

A Hardy inequality and applications to reverse Hölder inequalities for weights on \mathbb{R}

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Abstract. We prove a sharp integral inequality valid for non-negative functions defined on $[0, 1]$, with given L^1 norm. This is in fact a generalization of the well known integral Hardy inequality. We prove it as a consequence of the respective weighted discrete analogue inequality whose proof is presented in this paper. As an application we find the exact best possible range of $p > q$ such that any non-increasing g which satisfies a reverse Hölder inequality with exponent q and constant c upon the subintervals of $(0, 1]$, should additionally satisfy a reverse Hölder inequality with exponent p and in general a different constant c' . The result has been treated in [1] but here we give an alternative proof based on the above mentioned inequality.

1. Introduction.

During his efforts to simplify the proof of Hilbert's double series theorem, Hardy [7], first proved in 1920 the most famous inequality which is known in the literature as Hardy's inequality (see also [10], Theorem 3.5). This is stated as

THEOREM A. *If $p > 1$, $a_n > 0$, and $A_n = a_1 + a_2 + \cdots + a_n$, $n \in \mathbb{N}$, then*

$$\sum_{n=1}^{\infty} \left(\frac{A_n}{n} \right)^p < \left(\frac{p}{p-1} \right)^p \sum_{n=1}^{\infty} a_n^p. \quad (1.1)$$

Moreover, inequality (1.1) is best possible, that is the constant on the right side cannot be decreased.

In 1926, E. Copson, generalized Theorem A (see [3]) by replacing the arithmetic mean of a sequence with a weighted arithmetic mean. More precisely he proved the following

THEOREM B. *Let $p > 1$, $a_n, \lambda_n > 0$, for $n = 1, 2, \dots$.*

Further suppose that $A_n = \sum_{i=1}^n \lambda_i$ and $A_n = \sum_{i=1}^n \lambda_i a_i$. Then

$$\sum_{n=1}^{\infty} \lambda_n \left(\frac{A_n}{A_n} \right)^p \leq \left(\frac{p}{p-1} \right)^p \sum_{n=1}^{\infty} \lambda_n a_n^p, \quad (1.2)$$

where the constant involved in (1.2) is best possible.

In [3], Copson proves also a second weighted inequality, which as Hardy noted in [8], can be derived from Theorem B. From then and until now there have been given several generalizations of the above two inequalities. The first one is given by Hardy and Littlewood who generalized Theorem B in a specific direction (see [9]). This was generalized further by Leindler in [14], and by Nemeth in [17]. Also in [16] one can see further generalizations of Hardy’s and Copson’s series inequalities by replacing means by more general linear transforms. For the study of Copson’s inequality one can also see [4]. Additionally, in [5], Elliot has already proved inequality (1.2) by similar methods such as those that appear in [3].

There is a continued analogue of Theorem A (see [10]) which can be stated as

THEOREM C. *If $p > 1$, $f(x) \geq 0$ for $x \in [0, +\infty)$ then*

$$\int_0^\infty \left(\frac{1}{x} \int_0^x f(t) dt \right)^p dx \leq \left(\frac{p}{p-1} \right)^p \int_0^\infty f^p(x) dx. \tag{1.3}$$

Further generalizations of (1.3) can be seen in [8]. Other authors have also studied these inequalities in more general forms as it may be seen in [15] and [20]. E. Landau has also studied the above inequality and his work appears in [13]. For a complete discussion of the topic one can consult [12] and [19]. In this paper we generalize (1.3) by proving the following

THEOREM 1. *Let $g : [0, 1] \rightarrow \mathbb{R}^+$ be integrable function, $p > 1$, and additionally assume that $\int_0^1 g = f$. Then the following inequality is true, for any q such that $1 \leq q \leq p$*

$$\int_0^1 \left(\frac{1}{t} \int_0^t g \right)^p dt \leq \left(\frac{p}{p-1} \right)^q \int_0^1 \left(\frac{1}{t} \int_0^t g \right)^{p-q} g^q(t) dt - \frac{q}{p-1} f^p. \tag{1.4}$$

Moreover, inequality (1.4) is sharp in the sense that, the constant $(p/(p-1))^q$ cannot be decreased, while the constant $q/(p-1)$ cannot be increased for any fixed f .

In fact we are going to prove, an even more general inequality which is the discrete analogue of (1.4) for the case $q = 1$, which is weighted. This is a generalization of (1.2) and is described by the following:

THEOREM 2. *Let $(a_n)_n$ be a sequence of non-negative real numbers. We define for every sequence $(\lambda_n)_n$ of positive numbers the following quantities $A_n = \lambda_1 a_1 + \dots + \lambda_n a_n$ and $\Lambda_n = \lambda_1 + \dots + \lambda_n$. Then the following inequality is true:*

$$\sum_{n=1}^N \lambda_n \left(\frac{A_n}{\Lambda_n} \right)^p \leq \left(\frac{p}{p-1} \right) \sum_{n=1}^N \lambda_n a_n \left(\frac{A_n}{\Lambda_n} \right)^{p-1} - \frac{1}{p-1} \Lambda_N \left(\frac{A_N}{\Lambda_N} \right)^p, \tag{1.5}$$

for any $N \in \mathbb{N}$.

It is obvious that by setting $\lambda_n = 1$ for every $n \in \mathbb{N}$, in Theorem 2, we reach to

the discrete analogue of (1.4) for $q = 1$. Then Theorem 1 is an easy consequence, for the case $q = 1$, by the use of a standard approximation argument, of L^1 functions on $(0, 1]$, by simple functions. We then use this result (as can be seen in the sequel) in an effective way to provide a proof of Theorem 1, for any $q \in [1, p]$. We mention also that an opposite problem for negative exponents involving only the parameter q and not f , is treated in [18].

We believe that Theorem 1 has many applications in many fields and especially in the theory of weights. Our intention in this paper is to describe one of them. We mention the related details. Let $Q_0 \subseteq \mathbb{R}^d$ be a given cube. Let also $p > 1$ and $h : Q_0 \rightarrow \mathbb{R}^+$ be such that $h \in L^p(Q_0)$. Then, as is well known, the following, named as Hölder's inequality is satisfied

$$\left(\frac{1}{|Q|} \int_Q h\right)^p \leq \frac{1}{|Q|} \int_Q h^p, \text{ for any cube } Q \subseteq Q_0.$$

In this paper we are interested for functions that satisfy a reverse Hölder inequality. More precisely we say that h satisfies the reverse Hölder inequality with exponent $q > 1$ and constant $c \geq 1$ if the following holds

$$\frac{1}{|Q|} \int_Q h^q \leq c \cdot \left(\frac{1}{|Q|} \int_Q h\right)^q, \text{ for every cube } Q \subseteq Q_0. \tag{1.6}$$

Now in [6] it is proved the following.

THEOREM D. *Let $1 < q < \infty$ and $h : Q_0 \rightarrow \mathbb{R}^+$ such that (1.6) holds. Then there exists $\varepsilon = \varepsilon(d, q, c)$ such that $h \in L^p$ for any p such that $p \in [q, q + \varepsilon)$. Moreover the following inequality holds*

$$\frac{1}{|Q|} \int_Q h^p \leq c' \left(\frac{1}{|Q|} \int_Q h\right)^p,$$

for any cube $Q \subseteq Q_0$, $p \in [q, q + \varepsilon)$ and some constant $c' = c'(d, p, q, c)$.

As a consequence the following question naturally arises and is posed in [2]. What is the best possible value of ε ? The problem for the case $d = 1$ was solved in [1] for non-increasing functions g and was completed for arbitrary functions in [11]. More precisely in [1] it is shown the following:

THEOREM E. *Let $g : (0, 1] \rightarrow \mathbb{R}^+$ be non-increasing which satisfies the following inequality*

$$\frac{1}{b-a} \int_a^b g^q \leq c \left(\frac{1}{b-a} \int_a^b g\right)^q, \tag{1.7}$$

for every $(a, b) \subseteq (0, 1]$, where $q > 1$ is fixed, and c independent of a, b . If we define $p_0 > q$ as the root of the following equation

$$\frac{p_0 - q}{p_0} \cdot \left(\frac{p_0}{p_0 - 1} \right)^q \cdot c = 1, \quad (1.8)$$

we have that $g \in L^p((0, 1])$ and g satisfies a reverse Hölder inequality with exponent p , for every p such that $p \in [q, p_0)$. Moreover the result is sharp, that is the value of p_0 cannot be increased.

The problem that is mentioned above was completely solved in [11], where the notion of the non-increasing rearrangement of h was used and it is defined as follows:

$$h^*(t) = \sup_{\substack{e \subseteq (0, 1] \\ |e| \geq t}} \left[\inf_{x \in e} h(x) \right].$$

More precisely the following appears in [11].

THEOREM F. *Let $h : (0, 1] \rightarrow \mathbb{R}^+$, be a function that satisfies (1.7), for every $(a, b) \subseteq [0, 1]$ for some $q > 1$ and $c \geq 1$. Then the same inequality is true if we replace h by it's non-increasing rearrangement.*

It is immediate now that Theorems E and F answer the question as it was posed in [2], for the case $d = 1$.

Our aim in this paper is to give an alternative proof of Theorem E by using Theorem 1. We will prove the following variant of Theorem E which we state as

THEOREM 3. *Let $g : (0, 1] \rightarrow \mathbb{R}^+$ be non-increasing satisfying a reverse Hölder inequality with exponent $q > 1$ and constant $c \geq 1$ for all intervals of the form $(0, t]$. That is the following holds*

$$\frac{1}{t} \int_0^t g^q \leq c \cdot \left(\frac{1}{t} \int_0^t g \right)^q, \quad (1.9)$$

for any $t \in (0, 1]$. Then for every $p \in [q, p_0)$ the inequality

$$\frac{1}{t} \int_0^t g^p \leq c' \left(\frac{1}{t} \int_0^t g \right)^p \quad (1.10)$$

is true for any $t \in (0, 1]$, where $c' = c'(p, q, c)$ and p_0 is defined by (1.8). As a consequence $g \in L^p$ for every $p \in [q, p_0)$. Moreover the result is sharp, that is, the value of p_0 cannot be decreased.

By the same reasoning we can prove the analogue of Theorem 3, for intervals of the form $(t, 1]$. Ending this discussion we mention that in [11] it is proved the following:

THEOREM G. *Let $g : (0, 1] \rightarrow \mathbb{R}^+$ be non-increasing. Then (1.7) is satisfied for all subintervals of $(0, 1]$ iff it is satisfied for all subintervals of the form $(0, t]$ and $[t, 1]$.*

By the above results we conclude that Theorem 3, and its analogue for the intervals of the form $(t, 1]$, imply Theorem E.

2. The Hardy type inequality.

We first present the following which can be seen in [3].

PROOF OF THEOREM 2. For each $n \in \mathbb{N}$ define

$$\Delta_n = \lambda_n \left(\frac{A_n}{\lambda_n} \right)^p - \frac{p}{p-1} \lambda_n \left(\frac{A_n}{\lambda_n} \right)^{p-1} a_n = \lambda_n \Delta'_n,$$

where

$$\Delta'_n = \left(\frac{A_n}{\lambda_n} \right)^p - \frac{p}{p-1} \left(\frac{A_n}{\lambda_n} \right)^{p-1} a_n.$$

Obviously, $a_n = (A_n - A_{n-1})/\lambda_n$ for every $n \in \mathbb{N}$, so we have

$$\begin{aligned} \Delta'_n &= \left(\frac{A_n}{\lambda_n} \right)^p - \frac{p}{p-1} \left(\frac{A_n}{\lambda_n} \right)^{p-1} \frac{A_n - A_{n-1}}{\lambda_n} \\ &= \left(\frac{A_n}{\lambda_n} \right)^p - \frac{p}{p-1} \left(\frac{A_n}{\lambda_n} \right)^p \frac{A_n}{\lambda_n} + \frac{p}{p-1} \left(\frac{A_n}{\lambda_n} \right)^{p-1} \frac{A_{n-1}}{\lambda_n} \\ &= \left(\frac{A_n}{\lambda_n} \right)^p \left[1 - \frac{p}{p-1} \cdot \frac{A_n}{\lambda_n} \right] + \frac{1}{p-1} \left\{ p \cdot \left(\frac{A_n}{\lambda_n} \right)^{p-1} \frac{A_{n-1}}{\lambda_{n-1}} \right\} \frac{A_{n-1}}{\lambda_n}. \end{aligned} \tag{2.11}$$

We now use the following elementary inequality

$$px^{p-1}y \leq (p-1)x^p + y^p,$$

which holds for any $p > 1$ and $x, y \geq 0$.

We apply it for $x = A_n/\lambda_n$, $y = A_{n-1}/\lambda_{n-1}$, so using (2.11) we have that:

$$\begin{aligned} \Delta'_n &\leq \left(\frac{A_n}{\lambda_n} \right)^p \left[1 - \frac{p}{p-1} \frac{A_n}{\lambda_n} \right] + \frac{1}{p-1} \left[(p-1) \left(\frac{A_n}{\lambda_n} \right)^p + \left(\frac{A_{n-1}}{\lambda_{n-1}} \right)^p \right] \cdot \frac{A_{n-1}}{\lambda_n} \\ &= \left(\frac{A_n}{\lambda_n} \right)^p \left[1 - \frac{p}{p-1} \frac{A_n}{\lambda_n} + \frac{A_{n-1}}{\lambda_n} \right] + \frac{1}{p-1} \left(\frac{A_{n-1}}{\lambda_{n-1}} \right)^p \frac{A_{n-1}}{\lambda_n} \\ &= -\frac{1}{p-1} \cdot \frac{A_n}{\lambda_n} \left(\frac{A_n}{\lambda_n} \right)^p + \frac{1}{p-1} \frac{A_{n-1}}{\lambda_n} \left(\frac{A_{n-1}}{\lambda_{n-1}} \right)^p. \end{aligned} \tag{2.12}$$

Thus from (2.12) and the definition of Δ_n we conclude

$$\Delta_n \leq \frac{1}{p-1} A_{n-1} \left(\frac{A_{n-1}}{\lambda_{n-1}} \right)^p - \frac{1}{p-1} A_n \left(\frac{A_n}{\lambda_n} \right)^p. \tag{2.13}$$

This holds for every $n \in \mathbb{N}$, $n \geq 2$.

It is immediate now that for $n = 1$ we have the following equality

$$\Delta_1 = -\frac{1}{p-1} A_1 \left(\frac{A_1}{\lambda_1} \right)^p. \tag{2.14}$$

For any $N \in \mathbb{N}$ we sum (2.13) from $n = 2$ to N and add also the equality (2.14), so we conclude after making the appropriate cancellations, inequality (1.5) of Theorem 2. \square

The following now is an easy consequence of the above result.

COROLLARY 1. *Let $g : [0, 1] \rightarrow \mathbb{R}^+$ be integrable function, $p > 1$ and additionally assume that $\int_0^1 g = f$. Then the following inequality is true*

$$\int_0^1 \left(\frac{1}{t} \int_0^t g \right)^p dt \leq \left(\frac{p}{p-1} \right) \int_0^1 \left(\frac{1}{t} \int_0^t g \right)^{p-1} g(t) dt - \frac{1}{p-1} f^p. \quad (2.15)$$

We proceed now to the following proof.

PROOF OF THEOREM 1. For any $s \in [0, p]$ we define I_s by

$$I_s = \int_0^1 \left(\frac{1}{t} \int_0^t g \right)^{p-s} g^s(t) dt,$$

for any $g : [0, 1] \rightarrow \mathbb{R}^+$ integrable function, such that $\int_0^1 g = f$. Then, for the proof of inequality (1.4), we just need to prove that

$$I_0 \leq \left(\frac{p}{p-1} \right)^q I_q - \frac{q}{p-1} f^p,$$

for any $q \in (1, p]$.

We write

$$I_1 = \int_0^1 g(t) \left(\frac{1}{t} \int_0^t g \right)^{(p-q)/q} \left(\frac{1}{t} \int_0^t g \right)^{p-(p/q)} dt.$$

We then apply in the above inequality the integral form of Hölder's inequality, with exponents $q, q/(q-1)$, and we have as a consequence that

$$I_1 \leq I_q^{1/q} I_0^{(q-1)/q}. \quad (2.16)$$

Additionally from Corollary 1 we obtain

$$I_0 \leq \frac{p}{p-1} I_1 - \frac{1}{p-1} f^p. \quad (2.17)$$

We consider now the difference $L_q = I_0 - (p/(p-1))^q I_q$. We need to prove that $L_q \leq -q/(p-1) f^p$. By using the inequalities (2.16) and (2.17) we have that

$$\begin{aligned} L_q &\leq I_0 - \left(\frac{p}{p-1} \right)^q \frac{I_1^q}{I_0^{q-1}} \\ &\leq I_0 - \left(\frac{p}{p-1} \right)^q I_0^{-q+1} \left(\frac{p-1}{p} I_0 + \frac{1}{p} f^p \right)^q. \end{aligned} \quad (2.18)$$

We define now the following function of the variable $x > 0$:

$$G(x) = x - \left(\frac{p}{p-1}\right)^q x^{-q+1} \left(\frac{p-1}{p}x + \frac{1}{p}f^p\right)^q.$$

Then

$$G(x) = x - x^{-q+1} \left(x + \frac{1}{p-1}f^p\right)^q,$$

so that

$$G'(x) = 1 + (q-1) \left(1 + \frac{f^p}{(p-1)x}\right)^q - q \left(1 + \frac{f^p}{(p-1)x}\right)^{q-1}.$$

Now we consider the following function of the variable $t \geq 1$: $F(t) = 1 + (q-1)t^q - qt^{q-1}$. Then $F'(t) = q(q-1)t^{q-2}(t-1) > 0$, for every $t > 1$. Thus F is strictly increasing on its domain, so that $F(t) > F(1) = 0$, for any $t > 1$. We immediately conclude that $G'(x) > 0$, for every $x > 0$. As a consequence G is strictly increasing on $(0, +\infty)$. We evaluate now $\lim_{x \rightarrow +\infty} G(x) = l$. We have that

$$\begin{aligned} l &= \lim_{x \rightarrow +\infty} x \left[1 - \left(1 + \frac{f^p}{(p-1)x}\right)^q\right] \\ &= \lim_{y \rightarrow 0^+} \frac{1 - (1 + (yf^p/(p-1)))^q}{y} = -\frac{q}{p-1}f^p, \end{aligned} \tag{2.19}$$

by using De'l Hospital's rule. Thus since G is strictly increasing on $(0, +\infty)$, we have that $G(x) < -q/(p-1)f^p$, for any $x > 0$. Thus (2.18) yields $L_q < -q/(p-1)f^p$, which is inequality (1.4). We now prove its sharpness.

We let

$$J'_0 = \int_0^1 \left(\frac{1}{t} \int_0^t g\right)^p dt, \quad \text{and} \quad J'_q = \int_0^1 \left(\frac{1}{t} \int_0^t g\right)^{p-q} g^q(t) dt, \quad \text{for any } 1 \leq q \leq p.$$

Let also $g = g_a$, where g_a is defined for any $a \in (0, 1/p)$, by $g_a(t) = t^{-a}$, $t \in (0, 1]$.

Then for every $t \in (0, 1]$ we have that

$$\frac{1}{t} \int_0^t g_a = \frac{1}{1-a} g_a(t)$$

and so

$$\frac{J'_0}{J'_q} = \frac{(1/(1-a))^p \int_0^1 g_a^p dt}{(1/(1-a))^{p-q} \int_0^1 g_a^p dt} = \left(\frac{1}{1-a}\right)^q. \tag{2.20}$$

Letting $a \rightarrow (1/p)^-$ in (2.20) we obtain that the constant $(p/(p-1))^q$, on the right of inequality (1.4), cannot be decreased. We now prove the second part of the sharpness of Theorem 1. For this purpose we define for any fixed constant $f > 0$, and any $a \in (0, 1/p)$, the function $g_a(t) = f(1-a)t^{-a}$, for every $t \in (0, 1]$. Then it is easy to see that $\int_0^1 g_a = f$,

$1/t \int_0^t g_a(u)du = 1/(1-a)g_a(t)$, and $\int_0^1 g_a^p = f^p(1-a)^p/(1-ap)$. We consider now the difference

$$\begin{aligned} L_q(a) &= \int_0^1 \left(\frac{1}{t} \int_0^t g_a\right)^p dt - \left(\frac{p}{p-1}\right)^q \int_0^1 \left(\frac{1}{t} \int_0^t g_a\right)^{p-q} g_a^q(t) dt \\ &= \left(\frac{1}{1-a}\right)^p \int_0^1 g_a^p - \left(\frac{p}{p-1}\right)^q \left(\frac{1}{1-a}\right)^{p-q} \int_0^1 g_a^p \\ &= \frac{(1-a)^q f^p [(1/(1-a))^q - (p/(p-1))^q]}{1-ap}. \end{aligned} \tag{2.21}$$

Letting now $a \rightarrow (1/p)^-$, we immediately see, by an application of De'l Hospital's rule that $L_q(a) \rightarrow -q/(p-1)f^p$. We have just proved that the constant $q/(p-1)$, appearing in front of f^p , cannot be increased. That is, both constants appearing on the right of (1.4) are best possible. □

3. Applications to reverse Hölder inequalities.

We will need first a preliminary lemma which in fact holds under some additional hypothesis for g even if it is not decreasing, and which can be proved by using the integration by parts formula. We present a version that we will need below which is proved by measure theoretic techniques. More precisely we will prove the following:

LEMMA 1. *Let $g : (0, 1] \rightarrow \mathbb{R}^+$ be a non-increasing function. Then the following inequality is true for any $p > 1$ and every $\delta \in (0, 1)$*

$$\int_0^\delta \left(\frac{1}{t} \int_0^t g\right)^p dt = -\frac{1}{p-1} \left(\int_0^\delta g\right)^p \frac{1}{\delta^{p-1}} + \frac{p}{p-1} \int_0^\delta \left(\frac{1}{t} \int_0^t g\right)^{p-1} g(t) dt. \tag{3.22}$$

PROOF. By using Fubini's theorem it is easy to see that

$$\int_0^\delta \left(\frac{1}{t} \int_0^t g\right)^p dt = \int_{\lambda=0}^{+\infty} p\lambda^{p-1} \left| \left\{ t \in (0, \delta] : \frac{1}{t} \int_0^t g \geq \lambda \right\} \right| d\lambda. \tag{3.23}$$

Since now g is non-increasing we have that $1/\delta \int_0^\delta g = f_\delta \geq f = \int_0^1 g$. Also for the same reason we obtain

$$\begin{aligned} \frac{1}{t} \int_0^t g &\geq f_\delta, \quad \forall t \in (0, \delta) \quad \text{while} \\ \frac{1}{t} \int_0^t g &\leq f_\delta, \quad \forall t \in [\delta, 1]. \end{aligned}$$

Let λ be such that $0 < \lambda < f_\delta$. Then for every $t \in (0, \delta]$ we take $1/t \int_0^t g \geq 1/\delta \int_0^\delta g = f_\delta > \lambda$. Thus

$$\left| \left\{ t \in (0, \delta] : \frac{1}{t} \int_0^t g \geq \lambda \right\} \right| = |(0, \delta]| = \delta.$$

Now for every $\lambda > f_\delta$ there exists unique $a(\lambda) \in (0, \delta)$ such that $1/a(\lambda) \int_0^{a(\lambda)} g = \lambda$. It's existence is guaranteed by the fact that $\lambda > f_\delta$, that g is non-increasing and that $g(0^+) = +\infty$ which may without loss of generality be assumed (otherwise we work for the λ 's on the interval $(0, \|g\|_\infty]$). Then

$$\left\{ t \in (0, \delta] : \frac{1}{t} \int_0^t g \geq \lambda \right\} = (0, a(\lambda)].$$

Thus, from the above and (3.23) we conclude that

$$\begin{aligned} \int_0^\delta \left(\frac{1}{t} \int_0^t g \right)^p dt &= \int_{\lambda=0}^{f_\delta} p\lambda^{p-1} \cdot \delta \cdot d\lambda + \int_{\lambda=f_\delta}^{+\infty} p\lambda^{p-1} a(\lambda) d\lambda \\ &= \delta(f_\delta)^p + \int_{\lambda=f_\delta}^{+\infty} p\lambda^{p-1} \frac{1}{\lambda} \left(\int_0^{a(\lambda)} g(u) du \right) d\lambda \end{aligned} \tag{3.24}$$

by the definition of $a(\lambda)$. As a consequence, (3.24) gives

$$\begin{aligned} \int_0^\delta \left(\frac{1}{t} \int_0^t g \right)^p dt &= \frac{1}{\delta^{p-1}} \left(\int_0^\delta g \right)^p + \int_{\lambda=f_\delta}^{+\infty} p\lambda^{p-2} \left(\int_0^{a(\lambda)} g(u) du \right) d\lambda \\ &= \frac{1}{\delta^{p-1}} \left(\int_0^\delta g \right)^p + \int_{\lambda=f_\delta}^{+\infty} p\lambda^{p-2} \left(\int_{\substack{u \in (0, \delta] \\ \frac{1}{u} \int_0^u g \geq \lambda}} g \right) d\lambda \\ &= \frac{1}{\delta^{p-1}} \left(\int_0^\delta g \right)^p + \frac{p}{p-1} \int_0^\delta g(t) \left[\lambda^{p-1} \right]_{\lambda=f_\delta}^{\frac{1}{t} \int_0^t g} dt \\ &= \frac{1}{\delta^{p-1}} \left(\int_0^\delta g \right)^p + \frac{p}{p-1} \left[\int_0^\delta \left(\frac{1}{t} \int_0^t g \right)^{p-1} g(t) - \left(\int_0^\delta g(t) dt \right) f_\delta^{p-1} \right] \\ &= -\frac{1}{p-1} \frac{1}{\delta^{p-1}} \left(\int_0^\delta g \right)^p + \frac{p}{p-1} \int_0^\delta \left(\frac{1}{t} \int_0^t g \right)^{p-1} g(t) dt, \end{aligned}$$

where in the third equality we have used Fubini's theorem and the fact that $1/\delta \int_0^\delta g = f_\delta$. In this way we derived (3.22). \square

We are now able to give the following proof.

PROOF OF THEOREM 3. Suppose we are given $g : (0, 1] \rightarrow \mathbb{R}^+$ non-increasing and $\delta \in (0, 1]$. Our hypothesis for g is (1.9) or that:

$$\frac{1}{t} \int_0^t g^q \leq c \cdot \left(\frac{1}{t} \int_0^t g \right)^q, \text{ for every } t \in (0, 1].$$

Let now $p > q$ and set $a = p/q > 1$.

We apply Lemma 1 with g^q in place of g and a in that of p . We conclude that:

$$\int_0^\delta \left(\frac{1}{t} \int_0^t g^q \right)^{p/q} dt = -\frac{q}{p-q} \frac{1}{\delta^{(p/q)-1}} \left(\int_0^\delta g^q \right)^{p/q} + \frac{p}{p-q} \int_0^\delta \left(\frac{1}{t} \int_0^t g^q \right)^{(p/q)-1} g^q(t) dt \Rightarrow$$

$$\frac{1}{\delta} \int_0^\delta \left[\left(\frac{1}{t} \int_0^t g \right)^{(p/q)-1} g^q(t) - \frac{p-q}{p} \left(\frac{1}{t} \int_0^t g^q \right)^{p/q} \right] dt = \frac{q}{p} \left(\frac{1}{\delta} \int_0^\delta g^q \right)^{p/q}. \quad (3.25)$$

Define now for every $y > 0$ the function ϕ_y of the variable x , by

$$\phi_y(x) = x^{(p/q)-1} y - \frac{p-q}{p} x^{p/q},$$

for $x \geq y$.

Then

$$\phi'_y(x) = [(p/q)-1]x^{(p/q)-2}y - [(p/q)-1]x^{(p/q)-1} = [(p/q)-1]x^{(p/q)-2}(y-x) \leq 0, \text{ for } x \geq y.$$

Thus

$$y \leq x \leq z \Rightarrow \phi_y(x) \geq \phi_y(z). \quad (3.26)$$

Let us now set in (3.26)

$$x = \frac{1}{t} \int_0^t g^q, \quad y = g^q(t), \quad z = c \left(\frac{1}{t} \int_0^t g \right)^q, \text{ for any } t \in (0, 1].$$

Then $y \leq x \leq z \Rightarrow$

$$\begin{aligned} \left(\frac{1}{t} \int_0^t g^q \right)^{(p/q)-1} g^q(t) - \frac{p-q}{p} \left(\frac{1}{t} \int_0^t g^q \right)^{p/q} &\geq c^{(p/q)-1} \left(\frac{1}{t} \int_0^t g \right)^{p-q} g^q(t) \\ &\quad - \frac{p-q}{p} c^{p/q} \left(\frac{1}{t} \int_0^t g \right)^p, \quad \forall t \in (0, 1]. \end{aligned}$$

As a consequence (3.25) gives, by using the hypothesis and the last inequality the following

$$\frac{1}{\delta} \int_0^\delta \left(\frac{1}{t} \int_0^t g \right)^{p-q} g^q(t) dt \leq c \cdot \frac{p-q}{p} \cdot \frac{1}{\delta} \int_0^\delta \left(\frac{1}{t} \int_0^t g \right)^p dt + \frac{q}{p} c \left(\frac{1}{\delta} \int_0^\delta g \right)^p.$$

We use now the inequality,

$$\frac{1}{\delta} \int_0^\delta \left(\frac{1}{t} \int_0^t g \right)^p dt \leq \left(\frac{p}{p-1} \right)^q \frac{1}{\delta} \int_0^\delta \left(\frac{1}{t} \int_0^t g \right)^{p-q} g^q(t) dt$$

which is a consequence of Theorem 1.

We conclude that if p_0 is defined by (1.8), for any $p \in [q, p_0)$, the following holds

$$\left[1 - c \frac{p-q}{p} \left(\frac{p}{p-1} \right)^q \right] \frac{1}{\delta} \int_0^\delta \left(\frac{1}{t} \int_0^t g \right)^{p-q} g^q(t) dt \leq \frac{q}{p} c \left(\frac{1}{\delta} \int_0^\delta g \right)^p,$$

where $1 - c((p-q)/p)(p/(p-1))^q = k_p > 0$, for every such p . This becomes

$$\frac{1}{\delta} \int_0^\delta \left(\frac{1}{t} \int_0^t g \right)^{p-q} g^q(t) dt \leq \frac{q \cdot c}{p \cdot k_p} \left(\frac{1}{\delta} \int_0^\delta g \right)^p, \quad (3.27)$$

for any $\delta \in (0, 1]$, and any $p \in [q, p_0)$.

On the other hand $1/t \int_0^t g \geq g(t)$, since g is non-increasing, thus (3.27) \Rightarrow

$$\frac{1}{\delta} \int_0^\delta g^p \leq \frac{q \cdot c}{p \cdot k_p} \left(\frac{1}{\delta} \int_0^\delta g \right)^p,$$

for any $\delta \in (0, 1]$ and p such that $q \leq p < p_0$, which is an inequality of the form of (1.10), for suitable $c' > 1$.

So the first part of Theorem 3 is now proved. We continue with the sharpness of the result. For this reason we define for any fixed $c \geq 1$ and $q > 1$ the following function $g_a(t) = t^{-a}$ where $a = 1/p_0$, where p_0 is defined by (1.5). Then it is easy to see that $1/t \int_0^t g^q = c(1/t \int_0^t g)^q$, for every $t \in (0, 1]$. It is obvious now that $g \notin L^{p_0}((0, 1])$. Thus, p_0 cannot be increased and Theorem 3 is proved. \square

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