J. Math. Soc. Japan
Vol. 69, No. 4 (2017) pp. 1519–1554
doi: 10.2969/jmsj/06941519

Stability and bifurcation for surfaces with constant mean curvature

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(Received Aug. 23, 2015)
(Revised Jan. 29, 2016)

Abstract. We give criteria for the existence of smooth bifurcation branches of fixed boundary CMC surfaces in $\mathbb{R}^3$, and we discuss stability/instability issues for the surfaces in bifurcating branches. To illustrate the theory, we discuss an explicit example obtained from a bifurcating branch of fixed boundary unduloids in $\mathbb{R}^3$.

1. Introduction.

Plateau was already aware of bifurcation phenomena in the geometry of surfaces of constant mean curvature (CMC) [**16**]. In his renowned experimental investigation of the Delaunay surfaces, he observed how surfaces belonging to one family could be created experimentally until a "limit of stability" was reached, whence a spontaneous transformation into surfaces from an adjacent family would take place. Numerous investigation of bifurcation in CMC surfaces have followed, the vast majority of which, in our opinion, rely on empirical or heuristic arguments to justify that bifurcation has taken place. More rigorous bifurcation results for CMC hypersurfaces in Riemannian manifolds have been given in terms of Morse index, or abstract equivariant bifurcation results, see for instance [**3**], [**11**], [**12**], [**14**] and the references therein. On the one hand, all of these results use rather weak assumptions to guarantee the existence of a bifurcating branch. On the other hand, these types of results do not provide information about the cardinality of the bifurcation branch which may only consist of a sequence of bifurcating surfaces. In particular, they do not guarantee the existence of what is observed; a smooth bifurcating branch.

Our first goal here is to develop a mathematically rigorous approach to the existence of smooth bifurcating families of CMC surfaces. The main ingredients of our theory are stability/instability criteria proved by the first author in [**8**], see Section 2.3, for smooth perturbations of fixed boundary CMC surfaces. The other primary ingredient involves an abstract bifurcation theory of Crandall and Rabinowitz [**4**], [**5**] which can be applied, under appropriate conditions, to produce a smooth family of fixed boundary CMC surfaces via an implicit function theorem when the kernel of the Jacobi operator L (see (2)) is non-trivial. We use this to obtain two results on the existence of bifurcations:

2010 *Mathematics Subject Classification.* Primary 58E12; Secondary 53A10, 49Q10, 49R05.
Key Words and Phrases. bifurcation, constant mean curvature surfaces, stability.
The first author is supported in part by JSPS KAKENHI Grant Numbers JP25287012, JP26520205, and JP26610016. The third author is partially supported by CNPq and Fapesp, Brazil.

Let Σ be a two-dimensional oriented compact connected smooth manifold with smooth boundary $\partial\Sigma$, and consider a one-parameter family $X_t : \Sigma \to \mathbb{R}^3$ of CMC immersions. We assume that $\{X_t\}$ depend smoothly on $t \in (-\varepsilon, \varepsilon)$, and that each X_t is of the form $X_t = X + \varphi(t)\nu$ for some smooth function $\varphi(t) : \Sigma \to \mathbb{R}$, with $X = X_0$ (i.e., $\varphi(0) = 0$) and $X\big|_{\partial\Sigma} = X_t|_{\partial\Sigma}$ (i.e., $\varphi(t)\big|_{\partial\Sigma} = 0$). Here ν denotes a smooth unit normal vector field along X. Assuming that X_t has the form $X + \varphi(t)\nu$ is reasonable, because, for an immersion $X : \Sigma \to \mathbb{R}^3$, for any immersion $Y : \Sigma \to \mathbb{R}^3$ having the same boundary values as X in a sufficiently small neighborhood of X in the $C^{2+\alpha}$ topology, there exists a diffeomorphism $f : \Sigma \to \Sigma$ and a $C^{2+\alpha}$ function $\varphi : \Sigma \to \mathbb{R}$ with $\varphi|_{\partial\Sigma} = 0$ such that $Y \circ f = X + \varphi\nu$ holds. We denote the mean curvature of X_t and the volume of X_t, by $H(t)$, $V(t)$, respectively (The definition of an immersion will be given in Section 2.1.). We denote by L_t, $\widetilde{L}_t$ the symmetric operators associated with the second variation of the area for X_t (see (2) and (5)). Moreover, we denote by E the kernel of L_0. The eigenvalues of the eigenvalue problem (3) associated with the Jacobi operator L are denoted by λ_n, and the eigenvalues of the eigenvalue problem (6) associated with the operator $\widetilde{L}$ are denoted by $\widetilde{\lambda}_n$.

The first result on the existence of bifurcations furnishes a smooth bifurcating family of CMC surfaces with fixed boundaries, whose mean curvatures coincide with the mean curvatures of the original family, that is the bifurcation parameter is the mean curvature:

Theorem 1.1. *Assume*:

(i) $H'(0) \neq 0$.

(ii) $E = \{ae : a \in \mathbb{R}\}$, *for some* $e \in C_0^{2+\alpha}(\Sigma) \setminus \{0\}$.

Then, $\int_\Sigma e \, d\Sigma = 0$, *and there exists a differentiable map* $(-\varepsilon_0, \varepsilon_0) \ni t \mapsto \lambda(t) \in \mathbb{R}$, *with* $0 < \varepsilon_0 \leq \varepsilon$, *such that* $\lambda(0) = 0$, $\lambda(t)$ *is a simple eigenvalue of* L_t, *and there is no other eigenvalue of* L_t *near* 0.

Assume further that $\lambda'(0) \neq 0$ *holds. Then there is an analytic bifurcation branch of fixed boundary CMC immersions issuing at* X. *More precisely, let* $E^\perp$ *be the orthogonal complement of* E *in* $C_0^{2+\alpha}$ *with respect to the* L^2 *inner product. Then, there exist an open interval* $\hat{I} \subset \mathbb{R}$, *with* $0 \in \hat{I}$, *and* C^1 *functions* $\zeta : \hat{I} \to E^\perp$ *and* $t : \hat{I} \to \mathbb{R}$, *such that* $t(0) = 0$, $\zeta(0) = 0$, *and* $Y(s) := X + (\varphi(t(s)) + se + s\zeta(s))\nu$ *is a CMC immersion with mean curvature* $\hat{H}(s) := H\big(t(s)\big)$.

Moreover, every CMC immersion with the same boundary values as X *and sufficiently* $C^{2+\alpha}$*-close to* X *is equal, up to parameterization, to some element of the families* $\{X_t : t \in I\}$ *and* $\{Y(s) : s \in \hat{I}\}$, *where* I *is an open interval satisfying* $0 \in I \subset (-\epsilon_0, \epsilon_0)$. *Furthermore, the surfaces* $\{X_t : t \in I\}$ *and* $\{Y(s) : s \in \hat{I}\}$ *are pairwise distinct except for* $X_0 = Y(0)$.

There are some previous works on the existence of bifurcation from critical cylinders with specific boundary conditions with mean curvature as bifurcation parameter ([**13**], [**17**]). They also use the bifurcation theory of Crandall and Rabinowitz [**4**].

Our second result on the existence of bifurcations provides a smooth bifurcating branch of CMC surfaces, having the same boundaries whose volumes coincide with the volumes of surfaces in the original family, that is the bifurcation parameter is the volume.

The proof is much more involved than the proof of Theorem 1.1. This second result is inspired by unpublished material of Patnaik from the reference [**15**] (see Remark 4.4).

Theorem 1.2. *Assume*:

(i) $H'(0) \neq 0$.

(ii) $E = \{ae : a \in \mathbb{R}\}$, *for some* $e \in C_0^{2+\alpha}(\Sigma) \setminus \{0\}$.

Then, $\int_\Sigma e \, d\Sigma = 0$, *and there exist* $j \geq 2$ *and* $k \geq 1$ *such that* $\lambda_j = \widetilde{\lambda}_k = 0$. *Moreover, there exists a differentiable function* $(-\varepsilon_0, \varepsilon_0) \ni t \mapsto \widetilde{\lambda}(t) \in \mathbb{R}$, $0 < \varepsilon_0 \leq \varepsilon$, *with* $\widetilde{\lambda}(0) = 0$, *such that* $\widetilde{\lambda}(t)$ *is a simple eigenvalue of* $\widetilde{L}_t$, *which is the unique eigenvalue of* $\widetilde{L}_t$ *near* 0.

Assume further that $V'(0) \neq 0$ *and* $\widetilde{\lambda}'(0) \neq 0$ *hold. Then, there is a smooth bifurcating branch of CMC immersions, issuing from* X *that is described as follows. Let* $E^\perp$ *be the orthogonal complement of* E *in* $C_0^{2+\alpha}(\Sigma)$ *with respect to the* L^2 *inner product. Then, there exists an open interval* $\hat{I} \subset \mathbb{R}$, *with* $0 \in \hat{I}$, *and* C^1 *mappings* $\eta : \hat{I} \to C_0^{2+\alpha}(\Sigma)$ *and* $\tau : \hat{I} \to \mathbb{R}$, *such that* $\tau(0) = 0$, $\eta(0) = 0$, *and* $Y(s) := X + (\varphi(\tau(s)) + se + s\eta(s))\nu$ *is a CMC immersion with volume* $\hat{V}(s) := V(\tau(s))$. $\eta(s)$ *can be written as* $\eta(s) = c(s)\varphi'(0) + \xi(s)$, *where* $c : \hat{I} \to \mathbb{R}$ *and*

$$\xi : \hat{I} \longrightarrow \left\{u \in C_0^{2+\alpha}(\Sigma) : \int_\Sigma u \, d\Sigma = 0\right\} \cap E^\perp$$

are C^1 *mappings such that* $c(0) = 0$, $\xi(0) = 0$.

Moreover, every CMC immersion with the same boundary values as X *and sufficiently* $C^{2+\alpha}$*-close to* X *is equal, up to parameterization, to some element of the families* $\{X_t : t \in I\}$ *and* $\{Y(s) : s \in \hat{I}\}$, *where* I *is an open interval satisfying* $0 \in I \subset (-\epsilon_0, \epsilon_0)$. *Furthermore, the surfaces* $\{X_t : t \in I\}$ *and* $\{Y(s) : s \in \hat{I}\}$ *are pairwise distinct except for* $X_0 = Y(0)$.

Recall that stable CMC surfaces occur as local minima for the area functional subject to a constraint on the enclosed three dimensional volume. If a family of such surfaces, all having the same boundary, is given, bifurcation commonly takes place at a surface where stability is lost (for the definition of stability, see Definition 2.2). It is then an interesting question as to whether or not the property of being stable is transferred to the bifurcating branch. Closely related to this is the phenomena of symmetry breaking. In nature, the most stable equilibria are typically those possessing the greatest possible degree of symmetry compatible with the boundary configuration. When bifurcation takes place with a loss of stability, it is common for the bifurcating branch to possess less symmetry than the surfaces of the original family.

Koiso's stability/instability criteria are applied to the bifurcating branches obtained from the Crandall and Rabinowitz theory, obtaining conditions for pitchfork or transcritical bifurcation:

Corollary 1.3. *Assume the hypotheses of Theorem* 1.2 *and assume additionally that* $\lambda_2 = 0$ *holds* (*equivalently,* $\tilde{\lambda}_1 = 0$ *holds*). *Denote by* $\{Y(s)\}_{s \in \hat{I}}$ *the bifurcating branch of fixed boundary CMC surfaces given in Theorem* 1.2. *Assume also that* $H'(0) >$

0, $V'(0) > 0$ and $\widetilde{\lambda}_1'(0) > 0$ hold. Then, there exist positive constants $t_0 \in (0, \epsilon)$, $s_0 \in \hat{I}$ such that the following statements hold.

(I) *X_t is stable for all $t \in [0, t_0]$ and unstable for all $t \in [-t_0, 0)$.*

(II) *If $\hat{V}'(0) \neq 0$, then we have transcritical bifurcation for the branch $Y(s)$ (see the right hand picture in Figure 1). More precisely,*

- *if $\hat{V}'(0) > 0$, then $Y(s)$ is stable for $s \in [-s_0, 0]$ and unstable for $s \in (0, s_0]$;*
- *if $\hat{V}'(0) < 0$, then $Y(s)$ is stable for $s \in [0, s_0]$ and unstable for $s \in [-s_0, 0)$.*

(III) *Assume that $\hat{V}'(0) = 0$. Assume also that $\hat{V}$ is twice differentiable at $s = 0$. If $\hat{V}''(0) < 0$ holds, then $Y(s)$ is stable for all $s \in [-s_0, s_0]$ (supercritical pitchfork bifurcation. See the left-hand picture in Figure 1). If $\hat{V}''(0) > 0$, then $Y(s)$ is unstable for all $s \in [-s_0, 0) \bigcup (0, s_0]$ (subcritical pitchfork bifurcation. See the center picture in Figure 1).*

Similar conclusions hold in the case when $H'(0) < 0$ and $V'(0) < 0$, by reversing the parameterization of X_t, $t \mapsto -t$.

This result is derived from a more general result Theorem 6.2, where weaker assumptions on $H'(0)V'(0)$, $\hat{V}'$ are posed.

As to the bifurcation branch given in Theorem 1.1, we obtain conditions on the eigenvalues of the Jacobi operator for pitchfork or transcritical bifurcation (Theorem 6.4).

In order to illustrate our stability/instability criteria for bifurcating branches of CMC surfaces, in Section 7, we work out an explicit example obtained by considering a family of fixed boundary unduloids in $\mathbb{R}^3$ that are perturbation of a critical cylinder, and that are symmetric with respect to a horizontal plane (i.e., orthogonal to the axis of symmetry). In this case, one obtains a bifurcating branch of CMC surfaces, that consists again of unduloids, but which are not symmetric with respect to the horizontal plane. This gives an example of subcritical pitchfork bifurcation, with a partial break of symmetry.

Here we further give a remark on the symmetry of CMC surfaces in the bifurcation branch given in Theorems 1.1, 1.2. Denote by "dot" the differentiation with respect to t. In Theorem 1.1, the variation vector field of X_t at $t = 0$ is $(\dot{\varphi}(0))\nu$, and the variation vector field of $Y(s)$ is given by $(t'(0)\dot{\varphi}(0) + e)\nu$, and $\int_\Sigma e \, d\Sigma = 0$. In Theorem 1.2, the same formulas hold by exchanging t for τ. For the example given in Section 7, the original CMC surfaces X_t (and so also $(\dot{\varphi}(0))\nu$) are axially symmetric and they have a reflectional symmetry with respect to a plane Π orthogonal to the axis, while for the surfaces $Y(s)$ in the bifurcation branch this reflectional symmetry is broken. In fact, for these examples, $t'(0) = 0$ and $e\nu$ does not have the reflectional symmetry with respect to Π.

For the sake of clarity and brevity, we have limited ourselves in this paper to the study of two dimensional constant mean curvature surfaces satisfying Dirichlet boundary conditions in the three dimensional Euclidean space. However, the tools we employ are functional analytic in nature and with some modifications the results could be extended to a plethora of more general situations; higher dimensions, more general boundary conditions, more general ambient spaces, other functionals besides area, e.g. anisotropic surface energy with volume constraint (cf. Arroyo–Koiso–Palmer[**1**]). We hope to investigate some of these applications in the future.

This paper is organized as follows. Section 2 gives preliminary results on the nonexistence of bifurcation and stability/instability criteria. In Section 3 and in Section 4, Theorem 1.1 and Theorem 1.2 are proved respectively. Some results about estimates for the eigenvalues of the Jacobi operators of CMC surfaces in the bifurcation branches obtained from Theorems 1.1 and 1.2 are given in Section 5, and they are used in Section 6. In Section 6, two sets of criteria are given for the stability/instability of bifurcating branches of CMC surfaces. The first of these, Theorem 6.2 applies to the type of bifurcation given by Theorem 1.2, while the second Theorem 6.4, applies to the bifurcations provided by Theorem 1.1. In particular, we deduce from Theorem 6.2 that, from a stable CMC surface with nullity one, only three types of bifurcations can occur (see also Corollary 1.3): a supercritical pitchfork bifurcation, a subcritical pitchfork bifurcation, and a transcritical bifurcation. Also Corollary 1.3 is proved in Section 6. An explicit example that gives a bifurcation from the critical cylinder is described in Section 7. For the reader's convenience, the Crandall–Rabinowitz theory is summarized in Appendix A. Finally details of the proofs of some of the statements in Section 7 are given in Appendix B.

2. Perturbation of fixed boundary CMC immersions. Stability/instability criteria.

2.1. Preliminaries.

Let Σ be a two-dimensional, oriented, compact, connected smooth manifold with smooth boundary $\partial\Sigma$, and assume that $X\colon \Sigma \to \mathbb{R}^3$ is a smooth immersion having constant mean curvature equal to H_0. Denote by $\nu : \Sigma \to S^2 \subset \mathbb{R}^3$ the Gauss map of X. Define the *volume* of X as the volume of the cone over X which is explicitly given by $(1/3)\int_\Sigma \langle X, \nu\rangle\, d\Sigma$, where $\langle\ ,\ \rangle$ stands for the canonical inner product in $\mathbb{R}^3$, and $d\Sigma$ is the area element of the induced metric on Σ. Recall that the mean curvature of an immersion $X\colon \Sigma \to \mathbb{R}^3$ is constant if and only if X is a critical point of the area functional for all volume-preserving variations of X that fix the boundary values.

In order to satisfy appropriate Fredholm assumptions required in our theory, we will consider variations of X of class $C^{2+\alpha}$. For a volume-preserving, boundary preserving variation X_t of X, the second variation of the area functional A is given by (cf. [**2**]):

$$\delta^2 A = -\int_\Sigma \varphi L[\varphi]\, d\Sigma =: I(\varphi), \quad \varphi := \left\langle \left.\frac{\partial X_t}{\partial t}\right|_{t=0},\ \nu \right\rangle, \tag{1}$$

where L is the self-adjoint operator associated with $\delta^2 A$ (Jacobi operator) which is defined as follows:

$$L[\varphi] = 2\,\delta H = \Delta\varphi + \|d\nu\|^2\varphi, \tag{2}$$

where δH is the first variation of the mean curvature, and Δ is the Laplacian [1] on Σ

[1]For the Euclidean metric $ds^2 = \sum_{i,j}\delta_{ij}du^i du^j$, $\Delta\varphi = \varphi_{u^1u^1} + \varphi_{u^2u^2}$.

with the metric induced by X. Define:

$$E := \mathrm{Ker}(L) = \big\{ e \in C_0^{2+\alpha}(\Sigma) \;:\; L[e] = 0 \big\};$$

we will consider on $C_0^{2+\alpha}(\Sigma)$ the L^2-inner product $\langle \psi_1, \psi_2 \rangle_{L^2} = \int_\Sigma \psi_1 \psi_2 \, d\Sigma$, and we will denote by $E^\perp$ the orthogonal space of E. Observe that, by standard elliptic theory, every $e \in \mathrm{Ker}(L)$ is smooth.

Using an implicit function theorem, it is easy to see that (see [**8**, Theorem 1.1]), when $E = \{0\}$, X admits smooth deformations of the form $X_t = X + \phi(t) \cdot \nu$, for some smooth 1-parameter family $(-\varepsilon, \varepsilon) \ni t \mapsto \phi(t) \in C_0^{2+\alpha}(\Sigma)$, with $\phi(0) = 0$, such that the mean curvature of X_t is constant and equal to $H_0 + t$ for all t. Moreover, given any CMC immersion $Y : \Sigma \to \mathbb{R}^3$, with $Y\big|_{\partial\Sigma} = X\big|_{\partial\Sigma}$, sufficiently close to X, Y must be equal to some X_t, up to parameterization. We will say that two immersions $X : \Sigma \to \mathbb{R}^3$ and $X' : \Sigma \to \mathbb{R}^3$ are *equal up to parameterization* if there exists a diffeomorphism $F : \Sigma \to \Sigma$ such that $X' = X \circ F$. We will call two CMC immersions *different* if they are not equal up to parameterization.

A similar, though weaker, result on the existence of CMC perturbations of X was obtained by the first author under the assumption that $\dim(E) = 1$.

THEOREM 2.1 (Existence and uniqueness of CMC deformation [**8**, Theorem 1.2]). *Let $X : \Sigma \to \mathbb{R}^3$ be an immersion with constant mean curvature H_0, and assume the following*:

(i) $\dim E = 1$;

(ii) $\int_\Sigma e \, d\Sigma \neq 0$ *for some (hence, for all) $e \in E \setminus \{0\}$.*

Fix $e_0 \in E \setminus \{0\}$. Then, there exists $\varepsilon > 0$, and a map of class C^1:

$$(\xi, \eta) : (-\varepsilon, \varepsilon) \longrightarrow E^\perp \times \mathbb{R}$$

such that, $(\xi(0), \eta(0)) = (0, H_0)$, and, for all $t \in (-\varepsilon, \varepsilon)$, the map $X_t := X + \big(te + \xi(t)\big)\nu : \Sigma \to \mathbb{R}^3$ is a $C^{2+\alpha}$-immersion having constant mean curvature equal to $\eta(t)$, and with $X_t\big|_{\partial\Sigma} = X\big|_{\partial\Sigma}$.

Moreover, if $Y : \Sigma \to \mathbb{R}^3$ is a CMC immersion with $Y\big|_{\partial\Sigma} = X\big|_{\partial\Sigma}$ which is sufficiently $C^{2+\alpha}$-close to X, then Y must be equal to some X_t, up to parameterization.

In this paper, we will be interested in determining CMC immersions that *bifurcate* from a given one-parameter family $(X_t)_{t \in (-\varepsilon, \varepsilon)}$ and satisfy a fixed boundary condition. The uniqueness statements in [**8**, Theorem 1.1] and in Theorem 2.1 imply that when either $E = \{0\}$ or $\dim(E) = 1$ and $\int_\Sigma e \, d\Sigma \neq 0$ for some $e \in E \setminus \{0\}$, then no bifurcation from the family $(X_t)_{t \in (-\varepsilon, \varepsilon)}$ can occur at $t = 0$.

Consider the eigenvalue problem:[2]

[2] We denote by $H_0^1(\Sigma)$ the completion of $C_0^\infty(\Sigma)$ with respect to the norm defined by the inner product

$$(u, v)_{H^1} = \int_\Sigma \nabla u \nabla v \, d\Sigma,$$

where $\nabla u \nabla v$ denotes the inner product of the gradient of u and that of v with respect to the Riemannian metric of Σ induced by X.

$$L[\varphi] = -\lambda\varphi, \quad \varphi|_{\partial\Sigma} = 0, \qquad \varphi \in H_0^1(\Sigma). \tag{3}$$

Denote by λ_n, $n \geq 1$, the n-th eigenvalues of (3).

Theorem 2.1 applies, in particular, when $\lambda_1 = 0$, in which case E is the λ_1-eigenspace of L. In this case, it is well-known that the multiplicity of λ_1 is one, and that any λ_1-eigenfunction does not change sign. In particular, assumption (ii) in Theorem 2.1 is satisfied. Therefore, bifurcation may occur for the family $(X_t)_t$ only in the case where $\lambda_k = 0$ for some $k \geq 2$.

2.2. Stability under volume preserving variations.

Since a CMC immersion is a critical point of the area functional for volume-preserving variations that fix the boundary, the following definition is a natural one.

Definition 2.2. *A CMC immersion $X : \Sigma \to \mathbb{R}^3$ is said to be* stable *if the second variation of the area functional is nonnegative for all volume-preserving variations that fix the boundary. X is said to be unstable if it is not stable.*

Assume that $X : \Sigma \to \mathbb{R}^3$ is CMC. Consider the quadratic form (1) on $C_0^{2+\alpha}(\Sigma)$. It is not hard to show that it admits a continuous extension to the Hilbert space $H_0^1(\Sigma)$. We will consider the closed subspace:

$$F_0 := \left\{ \varphi \in H_0^1(\Sigma) : \int_\Sigma \varphi \, d\Sigma = 0 \right\}. \tag{4}$$

Then:

Fact 2.3 ([**2**]). A CMC immersion X is stable if and only if the quadratic form I defined in (1) is positive semi-definite on the space F_0.

The restriction of the quadratic form (1) to $F_0 \cap H^2(\Sigma)$ is represented by a self-adjoint operator $\widetilde{L}$:

$$\tilde{L}[\varphi] := L[\varphi] - \left(\int_\Sigma d\Sigma \right)^{-1} \int_\Sigma L[\varphi] d\Sigma. \tag{5}$$

The operator $\widetilde{L}$ is a compact (finite-rank) perturbation of L; observe that

$$\varphi \longmapsto |\Sigma|^{-1} \int_\Sigma (L\varphi) \, d\Sigma$$

is a rank one operator. Thus, $\widetilde{L}$ admits an unbounded sequence of eigenvalues, $\widetilde{\lambda}_1 \leq \widetilde{\lambda}_2 \leq \widetilde{\lambda}_3 \leq \ldots$, each of which has finite multiplicity, with corresponding eigenfunctions:

$$\widetilde{L}[\varphi] = -\widetilde{\lambda}\varphi, \qquad \varphi \in F_0 \setminus \{0\}, \tag{6}$$

that form an orthonormal basis of F_0. By elliptic regularity, every eigenfunction of $\widetilde{L}$ is of class C^2.

Note that the equation in (6) is equivalent to

$$\int_{\Sigma}(L[\varphi]+\widetilde{\lambda}\varphi)u\,d\Sigma=0,\quad \forall u\in F_0, \tag{7}$$

i.e.,

$$L[\varphi]+\widetilde{\lambda}\varphi=\text{const}.$$

From Fact 2.3, we have the following:

Lemma 2.4. *X is stable if and only if $\widetilde{\lambda}_1 \geq 0$.* □

The computation of the eigenvalues $\widetilde{\lambda}_k$ is rather difficult, and in many situations Lemma 2.4 is not used to establish the stability or instability of a CMC immersion.

2.3. Two stability criteria.

We recall two stability criteria for CMC surfaces, proved in [**8**], that use only the eigenvalues of the problem (3) and some information on the corresponding eigenfunctions and solutions of an associated inhomogeneous elliptic partial differential equation. The first one is based on 1-parameter families of deformations. For a one-parameter family $\{X_t:\Sigma\to\mathbb{R}^3\}$ of CMC immersions, denote by $H(t)$ and $V(t)$, the mean curvature and the volume of X_t, respectively. Also denote by Δ_t, ν_t the Laplacian, the Gauss map of X_t, respectively, and let L_t be the Jacobi operator for X_t, given by $L_t[\varphi]:=\Delta_t\varphi+\|d\nu_t\|^2\varphi$.

Theorem 2.5 (First stability criterion [**8**, Corollary 1.1]). *Let X be a CMC immersion.*

(I) *If $\lambda_1\geq 0$, then X is stable.*

(II) *Assume $\lambda_1<0\leq\lambda_2$. If there is a deformation X_t of X such that $H'(0)=$ constant $\neq 0$, then the following statements hold.*

  (i) *If $H'(0)V'(0)\geq 0$, then X is stable.*

  (ii) *If $H'(0)V'(0)<0$, then X is unstable.*

  If there is no such deformation, then X is unstable.

(III) *If $\lambda_2<0$, then X is unstable.*

Let us now recall a second stability criterion for CMC immersions, which uses eigenfunctions of the operator L in $H_0^1(\Sigma)$.

Theorem 2.6 (Second stability criterion [**8**, Theorem 1.3]). *Let $X:\Sigma\to\mathbb{R}^3$ be a CMC immersion.*

(I) *If $\lambda_1\geq 0$, then X is stable.*

(II) *If $\lambda_1<0<\lambda_2$, then there exists a uniquely determined function $u\in C_0^{2+\alpha}(\Sigma)$ (smooth, in fact) which satisfies $Lu=1$, and the following statements hold.*

  (II-1) *If $\int_{\Sigma}u\,d\Sigma\geq 0$, then X is stable.*

  (II-2) *If $\int_{\Sigma}u\,d\Sigma<0$, then X is unstable.*

(III) *If* $\lambda_1 < 0 = \lambda_2$*, then the following statements hold*:

(III-A) *If there exists a* λ_2*-eigenfunction* e *which satisfies* $\int_\Sigma e\ d\Sigma \neq 0$*, then* X *is unstable.*

(III-B) *If* $\int_\Sigma e\ d\Sigma = 0$ *for any* λ_2*-eigenfunction* e *, then there exists a uniquely determined function* $u \in E^\perp$ *which satisfies* $Lu = 1$*, and the following statements hold*:

(III-B1) *If* $\int_\Sigma u\ d\Sigma \geq 0$*, then* X *is stable.*

(III-B2) *If* $\int_\Sigma u\ d\Sigma < 0$*, then* X *is unstable.*

(IV) *If* $\lambda_2 < 0$*, then* X *is unstable.*

It is worth observing that Theorem 2.6 and Theorem 2.5 are equivalent statements, see [**8**] for details.

3. Proof of Theorem 1.1.

First, let us observe that:

$$0 \neq 2H'(0) = L[\varphi'(0)].$$

Hence,

$$2H'(0)\int_\Sigma e\ d\Sigma = \int_\Sigma eL[\varphi'(0)]\ d\Sigma = \int_\Sigma \varphi'(0)L[e]\ d\Sigma = 0.$$

Therefore,

$$\int_\Sigma e\ d\Sigma = 0$$

holds. Now we will apply Lemma A.3; set

$$Y := C_0^{2+\alpha}(\Sigma), \quad Z := C^\alpha(\Sigma), \quad W = E^\perp$$

and let $\iota : Y \to Z$ be the inclusion map. Define a mapping $L : Y \to Z$ by

$$L[u] = \Delta u + \|d\nu\|^2 u.$$

Now

$$R(L) := \text{Image of } L = \{Lu\ :\ u \in Y\} \subset Z,$$

and by the Fredholm Alternative

$$v \in R(L) \Longleftrightarrow v = Lu \text{ for some } u \in Y \ \Longleftrightarrow \int_\Sigma ve\ d\Sigma = 0 \ \Longleftrightarrow v \in E^\perp.$$

Hence,

$$R(L) = E^{\perp},$$

therefore

$$\operatorname{codim}\big(R(L)\big) = 1,$$

and

$$\iota e = e \notin R(L).$$

Hence 0 is an ι-simple eigenvalue of L in the sense of [**5**] (see Definition A.2 below). Therefore, by Lemma A.3, there exists a differentiable map $(-\varepsilon_0, \varepsilon_0) \ni t \mapsto \lambda(t)$, with $\lambda(0) = 0$, such that $\lambda(t)$ is an ι-simple eigenvalue of L_t, and there are no other eigenvalues of L_t near 0. The first part of Theorem 1.1 has been proved.

Denote by L_t the corresponding Jacobi operator for X_t, in particular, $L = L_0$. Using the eigenvalues $\lambda(t)$, we can write:

$$L_t(e(t)) = -\lambda(t)e(t), \quad \text{for some } e(t) \in C_0^{2+\alpha}(\Sigma) \setminus \{0\}. \tag{8}$$

The bifurcation result is obtained as an application of Theorem A.1. There exists a neighborhood U of 0 in $C^{2+\alpha}(\Sigma)$, such that $X + u\nu$ is an immersion for any $u \in U$. Also, we note that, in a neighborhood of X, all immersions $Y : \Sigma \to \mathbb{R}^3$ satisfying $Y|_{\partial\Sigma} = X|_{\partial\Sigma}$ are represented as $Y = X + u\nu$, $u \in C_0^{2+\alpha}(\Sigma)$. For $u \in U$, we denote by H_u the mean curvature of $X + u\nu$. In our case,

$$H_{\varphi(t)} = H(t), \qquad \forall t \in I.$$

Also, we denote by A_u, V_u the area and the volume of $X + u\nu$, respectively. We define a mapping $F : I \times C_0^{2+\alpha}(\Sigma) \to C_0^{\alpha}(\Sigma)^*$ as follows.[3]

$$F(t, \psi) = D_\psi(A_{\varphi(t)+\psi} + 2H(t)V_{\varphi(t)+\psi}).$$

F is a twice continuously Fréchet differentiable mapping. If we denote the Gauss map of $X + u\nu$ by ν_u, and by $d\Sigma_u$ the area element of Σ induced by of $X + u\nu$, then

$$F(t, \psi)u = -2\int_\Sigma (H_{\varphi(t)+\psi} - H(t))u\langle \nu, \nu_{\varphi(t)+\psi}\rangle \, d\Sigma_{\varphi(t)+\psi}.$$

Therefore,

$$F(t, 0) = 0, \qquad \forall t \in I$$

holds. Moreover, $H_{\varphi(t)+\psi}$ is identically equal to the constant $H(t)$ if $F(t, \psi) = 0$.

Conversely assume $H_u =$ constant$(=: c)$ and $|c - H_0|$ is sufficiently small. Then, $c = H(t_c)$ for some $t_c \in I$. Set $\psi = u - \varphi(t_c)$. Then, $H_{\varphi(t_c)+\psi}$ is constant $H(t_c)$, and $F(t_c, \psi) = 0$. Therefore, in a neighborhood of X, any CMC immersion with the same boundary values of X is obtained as a solution of $F(t, \psi) = 0$.

[3] $C_0^{\alpha}(\Sigma)^*$ denotes the dual space of $C_0^{\alpha}(\Sigma)$.

Now, the map $D_\psi F(0,0) : C_0^{2+\alpha}(\Sigma) \longrightarrow C_0^{2+\alpha}(\Sigma)^*$, is given by:

$$D_\psi F(0,0)(v)(u) = -\int_\Sigma uL[v]\, d\Sigma = -\int_\Sigma vL[u]\, d\Sigma, \tag{9}$$

where

$$L[u] = \Delta u + \|d\nu\|^2 u,$$

hence, we have

$$\operatorname{Ker}(D_\psi F(0,0)) = E,$$

so by assumption,

$$\dim(\operatorname{Ker}(D_\psi F(0,0))) = 1.$$

On the other hand, $\Phi \in \operatorname{Image}(D_\psi F(0,0)) \subset C_0^{2+\alpha}(\Sigma)^*$ if and only if there exists some $u \in C_0^{2+\alpha}(\Sigma)$ such that

$$\Phi(v) = -\int_\Sigma vL[u]\, d\Sigma, \quad \forall v \in C_0^{2+\alpha}(\Sigma).$$

Therefore, we have a one to one correspondence between $\operatorname{Image}(D_\psi F(0,0))$ and $\{L[u] : u \in C_0^{2+\alpha}(\Sigma)\} = E^\perp$. This means that

$$\operatorname{codim}\big(\operatorname{Image}(D_\psi F(0,0))\big) = 1$$

holds.

Now, in order to apply Theorem A.1, we will show that $D_{t\psi}F(0,0)e \notin R(D_\psi F(0,0))$ holds. We compute

$$D_\psi F(t,0)(v)(u) = -\int_\Sigma u\langle \nu, \nu_{\varphi(t)}\rangle L_t[v\langle \nu, \nu_{\varphi(t)}\rangle]\, d\Sigma_{\varphi(t)},$$

where

$$L_t[\omega] = \Delta_{\varphi(t)}\omega + \|d\nu_{\varphi(t)}\|^2\omega.$$

Hence,

$$D_{t\psi}F(0,0)(v)(u) = -\int_\Sigma u\left(\frac{\partial}{\partial t}\bigg|_{t=0} L_t[v\langle \nu, \nu_{\varphi(t)}\rangle] - 2H(0)\varphi'(0)L[v]\right)\, d\Sigma.$$

Therefore,

$$D_{t\psi}F(0,0)(e)(u) = -\int_\Sigma u\left(\frac{\partial}{\partial t}\bigg|_{t=0} L_t[e\langle \nu, \nu_{\varphi(t)}\rangle]\right)\, d\Sigma. \tag{10}$$

Recall

$$L_t[e(t)] = -\lambda(t)e(t), \quad \exists e(t) \in C_0^{2+\alpha}(\Sigma) \setminus \{0\}, \quad \lambda'(0) \neq 0. \tag{11}$$

We may assume that

$$\int_\Sigma (e(t))^2 \, d\Sigma_{\varphi(t)} = 1$$

holds. Then,

$$\lambda(t) = -\int_\Sigma e(t) L_t[e(t)] \, d\Sigma_{\varphi(t)} = D_\psi F(t,0)(v(t))(v(t)), \tag{12}$$

where

$$v(t) := \frac{e(t)}{\langle \nu, \nu_{\varphi(t)} \rangle}.$$

Assume now that $D_{t\psi}F(0,0)e \in R(D_\psi F(0,0))$ holds. Then, for some $w \in C_0^{2+\alpha}(\Sigma)$:

$$D_\psi F(0,0)w = D_{t\psi}F(0,0)e \tag{13}$$

holds. By using (9), (12), (13), and $v(0) = e$, we obtain

$$\begin{aligned} 0 \neq \lambda'(0) &= \left.\frac{d}{dt}\right|_{t=0} D_\psi F(t,0)(v(t))(v(t)) \\ &= D_{t\psi}F(0,0)(v(0))(v(0)) \\ &\quad + D_\psi F(0,0)(v'(0))(v(0)) + D_\psi F(0,0)(v(0))(v'(0)) \\ &= D_{t\psi}F(0,0)(e)(e) \\ &= -\int_\Sigma eL[w] \, d\Sigma = -\int_\Sigma wL[e] \, d\Sigma = 0, \end{aligned} \tag{14}$$

which is a contradiction. Hence, $D_{t\psi}F(0,0)e \notin R(D_\psi F(0,0))$ holds, and all of the assumptions in Theorem A.1 hold. Consequently, we obtain the existence of unique CMC immersions $Y(s) := X + (\varphi(t(s)) + se + s\zeta(s))\nu$ with mean curvature $H(t(s))$, $(s \in \hat{I})$.

Set

$$\psi_0 := \left.\frac{\partial \varphi(t)}{\partial t}\right|_{t=0}.$$

The variation vector field of X_t at X_0 is $\psi_0\nu$, and the variation vector field of $Y(s)$ at X_0 is $(t'(0)\psi_0 + e)\nu$. Since $L[\psi_0] = \text{constant} \neq 0$ and $L[e] = 0$, ψ_0 and e are linearly independent. Therefore, surfaces $\{X_t \, ; \, t \in I\}$ and $\{Y(s) \, ; \, s \in \hat{I}\}$ are all different except for $X_0 = Y(0)$. The proof of Theorem 1.1 is completed.

4. Proof of Theorem 1.2.

We will employ Theorem A.1 and Lemma A.6. First let us assume that $V'(0) \neq 0$ holds.

Set $V_0 := V(0)$, $b := V'(0) = \int_\Sigma \varphi'(0)\, d\Sigma \neq 0$. From now on, we denote by $A(\varphi)$ and $V(\varphi)$ the area and the volume of $X + \varphi\nu$, respectively.

Set

$$\ell := D_\varphi V(0) : C_0^{2+\alpha}(\Sigma) \longrightarrow \mathbb{R}.$$

Then,

$$\ell(\psi) = D_\varphi V(0)\psi = \int_\Sigma \psi \, d\Sigma, \quad \psi \in C_0^{2+\alpha}(\Sigma). \tag{15}$$

Set $\psi_0 := b^{-1}\varphi'(0)$. Then there holds,

$$\ell(\psi_0) = 1. \tag{16}$$

Also, set

$$B := C_0^{2+\alpha}(\Sigma), \quad B_0 := \big\{h \in B \ : \ \ell(h) = 0\big\}.$$

Then

$$B_0 = \left\{ h \in B \ : \ \int_\Sigma h \, d\Sigma = 0 \right\}.$$

For any $\varphi \in B$, φ can be represented uniquely as

$$\varphi = \ell(\varphi)\psi_0 + h, \quad h \in B_0.$$

In fact, if we set $h := \varphi - \ell(\varphi)\psi_0$, then $\ell(h) = \int_\Sigma \varphi \, d\Sigma - \big(\int_\Sigma \varphi \, d\Sigma\big)\, \ell(\psi_0) = 0$.

Now, define a mapping $P : B \to \mathbb{R} \times B_0$ as follows.

$$P(\varphi) := (P_1(\varphi), P_2(\varphi)) := (V(\varphi), h) = (V(\varphi), \varphi - \ell(\varphi)\psi_0). \tag{17}$$

Then, $P(0) = (V_0, 0)$, and P is nonsingular at 0. In fact, since

$$\begin{aligned} D_\varphi P(\varphi)w &= \left(\int_\Sigma w\langle \nu, \nu_\varphi \rangle \, d\Sigma_\varphi, w - \ell(w)\psi_0 \right), \\ D_\varphi P(0)w &= (\ell(w), w - \ell(w)\psi_0). \end{aligned} \tag{18}$$

$D_\varphi P(0)w = 0$ if and only if $w = 0$. Hence, P is invertible in a neighborhood of 0. Set $Q := P^{-1}$. Then for a neighborhood U of 0 in B_0 and a neighborhood J of 0 in $\mathbb{R}$,

$$Q = P^{-1} : J \times U \longrightarrow B.$$

Set $\widetilde{A} = A \circ Q$, $\widetilde{V} = V \circ Q$. Then,

$$\widetilde{A}, \widetilde{V} : J \times U \longrightarrow \mathbb{R}, \quad 0 \in U \subset B_0,\ 0 \in J \subset \mathbb{R}.$$

Set

$$P(\varphi(t)) =: (V(\varphi(t)), \varphi_0(t)).$$

Then,

$$\varphi_0(t) = \varphi(t) - \ell(\varphi(t))\psi_0 = \varphi(t) - b^{-1}\varphi'(0) \int_\Sigma \varphi(t)\, d\Sigma. \tag{19}$$

Hence,

$$\varphi_0(0) = 0, \quad \varphi_0'(0) = 0. \tag{20}$$

Denote by B_0^* the dual space of B_0. Define a mapping $F : J \times U_0 \to B_0^*$ as below, where U_0 is a neighborhood of 0 in B_0:

$$F(t,h) := D_h \widetilde{A}(V(\varphi(t)), \varphi_0(t) + h).$$

Note that, in a neighborhood Ω of X in $C^{2+\alpha}(\Sigma, \mathbb{R}^3)$, any immersion $Y : \Sigma \to \mathbb{R}^3$ satisfying $Y|_{\partial\Sigma} = X|_{\partial\Sigma}$ and having volume ω is represented as

$$Y = X + Q(\omega, h)\nu, \quad \exists h \in B_0.$$

Lemma 4.1. *In a neighborhood Ω of X, $Y \in \Omega$ is CMC with volume ω if and only if $Y = X + Q(V(\varphi(t)), \varphi_0(t) + h)\nu$, $(V(\varphi(t)) = \omega,\ \exists h \in B_0)$, and $F(t,h) = 0$. This is equivalent to $Y = X + \varphi\nu$, $(\varphi \in P^{-1}(F^{-1}(0) + (0, \varphi_0(t))) \subset B)$.*

In order to prove Lemma 4.1, we need some preparation.
Set $g(t,h) := Q(V(\varphi(t)), \varphi_0(t) + h)$. Then, $F(t,h) = D_h A(g(t,h))$. Note that

$$g(0,0) = Q(V_0, \varphi_0(0)) = \varphi(0) = 0.$$

We have

$$\begin{aligned} &F(t,h)k \\ &\quad = \lim_{\sigma\to 0} \frac{A(g(t,h+\sigma k)) - A(g(t,h))}{\sigma} \\ &\quad = -2\int_\Sigma H(g(t,h)) \frac{\partial g(t,h+\sigma k)}{\partial \sigma}\bigg|_{\sigma=0} \langle \nu, \nu_{g(t,h)}\rangle \, d\Sigma_{g(t,h)} \\ &\quad = -2\int_\Sigma H(g(t,h))(D_h Q(V(\varphi(t)), \varphi_0(t) + h)k)\langle \nu, \nu_{g(t,h)}\rangle \, d\Sigma_{g(t,h)}. \end{aligned} \tag{21}$$

Since PQ is the identity of $J \times U$, $(0 \in J \subset \mathbb{R},\ 0 \in U \subset B_0)$,

$$PQ(V(\varphi(t)), \varphi_0(t) + h) = (V(\varphi(t)), \varphi_0(t) + h), \quad 0 \in U \subset B_0,\ t \in J \subset \mathbb{R}. \tag{22}$$

Differentiating (22) with respect h, we get

$$D_\varphi P(g(t,h))(D_h Q(V(\varphi(t)),\varphi_0(t)+h)u) = (0,u), \quad \forall u \in B_0. \tag{23}$$

Set

$$\widetilde{u} := D_h Q(V(\varphi(t)),\varphi_0(t)+h)u.$$

By using (18), we obtain

$$D_\varphi P(g(t,h))\widetilde{u} = \left(\int_\Sigma \widetilde{u}\langle \nu, \nu_{g(t,h)}\rangle \, d\Sigma_{g(t,h)}, \widetilde{u} - \left(\int_\Sigma \widetilde{u}\, d\Sigma\right)\psi_0\right). \tag{24}$$

Hence, for any $u \in B_0$,

$$\int_\Sigma \widetilde{u}\langle \nu, \nu_{g(t,h)}\rangle \, d\Sigma_{g(t,h)} = 0, \quad \widetilde{u} - \left(\int_\Sigma \widetilde{u}\, d\Sigma\right)\psi_0 = u. \tag{25}$$

Therefore, in the special case where $(t,h)=(0,0)$, we have, for any $u \in B_0$,

$$\int_\Sigma D_h Q(V_0,0)u\, d\Sigma = 0, \quad D_h Q(V_0,0)u = u. \tag{26}$$

Next, we differentiate (22) with respect t, obtaining

$$D_\varphi P(g(t,h))(D_t Q(V(\varphi(t)),\varphi_0(t)+h)\tau) = (V(\varphi'(t))\tau, \varphi_0'(t)\tau) \in B_0 \times \mathbb{R}, \quad \forall \tau \in \mathbb{R}, \tag{27}$$

where $V'(\varphi(t)) = (d/dt)V(\varphi(t))$. Hence, using (18), we have, for every $\tau \in \mathbb{R}$

$$\int_\Sigma \hat{\tau}\langle \nu, \nu_{g(t,h)}\rangle \, d\Sigma_{g(t,h)} = V'(\varphi(t))\tau, \quad \hat{\tau} - \left(\int_\Sigma \hat{\tau}\, d\Sigma\right)\psi_0 = \varphi_0'(t)\tau, \tag{28}$$
$$\hat{\tau} := D_t Q(V(\varphi(t)),\varphi_0(t)+h)\tau.$$

Therefore, in the special case where $(t,h)=(0,0)$, by (20) and (28), we have, for any $\tau \in \mathbb{R}$,

$$D_t Q(V_0,0)\tau = b\tau\psi_0. \tag{29}$$

Since Q is nonsingular,

$$D_{(t,h)}Q(V(\varphi(t)),\varphi_0(t)+h) : B_0 \times \mathbb{R} \to B$$

is an isomorphism. Note

$$\begin{aligned} &D_{(t,h)}Q(V(\varphi(t)),\varphi_0(t)+h)(\tau,u) \\ &\quad = D_t Q(V(\varphi(t)),\varphi_0(t)+h)(\tau) + D_h Q(V(\varphi(t)),\varphi_0(t)+h)(u). \end{aligned}$$

Hence, for any $w \in B$, there exists a unique $(\tau,u) \in \mathbb{R} \times B_0$ such that

$$D_t Q(V(\varphi(t)),\varphi_0(t)+h)(\tau) + D_h Q(V(\varphi(t)),\varphi_0(t)+h)(u) = \frac{w}{\langle \nu, \nu_{g(t,h)}\rangle}. \tag{30}$$

If $\int_\Sigma w \, d\Sigma_{g(t,h)} = 0$, we have, for (τ, u) in (30),

$$\begin{aligned} 0 &= \int_\Sigma w \, d\Sigma_{g(t,h)} \\ &= \int_\Sigma \hat{\tau} \langle \nu, \nu_{g(t,h)} \rangle \, d\Sigma_{g(t,h)} + \int_\Sigma \widetilde{u} \langle \nu, \nu_{g(t,h)} \rangle \, d\Sigma_{g(t,h)} \\ &= V'(\varphi(t))\tau, \end{aligned}$$

here we used the first equalities of (25) and (28). Since $V'(\varphi(t))|_{t=0} = b \neq 0$, $V'(\varphi(t)) \neq 0$ when $|t|$ is small. Hence $\tau = 0$. Therefore, we have obtained the following result.

Lemma 4.2. *Let $w \in B$. The integral $\int_\Sigma w \, d\Sigma_{g(t,h)}$ vanishes if and only if there exists some $w_0 \in B_0$ such that*

$$w = (D_h Q(V(\varphi(t)), \varphi_0(t) + h) w_0) \langle \nu, \nu_{g(t,h)} \rangle \tag{31}$$

holds. Moreover, w_0 is uniquely determined by w. Conversely, if $w \in B_0$, then w which is defined by (31) satisfies $\int_\Sigma w \, d\Sigma_{g(t,h)} = 0$.

Proof of Lemma 4.1. The mean curvature $H(g(t,h))$ of

$$Y = X + Q\big(V(\varphi(t)), \varphi_0(t) + h\big)\nu$$

is constant if and only if

$$\int_\Sigma H(g(t,h)) w \, d\Sigma_{g(t,h)} = 0$$

holds for all $w \in B$ which satisfies $\int_\Sigma w \, d\Sigma_{g(t,h)} = 0$. Hence, from (21) and Lemma 4.2, we obtain the lemma. $\square$

Now, we will apply Theorem A.1 and Lemma A.6 to $Y := B_0$, $Z := B_0^*$, $F : J \times U_0 \to B_0^*$.

By Lemma 4.1, $F(t, 0) = 0$ holds for all $t \in J$. Next we will show that $\dim\big(N(D_h F(0,0))\big) = 1$. Since $V(g(t, h + \sigma k)) = V(\varphi(t))$ is independent of σ,

$$\begin{aligned} &\int_\Sigma (D_h Q(V(\varphi(t)), \varphi_0(t) + h) k) \langle \nu, \nu_{g(t,h)} \rangle \, d\Sigma_{g(t,h)} \\ &\quad = \left. \frac{dV(g(t, h + \sigma k))}{d\sigma} \right|_{\sigma = 0} = 0. \end{aligned}$$

Hence, from (21), and since $H(g(t,0)) \equiv$ constant on Σ, we have

$$\begin{aligned} F(t,h)k = &- 2 \int_\Sigma \Big(H(g(t,h)) - H(g(t,0)) \Big) \\ &\times (D_h Q(V(\varphi(t)), \varphi_0(t) + h) k) \langle \nu, \nu_{g(t,h)} \rangle \, d\Sigma_{g(t,h)}. \end{aligned}$$

Therefore, for any $u, v \in B_0$,

$$
\begin{aligned}
&D_hF(t,0)(v)(u)\\
&\quad = -2\int_\Sigma \left(\lim_{\sigma\to 0}\frac{H(g(t,\sigma v)) - H(g(t,0))}{\sigma}\right)\\
&\qquad\qquad \times (D_hQ(V(\varphi(t)),\varphi_0(t))u)\langle \nu, \nu_{g(t,0)}\rangle \, d\Sigma_{g(t,0)}\\
&\quad = -\int_\Sigma L_{\varphi(t)}[(D_hQ(V(\varphi(t)),\varphi_0(t))v)\langle \nu, \nu_{\varphi(t)}\rangle]\\
&\qquad\qquad \times (D_hQ(V(\varphi(t)),\varphi_0(t))u)\langle \nu, \nu_{\varphi(t)}\rangle d\Sigma_{\varphi(t)},
\end{aligned} \tag{32}
$$

here we used $g(t,0) = \varphi(t)$. Hence,

$$
\begin{aligned}
D_hF(0,0)(v)(u) &= -\int_\Sigma L[D_hQ(V_0,0)v](D_hQ(V_0,0)u)\, d\Sigma\\
&= -\int_\Sigma uL[v]\, d\Sigma,
\end{aligned} \tag{33}
$$

here we used (26). Therefore, $v \in N(D_hF(0,0)) \subset B_0$ if and only if

$$
-\int_\Sigma uL[v]\, d\Sigma = 0
$$

for all $u \in B_0$, which means that $L[v] \equiv \text{constant}$ on Σ. Now we need the following lemma.

Lemma 4.3. *In the notation above, the following statements hold.*

(i) *If $v \in B$ satisfies $L[v] \equiv \text{constant}$ on Σ, then v can be represented as $v = \alpha e + \beta\varphi'(0)$, $(\alpha,\beta \in \mathbb{R})$. Moreover, if $v \in B_0$, then $v = \alpha e$, $(\alpha \in \mathbb{R})$.*

(ii) *$v = \alpha e + \beta\varphi'(0)$, $(\alpha,\beta \in \mathbb{R})$, satisfies $v \in B$ and $L[v] \equiv \text{constant}$ on Σ.*

Proof. Note that $L[\varphi'(0)] = 2H'(0) =: \gamma \neq 0$.

(i) If $L[v] \equiv \text{constant} = c$, then $L[v - (c/\gamma)\varphi'(0)] = 0$. Therefore, $v - (c/\gamma)\varphi'(0) = \alpha e$, $(\exists \alpha \in \mathbb{R})$. The second statement is obvious.

(ii) $L[\alpha e + \beta\varphi'(0)] = \alpha L[e] + \beta L[\varphi'(0)] = \beta\gamma$. □

If $v \in B_0$ satisfies $L[v] \equiv \text{constant}$, by Lemma 4.3, $v = \alpha e + \beta\varphi'(0)$, $(\alpha,\beta \in \mathbb{R})$. Since

$$
\int_\Sigma e\, d\Sigma = 0, \quad \int_\Sigma \varphi'(0)\, d\Sigma = V'(0) \neq 0,
$$

we have $v = \alpha e$. Therefore,

$$
N(D_hF(0,0)) = \{ae \,:\, a \in \mathbb{R}\},
$$

which implies that $\dim\big(N(D_hF(0,0))\big) = 1$.

Next, we will show that $\operatorname{codim}\big(R(D_hF(0,0))\big) = 1$. $\Phi \in R(D_hF(0,0)) \subset B_0^*$ if and only if there exists some $v \in B_0$ such that

$$\Phi(u) = -\int_\Sigma uL[v]\, d\Sigma, \quad \forall u \in B_0.$$

Therefore, by the second statement of Lemma 4.3 (i), we have a one to one correspondence between $R(D_hF(0,0))$ and $E^\perp \cap B_0$. This means that

$$\operatorname{codim}\bigl(R(D_hF(0,0))\bigr) = 1.$$

Now we prove the first part of Theorem 1.2. For $u \in B_0$, set

$$\bar{u} := (D_hQ(V(\varphi(t)), \varphi_0(t))u)\langle \nu, \nu_{\varphi(t)}\rangle.$$

Set $T_t := D_hF(t,0) : B_0 \to B_0^*$. Then, from (32),

$$(T_t(v))(u) = -\int_\Sigma \bar{u}L_{\varphi(t)}[\bar{v}]\, d\Sigma_{\varphi(t)}, \quad u, v \in B_0. \tag{34}$$

Also define a linear mapping $K_t : B_0 \to B_0^*$ as follows.

$$(K_t(v))(u) := \int_\Sigma \bar{v}\bar{u}\, d\Sigma_{\varphi(t)}, \quad u, v \in B_0. \tag{35}$$

We will show that $K_0e \notin R(T_0)$. Suppose $K_0e \in R(T_0)$. Then, there exists some $w \in B_0$ such that $K_0e = T_0w$. By the second equality of (26) and (33),

$$\int_\Sigma eu\, d\Sigma = -\int_\Sigma uL[w]\, d\Sigma, \quad \forall u \in B_0.$$

This is equivalent to

$$e + L[w] = c, \quad \exists c \in \mathbb{R}.$$

Hence,

$$\begin{aligned} 0 > -\int_\Sigma e^2\, d\Sigma &= c\int_\Sigma e\, d\Sigma - \int_\Sigma e^2\, d\Sigma = \int_\Sigma e(c-e)\, d\Sigma \\ &= \int_\Sigma eL[w]\, d\Sigma = \int_\Sigma wL[e]\, d\Sigma = 0, \end{aligned}$$

which is a contradiction. This implies that $K_0e \notin R(T_0)$. Therefore, 0 is a K_0-simple eigenvalue of T_0 in the sense of [**5**] (see Definition A.2 below).

Therefore, by Lemma A.6, there exists a C^1-map $(-\varepsilon_0, \varepsilon_0) \ni t \mapsto \widetilde{\lambda}(t) \in \mathbb{R}$ such that $\widetilde{\lambda}(0) = 0$, $\widetilde{\lambda}(t)$ is a K_t-simple eigenvalue of T_t, and there is no other eigenvalue of T_t near 0. Note here that we have not used the assumption $V'(0) \neq 0$. This concludes the proof of the first part of Theorem 1.2.

We now complete the proof of Theorem 1.2. Using the eigenvalue function $\widetilde{\lambda}(t)$, we have:

$$T_t(e(t)) = \widetilde{\lambda}(t) K_t e(t), \quad \exists e(t) \in B_0 \setminus \{0\}, \tag{36}$$

that is

$$-\int_\Sigma \bar{u} L_{\varphi(t)}[\overline{e(t)}]\, d\Sigma_{\varphi(t)} = \widetilde{\lambda}(t) \int_\Sigma \overline{e(t)}\bar{u}\, d\Sigma_{\varphi(t)}, \quad \forall u \in B_0. \tag{37}$$

By (37) and Lemma 4.2, (36) is equivalent to

$$L_{\varphi(t)}[\overline{e(t)}] = -\widetilde{\lambda}(t)\overline{e(t)} + c(t), \tag{38}$$

where $c(t)$ is a constant which depends on t.

In order to obtain the existence of the bifurcation, we will apply Theorem A.1. In order to do this, we need to prove $D_{th}F(0,0)e \notin R(D_hF(0,0))$. We normalize $\overline{e(t)}$ so that

$$\int_\Sigma \overline{e(t)}^2\, d\Sigma_{\varphi(t)} = 1 \tag{39}$$

holds. Assume that $D_{th}F(0,0)e \in R(D_hF(0,0))$. Then, there exists some $v \in B_0$ such that

$$D_{th}F(0,0)e = D_hF(0,0)(v). \tag{40}$$

Note that, by the first equality in (25),

$$\int_\Sigma \overline{e(t)}\, d\Sigma_{\varphi(t)} = 0 \tag{41}$$

holds. From (32), (38), (39), and (41), we have

$$(D_hF(t,0)(e(t)))(e(t)) = -\int_\Sigma \overline{e(t)} L_{\varphi(t)}[\overline{e(t)}]\, d\Sigma_{\varphi(t)} = \widetilde{\lambda}(t). \tag{42}$$

Hence,

$$\begin{aligned}
0 &\neq \widetilde{\lambda}'(0) \\
&= D_{th}F(0,0)(e)(e) + (D_hF(0,0)(e'(0)))(e) + (D_hF(0,0)(e))(e'(0)) \\
&= D_{th}F(0,0)(e)(e) = D_hF(0,0)(v)(e) \\
&= -\int_\Sigma eL[v]\, d\Sigma = -\int_\Sigma vL[e]\, d\Sigma = 0,
\end{aligned}$$

which is a contradiction.

Therefore, $D_{th}F(0,0)e \notin R(D_hF(0,0))$, hence all of the assumptions in Theorem A.1 hold. Consequently, there exists an open interval $\hat{I}$ containing 0 and C^1 functions $\xi : \hat{I} \to E^\perp \cap B_0$ and $\tau : \hat{I} \to \mathbb{R}$, such that $\tau(0) = 0$, $\xi(0) = 0$, and $F(\tau(s), y(s)) = 0$ for $y(s) := se + s\xi(s)$, $(\forall s \in \hat{I})$. That is,

$$Y(s) := X + Q\big(V(\varphi(\tau(s))), \varphi_0(\tau(s)) + se + s\xi(s)\big)\nu$$

is an immersion with constant mean curvature with volume $\hat{V}(s) := V(\varphi(\tau(s)))$. By using (20), the second equality of (26), and (29), we obtain the following information about the variation vector field of $Y(s)$:

$$\begin{aligned}&\left.\frac{\partial Q(V(\varphi(\tau(s))), \varphi_0(\tau(s)) + se + s\xi(s))}{\partial s}\right|_{s=0}\\&\quad = D_t Q(V_0, 0)\tau'(0) + D_h Q(V_0, 0)(\varphi_0'(0)\tau'(0) + e) = b\tau'(0)\psi_0 + e.\end{aligned}$$

Here,

$$b\tau'(0)\psi_0 = b\tau'(0)b^{-1}\varphi'(0) = \left.\frac{d}{ds}\right|_{s=0}\varphi(\tau(s)).$$

Hence, $Y(s)$ can be represented as $Y(s) = X + (\varphi(\tau(s)) + se + s\eta(s))\nu$, for some $\eta(s) \in C_0^{2+\alpha}(\Sigma)$.

Now since

$$Q(V(\varphi(\tau(s))), \varphi_0(\tau(s)) + se + s\xi(s)) = \varphi(\tau(s)) + se + s\eta(s),$$

by using (17), we obtain

$$\varphi(\tau(s)) + se + s\eta(s) - \psi_0 \int_\Sigma (\varphi(\tau(s)) + se + s\eta(s))\, d\Sigma = \varphi_0(\tau(s)) + se + s\xi(s). \tag{43}$$

From (19), we have

$$\varphi_0(\tau(s)) = \varphi(\tau(s)) - \psi_0 \int_\Sigma \varphi(\tau(s))\, d\Sigma. \tag{44}$$

Substituting (44) to (43), we obtain

$$\eta(s) = \psi_0 \int_\Sigma \eta(s)\, d\Sigma + \xi(s) = \varphi'(0)\left(\int_\Sigma \varphi'(0)\, d\Sigma\right)^{-1} \int_\Sigma \eta(s)\, d\Sigma + \xi(s). \tag{45}$$

By the same arguments in the proof of Theorem 1.1, the surfaces $\{X_t\,;\, t \in I\}$ and $\{Y(s)\,;\, s \in \hat{I}\}$ are all different except for $X_0 = Y(0)$. We have proved Theorem 1.2.

Remark 4.4. Some of the ideas in this Section are inspired from the work of Patnaik in [**15**]. The map P defined in (17) is a modification of a similar function employed in [**15**].

5. Eigenvalue estimates.

By applying a general result on bifurcation by Crandall–Rabinowitz [**5**], we obtain the following lemmas.

Lemma 5.1. *Under the assumptions of Theorem* 1.1, *there exist an open interval* $J \subset \hat{I}$, *with* $0 \in J$, *and continuously differentiable functions* $\mu : J \to \mathbb{R}$ *and* $w : J \to C_0^{2+\alpha}(\Sigma)$ *such that*

$$L_{Y(s)}[w(s)] = -\mu(s)w(s), \quad \forall s \in J. \tag{46}$$

Moreover,

$$\big|st'(s)\lambda'(0) + \mu(s)\big| \le o(1)\left(\big|st'(s)\big| + \big|\mu(s)\big|\right) \text{ as } s \to 0, \tag{47}$$

$$\lim_{\substack{s\to 0\\ \mu(s)\neq 0}} \frac{-st'(s)\lambda'(0)}{\mu(s)} = 1, \tag{48}$$

and

$$\big\|y'(s) - w(s)\big\| \le C \min\left\{\big|st'(s)\big|, \big|\mu(s)\big|\right\}, \quad \forall s \in J, \tag{49}$$

hold, where $y(s) := se + s\zeta(s)$, and C is some constant. From (48) *we obtain, in particular, that $\mu(s)$ and the quantity*

$$-st'(s)\lambda'(0) = -s\hat{H}'(s)(H'(t(s)))^{-1}\lambda'(0)$$

have the same zeroes and, where $\mu(s) \neq 0$, the same sign.

Lemma 5.2. *Under the assumptions of Theorem* 1.2, *there exist an open interval $J \subset \hat{I}$ with $0 \in J$ and two continuously differentiable functions $\mu : J \to \mathbb{R}$ and $w : J \to F_0$ such that*

$$L_{Y(s)}[w(s)] = -\mu(s)w(s) + c(s), \quad \forall s \in J, \tag{50}$$

where $c(s)$ is a constant. Moreover,

$$\big|s\tau'(s)\widetilde{\lambda}'(0) + \mu(s)\big| \le o(1)\left(\big|s\tau'(s)\big| + \big|\mu(s)\big|\right) \text{ as } s \to 0, \tag{51}$$

$$\lim_{\substack{s\to 0\\ \mu(s)\neq 0}} \frac{-s\tau'(s)\widetilde{\lambda}'(0)}{\mu(s)} = 1, \tag{52}$$

and

$$\big\|y'(s) - w(s)\big\| \le C \min\left\{\big|s\tau'(s)\big|, \big|\mu(s)\big|\right\}, \quad \forall s \in J, \tag{53}$$

where $y(s) := se + s\eta(s)$, and C is a positive constant. From (52), *we obtain that $\mu(s)$ and $-s\tau'(s)\widetilde{\lambda}'(0) = -s\hat{V}'(s)(V'(\tau(s)))^{-1}\widetilde{\lambda}'(0)$ have the same zeroes and, where $\mu(s) \neq 0$, the same sign.*

Proof of Lemma 5.1. We will use Theorem A.8. In the proof of Theorem 1.1 (we use the same notation here), we proved that the assumptions in Theorem A.1 hold for $Y = C_0^{2+\alpha}(\Sigma)$, $Z = C_0^{\alpha}(\Sigma)^*$, $F : I \times V \to Z$, $F(t,\psi) = D_\psi(A_{\varphi(t)+\psi} + 2H(t)V_{\varphi(t)+\psi})$.

For $(t,\psi) \in I \times V$ and $u \in C_0^{\alpha}(\Sigma)$, set

$$\hat{u}(t,\psi) = u\langle \nu, \nu_{\varphi(t)+\psi}\rangle. \tag{54}$$

Also, define a linear mapping $K(t,\psi) : Y \to Z$ by

$$(K(t,\psi)v)u = \int_\Sigma \hat{v}(t,\psi)\hat{u}(t,\psi)\, d\Sigma_{\varphi(t)+\psi}\,. \tag{55}$$

Then, the assumptions in Corollary A.7 hold. Hence, there exist an open interval $J \subset \hat{I}$ with $0 \in J$ and continuously differentiable functions $\mu : J \to \mathbb{R}$ and $v : J \to Y$ such that

$$D_\psi F(t(s), y(s))v(s) = \mu(s)K(t(s), y(s))v(s), \quad \forall s \in J. \tag{56}$$

Note that

$$F(t,\psi)u = -2\int_\Sigma (H_{\varphi(t)+\psi} - H_{\varphi(t)})u\langle \nu, \nu_{\varphi(t)+\psi}\rangle\, d\Sigma_{\varphi(t)+\psi}, \quad \forall u \in C_0^\alpha(\Sigma) \tag{57}$$

and

$$H_{\varphi(t(s))+y(s)} - H_{\varphi(t(s))} = 0, \quad \forall s \in J \tag{58}$$

hold. Hence, we obtain

$$D_\psi F(t(s), y(s))(v)(u) = -\int_\Sigma L_{Y(s)}\big[\hat{v}\big(t(s), y(s)\big)\big]\hat{u}(t(s), y(s))\, d\Sigma_{Y(s)}, \tag{59}$$

for $v \in Y$, $u \in C_0^\alpha(\Sigma)$. From (55) and (59), (56) is equivalent to

$$L_{Y(s)}[w(s)] = -\mu(s)w(s), \quad w(s) := \hat{u}(s)(t(s), y(s)).$$

The other statements follow from Theorem A.8 and the fact that

$$t'(s) = \hat{H}'(s)(H'(t(s)))^{-1}. \qquad \square$$

Proof of Lemma 5.2. We will use Theorem A.8. In the proof of Theorem 1.2 (we use the same notations here), we proved that the assumptions in Theorem A.1 hold for $Y = B_0$, $Z = B_0^*$, $F : J \times U_0 \to Z$, and $F(t,h) = D_h\widetilde{A}(V(\varphi(t)), \varphi_0(t) + h)$.

For $(t,h) \in J \times U_0$ and $u \in B_0$, set

$$\bar{u}(t,h) = (D_h Q(V(\varphi(t)), \varphi_0(t) + h)u)\langle \nu, \nu_{g(t,h)}\rangle. \tag{60}$$

Also, define a linear mapping $K(t,h) : Y \to Z$ as

$$(K(t,h)v)u = \int_\Sigma \bar{v}(t,h)\bar{u}(t,h)\, d\Sigma_{g(t,h)}. \tag{61}$$

Then, the assumptions in Corollary A.7 hold. Hence, there exist an open interval $J' \subset J$ with $0 \in J'$ and continuously differentiable functions $\mu : J' \to \mathbb{R}$ and $v : J' \to Y$ such that

$$D_h F(\tau(s), y(s))v(s) = \mu(s)K(\tau(s), y(s))v(s), \quad \forall s \in J'. \tag{62}$$

By the same arguments used to prove (32), we obtain

$$D_hF(\tau(s),y(s))(v)(u) = -\int_\Sigma L_{g(\tau(s),y(s))}[\bar{v}(\tau(s),y(s))]\bar{u}(\tau(s),y(s))\,d\Sigma_{Y(s)}, \tag{63}$$

for $u, v \in B_0$. From (61) and (63), (62) is equivalent to

$$L_{Y(s)}[w(s)] = -\mu(s)w(s) + c(s), \quad w(s) := \overline{u(s)}(\tau(s),y(s)),$$

for some constant $c(s)$. The other statements follow from Theorem A.8 and the fact that $\tau'(s) = \hat{V}'(s)(V'(\tau(s)))^{-1}$. $\square$

6. Stability of surfaces in the bifurcation branch.

In view of Theorem 2.5 and Theorem 2.6, in order to study the stability of CMC surfaces in a bifurcation branch, we need to study only the case where $\lambda_2 = 0$. We will consider throughout the setup described in Theorems 1.1, 1.2, using the same notation. In particular, $\hat{I}$ will denote an interval of $\mathbb{R}$ containing 0, $\hat{I} \ni s \mapsto Y(s)$ is the branch of CMC surfaces issuing from the family $t \mapsto X_t$ at $t = 0$, with $X = X_0 = Y(0)$, $V(t)$ and $H(t)$ are respectively the volume and the mean curvature of X_t, while $\hat{V}(s)$ and $\hat{H}(s)$ are the volume and the mean curvature of $Y(s)$.

We will give a stability criterion for surfaces in the bifurcating branch obtained in Theorem 1.2 (Theorem 6.2), and a stability criterion for surfaces in the bifurcating branch obtained in Theorem 1.1 (Theorem 6.4).

We first establish a result on the stability of X_0 when $\lambda_2 = 0$. As a corollary of Theorem 2.5, we obtain easily the following stability result.

Proposition 6.1. *Assume*:

(i) $H'(0) \neq 0$.

(ii) $E = \{ae : a \in \mathbb{R}\}$, *for some* $e \in C_0^{2+\alpha}(\Sigma) \setminus \{0\}$.

If $\lambda_2 = 0$, *then the following statements hold.*

(A) *If* $H'(0)V'(0) \geq 0$, *then* X *is stable and* $\widetilde{\lambda}_1 = 0$.

(B) *If* $H'(0)V'(0) < 0$, *then* X *is unstable and* $\widetilde{\lambda}_2 = 0$.

Next, we study the stability in the bifurcation branch of Theorem 1.2. Denote by $\tilde{\lambda}_n(t)$ the nth eigenvalue of (6) for $\tilde{L}_t$ instead of $\tilde{L}$.

Theorem 6.2. *Assume the hypotheses of Theorem* 1.2 *and assume additionally that* $\lambda_2 = 0$ *holds. We may assume that* $V'(0) > 0$ *holds, by changing the parameter* t *to* $-t$ *if necessary. Denote by* $\{Y(s)\}_{s\in\hat{I}}$ *the bifurcating branch of fixed boundary CMC surfaces given in Theorem* 1.2.

Then, the following statements are true.

(A) *Assume* $H'(0) > 0$ (*so* X *is stable by Proposition* 6.1, (A), *and so* $\tilde{\lambda}_1 = 0$).

(A-1) *If* $\hat{V}'(s) = 0$ *for* s *near* 0 (*i.e., if* $\hat{V}$ *is locally constant*), *then* $Y(s)$ *is stable for* s *near* 0;

(A-2) *If $\hat{V}'(s) \neq 0$ for $|s| > 0$ small, then, for a sufficiently small $s_0 > 0$, on each interval $[-s_0, 0)$ and $(0, s_0]$, the branch $Y(s)$ consists of stable CMC immersions if $\tilde{\lambda}_1'(0)s\hat{V}'(s) < 0$, and of unstable CMC immersions if $\tilde{\lambda}_1'(0)s\hat{V}'(s) > 0$. In particular, supercritical and subcritical pitchfork bifurcations correspond to the cases where $s\hat{V}'(s)$ does not change sign at $s = 0$, and transcritical bifurcation occurs when $s\hat{V}'(s)$ changes sign at $s = 0$.*

(B) *If $H'(0) < 0$ (so X is unstable by Proposition* 6.1, (B)), *then $Y(s)$ is unstable for small $|s|$.*

Theorem 6.2 will be proved by using Theorem 1.2, Lemma 5.2, and Proposition 6.1.

Remark 6.3. Theorem 6.2 implies that, if $H'(0)V'(0) > 0$ (that is, the original surface X is stable), then, only the following three types of bifurcations can occur: a supercritical pitchfork bifurcation, a subcritical pitchfork bifurcation, and a transcritical bifurcation (Figure 1). Corollary 1.3 gives criteria for these three types of bifurcations in a simple case. There are many interesting examples which have symmetry: if, for the surfaces $Y(s)$ in Theorem 6.2, $Y(-s)$ and $Y(s)$ are congruent to each other, then if $H'(0)V'(0) > 0$ holds, only pitchfork bifurcations can occur. In Section 7, we will give an example for subcritical pitchfork bifurcations.

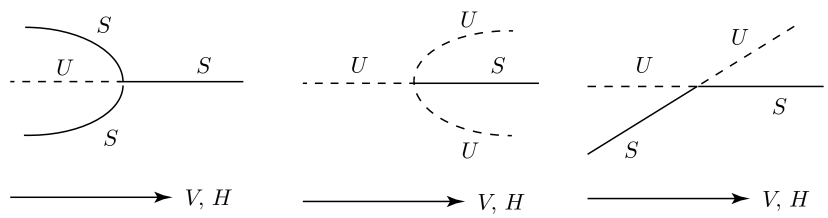

Figure 1. Solid lines represent stable surfaces, while dotted lines represent unstable ones. Left: Supercritical pitchfork bifurcation. Center: Subcritical pitchfork bifurcation. Right: Transcritical bifurcation.

Proof of Theorem 6.2. Let $\mu(s)$ be the eigenvalue of $L_{Y(s)}$ which is obtained in Lemma 5.2. First we will prove (A). If $H'(0)V'(0) > 0$, by Proposition 6.1, $\widetilde{\lambda}_1 = 0$. Hence, for $|s|$ sufficiently small, $\mu(s)$ is the smallest eigenvalue of the eigenvalue problem (6) for $Y(s)$. This follows easily from the fact that the condition $\widetilde{\lambda}_2 > 0$ is open in the set of operators of the form (2) (i.e., sum of a positive isomorphism and a compact operator). Therefore, $Y(s)$ is stable if and only if $\mu(s) \geq 0$. Since $V'(0) > 0$, by Lemma 5.2, $\mu(s)$ and $-s\hat{V}'(s)\widetilde{\lambda}_1'(0)$ have the same zeroes and, where $\mu(s) \neq 0$, the same sign. Since $\widetilde{\lambda}_1'(0) \neq 0$, if $\hat{V}'(s) = 0$, then $\mu(s) = 0$ and $Y(s)$ is stable. The other statements follow from the fact that $\mu(s)$ and $-s\hat{V}'(s)\widetilde{\lambda}_1'(0)$ have the same sign when $\mu(s) \neq 0$. This proves the statement (A).

Next, we will prove (B). If $H'(0)V'(0) < 0$, by Proposition 6.1, $\widetilde{\lambda}_2 = 0$. Hence, $\widetilde{\lambda}_1(Y(s)) < 0$ holds for $|s|$ sufficiently small (because the condition $\widetilde{\lambda}_1 < 0$ is open), and therefore $Y(s)$ is unstable. □

Proof of Corollary 1.3. The proof of (I) follows readily from Lemma 2.4, keeping in mind that $\tilde{\lambda}_1(0) = 0$ by Proposition 6.1, part (A), and the assumption $\tilde{\lambda}_1'(0) > 0$.

Part (II) follows immediately from Theorem 6.2, part (A-2). Also part (III) follows readily from Theorem 6.2, part (A-2), studying the sign of $s\hat{V}'(s)$ near $s = 0$. □

The following result is proved using Theorem 1.1 and Lemma 5.1. The proof is totally analogous to that of Theorem 6.2, and it will be omitted.

Theorem 6.4. *Under the assumptions of Theorem* 1.1, *denote by* $\{Y(s)\}_{s\in\hat{I}}$ *the bifurcating branch of fixed boundary CMC surfaces given in Theorem* 1.1. *Let* $\hat{H}(s)$ *be the mean curvature of* $Y(s)$, *and* $\mu(s)$ *the eigenvalue for the Jacobi operator* $L_{Y(s)}$ *which is obtained in Lemma* 5.1. *We may assume that* $H'(0) > 0$ *holds, by changing the parameter* t *to* $-t$ *if necessary.*

Then, the following statements are true.

(i) *If* $\hat{H}'(s) = 0$ *for* s *near* 0 (*i.e., if* $\hat{H}$ *is locally constant*), *then* $\mu(s) = 0$ *for* s *near* 0;

(ii) *If* $\hat{H}'(s) \neq 0$ *for* $|s| > 0$ *small, then, for a sufficiently small* $s_0 > 0$, *on each interval* $[-s_0, 0)$ *and* $(0, s_0]$, $\mu(s) > 0$ *if* $\lambda'(0)s\hat{H}'(s) < 0$, *and* $\mu(s) < 0$ *if* $\lambda'(0)s\hat{H}'(s) > 0$. *In particular, supercritical and subcritical pitchfork bifurcations correspond to the cases where* $s\hat{H}'(s)$ *does not change sign at* $s = 0$, *and transcritical bifurcation occurs when* $s\hat{H}'(s)$ *changes sign at* $s = 0$.

A result totally analogous to Corollary 1.3 holds for the situation described in Theorem 6.4.

7. An explicit example: bifurcation from the critical cylinder.

In this section we work out an explicit example to illustrate our stability/instability criteria (Theorem 6.2 Theorem 6.4) for the CMC surfaces in a bifurcating branch. Details of the proofs are given in Appendix B. The example is constructed by considering a 1-parameter family $\{X_t\}_{t\in]-\varepsilon,\varepsilon[}$ of unduloids in $\mathbb{R}^3$ bounded by two parallel coaxial circles of radius 1, and lying on two parallel planes separated by a distance 2π . Figure 2 shows the generating curves of these surfaces. X_0 is the cylinder of radius 1 and height 2π. We choose the outward-pointing unit normal ν along each surface of the family, hence the mean curvature H_t of X_t is negative for t near 0.

Set $\Sigma = \mathbb{R}/2\pi\mathbb{Z} \times [-\pi, \pi]$; recall that for a surface of revolution $X : \Sigma \to \mathbb{R}^3$:

$$X(\theta, z) = \big(p(z)\cos\theta, p(z)\sin\theta, z\big), \qquad p > 0,$$

the mean curvature H of X is

$$H = \frac{p^2p'' - p\{1 + (p')^2\}}{2p^2\{1 + (p')^2\}^{3/2}}. \tag{64}$$

If H is constant, then the following relation holds:

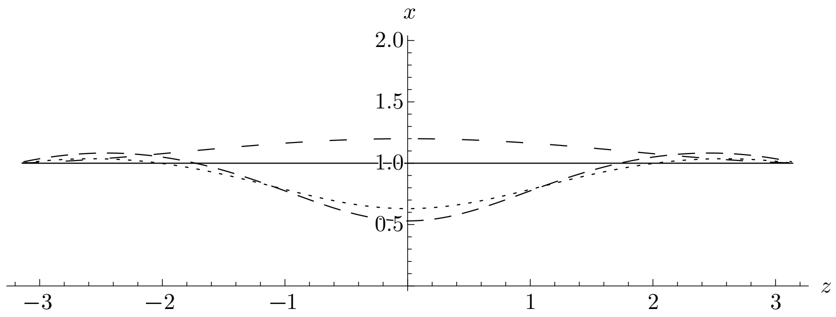

Figure 2. Generating curves of unduloids X_t with the same boundary $\Gamma := \{(x, y, z) \in \mathbf{R}^3 \mid x^2 + y^2 = 1, z = \pm\pi\}$. The horizontal axis is the z-axis. The mean curvature H and the sign of t of each surface is as follows. Dashed spaced curve: $H = -0.455$, $t > 0$; Solid line: $H = -0.5$, $t = 0$ (this corresponds to the cylinder X_0); Dotted curve: $H = -0.525$, $t < 0$; Dashed curve: $H = -0.55$, $t < 0$. The surfaces generated by the dashed spaced curve and the solid line are stable, and the surfaces generated by the dotted curve and the dashed curve are unstable.

$$p\{1 + (p')^2\}^{-1/2} = -Hp^2 + c, \tag{65}$$

where c is some constant.

The cylinder X_0 is parameterized:

$$X_0(\theta, z) = (\cos\theta, \sin\theta, z), \qquad (\theta, z) \in \Sigma.$$

Using the formula above, one easily computes the mean curvature of X_0 is $H_0 = -1/2$ and the constant $c = 1/2$. The Jacobi operator L along X_0 is given by

$$L\varphi = \Delta_0\varphi + (4H_0^2 - 2K_0)\varphi = \varphi_{\theta\theta} + \varphi_{zz} + \varphi, \quad \varphi \in C_0^{2+\alpha}(\Sigma),$$

where $K_0 = 0$ is the Gaussian curvature of X_0. Therefore, the function $e : \Sigma \to \mathbb{R}$ given by $e(\theta, z) = \sin z$ is a Jacobi field along X_0. One can show that e spans the kernel of the Jacobi operator (the eigenspace of the eigenvalue $\lambda_2 = 0$), and that the CMC embedding X_0 is stable (Lemma B.1).

The family X_t consists of fixed boundary unduloids, having the following properties (see Section B.2 for details on the construction of X_t).

(a) Each X_t is symmetric with respect to the horizontal plane $z = 0$.

(b) For $t > 0$:

 (b1) X_t is a proper subset of one period of an unduloid going from one neck to the next neck, and X_t contains the bulge;

 (b2) $-1/2 = H_0 < H_t < 0$, $0 < V(X_0) < V(X_t)$;

 (b3) $\lambda_2(X_t) > 0$, and X_t is stable (Theorem B.2).

(c) For $t < 0$:

(c1) X_t contains properly one period of an unduloid going from a bulge to the next bulge;

(c2) $H_t < H_0 = -1/2$, $0 < V(X_t) < V(X_0)$;

(c3) $\lambda_2(X_t) < 0$, and therefore X_t is unstable (Theorem B.2).

By applying Theorem 1.1 (or Theorem 1.2), there is a bifurcating branch $Y(s)$ of fixed boundary CMC surfaces issuing from X_t at $t = 0$. The surfaces $Y(s)$ can be determined explicitly (see Figure 3) as follows. Denote by $\hat{H}(s)$ and $\hat{V}(s)$ respectively the mean curvature and the volume of $Y(s)$.

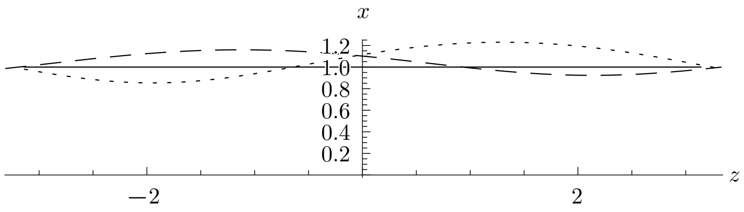

Figure 3. Generating curves of unduloids $Y(s)$ in the bifurcating branch. The mean curvature H and the sign of s of each surface is as follows. Dotted curve: $H = -0.48$, $s > 0$, Solid line: $H = -0.5$, $s = 0$, Dashed curve: $H = -0.48$, $s < 0$. The surface generated by the solid line is stable, and the surfaces generated by the dotted curve and the dashed curve are unstable.

(i) $Y(s)$ is axially symmetric with respect to the z axis. This follows from Alexandrov reflection method, using the fact that $Y(s)$ is "close" to the cylinder X_0. It follows in particular that $Y(s)$ is part of an unduloid.

(ii) Each $Y(s)$ is one period of an unduloid. This follows from the fact that the boundary of $Y(s)$ consists of two coaxial circles having the same radius, that $Y(s)$ is not symmetric with respect to the plane $\{z = 0\}$, and that $Y(s)$ is close to "one period of a cylinder" (about the last statement, see the fourth paragraph of Section B.2).

(iii) The parameter s can be chosen in such a way that $Y(s)$ is the reflection of $Y(-s)$ around the horizontal plane $z = 0$. It follows, in particular, that $\hat{H}'(0) = \hat{V}'(0) = 0$.

(iv) $\hat{H}(s) > \hat{H}(0)$ holds for $s \neq 0$ sufficiently small (Lemma B.3). Thus, by Theorem 6.4, $\lambda_2\big(Y(s)\big) < 0$ for $s \neq 0$, and therefore $Y(s)$ is unstable for $s \neq 0$ (Theorem 2.6, part (IV)).

The same conclusion on the instability in the bifurcating branch can be obtained using Theorem 6.2:

(v) $\hat{V}''(0) > 0$ (Proposition B.6). Thus, $\hat{V}(s) > \hat{V}(0)$ for $s \neq 0$. Hence, by Theorem 6.2, $Y(s)$ is unstable for $s \neq 0$.

This means that we are in the situation described by the picture in the middle of Figure 1: subcritical bifurcation.

A. Crandall and Rabinowitz bifurcation criterion.

We will recall some results from Crandall and Rabinowitz [**4**] and [**5**], and we will present a generalization that is used in the proofs of our results (Theorems 1.1 and 1.2). Proofs follow the same lines as the proofs of the original results, and they will be omitted.

Let Y, Z be real Banach spaces, let V be an open neighborhood of 0 in Y, let $I = (a, b)$ be a non-empty open interval, and let $F : I \times V \to Z$ be a twice continuously Fréchet differentiable mapping. For a linear mapping T, denote by $N(T)$ the kernel of T, and by $R(T)$ the image of T.

Theorem A.1 ([**4**, Theorem 1.7]). *Assume that $t_0 \in I$ and that the following statements hold.*

(i) $F(t, 0) = 0$ *for all* $t \in I$,

(ii) $\dim\big(N(D_yF(t_0, 0))\big) = \operatorname{codim}\big(R(D_yF(t_0, 0))\big) = 1$,

(iii) $D_{ty}F(t_0, 0)y_0 \notin R(D_yF(t_0, 0))$, *where* $y_0 \in Y$ *spans* $N(D_yF(t_0, 0))$.

Let W be any complement of $\operatorname{span}\{y_0\}$ *in Y. Then there exists an open interval $\hat{I}$ containing 0 and continuously differentiable functions $t : \hat{I} \to \mathbb{R}$ and $\zeta : \hat{I} \to W$ such that $t(0) = t_0$, $\zeta(0) = 0$, and if $y(s) = sy_0 + s\zeta(s)$, then $F(t(s), y(s)) = 0$. Moreover, $F^{-1}(\{0\})$ near $(t_0, 0)$ consists precisely of the curves $(t, 0)$, $t \in I$, and $(t(s), y(s))$, $s \in \hat{I}$.*

Denote by $B(Y, Z)$ the set of bounded linear maps of Y into Z.

Definition A.2 ([**5**, Definition 1.2]). *Let T, $K \in B(Y, Z)$. Then $\mu \in \mathbb{R}$ is a K-simple eigenvalue of T if*

$$\dim(N(T - \mu K)) = \operatorname{codim}\big(R(T - \mu K)\big) = 1,$$

and, if $N(T - \mu K) = \operatorname{span}\{e\}$,

$$Ke \notin R(T - \mu K).$$

Lemma A.3 ([**5**, Lemma 1.3]). *Let T_0, $K \in B(Y, Z)$ and assume that r_0 is a K-simple eigenvalue of T_0. Then there exists a value $\delta > 0$ such that, whenever $T \in B(Y, Z)$ and $\|T - T_0\| < \delta$, there is a unique $r(T) \in \mathbb{R}$ satisfying $|r(T) - r_0| < \delta$ for which $T - r(T)K : Y \to Z$ is not an isomorphism. The map $T \mapsto r(T)$ is analytic and $r(T)$ is a K-simple eigenvalue of T. Finally, if $N(T_0 - r_0K) =$ $\operatorname{span}\{y_0\}$ and W is any complement of* $\operatorname{span}\{y_0\}$ *in Y, there is a unique vector $x(T) \in N\big(T - r(T)K\big)$ satisfying $x(T) - y_0 \in W$. The map $T \mapsto x(T)$ is also analytic.*

Corollary A.4 ([**5**, Corollary 1.13]). *Under the same assumptions of Theorem* A.1, *let $K \in B(Y, Z)$ and assume that 0 is a K-simple eigenvalue of $D_yF(t_0, 0)$. Then, there exist open intervals J_1, J_2 with $t_0 \in J_1$, $0 \in J_2$ and continuously differentiable functions $\lambda : J_1 \to \mathbb{R}$, $\mu : J_2 \to \mathbb{R}$, $u : J_1 \to Y$, $w : J_2 \to Y$ such that*

(i) $D_yF(t, 0)u(t) = \lambda(t)Ku(t)$, $\forall t \in J_1$,

(ii) $D_yF(t(s), y(s))w(s) = \mu(s)Kw(s)$, $\forall s \in J_2$.

Moreover,

$$\lambda(t_0) = \mu(0) = 0, \quad u(t_0) = y_0 = w(0), \quad u(t) - y_0 \in W, \quad w(s) - y_0 \in W.$$

REMARK A.5. Corollary A.4 follows from Lemma A.3. Hence, λ and μ are unique, and u and w are unique up to constant multiple.

We need the following extensions of Lemma A.3 and Corollary A.4, where we allow to also perturb the operator K. A proof can be obtained following the same lines of the original results, see [**5**, Remark 1.11].

LEMMA A.6. *Let T_0, $K_0 \in B(Y, Z)$ and let r_0 be a K_0-simple eigenvalue of T_0. Then there exists $\delta > 0$ such that whenever $T, K \in B(Y, Z)$, $\|K - K_0\| < \delta$ and $\|T - T_0\| < \delta$, there is a unique $r(T, K) \in \mathbb{R}$ satisfying $|r(T, K) - r_0| < \delta$ for which $T - r(T, K)K : Y \to Z$ is not an isomorphism. The map $(T, K) \mapsto r(T, K)$ is smooth, and $r(T, K)$ is a K-simple eigenvalue of T. Finally, if $N(T_0 - r_0K_0) = \mathrm{span}\{y_0\}$ and W is any complement of $\mathrm{span}\{y_0\}$ in Y, there is a unique $x(T, K) \in N\big(T - r(T, K)K\big)$ satisfying $x(T, K) - y_0 \in W$. The map $(T, K) \mapsto x(T, K)$ is smooth.*

COROLLARY A.7. *Under the same assumptions of Theorem* A.1*, let $K(t, y) \in B(Y, Z)$ be differentiable with respect to $(t, y) \in I \times V$, and assume that 0 is a $K(t_0, 0)$-simple eigenvalue of $D_yF(t_0, 0)$. Then, there exist open intervals J_1, J_2 with $t_0 \in J_1$, $0 \in J_2$ and continuously differentiable functions $\lambda : J_1 \to \mathbb{R}$, $\mu : J_2 \to \mathbb{R}$, $u : J_1 \to Y$, $w : J_2 \to Y$ such that*

(i) $D_yF(t, 0)u(t) = \lambda(t)K(t, 0)u(t)$, $\forall t \in J_1$,

(ii) $D_yF(t(s), y(s))w(s) = \mu(s)K(t(s), y(s))w(s)$, $\forall s \in J_2$.

Moreover,

$$\lambda(t_0) = \mu(0) = 0, \quad u(t_0) = y_0 = w(0), \quad u(t) - y_0 \in W, \quad w(s) - y_0 \in W.$$

Here, λ and μ are unique, and u and w are unique up to constant multiple.

Similarly, we have the following extension of [**5**, Theorem 1.16].

THEOREM A.8. *Under the same assumptions of Theorem* A.1*, let λ, μ be the functions provided by Corollary* A.7*. Then, $\lambda'(t_0) \neq 0$, and near $s = 0$ the functions $\mu(s)$ and $-st'(s)\lambda'(t_0)$ have the same zeroes, and, whenever $\mu(s) \neq 0$, the same sign. More precisely,*

$$\big|st'(s)\lambda'(t_0) + \mu(s)\big| \leq o(1)\left(\big|st'(s)\big| + \big|\mu(s)\big|\right) \text{ as } s \to 0,$$
$$\lim_{\substack{s \to 0\\ \mu(s) \neq 0}} \frac{-st'(s)\lambda'(t_0)}{\mu(s)} = 1.$$

Moreover, there is a constant C such that

$$\big\|y'(s) - w(s)\big\| \leq C \min\big\{\big|st'(s)\big|, \big|\mu(s)\big|\big\}$$

near $s = 0$.

B. Detailed proofs from Section 7.

We present here some details about the proofs of the material in Section 7; we use the same notations.

B.1. Stability of a cylinder and unduloids.

Lemma B.1. *For* X_0*, we have the followings.*

(i) $\lambda_2 = 0$.

(ii) $e(\theta, z) = \sin z$ *spans the kernel of the Jacobi operator.*

(iii) X_0 *is stable and* $\tilde{\lambda}_1 = 0$.

Proof. In order to prove the stability of X_0, it is sufficient to prove stability only for axially symmetric variations. This follows easily using Schwarz symmetrization.

If X_ϵ is an axially symmetric variation of X_0 fixing the boundary, then X_ϵ is represented as a variation C_ϵ of the generating curve

$$C_0(s) = (1, z), \quad -\pi \leq z \leq \pi$$

of X_0. Set

$$C_\epsilon = C_0 + \epsilon\psi\tilde{\nu} + \mathcal{O}(\epsilon^2),$$

where

$$\tilde{\nu} = (1, 0).$$

The second variation of the area is given by

$$\mathcal{I}[\psi] = -2\pi \int_0^l \psi L_0[\psi] x \, dz, \tag{66}$$

where

$$L_0[\psi] = \psi'' + \psi. \tag{67}$$

It is clear that $L_0[\sin z] = 0$ on $[-\pi, \pi]$ and $\sin z|_{\partial[-\pi,\pi]} = 0$ hold. Since the problem is one-dimensional and the number of the nodal domains of $\sin z$ is two, it follows that 0 is the second eigenvalue of L_0. Now set $\psi(z) = 1 + \cos z$. Then, $L_0[\psi] = 1$ on $[-\pi, \pi]$, and $\psi|_{\partial[-\pi,\pi]} = 0$. Because

$$\int_{-\pi}^{\pi} \psi \, dz = 2\pi > 0,$$

X_0 is stable (Theorem 2.6 (III-B1)). Therefore, $\tilde{\lambda}_1 = 0$ (Lemma 2.4) and the function $e(\theta, z) = \sin z$ spans the corresponding eigenspace. This also shows that $\lambda_2 = 0$ and e spans the corresponding eigenspace. □

THEOREM B.2. *One period $\mathcal{U}_0$ of an unduloid $\mathcal{U}$ (from neck to neck) is stable as fixed boundary CMC surface. Moreover, any larger part of $\mathcal{U}$ is unstable, and any proper subset of $\mathcal{U}_0$ is stable.*

PROOF (Sketch). Let

$$X(s, \theta) = (x(s)e^{i\theta}, z(s)), \quad (s, \theta) \in \Sigma$$

be an immersion of one period $\mathcal{U}_0$ (from neck to neck) of an unduloid $\mathcal{U}$. Denote by L the Jacobi operator. Similarly to the case of cylinder (see the proof of Lemma B.1), in order to prove the stability of $\mathcal{U}_0$, it is sufficient to prove only the stability for axially symmetric variations. Since $\varphi(s, \theta) := x'(s)$ satisfies $L[\varphi] = 0$ on Σ ([**9**, Lemma 7.1]) and $\varphi|_{\partial\Sigma} = 0$, φ is an eigenfunction belonging to zero eigenvalue. Because the number of nodal domains of φ is two, and X is stable (because of its length, the proof of the stability will be given in another paper [**10**, Theorem 5.1]), λ_2 must be zero (Theorem 2.6). Therefore, for any larger part, $\lambda_2 < 0$ holds and it is unstable (Theorem 2.6). Because of the definition of the stability, it is obvious that any proper subset of $\mathcal{U}_0$ is stable. □

B.2. Construction of the family X_t.

In this subsection, we show the existence of the one-parameter family of unduloids X_t used in Section 7.

Note that generating curves of unduloids with $H = -1/2$ with a neck at the center which are close to the cylinder are known to have the shapes as in Figure 4. The length of the corresponding z-axis for each one period is shorter than 2π. Therefore, by rescaling these unduloids by a factor $r > 1$ and taking appropriate subdomains, we obtain one-parameter family of unduloids X_t $(t < 0)$ containing the circles

$$C_\pm := \left\{x^2 + y^2 = 1,\ z = \pm\pi\right\}$$

and the mean curvature $H(t) < -1/2$. The part bounded by C_+, C_- includes one period as a proper subset.

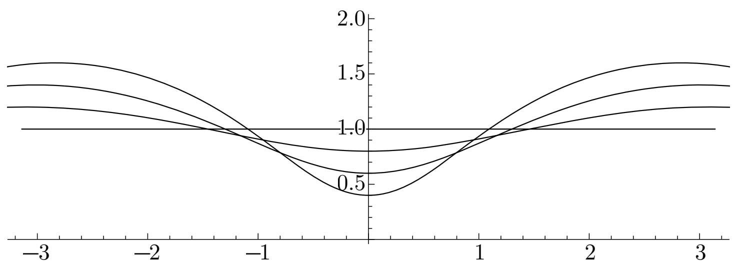

Figure 4. Generating curves of unduloids with $H = -1/2$ with a neck at the center.

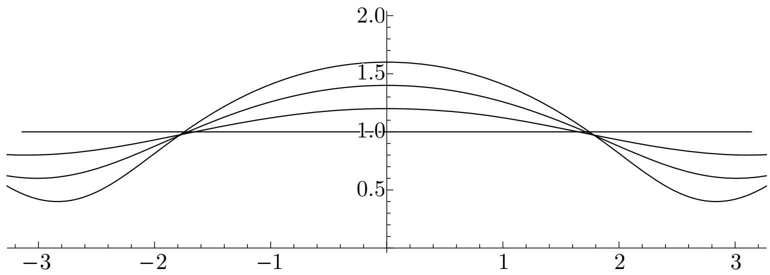

Figure 5. Generating curves of unduloids with $H = -1/2$ with a bulge at the center.

Next we consider generating curves of unduloids with a bulge at the center which are close to the cylinder (Figure 5 shows the case where $H = -1/2$).

Recall that an unduloid is the rotational surface obtained generated by the trace of a focus of an ellipse that rolls along the z-axis without slipping (cf. [**6**], [**7**]). Denote by $\mathcal{U}(a, b)$ the unduloid given by an ellipse $C(a, b)$ with the longer axis $2a$ and the shorter axis $2b$ ($a \geq b > 0$). The "height", the length along the axis of rotation of one period of $\mathcal{U}(a, b)$, is equal to the length of $C(a, b)$. Note that $C(a, a)$ is a circle with radius a and $\mathcal{U}(a, a)$ is a cylinder with radius a. We may therefore regard a part of the cylinder with height $2\pi a$ as one period of $\mathcal{U}(a, a)$. The mean curvature of $\mathcal{U}(a, b)$ for the outward-pointing normal is $H(a, b) = -1/(2a)$.

The length of $C(a, b)$ is

$$L(a, b) = 4 \int_0^{\pi/2} (a^2 \sin^2\theta + b^2 \cos^2\theta)^{1/2}\, d\theta. \tag{68}$$

Note that $L(1, 1) = 2\pi$.

Let us consider the case where $a > b > 1$, and b satisfies $b < \sqrt{2a - 1}$. Then, $L(a, b) > L(1, 1)$, and the radius B of the bulge and that N of the neck satisfy

$$B = a + \sqrt{a^2 - b^2} > 1 > a - \sqrt{a^2 - b^2} = N > 0.$$

Denote by $x = \gamma_{(a,b)}(z)$ the generating curve. If $\gamma_{(a,b)}(\pm\pi) > 1$, then by reducing $\mathcal{U}(a, b)$ we obtain an unduloid $\mathcal{U}(a', b')$ which passes the circles $C_\pm$. Since $L(a', b') > L(1, 1)$, $a' > 1$ must hold and this means that the mean curvature H of $\mathcal{U}(a', b')$ satisfies $H = -1/(2a') > -1/2$. If $\gamma_{(a,b)}(\pm\pi) < 1$, then by expanding $\mathcal{U}(a, b)$ we obtain an unduloid $\mathcal{U}(a', b')$ which passes $C_\pm$. Since $a' > a > 1$, the mean curvature H of $\mathcal{U}(a', b')$ satisfies $H = -1/(2a') > -1/2$. Therefore, we obtain one-parameter family of unduloids X_t ($t > 0$) each of which passes the circles $C_\pm$ and the mean curvature $H(t) > -1/2$, and the part bounded by C_+, C_- is a proper subset of one period.

B.3. On the bifurcating branch $Y(s)$.

Using the same notations above, consider the family $\mathcal{U}(a, b)$ of unduloids. The family $Y(s) = \mathcal{U}(a_s, b_s)$ ($s \neq 0$) is one period of an unduloid, with $L(a_s, b_s) = 2\pi$, and so

$$a_s > 1 > b_s$$

must be satisfied. This implies that

$$H(a_s, b_s) = -\frac{1}{2a_s} > -\frac{1}{2} \tag{69}$$

holds. We have obtained the following:

LEMMA B.3. *$\hat{H}(s) > \hat{H}(0) = H(0)$ holds for $s \neq 0$.*

Let us now prove that $\hat{V}''(0) > 0$.

LEMMA B.4. *Let $X(0) : \Sigma \to \mathbb{R}^3$ be an immersed surface with Gauss map ν and mean curvature H. Let*

$$X(t) = X(0) + \big(t\varphi + t^2 f + \mathcal{O}(t^3)\big)\nu$$

be any normal variation of $X(0)$. Then,

$$V''(0) := \frac{d^2}{dt^2}\bigg|_{t=0} V(X(t)) = -2\int_\Sigma H\varphi^2\, d\Sigma + 2\int_\Sigma f\, d\Sigma. \tag{70}$$

PROOF. Since

$$V' = \int_\Sigma \langle \delta X, \nu\rangle\, d\Sigma,$$

we obtain

$$V'' = \int_\Sigma (\delta\langle \delta X, \nu\rangle - \langle \delta X, \nu\rangle \cdot 2H\varphi)\, d\Sigma. \tag{71}$$

Now

$$\delta\langle \delta X, \nu\rangle = \langle \delta^2 X, \nu\rangle + \langle \delta X, \delta\nu\rangle = \langle 2f\nu, \nu\rangle + \langle \varphi\nu, \delta\nu\rangle = 2f + \varphi\langle \nu, \delta\nu\rangle = 2f. \tag{72}$$

(71) combined with (72) gives the desired result. □

LEMMA B.5. *$Y = Y(s)$ is given by parametric equations:*

$$Y(\theta, z) = \big(p_{(s)}(z)\cos\theta, p_{(s)}(z)\sin\theta, z\big),$$

where

$$p_{(s)} = 1 + s\sin z + s^2 F(s, z).$$

PROOF. From Theorem 1.1 (or equivalently Theorem 1.2) and Lemma B.1, $Y(s)$ is represented as

$$Y(s) = X_0 + \big(\varphi(t(s)) + se + s\zeta(s)\big)\nu,$$

where $\nu(\theta, z) = (\cos\theta, \sin\theta, 0)$ is the Gauss map of the cylinder X_0, and

$$e = \sin z, \quad t(0) = 0, \quad \zeta(0) = 0.$$

Hence,

$$Y'(0) = \big(t'(0)\dot{\varphi}(0) + \sin z\big)\nu. \tag{73}$$

Since $Y(s)$ is the reflection of $Y(-s)$ with respect to the plane $\{z = 0\}$, it holds that

$$Y(s, z) = Y(-s, -z).$$

Hence we have

$$Y_s(0, z) = -Y_s(0, -z).$$

This means, by (73), that

$$t'(0)\dot{\varphi}(0)(\theta, z) + \sin z = -(t'(0)\dot{\varphi}(0)(\theta, -z) + \sin(-z)).$$

Hence,

$$t'(0)(\dot{\varphi}(0)(\theta, z) + \dot{\varphi}(0)(\theta, -z)) = 0 \tag{74}$$

holds. Since X_t is symmetric with respect to the plane $\{z = 0\}$,

$$\dot{\varphi}(0)(\theta, z) = \dot{\varphi}(0)(\theta, -z), \quad \forall\theta,\ \forall z \tag{75}$$

holds. Formula (74) combined with (75) yields

$$t'(0)\dot{\varphi}(0) = 0,$$

which gives the desired result. $\square$

Proposition B.6. $\hat{V}''(0) > 0$.

Proof. Let

$$Y(\theta, z) = \big(p_{(s)}(z)\cos\theta, p_{(s)}(z)\sin\theta, z\big), \qquad p > 0$$

be our surfaces in the bifurcation branch. Note that Y depends on s. Equation (65) gives

$$p_{(s)}^2 = (-\hat{H}p_{(s)}^2 + \hat{c})^2\{1 + (p'_{(s)})^2\}. \tag{76}$$

By Lemma B.5, $p_{(s)}$ can be represented as

$$p_{(s)} = 1 + s\sin z + s^2 F(s, z), \tag{77}$$

and

$$\hat{H} = \hat{H}(s), \quad \hat{c} = \hat{c}(s).$$

Since $Y(s)$ and $Y(-s)$ are congruent, we obtain

$$\hat{H}(s) = -\frac{1}{2} + \frac{1}{2}s^2\hat{H}_{ss}(0) + \mathcal{O}(s^4), \tag{78}$$

$$\hat{c}(s) = \frac{1}{2} + \frac{1}{2}s^2\hat{c}_{ss}(0) + \mathcal{O}(s^4). \tag{79}$$

For simplicity, write

$$F(s) = F(s, z).$$

Inserting (77), (78), and (79) into (76), we obtain

$$\begin{aligned}
&1 + 2s\sin z + s^2(\sin^2 z + 2F(0)) + s^3(2F_s(0) + 2F(0)\sin z) + \mathcal{O}(s^4) \\
&\quad = 1 + 2s\sin z + s^2\{1 + \sin^2 z + 2F(0) - \hat{H}_{ss}(0) + \hat{c}_{ss}(0)\} \\
&\qquad + s^3\Big[2F_z(0)\cos z + 2F_s(0) + (\sin z)\{1 + 4F(0) + \cos^2 z - 3\hat{H}_{ss}(0) + \hat{c}_{ss}(0)\}\Big] + \mathcal{O}(s^4).
\end{aligned}$$

Comparing the coefficients of s^2 and s^3 of the left and the right hand sides of the above equality, we obtain

$$\begin{aligned}
\sin^2 z + 2F(0) &= 1 + \sin^2 z + 2F(0) - \hat{H}_{ss}(0) + \hat{c}_{ss}(0), \\
2F_s(0) + 2F(0)\sin z &= 2F_z(0)\cos z + 2F_s(0) \\
&\quad + (\sin z)\{1 + 4F(0) + \cos^2 z - 3\hat{H}_{ss}(0) + \hat{c}_{ss}(0)\}.
\end{aligned}$$

Hence,

$$1 - \hat{H}_{ss}(0) + \hat{c}_{ss}(0) = 0, \tag{80}$$

$$2F_z(0)\cos z + (\sin z)\{1 + 2F(0) + \cos^2 z - 3\hat{H}_{ss}(0) + \hat{c}_{ss}(0)\} = 0. \tag{81}$$

These equations give

$$2F_z(0)\cos z + (\sin z)(2F(0) + \cos^2 z - 2\hat{H}_{ss}(0)) = 0.$$

Solving this ODE with boundary condition $F(0)|_{z=\pm\pi} = 0$, we obtain

$$F(0) = \frac{1}{4}\cos 2z + \frac{1}{2}(1 + 2\hat{H}_{ss}(0))\cos z + \hat{H}_{ss}(0) + \frac{1}{4}. \tag{82}$$

Since $\hat{H}(0) = H(0) = -1/2$, by Lemma B.4,

$$\hat{V}''(0) = \int_\Sigma \sin^2 z\, d\Sigma + 2\int_\Sigma F(0)\, d\Sigma = 2\pi\left[\int_{-\pi}^{\pi} (\sin^2 z + 2F(0))\, dz\right]. \tag{83}$$

Using (82) and (83), we then obtain:

$$\hat{V}''(0) = 4\pi^2(1 + 2\hat{H}_{ss}(0)).$$

By Lemma B.3, $\hat{H}_{ss}(0) \geq 0$, and we obtain $\hat{V}''(0) > 0$, which is what we wanted to prove. □

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