];L^2)}\|w_{(\infty)}(\tau)\|^2_{H^s_2} \\ &+\|\partial_t w_{(\infty)}(\tau)\|^2_{L^2_1}\Big)d\tau.\end{aligned} \tag{4.8}$$

(ii) *Let* s *be an integer satisfying* $s \ge 3$ *and* $u_{(1),m} = {}^\top(\phi_{(1)}, m_{(1)})$ *and* $u_{(\infty)} = {}^\top(\phi_{(\infty)}, w_{(\infty)})$ *satisfy* $\{u_{(1),m}, u_{(\infty)}\} \in X^s_{sym}(a,b)$. *We suppose that* $\phi = \phi_{(1)} + \phi_{(\infty)}$ *satisfies* $\sup_{t\in[a,b]}(\|\phi\|_{L^\infty_1} + \|\nabla\phi\|_{L^2}) \le \delta_0$ *and* $\{u_{(1),m}, u_{(\infty)}\}$ *satisfies*

$$\begin{aligned}\partial_t u_{(1),m} + Au_{(1),m} &= F_{low,m}(u_{(1)} + u_{(\infty)}, g), \\ w_{(1)} &= m_{(1)} - P_1(\phi w), \\ \partial_t u_{(\infty)} + Au_{(\infty)} + P_\infty(B[u_{(1)} + u_{(\infty)}]u_{(\infty)}) &= F_{high}(u_{(1)} + u_{(\infty)}, g).\end{aligned}$$

Here $w = w_{(1)} + w_{(\infty)}$ *and* $w_{(1)}$ *defined by* (4.6). *Then* $\{u_{(1)}, u_{(\infty)}\}$ *satisfies* (4.1)–(4.2) *with* $u_{(1)} = {}^\top(\phi_{(1)}, w_{(1)})$.

By using Lemma 2.1 and Lemma 4.4, Lemma 4.9 can be proved by the same way as the proof of [**10**, Lemma 4.6] and we omit the details.

Therefore, we consider (4.2), (4.4) and (4.6) because if we show the existence of a solution $\{u_{(1),m}, u_{(\infty)}\} \in X^s_{sym}(a,b)$ satisfying (4.2), (4.4) and (4.6), then by Lemma 4.9, we obtain a solution $\{u_{(1)}, u_{(\infty)}\} \in X^s_{sym}(a,b)$ satisfying (4.1)–(4.2).

As in [**10**], we formulate (4.2), (4.4) and (4.6) by using time-T-mapping to solve the time periodic problem. We consider the following linear problems for the low frequency part and the high frequency part respectively:

$$\begin{cases}\partial_t u_{(1),m} + Au_{(1),m} = F_{(1),m}, \\ u_{(1),m}|_{t=0} = u_{01,m},\end{cases} \tag{4.9}$$

and

$$\begin{cases}\partial_t u_{(\infty)} + Au_{(\infty)} + P_\infty(B[\tilde{u}]u_{(\infty)}) = F_{(\infty)}, \\ u_{(\infty)}|_{t=0} = u_{0\infty},\end{cases} \tag{4.10}$$

where $\tilde{u} = {}^\top(\tilde{\phi}, \tilde{w}), u_{01,m}, u_{0\infty}, F_{(1),m}$ and $F_{(\infty)}$ are given functions.

The solution operators are introduced as follows. (The precise definition of these operators will be given later.) $S_1(t)$ stands for the solution operator for (4.9) with $F_{(1),m} = 0$, and $\mathscr{S}_1(t)$ stands for the solution operator for (4.9) with $u_{01,m} = 0$. On the other hand, $S_{\infty,\tilde{u}}(t)$ stands for the solution operator for (4.10) with $F_{(\infty)} = 0$ and $\mathscr{S}_{\infty,\tilde{u}}(t)$ stands for the solution operator for (4.10) with $u_{0\infty} = 0$.

As in [**10**], we will look for $\{u_{(1),m}, u_{(\infty)}\}$ satisfying

$$\begin{cases} u_{(1),m}(t) = S_1(t)u_{01,m} + \mathscr{S}_1(t)[F_{low,m}(u,g)], \\ u_{(\infty)}(t) = S_{\infty,u}(t)u_{0\infty} + \mathscr{S}_{\infty,u}(t)[F_{high}(u,g)], \end{cases} \tag{4.11}$$

where

$$\begin{cases} u_{01,m} = (I - S_1(T))^{-1}\mathscr{S}_1(T)[F_{low,m}(u,g)], \\ u_{0\infty} = (I - S_{\infty,u}(T))^{-1}\mathscr{S}_{\infty,u}(T)[F_{high}(u,g)], \end{cases} \tag{4.12}$$

$u = {}^\top(\phi, w)$ is a function given by $u_{(1),m} = {}^\top(\phi_{(1)}, m_{(1)})$ and $u_{(\infty)} = {}^\top(\phi_{(\infty)}, w_{(\infty)})$ through the relation

$$\phi = \phi_{(1)} + \phi_{(\infty)}, \quad w = w_{(1)} + w_{(\infty)}, \quad w_{(1)} = m_{(1)} - P_1(\phi w).$$

From (4.11) and (4.12), it holds that $u_{(1),m}(T) = u_{(1),m}(0)$, $u_{(\infty)}(T) = u_{(\infty)}(0)$. Hence we look for a pair of functions $\{u_{(1),m}, u_{(\infty)}\}$ satisfying (4.11)–(4.12). The solution operators $S_1(t)$ and $\mathscr{S}_1(t)$ are investigated and we state the estimate of a solution for the low frequency part in Section 5; Some properties of $S_{\infty,u}(t)$ and $\mathscr{S}_{\infty,u}(t)$ will be stated and we estimate a solution for the high frequency part in Section 6.

In the remaining of this section some lemmas are stated which will be used in the proof of Theorem 3.1.

We will estimate integral kernels which will appear in the analysis of the low frequency part. Then we use the following lemma.

LEMMA 4.10. [**10**, Lemma 4.8] *Let ℓ be an integer satisfying that $\ell \geq 1$ and let $E(x) := \Phi_\ell = \mathscr{F}^{-1}\hat{\Phi}_\ell$ ($x \in \mathbb{R}^2$), where $\hat{\Phi}_\ell \in C^\infty(\mathbb{R}^2 - \{0\})$ is a function satisfying*

$$\partial_\xi^\alpha \hat{\Phi}_\ell \in L^1 \quad (|\alpha| \leq -1 + \ell),$$
$$|\partial_\xi^\beta \hat{\Phi}_\ell| \leq C|\xi|^{-2-|\beta|+\ell} \quad (\xi \neq 0, |\beta| \geq 0).$$

Then the following estimate holds for $x \neq 0$,

$$|E(x)| \leq C|x|^{-\ell}.$$

The following lemma plays important roles to estimate a convolution with antisymmetry for the low frequency part.

LEMMA 4.11. *Let $E(x)$ ($x \in \mathbb{R}^2$) be a scalar function satisfying*

$$|\partial_x^\alpha E(x)| \leq \frac{C}{(1+|x|)^{|\alpha|+1}} \quad (|\alpha| \geq 0) \tag{4.13}$$

and let f be a scalar function satisfying $f \in L_2^\infty$. We assume that f satisfies

$$f(-x_1,x_2)=-f(x_1,x_2) \ \textit{or}\ f(x_1,-x_2)=-f(x_1,x_2) \ \textit{or}\ f(x_2,x_1)=-f(x_1,x_2). \tag{4.14}$$

Then there holds the following estimate.

$$|E*f(x)|\le \frac{C\|f\|_{L^\infty_2}}{(1+|x|)}. \tag{4.15}$$

Proof. We first assume that $|x|\ge 1$. We set $R:=|x|/2$. Then we see that

$$\begin{aligned}
E*f(x) &= \int_{\mathbb{R}^2} E(x-y)f(y)dy\\
&= \int_{|x-y|\ge R,\,|y|\ge R} E(x-y)f(y)dy\\
&\qquad + \int_{|x-y|\le R} E(x-y)f(y)dy + \int_{|y|\le R} E(x-y)f(y)dy\\
&=: I_1+I_2+I_3,
\end{aligned}$$

where,

$$\begin{aligned}
I_1 &:= \int_{|x-y|\ge R,\,|y|\ge R} E(x-y)f(y)dy,\ I_2 := \int_{|x-y|\le R} E(x-y)f(y)dy,\\
I_3 &:= \int_{|y|\le R} E(x-y)f(y)dy.
\end{aligned}$$

Concerning the estimate for I_1, since $|y|\le |x|+|x-y|\le 3|x-y|$ if $|x-y|\ge R$ and $|y|\ge R$, it follows from (4.13) that

$$|I_1|\le C\|f\|_{L^\infty_2}\int_{|y|\ge R}\frac{1}{(1+|y|)^3}dy\le \frac{C\|f\|_{L^\infty_2}}{1+|x|}.$$

We next derive the estimate of I_2. Since it holds that $|y|\ge |x|-|x-y|\ge R$ if $|x-y|\le R$, we obtain from (4.13) that

$$|I_2|\le \frac{C\|f\|_{L^\infty_2}}{R^2}\int_{|x-y|\le R}\frac{1}{(1+|x-y|)}dy\le \frac{C\|f\|_{L^\infty_2}}{1+|x|}.$$

As for the estimate of I_3, we consider the case such that f satisfies $f(-x_1,x_2)=-f(x_1,x_2)$. We define $\tilde{y}:={}^\top(-y_1,y_2)$ for $y={}^\top(y_1,y_2)$ on $\mathbb{R}^2$ satisfying $y_1\ge 0$. Note that $f(\tilde{y})=-f(y)$. This implies that

$$\begin{aligned}
I_3 &= \int_{|y|\le R,\,y_1\ge 0} E(x-y)f(y)dy + \int_{|y|\le R,\,y_1\ge 0} E(x-\tilde{y})f(\tilde{y})dy\\
&= \int_{|y|\le R,\,y_1\ge 0}\{E(x-y)-E(x-\tilde{y})\}f(y)dy.
\end{aligned}$$

In addition, we see from (4.13) that

$$|E(x-y)-E(x-\tilde{y})| \le \frac{C|y|}{1+|x-y|^2} \le \frac{C|y|}{(1+R)^2} \tag{4.16}$$

for $|y| \le R$. Hence we arrive at

$$|I_3| \le \frac{C\|f\|_{L_2^\infty}}{(1+R)^2}\int_{|y|\le R}\frac{1}{1+|y|}dy \le \frac{C\|f\|_{L_2^\infty}}{1+|x|}.$$

Similarly, we obtain (4.15) in the case such that f satisfies $f(x_1,-x_2) = -f(x_1,x_2)$. If f satisfies $f(x_2,x_1) = -f(x_1,x_2)$, by setting $\tilde{y} := {}^\top(y_2,y_1)$ for $y = {}^\top(y_1,y_2)$ on $\mathbb{R}^2$, $|I_3|$ is written as

$$\begin{aligned}|I_3| &= \left|\int_{|y|\le R,\, y_2\ge y_1} E(x-y)f(y)dy + \int_{|y|\le R,\, y_2\ge y_1} E(x-\tilde{y})f(\tilde{y})dy\right| \\ &= \left|\int_{|y|\le R,\, y_2\ge y_1}\{E(x-y)-E(x-\tilde{y})\}f(y)dy\right|.\end{aligned}$$

This together with (4.16) yields the required estimate (4.15). By using the estimates for I_j $(j = 1,2,3)$, we get the required estimate (4.15) for $|x| \ge 1$.

As for the case $|x| \le 1$, the required estimate (4.15) can be verified by direct computations and we omit the details. This completes the proof. □

In addition, we have the following estimates for a convolution.

LEMMA 4.12. (i) *Let $E(x)$ $(x \in \mathbb{R}^2)$ be a scalar function satisfying* (4.13) *and let f be a scalar function which is written as $f = \partial_{x_j} f_1$ for $j = 1$ or 2 and satisfy $\|\partial_{x_j} f_1\|_{L_3^\infty} + \|f_1\|_{L_2^\infty} < \infty$. We assume that f_1 satisfies* (4.14). *Then the following estimate is true.*

$$|E * f(x)| \le \frac{C}{(1+|x|)^2}(\|\partial_{x_j} f_1\|_{L_3^\infty} + \|f_1\|_{L_2^\infty}).$$

(ii) *Let $E(x)$ $(x \in \mathbb{R}^2)$ be a scalar function satisfying* (4.13) *and let f be a scalar function of the form: $f = \partial_{x_j} f_1$ for $j = 1$ or 2 and it holds that $\|\partial_{x_j} f_1\|_{L_3^\infty} + \|f_1\|_{L_2^\infty} < \infty$. Then we have the following estimate.*

$$|\partial_x^\alpha E * f(x)| \le \frac{C}{(1+|x|)^{1+|\alpha|}}(\|\partial_{x_j} f_1\|_{L_3^\infty} + \|f_1\|_{L_2^\infty}).$$

Lemma 4.12 yields in a similar manner to the proof of Lemma 4.11 and we omit the proofs.

The following L^2 estimates hold for the low frequency part.

LEMMA 4.13. (i) *Let $E(\xi)$ $(\xi \in \mathbb{R}^2)$ be a scalar function satisfying* $\operatorname{supp} E \subset \{|\xi| \le r_\infty\}$ *and*

$$|\partial_\xi^\alpha E(\xi)| \le \frac{C}{|\xi|^{2+|\alpha|}} \text{ for } |\xi| \le r_\infty,\ |\xi| \ne 0,\ |\alpha| \ge 0.$$

Let f belong to $L^2_{(1),1} \cap L^1_1$ and we assume that the following case (1) *or* (2) *hold*;

$$\begin{aligned}&(1)\quad f(-x_1,x_2)=-f(x_1,x_2),\quad f(x_1,-x_2)=f(x_1,x_2),\\&(2)\quad f(-x_1,x_2)=f(x_1,x_2),\quad f(x_1,-x_2)=-f(x_1,x_2).\end{aligned}$$

Then we have the estimate

$$\|\mathcal{F}^{-1}(E\hat{f})\|_{\mathscr{Y}_{(1),L^2}} \le C\|f\|_{L^2_1\cap L^1_1}.$$

(ii) *We suppose that $E(\xi)$ ($\xi \in \mathbb{R}^2$) is a scalar function satisfying* $\operatorname{supp} E \subset \{|\xi| \le r_\infty\}$ *and*

$$|\partial_\xi^\alpha E(\xi)| \le \frac{C}{|\xi|^{1+|\alpha|}} \ \textit{for}\ |\xi|\le r_\infty,\ |\xi|\ne 0,\ |\alpha|\ge 0.$$

and f belongs to $L^2_{(1),1} \cap L^1_1$ which satisfies the following case (1) *or* (2);

$$\begin{aligned}&(1)\quad f(-x_1,x_2)=-f(x_1,x_2),\quad f(x_1,-x_2)=f(x_1,x_2),\\&(2)\quad f(-x_1,x_2)=f(x_1,x_2),\quad f(x_1,-x_2)=-f(x_1,x_2).\end{aligned}$$

Then there holds the estimate

$$\|\mathcal{F}^{-1}(E\hat{f})\|_{\mathscr{X}_{(1),L^2}} \le C\|f\|_{L^2_1\cap L^1_1}.$$

Proof. (i) We assume that f satisfies (1) without loss of generality. Since $\hat{f}(\xi_1,-\xi_2) = -\hat{f}(\xi_1,\xi_2)$, it holds that $\hat{f}(\xi_1,0)=0$. Hence we see that

$$\begin{aligned}\|\nabla\{\mathcal{F}^{-1}(E\hat{f})\}\|_{L^2} &\le C\Big\|\frac{1}{|\xi|}\hat{f}\Big\|_{L^2}\\&\le C\Big\|\xi_2\frac{1}{|\xi|}\Big\|_{L^2(|\xi|\le r_\infty)}\Big\|\int_0^1 \partial_{\xi_2}\hat{f}(\xi_1,\tau\xi_2)d\tau\Big\|_{L^\infty(|\xi|\le r_\infty)}\\&\le C\|xf\|_{L^1}.\end{aligned}$$

Similarly, we obtain the estimate

$$\|\nabla^2\{\mathcal{F}^{-1}(E\hat{f})\}\|_{L^2_1} \le C\|f\|_{L^1_1\cap L^2_1}.$$

The assertion (ii) can be proved by the same way as that for (i). This completes the proof. □

We find the following estimate for the nonlinear term on the low frequency part in weighted L^2 spaces.

Lemma 4.14. (i) *Let $w_{(1)} \in \mathscr{Y}_{(1),\#}$. Then, it holds that*

$$\|(w_{(1)})^2\|_{L^2} + \|w_{(1)}\partial_{x_j}w_{(1)}\|_{L^2_1} \le C\|w_{(1)}\|^2_{\mathscr{Y}_{(1)}} \quad (j=1,2).$$

(ii) *Let* $\phi \in \mathscr{X}_{(1)}$ *and* $w_{(1)} \in \mathscr{Y}_{(1),\#}$. *Then, there holds the estimate*

$$\|\phi w_{(1)}\|_{L^2} + \|\partial_{x_j}(\phi w_{(1)})\|_{L^2_1} \leq C\|\phi\|_{\mathscr{X}_{(1)}}\|w_{(1)}\|_{\mathscr{Y}_{(1)}} \quad (j = 1, 2).$$

PROOF. Concerning the assertion (i), applying Lemma 2.2, we see that

$$\|(w_{(1)})^2\|_{L^2} \leq C\|w_{(1)}\|_{L^\infty_1}\left\|\frac{w_{(1)}}{|x|}\right\|_{L^2} \leq C\|w_{(1)}\|_{L^\infty_1}\|\nabla w_{(1)}\|_{L^2}.$$

Similarly we derive that

$$\|w_{(1)}\partial_{x_j}w_{(1)}\|_{L^2_1} \leq C\|w_{(1)}\|^2_{\mathscr{Y}_{(1)}}.$$

The assertion (ii) yields similarly to the proof of the estimate for (i). This completes the proof. □

The following inequalities will be used for the analysis of the high frequency part.

LEMMA 4.15. [**4**, Lemma 4.4] (i) *Let* k *be a nonnegative integer. Then* P_∞ *is a bounded linear operator on* H^k.

(ii) *There hold the inequalities*

$$\|P_\infty f\|_{L^2} \leq C\|\nabla f\|_{L^2} \quad (f \in H^1),$$

$$\|F_{(\infty)}\|_{L^2} \leq C\|\nabla F_{(\infty)}\|_{L^2} \quad (F_{(\infty)} \in H^1_{(\infty)}).$$

LEMMA 4.16. [**10**, Lemma 4.13] *Let* $\ell \in \mathbb{N}$. *Then there exists a positive constant* C *depending only on* ℓ *such that*

$$\|P_\infty f\|_{L^2_\ell} \leq C\|\nabla f\|_{L^2_\ell}.$$

5. Estimates for solution on the low frequency part.

In this section we estimate a solution $u_{(1)}$ satisfying $u_{(1)}(0) = u_{(1)}(T)$ and

$$\partial_t u_{(1)} + Au_{(1)} = F_{(1)}, \tag{5.1}$$

where $F_{(1)} = {}^\top(0, \tilde{F}_{(1)})$.

We define A_1 by the restriction of A on $\mathscr{X}_{(1)} \times \mathscr{Y}_{(1)}$. The symbol S_1 and $\mathscr{S}_1(t)$ are defined by $S_1(t) := e^{-tA_1}$ and

$$\mathscr{S}_1(t)F_{(1)} := \int_0^t S_1(t-\tau)F_{(1)}(\tau)\,d\tau.$$

Recall that Γ_j $(j = 1, 2, 3)$ are defined by

$$(\Gamma_1 u)(x) := {}^\top(\phi(-x), -w_1(-x), w_2(-x)), \quad (\Gamma_2 u)(x) := {}^\top(\phi(-x), w_1(-x), -w_2(-x)),$$

$$(\Gamma_3 u)(x_1, x_2) := {}^\top(\phi(x_2, x_1), w_2(x_2, x_1), w_1(x_2, x_1))$$

for $u(x) = {}^\top(\phi(x), w_1(x), w_2(x))$ and $x \in \mathbb{R}^2$. We have the following.

Proposition 5.1. (i) *A_1 is a bounded linear operator on $\mathscr{X}_{(1)} \times \mathscr{Y}_{(1)}$. Moreover, $S_1(t)$ is a uniformly continuous semigroup on $\mathscr{X}_{(1)} \times \mathscr{Y}_{(1)}$ and $S_1(t)$ satisfies the following estimates for all $T' > 0$;*

$$S_1(t)u_{(1)} \in C^1([0,T']; \mathscr{X}_{(1)} \times \mathscr{Y}_{(1)}), \quad \partial_t S_1(\cdot)u_{(1)} \in C([0,T']; L^2),$$

$$\partial_t S_1(t)u_{(1)} = -A_1 S_1(t)u_{(1)} \, (= -AS_1(t)u_{(1)}), \; S_1(0)u_{(1)} = u_{(1)} \;\; \text{for} \;\; u_{(1)} \in \mathscr{X}_{(1)} \times \mathscr{Y}_{(1)},$$

$$\|\partial_t^k S_1(\cdot)u_{(1)}\|_{C([0,T'];\mathscr{X}_{(1)} \times \mathscr{Y}_{(1)})} \le C\|u_{(1)}\|_{\mathscr{X}_{(1),L^\infty} \times \mathscr{Y}_{(1),L^\infty}},$$

for $u_{(1)} \in \mathscr{X}_{(1)} \times \mathscr{Y}_{(1)}$, $k = 0, 1$,

$$\|\partial_t S_1(t)u_{(1)}\|_{C([0,T'];L^2)} \le C\|u_{(1)}\|_{\mathscr{X}_{(1)} \times \mathscr{Y}_{(1)}},$$

and

$$\|\partial_t \nabla S_1(t)u_{(1)}\|_{C([0,T'];L^2_1)} \le C\|u_{(1)}\|_{\mathscr{X}_{(1)} \times \mathscr{Y}_{(1)}},$$

for $u_{(1)} \in \mathscr{X}_{(1)} \times \mathscr{Y}_{(1)}$, where C is a positive constant depending on T'.

(ii) *It holds for each $F_{(1)} \in C([0,T]; \mathscr{X}_{(1)}) \times L^2(0,T; \mathscr{Y}_{(1)})$ that*

$$\mathscr{S}_1(\cdot)F_{(1)} \in C^1([0,T]; \mathscr{X}_{(1)}) \times [C([0,T]; \mathscr{Y}_{(1)}) \times H^1(0,T; \mathscr{Y}_{(1)})],$$

and

$$\partial_t \mathscr{S}_1(t)F_{(1)} + A_1 \mathscr{S}_1(t)F_{(1)} = F_{(1)}(t), \; \mathscr{S}_1(0)F_{(1)} = 0,$$

$$\|\mathscr{S}_1(\cdot)F_{(1)}\|_{C([0,T];\mathscr{X}_{(1)} \times \mathscr{Y}_{(1)})} \le C\|F_{(1)}\|_{C([0,T];\mathscr{X}_{(1)}) \times L^2(0,T;\mathscr{Y}_{(1)})},$$

$$\|\partial_t \mathscr{S}_1(\cdot)F_{(1)}\|_{C([0,T];\mathscr{X}_{(1)}) \times L^2(0,T;\mathscr{Y}_{(1)})} \le C\|F_{(1)}\|_{C([0,T];\mathscr{X}_{(1)}) \times L^2(0,T;\mathscr{Y}_{(1)})},$$

where C is a positive constant depending on T. In addition, $\partial_t \mathscr{S}_1(\cdot)F_{(1)} \in C([0,T]; L^2)$, $\partial_t \nabla \mathscr{S}_1(\cdot)F_{(1)} \in C([0,T]; L^2_1)$ for $F_{(1)} \in C([0,T]; L^2_1)$ and we have

$$\|\partial_t \mathscr{S}_1(\cdot)F_{(1)}\|_{C([0,T];L^2)} \le C\|F_{(1)}\|_{C([0,T];L^2)},$$

and

$$\|\partial_t \nabla \mathscr{S}_1(\cdot)F_{(1)}\|_{C([0,T];L^2_1)} \le C\|\nabla F_{(1)}\|_{C([0,T];L^2_1)},$$

where C is a positive constant depending on T.

(iii) *There holds the following relation between S_1 and $\mathscr{S}_1$.*

$$S_1(t)\mathscr{S}_1(t')F_{(1)} = \mathscr{S}_1(t')[S_1(t)F_{(1)}]$$

for any $t \geq 0$, $t' \in [0,T]$ *and* $F_{(1)} \in C([0,T];\mathscr{X}_{(1)}) \times L^2(0,T;\mathscr{Y}_{(1)})$.

(iv) $\Gamma_j S_1(t) = S_1(t)\Gamma_j$ *and* $\Gamma_j \mathscr{S}_1(t) = \mathscr{S}_1(t)\Gamma_j$ *for* $j = 1,2,3$. *Therefore the assertions* (i)–(iii) *above hold with function spaces* $\mathscr{X}_{(1)}$ *and* $\mathscr{Y}_{(1)}$ *replaced by* $(\mathscr{X}_{(1)})_\diamond$ *and* $(\mathscr{Y}_{(1)})_\#$, *respectively.*

The assertions (i)–(iii) follows by the same way as that in [**10**, Proposition 5.1]. The assertion (iv) is verified by the fact $\Gamma_j A_1 = A_1 \Gamma_j$, which derive that $\Gamma_j S_1(t) = S_1(t)\Gamma_j$ for $j = 1,2,3$.

We next investigate invertibility of $I - S_1(T)$.

Proposition 5.2. *There uniquely exists* $u \in (\mathscr{X}_{(1)} \times \mathscr{Y}_{(1)})_{sym}$ *that satisfies* $(I - S_1(T))u = F_{(1)}$ *and* u *satisfies the estimate in each* (i)–(ii) *for* $F_{(1)}$ *satisfying the conditions given in either* (i)–(iii), *respectively.*

(i) $F_{(1)} = \partial_x^\alpha f_{(1)} \in L^\infty_{3,sym} \cap L^2_{(1),1}$ *with* $f_{(1)} \in L^2_{(1)} \cap L^\infty_2$ *for some* α *satisfying* $|\alpha| = 1$ *and* $f_{(1)}$ *satisfies the following condition*

$$\begin{aligned} &f_{(1)}(-x_1,x_2) = -f_{(1)}(x_1,x_2) \text{ or } f_{(1)}(x_1,-x_2) = -f_{(1)}(x_1,x_2) \\ &\quad \text{or } f_{(1)}(x_2,x_1) = -f_{(1)}(x_1,x_2); \end{aligned} \tag{5.2}$$

$$\|u\|_{\mathscr{X}_{(1)}\times\mathscr{Y}_{(1)}} \leq C\{\|F_{(1)}\|_{L^\infty_3} + \|f_{(1)}\|_{L^\infty_2} + \|f_{(1)}\|_{L^2} + \|F_{(1)}\|_{L^2_1}\}. \tag{5.3}$$

(ii) $F_{(1)} = {}^\top(0, \nabla f_{(1)}) \in L^\infty_{3,sym} \cap L^2_{(1),1}$ *with* $f_{(1)} \in L^2_{(1)} \cap L^\infty_2$;

$$\|u\|_{\mathscr{X}_{(1)}\times\mathscr{Y}_{(1)}} \leq C\{\|F_{(1)}\|_{L^\infty_3} + \|f_{(1)}\|_{L^\infty_2} + \|f_{(1)}\|_{L^2} + \|F_{(1)}\|_{L^2_1}\}. \tag{5.4}$$

(iii) $F_{(1)} = \partial_x^\alpha f_{(1)} \in L^\infty_{3,sym} \cap L^2_{(1),1}$ *with* $f_{(1)} \in L^2_{(1)} \cap L^\infty_2$ *for some* α *satisfying* $|\alpha| \geq 2$;

$$\|u\|_{\mathscr{X}_{(1)}\times\mathscr{Y}_{(1)}} \leq C\{\|F_{(1)}\|_{L^\infty_3} + \|f_{(1)}\|_{L^\infty_2} + \|f_{(1)}\|_{L^2} + \|F_{(1)}\|_{L^2_1}\}. \tag{5.5}$$

To prove Proposition 5.2, we use the following lemmas.

Lemma 5.3. [**10**, Lemma 5.3] (i) *The set of all eigenvalues of* $-\hat{A}_\xi$ *consists of* $\lambda_j(\xi)\,(j = 1, \pm)$, *where*

$$\begin{cases} \lambda_1(\xi) = -\nu|\xi|^2, \\ \lambda_\pm(\xi) = -\dfrac{1}{2}(\nu+\tilde{\nu})|\xi|^2 \pm \dfrac{1}{2}\sqrt{(\nu+\tilde{\nu})^2|\xi|^4 - 4\gamma^2|\xi|^2}. \end{cases}$$

If $|\xi| < 2\gamma/(\nu+\tilde{\nu})$, *then*

$$\operatorname{Re}\lambda_\pm = -\frac{1}{2}(\nu+\tilde{\nu})|\xi|^2, \quad \operatorname{Im}\lambda_\pm = \pm\gamma|\xi|\sqrt{1 - \frac{(\nu+\tilde{\nu})^2}{4\gamma^2}|\xi|^2}.$$

(ii) *For* $|\xi| < 2\gamma/(\nu + \tilde{\nu})$, $e^{-t\hat{A}_\xi}$ *has the spectral resolution*

$$e^{-t\hat{A}_\xi} = \sum_{j=1,\pm} e^{t\lambda_j(\xi)}\Pi_j(\xi),$$

where $\Pi_j(\xi)$ *are eigenprojections for* $\lambda_j(\xi)\,(j = 1, \pm)$, *and* $\Pi_j(\xi)\,(j = 1, \pm)$ *satisfy*

$$\Pi_1(\xi) = \begin{pmatrix} 0 & 0 \\ 0 & I_2 - \xi^\top\xi/|\xi|^2 \end{pmatrix},$$
$$\Pi_\pm(\xi) = \pm\frac{1}{\lambda_+ - \lambda_-}\begin{pmatrix} -\lambda_\mp & -i\gamma^\top\xi \\ -i\gamma\xi & \lambda_\pm\xi^\top\xi/|\xi|^2 \end{pmatrix}.$$

Furthermore, if $0 < r_\infty < 2\gamma/(\nu + \tilde{\nu})$, *then there exists a constant* $C > 0$ *such that the estimates*

$$\|\Pi_j(\xi)\| \leq C\,(j = 1, \pm), \tag{5.6}$$

hold for $|\xi| \leq r_\infty$.

Hereafter we fix $0 < r_1 < r_\infty < 2\gamma/(\nu + \tilde{\nu})$ so that (5.6) in Lemma 5.3 holds for $|\xi| \leq r_\infty$.

Lemma 5.4. [**10**, Lemma 5.4] *Let* α *be a multi-index. Then the following estimates hold true uniformly for* ξ *with* $|\xi| \leq r_\infty$ *and* $t \in [0, T]$.

(i) $|\partial_\xi^\alpha \lambda_1| \leq C|\xi|^{2-|\alpha|}$, $|\partial_\xi^\alpha \lambda_\pm| \leq C|\xi|^{1-|\alpha|}$ $(|\alpha| \geq 0)$.

(ii) $|(\partial_\xi^\alpha \Pi_1)\hat{F}_{(1)}| \leq C|\xi|^{-|\alpha|}|\hat{\tilde{F}}_{(1)}|$, $|(\partial_\xi^\alpha \Pi_\pm)\hat{F}_{(1)}| \leq C|\xi|^{-|\alpha|}|\hat{F}_{(1)}|$ $(|\alpha| \geq 0)$, *where* $F_{(1)} = {}^\top(F^0_{(1)}, \tilde{F}_{(1)})$.

(iii) $|\partial_\xi^\alpha(e^{\lambda_1 t})| \leq C|\xi|^{2-|\alpha|}$ $(|\alpha| \geq 1)$.

(iv) $|\partial_\xi^\alpha(e^{\lambda_\pm t})| \leq C|\xi|^{1-|\alpha|}$ $(|\alpha| \geq 1)$.

(v) $|(\partial_\xi^\alpha e^{-t\hat{A}_\xi})\hat{F}_{(1)}| \leq C(|\xi|^{1-|\alpha|}|\hat{F^0_{(1)}}| + |\xi|^{-|\alpha|}|\hat{\tilde{F}}_{(1)}|)$ $(|\alpha| \geq 1)$, *where* $F_{(1)} = {}^\top(F^0_{(1)}, \tilde{F}_{(1)})$.

(vi) $|\partial_\xi^\alpha(I - e^{\lambda_1 t})^{-1}| \leq C|\xi|^{-2-|\alpha|}$ $(|\alpha| \geq 0)$.

(vii) $|\partial_\xi^\alpha(I - e^{\lambda_\pm t})^{-1}| \leq C|\xi|^{-1-|\alpha|}$ $(|\alpha| \geq 0)$.

We define

$$E_{1,j}(x) := \mathcal{F}^{-1}(\hat{\chi}_0(I - e^{\lambda_j T})^{-1}\Pi_j) \quad (j = 1, \pm) \quad (x \in \mathbb{R}^2), \tag{5.7}$$

where χ_0 is a cut-off function defined by $\chi_0 := \mathcal{F}^{-1}\hat{\chi}_0$ with $\hat{\chi}_0$ satisfying

$$\hat{\chi}_0 \in C^\infty(\mathbb{R}^2), \quad 0 \leq \hat{\chi}_0 \leq 1, \quad \hat{\chi}_0 = 1 \ \text{ on } \ \{|\xi| \leq r_\infty\}, \quad \operatorname{supp}\hat{\chi}_0 \subset \{|\xi| \leq 2r_\infty\}. \tag{5.8}$$

We have the following estimates for $E_{1,j}$.

LEMMA 5.5. *There hold*

$$|\partial_x^\alpha E_{1,1}(x)| \le C(1+|x|)^{-(1+|\alpha|)}$$

for $|\alpha| \ge 1, x \in \mathbb{R}^2$ *and*

$$|\partial_x^\alpha E_{1,\pm}(x)| \le C(1+|x|)^{-(1+|\alpha|)}$$

for $|\alpha| \ge 0, x \in \mathbb{R}^2$.

By using Lemma 4.10 and Lemma 5.4, Lemma 5.5 can be proved in a similar manner to the proof of [**10**, Lemma 5.5] and we omit the details.

Since Π_1 is the projection to the solenoidal vector space on $\mathbb{R}^2$, we have the following property for Π_1.

LEMMA 5.6. *It holds that*

$$\Pi_1(\xi)\widehat{\nabla F}(\xi) = 0 \quad (\xi \ne 0,\, |\xi| \le r_\infty),$$

where F *is a scalar function in* H^1.

We are now in a position to prove Proposition 5.2.

PROOF OF PROPOSITION 5.2. (i) We suppose that $F_{(1)} = \partial_{x_2} f_{(1)}$ without loss of generality. We define $u = {}^\top(\phi, w)$ by

$$\begin{aligned} u &:= \mathcal{F}^{-1}((I - e^{-T\hat{A}_\xi})^{-1}\hat{F}_{(1)}) \\ &= \mathcal{F}^{-1}((i\xi_2)(I - e^{-T\hat{A}_\xi})^{-1}\hat{f}_{(1)}) = \mathcal{E} * f_{(1)}, \end{aligned}$$

where

$$\mathcal{E} := \mathcal{F}^{-1}\{(i\xi_2 \sum_{j\in\{1,\pm\}} \hat{E}_{1,j})\},$$

$E_{1,j}$ are the ones defined in (5.7). We obtain from Lemma 5.5 that

$$|\partial_x^\alpha \mathcal{E}(x)| \le C(1+|x|)^{-(1+|\alpha|)} \tag{5.9}$$

for $|\alpha| \ge 0$, $x \in \mathbb{R}^2$. Therefore, by Lemma 4.11, Lemma 4.12 (i) and (5.9), we find that

$$\|w\|_{L_1^\infty} + \|\nabla w\|_{L_2^\infty} \le C\{\|F_{(1)}\|_{L_3^\infty} + \|f_{(1)}\|_{L_2^\infty}\}. \tag{5.10}$$

Concerning the weighted L^∞ estimate for ϕ, We also obtain from Lemma 4.4, Lemma 4.12 (ii) and Lemma 5.5 that

$$\|\phi\|_{L_1^\infty} + \|\nabla \phi\|_{L_2^\infty} \le C\{\|F_{(1)}\|_{L_3^\infty} + \|f_{(1)}\|_{L_2^\infty}\}.$$

This together with Lemma 5.4 and (5.10), we get that $u \in \mathscr{X}_{(1)} \times \mathscr{Y}_{(1)}$, $(I - S_1(T))u = F_{(1)}$ and u satisfies the estimate (5.3). By the assumption of $F_{(1)}$ and Proposition 5.1 (i) and (iii) we see that $\Gamma_j u = u$ $(j = 1,2,3)$, i.e., $u \in (\mathscr{X}_{(1)} \times \mathscr{Y}_{(1)})_{sym}$.

(ii) By Lemma 5.6, we derive that

$$u := \mathcal{F}^{-1}((I - e^{-T\hat{A}_\xi})^{-1}\hat{F}_{(1)}) = \mathcal{F}^{-1}\{\sum_{j\in\{\pm\}} \hat{E}_{1,j}\hat{F}_{(1)}\}$$

for $F_{(1)} = {}^\top(0, \nabla f_{(1)}) \in L^\infty_{3,sym} \cap L^2_{(1),1}$ with $f_{(1)} \in L^2_{(1)} \cap L^\infty_2$. It then follows from Lemma 4.12 (ii), Proposition 5.1, Lemma 5.4 and Lemma 5.5 that $u \in (\mathscr{X}_{(1)} \times \mathscr{Y}_{(1)})_{sym}$, $(I - S_1(T))u = F_{(1)}$ and u satisfies the estimate

$$\|u\|_{\mathscr{X}_{(1)} \times \mathscr{Y}_{(1)}} \le C\{\|F_{(1)}\|_{L^\infty_3} + \|f_{(1)}\|_{L^\infty_2} + \|f_{(1)}\|_{L^2} + \|F_{(1)}\|_{L^2_1}\}.$$

We arrive at the assertion (iii) from Lemma 4.12 (ii), Lemma 5.4 and Lemma 5.5 similarly to the assertion (ii). This completes the proof. □

In view of Proposition 5.2, if $F_{(1)}$ satisfies the each condition (i)–(iii) below, the $I - S_1(T)$ has bounded inverse $(I - S_1(T))^{-1}$ in $(\mathscr{X}_{(1)} \times \mathscr{Y}_{(1)})_{sym}$ satisfying the estimate in (i)–(iii) respectively;

(i) $F_{(1)} = \partial_x^\alpha f_{(1)} \in L^\infty_{3,sym} \cap L^2_{(1),1}$ with $f_{(1)} \in L^2_{(1)} \cap L^\infty_2$ for some α satisfying $|\alpha| = 1$ and $f_{(1)}$ satisfies (5.2);

$$\|(I - S_1(T))^{-1}F_{(1)}\|_{\mathscr{X}_{(1)} \times \mathscr{Y}_{(1)}} \le C\{\|F_{(1)}\|_{L^\infty_3} + \|f_{(1)}\|_{L^\infty_2} + \|f_{(1)}\|_{L^2} + \|F_{(1)}\|_{L^2_1}\}.$$

(ii) $F_{(1)} = {}^\top(0, \nabla f_{(1)}) \in L^\infty_{3,sym} \cap L^2_{(1),1}$ with $f_{(1)} \in L^2_{(1)} \cap L^\infty_2$;

$$\|(I - S_1(T))^{-1}F_{(1)}\|_{\mathscr{X}_{(1)} \times \mathscr{Y}_{(1)}} \le C\{\|F_{(1)}\|_{L^\infty_3} + \|f_{(1)}\|_{L^\infty_2} + \|f_{(1)}\|_{L^2} + \|F_{(1)}\|_{L^2_1}\}.$$

(iii) $F_{(1)} = \partial_x^\alpha f_{(1)} \in L^\infty_{3,sym} \cap L^2_{(1),1}$ with $f_{(1)} \in L^2_{(1)} \cap L^\infty_2$ for some α satisfying $|\alpha| \ge 2$;

$$\|(I - S_1(T))^{-1}F_{(1)}\|_{\mathscr{X}_{(1)} \times \mathscr{Y}_{(1)}} \le C\{\|F_{(1)}\|_{L^\infty_3} + \|f_{(1)}\|_{L^\infty_2} + \|f_{(1)}\|_{L^2} + \|F_{(1)}\|_{L^2_1}\}.$$

We can write $\mathscr{S}_1(t)F_{(1)}$ and $S_1(t)\mathscr{S}_1(T)(I - S_1(T))^{-1}F_{(1)}$ as

$$S_1(t)\mathscr{S}_1(T)(I - S_1(T))^{-1}F_{(1)} = \int_0^T E_1(t,\sigma) * F_{(1)}(\sigma)d\sigma, \tag{5.11}$$

$$\mathscr{S}_1(t)F_{(1)} = \int_0^t S_1(t-\tau)F_{(1)}(\tau)d\tau = \int_0^t E_2(t,\tau) * F_{(1)}(\tau)d\tau, \tag{5.12}$$

where $E_1(t,\sigma)$ and $E_2(t,\tau)$ are defined by

$$E_1(t,\sigma) := \mathcal{F}^{-1}\{\hat{\chi}_0 e^{-t\hat{A}_\xi}(I - e^{-T\hat{A}_\xi})^{-1}e^{-(T-\sigma)\hat{A}_\xi}\},$$
$$E_2(t,\tau) := \mathcal{F}^{-1}\{\hat{\chi}_0 e^{-(t-\tau)\hat{A}_\xi}\}$$

for $\sigma \in [0,T]$, $0 \le \tau \le t \le T$, $\hat{\chi}_0$ is the cut-off function defined by (5.8). Then $E_1(t,\sigma) * F_{(1)}$ and $E_2(t,\tau) * F_{(1)}$ are estimated as follows.

LEMMA 5.7. *$E_1(t,\sigma) * F_{(1)} \in (\mathscr{X}_{(1)} \times \mathscr{Y}_{(1)})_{sym}$ and $E_2(t,\tau) * F_{(1)} \in (\mathscr{X}_{(1)} \times \mathscr{Y}_{(1)})_{sym}$ $(t,\sigma,\tau \in [0,T], j = 1,2)$ if $F_{(1)}$ satisfies the conditions given in either* (i)–(iii) *and $E_1(t,\sigma) * F_{(1)}, E_2(t,\tau) * F_{(1)}$ satisfy the following estimate in each* (i)–(iii).

(i) *$F_{(1)} = \partial_x^\alpha f_{(1)} \in L^\infty_{3,sym} \cap L^2_{(1),1}$ with $f_{(1)} \in L^2_{(1)} \cap L^\infty_2$ for some α satisfying $|\alpha| = 1$ and $f_{(1)}$ satisfies* (5.2);

$$\|E_1(t,\sigma) * F_{(1)}\|_{\mathscr{X}_{(1)} \times \mathscr{Y}_{(1)}} + \|E_2(t,\tau) * F_{(1)}\|_{\mathscr{X}_{(1)} \times \mathscr{Y}_{(1)}}$$
$$\le C\{\|F_{(1)}\|_{L^\infty_3} + \|f_{(1)}\|_{L^\infty_2} + \|f_{(1)}\|_{L^2} + \|F_{(1)}\|_{L^2_1}\}$$

uniformly for $\sigma \in [0,T]$ and $0 \le \tau \le t \le T$.

(ii) *$F_{(1)} = {}^\top(0, \nabla f_{(1)}) \in L^\infty_{3,sym} \cap L^2_{(1),1}$ with $f_{(1)} \in L^2_{(1)} \cap L^\infty_2$;*

$$\|E_1(t,\sigma) * F_{(1)}\|_{\mathscr{X}_{(1)} \times \mathscr{Y}_{(1)}} + \|E_2(t,\tau) * F_{(1)}\|_{\mathscr{X}_{(1)} \times \mathscr{Y}_{(1)}}$$
$$\le C\{\|F_{(1)}\|_{L^\infty_3} + \|f_{(1)}\|_{L^\infty_2} + \|f_{(1)}\|_{L^2} + \|F_{(1)}\|_{L^2_1}\}$$

uniformly for $\sigma \in [0,T]$ and $0 \le \tau \le t \le T$.

(iii) *$F_{(1)} = \partial_x^\alpha f_{(1)} \in L^\infty_{3,sym} \cap L^2_{(1),1}$ with $f_{(1)} \in L^2_{(1)} \cap L^\infty_2$ for some α satisfying $|\alpha| \ge 2$;*

$$\|E_1(t,\sigma) * F_{(1)}\|_{\mathscr{X}_{(1)} \times \mathscr{Y}_{(1)}} + \|E_2(t,\tau) * F_{(1)}\|_{\mathscr{X}_{(1)} \times \mathscr{Y}_{(1)}}$$
$$\le C\{\|F_{(1)}\|_{L^\infty_3} + \|f_{(1)}\|_{L^\infty_2} + \|f_{(1)}\|_{L^2} + \|F_{(1)}\|_{L^2_1}\}$$

uniformly for $\sigma \in [0,T]$ and $0 \le \tau \le t \le T$.

PROOF OF LEMMA 5.7. It follows from Lemmas 5.3 and 5.4 that

$$|\partial_\xi^\beta(\hat{\chi}_0 (i\xi)^\alpha e^{-t\hat{A}_\xi}(I - e^{-T\hat{A}_\xi})^{-1} e^{-(T-\sigma)\hat{A}_\xi})| \le C|\xi|^{-2+|\alpha|-|\beta|},$$
$$|\partial_\xi^\beta(\hat{\chi}_0 (i\xi)^\alpha e^{-(t-\tau)\hat{A}_\xi})| \le C|\xi|^{|\alpha|-|\beta|},$$

for $\sigma \in [0,T]$, $0 \le \tau \le t \le T$ and $|\alpha|$, $|\beta| \ge 0$. Hence by Lemma 4.10 we see that

$$|\partial_x^\alpha E_1(x)| \le C(1+|x|)^{-|\alpha|} \quad (|\alpha| \ge 1), \tag{5.13}$$
$$|\partial_x^\alpha E_2(x)| \le C(1+|x|)^{-(2+|\alpha|)} \quad (|\alpha| \ge 0). \tag{5.14}$$

This together with Lemma 4.11 and Lemma 4.12 we obtain the desired estimate in a similar manner to the proof of Proposition 5.2. This completes the proof. □

The symbol Ψ_1 and Ψ_2 stand for

$$\Psi_1[\tilde{F}_{(1)}](t) := S_1(t)\mathscr{S}_1(T)(I - S_1(T))^{-1}\begin{pmatrix} 0 \\ \tilde{F}_{(1)} \end{pmatrix}, \ \Psi_2[\tilde{F}_{(1)}](t) := \mathscr{S}_1(t)\begin{pmatrix} 0 \\ \tilde{F}_{(1)} \end{pmatrix}. \tag{5.15}$$

For Ψ_1 and Ψ_2 we derive the following estimates.

PROPOSITION 5.8. (i) *If $\tilde{F}_{(1)}$ satisfies $\tilde{F}_{(1)} = \partial_x^\alpha f_{(1)} \in L^2(0,T; L^\infty_{3,\#} \cap L^2_{(1),1})$*

with $f_{(1)} \in L^2(0,T;L^2_{(1)} \cap L^\infty_2)$ *for some* α *satisfying* $|\alpha| = 1$ *and* $f_{(1)}$ *satisfies* (5.2), *then* $\Psi_j[\tilde{F}_{(1)}] \in C^1([0,T];\mathscr{X}_{(1),\diamond}) \times [C([0,T];\mathscr{Y}_{(1),\#}) \cap H^1(0,T;\mathscr{Y}_{(1),\#})]$ $(j = 1,2)$ *and* $\Psi_j[\tilde{F}_{(1)}]$ *satisfy the following estimates.*

$$\|\partial_t^k \Psi_j[\tilde{F}_{(1)}]\|_{C([0,T];\mathscr{X}_{(1)}) \times L^2(0,T;\mathscr{Y}_{(1)})} \le C(\|\tilde{F}_{(1)}\|_{L^2(0,T;L^\infty_3 \cap L^2_1)} + \|f_{(1)}\|_{L^2(0,T;L^\infty_2 \cap L^2)})$$

for $k = 0,1$ *and* $j = 1,2$.

(ii) *We have that* $\Psi_j[\tilde{F}_{(1)}] \in C^1([0,T];\mathscr{X}_{(1),\diamond}) \times [C([0,T];\mathscr{Y}_{(1),\#}) \cap H^1(0,T;\mathscr{Y}_{(1),\#})]$ $(j = 1,2)$ *for* $\tilde{F}_{(1)} = \nabla f_{(1)} \in L^2(0,T;L^\infty_{3,\#} \cap L^2_{(1),1})$ *with* $f_{(1)} \in L^2(0,T;L^2_{(1)} \cap L^\infty_2)$ *and* $\Psi_j[\tilde{F}_{(1)}]$ *satisfy the estimates*

$$\|\partial_t^k \Psi_j[\tilde{F}_{(1)}]\|_{C([0,T];\mathscr{X}_{(1)}) \times L^2(0,T;\mathscr{Y}_{(1)})} \le C(\|\tilde{F}_{(1)}\|_{L^2(0,T;L^\infty_3 \cap L^2_1)} + \|f_{(1)}\|_{L^2(0,T;L^\infty_2 \cap L^2)})$$

for $k = 0,1$ *and* $j = 1,2$.

(iii) *Let* $\tilde{F}_{(1)} = \partial_x^\alpha f_{(1)} \in L^2(0,T;L^\infty_{3,\#} \cap L^2_{(1),1})$ *with* $f_{(1)} \in L^2(0,T;L^2_{(1)} \cap L^\infty_2)$ *for some* α *satisfying* $|\alpha| \ge 2$. *Then* $\Psi_j[\tilde{F}_{(1)}] \in C^1([0,T];\mathscr{X}_{(1),\diamond}) \times [C([0,T];\mathscr{Y}_{(1),\#}) \cap H^1(0,T;\mathscr{Y}_{(1),\#})]$ $(j = 1,2)$ *and* $\Psi_j[\tilde{F}_{(1)}]$ *satisfy the estimates*

$$\|\partial_t^k \Psi_j[\tilde{F}_{(1)}]\|_{C([0,T];\mathscr{X}_{(1)}) \times L^2(0,T;\mathscr{Y}_{(1)})} \le C(\|\tilde{F}_{(1)}\|_{L^2(0,T;L^\infty_3 \cap L^2_1)} + \|f_{(1)}\|_{L^2(0,T;L^\infty_2 \cap L^2)})$$

for $k = 0,1$ *and* $j = 1,2$.

Proof. As for the assertion (i), it follows from Proposition 5.1 (i), (ii) and Lemma 5.7 that

$$\|\Psi_j[\tilde{F}_{(1)}]\|_{C([0,T];\mathscr{X}_{(1)}) \times L^2(0,T;\mathscr{Y}_{(1)})} \le C(\|\tilde{F}_{(1)}\|_{L^2(0,T;L^\infty_3 \cap L^2_1)} + \|f_{(1)}\|_{L^2(0,T;L^\infty_2 \cap L^2)})$$

for $j = 1,2$,

$$\|\partial_t \Psi_1[\tilde{F}_{(1)}]\|_{C([0,T];\mathscr{X}_{(1)}) \times L^2(0,T;\mathscr{Y}_{(1)})} \le C(\|\tilde{F}_{(1)}\|_{L^2(0,T;L^\infty_3 \cap L^2_1)} + \|f_{(1)}\|_{L^2(0,T;L^\infty_2 \cap L^2)}),$$

and

$$\begin{aligned}
&\|\partial_t \Psi_2[\tilde{F}_{(1)}]\|_{C([0,T];\mathscr{X}_{(1)}) \times L^2(0,T;\mathscr{Y}_{(1)})} \\
&\le C(\|\tilde{F}_{(1)}\|_{L^2(0,T;L^\infty_3 \cap L^2_1)} + \|f_{(1)}\|_{L^2(0,T;L^\infty_2 \cap L^2)}) + \|\tilde{F}_{(1)}\|_{L^2(0,T;\mathscr{Y}_{(1)})}).
\end{aligned}$$

Note that $\tilde{F}_{(1)} = \chi_0 * \tilde{F}_{(1)}$, where $\chi_0 = \mathcal{F}^{-1}\hat{\chi}_0$, $\hat{\chi}_0$ is the cut-off function defined by (5.8). Since $\hat{\chi}_0$ belongs to the Schwartz space on $\mathbb{R}^2$, we get that

$$|\partial_x^\alpha \chi_0(x)| \le C(1+|x|)^{-(2+|\alpha|)} \quad \text{for} \quad |\alpha| \ge 0.$$

Therefore, we derive the following estimate for $\|\tilde{F}_{(1)}\|_{L^2(0,T;\mathscr{Y}_{(1)})}$ in a similar manner to the proof of Proposition 5.2.

$$\|\tilde{F}_{(1)}\|_{L^2(0,T;\mathscr{Y}_{(1)})} \le C(\|\tilde{F}_{(1)}\|_{L^2(0,T;L^\infty_3 \cap L^2_1)} + \|f_{(1)}\|_{L^2(0,T;L^\infty_2 \cap L^2)}).$$

Consequently, we obtain the desired estimate in (i). Similarly, we can verify the assertion (ii)–(iii). This completes the proof. □

By using Proposition 5.8, we give estimates for a solution of (5.1) satisfying $u_{(1)}(0) = u_{(1)}(T)$.

PROPOSITION 5.9. *Set*

$$\Psi[\tilde{F}_{(1)}](t) := \Psi_1[\tilde{F}_{(1)}] + \Psi_2[\tilde{F}_{(1)}], \tag{5.16}$$

for $F_{(1)} = {}^{\top}(0, \tilde{F}_{(1)})$, *where* Ψ_1 *and* Ψ_2 *were defined by* (5.15). *If* $\tilde{F}_{(1)}$ *satisfies the conditions given in either* (i)–(iii), *then* $\Psi[\tilde{F}_{(1)}]$ *is a solution of* (5.1) *with* $F_{(1)} = {}^{\top}(0, \tilde{F}_{(1)})$ *in* $\mathscr{Z}_{(1),sym}(0,T)$ *satisfying* $\Psi[\tilde{F}_{(1)}](0) = \Psi[\tilde{F}_{(1)}](T)$ *and* $\Psi[\tilde{F}_{(1)}]$ *satisfies the estimate in each* (i)–(iii), *respectively.*

(i) $\tilde{F}_{(1)} = \partial_x^{\alpha} f_{(1)} \in L^2(0,T; L^{\infty}_{3,\#} \cap L^2_{(1),1})$ *with* $f_{(1)} \in L^2(0,T; L^2_{(1)} \cap L^{\infty}_2)$ *for some* α *satisfying* $|\alpha| = 1$ *and* $f_{(1)}$ *satisfies* (5.2);

$$\|\Psi[\tilde{F}_{(1)}]\|_{\mathscr{Z}_{(1)}(0,T)} \leq C(\|\tilde{F}_{(1)}\|_{L^2(0,T;L^{\infty}_3 \cap L^2_1)} + \|f_{(1)}\|_{L^2(0,T;L^{\infty}_2 \cap L^2)}). \tag{5.17}$$

(ii) $\tilde{F}_{(1)} = \nabla f_{(1)} \in L^2(0,T; L^{\infty}_{3,\#} \cap L^2_{(1),1})$ *with* $f_{(1)} \in L^2(0,T; L^2_{(1)} \cap L^{\infty}_2)$;

$$\|\Psi[\tilde{F}_{(1)}]\|_{\mathscr{Z}_{(1)}(0,T)} \leq C(\|\tilde{F}_{(1)}\|_{L^2(0,T;L^{\infty}_3 \cap L^2_1)} + \|f_{(1)}\|_{L^2(0,T;L^{\infty}_2 \cap L^2)}). \tag{5.18}$$

(iii) $\tilde{F}_{(1)} = \partial_x^{\alpha} f_{(1)} \in L^2(0,T; L^{\infty}_{3,\#} \cap L^2_{(1),1})$ *with* $f_{(1)} \in L^2(0,T; L^2_{(1)} \cap L^{\infty}_2)$ *for some* α *satisfying* $|\alpha| \geq 2$;

$$\|\Psi[\tilde{F}_{(1)}]\|_{\mathscr{Z}_{(1)}(0,T)} \leq C(\|\tilde{F}_{(1)}\|_{L^2(0,T;L^{\infty}_3 \cap L^2_1)} + \|f_{(1)}\|_{L^2(0,T;L^{\infty}_2 \cap L^2)}). \tag{5.19}$$

PROOF. By Proposition 5.1 (iii) and Proposition 5.2 we see that $\Psi[\tilde{F}_{(1)}]$ is a solution of (5.1) with $F_{(1)} = {}^{\top}(0, \tilde{F}_{(1)})$ and satisfies $\Psi[\tilde{F}_{(1)}](0) = \Psi[\tilde{F}_{(1)}](T)$. The estimates and antisymmetry of $\Psi[\tilde{F}_{(1)}]$ in (i)–(iii) are verified by Proposition 5.8. This completes the proof. □

6. Estimates for solution on the high frequency part.

In this section we estimate a solution for the high frequency part. We begin with some properties of $S_{\infty,\tilde{u}}(t)$ and $\mathscr{S}_{\infty,\tilde{u}}(t)$.

As for the solvability of (4.10), we state the following proposition.

PROPOSITION 6.1. *Let* s *be an integer satisfying* $s \geq 3$. *Set* $k = s-1$ *or* s. *Assume that*

$$\begin{aligned}
&\nabla \tilde{w} \in C([0,T']; H^{s-1}) \cap L^2(0,T'; H^s),\\
&u_{0\infty} = {}^{\top}(\phi_{0\infty}, w_{0\infty}) \in H^k_{(\infty)},\\
&F_{(\infty)} = {}^{\top}(F^0_{(\infty)}, \tilde{F}_{(\infty)}) \in L^2(0,T'; H^k_{(\infty)} \times H^{k-1}_{(\infty)}).
\end{aligned}$$

Here T' is a given positive number. Then there exists a unique solution $u_{(\infty)} = {}^\top(\phi_{(\infty)}, w_{(\infty)})$ of (4.10) satisfying

$$\phi_{(\infty)} \in C([0,T']; H^k_{(\infty)}),$$
$$w_{(\infty)} \in C([0,T']; H^k_{(\infty)}) \cap L^2(0,T'; H^{k+1}_{(\infty)}) \cap H^1(0,T'; H^{k-1}_{(\infty)}).$$

One can verify Proposition 6.1 in a similar manner to the proof of [**4**, Proposition 6.4] and we omit the details.

Remark 6.2. Concerning the space dimension n, in [**4**, Proposition 6.4] we assume that $n \geq 3$. But we can replace the space dimension to $n = 2$ by taking a look at the fact that [**2**, Theorem 4.1] holds for the space dimension $n = 2$ and the proof of [**4**, Proposition 6.4]. See also [**10**, Remark 6.2] for the condition of $\tilde{w}$.

Therefore, it follows from Proposition 6.1 that we can define $S_{\infty,\tilde{u}}(t)$ $(t \geq 0)$ and $\mathscr{S}_{\infty,\tilde{u}}(t)$ $(t \in [0,T])$ as follows.

Let an integer s satisfy $s \geq 3$ and a function $\tilde{u} = {}^\top(\tilde{\phi}, \tilde{w})$ satisfy

$$\tilde{\phi} \in C_{per}(\mathbb{R}; H^s),\ \nabla\tilde{w} \in C_{per}(\mathbb{R}; H^{s-1}) \cap L^2_{per}(\mathbb{R}; H^s). \tag{6.1}$$

Let $k = s-1$ or s. We define an operator $S_{\infty,\tilde{u}}(t) : H^k_{(\infty)} \longrightarrow H^k_{(\infty)}$ $(t \geq 0)$ by

$$u_{(\infty)}(t) = S_{\infty,\tilde{u}}(t)u_{0\infty} \ \text{ for } \ u_{0\infty} = {}^\top(\phi_{0\infty}, w_{0\infty}) \in H^k_{(\infty)},$$

where $u_{(\infty)}(t)$ is the solution of (4.10) with $F_{(\infty)} = 0$. Moreover, we define an operator $\mathscr{S}_{\infty,\tilde{u}}(t) : L^2(0,T; H^k_{(\infty)} \times H^{k-1}_{(\infty)}) \longrightarrow H^k_{(\infty)}$ $(t \in [0,T])$ by

$$u_{(\infty)}(t) = \mathscr{S}_{\infty,\tilde{u}}(t)[F_{(\infty)}] \ \text{ for } \ F_{(\infty)} = {}^\top(F^0_{(\infty)}, \tilde{F}_{(\infty)}) \in L^2(0,T; H^k_{(\infty)} \times H^{k-1}_{(\infty)}),$$

where $u_{(\infty)}(t)$ is the solution of (4.10) with $u_{0\infty} = 0$.

We have the following properties for $S_{\infty,\tilde{u}}(t)$ and $\mathscr{S}_{\infty,\tilde{u}}(t)$ in the weighted L^2-Sobolev spaces.

Proposition 6.3. *Let s be a nonnegative integer satisfying $s \geq 3$ and let $k = s-1$ or s. We suppose that $\tilde{u} = {}^\top(\tilde{\phi}, \tilde{w})$ satisfies (6.1). Then there exists a constant $\delta > 0$ such that if $\|\nabla\tilde{w}\|_{C([0,T];H^{s-1}) \cap L^2(0,T;H^s)} \leq \delta$, then the following assertions hold true.*

(i) *For $u_{0\infty} = {}^\top(\phi_{0\infty}, w_{0\infty}) \in H^k_{(\infty),2}$, there holds $S_{\infty,\tilde{u}}(\cdot)u_{0\infty} \in C([0,\infty); H^k_{(\infty),2})$ and there exist constants $a > 0$ and $C > 0$ such that $S_{\infty,\tilde{u}}(t)$ satisfies the following estimate for all $t \geq 0$ and $u_{0\infty} \in H^k_{(\infty),2}$.*

$$\|S_{\infty,\tilde{u}}(t)u_{0\infty}\|_{H^k_{(\infty),2}} \leq Ce^{-at}\|u_{0\infty}\|_{H^k_{(\infty),2}}.$$

(ii) *For $F_{(\infty)} = {}^\top(F^0_{(\infty)}, \tilde{F}_{(\infty)}) \in L^2(0,T; H^k_{(\infty),2} \times H^{k-1}_{(\infty),2})$, there holds $\mathscr{S}_{\infty,\tilde{u}}(\cdot)F_{(\infty)} \in C([0,T]; H^k_{(\infty),2})$ and $\mathscr{S}_{\infty,\tilde{u}}(t)$ satisfies the following estimate for $t \in [0,T]$ and $F_{(\infty)} \in L^2(0,T; H^k_{(\infty),2} \times H^{k-1}_{(\infty),2})$ with a positive constant C depending on T.*

$$\|\mathscr{S}_{\infty,\tilde{u}}(t)[F_{(\infty)}]\|_{H^k_{(\infty),2}} \le C\left\{\int_0^t e^{-a(t-\tau)}\|F_{(\infty)}\|^2_{H^k_{(\infty),2}\times H^{k-1}_{(\infty),2}}d\tau\right\}^{1/2}.$$

(iii) *We define* $r_{H^k_{(\infty),2}}(S_{\infty,\tilde{u}}(T))$ *by the spectral radius of* $S_{\infty,\tilde{u}}(T)$ *on* $H^k_{(\infty),2}$. *Then it holds that* $r_{H^k_{(\infty),2}}(S_{\infty,\tilde{u}}(T)) < 1$.

(iv) $I - S_{\infty,\tilde{u}}(T)$ *has a bounded inverse* $(I - S_{\infty,\tilde{u}}(T))^{-1}$ *on* $H^k_{(\infty),2}$ *satisfying*

$$\|(I - S_{\infty,\tilde{u}}(T))^{-1}u\|_{H^k_{(\infty),2}} \le C\|u\|_{H^k_{(\infty),2}} \quad \text{for} \quad u \in H^k_{(\infty),2}.$$

(v) *Suppose that* $\Gamma_j\tilde{u} = \tilde{u}$ *for* $j = 1,2,3$. *Then it holds that* $\Gamma_j S_{\infty,\tilde{u}}(t) = S_{\infty,\tilde{u}}(t)\Gamma_j$ *and* $\Gamma_j\mathscr{S}_{\infty,\tilde{u}}(t) = \mathscr{S}_{\infty,\tilde{u}}(t)\Gamma_j$. *Accordingly, the assertions* (i)–(iv) *hold true in function spaces* $H^k_{(\infty),2}$ *and* $H^k_{(\infty),2} \times H^{k-1}_{(\infty),2}$ *replaced by* $(H^k_{(\infty),2})_{sym}$ *and* $(H^k_{(\infty),2} \times H^{k-1}_{(\infty),2})_{sym}$, *respectively if* $\Gamma_j\tilde{u} = \tilde{u}$ $(j = 1,2,3)$.

We can verify Proposition 6.3 in a similar manner to the proof of [**4**, Proposition 6.5] and we omit the proof.

Remark 6.4. As for the space dimension n, in [**4**, Proposition 6.5] it is assumed that $n \ge 3$. But it is replaced by $n = 2$ due to taking a look at the proof of [**4**, Proposition 6.4]. See also [**10**, Remark 6.2] for the condition of $\tilde{w}$.

We are now in a position to give the following estimate for a solution $u_{(\infty)}$ of (4.10) satisfying $u_{(\infty)}(0) = u_{(\infty)}(T)$.

Proposition 6.5. *Let* s *be a nonnegative integer satisfying* $s \ge 3$. *We suppose that*

$$F_{(\infty)} = {}^\top(F^0_{(\infty)}, \tilde{F}_{(\infty)}) \in L^2(0,T;(H^k_{(\infty),2} \times H^{k-1}_{(\infty),2})_{sym}),$$

with $k = s-1$ *or* s. *We also assume that* $\tilde{u} = {}^\top(\tilde{\phi}, \tilde{w})$ *satisfies* (6.1). *Then there exists a positive constant* δ *such that if*

$$\|\nabla\tilde{w}\|_{C([0,T];H^{s-1})\cap L^2(0,T;H^s)} \le \delta,$$

then the following assertion holds true.

The function

$$u_{(\infty)}(t) := S_{\infty,\tilde{u}}(t)(I - S_{\infty,\tilde{u}}(T))^{-1}\mathscr{S}_{\infty,\tilde{u}}(T)[F_{(\infty)}] + \mathscr{S}_{\infty,\tilde{u}}(t)[F_{(\infty)}] \qquad (6.2)$$

is a solution of (4.10) *in* $\mathscr{Z}^k_{(\infty),2,sym}(0,T)$ *satisfying* $u_{(\infty)}(0) = u_{(\infty)}(T)$ *and the estimate*

$$\|u_{(\infty)}\|_{\mathscr{Z}^k_{(\infty),2}(0,T)} \le C\|F_{(\infty)}\|_{L^2(0,T;H^k_{(\infty),2}\times H^{k-1}_{(\infty),2})}.$$

Proposition 6.5 is directly derived by Proposition 6.3.

7. Proof of Theorem 3.1.

In this section we prove Theorem 3.1.

The estimates for the nonlinear and inhomogeneous terms are established here. We set $F_{low,m}(u,g)$ and $F_{high}(u,g)$ by

$$F_{low,m}(u,g) := \begin{pmatrix} 0 \\ \tilde{F}_{low,m}(u,g) \end{pmatrix},$$

$$F_{high}(u,g) = \begin{pmatrix} F^0_{high}(u) \\ \tilde{F}_{high}(u,g) \end{pmatrix} := P_\infty \begin{pmatrix} -\gamma w \cdot \nabla\phi_{(1)} + F^0(u) \\ \tilde{F}(u,g) \end{pmatrix},$$

where $u = {}^\top(\phi, w)$ is given by $u_{(1),m} = {}^\top(\phi_{(1)}, m_{(1)})$ and $u_{(\infty)} = {}^\top(\phi_{(\infty)}, w_{(\infty)})$ through the relation

$$\phi = \phi_{(1)} + \phi_{(\infty)}, \quad w = w_{(1)} + w_{(\infty)}, \quad w_{(1)} = m_{(1)} - P_1(\phi w),$$

$\tilde{F}_{low,m}(u,g)$, $F^0(u)$ and $\tilde{F}(u,g)$ were given in (4.5), (3.5) and (3.6), respectively,

As for the estimate $F_{low,m}(u,g)$, we use the notation Ψ introduced in section 5, i.e.,

$$\Psi[\tilde{F}_{(1)}](t) := S_1(t)\mathscr{S}_1(T)(I - S_1(T))^{-1}\begin{pmatrix} 0 \\ \tilde{F}_{(1)} \end{pmatrix} + \mathscr{S}_1(t)\begin{pmatrix} 0 \\ \tilde{F}_{(1)} \end{pmatrix}.$$

We have the following estimate for $\Psi[\tilde{F}_{low,m}(u,g)]$ in $\mathscr{Z}_{(1),sym}(0,T)$.

Proposition 7.1. *Let $u_{(1),m} = {}^\top(\phi_{(1)}, m_{(1)}) \in (\mathscr{X}_{(1)} \times \mathscr{Y}_{(1)})_{sym}$ and $u_{(\infty)} = {}^\top(\phi_{(\infty)}, w_{(\infty)}) \in H^s_{(\infty),2,sym}$ satisfying*

$$\sup_{0\le t\le T}\|u_{(1),m}(t)\|_{\mathscr{X}_{(1)}\times\mathscr{Y}_{(1)}} + \sup_{0\le t\le T}\|u_{(\infty)}(t)\|_{H^s_2}$$
$$+ \sup_{0\le t\le T}\|\phi(t)\|_{L^\infty_1} + \sup_{0\le t\le T}\|\nabla\phi(t)\|_{L^2_1} \le \min\{\delta_0, \delta, \frac{1}{2}\},$$

where δ_0, δ are the ones in Lemma 4.9 (i) *and Proposition* 6.5 *respectively and $\phi = \phi_{(1)} + \phi_{(\infty)}$. Then we obtain the following estimate*

$$\|\Psi[\tilde{F}_{low,m}(u,g)]\|_{\mathscr{Z}_{(1)}(0,T)}$$
$$\le C\|\{u_{(1),m}, u_{(\infty)}\}\|^2_{X^s(0,T)} + C\Big(1 + \|\{u_{(1),m}, u_{(\infty)}\}\|_{X^s(0,T)}\Big)[g]_s,$$

uniformly for $u_{(1),m}$ and $u_{(\infty)}$.

Proof. Let $u^{(j)} = {}^\top(\phi^{(j)}, w^{(j)})$ $(j = 1, \infty)$, $w^{(j)} = {}^\top(w^{(j)}_1, w^{(j)}_2)$ and we define

$$G_1(u^{(1)}, u^{(2)}) := -P_1\left\{\gamma\partial_{x_2}\begin{pmatrix} w^{(1)}_1 w^{(2)}_2 \\ w^{(1)}_2 w^{(2)}_2 - w^{(1)}_1 w^{(2)}_1 \end{pmatrix} + \gamma\partial_{x_1}\begin{pmatrix} 0 \\ w^{(1)}_1 w^{(2)}_2 \end{pmatrix}\right\},$$

$$G_2(u^{(1)}, u^{(2)}) := -P_1(\gamma\nabla(w_1^{(1)}w_1^{(2)})),$$
$$G_3(u^{(1)}, u^{(2)}) := -P_1(\mu\Delta(\phi^{(1)}w^{(2)}) + \tilde{\mu}\nabla\mathrm{div}\,(\phi^{(1)}w^{(2)})),$$
$$G_4(\phi, u^{(1)}, u^{(2)}) := -P_1\left(\frac{\rho_*}{\gamma}\nabla(\tilde{p}(\phi)\phi^{(1)}\phi^{(2)})\right),$$
$$G_5(\phi, u^{(1)}, u^{(2)}) := -P_1(\gamma\mathrm{div}\,(\phi w^{(1)} \otimes w^{(2)})),$$
$$H_k(u^{(1)}, u^{(2)}) := G_k(u^{(1)}, u^{(2)}) + G_k(u^{(2)}, u^{(1)}), \quad (k = 1, 2, 3),$$
$$H_k(\phi, u^{(1)}, u^{(2)}) := G_k(\phi, u^{(1)}, u^{(2)}) + G_k(\phi, u^{(2)}, u^{(1)}), \quad (k = 4, 5).$$

And we then write $\Psi[\tilde{F}_{low,m}(u, g)]$ as

$$\begin{aligned}\Psi[\tilde{F}_{low,m}(u, g)] = &\sum_{k=1}^{3}\big(\Psi[G_k(u_{(1)}, u_{(1)})] + \Psi[H_k(u_{(1)}, u_{(\infty)})] + \Psi[G_k(u_{(\infty)}, u_{(\infty)})]\big)\\ &+\sum_{k=4}^{5}\Psi[G_k(\phi, u_{(1)}, u_{(1)})] + \Psi[H_k(\phi, u_{(1)}, u_{(\infty)})] + \Psi[G_k(\phi, u_{(\infty)}, u_{(\infty)})]\\ &+\Psi\left[\frac{1}{\gamma}(1+\phi_{(1)})g\right] + \Psi\left[\frac{1}{\gamma}\phi_{(\infty)}g\right].\end{aligned}$$

Using Lemma 4.14 and (5.17) we have the following estimate for $\Psi[G_1(u_{(1)}, u_{(1)})]$.

$$\|\Psi[G_1(u_{(1)}, u_{(1)})]\|_{\mathscr{Z}_{(1)}(0,T)} \le C\|\{u_{(1)}, u_{(\infty)}\}\|^2_{X^s(0,T)}.$$

Concerning the estimates $\Psi[G_2(u_{(1)}, u_{(1)})]$ and $\Psi[G_4(\phi, u_{(1)}, u_{(1)})]$, applying Lemma 4.14 and (5.18) with $f_{(1)} = (w_{(1)})^2$ and $f_{(1)} = \tilde{p}(\phi)\phi_{(1)}^2$ we obtain the estimates

$$\|\Psi[G_2(u_{(1)}, u_{(1)})]\|_{\mathscr{Z}_{(1)}(0,T)} \le C\|\{u_{(1)}, u_{(\infty)}\}\|^2_{X^s(0,T)},$$
$$\|\Psi[G_4(\phi, u_{(1)}, u_{(1)})]\|_{\mathscr{Z}_{(1)}(0,T)} \le C\|\{u_{(1)}, u_{(\infty)}\}\|^2_{X^s(0,T)}.$$

By using Lemma 4.14 and (5.19) we arrive at the following estimate for $\Psi[G_3(u_{(1)}, u_{(1)})]$.

$$\|\Psi[G_3(u_{(1)}, u_{(1)})]\|_{\mathscr{Z}_{(1)}(0,T)} \le C\|\{u_{(1)}, u_{(\infty)}\}\|^2_{X^s(0,T)}.$$

It follows from Lemma 4.4, Lemma 4.14 and (5.17) that we get

$$\|\Psi[G_1(u_{(1)}, u_{(\infty)})]\|_{\mathscr{Z}_{(1)}(0,T)} \le C\|\{u_{(1)}, u_{(\infty)}\}\|^2_{X^s(0,T)},$$
$$\|\Psi[G_1(u_{(\infty)}, u_{(\infty)})]\|_{\mathscr{Z}_{(1)}(0,T)} \le C\|\{u_{(1)}, u_{(\infty)}\}\|^2_{X^s(0,T)}.$$

Similarly, by Lemma 4.4, Lemma 4.14, (5.18) and (5.19) we obtain for $k = 2, 3$ that

$$\begin{aligned}&\|\Psi[G_k(u_{(1)}, u_{(\infty)})]\|_{\mathscr{Z}_{(1)}(0,T)} + \|\Psi[G_4(\phi, u_{(1)}, u_{(\infty)})]\|_{\mathscr{Z}_{(1)}(0,T)}\\ &+\|\Psi[G_k(u_{(\infty)}, u_{(\infty)})]\|_{\mathscr{Z}_{(1)}(0,T)}\|\Psi[G_4(\phi, u_{(\infty)}, u_{(\infty)})]\|_{\mathscr{Z}_{(1)}(0,T)}\end{aligned}$$

$$\leq C\|\{u_{(1)}, u_{(\infty)}\}\|^2_{X^s(0,T)}.$$

$G_5(\phi, u, u)$ is estimated by same way as that in the estimate for $\Psi[G_1(u_{(1)}, u_{(1)})]$ and we see that

$$\|\Psi[G_5(\phi, u, u)]\|_{\mathscr{Z}_{(1)}(0,T)} \leq C\|\{u_{(1)}, u_{(\infty)}\}\|^2_{X^s(0,T)}.$$

As for the estimates for $\Psi[(1+\phi_{(1)})g]$ and $\Psi[\phi_{(\infty)}g]$, it holds from Lemma 4.13 and (5.17) that

$$\|\Psi[(1+\phi_{(1)})g]\|_{\mathscr{Z}_{(1)}(0,T)} + \|\Psi[\phi_{(\infty)}g]\|_{\mathscr{Z}_{(1)}(0,T)} \leq C(1 + \|\{u_{(1)}, u_{(\infty)}\}\|_{X^s(0,T)})[g]_s.$$

Therefore, we find that

$$\|\Psi[\tilde{F}_{low,m}(u,g)]\|_{\mathscr{Z}_{(1)}(0,T)} \leq C\|\{u_{(1)}, u_{(\infty)}\}\|^2_{X^s(0,T)} + C\Big(1 + \|\{u_{(1)}, u_{(\infty)}\}\|_{X^s(0,T)}\Big)[g]_s.$$

Consequently, we obtain the desired estimate by applying Lemma 4.9 (i). This completes the proof. □

We state the estimates for the nonlinear and inhomogeneous terms of the high frequency part.

Proposition 7.2. *Let* $u_{(1),m} = {}^\top(\phi_{(1)}, m_{(1)}) \in (\mathscr{X}_{(1)} \times \mathscr{Y}_{(1)})_{sym}$ *and* $u_{(\infty)} = {}^\top(\phi_{(\infty)}, w_{(\infty)}) \in H^s_{(\infty),2,sym}$ *satisfying*

$$\sup_{0\leq t\leq T} \|u_{(1),m}(t)\|_{\mathscr{X}_{(1)}\times\mathscr{Y}_{(1)}} + \sup_{0\leq t\leq T} \|u_{(\infty)}(t)\|_{H^s_2} + \sup_{0\leq t\leq T} \|\phi(t)\|_{L^\infty_1} + \sup_{0\leq t\leq T} \|\nabla\phi(t)\|_{L^2_1} \leq \min\left\{\delta_0, \delta, \frac{1}{2}\right\},$$

where δ_0, δ *are the ones in Lemma* 4.9 (i) *and Proposition* 6.5 *respectively and* $\phi = \phi_{(1)} + \phi_{(\infty)}$. *Then we have the estimate*

$$\|F_{high}(u,g)\|_{L^2(0,T;H^s_2\times H^{s-1}_2)} \leq C\|\{u_{(1),m}, u_{(\infty)}\}\|^2_{X^s(0,T)} + C\Big(1 + \|\{u_{(1),m}, u_{(\infty)}\}\|_{X^s(0,T)}\Big)[g]_s,$$

uniformly for $u_{(1),m}$ *and* $u_{(\infty)}$.

Proposition 7.2 follows in a similar manner to the proof of [**10**, Proposition 7.2] and we omit the details.

By the same way as that in the proof of Proposition 7.1, we have the following estimate for $F_{low,m}(u^{(1)}, g) - F_{low,m}(u^{(2)}, g)$.

Proposition 7.3. *Let* $u^{(k)}_{(1),m} = {}^\top(\phi^{(k)}_{(1)}, m^{(k)}_{(1)}) \in (\mathscr{X}_{(1)} \times \mathscr{Y}_{(1)})_{sym}$ *and* $u^{(k)}_{(\infty)} = {}^\top(\phi^{(k)}_{(\infty)}, w^{(k)}_{(\infty)}) \in H^s_{(\infty),2,sym}$ *satisfying*

$$\sup_{0\le t\le T}\|u^{(k)}_{(1),m}(t)\|_{\mathscr{X}_{(1)}\times\mathscr{Y}_{(1)}}+\sup_{0\le t\le T}\|u^{(k)}_{(\infty)}(t)\|_{H^s_2}$$
$$+\sup_{0\le t\le T}\|\phi^{(k)}(t)\|_{L^\infty_1}+\sup_{0\le t\le T}\|\nabla\phi^{(k)}(t)\|_{L^2_1}\le\min\left\{\delta_0,\delta,\frac{1}{2}\right\},$$

where δ_0,δ *are the ones in Lemma* 4.9 (i) *and Proposition* 6.5 *respectively and* $\phi^{(k)}=\phi^{(k)}_{(1)}+\phi^{(k)}_{(\infty)}$ $(k=1,2)$. *Then it holds that*

$$\|\Psi[\tilde F_{low,m}(u^{(1)},g)-\tilde F_{low,m}(u^{(2)},g)]\|_{\mathscr{Z}_{(1)}(0,T)}$$
$$\le C\sum_{k=1}^{2}\left\|\left\{u^{(k)}_{(1),m},u^{(k)}_{(\infty)}\right\}\right\|_{X^s(0,T)}\left\|\left\{u^{(1)}_{(1),m}-u^{(2)}_{(1),m},u^{(1)}_{(\infty)}-u^{(2)}_{(\infty)}\right\}\right\|_{X^{s-1}(0,T)}$$
$$+C[g]_s\left\|\left\{u^{(1)}_{(1),m}-u^{(2)}_{(1),m},u^{(1)}_{(\infty)}-u^{(2)}_{(\infty)}\right\}\right\|_{X^{s-1}(0,T)},$$

uniformly for $u^{(k)}_{(1),m}$ *and* $u^{(k)}_{(\infty)}$.

We next estimate $F_{high}(u^{(1)},g)-F_{high}(u^{(2)},g)$.

Proposition 7.4. *Let* $u^{(k)}_{(1),m}={}^\top(\phi^{(k)}_{(1)},m^{(k)}_{(1)})\in(\mathscr{X}_{(1)}\times\mathscr{Y}_{(1)})_{sym}$ *and* $u^{(k)}_{(\infty)}={}^\top(\phi^{(k)}_{(\infty)},w^{(k)}_{(\infty)})\in H^s_{(\infty),2,sym}$ *satisfying*

$$\sup_{0\le t\le T}\|u^{(k)}_{(1),m}(t)\|_{\mathscr{X}_{(1)}\times\mathscr{Y}_{(1)}}+\sup_{0\le t\le T}\|u^{(k)}_{(\infty)}(t)\|_{H^s_2}$$
$$+\sup_{0\le t\le T}\|\phi^{(k)}(t)\|_{L^\infty_1}+\sup_{0\le t\le T}\|\nabla\phi^{(k)}(t)\|_{L^2_1}\le\min\left\{\delta_0,\delta,\frac{1}{2}\right\},$$

where δ_0,δ *are the ones in Lemma* 4.9 (i) *and Proposition* 6.5 *respectively and* $\phi^{(k)}=\phi^{(k)}_{(1)}+\phi^{(k)}_{(\infty)}$ $(k=1,2)$. *Then it holds that*

$$\|F_{high}(u^{(1)},g)-F_{high}(u^{(2)},g)]\|_{L^2(0,T;H^{s-1}_2\times H^{s-2}_2)}$$
$$\le C\sum_{k=1}^{2}\left\|\left\{u^{(k)}_{(1),m},u^{(k)}_{(\infty)}\right\}\right\|_{X^s(0,T)}\left\|\left\{u^{(1)}_{(1),m}-u^{(2)}_{(1),m},u^{(1)}_{(\infty)}-u^{(2)}_{(\infty)}\right\}\right\|_{X^{s-1}(0,T)}$$
$$+C[g]_s\left\|\left\{u^{(1)}_{(1),m}-u^{(2)}_{(1),m},u^{(1)}_{(\infty)}-u^{(2)}_{(\infty)}\right\}\right\|_{X^{s-1}(0,T)},$$

uniformly for $u^{(k)}_{(1),m}$ *and* $u^{(k)}_{(\infty)}$.

Proposition 7.4 easily follows from Lemmas 2.1–2.4, Lemma 4.4, Lemma 4.15 and Lemma 4.16 in a similar manner to the proof of Proposition 7.2.

The following estimate is concerned with Proposition 7.6.

Proposition 7.5. (i) *Let* $u_{(1),m}={}^\top(\phi_{(1)},m_{(1)})\in(\mathscr{X}_{(1)}\times\mathscr{Y}_{(1)})_{sym}$ *and* $u_{(\infty)}={}^\top(\phi_{(\infty)},w_{(\infty)})\in H^s_{(\infty),2,sym}$ *satisfying*

$$\sup_{0\le t\le T}\|u_{(1),m}(t)\|_{\mathscr{X}_{(1)}\times\mathscr{Y}_{(1)}}+\sup_{0\le t\le T}\|u_{(\infty)}(t)\|_{H^s_2}$$
$$+\sup_{0\le t\le T}\|\phi(t)\|_{L^\infty_1}+\sup_{0\le t\le T}\|\nabla\phi(t)\|_{L^2_1}\le \min\left\{\delta_0,\delta,\frac{1}{2}\right\},$$

where δ_0,δ *are the ones in Lemma* 4.9 (i) *and Proposition* 6.5 *respectively and* $\phi=\phi_{(1)}+\phi_{(\infty)}$. *Then it holds that*

$$\|F_{low,m}(u,g)\|_{C([0,T];L^2)}+\|\nabla F_{low,m}(u,g)\|_{C([0,T];L^2_1)}$$
$$\le C\|\{u_{(1),m},u_{(\infty)}\}\|^2_{X^s(0,T)}+C\Big(1+\|\{u_{(1),m},u_{(\infty)}\}\|_{X^s(0,T)}\Big)[g]_s,$$

uniformly for $u_{(1),m}$ *and* $u_{(\infty)}$.

(ii) *Let* $u^{(k)}_{(1),m}={}^\top(\phi^{(k)}_{(1)},m^{(k)}_{(1)})\in(\mathscr{X}_{(1)}\times\mathscr{Y}_{(1)})_{sym}$ *and* $u^{(k)}_{(\infty)}={}^\top(\phi^{(k)}_{(\infty)},w^{(k)}_{(\infty)})\in H^s_{(\infty),2,sym}$ *satisfying*

$$\sup_{0\le t\le T}\|u^{(k)}_{(1),m}(t)\|_{\mathscr{X}_{(1)}\times\mathscr{Y}_{(1)}}+\sup_{0\le t\le T}\|u^{(k)}_{(\infty)}(t)\|_{H^s_2}$$
$$+\sup_{0\le t\le T}\|\phi^{(k)}(t)\|_{L^\infty_1}+\sup_{0\le t\le T}\|\nabla\phi^{(k)}(t)\|_{L^2_1}\le \min\left\{\delta_0,\delta,\frac{1}{2}\right\},$$

where δ_0,δ *are the ones in Lemma* 4.9 (i) *and Proposition* 6.5 *respectively and* $\phi^{(k)}=\phi^{(k)}_{(1)}+\phi^{(k)}_{(\infty)}$ $(k=1,2)$. *Then it holds that*

$$\|F_{low,m}(u^{(1)},g)-F_{low,m}(u^{(2)},g)\|_{L^2}+\|\nabla F_{low,m}(u^{(1)},g)-F_{low,m}(u^{(2)},g)\|_{L^2_1}$$
$$\le C\sum_{k=1}^{2}\left\|\left\{u^{(k)}_{(1),m},u^{(k)}_{(\infty)}\right\}\right\|_{X^s(0,T)}\left\|\left\{u^{(1)}_{(1),m}-u^{(2)}_{(1),m},u^{(1)}_{(\infty)}-u^{(2)}_{(\infty)}\right\}\right\|_{X^{s-1}(0,T)}$$
$$+C[g]_s\left\|\left\{u^{(1)}_{(1),m}-u^{(2)}_{(1),m},u^{(1)}_{(\infty)}-u^{(2)}_{(\infty)}\right\}\right\|_{X^{s-1}(0,T)},$$

uniformly for $u^{(k)}_{(1),m}$ *and* $u^{(k)}_{(\infty)}$.

Proposition 7.5 follows from direct computations based on Lemma 4.14.

We obtain the existence of a solution $\{u_{(1),m},u_{(\infty)}\}$ of (4.2), (4.4) and (4.6) on $[0,T]$ satisfying $u_{(1),m}(0)=u_{(1),m}(T)$ and $u_{(\infty)}(0)=u_{(\infty)}(T)$ by similar iteration argument to that in [**10**].

$u^{(0)}_{(1),m}={}^\top(\phi^{(0)}_{(1)},m^{(0)}_{(1)})$ and $u^{(0)}_{(\infty)}={}^\top(\phi^{(0)}_{(\infty)},w^{(0)}_{(\infty)})$ are defined by

$$\begin{cases} u^{(0)}_{(1),m}(t):=S_1(t)\mathscr{S}_1(T)[(I-S_1(T))^{-1}\mathbb{G}_1]+\mathscr{S}_1(t)[\mathbb{G}_1],\\ w^{(0)}_{(1)} \quad\;\; =m^{(0)}_{(1)}-P_1(\phi^{(0)}w^{(0)}),\\ u^{(0)}_{(\infty)}(t)\;\;:=S_{\infty,0}(t)(I-S_{\infty,0}(T))^{-1}\mathscr{S}_{\infty,0}(T)[\mathbb{G}_\infty]+\mathscr{S}_{\infty,0}(t)[\mathbb{G}_\infty],\end{cases}\tag{7.1}$$

where $t\in[0,T]$, $\mathbb{G}={}^\top(0,\frac{1}{\gamma}g(x,t))$, $\mathbb{G}_1=P_1\mathbb{G}$, $\mathbb{G}_\infty=P_\infty\mathbb{G}$, $\phi^{(0)}=\phi^{(0)}_{(1)}+\phi^{(0)}_{(\infty)}$ and

$w^{(0)} = w^{(0)}_{(1)} + w^{(0)}_{(\infty)}$. Note that $u^{(0)}_{(1),m}(0) = u^{(0)}_{(1),m}(T)$ and $u^{(0)}_{(\infty)}(0) = u^{(0)}_{(\infty)}(T)$.

$u^{(N)}_{(1),m} = {}^\top(\phi^{(N)}_{(1)}, m^{(N)}_{(1)})$ and $u^{(N)}_{(\infty)} = {}^\top(\phi^{(N)}_{(\infty)}, w^{(N)}_{(\infty)})$ are defined, inductively for $N \geq 1$, by

$$\begin{cases} u^{(N)}_{(1),m}(t) := S_1(t)\mathscr{S}_1(T)[(I - S_1(T))^{-1}F_{low,m}(u^{(N-1)}, g)] + \mathscr{S}_1(t)[F_{low,m}(u^{(N-1)}, g)], \\ w^{(N)}_{(1)} \quad\;\; = m^{(N)}_{(1)} - P_1(\phi^{(N)} w^{(N)}), \\ u^{(N)}_{(\infty)}(t) \;\; := S_{\infty,u^{(N-1)}}(t)(I - S_{\infty,u^{(N-1)}}(T))^{-1}\mathscr{S}_{\infty,u^{(N-1)}}(T)[F_{high}(u^{(N-1)}, g)] \\ \qquad\qquad + \mathscr{S}_{\infty,u^{(N-1)}}(t)[F_{high}(u^{(N-1)}, g)], \end{cases} \tag{7.2}$$

where $t \in [0, T]$, $u^{(N-1)} = u^{(N-1)}_{(1)} + u^{(N-1)}_{(\infty)}$, $u^{(N-1)}_{(1)} = {}^\top(\phi^{(N-1)}_{(1)}, w^{(N-1)}_{(1)})$, $\phi^{(N)} = \phi^{(N)}_{(1)} + \phi^{(N)}_{(\infty)}$ and $w^{(N)} = w^{(N)}_{(1)} + w^{(N)}_{(\infty)}$. Note that $u^{(N)}_{(1),m}(0) = u^{(N)}_{(1),m}(T)$ and $u^{(0)}_{(\infty)}(0) = u^{(0)}_{(\infty)}(T)$.

The symbol $B_{X^k_{sym}(a,b)}(r)$ stands for the closed unit ball in $X^k_{sym}(a, b)$ centered at 0 with radius r, i.e.,

$$B_{X^k_{sym}(a,b)}(r) := \left\{\{u_{(1),m}, u_{(\infty)}\} \in X^k_{sym}(a, b); \|\{u_{(1),m}, u_{(\infty)}\}\|_{X^k(a,b)} \leq r\right\}.$$

We have the following proposition from Propositions 5.1, 6.5, 7.1, 7.2, and 7.5 by the same argument as that in [**10**].

Proposition 7.6. *There exists a constant $\delta_1 > 0$ such that if $[g]_s \leq \delta_1$, then it holds that*

(i) $$\left\|\left\{u^{(N)}_{(1),m}, u^{(N)}_{(\infty)}\right\}\right\|_{X^s(0,T)} \leq C_1[g]_s,$$

for all $N \geq 0$, and

(ii) $$\begin{aligned} &\left\|\left\{u^{(N+1)}_{(1),m} - u^{(N)}_{(1),m}, u^{(N+1)}_{(\infty)} - u^{(N)}_{(\infty)}\right\}\right\|_{X^{s-1}(0,T)} \\ &\leq C_1[g]_s \left\|\left\{u^{(N)}_{(1),m} - u^{(N-1)}_{(1),m}, u^{(N)}_{(\infty)} - u^{(N-1)}_{(\infty)}\right\}\right\|_{X^{s-1}(0,T)}, \end{aligned}$$

for $N \geq 1$. Here C_1 is a constant independent of g and N.

Concerning the existence of a solution $\{u_{(1),m}, u_{(\infty)}\}$ of (4.2), (4.4) and (4.6) on $[0, T]$ satisfying $u_{(1),m}(0) = u_{(1),m}(T)$ and $u_{(\infty)}(0) = u_{(\infty)}(T)$, we state the following

Proposition 7.7. *There exists a constant $\delta_2 > 0$ such that if $[g]_s \leq \delta_2$, then the system (4.2), (4.4) and (4.6) has a unique solution $\{u_{(1),m}, u_{(\infty)}\}$ on $[0, T]$ in $B_{X^s_{sym}(0,T)}(C_1[g]_s)$ satisfying $u_{(1),m}(0) = u_{(1),m}(T)$ and $u_{(\infty)}(0) = u_{(\infty)}(T)$. The uniqueness of solutions of (4.2), (4.4) and (4.6) on $[0, T]$ satisfying $u_{(1),m}(0) = u_{(1),m}(T)$ and $u_{(\infty)}(0) = u_{(\infty)}(T)$ holds in $B_{X^s_{sym}(0,T)}(C_1\delta_2)$.*

Corollary 7.8. *There exists a constant $\delta_3 > 0$ such that if $[g]_s \leq \delta_3$, then the system (4.1)–(4.2) has a unique solution $\{u_{(1)}, u_{(\infty)}\}$ on $[0, T]$ in $B_{X^s_{sym}(0,T)}(C_2[g]_s)$ satisfying $u_{(j)}(0) = u_{(j)}(T)$ $(j = 1, \infty)$ where $u_{(j)} = {}^\top(\phi_{(j)}, w_{(j)})$ $(j = 1, \infty)$ and C_2 is a* *constant independent of* g. *The uniqueness of solutions of* (4.1)–(4.2) *on* $[0,T]$ *satisfying* $u_{(j)}(0) = u_{(j)}(T)$ $(j = 1, \infty)$ *holds in* $B_{X^s_{sym}(0,T)}(C_2\delta_3)$.

Proposition 7.7 and Corollary 7.8 follow from Lemma 4.9 (i) and Proposition 7.7 by the same way as that in [**10**] and we omit the proofs.

As for the unique existence of solutions of the initial value problem, (4.1)–(4.2), the following proposition can be proved from the estimates in sections 5–7, as in [**4**], [**10**].

Proposition 7.9. *Let* $h \in \mathbb{R}$ *and let* $U_0 = U_{01} + U_{0\infty}$ *with* $U_{01} \in \mathscr{X}_{(1),sym} \times \mathscr{Y}_{(1),syn}$ *and* $U_{0\infty} \in H^s_{(\infty),2,sym}$. *Then there exist constants* $\delta_4 > 0$ *and* $C_3 > 0$ *such that if*

$$M(U_{01}, U_{0\infty}, g) := \|U_{01}\|_{\mathscr{X}_{(1)} \times \mathscr{Y}_{(1)}} + \|U_{0\infty}\|_{H^s_{(\infty),2}} + [g]_s \leq \delta_4,$$

there exists a solution $\{u_{(1)}, u_{(\infty)}\}$ *of the initial value problem for* (4.1)–(4.2) *on* $[h, h+T]$ *in* $B_{X^s_{sym}(h,h+T)}(C_3 M(U_{01}, U_{0\infty}, g))$ *satisfying the initial condition* $u_{(j)}|_{t=h} = U_{0j}$ $(j = 0, \infty)$. *The uniqueness for this initial value problem holds in* $B_{X^s_{sym}(h,h+T)}(C_3\delta_4)$.

Therefore, we can extend $\{u_{(1)}, u_{(\infty)}\}$ periodically on $\mathbb{R}$ as a time periodic solution of (4.1)–(4.2) by using Corollary 7.8 and Proposition 7.9 in the same argument as that given in [**4**]. Consequently, we obtain Theorem 3.1. This completes the proof.

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