J. Math. Soc. Japan
Vol. 70, No. 1 (2018) pp. 325–344
doi: 10.2969/jmsj/07017447

Basic relative invariants of homogeneous cones and their Laplace transforms

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(Received Feb. 29, 2016)
(Revised May 19, 2016)

Abstract. The purpose of this paper is to show that it is characteristic of symmetric cones among irreducible homogeneous cones that there exists a non-constant relatively invariant polynomial such that its Laplace transform is the reciprocal of a certain polynomial. To prove our theorem, we need the inductive structure of the basic relative invariants of a homogeneous cone. However, we actually work in a more general setting, and consider the inducing of the basic relative invariants from lower rank cones.

Introduction.

It is well known that the Fourier–Laplace transform of a complex power x^s of positive reals x, or of a complex power $P(x)^s$ of a positive-definite quadratic form $P(x)$, is essentially given by a complex power of a certain polynomial (see [**2**, Sections II.2 and III.2], [**10**], for example). These facts are the fundamental principles of the theory of prehomogeneous vector spaces constructed by M. Sato (see [**10**, Introduction]). On the other hand, Wishart [**17**] and Ingham [**5**] consider the gamma functions of positive-definite symmetric matrices in statistics, and Siegel [**15**] in number theory independently. The gamma functions are also studied by Riesz [**14**] for Lorentz cones, and by Koecher [**11**] for general symmetric cones. As is computed in the book Faraut and Korányi [**1**], once the gamma functions associated with symmetric cones are calculated, we are able to obtain the Laplace transforms of relative invariants by using the transitivity of group actions. Looking at the formula closely, we find that the Laplace transform of a complex power of the determinant function $\det x$ (Minkowski metric in the case of Lorentz cones) of symmetric cones Ω is expressed, up to gamma factors, as a complex power of the reciprocal of the same determinant function (see [**1**, Chapter VII], for example):

$$\int_\Omega e^{-\langle x\,|\,y\rangle}(\det x)^s\,d\mu(x)=\frac{(\text{gamma factors})}{(\det y)^s}\quad (y\in\Omega),$$

where $d\mu$ is a suitable invariant measure on Ω. Gindikin [**3**] considers the Laplace transform in a more general setting where the integration domains are homogeneous cones which form a class of prehomogeneous vector spaces. Then a natural question is whether or not, in the case of general homogeneous cones, there exists a non-constant relatively

2010 *Mathematics Subject Classification.* Primary 44A10; Secondary 22E25, 43A85, 11S90.
Key Words and Phrases. Laplace transforms, symmetric cones, homogeneous cones, basic relative invariants.

invariant polynomial such that its Laplace transform is the reciprocal of a certain polynomial. Here, the groups that we consider for relative invariance are the split solvable Lie groups acting simply transitively on the cones. In this paper, we give an answer to this question by showing that this property is characteristic of symmetric cones, that is, this property holds if and only if the cone is symmetric.

We now describe the contents of this paper in more detail. Let Ω be a homogeneous cone of rank r in a finite-dimensional real vector space V. Then, there exists a split solvable Lie group H acting on Ω linearly and simply transitively. By differentiating the action of H on Ω, we see that V admits an algebraic structure $\triangle$, called a Vinberg algebra, having a unit element e_0. We have the normal decomposition $V=\bigoplus_{1\le j\le k\le r} V_{kj}$ with respect to a complete system of orthogonal primitive idempotents. The cone Ω has r irreducible relatively H-invariant polynomial functions $\Delta_1,\dots,\Delta_r$, called the basic relative invariants of Ω. We place the multipliers of these Δ_j in an $r\times r$ matrix σ which is called the *multiplier matrix* of Ω in this paper (see (1.5) for detail). An algorithm for calculating σ is given by using the data of Ω (cf. [**12**]). The dual cone Ω^* of Ω is defined through an inner product $\langle\cdot|\cdot\rangle$ in V given by an admissible linear form (see (V2) in this paper), and the corresponding Vinberg algebra is denoted by $(V,\triangledown)$.

In order to work out the problem above, we need an inductive structure of Vinberg algebra, but in this paper, we deal with a rather general situation with a view of future studies. Let p,q be positive integers such that $p+q=r$, and we put

$$V_- := \bigoplus_{1\le j\le k\le p} V_{kj},\quad E := \bigoplus_{1\le j\le p<k\le r} V_{kj},\quad V_+ := \bigoplus_{p<j\le k\le r} V_{kj}. \tag{0.1}$$

Then, V is decomposed into a direct sum $V=V_-\oplus E\oplus V_+$ of these vector subspaces. We denote general elements x in V by $x_-+\xi+x_+\in V$ ($x_\pm\in V_\pm$, $\xi\in E$) without any comments. We note that $V_\pm$ are subalgebras of V, and hence there exist homogeneous cones $\Omega_\pm$ corresponding to $V_\pm$, respectively. On E, the linear operators $\psi(x_-)$ and $\varphi(x_+)$ ($x_\pm\in V_\pm$) are defined, respectively, by

$$\psi(x_-)\xi := \xi\mathbin{\triangle} x_-,\quad \varphi(x_+)\xi := \xi\mathbin{\triangledown} x_+\quad (\xi\in E).$$

It is then shown in Lemma 2.2 that ψ (resp. φ) is a selfadjoint representation of $(V_-,\triangle)$ (resp. $(V_+,\triangledown)$). Let Q be the symmetric V_+-valued bilinear map associated with φ, so that $Q(\xi,\eta)=\xi\mathbin{\triangle}\eta$ ($\xi,\eta\in E$). Then, the determinants of the right multiplication operators R_x, defined by $R_xy=y\mathbin{\triangle}x$ ($x,y\in V$) are described in Proposition 2.5 as

$$\operatorname{Det} R_x = \operatorname{Det} R^-_{x_-}\cdot\operatorname{Det}\psi(x_-)\cdot\operatorname{Det} R^+_{x_+-(1/2)Q(\psi(x_-)^{-1}\xi,\ \xi)}\quad (x\in\Omega),$$

where $R^\pm$ denote the right multiplication operators of $V_\pm$, respectively. By putting $\widetilde{x}:=x_+-(1/2)Q(\psi(x_-)^{-1}\xi,\xi)\in V_+$, we see in Lemma 2.6 that $(h\cdot x)^{\sim}=h_+\cdot\widetilde{x}$, where h_+ is the "V_+-part" of $h\in H$ (see (2.6) for detail). Let $\sigma_\pm$ be the multiplier matrices of $\Omega_\pm$, respectively. We denote by $\Delta^-_1,\dots,\Delta^-_p$ the basic relative invariants of Ω_-, and by $\Delta^+_1,\dots,\Delta^+_q$ those of Ω_+. Then, by calculating the irreducible factors in $\operatorname{Det} R_x$ ($x\in\Omega$), Theorem 2.8 shows that there exists a unique matrix $\Xi\in\operatorname{Mat}(q,p;\{0,1\})$ such that, by putting $\Gamma=(\gamma_{jk}):=\sigma_+\Xi$, we have

$$\begin{cases} \Delta_i(x) = \Delta_i^-(x_-) & (i = 1, \dots, p), \\ \Delta_{p+j}(x) = \Delta_1^-(x_-)^{\gamma_{j1}} \cdots \Delta_p^-(x_-)^{\gamma_{jp}} \Delta_j^+\left(x_+ - \dfrac{1}{2} Q(\psi(x_-)^{-1}\xi, \xi)\right) & (j = 1, \dots, q) \end{cases}$$

for any $x \in \Omega$. Moreover, the multiplier matrix σ of Ω is described as

$$\sigma = \begin{pmatrix} \sigma_- & 0 \\ \sigma_+ \Xi \sigma_- & \sigma_+ \end{pmatrix} = \begin{pmatrix} I_p & 0 \\ 0 & \sigma_+ \end{pmatrix} \begin{pmatrix} I_p & 0 \\ \Xi & I_q \end{pmatrix} \begin{pmatrix} \sigma_- & 0 \\ 0 & I_q \end{pmatrix}.$$

We now consider the Laplace transform of relatively H-invariant functions on homogeneous cones. Let $d\mu$ be an H-invariant measure on Ω, and $\Delta_{\underline{s}}$ the relatively H-invariant function whose multiplier is $\underline{s} \in \mathbb{R}^r$ with $\Delta_{\underline{s}}(e_0) = 1$. Let $\Gamma_\Omega(\underline{s})$ be the gamma function of Ω (see (3.1) for definition). Then, the Laplace transform $\mathcal{L}[\Delta_{\underline{s}}]$ of $\Delta_{\underline{s}}$ is defined as

$$\mathcal{L}[\Delta_{\underline{s}}](y) := \frac{1}{\Gamma_\Omega(\underline{s})} \int_\Omega e^{-\langle x \,|\, y \rangle} \Delta_{\underline{s}}(x) \, d\mu(x) \quad (y \in \Omega^*),$$

where Ω^* is the dual cone of Ω. Note that, in our definition, the Laplace transform is normalized as $\mathcal{L}[\Delta_{\underline{s}}](e_0) = 1$. We denote by $\Delta_1^*, \dots, \Delta_r^*$ the basic relative invariants of Ω^*. For any $\underline{\nu}, \underline{\mu} \in \mathbb{Z}^r$, let $\Delta^{\underline{\nu}}(x)$ $(x \in \Omega)$ and $\Delta_*^{\underline{\mu}}(y)$ $(y \in \Omega^*)$ be rational functions defined, respectively, by

$$\Delta^{\underline{\nu}}(x) := \Delta_1(x)^{\nu_1} \cdots \Delta_r(x)^{\nu_r}, \quad \Delta_*^{\underline{\mu}}(y) := \Delta_1^*(y)^{\mu_1} \cdots \Delta_r^*(y)^{\mu_r},$$

and we put $p_k := \sum_{j<k} \dim V_{kj}$ for $k = 1, \dots, r$. Moreover, let σ_* be the multiplier matrix of Ω^*. Then, Gindikin [**3**] tells us that, if $(\underline{\nu}\sigma)_k > p_k$ for any $k = 1, \dots, r$, then we have

$$\mathcal{L}[\Delta^{\underline{\nu}}](y) = \frac{1}{\Delta_*^{\underline{\nu}'}(y)} \quad (y \in \Omega^*;\ \underline{\nu}' := \underline{\nu}\sigma\sigma_*^{-1}).$$

The decomposition $V = V_- \oplus E \oplus \mathbb{R}c_r$ where we put $p = r - 1$ in (0.1) describes an inductive structure of V. Investigating the matrix $\sigma\sigma_*^{-1}$ in detail by using this inductive structure (Lemma 3.1 and Proposition 3.2), we prove our main theorem in Theorem 3.4, that is, an irreducible homogeneous cone is symmetric if and only if there exists a non-constant polynomial $\Delta^{\underline{\nu}}(x)$ such that the reciprocal $\Delta_*^{\underline{\nu}'}(y) = (\mathcal{L}[\Delta^{\underline{\nu}}](y))^{-1}$ of the Laplace transform of $\Delta^{\underline{\nu}}(x)$ is also a non-constant polynomial.

We organize this paper as follows. Section 1 contains the fundamental facts about homogeneous cones and Vinberg algebras. In Section 2, we study the basic relative invariants of a homogeneous cone. By decomposing the corresponding Vinberg algebra as a direct sum of three vector subspaces of which two are subalgebras, the basic relative invariants are induced from the two cones associated with the subalgebras. Section 3 is devoted to giving a characterization of symmetric cones among irreducible homogeneous cones by using the reciprocals of the Laplace transforms on homogeneous cones. To do so, we consider inductive structures of the multiplier matrices of homogeneous cones and of the dual cones.

Acknowledgements. The author would like to thank Professor Takaaki Nomura for the encouragement and the comments in writing this paper. The author is also grateful to the referee for invaluable comments for an improvement of this paper.

1. Preliminaries.

Let V be a finite-dimensional real vector space, and Ω an open convex cone in V containing no entire line. The cone Ω is said to be homogeneous if the group $G(\Omega) = \{g \in GL(V);\ g(\Omega) = \Omega\}$ acts on Ω transitively. In this paper, we always assume that Ω is homogeneous, and call it a homogeneous cone for short. By Vinberg [**16**], there exists a split solvable Lie subgroup H of $G(\Omega)$ such that H acts on Ω simply transitively. Let $\mathfrak{h}$ be the Lie algebra corresponding to H, and fix a point $e_0 \in \Omega$. Then, we have a linear isomorphism $\mathfrak{h} \ni X \mapsto Xe_0 \in V$ obtained by differentiating the orbit map $h \mapsto he_0$ at the unit element of H. We denote the inverse map by $L\colon V \ni x \mapsto L_x \in \mathfrak{h}$. According to [**16**], we introduce a product $\triangle$ in V by

$$x \triangle y := L_x y \quad (x, y \in V).$$

The product $\triangle$ is neither commutative nor associative in general. We know that e_0 is a unit element of V. Moreover, $(V, \triangle)$ satisfies the following three conditions:

(V1) $L_{x \triangle y - y \triangle x} = L_x L_y - L_y L_x$ for any $x, y \in V$,

(V2) there exists a linear form s such that $s(x \triangle y)$ defines an inner product in V,

(V3) the linear operator L_x has only real eigenvalues for each $x \in V$.

We call $(V, \triangle)$ (or simply V) a *Vinberg algebra* in this paper. Linear forms with the property (V2) are said to be admissible. The rank r of the cone Ω is defined by the dimension of a maximal connected commutative subgroup of $G(\Omega)$. Let $c_1, \dots, c_r$ be a complete system of orthogonal primitive idempotents of V. They satisfy $c_j \triangle c_k = \delta_{jk} c_j$ and $e_0 = c_1 + \cdots + c_r$. We denote by R_x $(x \in V)$ the right multiplication operators $R_x y := y \triangle x$ $(y \in V)$. Let V_{jj} be the one-dimensional subspace $\mathbb{R}c_j$ $(j = 1, \dots, r)$, and we put for $j < k$

$$V_{kj} := \left\{ x \in V;\ L_{c_i} x = \frac{1}{2}(\delta_{ij} + \delta_{ik}) x,\ R_{c_i} x = \delta_{ij} x\ (i = 1, \dots, r) \right\}.$$

Then, V is decomposed into

$$V = \bigoplus_{1 \le j \le k \le r} V_{kj}, \tag{1.1}$$

which is called the normal decomposition of V. The complete system $c_1, \dots, c_r$ is said to be a *Vinberg frame* in this paper. With respect to this decomposition, we have the following multiplication rules:

$$\begin{gathered} V_{ji} \triangle V_{lk} = \{0\} \quad (\text{if } i \neq k, l), \quad V_{kj} \triangle V_{ji} \subset V_{ki}, \\ V_{ji} \triangle V_{ki} \subset V_{jk} \text{ or } V_{kj} \quad (\text{if } j \ge k \text{ or } j \le k, \text{ respectively}). \end{gathered} \tag{1.2}$$

A homogeneous cone Ω is said to be irreducible if there exist no non-trivial subspaces $V_1, V_2 \subset V$ and homogeneous cones $\Omega_j \subset V_j$ $(j = 1, 2)$ such that V is a direct sum of V_1 and V_2, and $\Omega = \Omega_1 \oplus \Omega_2$. It is easily verified that any homogeneous cone Ω in V is decomposed as a direct sum of its irreducible components Ω_a $(a = 1, \ldots, q)$, where q is the number of the irreducible components of Ω. Vinberg [**16**, Proposition 10] tells us that there exists a permutation $\sigma \in \mathfrak{S}_r$ such that the normal decomposition $V = \bigoplus_{j \le k} V_{\sigma(k)\sigma(j)}$ is of the diagonal form

$$\begin{pmatrix} U_1 & 0 & \cdots & 0 \\ 0 & U_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & U_q \end{pmatrix}, \tag{1.3}$$

where $U_a \subset V$ is the ambient vector space of Ω_a. In other words, the irreducible components can be placed on the diagonal blocks.

A typical example of homogeneous cone is a positive-definite matrices $\mathcal{S}_N^+$ in the space $\mathcal{S}_N$ of symmetric matrices of rank N. The group $GL(N, \mathbb{R})$ acts on $\mathcal{S}_N^+$ transitively by $\rho(g)x := gx\,{}^t g$ where $g \in GL(N, \mathbb{R})$ and $x \in \mathcal{S}_N^+$. Let $\mathcal{H}_N$ be the group of lower triangular matrices with positive diagonals. Then, $\mathcal{H}_N$ is a split solvable Lie subgroup of $GL(N, \mathbb{R})$, and acts on $\mathcal{S}_N^+$ simply transitively. For $x = (x_{ij}) \in \mathcal{S}_N$, let us define a lower triangular matrix $\underline{x}$ by

$$(\underline{x})_{ij} = \begin{cases} x_{ij} & (i > j), \\ \dfrac{1}{2} x_{jj} & (i = j), \\ 0 & (i < j), \end{cases}$$

and put $\overline{x} := {}^t(\underline{x})$. Obviously we have $x = \underline{x} + \overline{x}$. We choose the unit matrix I_N as the element e_0 in the construction of a Vinberg algebra $(\mathcal{S}_N, \triangle)$ from $\mathcal{S}_N^+$. Then, the product $\triangle$ is given as

$$x \triangle y := \underline{x} y + y \overline{x} \quad (x, y \in \mathcal{S}_N).$$

According to Ishi [**7**, Section 3.1], we realize a homogeneous cone Ω of rank r in a subspace $\mathcal{Z}_{\mathcal{V}}$ of $\mathcal{S}_N$, defined below, with a suitable N (see also [**4**, Section 3.2]). With respect to Ω, there exist a partition $N = n_1 + \cdots + n_r$ of N and a system of vector spaces $\mathcal{V}_{kj} \subset \mathrm{Mat}(n_k, n_j; \mathbb{R})$ $(1 \le j < k \le r)$ satisfying the following conditions:

(1) if $A \in \mathcal{V}_{lk}$ and $B \in \mathcal{V}_{kj}$, then one has $AB \in \mathcal{V}_{lj}$ $(1 \le j < k < l \le r)$,

(2) if $A \in \mathcal{V}_{lj}$ and $B \in \mathcal{V}_{kj}$, then one has $A\,{}^t B \in \mathcal{V}_{lk}$ $(1 \le j < k < l \le r)$,

(3) if $A \in \mathcal{V}_{kj}$, then one has $A\,{}^t A \in \mathbb{R} I_{n_k}$ $(1 \le j < k \le r)$.

Let $\mathcal{Z}_{\mathcal{V}}$ be the subspace of $\mathcal{S}_N$ defined by

$$\mathcal{Z}_{\mathcal{V}} := \left\{ x = \begin{pmatrix} x_1 I_{n_1} & {}^t X_{21} & \cdots & {}^t X_{r1} \\ X_{21} & x_2 I_{n_2} & & {}^t X_{r2} \\ \vdots & & \ddots & \\ X_{r1} & X_{r2} & \cdots & x_r I_{n_r} \end{pmatrix};\ \begin{array}{ll} x_j \in \mathbb{R} & (j = 1, \dots, r), \\ X_{kj} \in \mathcal{V}_{kj} & (1 \leq j < k \leq r) \end{array} \right\} \subset \mathcal{S}_N,$$

and let $H_{\mathcal{V}}$ be the split solvable subgroup of $\mathcal{H}_N$ defined by

$$H_{\mathcal{V}} := \left\{ h = \begin{pmatrix} h_1 I_{n_1} & & & \\ T_{21} & h_2 I_{n_2} & & \\ \vdots & & \ddots & \\ T_{r1} & T_{r2} & \cdots & h_r I_{n_r} \end{pmatrix};\ \begin{array}{ll} h_j \in \mathbb{R}^+ & (j = 1, \dots, r), \\ T_{kj} \in \mathcal{V}_{kj} & (1 \leq j < k \leq r) \end{array} \right\}. \tag{1.4}$$

Then, Ω is linearly isomorphic to $\mathcal{P}_{\mathcal{V}} := \mathcal{Z}_{\mathcal{V}} \cap \mathcal{S}_N^+$ on which $H_{\mathcal{V}}$ acts simply transitively by the action $\rho(h)x = hx\,{}^t h$ for $h \in H_{\mathcal{V}}$ and $x \in \mathcal{P}_{\mathcal{V}}$. Moreover, a split solvable Lie group H acting on Ω simply transitively is isomorphic to $H_{\mathcal{V}}$. Although this realization is not unique in general, each realization corresponding to the same Ω is mutually linearly isomorphic. We note that there exists a minimal realization in the sense of Yamasaki and Nomura [**18**], which is unique up to the order of Vinberg frame. In this paper, we assume that Ω is realized as a matrix form, and so is H.

Let $\chi\colon H \to \mathbb{R}^\times$ be a character of H. Since H is split solvable, there exists $\underline{\nu} = (\nu_1, \dots, \nu_r) \in \mathbb{R}^r$ such that

$$\chi(h) = \chi_{\underline{\nu}}(h) = h_1^{2\nu_1} \cdots h_r^{2\nu_r} \quad (h \in H),$$

where $h \in H$ is described as in (1.4). A function f on Ω is said to be relatively H-invariant if there exists a character $\chi_{\underline{\nu}}$ with some $\underline{\nu} \in \mathbb{R}^r$ such that $f(\rho(h)x) = \chi_{\underline{\nu}}(h) f(x)$ for any $h \in H$ and $x \in \Omega$. The vector $\underline{\nu} \in \mathbb{R}^r$ is called the multiplier of f, and we write $f(x) = \Delta_{\underline{\nu}}(x)$ if $f(e_0) = 1$. Among relatively H-invariant polynomial functions, there exist exactly r irreducible ones $\Delta_1, \dots, \Delta_r$, by which any relatively H-invariant polynomial function p is described as

$$p(x) = (\mathrm{const}) \Delta_1(x)^{m_1} \cdots \Delta_r(x)^{m_r} \quad (x \in \Omega;\ \text{for some } m_1, \dots, m_r \in \mathbb{Z}_{\geq 0})$$

(cf. Ishi [**6**]). Moreover, Ω is written as

$$\Omega = \{ x \in V;\ \Delta_1(x) > 0, \dots, \Delta_r(x) > 0 \}.$$

The polynomials $\Delta_1(x), \dots, \Delta_r(x)$ are called the *basic relative invariants* of Ω. Let $\underline{\sigma}_j = (\sigma_{j1}, \dots, \sigma_{jr})$ be the multiplier of Δ_j $(j = 1, \dots, r)$, and we place them in an $r \times r$ matrix σ as follows:

$$\sigma = \begin{pmatrix} \underline{\sigma}_1 \\ \vdots \\ \underline{\sigma}_r \end{pmatrix} = (\sigma_{jk})_{1 \leq j,k \leq r}, \tag{1.5}$$

which is called the *multiplier matrix* of Ω in this paper. If we determine the numbering of the basic relative invariants by the procedure of Ishi [**6**] according to the Vinberg frame, then we see that σ is described as a lower triangular matrix with ones on the main diagonals. Thus, we always assume that the basic relative invariants are labeled in this order. Put $d_{kj} := \dim V_{kj}$ for $1 \le j < k \le r$ and $\boldsymbol{d}_i := {}^t(0, \ldots, 0, d_{i+1,i}, \ldots, d_{ri})$ for $i = 1, \ldots, r-1$. Let us recall the algorithm for calculating σ given in [**12**].

Lemma 1.1. *For $i = 1, \ldots, r-1$, defining $\boldsymbol{l}_i^{(j)} = {}^t(l_{1i}^{(j)}, \ldots, l_{ri}^{(j)})$ $(j = i, \ldots, r)$ inductively by $\boldsymbol{l}_i^{(i)} := \boldsymbol{d}_i$ and, for $k = i+1, \ldots, r-1$,*

$$\boldsymbol{l}_i^{(k)} = \begin{cases} \boldsymbol{l}_i^{(k-1)} - \boldsymbol{d}_k & (\boldsymbol{l}_{ki}^{(k-1)} > 0), \\ \boldsymbol{l}_i^{(k-1)} & (\boldsymbol{l}_{ki}^{(k-1)} = 0), \end{cases}$$

one sets $\boldsymbol{\varepsilon}^{[i]} = {}^t(\varepsilon_{i+1,i}, \ldots, \varepsilon_{ri}) \in \{0,1\}^{r-i}$ $(i = 1, \ldots, r-1)$ where

$$\varepsilon_{ji} = \begin{cases} 1 & (\textit{if } \boldsymbol{l}_{ji}^{(r-1)} > 0), \\ 0 & (\textit{if } \boldsymbol{l}_{ji}^{(r-1)} = 0) \end{cases} \qquad (j = i+1, \ldots, r).$$

Then, the multiplier matrix σ is calculated as

$$\sigma = \mathcal{E}_{r-1}\mathcal{E}_{r-2}\cdots\mathcal{E}_1, \quad \mathcal{E}_i := \begin{pmatrix} I_{i-1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \boldsymbol{\varepsilon}^{[i]} & I_{r-i} \end{pmatrix} \quad (i = 1, \ldots, r-1). \tag{1.6}$$

Note that, if we write $\sigma^{-1} = (\sigma^{jk})_{1 \le j,k \le r}$, then we find that $\sigma^{jk} = -\varepsilon_{jk}$ for $j > k$ by taking the inverse of both sides of (1.6).

Let $\langle \cdot | \cdot \rangle$ be the inner product of V given by (V2). Let ρ^* be the contragredient representation of ρ defined by $\langle \rho(h)x \,|\, \rho^*(h)y \rangle = \langle x \,|\, y \rangle$ for any $h \in H$ and $x, y \in V$. Through the inner product $\langle \cdot | \cdot \rangle$, we define

$$\Omega^* := \left\{ y \in V;\ \langle x \,|\, y \rangle > 0 \text{ for all } x \in \overline{\Omega} \setminus \{0\} \right\}.$$

Then, Ω^* is an open convex cone on which H acts simply transitively by ρ^*, and the homogeneous cone Ω^* is called the dual cone of Ω. A function f^* on Ω^* is said to be relatively H-invariant if there exists a character $\chi_{\underline{\mu}}$ with some $\underline{\mu} \in \mathbb{R}^r$ such that $f^*(\rho^*(h)y) = \chi_{\underline{\mu}}^{-1}(h) f^*(y)$ for any $h \in H$ and $y \in \Omega^*$. The vector $\underline{\mu}$ is called the multiplier of f^* of Ω^*, and write $f^*(y) = \Delta^*_{\underline{\mu}}(y)$ when $f^*(e_0) = 1$. The Vinberg algebra $(V, \triangledown)$ corresponding to Ω^* is called the dual Vinberg algebra of $(V, \triangle)$, and the product $\triangledown$ satisfies

$$\left\langle x \triangledown y \,|\, z \right\rangle = \langle y \,|\, x \triangle z \rangle \quad (x, y, z \in V). \tag{1.7}$$

We summarize the properties of $(V, \triangledown)$ (cf. [**12**]). For a Vinberg frame of $(V, \triangledown)$, we choose $c_r, \ldots, c_1$. Then, the normal decomposition of $(V, \triangledown)$ with respect to the frame is given also by (1.1) with the multiplication rules below:

$$
\begin{gathered}
V_{ji} \triangledown V_{lk} = \{0\} \quad (\text{if } j \neq k, l), \quad V_{ji} \triangledown V_{kj} \subset V_{ki}, \\
V_{ki} \triangledown V_{kj} \subset V_{ji} \text{ or } V_{ij} \quad (\text{if } i \leq j \text{ or } i \geq j, \text{ respectively}).
\end{gathered} \tag{1.8}
$$

Moreover, we have the following relationship between the products $\triangle$ and $\triangledown$:

$$
x \triangle y + x \triangledown y = y \triangle x + y \triangledown x \quad (x, y \in V). \tag{1.9}
$$

Let $\Delta_1^*(x), \ldots, \Delta_r^*(x)$ be the basic relative invariants of Ω^*. Here, they are given inductively by the procedure of [**6**] according to the Vinberg frame $c_r, \ldots, c_1$, and the indices are labeled from r down to 1. Then, the multiplier matrix σ_* of Ω^* is described as an upper triangular form, whereas σ is lower triangular.

For a linear map φ on V to $\mathcal{S}_N$, the lower part $\underline{\varphi}$ and the upper part $\overline{\varphi}$ are defined, respectively, by

$$
\underline{\varphi}(x) = \underline{\varphi(x)}, \quad \overline{\varphi}(x) = \overline{\varphi(x)} \quad (x \in V).
$$

Then, we easily find that

$$
{}^t(\underline{\varphi}(x)) = \overline{\varphi}(x), \quad \varphi(x) = \underline{\varphi}(x) + \overline{\varphi}(x) \qquad (x \in V).
$$

When φ is an algebra homomorphism of Vinberg algebras, that is, φ satisfies

$$
\varphi(x \triangle y) = \underline{\varphi}(x)\varphi(y) + \varphi(y)\overline{\varphi}(x) \quad (x, y \in V),
$$

we call φ a selfadjoint representation of $(V, \triangle)$ (cf. Ishi [**8**]). We require also $\varphi(e_0) = I_N$. Similarly, φ is called a selfadjoint representation of $(V, \triangledown)$ if φ satisfies

$$
\varphi(x \triangledown y) = \overline{\varphi}(x)\varphi(y) + \varphi(y)\underline{\varphi}(x) \quad (x, y \in V),
$$

and also $\varphi(e_0) = I_N$.

2. Inducing the basic relative invariants from lower rank cones.

Let Ω be a homogeneous cone of rank $r \geq 2$, and V the corresponding Vinberg algebra. We fix a matrix realization $\Omega = \mathcal{P}_{\mathcal{V}}$, and keep to the notation used in Section 1. Let us take positive integers p, q such that $r = p + q$. According to the normal decomposition $V = \bigoplus_{j \leq k} V_{kj}$, the subspaces $V_{\pm}$ and E are defined respectively by

$$
V_- = \bigoplus_{1 \leq j \leq k \leq p} V_{kj}, \quad E = \bigoplus_{1 \leq j \leq p < k \leq r} V_{kj}, \quad V_+ = \bigoplus_{p < j \leq k \leq r} V_{kj}.
$$

Then, V is decomposed into the direct sum

$$
V = V_+ \oplus E \oplus V_-. \tag{2.1}
$$

With respect to this decomposition, we have the following multiplication tables by the multiplication rules (1.2) and (1.8):

$$\text{(a)}\quad \begin{array}{c|ccc} \triangle & V_- & E & V_+ \\ \hline V_- & V_- & E & 0 \\ E & E & V_+ & 0 \\ V_+ & 0 & E & V_+ \end{array}, \qquad \text{(b)}\quad \begin{array}{c|ccc} \nabla & V_- & E & V_+ \\ \hline V_- & V_- & E & 0 \\ E & 0 & V_- & E \\ V_+ & 0 & E & V_+ \end{array}, \tag{2.2}$$

where left factors of the products are placed in row entries, and right ones are placed in column entries. These tables immediately lead us to the fact that $V_\pm$ are subalgebras of V. We denote general elements x of V by

$$x = x_+ + \xi + x_- \quad (x_+ \in V_+,\ \xi \in E,\ x_- \in V_-)$$

without any comments. We note that the special case $(p, q) = (1, r-1)$ is dealt with in the previous paper [**12**]. We shall generalize its results to a general situation in this section, and our result for the case $(p, q) = (r-1, 1)$ will be used in the proof of the main theorem given in the next section.

We denote by $\mathcal{L}(E)$ the space of linear operators on E. Let φ be the linear operator on V_+ to $\mathcal{L}(E)$ defined by

$$\varphi(x_+)\xi := \xi \mathbin{\nabla} x_+ \quad (x_+ \in V_+,\ \xi \in E),$$

and Q the symmetric V_+-valued bilinear map associated with φ, that is,

$$\langle Q(\xi, \eta) \,|\, x_+ \rangle = \langle \varphi(x_+)\xi \,|\, \eta \rangle \quad (\xi, \eta \in E,\ x_+ \in V_+).$$

Since $\langle \varphi(x_+)\xi \,|\, \eta \rangle = \big\langle \xi \mathbin{\nabla} x_+ \,|\, \eta \big\rangle = \langle x_+ \,|\, \xi \mathbin{\triangle} \eta \rangle$ by (1.7), we have

$$Q(\xi, \eta) = \xi \mathbin{\triangle} \eta \quad (\xi, \eta \in E). \tag{2.3}$$

Moreover, (1.9) together with the multiplication table (a) in (2.2) implies that

$$\varphi(x_+)\xi = x_+ \mathbin{\triangle} \xi + x_+ \mathbin{\nabla} \xi = (L_{x_+} + {}^t L_{x_+})\xi,$$

and hence we obtain

$$\underline{\varphi}(x_+)\xi = x_+ \mathbin{\triangle} \xi, \quad \overline{\varphi}(x_+)\xi = x_+ \mathbin{\nabla} \xi \quad (x_+ \in V_+,\ \xi \in E). \tag{2.4}$$

Similarly to φ, the linear operator ψ on V_- to $\mathcal{L}(E)$ is defined by

$$\psi(x_-)\xi := \xi \mathbin{\triangle} x_- \quad (x_- \in V_-,\ \xi \in E),$$

and its lower and upper parts are given, respectively, by

$$\underline{\psi}(x_-)\xi = x_- \mathbin{\triangle} \xi, \quad \overline{\psi}(x_-)\xi = x_- \mathbin{\nabla} \xi. \tag{2.5}$$

Lemma 2.1. *For any $x_\pm \in V_\pm$ and $\xi, \eta \in E$, one has the following*:

(i) $Q(\psi(x_-)\xi, \eta) = Q(\xi, \psi(x_-)\eta)$,
(ii) $\psi(x_-)\underline{\varphi}(x_+) = \underline{\varphi}(x_+)\psi(x_-)$.

Proof. (i) Using (2.3) and (V1), we obtain

$$Q(\xi, \psi(x_-)\eta) = \eta \vartriangle (\xi \vartriangle x_-) + (\xi \vartriangle \eta - \eta \vartriangle \xi) \vartriangle x_- = \eta \vartriangle (\xi \vartriangle x_-),$$

and thus we get the assertion.
(ii) The multiplication table (a) in (2.2) and (V1) together with (2.4) yield that

$$\psi(x_-)\underline{\varphi}(x_+)\xi = (\xi \vartriangle x_+) \vartriangle x_- + x_+ \vartriangle (\xi \vartriangle x_-) - \xi \vartriangle (x_+ \vartriangle x_-) = L_{x_+}\psi(x_-)\xi,$$

and the proof is completed. □

These operators φ and ψ are fundamental because we have the following lemma.

Lemma 2.2. (i) (φ, E) *is a selfadjoint representation of* $(V_+, \triangledown)$,
(ii) (ψ, E) *is a selfadjoint representation of* $(V_-, \vartriangle)$.

Proof. (V1) and (1.9) together with the table (a) in (2.2) imply that

$$\begin{aligned}\varphi(x_+ \triangledown y_+)\xi &= x_+ \triangledown (\xi \triangledown y_+) + (\xi \triangledown x_+ - x_+ \triangledown \xi) \triangledown y_+ \\ &= x_+ \triangledown (\xi \triangledown y_+) + (x_+ \vartriangle \xi) \triangledown y_+,\end{aligned}$$

and hence the assertion (i) holds. (ii) is proved in a similar way. □

Let $\Omega_\pm$ be the homogeneous cones corresponding to $V_\pm$, respectively. Let us put

$$H_- := \left\{ h_- = \begin{pmatrix} h_1 I_{n_1} & & \\ \vdots & \ddots & \\ T_{p,1} & \cdots & h_p I_{n_p} \end{pmatrix}; \begin{array}{l} h_i \in \mathbb{R}^+ \ (i = 1, \ldots, p), \\ T_{kj} \in \mathcal{V}_{kj} \ (1 \le j < k \le p) \end{array} \right\},$$
$$H_+ := \left\{ h_+ = \begin{pmatrix} h_{p+1} I_{n_{p+1}} & & \\ \vdots & \ddots & \\ T_{r,p+1} & \cdots & h_r I_{n_r} \end{pmatrix}; \begin{array}{l} h_j \in \mathbb{R}^+ \ (j = p+1, \ldots, r), \\ T_{kj} \in \mathcal{V}_{kj} \ (p+1 \le j < k \le r) \end{array} \right\}.$$

Then, $H_\pm$ act simply transitively on $\Omega_\pm$, respectively. These are embedded into H respectively, by putting $n_- := n_1 + \cdots + n_p$ and $n_+ := n_{p+1} + \cdots + n_r$, as

$$H_- \ni h_- \mapsto \begin{pmatrix} h_- & 0 \\ 0 & I_{n_+} \end{pmatrix} \in H, \qquad H_+ \ni h_+ \mapsto \begin{pmatrix} I_{n_-} & 0 \\ 0 & h_+ \end{pmatrix} \in H, \tag{2.6}$$

and we identify those in this manner. Since we have a matrix realization (1.4) of H, any $h \in H$ can be written by using some $h_\pm \in H_\pm$ and $\eta_h \in E$ as

$$h = \begin{pmatrix} h_- & 0 \\ \eta_h & h_+ \end{pmatrix} = h_- \cdot \begin{pmatrix} I_{n_-} & 0 \\ \eta_h & I_{n_+} \end{pmatrix} \cdot h_+,$$

and we use this decomposition without comments in this section. In what follows, we write the action by $h \cdot x$ instead of $\rho(h)x$ to avoid complexity. We note that since $H_\pm$ act on $\Omega_\pm$ simply transitively, for any $h_\pm \in H_\pm$ there exist $y_\pm \in V_\pm$ such that $h_\pm = \exp L_{y_\pm}$.

Lemma 2.3. *For any $h_\pm \in H_\pm$ and $\xi, \eta \in E$, one has the following formulas*:

$$\begin{aligned}
\text{(i)}\ h_+ \cdot Q(\xi,\eta) &= Q(h_+\cdot\xi, h_+\cdot\eta),\\
\text{(ii)}\quad \psi(x_-)h_+ &= h_+\psi(x_-),\\
\text{(iii)}\ \psi(h_-\cdot x_-) &= h_-\psi(x_-)\,{}^t h_-,\\
\text{(iv)}\ Q(\xi, h_-\cdot\eta) &= Q({}^t h_-\cdot\xi,\eta).
\end{aligned}$$

Proof. (i) The axiom (V1) and the table (a) in (2.2) together with (2.3) imply that

$$\begin{aligned}
L_{y_+}Q(\xi,\eta) &= \xi \triangle (y_+ \triangle \eta) + (y_+ \triangle \xi - \xi \triangle y_+) \triangle \eta\\
&= Q(L_{y_+}\xi,\eta) + Q(\xi, L_{y_+}\eta).
\end{aligned} \tag{2.7}$$

Taking exponential in both sides with respect to L_{y_+}, we get the assertion.
(ii) This comes from Lemma 2.1 (ii).
(iii) By (V1), (2.4) and Lemma 2.1 (ii), we obtain

$$\psi(L_{y_-}x_-) = \underline{\psi}(y_-)\psi(x_-) + \psi(x_-)\overline{\psi}(y_-) = L_{y_-}\psi(x_-) + \psi(x_-)\,{}^t L_{y_-}.$$

Taking exponential in both sides with respect to L_{y_-}, we conclude $\psi(h_-\cdot x_-) = h_-\psi(x_-)\,{}^t h_-$.
(iv) (1.9) and (2.2) yield that $y_- \triangle \xi + y_- \triangledown \xi = \xi \triangle y_-$. Using (V1) we obtain

$$Q(\xi, L_{y_-}\eta) = y_- \triangle (\xi \triangle \eta) + (\xi \triangle y_- - y_- \triangle \xi) \triangle \eta = Q({}^t L_{y_-}\xi, \eta),$$

and hence the assertion is proved. □

Lemma 2.4. *$h\cdot x$ is calculated as*

$$h_-\cdot x_- + h_-\cdot(\psi(x_-)\eta_h + h_+\cdot\xi) + \left(\frac{1}{2}Q(\psi(x_-)\eta_h,\eta_h) + Q(\eta_h, h_+\cdot\xi) + h_+\cdot x_+\right).$$

Proof. The multiplication table (a) in (2.2) implies

$$h_+\cdot x_- = x_-,\quad h_+\cdot\xi \in E,\quad h_+\cdot x_+ \in V_+,$$

and

$$L_{\eta_h}(V_-) \subset E,\quad L_{\eta_h}(E) \subset V_+,\quad L_{\eta_h}(V_+) = \{0\}.$$

Thus, taking exponential we have

$$\begin{aligned}
(\exp L_{\eta_h})x_- &= x_- + L_{\eta_h}x_- + \frac{1}{2}L_{\eta_h}^2 x_- = x_- + \psi(x_-)\eta_h + \frac{1}{2}Q(\psi(x_-)\eta_h,\eta_h),\\
(\exp L_{\eta_h})\xi &= \xi + L_{\eta_h}\xi = \xi + Q(\eta_h,\xi),\quad (\exp L_{\eta_h})x_+ = x_+.
\end{aligned}$$

Finally, again by the table (a) in (2.2) we have

$$h_-\cdot x_- \in V_-,\quad h_-\cdot\xi \in E,\quad h_-\cdot x_+ = x_+,$$

and these observations yield that

$$\begin{aligned}
h\cdot x_- &= h_-(\exp L_{\eta_h})\cdot x_- = h_-\cdot\left(x_- + \psi(x_-)\eta_h + \frac{1}{2}Q(\psi(x_-)\eta_h,\eta_h)\right),\\
h\cdot\xi &= h_-\cdot(h_+\cdot\xi + Q(\eta_h, h_+\cdot\xi)),\\
h\cdot x_+ &= h_+\cdot x_+.
\end{aligned}$$

Thus, we have proved the assertion. □

The multiplication table (a) in (2.2) together with (2.3), (2.4) and (2.5) yields that the product $x \triangle y$ is rewritten by using the operators φ and ψ as

$$x \triangle y = x_- \triangle y_- + \big(\psi(y_-)\xi + \underline{\varphi}(x_+)\eta + \underline{\psi}(x_-)\eta\big) + (Q(\xi,\eta) + x_+ \triangle y_+)\,, \tag{2.8}$$

where $y = y_- + \eta + y_+$ with $y_\pm \in V_\pm$ and $\eta \in E$. The identity (5.3) in [**9**] tells us that $\operatorname{Det} R_x$ is a relatively H-invariant polynomial and that the basic relative invariants of Ω are given as the irreducible factors of $\operatorname{Det} R_x$. Let $R^\pm$ be the right multiplication operators of $V_\pm$, respectively. Then, (2.8) implies

$$R_x = \begin{pmatrix} R^-_{x_-} & 0 & 0 \\ \underline{\psi}(\,\cdot\,)\xi & \psi(x_-) & \underline{\varphi}(\,\cdot\,)\xi \\ 0 & Q(\xi,\,\cdot\,) & R^+_{x_+} \end{pmatrix},$$

where the basis is taken in the order V_-, E, and V_+.

Proposition 2.5. *For $x \in \Omega$, one has*

$$\operatorname{Det} R_x = \operatorname{Det} R^-_{x_-}\cdot\operatorname{Det}\psi(x_-)\cdot\operatorname{Det} R^+_{x_+-(1/2)Q(\psi(x_-)^{-1}\xi,\ \xi)}. \tag{2.9}$$

Proof. First, Lemma 2.4 tells us that $x \in \Omega$ implies $x_- \in \Omega_-$, and hence $\psi(x_-)$ is invertible by Lemma 2.3 (iii). Using the following elementary determinant formula

$$\operatorname{Det}\begin{pmatrix} A & B \\ C & D\end{pmatrix} = \operatorname{Det}(A)\operatorname{Det}(D - CA^{-1}B) \quad (\operatorname{Det} A \neq 0),$$

we obtain

$$\begin{aligned}
\operatorname{Det} R_x &= \operatorname{Det} R^-_{x_-}\operatorname{Det}\begin{pmatrix}\psi(x_-) & \underline{\varphi}(\cdot)\xi \\ Q(\cdot,\xi) & R^+_{x_+}\end{pmatrix}\\
&= \operatorname{Det} R^-_{x_-}\operatorname{Det}\psi(x_-)\operatorname{Det}(R^+_{x_+} - Q(\psi(x_-)^{-1}\underline{\varphi}(\,\cdot\,)\xi,\xi)).
\end{aligned}$$

By Lemma 2.1, we have

$$\psi(x_-)^{-1}\underline{\varphi}(y_+)\xi = \underline{\varphi}(y_+)\psi(x_-)^{-1}\xi, \quad Q(\psi(x_-)^{-1}\underline{\varphi}(y_+)\xi,\xi) = Q(\underline{\varphi}(y_+)\xi,\psi(x_-)^{-1}\xi),$$

and these formulas together with (2.7) yield that

$$
\begin{aligned}
Q(\psi(x_-)^{-1}\underline{\varphi}(y_+)\xi,\xi) &= \frac{1}{2}\left(Q(\psi(x_-)^{-1}\underline{\varphi}(y_+)\xi,\xi)+Q(\psi(x_-)^{-1}\underline{\varphi}(y_+)\xi,\xi)\right)\\
&= \frac{1}{2}\left(Q(\underline{\varphi}(y_+)\psi(x_-)^{-1}\xi,\xi)+Q(\underline{\varphi}(y_+)\xi,\psi(x_-)^{-1}\xi)\right)\\
&= \frac{1}{2}y_+ \bigtriangleup Q(\psi(x_-)^{-1}\xi,\xi) = R^+_{(1/2)Q(\psi(x_-)^{-1}\xi,\ \xi)}(y_+).
\end{aligned}
$$

Since $R^+_{x_+}$ is linear with respect to x_+, the proof is now completed. $\square$

For $x \in \Omega$, we put $\widetilde{x} := x_+ - \dfrac{1}{2}Q(\psi(x_-)^{-1}\xi,\xi) \in V_+$.

LEMMA 2.6. *One has* $(h\cdot x)^{\sim} = h_+\cdot\widetilde{x}$.

PROOF. We shall calculate the Q-part of $(h\cdot x)^{\sim}$. Lemma 2.3 (iii) tells us that

$$
\begin{aligned}
\psi(h_-\cdot x_-)^{-1}h_-\cdot(\psi(x_-)\eta+h_+\cdot\xi) &= {}^t h_-^{-1}\cdot\psi(x_-)^{-1}(\psi(x_-)\eta+h_+\cdot\xi)\\
&= {}^t h_-^{-1}\cdot(\eta+\psi(x_-)^{-1}h_+\cdot\xi),
\end{aligned}
$$

and Lemma 2.1 (i) implies that $Q(\psi(x_-)^{-1}h_+\cdot\xi,\psi(x_-)\eta) = Q(h_+\cdot\xi,\eta)$. Using (ii) and (i) of Lemma 2.3, we have $Q(\psi(x_-)^{-1}h_+\cdot\xi,h_+\cdot\xi)) = h_+\cdot Q(\psi(x_-)^{-1}\xi,\xi)$, and thus we obtain by Lemma 2.3 (iv)

$$
\begin{aligned}
&Q\left({}^t h_-^{-1}\cdot(\eta+\psi(x_-)^{-1}h_+\cdot\xi),\ h_-\cdot(\psi(x_-)\eta+h_+\cdot\xi)\right)\\
&\quad= Q\left(\eta+\psi(x_-)^{-1}h_+\cdot\xi,\ \psi(x_-)\eta+h_+\cdot\xi\right)\\
&\quad= Q(\eta,\ \psi(x_-)\eta)+2Q(\eta,\ h_+\cdot\xi)+h_+\cdot Q(\psi(x_-)^{-1}\xi,\ \xi).
\end{aligned}
$$

Consequently, we get $(h\cdot x)^{\sim} = h_+\cdot(x_+-(1/2)Q(\psi(x_-)^{-1}\xi,\xi))$, and now there is nothing to prove. $\square$

Let us calculate the irreducible factors in (2.9). We first note that $\operatorname{Det} R^-_{x_-}$ is a relatively H_--invariant polynomial, and so is $\operatorname{Det}\psi(x_-)$ because we have Lemma 2.3 (iii). This means that, since these polynomials are relatively H-invariant by Lemma 2.4, the basic relative invariants $\Delta_1^-(x_-),\dots,\Delta_p^-(x_-)$ of Ω_- are still those of Ω. In the case of $\operatorname{Det} R^+_{\widetilde{x}}$, we have

$$
\operatorname{Det} R^+_{\widetilde{x}} = \Delta_1^+(\widetilde{x})^{m_1}\cdots\Delta_q^+(\widetilde{x})^{m_q},
$$

where $m_1,\dots,m_q$ are positive integers, and hence it is sufficient to consider $\Delta_j^+(\widetilde{x})$ for each $j=1,\dots,q$. By Lemma 2.6, $\Delta_j^+(\widetilde{x})$ is relatively H-invariant. For an operator T, we denote by ${}^{\mathrm{co}}T$ the cofactor of T, that is, we have ${}^{\mathrm{co}}T = (\operatorname{Det}T)T^{-1}$ for invertible T. Then, the inverse $\psi(x_-)^{-1}$ is described as

$$
\psi(x_-)^{-1} = \frac{1}{\operatorname{Det}\psi(x_-)}\,{}^{\mathrm{co}}\psi(x_-) \quad (x_-\in\Omega_-),
$$

and the elements of ${}^{\mathrm{co}}\psi(x_-)$ are polynomials of the elements of x_-. Since $\operatorname{Det}\psi(x_-)$ is relatively H_--invariant, there exist non-negative integers $\gamma_{j1},\dots,\gamma_{jp}$ such that

$$P_j(x) := \Delta_1^-(x_-)^{\gamma_{j1}} \cdots \Delta_p^-(x_-)^{\gamma_{jp}} \Delta_j^+(\widetilde{x}) \tag{2.10}$$

is a polynomial having no $\Delta_i^-(x_-)$-factors. We note that each $P_j(x)$ is relatively H-invariant because $\Delta_i^-(x_-)$ and $\Delta_j^+(\widetilde{x})$ are relatively H-invariant.

Before we confirm the irreducibility of $P_j(x)$, let us describe the multiplier matrix σ of Ω by using those of $\Omega_\pm$. By Theorem 5.1 of [**12**] and by the fact that $\Delta_1^-,\dots,\Delta_p^-$ are the basic relative invariants of Ω, we see that σ is expressed, by using a certain $q\times p$ matrix X, as

$$\sigma = \begin{pmatrix} \sigma_- & 0 \\ X & \sigma_+ \end{pmatrix}. \tag{2.11}$$

Proposition 2.7. *$P_1(x),\dots,P_q(x)$ are the basic relative invariants of Ω.*

Proof. Let $\Delta_1(x),\dots,\Delta_{p+q}(x)$ be the basic relative invariants of Ω. We already know that $\Delta_i(x) = \Delta_i^-(x)$ $(i=1,\dots,p)$. Since each $P_j(x)$ is a relatively H-invariant polynomial and has no $\Delta_i^-(x)$-factors, there exists $\underline{\tau} = (\tau_1,\dots,\tau_q) \in \mathbb{Z}^q_{\geq 0}$ such that

$$P_j(x) = \Delta_{p+1}(x)^{\tau_1}\cdots\Delta_{p+q}(x)^{\tau_q}.$$

Then, by (2.11), the multiplier of P_j is equal to $(0,\ \underline{\tau})\sigma = (\underline{\tau}X,\ \underline{\tau}\sigma_+)$. On the other hand, (2.10) means that the multiplier of P_j is given as $(\underline{\gamma}_j\sigma_-,\ \underline{\sigma}_j^+)$, and hence $\underline{\tau}$ satisfies $\underline{\tau}\sigma_+ = \underline{\sigma}_j^+$. Since $\tau_1,\dots,\tau_q$ are all non-negative, $\underline{\tau}$ needs to be equal to the row vector $\underline{e}_j$ having one in the j-th coordinate and zeros elsewhere, that is, $P_j(x) = \Delta_{p+j}(x)$, and the proposition is now proved. $\square$

Let us put $\Gamma := (\gamma_{jk}) \in \mathrm{Mat}(q,p;\,\mathbb{R})$ and $\Xi := \sigma_+^{-1}\Gamma$. Then equations (2.11) and (2.10) yield that $X = \Gamma\sigma_-$, and the inverse of σ is described as $\sigma^{-1} = \begin{pmatrix} \sigma_-^{-1} & 0 \\ -\Xi & \sigma_+^{-1}\end{pmatrix}$. By Lemma 1.1, elements of Ξ need to be equal to 0 or 1. Summing up these arguments, we arrive at the following theorem.

Theorem 2.8. *Assume that V is decomposed into $V_-\oplus E\oplus V_+$.*
(i) *There exists a unique matrix $\Xi\in\mathrm{Mat}(q,p;\,\{0,1\})$ such that, by putting $\Gamma = (\gamma_{jk}) := \sigma_+\Xi$, one has for any $x\in\Omega$*

$$\begin{cases} \Delta_i(x) = \Delta_i^-(x_-) & (i=1,\dots,p), \\ \Delta_{p+j}(x) = \Delta_1^-(x_-)^{\gamma_{j1}}\cdots\Delta_p^-(x_-)^{\gamma_{jp}}\Delta_j^+\left(x_+ - \dfrac{1}{2}Q(\psi(x_-)^{-1}\xi,\xi)\right) & (j=1,\dots,q). \end{cases}$$

(ii) *The multiplier matrix σ of Ω is given by*

$$\sigma = \begin{pmatrix} \sigma_- & 0 \\ \sigma_+\Xi\sigma_- & \sigma_+ \end{pmatrix} = \begin{pmatrix} I_p & 0 \\ 0 & \sigma_+ \end{pmatrix}\begin{pmatrix} I_p & 0 \\ \Xi & I_q \end{pmatrix}\begin{pmatrix} \sigma_- & 0 \\ 0 & I_q \end{pmatrix}.$$

3. Polynomial condition of the reciprocals of the Laplace transforms.

In this section, we consider the Laplace transforms of relatively H-invariant functions on homogeneous cones. The notation is continued from the previous sections. Let $\Delta_{\underline{s}}(x)$ be the relatively H-invariant function of Ω whose multiplier is $\underline{s} \in \mathbb{R}^r$ with $\Delta_{\underline{s}}(e_0) = 1$. Let $d\mu$ be an H-invariant measure on Ω, and put $p_k = \sum_{j<k} \dim V_{kj}$. Then, by Gindikin [**3**], the integral

$$\Gamma_\Omega(\underline{s}) = \int_\Omega e^{-\langle x \,|\, e_0 \rangle} \Delta_{\underline{s}}(x)\, d\mu(x) \tag{3.1}$$

converges if and only if $s_k > p_k/2$ for any $k = 1, \ldots, r$. The Laplace transform $\mathcal{L}[\Delta_{\underline{s}}]$ of $\Delta_{\underline{s}}$ is defined by

$$\mathcal{L}[\Delta_{\underline{s}}](y) := \frac{1}{\Gamma_\Omega(\underline{s})} \int_\Omega e^{-\langle x \,|\, y \rangle} \Delta_{\underline{s}}(x)\, d\mu(x) \quad (y \in \Omega^*). \tag{3.2}$$

Here, in our definition, the Laplace transform is normalized as $\mathcal{L}[\Delta_{\underline{s}}](e_0) = 1$. Then, Gindikin [**3**] also shows that the integral (3.2) converges absolutely if and only if $s_k > p_k/2$ for any $k = 1, \ldots, r$, and if converges, we have the formula

$$\mathcal{L}[\Delta_{\underline{s}}](y) = \frac{1}{\Delta^*_{\underline{s}}(y)} \quad (y \in \Omega^*). \tag{3.3}$$

For $\underline{\nu}, \underline{\mu} \in \mathbb{Z}^r$, let $\Delta^{\underline{\nu}}(x)$ $(x \in \Omega)$ and $\Delta^{\underline{\mu}}_*(y)$ $(y \in \Omega^*)$ be rational functions defined respectively by

$$\Delta^{\underline{\nu}}(x) := \Delta_1(x)^{\nu_1} \cdots \Delta_r(x)^{\nu_r}, \quad \Delta^{\underline{\mu}}_*(y) := \Delta^*_1(y)^{\mu_1} \cdots \Delta^*_r(y)^{\mu_r}.$$

By definition, it is easily verified that

$$\Delta^{\underline{\nu}}(x) = \Delta_{\underline{\nu}\sigma}(x), \quad \Delta^{\underline{\mu}}_*(y) = \Delta^*_{\underline{\mu}\sigma_*}(y) \quad (x \in \Omega,\ y \in \Omega^*), \tag{3.4}$$

and hence the formula (3.3) is rewritten, by putting $\underline{\nu}' := \underline{\nu}\sigma\sigma_*^{-1}$, as

$$\mathcal{L}[\Delta^{\underline{\nu}}](y) = \frac{1}{\Delta^{\underline{\nu}'}_*(y)} \quad (y \in \Omega^*). \tag{3.5}$$

We note that it is shown by (3.4) that a relatively H-invariant function $\Delta_{\underline{s}}$ is polynomial, in other words the vector $\underline{s}$ is Ω-integral in the sense of [**3**, p. 37], if and only if all the elements of $\underline{s}\sigma^{-1}$ are non-negative integers.

Let us assume that Ω is the symmetric cone $\mathcal{S}^+_r$ for a while. In this case, $\Delta_j(x)$ are the left upper corner principal minors of $x \in \Omega$, and $\Delta^*_k(y)$ are the right lower corner principal minors of $y \in \Omega$. By Propositions VI.3.10 and VII.1.5 of [**1**], for example, the multiplier matrices σ and σ_* are given, respectively, as

$$\sigma = \begin{pmatrix} 1 & & \\ \vdots & \ddots & \\ 1 & \cdots & 1 \end{pmatrix}, \quad \sigma_* = \begin{pmatrix} 1 & \cdots & 1 \\ & \ddots & \vdots \\ & & 1 \end{pmatrix}, \tag{3.6}$$

and this yields that $\underline{\nu}' = \underline{\nu}\sigma\sigma_*^{-1} = (\nu_1 + \cdots + \nu_r, -\nu_1, \ldots, -\nu_{r-1})$. Thus, we obtain by (3.5)

$$\mathcal{L}[\Delta^{\underline{\nu}}](y) = \frac{1}{\Delta_*^{\underline{\nu}'}(y)} = \frac{\Delta_2^*(y)^{\nu_1} \cdots \Delta_r^*(y)^{\nu_{r-1}}}{\Delta_1^*(y)^{\nu_1 + \cdots + \nu_r}} \quad (y \in \Omega). \tag{3.7}$$

If $\underline{\nu} = (0, \ldots, 0, n)$ with $n \in \mathbb{N}$, then we find that $\Delta^{\underline{\nu}}(x)$ and $\Delta_*^{\underline{\nu}'}(y) = (\mathcal{L}[\Delta^{\underline{\nu}}](y))^{-1}$ are both non-constant polynomials. This property is valid for any irreducible symmetric cone (cf. [**1**, Chapter VII]), and we shall show that, among irreducible homogeneous cones, symmetric cones are characterized by this property.

We now return to a general situation that Ω is an irreducible homogeneous cone. Let us introduce an inductive structure in V. We set $(p, q) = (r-1, 1)$ in the decomposition (2.1). Then, by putting $V' = V_-$, we have

$$V = V' \oplus E \oplus \mathbb{R}c_r. \tag{3.8}$$

Let Ω' be the cone corresponding to V', and we denote the multiplier matrices of Ω' and $(\Omega')^*$ by σ' and σ'_*, respectively. Theorem 2.8 implies that there exist $\underline{\gamma} \in \{0,1\}^{r-1}$ and $\boldsymbol{\varepsilon} \in \{0,1\}^{r-1}$ such that

$$\sigma = \begin{pmatrix} \sigma' & 0 \\ \underline{\gamma}\sigma' & 1 \end{pmatrix}, \quad \sigma_* = \begin{pmatrix} \sigma'_* & \sigma'_*\boldsymbol{\varepsilon} \\ 0 & 1 \end{pmatrix}.$$

Between $\underline{\gamma}$ and $\boldsymbol{\varepsilon}$, the following relationship holds.

Lemma 3.1. *There exists an integer j_0 such that $\gamma_{j_0} = \varepsilon_{j_0} = 1$.*

Proof. Since V is irreducible, we have $\{j;\ \gamma_j = 1\} \neq \emptyset$. Let j_0 be the maximal integer such that $\gamma_j = 1$.

Then, by Lemma 1.1, we obtain

$$\dim V_{rj_0} > 0, \quad \dim V_{rk} = 0 \quad (j_0 < k < r),$$

and, Lemma 1.1 for σ_* yields that

$$\varepsilon_{j_0} = 1, \quad \varepsilon_k = 0 \quad (k = j_0 + 1, \ldots, r-1),$$

so that the lemma is now proved. □

To go further, we introduce some notations. For row vectors $\boldsymbol{u} = {}^t(u_1, \ldots, u_r) \in \mathbb{R}^r$, we write $\boldsymbol{u} > 0$ when $u_j > 0$ for all j. We use similar notation for column vectors $\underline{\nu} \in \mathbb{R}^r$, and for other inequalities $\geq$, $<$, $\leq$.

Proposition 3.2. *Let Ω be an irreducible homogeneous cone. If Ω is not symmetric, then there exists $\boldsymbol{u} \in \{0,1\}^r$ such that $\sigma\sigma_*^{-1}\boldsymbol{u} < 0$. If Ω is symmetric, then there are no such $\boldsymbol{u} \in \{0,1\}^r$, and, at best, $\sigma\sigma_*^{-1}\boldsymbol{u} = -{}^t(1,\dots,1,0)$ for some $\boldsymbol{u}$.*

Proof. At first, we assume that Ω is symmetric. In this case, we know that σ and σ_* are expressed as in (3.6), and a simple calculation yields that

$$\sigma\sigma_*^{-1} = \begin{pmatrix} 1 & & & \\ \vdots & & -I_{r-1} & \\ 1 & & & \\ 1 & 0 & \cdots & 0 \end{pmatrix}.$$

Hence, it is easily verified that there are no $\boldsymbol{u} \in \{0,1\}^r$ such that $\sigma\sigma_*^{-1}\boldsymbol{u} < 0$, and $\sigma\sigma_*^{-1}\boldsymbol{u} = -{}^t(1,\dots,1,0)$ when $\boldsymbol{u} = {}^t(0,1,\dots,1)$. Next we assume that Ω is not symmetric. We shall prove the proposition in this case by the induction on the rank r. Let us decompose V as in (3.8). Since Ω' is not necessarily irreducible, we consider the irreducible decomposition $\Omega' = \bigoplus_{a=1}^q \Omega_a$, where q is the number of the irreducible components Ω_a of Ω'. We relabel the Vinberg frame, if necessary, such that the ambient vector space U_a of Ω_a are placed on the diagonal blocks (cf. (1.3)). Moreover, let $E = \bigoplus_{a=1}^q E_a$ be the decomposition of E with respect to the relabeled Vinberg frame. Then, V is of the form

$$V = \begin{pmatrix} U_1 & 0 & \cdots & 0 & E_1 \\ 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & U_{q-1} & 0 & E_{q-1} \\ 0 & \cdots & 0 & U_q & E_q \\ E_1 & \cdots & E_{q-1} & E_q & \mathbb{R}c_r \end{pmatrix}.$$

For each $a = 1,\dots,q$, let σ_a be the multiplier matrix of the cone Ω_a, and σ_a^* that of the dual cone of Ω_a. Since the subspace $U_a^0 := U_a \oplus E_a \oplus \mathbb{R}c_r$ is a subalgebra as Vinberg algebra, we have the cone Ω_a^0 corresponding to U_a^0. Applying Theorem 2.8 to Ω_a^0 and to its dual cone $(\Omega_a^0)^*$, we see that there exist zero-one vectors $\underline{\gamma}_a$ and $\boldsymbol{\varepsilon}_a$ such that the multiplier matrices of Ω_a^0 and of $(\Omega_a^0)^*$ are written, respectively, as

$$\begin{pmatrix} \sigma_a & 0 \\ \underline{\gamma}_a\sigma_a & 1 \end{pmatrix}, \quad \begin{pmatrix} \sigma_a^* & \sigma_a^*\boldsymbol{\varepsilon}_a \\ 0 & 1 \end{pmatrix}.$$

We note that, since V is irreducible, each U_a^0 needs to be irreducible, and hence $\underline{\gamma}_a$ and $\boldsymbol{\varepsilon}_a$ are both not zero-vectors for all a. Recalling Lemma 1.1, we see that σ and σ_* are described, respectively, as

$$\sigma = \left(\begin{array}{ccc|c} \sigma_1 & & & \\ & \ddots & & \\ 0 & & \sigma_q & \\ \hline \underline{\gamma}_1\sigma_1 & \cdots & \underline{\gamma}_q\sigma_q & 1 \end{array}\right), \quad \sigma_* = \left(\begin{array}{ccc|c} \sigma_1^* & & 0 & \sigma_1^*\boldsymbol{\varepsilon}_1 \\ & \ddots & & \vdots \\ & & \sigma_q^* & \sigma_q^*\boldsymbol{\varepsilon}_q \\ \hline & & & 1 \end{array}\right).$$

For any $a = 1, \dots, q$, since $\operatorname{rank} \Omega_a < r$, the hypothesis implies that there exists a zero-one vector $\boldsymbol{u}_a$ satisfying the property for Ω_a, and we place them in a vector form as

$$\boldsymbol{u} := \begin{pmatrix} \boldsymbol{u}_1 \\ \vdots \\ \boldsymbol{u}_q \\ 1 \end{pmatrix} \in \{0,1\}^r.$$

By a simple matrix calculation, we find that

$$\sigma\sigma_*^{-1}\boldsymbol{u} = \begin{pmatrix} \sigma_1(\sigma_1^*)^{-1}\boldsymbol{u}_1 - \sigma_1\boldsymbol{\varepsilon}_1 \\ \vdots \\ \sigma_q(\sigma_q^*)^{-1}\boldsymbol{u}_q - \sigma_q\boldsymbol{\varepsilon}_q \\ \sum_a \underline{\gamma}_a\sigma_a(\sigma_a^*)^{-1}\boldsymbol{u}_a + 1 - \sum_a \underline{\gamma}_a\sigma_a\boldsymbol{\varepsilon}_a \end{pmatrix},$$

and put

$$\boldsymbol{Y}_a = \sigma_a(\sigma_a^*)^{-1}\boldsymbol{u}_a - \sigma_a\boldsymbol{\varepsilon}_a, \quad X = \sum_a \underline{\gamma}_a\sigma_a(\sigma_a^*)^{-1}\boldsymbol{u}_a + 1 - \sum_a \underline{\gamma}_a\sigma_a\boldsymbol{\varepsilon}_a.$$

First, let us consider $\boldsymbol{Y}_a$. If Ω_a is not symmetric, then by the hypothesis we have $\sigma_a(\sigma_a^*)^{-1}\boldsymbol{u}_a < 0$, and hence $\boldsymbol{Y}_a < 0$. Let us assume that Ω_a is symmetric. In this case σ_a is described as in (3.6). For a vector $\underline{s} = (s_1, \dots, s_r)$, put $|\underline{s}| := s_1 + \cdots + s_r$. Then, the last factor of $\sigma_a\boldsymbol{\varepsilon}_a$ is equal to $|\boldsymbol{\varepsilon}_a|$, and we have $|\boldsymbol{\varepsilon}_a| > 0$ because $\boldsymbol{\varepsilon}_a \neq 0$. Since we have $\sigma_a(\sigma_a^*)^{-1}\boldsymbol{u}_a = -{}^t(1, \dots, 1, 0)$, we conclude $\boldsymbol{Y}_a < 0$.

Next, let us consider X. By Lemma 3.1, we have $\underline{\gamma}_a\sigma_a\boldsymbol{\varepsilon}_a \geq 1$ for each a, and thus

$$1 - \sum_a \underline{\gamma}_a\sigma_a\boldsymbol{\varepsilon}_a \leq 1 - q. \tag{3.9}$$

Therefore, if $q \geq 2$, we easily find that $X \leq 1 - q < 0$. Suppose that $q = 1$. In this case we have $\Omega' = \Omega_1$, so that we use prime symbols, like $\boldsymbol{u}' = \boldsymbol{u}_1$, for simplicity. Since (3.9) implies that $X \leq \underline{\gamma}'\sigma'(\sigma'_*)^{-1}\boldsymbol{u}'$, we first consider the value of $\underline{\gamma}'\sigma'(\sigma'_*)^{-1}\boldsymbol{u}'$. If Ω' is not symmetric, the hypothesis implies that $\sigma'(\sigma'_*)^{-1}\boldsymbol{u}' < 0$, and hence $X < 0$. We now assume that Ω' is symmetric. Then, we have $\sigma'(\sigma'_*)^{-1}\boldsymbol{u}' = -{}^t(1, \dots, 1, 0)$ with $\boldsymbol{u}' = {}^t(0, 1, \dots, 1)$. Thus, if $\underline{\gamma}' \neq (0, \dots, 0, 1)$, then we obtain $X \leq \underline{\gamma}'\sigma'(\sigma'_*)^{-1}\boldsymbol{u}' < 0$. Next, we put $\underline{\gamma}' = (0, \dots, 0, 1)$. In this case $\underline{\gamma}'\sigma'(\sigma'_*)^{-1}\boldsymbol{u}' = 0$, so that we consider the value of $1 - \underline{\gamma}'\sigma'\boldsymbol{\varepsilon}'$. Lemma 3.1 tells us that $\varepsilon_{r-1} = 1$, where we put $\boldsymbol{\varepsilon}' = {}^t(\varepsilon_1, \dots, \varepsilon_{r-1})$. Since $\underline{\gamma}'\sigma'\boldsymbol{\varepsilon}' = 1 + \varepsilon_1 + \cdots + \varepsilon_{r-2}$, if $\boldsymbol{\varepsilon} \neq {}^t(0, \dots, 0, 1)$ then we have $X \leq 1 - \underline{\gamma}'\sigma'\boldsymbol{\varepsilon}' < 0$. Finally, let $\boldsymbol{\varepsilon} = {}^t(0, \dots, 0, 1)$. Then, a simple calculation yields that

$$\sigma = \begin{pmatrix} \sigma' & 0 \\ \underline{\gamma}\sigma' & 1 \end{pmatrix} = \begin{pmatrix} 1 & & \\ \vdots & \ddots & \\ 1 & \cdots & 1 \end{pmatrix}, \quad \sigma_* = \begin{pmatrix} \sigma'_* & \sigma'_*\boldsymbol{\varepsilon} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \cdots & 1 \\ & \ddots & \vdots \\ & & 1 \end{pmatrix},$$

and therefore $\sigma^{-1} + \sigma_*^{-1}$ is equal to the Cartan matrix of type A_r. Theorem 4.3 of [**13**] tells us that Ω is nothing other than an irreducible symmetric cone. Hence, the proof is now completed. $\square$

LEMMA 3.3. *Let $\underline{\nu} \geq 0$. If there exists a matrix A such that $\underline{\nu} A \geq 0$ and $A\boldsymbol{u} < 0$ for some $\boldsymbol{u} \in \{0,1\}^r$, then $\underline{\nu} = 0$.*

PROOF. First, we can easily verify $\underline{\nu} A\boldsymbol{u} = (\underline{\nu} A)\boldsymbol{u} \geq 0$. On the other hand, a straightforward calculation yields that $\underline{\nu} A\boldsymbol{u} = \underline{\nu}(A\boldsymbol{u}) \leq 0$. Thus, we obtain $\underline{\nu} A\boldsymbol{u} = 0$. Since all elements of $A\boldsymbol{u}$ are strictly negative and $\underline{\nu} \geq 0$, we conclude that $\underline{\nu} = 0$. $\square$

Proposition 3.2 together with Lemma 3.3 leads us to the following theorem which gives a characterization of symmetric cones among irreducible homogeneous cones by using the Laplace transforms of relatively H-invariant functions on Ω.

THEOREM 3.4. *Let Ω be an irreducible homogeneous cone. Then, Ω is symmetric if and only if there exists a non-constant relatively H-invariant polynomial $\Delta^{\underline{\nu}}(x)$ such that the reciprocal $\Delta_*^{\underline{\nu}'}(y) = (\mathcal{L}[\Delta^{\underline{\nu}}](y))^{-1}$ of the Laplace transform of $\Delta^{\underline{\nu}}(x)$ is also a non-constant polynomial.*

PROOF. Since the multiplier matrices σ and σ_* of symmetric cones are described as in (3.6), the "only if" part is already proved in (3.7). We now assume that Ω is not symmetric. Then, by Proposition 3.2, there exists $\boldsymbol{u} \in \{0,1\}^r$ such that $\sigma\sigma_*^{-1}\boldsymbol{u} < 0$. Let us assume that $\Delta^{\underline{\nu}}(x)$ and $\Delta_*^{\underline{\nu}'}(y)$ are both polynomials, that is, we assume that $\underline{\nu} \geq 0$ and $\underline{\nu}' = \underline{\nu}\sigma\sigma_*^{-1} \geq 0$. Then, Lemma 3.3 shows that $\underline{\nu}$ needs to be equal to 0. Thus, there are no non-constant polynomials which satisfy the property when Ω is not symmetric. $\square$

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