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Weighted Bott–Chern and Dolbeault cohomology for LCK-manifolds with potential

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Abstract. A locally conformally Kähler (LCK) manifold is a complex manifold, with a Kähler structure on its universal covering $\widetilde{M}$, with the deck transform group acting on $\widetilde{M}$ by holomorphic homotheties. One could think of an LCK manifold as of a complex manifold with a Kähler form taking values in a local system L, called *the conformal weight bundle*. The L-valued cohomology of M is called *Morse–Novikov cohomology*; it was conjectured that (just as it happens for Kähler manifolds) the Morse–Novikov complex satisfies the dd^c-lemma, which (if true) would have far-reaching consequences for the geometry of LCK manifolds. In particular, this version of dd^c-lemma would imply existence of LCK potential on any LCK manifold with vanishing Morse–Novikov class of its L-valued Hermitian symplectic form. The dd^c-conjecture was disproved for Vaisman manifolds by Goto. We prove that the dd^c-lemma is true with coefficients in a sufficiently general power of L on any Vaisman manifold or LCK manifold with potential.

1. Introduction.

1.1. LCK manifolds and $d_\theta d^c_\theta$-lemma.

A locally conformally Kähler (LCK) manifold is a complex manifold which admits a Kähler metric on its universal covering $\widetilde{M}$ such that the monodromy acts on $\widetilde{M}$ by Kähler homotheties. For more details and the reference on this subject, please see Section 2.

The LCK property is equivalent to existence of a Hermitian form ω on M satisfying $d\omega = \omega \wedge \theta$, where θ is a closed 1-form. This form is called *the Lee form* of the LCK-manifold.

One can consider the Kähler form on $\widetilde{M}$ as a Kähler form on M taking values in a 1-dimensional local system, or, equivalently, in a flat line bundle L. This bundle is called *the weight bundle* of M.

The cohomology of this local system is known as *the Morse–Novikov cohomology* of an LCK manifold. In locally conformally Kähler geometry, the Morse–Novikov cohomology shares many properties of the Hodge decomposition with the usual cohomology of the complex manifolds. The locally conformally Kähler form represents a cohomology class (called the Morse–Novikov class) of an LCK manifold, encoding the topological

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properties of an LCK structure. However, the dd^c-lemma, which plays a crucial role for the Kähler geometry, is invalid in the Morse–Novikov setting. The main question of the locally conformally Kähler geometry is to find a replacement of the dd^c-lemma which would allow one to study the interaction between the complex geometry and the topology of a manifold.

The statement of the dd^c-lemma seems, on the first sight, to be technical. It says that on any compact Kähler manifold (M, I), one has $\operatorname{im} d \cap \ker d^c = \operatorname{im} dd^c$, where $d^c = IdI^{-1}$ is the twisted de Rham differential. However, it is used as a crucial step in the proof of the degeneration of the Dolbeault–Frölicher spectral sequence, and in the proof of homotopy formality of Kähler manifolds.

For an LCK manifold, one replaces the de Rham differential by its Morse–Novikov counterpart $d_\theta := d - \theta$, where θ is the connection form of its weight bundle; the twisted de Rham differential is replaced by $d^c_\theta = Id_\theta I^{-1}$. It was conjectured in [**OV1**] that the $d_\theta d^c_\theta$-lemma would hold on any LCK manifold, giving $\operatorname{im} d_\theta \cap \ker d^c_\theta = \operatorname{im} d_\theta d^c_\theta$. The implication of the $d_\theta d^c_\theta$-lemma would include the topological classification of LCK structures on some manifolds (such as nilmanifolds) and a construction of automorphic Kähler potentials on LCK manifolds with vanishing Morse–Novikov class. However, this conjecture was false, as shown by Goto ([**G**]).

1.2. Weighted Bott–Chern cohomology.

When the $d_\theta d^c_\theta$-lemma is false, one needs to study a more delicate cohomological invariant, called *the weighted Bott–Chern cohomology of a manifold*:

$$H^{p,q}_{BC}(M, L) := \frac{\ker d_\theta \cap \ker d^c_\theta}{\operatorname{im} d_\theta d^c_\theta}\Big|_{\Lambda^{p,q}(M)}.$$

In [**G**], Goto has shown that the Bott–Chern cohomology group is responsible for the deformational properties of an LCK manifold, and computed it for certain (p, q) and certain examples of LCK manifolds, called *the Vaisman manifolds* (see Subsection 2.2).

DEFINITION 1.1. The local system L associated to a LCK manifold M is a real, oriented line bundle over M with a flat connection. Trivializing this bundle, we can write its connection as $\nabla_L = d - \theta$, where θ is the Lee form of our LCK manifold. For arbitrary $a \in \mathbb{C}$, the connection $\nabla_{L_a} := d - a\theta$ is also flat. For $a \in \mathbb{Z}$, the corresponding line bundle is identified with the a-th tensor power of L, denoted as L^a. One may think of the flat line bundle (L, ∇_{L_a}) as of a real (or complex) power of L. We denote this line bundle and its local system by L_a, and call it *a-th power of the weight bundle.*

In this paper we compute the weighted Bott–Chern cohomology for L_a, on LCK manifolds with proper potential, and show that it vanishes for all a outside of a discrete countable subset of $\mathbb{R}$ (Corollary 4.2). This implies dd^c-lemma for forms with coefficients in L_a, for these values of a. This result is based on a computation of Dolbeault cohomology with coefficients in L_a, which also vanishes for all a but a discrete countable subset (Theorem 3.2).

1.3. LCK manifolds with potential.

DEFINITION 1.2. A compact LCK manifold (M, ω, θ) is called *LCK with potential* if $\omega = d_\theta d^c_\theta \psi$ for a positive function ψ which is called *LCK potential.*

An equivalent definition will be given in Subsection 2.3.

LCK manifolds with potential are understood very well now. The following results were proven in [**OV2**] and [**OV3**] (see also [**OV6**]). Recall that *a linear Hopf manifold* is a quotient of $\mathbb{C}^n \backslash 0$ by a $\mathbb{Z}$-action generated by a linear map with all eigenvalues $|\alpha_i| > 1$.

THEOREM 1.3. *Let M be a compact complex manifold. Then M admits an LCK metric with potential if and only if M admits an embedding to a linear Hopf manifold.*

THEOREM 1.4. *Let M be an LCK manifold with potential. Then M is a deformation of a Vaisman manifold* (*Definition* 2.2). *In particular, M is diffeomorphic to a principal $S^1 \times S^1$-bundle over a projective orbifold.*

It would be nice to have a topological characterization of LCK manifolds with potential. Since [**OV1**], we were extending much effort trying to prove the following conjecture, which has many geometric consequences.

CONJECTURE 1.5. *Let (M, ω, θ) be a compact LCK manifold. Assume that ω is d_θ-exact. Then ω is $d_\theta d^c_\theta$-exact, that is, M is a LCK manifold with potential.*

This conjecture is still open. It would trivially follow if the $d_\theta d^c_\theta$-lemma were true, but it is known now to be false. However, a weaker conjecture still stands.

CONJECTURE 1.6. *Let (M, ω, θ) be a compact LCK manifold, L its weight bundle, and L_a the weight bundle to the power of $a \in \mathbb{R}$* (*Definition* 1.1). *Then, for all a outside of a discrete countable set, $d_{a\theta} d^c_{a\theta}$-lemma is true*: *for any $d_{a\theta}$-exact* $(1,1)$*-form η, one has $\eta = d_{a\theta} d^c_{a\theta} f$* (*but this does not imply that the $d_{a\theta} d^c_{a\theta}$-lemma is true for other bidegrees*).

In this paper, we prove that Conjecture 1.6 is true for LCK manifolds with proper potential (Corollary 4.2). This is done by first proving a generic vanishing result for weighted Dolbeault cohomology (Theorem 3.2).

2. Locally conformally Kähler geometry.

In this section we give the necessary definitions and properties of locally conformally Kähler (LCK) manifolds.

2.1. LCK manifolds.

DEFINITION 2.1. A complex manifold (M, I) is LCK if it admits a Kähler covering $(\widetilde{M}, \widetilde{\omega})$, such that the covering group acts by holomorphic homotheties.

Equivalently, there exists on M a *closed* 1-form θ, called *the Lee form*, such that ω satisfies the integrability condition:

$$d\omega = \theta \wedge \omega.$$

Clearly, the metric $g := \omega(\cdot, I\cdot)$ on M is locally conformal to some Kähler metrics and its lift to the Kähler cover in the definition is globally conformal to the Kähler metric corresponding to $\widetilde{\omega}$.

To an LCK manifold one associates the *weight bundle* $L_{\mathbb{R}} \longrightarrow M$. It is a real line bundle associated to the representation[1]

$$GL(2n, \mathbb{R}) \ni A \mapsto \mid \det A \mid^{1/n} .$$

The Lee form induces a connection in $L_{\mathbb{R}}$ by the formula $\nabla = d - \theta$. ∇ is associated to the Weyl covariant derivative (also denoted ∇) determined on M by the LCK metric and the Lee form. As $d\theta = 0$, then $\nabla^2 = d\theta = 0$, and hence $L_{\mathbb{R}}$ is flat.

The complexification of the weight bundle will be denoted by L. The Weyl connection extends naturally to L and its $(0,1)$-part endows L with a holomorphic structure.

2.2. Vaisman manifolds.

DEFINITION 2.2. A Vaisman manifold is an LCK manifold with ∇^g-parallel Lee form, where ∇^g is the Levi–Civita connection.

The following definition is implicit in the work of Boyer and Galicki, see [**BG**]:

DEFINITION 2.3. A Sasakian manifold is an odd-dimensional contact manifold S such that its symplectic cone CS is equipped with a Kähler structure, compatible with its symplectic structure, and the standard symplectic homothety map $\rho_t : CS \longrightarrow CS$ is holomorphic.

Compact Vaisman manifolds can be described in terms of Sasakian geometry as follows.

THEOREM 2.4. *Let (M, I, g) be a compact Vaisman manifold. Then M admits a conic Kähler covering $(W \times \mathbb{R}_+, t^2 g_W + dt^2)$ such that the covering group is an infinite cyclic group, generated by the transformation $(w, t) \mapsto (\varphi(w), qt)$ for some Sasakian automorphism φ and $q \in \mathbb{Z}$.*

The typical example of a compact Vaisman manifold is the diagonal Hopf manifold $H_A := \mathbb{C}^n / \langle A \rangle$ with $A = \mathrm{diag}(\alpha_i)$, with $|\alpha_i| > 1$. An explicit construction of the Vaisman metric on H_A is given in [**OV5**]. Other Vaisman metrics appear on compact complex surfaces, [**Be**].

Among the LCK manifolds which do not admit Vaisman metrics are some of the Inoue surfaces (cf. [**Tr**], [**Be**]) and their generalizations to higher dimensions ([**OT**]). The rank 0 Hopf surfaces are also non-Vaisman ([**GO**]).

2.3. LCK manifolds with potential.

DEFINITION 2.5 ([**OV2**]). A compact complex manifold (M, I) is *LCK with potential* if it admits a Kähler cover $(\widetilde{M}, \widetilde{\omega})$ with global potential $\varphi : \widetilde{M} \to \mathbb{R}_+$, such that the monodromy map τ acts on φ by multiplication with a constant: $\tau(\varphi) = \mathrm{const} \cdot \varphi$.

[1] In conformal geometry, the weight bundle usually corresponds to $\mid \det A \mid^{1/2n}$. For LCK-geometry, $\mid \det A \mid^{1/n}$ is much more convenient.

If φ is *proper* (inverse images of compact sets are compact), then (M, I) is called *LCK with proper potential*.

Remark 2.6. In [**OV2**, Proposition 2.5] (see also [**OV6**]) it was proven that φ is proper if and only if the monodromy of the weight bundle is discrete in $\mathbb{R}_+$, that is, isomorphic to $\mathbb{Z}$.

Vaisman manifolds are LCK with potential (the potential is equal to the squared norm of the Lee field), which can be easily seen from the Sasakian description given above ([**Ve1**]). LCK metrics with potential are in one to one correspondence with strongly pseudoconvex shells in affine cones, as shown in [**OV5**].

We summarize the main properties of compact LCK manifolds with potential:

Theorem 2.7.

(i) ([**OV2**]) *The class of compact LCK manifolds with potential is stable to small deformations.*

(ii) ([**OV3**, Theorem 2.1]) *Any LCK manifold with potential can be deformed to a Vaisman manifold. Moreover, the set of points which correspond to Vaisman manifolds is dense in the moduli of compact LCK manifolds with potential.*

(iii) ([**OV2**]) *Any compact LCK manifold with potential can be holomorphically embedded into a Hopf manifold. Moreover, a compact Vaisman manifold can be holomorphically embedded in a diagonal Hopf manifold.*

2.4. Morse–Novikov complex and cohomology of local systems.

Let M be a smooth manifold, and θ a closed 1-form on M. Denote by $d_\theta : \Lambda^i(M) \longrightarrow \Lambda^{i+1}(M)$ the map $d - \theta$. Since $d\theta = 0$, $d_\theta^2 = 0$.

Consider the *the Morse–Novikov complex*, (see *e.g.* [**P**], [**Ra**], [**Mi**])

$$\Lambda^0(M) \xrightarrow{d_\theta} \Lambda^1(M) \xrightarrow{d_\theta} \Lambda^2(M) \xrightarrow{d_\theta} \cdots$$

Its cohomology is *the Morse–Novikov cohomology* of (M, θ).

In Jacobi and locally conformal symplectic geometry, this object is called *Lichnerowicz–Jacobi*, or *Lichnerowicz cohomology*, motivated by Lichnerowicz's work [**Li**] on Jacobi manifolds (see *e.g.* [**LLMP**] and [**B**]).

Obviously, the flat line bundle L can be viewed as a local system associated with the character $\chi : \pi_1(M) \longrightarrow \mathbb{R}^{>0}$ given by the exponential $e^\theta \in H^1(M, \mathbb{R}^{>0})$, considered as an element of $\mathbb{R}^{>0}$-valued cohomology. Then we have:

Proposition 2.8 (see *e.g.* [**N**]). *The cohomology of the local system L is naturally identified with the cohomology of the Morse–Novikov complex $(\Lambda^*(M), d_\theta)$.*

The following result was proven in [**LLMP**] and, with a different method, in [**OV1**]:

Theorem 2.9. *The Morse–Novikov cohomology of a compact Vaisman manifold vanishes identically.*

On the other hand, on one of the Inoue surfaces (which is LCK but non-Vaisman) the Morse–Novikov class of ω is non–zero, see [**B**, Theorem 1].

3. Weighted Dolbeault cohomology for LCK manifolds with potential.

Let M be an LCK manifold with proper potential, and $\widetilde{M}$ its $\mathbb{Z}$-covering equipped with the automorphic Kähler metric. In [**OV2**] it was shown that the metric completion $\widetilde{M}_c$ of M is a Stein variety with at most one isolated singularity. Moreover, $\widetilde{M}_c$ is obtained from $\widetilde{M}$ by adding one point, called "the origin". Denote this point by c, and let R be its local ring.

Remark 3.1. If M is Vaisman, then $\widetilde{M}$ is a true (Riemannian) cone and the fibres are Sasakian. In the general case, nothing more precise can be said neither on the metric of $\widetilde{M}$ nor on the contact metric structure of the fibres.

Since $\widetilde{M}_c$ is a singular variety, to control what happens in the neighbourhood of c we need some technique borrowed from algebraic geometry which we briefly explain below. Note that we could arrive at the same results by using L^2-estimates, but the computations and technicalities would have been much more involved.

3.1. Main result: the generic vanishing theorem.

The main result of this paper is:

Theorem 3.2. *Let M be an LCK manifold with proper potential, θ its Lee form, $\widetilde{M}$ its Kähler $\mathbb{Z}$-cover and denote by $t : \widetilde{M} \longrightarrow \widetilde{M}$ the monodromy action. Let $\alpha \in \mathbb{C}$ be arbitrary and let L_α be the flat line bundle on M corresponding to $\alpha\cdot\theta$.*

Then for any $q \in \mathbb{N}$

$$H^q(M, \Omega^p_M \otimes L_\alpha) = 0,$$

for all $\alpha \in \mathbb{C}$ but a discrete countable subset.

Remark 3.3. For some Hopf manifolds, stronger vanishing results were obtained by Ise [**Is**] and Mall [**Ma**]. In these cases, the set of exceptions is made explicit.

We describe the main steps of the proof and give the details in the next section.

Step 1: reduction to the local cohomology.

One has the following exact sequence, (see Corollary 3.7, which follows from Theorem 3.6):

$$\begin{aligned}0 \longrightarrow H^0(M, \Omega^i_M \otimes L_\alpha) \longrightarrow H^0(\widetilde{M}, \Omega^i_{\widetilde{M}}) &\xrightarrow{t-\alpha} \\ \xrightarrow{t-\alpha} H^0(\widetilde{M}, \Omega^i_{\widetilde{M}}) &\longrightarrow H^1(M, \Omega^i_M \otimes L_\alpha) \longrightarrow \cdots \end{aligned} \tag{3.1}$$

We are thus reduced to the study of the maps

$$H^j\left(\widetilde{M}, \Omega^i_{\widetilde{M}}\right) \xrightarrow{t-\alpha} H^j\left(\widetilde{M}, \Omega^i_{\widetilde{M}}\right).$$

Denote by $\Omega^i_{\widetilde{M}_c}$ be the exterior i-power of the sheaf of Kähler differentials on $\widetilde{M}_c$ and by S its stalk at c. Using cohomology with supports, we have an exact sequence

$$0 \longrightarrow H^0_{\mathfrak{m}}(S) \longrightarrow H^0\left(\widetilde{M}_c, \Omega^i_{\widetilde{M}_c}\right) \longrightarrow H^0\left(\widetilde{M}, \Omega^i_{\widetilde{M}}\right) \longrightarrow$$
$$\longrightarrow H^1_{\mathfrak{m}}(S) \longrightarrow H^1\left(\widetilde{M}_c, \Omega^i_{\widetilde{M}_c}\right) \longrightarrow \cdots$$

Since $\widetilde{M}_c$ is Stein, $H^j\left(\widetilde{M}_c, \Omega^i_{\widetilde{M}_c}\right) = 0$ for all $j \geq 1$, we obtain isomorphisms

$$H^j\left(\widetilde{M}, \Omega^i_{\widetilde{M}}\right) \simeq H^{j+1}_{\mathfrak{m}}(S),$$

and an exact sequence

$$0 \longrightarrow H^0_{\mathfrak{m}}(S) \longrightarrow H^0\left(\widetilde{M}_c, \Omega^i_{\widetilde{M}_c}\right) \xrightarrow{t-\alpha} H^0\left(\widetilde{M}, \Omega^i_{\widetilde{M}}\right) \longrightarrow H^1_{\mathfrak{m}}(S) \longrightarrow 0$$

These induce the commutative diagrams

$$\begin{array}{ccc} H^i\left(\widetilde{M}, \Omega^j_{\widetilde{M}}\right) & \xrightarrow{\simeq} & H^{i+1}_{\mathfrak{m}}(S) \\ \downarrow & & \downarrow \\ H^i\left(\widetilde{M}, \Omega^j_{\widetilde{M}}\right) & \xrightarrow{\simeq} & H^{i+1}_{\mathfrak{m}}(S) \end{array} \tag{3.2}$$

and respectively

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^0_{\mathfrak{m}}(S) & \longrightarrow & H^0\left(\widetilde{M}_c, \Omega^i_{\widetilde{M}_c}\right) & \longrightarrow & H^0\left(\widetilde{M}, \Omega^i_{\widetilde{M}}\right) & \longrightarrow & H^1_{\mathfrak{m}}(S) & \longrightarrow & 0 \\ & & {\scriptstyle t-\alpha}\downarrow & & {\scriptstyle t-\alpha}\downarrow & & {\scriptstyle t-\alpha}\downarrow & & {\scriptstyle t-\alpha}\downarrow & & \\ 0 & \longrightarrow & H^0_{\mathfrak{m}}(S) & \longrightarrow & H^0\left(\widetilde{M}_c, \Omega^i_{\widetilde{M}_c}\right) & \longrightarrow & H^0\left(\widetilde{M}, \Omega^i_{\widetilde{M}}\right) & \longrightarrow & H^1_{\mathfrak{m}}(S) & \longrightarrow & 0 \end{array} \tag{3.3}$$

Eventually, notice that $H^{i+1}_{\mathfrak{m}}(S)$ and $H^0\left(\widetilde{M}_c, \Omega^i_{\widetilde{M}_c}\right)$ are R-modules.

Step 2: algebraic proof of generic vanishing.

At this step we use the following result, which will be proven in section 3.3:

THEOREM 3.4. *For any local Noetherian $\mathbb{C}$-algebra R endowed with a $\mathbb{Z}$-action given by an automorphism of local $\mathbb{C}$-algebras t_R and for any R-module N endowed also with a $\mathbb{Z}$ action t_N which is t_R-equivariant, i.e.*

$$t_N(rm) = t_R(r)t_N(m), \quad \textit{for all } r \in R,\ m \in N,$$

the map $t_M - \alpha$ is a $\mathbb{C}$-linear isomorphism for all $\alpha \in \mathbb{C}$ but a countable subset.

Step 3.

Using the above commutative diagrams (3.2), (3.3), we conclude that for each $\alpha \in \mathbb{C}$ but a countable subset and any $i, j \geq 0$ the map

$$t-\alpha : H^i\left(\widetilde{M}, \Omega^j_{\widetilde{M}}\right) \longrightarrow H^i\left(\widetilde{M}, \Omega^j_{\widetilde{M}}\right)$$

is an isomorphism. From the exact sequence (3.1) we obtain $H^i(M, \Omega^j_M \otimes L_\alpha) = 0$, for all α in $\mathbb{C}$ but a countable set. Moreover, by upper-continuity on α, the set $\{\alpha \in \mathbb{C}\,;\, H^i(M, \Omega^j_M \otimes L_\alpha) = 0\}$ is analytically Zariski open, and hence its complement is discrete since it is countable.

3.2. Proof of Step 1: reduction to the local cohomology.

Definition 3.5. Let F be a sheaf of $\mathbb{C}$-vector spaces over a topological vector space. Denote by F_x the stalk of F in $x \in M$, and let $\operatorname{God}(F)$ be the sheaf defined by $\operatorname{God}(F)(U) := \prod_{x\in U} F_x$. The natural sheaf embedding $F \hookrightarrow \operatorname{God}(F)$ is apparent. The sheaves $\operatorname{God}_i(F)$ are defined inductively: set $\operatorname{God}_0(F) := F$, $\operatorname{God}_1(F) := \operatorname{God}(F)$, and then

$$\operatorname{God}_{i+1}(F) := \operatorname{God}(\operatorname{God}_i(F)/\operatorname{God}_{i-1}(F)).$$

This gives an exact sequence

$$0 \longrightarrow F \longrightarrow \operatorname{God}_1(F) \longrightarrow \operatorname{God}_2(F) \longrightarrow \cdots$$

called *the Godement resolution of F*.

Theorem 3.6. *Let $\widetilde{M} \xrightarrow{\pi} M$ be a manifold equipped with a free action of $\mathbb{Z}$, $M := \widetilde{M}/\mathbb{Z}$ its quotient, and let F be a $\mathbb{Z}$-equivariant sheaf on $\widetilde{M}$. For any character $\alpha :\ \mathbb{Z} \longrightarrow \mathbb{R}$, denote by $F_\alpha \subset \pi_*F$ the sheaf of automorphic sections of π_*F, associated with the character α, considered as a sheaf on M.*

Then one has the exact sequence

$$0 \longrightarrow H^0(M, F_\alpha) \longrightarrow H^0(\widetilde{M}, F) \xrightarrow{t-\alpha} H^0(\widetilde{M}, F) \longrightarrow H^1(M, F_\alpha) \longrightarrow \cdots \tag{3.4}$$

where t is the associated action by the generator of $\mathbb{Z}$ acting on $\widetilde{M}$, and α is the multiplication by the number $\alpha(t)$.

Proof. Consider the Godement resolution $0 \longrightarrow F \longrightarrow F^1 \longrightarrow F^2 \longrightarrow \cdots$. Here $F^i = \operatorname{God}(F^{i-1}/\operatorname{im}(d_{i-1})) = \operatorname{God}(\operatorname{coker}(d_{i-1}))$, $F^0 = F$, and $d_i : F^{i-1} \longrightarrow F^i$. Then

$$0 \longrightarrow F^k_\alpha \longrightarrow \pi_*F^k \xrightarrow{t-\alpha} \pi_k F^* \longrightarrow 0 \tag{3.5}$$

is an exact sequence of complexes of flabby sheaves over M.

Indeed, $F^k_\alpha = \ker(t-\alpha)$ and we only have to show that $t-\alpha$ is surjective. It is enough to make the proof at the level of sections of F^k. The argument is combinatorial. We look at $\widetilde{M}$ as $\bigcup_{i\in\mathbb{Z}} \widetilde{M}_i$ where M_0 is a fundamental domain of the $\mathbb{Z}$ action and $\widetilde{M}_i = t^i(M_0)$.

Then, given $f \in F^k(U)$, $U \subset \widetilde{M}$, it is enough to solve the equation $(t-\alpha)g = f$ for each $f_i = f|_{U_i}$, $U_i = U \cap \widetilde{M}_i$; this will give as solution the section $g_{i-1} \in F(U_{i-1})$, $i \in \mathbb{Z}$. The equation is

$$tg_it^{-1} - \alpha g_{i-1} = f_{i-1},$$

which can be solved recursively once we have chosen arbitrarily $g_0 \in F(U_0)$.

The long exact sequence associated to (3.5) is precisely (3.4). □

Let now M be a locally conformally Kähler manifold with Kähler covering $\widetilde{M}$ and monodromy $\Gamma \cong \mathbb{Z}$. Consider the weight bundle L on M, and let L_α be its power associated with the character $\alpha \in \mathrm{Hom}(\Gamma, \mathbb{R})$. Since the automorphic forms on $\widetilde{M}$ can be identified with forms on M with values in L, from the above result we directly obtain:

Corollary 3.7. *For a compact LCK manifold with monodromy $\mathbb{Z}$ one has the exact sequence for the Dolbeault cohomology of M with values in L_α:*

$$0 \longrightarrow H^0(M, \Omega^i_M \otimes L_\alpha) \longrightarrow H^0\left(\widetilde{M}, \Omega^i_{\widetilde{M}}\right) \stackrel{t-\alpha}{\longrightarrow}$$
$$\stackrel{t-\alpha}{\longrightarrow} H^0\left(\widetilde{M}, \Omega^i_{\widetilde{M}}\right) \longrightarrow H^1(M, \Omega^i_M \otimes L_\alpha) \longrightarrow \cdots$$

3.3. Proof of Step 2: algebraic proof of generic vanishing.

Remark 3.8. Let $(V_n, t_n)_{n\geq 0}$ be a sequence of finite-dimensional vector spaces and endomorphisms $t_n : V_n \longrightarrow V_n$. Let $V = \prod_{n\geq 0} V_n$ and $t = \prod_{n\geq 0} t_n$. Then

$$\mathrm{Spec}(t) = \bigcup_{n\geq 0} \mathrm{Spec}(t_n)$$

In particular, $\mathrm{Spec}(t)$ is at most countable.

Here, for a $\mathbb{C}$-vector space V and $u \in \mathrm{End}(V)$, $\mathrm{Spec}(u) := \{\lambda \in \mathbb{C}\ ;\ u - \lambda \cdot \mathsf{id}$ is not an isomorphism$\}$.

This implies the following:

Lemma 3.9. *If (M, t_m) is a finitely generated complete R-module which is equivariant, then* $\mathrm{Spec}(t_M)$ *is at most countable.*

Proof. Since M is complete we have

$$M = \prod_{n\geq 0} \mathfrak{m}^n M/\mathfrak{m}^{n+1}M.$$

Since M is finitely generated, $\mathfrak{m}^n M/\mathfrak{m}^{n+1}M$ is finite dimensional $\mathbb{C}$-vector space for all $n \geq 0$, so Remark 3.8 applies. □

Unfortunately, the cohomology modules $H^i_{\mathfrak{m}}(M)$ are usually not finitely generated, so we need to elaborate further, by first reducing to the case of regular rings, and then using local duality and the explicit description of the injective hull of the residue field.

First, since local cohomology does not change under completion (cf [**Hun**], Proposition 2.15), we may assume that both R and M are complete.

Next, we reduce to the case when R is regular.

To do this, we choose a minimal system of generators for $\mathfrak{m}_R$, $m_1, \ldots, m_n$ and define a map

$$\pi : S = \mathbb{C}[[X_1, \dots, X_n]] \longrightarrow R,$$

by $X_i \mapsto m_i, i = 1, \dots, n$.

The action t_R on R lifts to an action t_S on S as follows. Choose lifts $s_i \in S$ of $t(m_i)$ for all $i = 1, \dots, n$, and define $t_S(X_i) = s_i$. Note that t_S is well-defined as a morphism of local $\mathbb{C}$-algebras by [**E**, Theorem 7.16].

So we can look at M as an equivariant S-module.

Also, the local cohomology is preserved, since $\mathfrak{m}_R = \mathfrak{m}_S R$ and using [**Hun**, Proposition 2.14 (2)], we have $H^i_{\mathfrak{m}_S}(M) \simeq H^i_{\mathfrak{m}_R}(M)$.

Denote by t^i the endomorphism of $H^i_{\mathfrak{m}}(M)$ induced by t_M and t_R.

By local duality ([**Hun**, Theorem 4.4]) we have:

$$H^i_{\mathfrak{m}}(M) \simeq \mathrm{Ext}^{n-i}_R(M, R)^\vee = \mathrm{Hom}_R(\mathrm{Ext}^{n-i}_R(M, R), E(k))$$

where $E(k)$ is the injective hull of the residue field.

For regular rings, the injective hull $E(k)$ is described by Lyubeznik ([**Ly**]):

$$E(k) = \mathcal{D}/\mathfrak{m}\mathcal{D}$$

where $\mathcal{D}$ is the space of differential operators.

Notice that $\mathcal{D}$ has a direct sum decomposition of the form $\mathcal{D} = \oplus_{n\geq 0}\mathcal{D}_n$ where $\mathcal{D}_n$ is the set of differential operators of order n with no lower-order terms. Note that $\mathcal{D}_n$ is invariant under the map induced by t_R and finitely generated over R. So

$$E(k) = \bigoplus_{n\geq 0} E(k)_n$$

where $E(k)_n = \mathcal{D}_m/\mathfrak{m}\mathcal{D}_n$ and each $E(k)_n$ is equivariant and finitely generated R-module. This gives a decomposition as follows:

$$H^i_{\mathfrak{m}}(M) \simeq \bigoplus_{n\geq 0} \mathrm{Hom}_R(\mathrm{Ext}^{n-i}_R(M, R), E(k)_n)$$

But each factor $\mathrm{Hom}_R(\mathrm{Ext}^{n-i}_R(M, R), E(k)_n)$ is finitely generated over R so Lemma 3.9 applies to it. Since there are countably many factors in the above decomposition, we see $\mathrm{Spec}(t^i)$ is countable.

Now Theorem 3.2 is completely proven. □

3.4. Degeneration of the Dolbeault–Frölicher spectral sequence with coefficients in a local system.

The next result, interesting in itself, proves that on compact LCK manifolds with proper potential, in the Dolbeault–Frölicher spectral sequence with coefficients in a local system L_α,

$$E_1^{p,q} := H^q(M, \Omega^p_M \otimes L_\alpha) \Rightarrow H^{p+q}(M, L_\alpha(\mathbb{C})),$$

all the terms vanish at E_2 level: $E_2^{p,q} = 0$ (where $L_\alpha(\mathbb{C})$ denotes the local system associated to L_α). This parallels the degeneration of this spectral sequence at E_1 level for compact Kähler manifolds (where L_α is taken to be trivial). In particular, this gives a new proof to Theorem 2.9 and produces new examples of compact complex manifolds that do not carry LCK metrics with potential. One of the approaches to finding such manifolds is due to S. Rollenske ([**Ro**]), who showed that on a nilmanifold, the Dolbeault–Frölicher spectral sequence does not necessarily degenerate, and gave examples when the n-th differential d_n is non-zero, for arbitrarily high n.

Proposition 3.10. *Let M be a compact LCK manifold with proper potential, $\alpha \in \operatorname{Hom}(\Gamma, \mathbb{R}_+)$ a positive character, and L_α the corresponding line bundle. For any p, q, consider the map*

$$\partial_{p.q}:\ H^p(M, \Omega^q_M \otimes L_\alpha) \longrightarrow H^p(M, \Omega^{q+1}_M \otimes L_\alpha).$$

Then $\ker \partial_{q,p+1} = \operatorname{im} \partial_{q,p}$, *for all* p, q.

Proof. The monodromy map $\widetilde{t}$ on $\widetilde{M}$ is the exponential of a holomorphic vector field X. This is proven in [**OV4**, Theorem 2.3] using the embedding of M in a Hopf manifold $\mathbb{C}^N \setminus \{0\})/\langle A\rangle$ where A is linear, with all eigenvalues smaller than 1. The holomorphic vector field is then $X = \log A$. In particular:

$$\widetilde{t}^*(\eta) = \operatorname{Lie}_X \eta.$$

Let now $[\eta] \in H^p(M, \Omega^{q+1}_M \otimes L_\alpha)$. A representative η can be seen as a $(q+1, p)$-form on $\widetilde{M}$ which is $\overline{\partial}$-closed and automorphic of weight α.

Suppose η is also ∂-closed. Then, since $\widetilde{t}^*(\eta) = \alpha \cdot \eta$, we obtain

$$\alpha \cdot \eta = \operatorname{Lie}_X(\eta) = d i_X \eta + i_X d\eta,$$

by Cartan's formula.

But $\overline{\partial}(\eta) = \partial\eta = 0$ by assumption, thus $i_X d\eta = 0$, and we are left with:

$$\alpha \cdot \eta = \partial(i_X\eta) + \overline{\partial}(i_X\eta).$$

As X is holomorphic, $i_X\eta$ is of type (q, p), and hence $\overline{\partial}(i_X\eta)$ is of type $(q, p+1)$. On the other hand both $\partial i_X(\eta)$ and $\alpha \cdot \eta$ are of type $(q+1, p)$, implying $\overline{\partial}(i_X\eta) = 0$ and

$$\alpha \cdot \eta = \partial(i_X\eta).$$

This yields $\eta = \partial i_X\left(\frac{1}{\alpha}\eta\right)$, and hence $\eta \in \operatorname{im}(\partial_{q,p})$. ☐

4. Weighted Bott–Chern cohomology for LCK manifolds with potential.

We now generalize [**OV1**, Theorem 4.7]. We have:

Proposition 4.1. *Let (M, I, g) be a compact LCK manifold. Then the following sequence is exact for all $\alpha \in \mathbb{C}$ but a discrete countable subset:*

$$H^{q-1}_{\overline{\partial}}(\Omega^p_M \otimes L_\alpha) \oplus \overline{H^{p-1}_{\overline{\partial}}(\Omega^q_M \otimes L_\alpha)} \xrightarrow{\partial_\theta + \overline{\partial}_\theta} H^{p,q}_{BC}(M, L_\alpha) \xrightarrow{\nu} H^{p+q}(M, L_\alpha(\mathbb{C})) \tag{4.1}$$

where ν is the tautological map, $\partial_\theta = \partial - \theta^{1,0}$ and $\overline{\partial}_\theta = \overline{\partial} - \theta^{0,1}$.

PROOF. We prove that $\operatorname{im}(\partial_\theta + \overline{\partial}_\theta) = \ker \nu$. Let η be a (p,q)-form with values in L_α whose class vanishes in the cohomology of the local system $L_\alpha(\mathbb{C})$. Then $\eta = d_\theta \beta$. Suppose that β has only two Hodge components, $\beta = \beta^{p,q-1} + \beta^{p-1,q}$. Then η decomposes as $\eta = \overline{\partial}_\theta \beta^{p,q-1} + \partial_\theta \beta^{p-1,q}$. On the other hand, as η is of bidegree (p,q), we have $\partial_\theta \beta^{p,q-1} = 0$ and $\overline{\partial}_\theta \beta^{p-1,q} = 0$, and hence $\beta^{p,q-1}$ and $\beta^{p-1,q}$ produce the cohomology classes in $[\beta^{p-1,q}] \in H^{q-1}_{\overline{\partial}}(\Omega^p_M \otimes L_\alpha)$ and $[\beta^{p,q-1}] \in \overline{H^{p-1}_{\overline{\partial}}(\Omega^q_M \otimes L_\alpha)}$. Then $[\eta]_{BC} = \partial_\theta[\beta^{p-1,q}] + \overline{\partial}_\theta[\beta^{p,q-1}]$.

It remains to reduce Proposition 4.1 to the case when β has only two Hodge components. We may already assume that $H^{p,q}(L_\alpha) = 0$ for all p, q (Theorem 3.2). We use induction by the number of Hodge components. Take the outermost Hodge component of β, say, $\beta^{p-d-1,q+d}$, with $d > 0$. Then $\overline{\partial}_\theta(\beta^{p-d-1,q+d}) = 0$, hence, by vanishing of the Dolbeault cohomology group $H^{p-d-1,q+d}(L_\alpha)$, we have $\beta^{p-d-1,q+d} = \overline{\partial}_\theta(\gamma)$, where $\gamma \in \Lambda^{p-d-1,q-1+d}(M, L_\alpha)$ is an L_α-valued $(p-d-1, q-1+d)$-form. Now if we replace β by $\beta - d_\theta\gamma$, we obtain another form β' such that $\eta = d_\theta\beta'$, and β' has a smaller number of Hodge components. $\square$

As compact LCK manifolds with potential are topologically equivalent with Vaisman manifolds, Theorem 2.7 (ii), by Theorem 2.9 their cohomology of the local system $L_\alpha(\mathbb{C})$ vanishes identically. Together with our main result (Theorem 3.2), this proves the following generic vanishing of Bott–Chern cohomology (we keep the notations in Section 3):

COROLLARY 4.2. *Let M be an LCK manifold with proper potential, $\alpha \in \mathbb{C}$ and L_α the flat line bundle corresponding to $\alpha \cdot \theta$. Then $H^{p,q}_{BC}(M, L_\alpha) = 0$ for all $\alpha \in \mathbb{C}$ but a discrete countable subset.*

REMARK 4.3. Note that $H^{p,q}_{BC}(M, L_\alpha) = 0$ implies the $d_{\alpha\theta} d^c_{\alpha\theta}$-lemma at the level (p,q), and hence our result says that, generically, a compact LCK manifold with proper potential satisfies the $d_{\alpha\theta} d^c_{\alpha\theta}$-lemma for all (p,q).

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