J. Math. Soc. Japan
Vol. 69, No. 4 (2017) pp. 1443–1474
doi: 10.2969/jmsj/06941443

Jacquet–Langlands–Shimizu correspondence for theta lifts to $GSp(2)$ and its inner forms I: An explicit functorial correspondence

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(Received Jan. 8, 2016)

Abstract. As was first essentially pointed out by Tomoyoshi Ibukiyama, Hecke eigenforms on the indefinite symplectic group $GSp(1,1)$ or the definite symplectic group $GSp^*(2)$ over $\mathbb{Q}$ right invariant by a (global) maximal open compact subgroup are conjectured to have the same spinor L-functions as those of paramodular new forms of some specified level on the symplectic group $GSp(2)$ (or $GSp(4)$). This can be viewed as a generalization of the Jacquet–Langlands–Shimizu correspondence to the case of $GSp(2)$ and its inner forms $GSp(1,1)$ and $GSp^*(2)$.

In this paper we provide evidence of the conjecture on this explicit functorial correspondence with theta lifts: a theta lift from $GL(2)\times B^\times$ to $GSp(1,1)$ or $GSp^*(2)$ and a theta lift from $GL(2)\times GL(2)$ (or $GO(2,2)$) to $GSp(2)$. Here B denotes a definite quaternion algebra over $\mathbb{Q}$. Our explicit functorial correspondence given by these theta lifts are proved to be compatible with archimedean and non-archimedean local Jacquet–Langlands correspondences. Regarding the non-archimedean local theory we need some explicit functorial correspondence for spherical representations of the inner form and non-supercuspidal representations of $GSp(2)$, which is studied in the appendix by Ralf Schmidt.

1. Introduction.

1.1. Background and aim of this paper.

According to Eichler [**7**], [**8**], Shimizu [**41**] and Jacquet–Langlands [**20**], automorphic L-functions of a multiplicative group of a definite quaternion algebra are those of holomorphic new forms on $GL(2)$. The Langlands principle of functoriality (cf. [**22**]) predicts that this should have more generality. More precisely, let G and H be two reductive algebraic groups, and ${}^LG$ and ${}^LH$ be their L-groups (for the definition see [**22**] or [**4**]). The principle says that, given an L-homomorphism between ${}^LG$ and ${}^LH$, there should be a correspondence between automorphic representations of G and those of H such that it preserves the relation of their L-functions induced by the L-homomorphism. For this we note that it is standard to assume that G or H is quasi-split.

2010 *Mathematics Subject Classification.* Primary 11F70; Secondary 11F27, 11F46, 11F55.
Key Words and Phrases. Jacquet–Langlands correspondence, spinor L-functions, theta lifts.
This work was partly supported by Grant-in-Aid for Young Scientists (B) 21740025, Japan Society for the Promotion of Science.

When H is a quasi-split group and G is an inner form of H we have ${}^LG = {}^LH$. As a natural L-morphism we can take the identity map for this case. Namely we expect that an automorphic L-function of G is some L-function of H. The original Jacquet–Langlands–Shimizu correspondence (cf. [**20**], [**41**]) is a typical example of this, which deals with the case where G is a multiplicative group of a quaternion algebra and $H = GL(2)$. We now have the Jacquet–Langlands correspondence for $GL(n)$ established by Badulescu–Renard (cf. [**2**], [**3**]). The aim of this paper is to provide examples of automorphic forms or automorphic representations satisfying the principle when G is the indefinite or definite symplectic group $GSp(1,1)$ or $GSp^*(2)$ respectively and H is the split symplectic group $GSp(2)$ of genus two (for these groups see Section 2.1). We note that $GSp(1,1)$ and $GSp^*(2)$ are non-split inner forms of $GSp(2)$.

We are inspired by Ibukiyama's conjecture [**17**] (see also [**18**]). He considers the case of $G = GSp^*(2)$, which is the compact inner form of $H = GSp(2)$. His conjecture says that spinor L-functions of Hecke eigenforms on this compact form right-invariant by some (global) maximal open compact subgroup of non-principal genus should be those of paramodular new forms on $GSp(2)$ of square free level. For this we should note that there is a quite established study on a non-archimedean local theory of paramodular forms by Roberts and Schmidt [**39**]. We can expect the conjecture not only for $GSp^*(2)$ but also for $GSp(1,1)$ since their L-group are the same. We will generalize Ibukiyama's conjecture for the cases of any (global) maximal open compact subgroups (cf. Conjecture 4.2), for which our result provides evidence.

1.2. Main result.

Throughout this paper, we work over the rational number field $\mathbb{Q}$ and assume that every automorphic form or automorphic representation has the trivial central character. In what follows, we denote the adele ring of $\mathbb{Q}$ by $\mathbb{A}$. By B we denote a definite quaternion algebra over $\mathbb{Q}$. Let d_B be the discriminant of B and D a (square free) divisor of d_B. To state our result we introduce the space $S_{\kappa_1}(D)$ of elliptic cusp forms of weight κ_1 and level D (cf. [**31**, Section 3.1]) and the space $\mathcal{A}_{\kappa_2}$ of automorphic forms on $B_{\mathbb{A}}^{\times}$ of weight σ_{κ_2} (cf. [**31**, Section 3.2]), for which we assume that $\kappa_2 > 0$.

For $(f, f') \in S_{\kappa_1}(D) \times \mathcal{A}_{\kappa_2}$ we define the theta lifting from (f, f') to an automorphic form $\mathcal{L}(f, f')$ on $GSp(1,1)(\mathbb{A})$ or $GSp^*(2)(\mathbb{A})$, following the formulation as in [**30**] (cf. Section 3.2.2). By a theta integral kernel with some specified Schwartz–Bruhat function, we define the theta lift $\mathcal{L}(f, f')$ on $GSp(1,1)(\mathbb{A})$ (respectively $GSp^*(2)(\mathbb{A})$) so that it is a cusp form on $GSp(1,1)(\mathbb{A})$ (respectively an automorphic form on $GSp^*(2)(\mathbb{A})$) right-invariant by a (global) maximal open compact subgroup. By $\pi(f, f')$ we denote the automorphic representation generated by $\mathcal{L}(f, f')$.

On the other hand, let $\mathrm{JL}(f') \in S_{\kappa_2+2}(d_B)$ be a primitive cusp form corresponding to a Hecke eigenform $f' \in \mathcal{A}_{\kappa_2}$ by Eichler or Jacquet–Langlands–Shimizu correspondence (cf. [**7**], [**8**], [**20**], [**41**]). We consider the cuspidal representation $\pi'(f, \mathrm{JL}(f'))$ given by a theta lift to $GSp(2)(\mathbb{A})$ from $(f, \mathrm{JL}(f'))$ (or from a cuspidal representation of $GO(2,2)$ defined by $(f, \mathrm{JL}(f'))$). This theta lift follows the formulation by Roberts [**38**] and Harris–Kudla [**12**].

Now we can consider the natural map $\Phi : \pi(f, f') \mapsto \pi'(f, \mathrm{JL}(f'))$, which makes the following diagram commute (if we assume the non-vanishing of $\pi(f, f')$ and $\pi'(f, \mathrm{JL}(f'))$):

$$\begin{array}{ccc} (f, f') & \longrightarrow & \pi(f, f') \\ {\scriptstyle \mathrm{id}\times\mathrm{JL}}\big\downarrow & & \big\downarrow{\scriptstyle \Phi} \\ (f, \mathrm{JL}(f')) & \longrightarrow & \pi'(f, \mathrm{JL}(f')) \end{array}$$

Our result (cf. Theorem 4.13) is that Φ satisfies the expected conditions of the Jacquet–Langlands–Shimizu correspondence for $GSp(2)$ and its inner forms $GSp^*(2)$, $GSp(1,1)$.

THEOREM. *Let $(f, f') \in S_{\kappa_1}(D) \times \mathcal{A}_{\kappa_2}$ be Hecke eigenforms and suppose that f is primitive. When $\mathcal{L}(f, f')$ is a theta lift to $GSp(1,1)(\mathbb{A})$ (respectively $GSp^*(2)(\mathbb{A})$) we assume that $1 < \kappa_1 < \kappa_2+2$ (respectively $1 < \kappa_2+2 < \kappa_1$). Let $\pi(f, f')$ and $\pi'(f, \mathrm{JL}(f'))$ be the automorphic representations above.*
(1) *The two representations $\pi(f, f')$ and $\pi'(f, \mathrm{JL}(f'))$ are irreducible and thus decompose into the restricted tensor product $\pi(f, f') = \bigotimes'_{v\le\infty} \pi_v$ and $\pi'(f, \mathrm{JL}(f')) = \bigotimes'_{v\le\infty} \pi'_v$.*

For $v = p \nmid d_B$, $\pi_p \simeq \pi'_p$ is a unique irreducible subquotient with a spherical vector for an unramified principal series representation, which we call type I. For $v = p | d_B/D$ (respectively $p|D$), π_p and π'_p are the representation of type IIa (respectively Va). Here see section A.4 *of the appendix for the representations of type I, IIa and Va. For $v = \infty$, π_∞ and π'_∞ are irreducible admissible representations square integrable modulo center (Section* 2.3.2*) with the same L-parameter.*

The map Φ is thus compatible with archimedean and non-archimedean local Jacquet–Langlands correspondences (for the non-archimedean correspondence, see the appendix).
(2) *We have the coincidence of the global spinor L-functions as follows:*

$$L(\pi(f, f'), spin, s) = L(\pi'(f, \mathrm{JL}(f')), spin, s).$$

REMARK. (1) We remark that Sorensen [**44**] dealt with some other special case of the global Jacquet–Langlands correspondence for $GSp^*(2)$ and $GSp(2)$ by a trace formula approach. As for the trace formula approach toward the Langlands functoriality, many specialists are interested in the recent remarkable progress due to Arthur (cf. [**1**]) and to Moeglin and Waldspurger (cf. [**26**]). However, the global Jacquet–Langlands correspondence of $GSp(2)$ and its inner forms seems still open in gerenal.
(2) For this theorem we note that all π_p's for $p|d_B$ are spherical representations of the non-split group $GSp(1,1)(\mathbb{Q}_p) \simeq GSp^*(2)(\mathbb{Q}_p)$. *We define local spinor L-functions for all spherical representations of this group in terms of Hecke eigenvalues.* We verify that such local L-functions are local spinor L-functions of irreducible admissible representations of $GSp_2(\mathbb{Q}_p)$ with the same L-parameters as the spherical representations (cf. Proposition 4.9). In Remark 4.10 it is remarked how the invariance conditions of spherical representations with respect to the maximal compact subgroups are related to the degrees of the local spinor L-functions.
(3) The non-archimedean local Jacquet–Langlands correspondence of $GSp(2)$ (or $GSp(4)$) and its inner forms has been essentially established by Gan–Takeda [**10**] and Gan–Tantono [**11**]. However, *the local Jacquet–Langlands correspondence we take up describes the correspondence between the paramodular level structures for the local representations of $GSp(2)$ and the invariance conditions mentioned in* (2) *for spherical representations of the inner forms*, which [**10**] and [**11**] do not point out. If one wants to understand

the Jacquet–Langlands correspondence in the context of modular forms or automorphic forms, this description is quite natural to study.
(4) Though the statement of the theorem is representation theoretic we remark that we are interested in the realization of the Jacquet–Langlands correspondence in terms of the explicit construction of automorphic forms like Shimizu [**41**]. The construction of the theta lifts $\mathcal{L}(f,f')$'s is due to such motivation. We remark that, when f is a primitive form, the cuspidal representation $\pi'(f,\mathrm{JL}(f'))$ has a paramodular newform of level Dd_B (cf. Section 4.6), which should correspond to $\mathcal{L}(f,f')$ by the Jacquet–Langlands correspondence.

1.3. A brief explanation of the paper.

Let us explain the outline of the paper. In Section 2 we provide basic notation and facts necessary for the later argument. In Section 3 we introduce automorphic forms on $GSp(1,1)(\mathbb{A})$ and $GSp^*(2)(\mathbb{A})$ in our concern and formulate the theta lifts to the two inner forms. We then discuss the global Jacquet–Langlands–Shimizu correspondence for representations $\pi(f,f')$ and $\pi'(f,\mathrm{JL}(f'))$ in Section 4. These representations are shown to have the same spinor L-function (cf. Corollary 4.12). The main theorem, Theorem 4.13, asserts that the correspondence is compatible with the local Jacquet–Langlands correspondences for $GSp(2)$ and its inner form.

In the second work [**35**] we discuss non-vanishing of the theta lifts $\mathcal{L}(f,f')$ to $GSp(1,1)$ and $GSp^*(2)$. For this we remark that the non-vanishing of the theta lifts $\pi'(f,\mathrm{JL}(f'))$ can be said to be known in view of Roberts [**38**, Theorem 8.3], Przebinda [**37**] and Gan–Takeda [**9**] (cf. Section 4.4). In [**35**] we provide an explicit formula for Bessel periods of $\mathcal{L}(f,f')$ in terms of central L-values for the case of $GSp(1,1)$, and we apply it to our study on the non-vanishing of $\mathcal{L}(f,f')$. This result is announced in [**34**]. The work [**35**] is viewed as a generalization of the papers [**31**] and [**32**], which already provided examples of non-vanishing $\mathcal{L}(f,f')$'s on $GSp(1,1)(\mathbb{A})$ for the case of $\kappa_1=\kappa_2$. In [**35**] we also remark that Ibukiyama–Ihara [**19**] have given several examples of non-vanishing $\mathcal{L}(f,f')$'s for $GSp^*(2)$.

Notation. For a number filed F we denote by $\mathbb{A}_F$ the adele ring of F. When $F=\mathbb{Q}$ we denote $\mathbb{A}_{\mathbb{Q}}$ simply by $\mathbb{A}$ and the ring of finite adeles in $\mathbb{A}$ by $\mathbb{A}_f$. Given a $\mathbb{Q}$-algebra R and a $\mathbb{Q}$-algebraic group $\mathcal{G}$, $\mathcal{G}(R)$ denotes the group of R-valued points. When R is the p-adic field $\mathbb{Q}_p$ (respectively the field $\mathbb{R}$ of real numbers), we sometimes write $\mathcal{G}_p$ (respectively $\mathcal{G}_\infty$) for $\mathcal{G}(R)$. By $\mathrm{diag}(a_1,a_2,\ldots,a_n)$ we denote the diagonal matrix with the i-th diagonal entry a_i for $1\le i\le n$. For a group H, Z_H denotes the center of H.

2. Basic notations and facts.

2.1. Algebraic groups.

Let B be a definite quaternion algebra over $\mathbb{Q}$ and denote the discriminant of B by d_B, which is defined as the product of finite primes p's such that $B_p=B\otimes_{\mathbb{Q}}\mathbb{Q}_p$ is a division algebra. The real division algebra $\mathbb{H}:=B\otimes_{\mathbb{Q}}\mathbb{R}$ is the Hamilton quaternion algebra. Let $B\ni x\mapsto \bar{x}\in B$ be the main involution of B. By n and tr we denote the reduced norm and the reduced trace of B respectively.

Let $G_{\rm nc} = GSp(1,1)$ and $Sp(1,1)$ be the $\mathbb{Q}$-algebraic groups defined by

$$G_{\rm nc}(\mathbb{Q}) := \{g \in M_2(B) \mid {}^t\bar{g}Q_{\rm nc}g = \nu(g)Q_{\rm nc},\ \nu(g) \in \mathbb{Q}^\times\},$$
$$Sp(1,1)(\mathbb{Q}) := \{g \in G_{\rm nc}(\mathbb{Q}) \mid \nu(g) = 1\},$$

where $Q_{\rm nc} := \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. Furthermore let $G_{\rm c} = GSp^*(2)$ and $Sp^*(2)$ be the $\mathbb{Q}$-algebraic groups defined by

$$G_{\rm c}(\mathbb{Q}) := \{g \in M_2(B) \mid {}^t\bar{g}Q_{\rm c}g = \mu(g)Q_{\rm c},\ \mu(g) \in \mathbb{Q}^\times\},\ Sp^*(2)(\mathbb{Q}) := \{g \in G_{\rm c}(\mathbb{Q}) \mid \mu(g) = 1\},$$

where $Q_{\rm c} := \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. In what follows, we often denote $G_{\rm nc}$ or $G_{\rm c}$ simply by G. In addition to this, we introduce a $\mathbb{Q}$-algebraic group O_4^* defined by the group of $\mathbb{Q}$-rational points as follows:

$$O_4^*(\mathbb{Q}) := \{g \in M_2(B) \mid {}^t\bar{g}Rg = R\},$$

where $R := \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. It is known that the accidental isomorphism

$$O_4^*(\mathbb{R}) \simeq (SL(2;\mathbb{R}) \times \mathbb{H}^1)/\{\pm(1_2, 1)\}$$

holds, where $\mathbb{H}^1 := \{u \in \mathbb{H} \mid n(u) = 1\}$. Later we need this and the fact that $Sp(1,1) \times O_4^*$ and $Sp^*(2) \times O_4^*$ form dual pairs in the symplectic group $Sp(8)$ of degree eight (see the proof of Proposition 3.3 (2)).

On the other hand, let $G' = GSp(2)$ and $Sp(2)$ be the $\mathbb{Q}$-algebraic groups defined by

$$G'(\mathbb{Q}) := \{g \in GL_4(\mathbb{Q}) \mid {}^tgSg = \lambda(g)S,\ \lambda(g) \in \mathbb{Q}^\times\},\ Sp(2)(\mathbb{Q}) := \{g \in G'(\mathbb{Q}) \mid \lambda(g) = 1\},$$

where $S = \left(\begin{smallmatrix} 0_2 & 1_2 \\ -1_2 & 0_2 \end{smallmatrix}\right)$. We should note that $G = G_{\rm nc}$ or $G_{\rm c}$ is a non-split inner form of G'.

2.2. Maximal compact subgroups at ∞ and open compact subgroups at finite places.

Let $Q = Q_{\rm nc}$ or $Q_{\rm c}$. We first introduce maximal compact subgroups at the archimedean place. We put $G^1_\infty := \{g \in M_2(\mathbb{H}) \mid {}^t\bar{g}Qg = Q\}$, namely $Sp(1,1)(\mathbb{R})$ or $Sp^*(2)(\mathbb{R})$. Then G^1_∞ is the compact Lie group itself when $Q = Q_{\rm c}$, and

$$K_\infty := \left\{ \begin{pmatrix} a & b \\ b & a \end{pmatrix} \in M_2(\mathbb{H}) \mid a \pm b \in \mathbb{H}^1 \right\}$$

forms a maximal compact subgroup of G^1_∞ when $Q = Q_{\rm nc}$, where recall that $\mathbb{H}^1 := \{u \in \mathbb{H} \mid n(u) = 1\}$. The map $K_\infty \ni \left(\begin{smallmatrix} a & b \\ b & a \end{smallmatrix}\right) \mapsto (a+b, a-b) \in \mathbb{H}^1 \times \mathbb{H}^1$ gives rise to an isomorphism $K_\infty \simeq \mathbb{H}^1 \times \mathbb{H}^1$.

We put ${G'}^1_\infty := \{g \in GL_4(\mathbb{R}) \mid {}^tgSg = S\}$, namely $Sp(2)(\mathbb{R})$. Then

$$K'_\infty := \left\{ \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \in GL_4(\mathbb{R}) \;\middle|\; A + \sqrt{-1}B \in U(2) \right\}$$

is a maximal compact subgroup of ${G'}^1_\infty$, where $U(2) := \{X \in M_2(\mathbb{C}) \mid {}^t\bar{X}X = 1_2\}$

denotes the unitary group of degree two. The map ${K'}_\infty \ni \left(\begin{smallmatrix} A & B \\ -B & A \end{smallmatrix}\right) \mapsto A+\sqrt{-1}B \in U(2)$ induces an isomorphism ${K'}_\infty \simeq U(2)$.

Let us next introduce open compact subgroups at non-archimedean places for $G = GSp(1,1)$ or $GSp^*(2)$ and $G' = GSp(2)$. We remark that $GSp(1,1)$ and $GSp^*(2)$ are isomorphic to each other over $\mathbb{Q}_p$ for any $p < \infty$, and that these are isomorphic to $GSp(2)$ over $\mathbb{Q}_p$ for $p \nmid d_B$. In what follows, we thus identify $GSp^*(2)(\mathbb{Q}_p)$ with $GSp(1,1)(\mathbb{Q}_p)$ for any $p < \infty$, and G_p with G'_p for $p \nmid d_B$.

We let D be a divisor of d_B and fix a maximal order $\mathfrak{O}$ of B. For $p|d_B$ let $\mathfrak{P}_p$ be the maximal ideal of the p-adic completion $\mathfrak{O}_p$ of $\mathfrak{O}$ and let

$$\begin{cases} L_{1,p} := {}^t(\mathfrak{O}_p \oplus \mathfrak{O}_p) & (p \nmid d_B/D), \\ L_{2,p} := {}^t(\mathfrak{O}_p \oplus \mathfrak{P}_p^{-1}) & (p|d_B/D). \end{cases}$$

We introduce $K_{i,p} := \{k \in G_p \mid kL_{i,p} = L_{i,p}\}$ for $i = 1,\ 2$. Then, up to G_p-conjugation, every maximal compact subgroup of G_p is isomorphic to $K_{1,p}$ or $K_{2,p}$ for each finite prime p when $G = GSp(1,1)$ or $GSp^*(2)$ (cf. Section A.2 of the appendix).

Let us deal with the case of $GSp(2)$. For a non-negative power p^n of a prime p we put

$$K'_p(p^n) := \left\{ g \in GSp(2)(\mathbb{Q}_p) \cap \begin{pmatrix} \mathbb{Z}_p & \mathbb{Z}_p & p^{-n}\mathbb{Z}_p & \mathbb{Z}_p \\ p^n\mathbb{Z}_p & \mathbb{Z}_p & \mathbb{Z}_p & \mathbb{Z}_p \\ p^n\mathbb{Z}_p & p^n\mathbb{Z}_p & \mathbb{Z}_p & p^n\mathbb{Z}_p \\ p^n\mathbb{Z}_p & \mathbb{Z}_p & \mathbb{Z}_p & \mathbb{Z}_p \end{pmatrix} \;\middle|\; \lambda(g) \in \mathbb{Z}_p^\times \right\}.$$

We call this open compact subgroup of $GSp(2)(\mathbb{Q}_p)$ a paramodular subgroup of level p^n. From now on K'_p denotes $K'_p(N_p)$ with

$$N_p = \begin{cases} 1 & (p \nmid d_B), \\ p & (p|d_B/D), \\ p^2 & (p|D). \end{cases}$$

We remark that $K_{1,p} \simeq K'_p = GSp(2)(\mathbb{Z}_p)(:= GSp(2)(\mathbb{Q}_p) \cap GL(4)(\mathbb{Z}_p))$ for $p \nmid d_B$.

We now introduce open compact subgroups $K_f(D) := \prod_{p \nmid d_B/D} K_{1,p} \prod_{p|d_B/D} K_{2,p}$ and $K'_f(D) := \prod_{p<\infty} K'_p$ of $G(\mathbb{A}_f)$ and $G'(\mathbb{A}_f)$ respectively. For this we note that $G_{\mathrm{nc}}(\mathbb{A}_f) \simeq G_{\mathrm{c}}(\mathbb{A}_f)$ since $G_{\mathrm{nc}}(\mathbb{Q}_p) \simeq G_{\mathrm{c}}(\mathbb{Q}_p)$ for each $p < \infty$ as is remarked above. We further note that the subgroup $K_f(D)$ is a maximal open compact subgroup of $G(\mathbb{A}_f)$ and that every maximal open compact subgroup of $G(\mathbb{A}_f)$ is conjugate to some $K_f(D)$.

2.3. Representations at the archimedean place.

We will need admissible representations of G^1_∞, ${G'}^1_\infty$, G_∞ and ${G'}_\infty$, called discrete series representations or square integrable representations (modulo center). For the definition of admissible representations, see [**48**, p. 81] for instance.

2.3.1. Discrete series representations of G^1_∞ and ${G'}^1_\infty$.

We give Harish-Chandra's parametrization of discrete series representations of G^1_∞ and ${G'}^1_\infty$ (cf. [**14**, Theorem 16]). For a detail on this see [**21**, Theorem 9.20, Theorem

12.21]. We can realize such parametrization in $\mathbb{Z}^2$. Let $\mathfrak{g}$ (respectively $\mathfrak{g}'$) be the Lie algebra of G^1_∞ (respectively ${G'}^1_\infty$). The complexifications $\mathfrak{g}_\mathbb{C}$ and $\mathfrak{g}'_\mathbb{C}$ of $\mathfrak{g}$ and $\mathfrak{g}'$ are isomorphic to each other. The complex root system of the complexified Lie algebras with respect to a (complexified) compact Cartan subalgebra is given by

$$\Delta := \{\pm 2e_1,\ \pm 2e_2,\ \pm(e_1 - e_2),\ \pm(e_1 + e_2)\},$$

where $\{e_1,\ e_2\}$ denotes the standard basis of $\mathbb{Z}^2$. For $\mathfrak{g}_\mathbb{C}$ (respectively $\mathfrak{g}'_\mathbb{C}$) the standard choice of the compact positive roots is

$$\Delta_c^+ = \begin{cases} \{2e_1,\ 2e_2,\ e_1 \pm e_2\} & (Q = Q_\mathrm{c}), \\ \{2e_1,\ 2e_2\} & (Q = Q_\mathrm{nc}) \end{cases}$$

(respectively ${\Delta'}_c^+ = \{e_1 - e_2\}$). Let ρ_c (respectively ρ_n) denote the half sum of compact positive roots (respectively non-compact positive roots). Given a Harish–Chandra parameter $\lambda \in \mathbb{Z}^2$, $\Lambda := \lambda + \rho_n - \rho_c$ parametrizes the highest weight of the minimal K-type of the discrete series representation with Harish–Chandra parameter λ, where K stands for a maximal compact subgroup of G^1_∞ or ${G'}^1_\infty$ (Λ is called the Blattner parameter).

When $Q = Q_\mathrm{nc}$ the Lie algebra $\mathfrak{g}_\mathbb{C}$ has two positive systems

$$\Delta_I^+ = \{2e_1,\ 2e_2,\ e_1 \pm e_2\},$$
$$\Delta_{II}^+ = \{2e_1,\ 2e_2,\ \pm e_1 + e_2\}$$

containing Δ_c^+, while it has one positive system Δ_I^+ containing Δ_c^+ when $Q = Q_\mathrm{c}$. We now introduce the following two sets of weights

$$\Xi_I := \{(\lambda_1, \lambda_2) \in \mathbb{Z}^2_{>0} \mid \lambda_1 > \lambda_2\}, \quad \Xi_{II} := \{(\lambda_1, \lambda_2) \in \mathbb{Z}^2_{>0} \mid \lambda_2 > \lambda_1\},$$

which are regular dominant with respect to Δ_I^+ and Δ_{II}^+ respectively. When $Q = Q_\mathrm{c}$, the equivalence classes of discrete series representations of G^1_∞ are parametrized by Ξ_I. On the other hand, when $Q = Q_\mathrm{nc}$, the equivalence classes of discrete series representations of G^1_∞ are parametrized by the union of Ξ_I and Ξ_{II}.

When $Q = Q_\mathrm{c}$ the discrete series parametrized by $\lambda = (\lambda_1, \lambda_2) \in \Xi_I$ is nothing but the irreducible representation of G^1_∞ with the highest weight given by the dominant weight $\Lambda = (\lambda_1 - 2, \lambda_2 - 1)$. When $Q = Q_\mathrm{nc}$ the minimal K_∞-type of the discrete series parametrized by $\lambda = (\lambda_1, \lambda_2) \in \Xi_I$ (respectively Ξ_{II}) has the highest weight $\Lambda = (\lambda_1, \lambda_2 - 1)$ (respectively $(\lambda_1 - 1, \lambda_2)$).

On the other hand, $\mathfrak{g}'_\mathbb{C}$ has four positive systems containing ${\Delta'}_c^+$:

$$\begin{aligned} {\Delta'}_I^+ &= \{e_1 - e_2,\ e_1 + e_2,\ 2e_1,\ 2e_2\}, \\ {\Delta'}_{II}^+ &= \{e_1 - e_2,\ 2e_1,\ -2e_2,\ e_1 + e_2\}, \\ {\Delta'}_{III}^+ &= \{e_1 - e_2,\ 2e_1,\ -2e_2,\ -e_1 - e_2\}, \\ {\Delta'}_{IV}^+ &= \{e_1 - e_2,\ -2e_1,\ -2e_2,\ -e_1 - e_2\}. \end{aligned}$$

The equivalence classes of holomorphic discrete series and anti-holomorphic discrete series of ${G'}^1_\infty$ are parametrized by

$$\Xi'_I := \{(\lambda'_1, \lambda'_2) \in \mathbb{Z}^2_{>0} \mid \lambda'_1 > \lambda'_2\}, \quad \Xi'_{IV} := \{(\lambda'_1, \lambda'_2) \in \mathbb{Z}^2_{<0} \mid \lambda'_1 > \lambda'_2\},$$

which are regular dominant with respect to ${\Delta'}^+_I$ and ${\Delta'}^+_{IV}$ respectively. The equivalence classes of non-holomorphic (or large) discrete series and their contragredients of ${G'}^1_\infty$ are parametrized by

$$\Xi'_{II} := \{(\lambda'_1, \lambda'_2) \in \mathbb{Z}_{>0} \times \mathbb{Z}_{<0} \mid \lambda'_1 > -\lambda'_2\}, \quad \Xi'_{III} := \{(\lambda'_1, \lambda'_2) \in \mathbb{Z}_{>0} \times \mathbb{Z}_{<0} \mid \lambda'_1 < -\lambda'_2\},$$

which are regular dominant with respect to Δ'_{II} and Δ'_{III} respectively. We remark that all discrete series of ${G'}^1_\infty$ are exhausted by these four sets, up to equivalence. We call Ξ_I or $\Xi_I \cup \Xi_{II}$ (respectively $\Xi'_I \cup \Xi'_{II} \cup \Xi'_{III} \cup \Xi'_{IV}$) the set of Harish–Chandra parameters for G^1_∞ (respectively ${G'}^1_\infty$).

2.3.2. Square integrable representations modulo center.
We will need some irreducible admissible representations of G_∞ and G'_∞ square integrable modulo center (cf. [**23**, Section 3]). To explain them we remark that the four sets Ξ_I, Ξ_{II}, Ξ'_{II} and Ξ'_{III} are conjugate to each other by the Weyl group of $\mathfrak{g}_{\mathbb{C}} \simeq \mathfrak{g}'_{\mathbb{C}}$. More precisely, given $(\lambda_1, \lambda_2) \in \Xi_I$, the set consisting of

$$\lambda = \lambda_I := (\lambda_1, \lambda_2) \in \Xi_I,\ \lambda_{II} := (\lambda_2, \lambda_1) \in \Xi_{II},\ \lambda'_{II} := (\lambda_1, -\lambda_2) \in \Xi'_{II},\ \lambda'_{III} := (\lambda_2, -\lambda_1) \in \Xi'_{III}$$

is included in an orbit of the Weyl group. The discrete series representations with these parameters have the same L-parameter. The following proposition is viewed as a special case of the general theory in [**23**, Section 3] for G_∞ and G'_∞.

Proposition 2.1. (1) *For $\lambda = (\lambda_1, \lambda_2) \in \Xi_I$ there is the irreducible admissible representation π_λ of G_∞ satisfying the following*:

- *it has the trivial central character and is square integrable modulo center,*
- *its restriction to G^1_∞ is the direct sum of two discrete series representations with Harish–Chandra parameters $\lambda = \lambda_I$ and λ_{II} when $G = G_{\mathrm{nc}}$ (respectively the discrete series representation with Harish–Chandra parameter $\lambda = \lambda_I$ when $G = G_{\mathrm{c}}$).*

On the other hand, there is the irreducible admissible representation π'_λ of G'_∞ satisfying the following:

- *it has the trivial central character and is square integrable modulo center,*
- *its restriction to ${G'}^1_\infty$ is the direct sum of two discrete series representations with Harish–Chandra parameters λ'_{II} and λ'_{III}.*

(2) *The two representations π_λ and π'_λ have the same L-parameter, namely these two are involved in the archimedean local Jacquet–Langlands correspondence.*

The first assertion is a special case of [**23**, Lemma 3.5]. For the second assertion, see [**23**, pp. 38–44], which discusses how to define L-packets for square integrable representations modulo center. We remark that π_λ and π'_λ are representations induced from discrete series representations of G^1_∞ and ${G'}^1_\infty$ with the same L-parameter (cf. [**23**, p. 41]).

3. Automorphic forms on $G_{\rm nc} = GSp(1,1)$ and $G_{\rm c} = GSp^*(2)$.

3.1. Automorphic forms generating representations square integrable modulo center at the archimedean place.

Let $\Lambda = (\Lambda_1, \Lambda_2) \in (\mathbb{Z}_{\geq 0})^2$. For such Λ we introduce an irreducible representation of G_∞^1 for $G = G_{\rm c}$ or an irreducible representation of K_∞ for $G = G_{\rm nc}$. We denote it by τ_Λ. We then define two spaces of automorphic forms of weight τ_Λ for $G = G_{\rm c}$ or $G_{\rm nc}$.

We first let $G = G_{\rm c}$ and assume that $\Lambda_1 \geq \Lambda_2$ for $\Lambda = (\Lambda_1, \Lambda_2)$. By $(\tau_\Lambda, W_\Lambda)$ we mean the irreducible representation of G_∞^1 with highest weight Λ, where W_Λ denotes the representation space of τ_Λ.

Definition 3.1. Let $G = G_c$ and $D|d_B$. For $\Lambda = (\Lambda_1, \Lambda_2) \in \mathbb{Z}^2$ with $\Lambda_1 \geq \Lambda_2$ we introduce the space $\mathcal{A}^{\rm c}_{\tau_\Lambda}(D)$ of W_Λ-valued functions on $G(\mathbb{A})$ satisfying

$$F(z\gamma g k_f k_\infty) = \tau_\Lambda(k_\infty)^{-1} F(g)$$

for $(z, \gamma, g, k_f, k_\infty) \in Z_G(\mathbb{A}) \times G(\mathbb{Q}) \times G(\mathbb{A}) \times K_f(D) \times G_\infty^1$.

For this we note that, with any fixed $g_f \in G(\mathbb{A}_f)$, the right translations of the coefficients of $F(g_f *)$ by G_∞ generates the irreducible representation π_λ (cf. Section 2.3.2) with $\lambda = (\Lambda_1 + 2, \Lambda_2 + 1) \in \Xi_I$ (if F is non-zero).

We next let $G = G_{\rm nc}$. For a non-negative integer κ we let $(\sigma'_\kappa, V_\kappa)$ be the κ-th symmetric tensor representation of $GL_2(\mathbb{C})$ with the representation space V_κ and let σ_κ be the pull-back of σ'_κ to $\mathbb{H}^\times$ via the standard embedding $\mathbb{H} \subset M_2(\mathbb{C})$ given by

$$\mathbb{H} \ni x_0 + x_1 i + x_2 j + x_3 k \mapsto \begin{pmatrix} x_0 + \sqrt{-1}x_1 & x_2 + \sqrt{-1}x_3 \\ -(x_2 - \sqrt{-1}x_3) & x_0 - \sqrt{-1}x_1 \end{pmatrix} \in M_2(\mathbb{C}).$$

Here $\{1, i, j, k\}$ is the basis of $\mathbb{H}$ defined by

$$i^2 = j^2 = -1, \quad ij = -ji = k.$$

For $\Lambda = (\Lambda_1, \Lambda_2) \in \mathbb{Z}_{\geq 0}^{\oplus 2}$ we define an irreducible representation $(\tau_\Lambda, V_{\Lambda_1} \otimes V_{\Lambda_2})$ of K_∞ by

$$\tau_\Lambda\left(\begin{pmatrix} a & b \\ b & a \end{pmatrix}\right) := \sigma_{\Lambda_1}(a+b) \boxtimes \sigma_{\Lambda_2}(a-b), \ \left(\begin{pmatrix} a & b \\ b & a \end{pmatrix} \in K_\infty\right).$$

Definition 3.2. Let $G = G_{\rm nc}$ and $D|d_B$. For $\lambda = (\lambda_1, \lambda_2) \in \Xi_I$ let π_λ be the irreducible admissible representation of G_∞ square integrable modulo center (cf. Section 2.3.2) and let $\Lambda = (\Lambda_1, \Lambda_2) := (\lambda_1, \lambda_2 - 1)$. We then introduce the space $\mathcal{S}^{\rm nc}_{\tau_\Lambda}(D)$ of $V_{\Lambda_1} \boxtimes V_{\Lambda_2}$-valued cusp forms F on $G(\mathbb{A})$ satisfying the following:

1. $F(z\gamma g k_f k_\infty) = \tau_\Lambda(k_\infty)^{-1} F(g)$ for $(z, \gamma, g, k_f, k_\infty) \in Z_G(\mathbb{A}) \times G(\mathbb{Q}) \times G(\mathbb{A}) \times K_f(D) \times K_\infty$,
2. for each fixed $g_f \in G(\mathbb{A}_f)$, the right translations of the coefficients of $F(g_f *)$ by G_∞ generate π_λ (cf. Section 2.3.2) as admissible representations of G_∞ (if F is non-zero).

For this definition we remark that τ_Λ is the minimal K_∞-type of the discrete series representation of G^1_∞ with Harish–Chandra parameter $\lambda = (\lambda_1, \lambda_2) \in \Xi_I$.

3.2. Theta lifts to G.

3.2.1. Automorphic forms on $GL(2)$ and $B^\times$.

Let H and H' be $\mathbb{Q}$-algebraic groups defined by

$$H(\mathbb{Q}) = GL_2(\mathbb{Q}),\ H'(\mathbb{Q}) := B^\times$$

respectively. For a positive even integer κ_1 we let $S_{\kappa_1}(D)$ be the space of elliptic cusp forms of weight κ_1 with level D (cf. [**31**, Section 3.1]), namely $f \in S_{\kappa_1}(D)$ is right-invariant with respect to $U_f(D) := \prod_{p<\infty} U_p$ with $U_p := \{(u_{ij}) \in GL_2(\mathbb{Z}_p) \mid u_{21} \in D\mathbb{Z}_p\}$. For a non-negative even integer κ_2 we let $\mathcal{A}_{\kappa_2}$ be the space of automorphic forms of weight σ_{κ_2} with respect to $\prod_{v<\infty} \mathfrak{O}_v^\times$ (cf. [**31**, Section 3.2]), where $\mathfrak{O}_v^\times$ denotes the unit group of $\mathfrak{O}_v$ (for $\mathfrak{O}$ and $\mathfrak{O}_v$ see Section 2.2). In what follows, we will often assume that $(f, f') \in S_{\kappa_1}(D) \times \mathcal{A}_{\kappa_2}$ are Hecke eigenforms (for the definition see [**31**, Sections 3.1, 3.2]).

We note that f (respectively $\bar{f}$) generates the discrete series representation of lowest weight $-\kappa_1$ (respectively lowest weight κ_1) as an admissible representation of $SL_2(\mathbb{R})$, and that f' generates the irreducible representation σ_{κ_2} of $\mathbb{H}^1$. Assuming that $(\kappa_1, \kappa_2) \in (2\mathbb{Z}_{>0})^2$ satisfies $1 < \kappa_1 < \kappa_2 + 2$ or $1 < \kappa_2 + 2 < \kappa_1$, we then know that $(\bar{f}, f')$ generates the discrete series representation of $O^*_4(\mathbb{R}) \simeq (SL_2(\mathbb{R}) \times \mathbb{H}^1)/\{\pm(1_2, 1)\}$ (for this isomorphism see Section 2.1) with Harish–Chandra parameter $((\kappa_1+\kappa_2)/2, -(\kappa_2-\kappa_1)/2-1)$ as an admissible representation of $O^*_4(\mathbb{R})$, where see [**24**, Section 2.2]) for Harish–Chandra's parametrization of discrete series of $O^*_4(\mathbb{R})$. We use this fact for the proof of Proposition 3.3.

3.2.2. Theta lifts to automorphic forms on G.

We shall introduce theta lifts from $H \times H'$ to G_{nc} and G_{c}, following the formulation as in [**30**]. The archimedean representations of our theta lifts are more general than those in [**30**].

For $v = p < \infty$ let $\mathbb{V}_p$ be the space of locally constant compactly supported functions on $B_p^2 \times \mathbb{Q}_p^\times$. We let $\varphi_{0,p} \in \mathbb{V}_p$ be the characteristic function of $L_p \times \mathbb{Z}_p^\times$, where see Section 2.2 for L_p.

Let $\mathcal{S}(\mathbb{H}^2)$ stand for the space of Schwartz functions on $\mathbb{H}^2$ and let $\mathcal{H}_{\kappa_1-4}$ denote the space of homogeneous harmonic polynomials of degree $\kappa_1 - 4$ on $\mathbb{H}^2$. For $v = \infty$ we then introduce the function space $\mathbb{V}_\infty$. When $G = G_{\mathrm{nc}}$ (respectively $G = G_{\mathrm{c}}$) it stands for the space of smooth functions φ on $\mathbb{H}^2 \times \mathbb{R}^\times$ such that, for each fixed $t \in \mathbb{R}^\times$, $\mathbb{H}^2 \ni X \mapsto \varphi(X, t)$ belongs to $\mathcal{S}(\mathbb{H}^2) \otimes \mathrm{End}(V_{(\kappa_1+\kappa_2)/2} \boxtimes V_{(\kappa_2-\kappa_1)/2})$ (respectively $\mathcal{S}(\mathbb{H}^2) \otimes \mathcal{H}_{\kappa_1-4}$) for $(\kappa_1, \kappa_2) \in (2\mathbb{Z}_{\geq 0})^2$ with $\kappa_1 \leq \kappa_2$ (respectively $\kappa_1 \in 2\mathbb{Z}_{\geq 0}$ with $\kappa_1 \geq 4$).

Let $G = G_{\mathrm{nc}}$. For $(\kappa_1, \kappa_2) \in (2\mathbb{Z}_{\geq 0})^2$ with $\kappa_1 \leq \kappa_2$ we define $\varphi^{\mathrm{nc}}_{0,\infty}(X, t)$ by

$$\varphi^{\mathrm{nc}}_{0,\infty}(X,t) := \begin{cases} t^{(\kappa_2+3)/2} \sigma_{(\kappa_1+\kappa_2)/2}(X_1 + X_2) \boxtimes \sigma_{(\kappa_2-\kappa_1)/2}(X_1 - X_2) \exp(-2\pi t {}^t\bar{X} X) & (t > 0), \\ 0 & (t < 0). \end{cases}$$

Let $G = G_{\mathrm{c}}$. We note that, given an inner product $(*, *)$ of $\mathcal{H}_{\kappa_1-4}$, there is a reproducing kernel function

$$\mathbb{H}^2 \ni X \mapsto C_X \in \mathcal{H}_{\kappa_1-4},$$

which satisfies $(C_X, \Phi) = \Phi(X)$ for $\Phi \in \mathcal{H}_{\kappa_1-4}$. Modifying [**25**, Definition 6.1], we define $\varphi^{\mathrm{c}}_{0,\infty}(X,t)$ by

$$\varphi^{\mathrm{c}}_{0,\infty}(X,t) := \begin{cases} t^{(\kappa_1-1)/2}\exp(-2\pi t\,{}^t\bar{X}X)C_X & (t>0), \\ 0 & (t<0). \end{cases}$$

For this case we have a remark necessary to consider the theta lift to G_{c} with this $\varphi^{\mathrm{c}}_{0,\infty}$. Recall that, for $(a,b) \in (\mathbb{Z}_{\geq 0})^2$ such that $a \geq b$, $(\tau_{(a,b)}, W_{(a,b)})$ denotes the irreducible representation of $G^1_\infty = Sp^*(2)(\mathbb{R})$ with highest weight (a,b) (cf. Section 3.1). As a unitary representation of $Sp^*(2)(\mathbb{R}) \times \mathbb{H}^1$, $\mathcal{H}_{\kappa_1-4}$ has a decomposition

$$\mathcal{H}_{\kappa_1-4} = \bigoplus_{\substack{a\geq b\geq 0 \\ a+b=\kappa_1-4}} W_{(a,b)} \boxtimes V_{a-b} \tag{3.1}$$

(cf. [**19**, Section 1.2]). We therefore know that there is a $Sp^*(2)(\mathbb{R})$-equivariant $W_{(a,b)}$-valued paring $(*,*)_{a,b}$ of $\mathcal{H}_{\kappa_1-4} \times V_{a-b}$, which is unique up to constant multiples.

Following [**30**, Section 3] we introduce a metaplectic representation $r = \bigotimes'_{v\leq\infty} r_v$ of $G(\mathbb{A}) \times H(\mathbb{A}) \times H'(\mathbb{A})$ on the restricted tensor product $\mathbb{V} = \bigotimes'_{v\leq\infty} \mathbb{V}_v$ with respect to $\{\varphi_{0,p}\}_{p<\infty}$. It is associated with the standard additive character ψ of $\mathbb{A}/\mathbb{Q}$, which satisfies $\psi(a) = \exp(2\pi\sqrt{-1}a)$ for $a \in \mathbb{R}$. For $G = G_{\mathrm{nc}}$ (respectively $G = G_{\mathrm{c}}$) we define the End $\big(V_{(\kappa_1+\kappa_2)/2} \boxtimes V_{(\kappa_2-\kappa_1)/2}\big)$-valued (respectively $\mathcal{H}_{\kappa_1-4}$-valued) theta function $\theta_{\kappa_1,\kappa_2}(g,h,h')$ by

$$\sum_{(X,t)\in B^2\times\mathbb{Q}^\times} r(g,h,h')\varphi_0(X,t),$$

where $\varphi_0 := \prod_{v\leq\infty} \varphi_{0,v}$ with

$$\varphi_{0,\infty} := \begin{cases} \varphi^{\mathrm{nc}}_{0,\infty} & (G = G_{\mathrm{nc}}), \\ \varphi^{\mathrm{c}}_{0,\infty} & (G = G_{\mathrm{c}}). \end{cases}$$

When $G = G_{\mathrm{nc}}$ (respectively $G = G_{\mathrm{c}}$) we consider the theta lift

$$S_{\kappa_1}(D) \times \mathcal{A}_{\kappa_2} \ni (f,f') \mapsto \mathcal{L}(f,f')(g)$$

for $(\kappa_1,\kappa_2) \in (2\mathbb{Z}_{>0})^2$ with $\kappa_1 \leq \kappa_2$ (respectively with $\kappa_1 \geq \kappa_2$ and $\kappa_1 \geq 4$), where $\mathcal{L}(f,f')(g) :=$

$$\begin{cases} \displaystyle\int_{\mathbb{R}_+^2(H\times H')(\mathbb{Q})\backslash(H\times H')(\mathbb{A})} \overline{f(h)}\theta_{\kappa_1,\kappa_2}(g,h,h')f'(h')dhdh' & (G=G_{\rm nc}),\\ \displaystyle\int_{\mathbb{R}_+^2(H\times H')(\mathbb{Q})\backslash(H\times H')(\mathbb{A})} \overline{f(h)}(\theta_{\kappa_1,\kappa_2}(g,h,h'),f'(h'))_{(\kappa_1+\kappa_2)/2-2,(\kappa_1-\kappa_2)/2-2}dhdh' & \\ & (G=G_{\rm c}).\end{cases}$$

For the case of $G_{\rm nc}$ we note that, as a representation of $\mathbb{H}^1$, σ_{κ_2} occurs with multiplicity one in the restriction of $\sigma_{(\kappa_1+\kappa_2)/2}\boxtimes\sigma_{(\kappa_2-\kappa_1)/2}$ to $\left\{\left(\begin{smallmatrix} a & 0\\ 0 & a\end{smallmatrix}\right)\mid a\in\mathbb{H}^1\right\}$ (cf. [**46**, Section 6.9.5, (2)]). As for the case of $G_{\rm c}$ note that the paring $(*,*)_{(\kappa_1+\kappa_2)/2-2,(\kappa_1-\kappa_2)/2-2}$ has been defined on $\mathcal{H}_{\kappa_1-4}\times V_{\kappa_2}$. We then know that the integral above with respect to the archimedean part h'_∞ of h' is well-defined. In addition, for $G_{\rm nc}$ and $G_{\rm c}$, note that $\theta_{\kappa_1,\kappa_2}(g,h,h')$ satisfies the invariance with respect to $\prod_{v<\infty}\mathfrak{O}_v^\times\times B^\times$ as f' satisfies (cf. [**30**, Sections 3, 4]). Hence we can say that the integral representing $\mathcal{L}(f,f')$ is well-defined with respect to h' for both of $G_{\rm nc}$ and $G_{\rm c}$.

We furthermore note that, as a function in h, $\theta_{\kappa_1,\kappa_2}(g,h,h')$ is shown to satisfy the same automorphy as $S_{\kappa_1}(D)$. The proof is similar to [**30**, Sections 3, 4]. For this we remark that the proof of the equivariance with respect to "U_∞"($\simeq SO(2)$) needs [**30**, Lemma 3.2] for $\varphi_{0,\infty}^{\rm nc}$ and $\varphi_{0,\infty}^{\rm c}$. It is obtained by considering [**33**, Proposition 3.3] for the case of "$(\nu_1,\nu_2)=((\kappa_1+\kappa_2)/2,(\kappa_2-\kappa_1)/2)$ and $(p,q)=(4,4)$" when $G=G_{\rm nc}$ (respectively for the case of "$(\nu_1,\nu_2)=(\kappa_1-4,0)$ and $(p,q)=(8,0)''$ when $G=G_{\rm c}$). This leads to [**33**, Lemma 3.8] and thus [**30**, Lemma 3.2] for $\varphi_{0,\infty}^{\rm nc}$ and $\varphi_{0,\infty}^{\rm c}$. We remark that the case of $G_{\rm c}$ is also due to [**15**, Section 6] or [**19**, Section 2.1]. As a result we can also say that the integral representing $\mathcal{L}(f,f')$ is well-defined with respect to h.

Proposition 3.3. *Let $(\kappa_1,\kappa_2)\in(2\mathbb{Z}_{\geq 0})^{\oplus 2}$.*
(1) *Suppose that (f,f') are Hecke eigenforms. Then $\mathcal{L}(f,f')$ is also a Hecke eigenform. Furthermore, for each $p|D$, let ϵ_p (respectively ϵ'_p) be the eigenvalue for the Atkin–Lehner involution, i.e. the involutive action of $\left(\begin{smallmatrix} 0 & 1\\ -p & 0\end{smallmatrix}\right)$ (respectively a prime element $\varpi_{B,p}\in B_p$) on f (respectively f'). Then $\mathcal{L}(f,f')\equiv 0$ unless $\epsilon_p=\epsilon'_p$.*
(2) *Assume that $1<\kappa_1<\kappa_2+2$ when $G=G_{\rm nc}$ (respectively $1<\kappa_2+2<\kappa_1$ when $G=G_{\rm c}$). Then we have $\mathcal{L}(f,f')\in\mathcal{S}_{\tau_\Lambda}^{\rm nc}(D)$ with Harish–Chandra parameter $\lambda=((\kappa_1+\kappa_2)/2,(\kappa_2-\kappa_1)/2+1)$ (respectively $\mathcal{L}(f,f')\in\mathcal{A}_{\tau_\Lambda}^{\rm c}(D)$ with Harish–Chandra parameter $\lambda=((\kappa_2+\kappa_1)/2,(\kappa_1-\kappa_2)/2-1)$).*

Proof. (1) The proof of [**30**, Theorem 5.1] is useful also for our situation, which implies the first assertion. The second assertion is due to [**30**, Remark 5.2 (ii)].
(2) Let $G=G_{\rm nc}$. Following the argument in [**30**, Section 4], we can show the convergence of $\theta_{\kappa_1,\kappa_2}(g,h,h')$ and further see that, as a function in $g\in G(\mathbb{A})$, $\theta_{\kappa_1,\kappa_2}(g,h,h')$ satisfies the automorphy of $\mathcal{S}_{\tau_\Lambda}^{\rm nc}(D)$ with $\Lambda=((\kappa_1+\kappa_2)/2,(\kappa_2-\kappa_1)/2)$, which implies such automorphy of $\mathcal{L}(f,f')(g)$. By the argument in [**30**, Section 4] we then verify that $\mathcal{L}(f,f')(g)$ is convergent on any compact subset of $G(\mathbb{A})$. In fact, as in the proof of [**30**, Theorem 4.1], we can reduce its convergence to that of the restriction of $\mathcal{L}(f,f')$ to G_∞^1. We can then show the convergence by following the proof of [**33**, Proposition 4.2]. For the proof we should note the convergence of the theta series introduced at [**33**, (3.1)], particularly those for the case of "$(\nu_1,\nu_2)=((\kappa_1+\kappa_2)/2,(\kappa_2-\kappa_1)/2)$ and

$V = \mathbb{H}^2 \simeq \mathbb{R}^8$". We can further verify that $\mathcal{L}(f,f')$ is cuspidal by the same reasoning as in the proof of [**31**, Proposition 2.4.1].

Let us note that $Sp(1,1) \times O_4^*$ forms a dual pair in the symplectic group $Sp(8)$ of degree eight. For the local metaplectic representation r_v the transformation law of $r_v|_{Sp(1,1)(\mathbb{Q}_v)\times O_4^*(\mathbb{Q}_v)}$ turns out to be that of the restriction of the Weil representation of $Sp(8)(\mathbb{Q}_v)$ (cf. [**47**]) to $Sp(1,1)(\mathbb{Q}_v)\times O_4^*(\mathbb{Q}_v)$. As we did in the proof of [**30**, Theorem 4.1], we can view $\mathcal{L}(f,f')|_{G^1_\infty}(g_f*)$ with any fixed $g_f \in G(\mathbb{A}_f)$ as a finite linear combination of non-adelic theta lifts from $(SL(2;\mathbb{R}) \times \mathbb{H}^1)/\{\pm(1_2,1)\} \simeq O^*(4)(\mathbb{R})$ to G^1_∞. We thereby see that, at the archimedean component, the coefficient functions of $\mathcal{L}(f,f')|_{G^1_\infty}(g_f*)$ as above generate an admissible G^1_∞-module isomorphic to a subquotient of the admissible G^1_∞-module in the theta correspondence with the discrete series representation of $O_4^*(\mathbb{R})$ with Harish–Chandra parameter $((\kappa_1+\kappa_2)/2, -(\kappa_2-\kappa_1)/2-1)$ (for "the admissible G^1_∞-module in the theta correspondence", see the notation "ρ_1'" in Howe [**16**, Theorem 1A]). Here see Section 3.2.1) for the discrete series of $O^*(4)$. Due to the general theory on the archimedean local theta correspondence by Howe [**16**, Theorem 1A], the admissible G^1-module in the theta correspondence is quasi-simple and has a unique irreducible subquotient. According to Li–Paul–Tan–Zhu [**24**, Theorem 5.1], such irreducible subquotient is the discrete series representation of G^1_∞ with Harish–Chandra parameter $((\kappa_1+\kappa_2)/2, (\kappa_2-\kappa_1)/2+1)$.

Now recall that, as is well-known, the cuspidal spectrum decomposes discretely into irreducible pieces with finite multiplicities and that the restriction of an irreducible admissible representation of G_∞ to G^1_∞ is a finite sum of irreducible admissible representation of G^1_∞, in fact, a sum of an irreducible admissible representation and its conjugation by $\left(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right)$ (cf. [**23**, Lemma 3.5]). The G^1_∞-module generated by coefficient functions of $\mathcal{L}(f,f')(g_f*)$ with any fixed $g_f \in G(\mathbb{A}_f)$ is (at most) a finite sum of irreducible admissible representations. We therefore see that, when $\mathcal{L}(f,f') \not\equiv 0$, such G^1_∞-module is irreducible and is isomorphic to the discrete series representation mentioned above as admissible representations of G^1_∞ since the K_∞-module generated by the coefficient functions is isomorphic to the minimal K_∞-type of the discrete series representation (which occurs with multiplicity one in the discrete series). Next let us consider the admissible G_∞-module generated by the coefficients of $\mathcal{L}(f,f')(g_f*)$. Its restriction to G^1_∞ is the sum of the discrete series representation above and its conjugation by $\left(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right)$, and we thus know that such G_∞-module is nothing but π_λ with $\lambda = ((\kappa_1+\kappa_2)/2, (\kappa_2-\kappa_1)/2+1)$ in the assertion.

On the other hand, let $G = G_{\mathrm{c}}$. The proof of the convergence for $\theta_{\kappa_1,\kappa_2}(g,h,h')$ is similar to the case of $G = G_{\mathrm{nc}}$. Let us see that $\theta_{\kappa_1,\kappa_2}(g,h,h')$ satisfies the automorphy of $\mathcal{A}^{\mathrm{c}}_{\tau_\Lambda}(D)$ as a function in $g \in G(\mathbb{A})$. Its right G^1_∞-equivariance by τ_Λ with $\Lambda = ((\kappa_2+\kappa_1)/2-2, (\kappa_1-\kappa_2)/2-2)$ follows from the decomposition of $\mathcal{H}_{\kappa_1-4}$ into irreducible pieces as a $Sp^*(2)(\mathbb{R})\times\mathbb{H}^1$-module (cf. (3.1)). Its right $K_f(D)$-invariance and left $G(\mathbb{Q})$-invariance are verified by the argument in [**30**, Section 4]. We next discuss the convergence of $\mathcal{L}(f,f')$ for this case. The function

$$H(\mathbb{A}) \ni h \mapsto \int_{\mathbb{R}_+H'(\mathbb{Q})\backslash H'(\mathbb{A})} (\theta_{\kappa_1,\kappa_2}(g,h,h'), f'(h'))_{(\kappa_1+\kappa_2)/2-2,(\kappa_1-\kappa_2)/2-2}\, dh'$$

with any fixed $g \in G_{\mathrm{c}}(\mathbb{A})$ is $W_{((\kappa_1+\kappa_2)/2-2,(\kappa_1-\kappa_2)/2-2)}$-valued and its coefficient functions turn out to be elliptic modular forms of weight κ_1 and level D (cf. [**15**, Section 6], [**19**, Section 2.1]), for which note that we have already remarked just before the proposition that $\theta_{\kappa_1,\kappa_2}(g,h,h')$ satisfies the same automorphy as that of the elliptic modular forms just mentioned. The convergence of the integral representing $\mathcal{L}(f,f')$ is thus reduced to that of the Petersson inner product of an elliptic modular form and an elliptic cusp form, which leads to the convergence of $\mathcal{L}(f,f')$ on any compact subset of $G_{\mathrm{c}}(\mathbb{A})$, in fact, on G^1_∞.

For this case note that $Sp^*(2) \times O^*_4$ forms a dual pair in $Sp(8)$. We then remark that [**24**, Theorem 5.1] determines the archimedean representation generated by $\mathcal{L}(f,f')$, which means another proof of the right G^1_∞-equivariance by τ_Λ mentioned above. □

Remark 3.4. (1) The condition $1 < \kappa_1 < \kappa_2 + 2$ (respectively $1 < \kappa_2 + 2 < \kappa_1$) means that the Harish–Chandra parameter $((\kappa_1+\kappa_2)/2, -(\kappa_2-\kappa_1)/2-1)$ of $O^*_4(\mathbb{R})$ and the parameter $((\kappa_1+\kappa_2)/2, (\kappa_2-\kappa_1)/2+1)$ of $Sp(1,1)(\mathbb{R})$ (respectively Harish–Chandra parameter $((\kappa_1+\kappa_2)/2, (\kappa_1-\kappa_2)/2-1)$ of $O^*_4(\mathbb{R})$ and $Sp^*(2)(\mathbb{R})$) are regular.
(2) If we consider the theta correspondence between discrete series representations of $G^1_\infty = Sp(1,1)(\mathbb{R})$ and $O^*_4(\mathbb{R})$ under the assumption that $\kappa_1 > \kappa_2$ we have to think of the discrete series representation of $O^*_4(\mathbb{R})$ with Harish–Chandra parameter $((\kappa_1+\kappa_2)/2, (\kappa_1-\kappa_2)/2-1)$, whose regularity condition means $1 < \kappa_2+2 < \kappa_1$. However, according to [**24**, Theorem 5.1], the theta lift from such discrete series representation of $O^*_4(\mathbb{R})$ to $Sp(1,1)(\mathbb{R})$ vanishes.

4. The Jacquet–Langlands–Shimizu correspondence for theta lifts.

In this section we study the automorphic representations generated by theta lifts from Hecke eigenforms $(f,f') \in S_{\kappa_1}(D) \times \mathcal{A}_{\kappa_2}$ to $G(\mathbb{A})$ and $G'(\mathbb{A})$. We understand them in terms of the global Jacquet–Langlands–Shimizu correspondence for $G' = GSp(2)$ and the inner form $G = GSp(1,1)$ or $GSp^*(2)$. We also show the coincidence of the global spinor L-functions for the lifts.

4.1. Conjecture.

We recall that the L-group ${}^LG$ of G is the same as the L-group ${}^LG'$ of G', where ${}^LG = {}^LG'$ is the direct product of $GSp(2)(\mathbb{C})$ and the Weil group of $\mathbb{Q}$ (for the definition of L-groups, see [**22**] and [**4**]). As the choice of the L-morphism between ${}^LG$ and ${}^LG'$ we can take the identity map. The Langlands principle of functoriality predicts the following:

Conjecture 4.1 (Langlands). *The L-morphism induced by the identity map would give rise to the natural transfer from equivalence classes of irreducible automorphic representations of $G(\mathbb{A})$ to those of $G'(\mathbb{A})$ which preserves L-functions. Namely an L-function of an irreducible automorphic representation of $G(\mathbb{A})$ is one of some irreducible automorphic representation of $G'(\mathbb{A})$.*

We shall formulate another kind of the functoriality conjecture for some automorphic representations of $G(\mathbb{A})$ and $G'(\mathbb{A})$, which was first essentially pointed out by Ibukiyama

[**17**].

Let $\mathcal{A}_G$ and $\mathcal{A}_{G'}$ denote the equivalence classes of irreducible automorphic representations of $G(\mathbb{A})$ and $G'(\mathbb{A})$ with trivial central characters, respectively. We recall that, for a divisor D of d_B, we have introduced a maximal open compact subgroup $K_f(D) := \prod_{p\nmid d_B/D} K_{1,p} \prod_{p\mid d_B/D} K_{2,p}$ of $G(\mathbb{A}_f) = G_{\rm nc}(\mathbb{A}_f) = G_{\rm c}(\mathbb{A}_f)$ and an open compact subgroup $K'_f(D) := \prod_{p<\infty} K'_p$ of $G'(\mathbb{A}_f) = GSp(\mathbb{A}_f)$ in Section 2.2. Let us introduce

$$\mathcal{A}_G(K_f(D)) := \left\{ \pi = {\bigotimes_{v\le\infty}}' \pi_v \in \mathcal{A}_G \;\middle|\; \begin{array}{c} \pi_p \text{ has a } K_{1,p}\text{-fixed vector for } p \nmid \dfrac{d_B}{D} \\ \text{and a } K_{2,p}\text{-fixed vector for } p \mid \dfrac{d_B}{D} \end{array} \right\},$$

$$\mathcal{A}^{\rm new}_{G'}(K'_f(D)) := \left\{ \pi' = {\bigotimes_{v\le\infty}}' \pi'_v \in \mathcal{A}_{G'} \;\middle|\; \begin{array}{c} \pi'_p \text{ has a } K'_p\text{-fixed vector for } v = p < \infty \\ \text{which is a paramodular new vector} \end{array} \right\}.$$

Here the levels of the paramodular new vectors above are given by N_p in Section 2.2. We now formulate the conjecture as follows:

Conjecture 4.2. *The transfer in Conjecture* 4.1 *would map* $\pi \in \mathcal{A}_G(K_f(D))$ *to some* $\pi' \in \mathcal{A}^{\rm new}_{G'}(K'_f(D))$*, and an L-function of* $\pi \in \mathcal{A}_G(K_f(D))$ *is one of* $\pi' \in \mathcal{A}^{\rm new}_{G'}(K'_f(D))$*.*

For this conjecture we remark that, as evidence in terms of non-archimedean local theory, we have [**39**, Section A.8, Table A.13] and the table of irreducible admissible representations of $G(\mathbb{Q}_p) = G_{\rm nc}(\mathbb{Q}_p) = G_{\rm c}(\mathbb{Q}_p)$ and $G'(\mathbb{Q}_p)$ in Section A.4 of the appendix. We provide "global evidence" of this conjecture with theta lifts to $G_{\rm nc}$, $G_{\rm c}$ and G' including those introduced in Section 3.2. We will see that the global spinor L-functions of the lifts coincide with each other.

4.2. Automorphic representations for $GL(2)$ and $B^\times$.

For Hecke eigenforms $(f, f') \in S_{\kappa_1}(D) \times \mathcal{A}_{\kappa_2}$ let $\pi(f)$ be the automorphic representation of $GL_2(\mathbb{A})$ generated by f and $\mathrm{JL}(\pi(f'))$ be the Jacquet–Langlands lift of the automorphic representation $\pi(f')$ generated by f'. The Hecke equivariant isomorphism between $\mathcal{A}_{\kappa_2}$ and the space spanned by primitive forms in $S_{\kappa_2+2}(d_B)$ sends a Hecke eigenform f' to a primitive form $\mathrm{JL}(f')$ (cf. [**7**], [**8**], [**41**, Section 6]). The automorphic representation $\mathrm{JL}(\pi(f'))$ is nothing but that generated by $\mathrm{JL}(f')$. We describe each local component of the automorphic representations $\pi(f)$, $\pi(f')$ and $\mathrm{JL}(\pi(f'))$.

4.2.1. Local representations at finite places.

Assume that f is primitive, and decompose $\pi(f)$ into the restricted tensor product $\bigotimes'_{v\le\infty} \pi(f)_v$ of local representations. Then, for $v = p \nmid D$, $\pi(f)_p$ is an unramified principal series representation of $GL_2(\mathbb{Q}_p)$. Let $\chi_{f,p}$ denote the unramified character of $\mathbb{Q}_p^\times$ which induces $\pi(f)_p$. Then the eigenvalue $\lambda(f)_p$ of f for the Hecke operator defined by the double coset $GL_2(\mathbb{Z}_p) \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) GL_2(\mathbb{Z}_p)$ can be written as

$$\lambda(f)_p = p^{1/2}(\chi_{f,p}(p) + \chi_{f,p}(p)^{-1}) \tag{4.1}$$

(cf. [**5**, Proposition 4.6.6]).

Let $p|D$. Now recall that ϵ_p denotes the eigenvalue of the Atkin Lehner involution of f at $p|D$ (see Proposition 3.3 (1)). Let $\lambda^+(f)_p$ (respectively $\lambda^-(f)_p$) be the eigenvalue of f for the Hecke operator defined by the double coset $U_p \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) U_p$ (respectively $U_p \left(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix}\right) U_p$), where see Section 3.2.1 for U_p. It is verified that

$$\lambda^+(f)_p = \lambda^-(f)_p = -\epsilon_p \tag{4.2}$$

(cf. [**27**, Theorem 4.6.17 (2)], [**29**, Section 3.3]), for which note that the conjugation by $\left(\begin{smallmatrix} 0 & 1 \\ -p & 0 \end{smallmatrix}\right)$ relates the two double cosets. Let δ'_p be the unramified character of $\mathbb{Q}_p^\times$ of order at most two such that $\delta'_p(p) = -\epsilon_p$. The representation $\pi(f)_p$ is a special representation of $GL_2(\mathbb{Q}_p)$ given by the irreducible subrepresentation or subquotient of the induced representation of $GL_2(\mathbb{Q}_p)$ associated with two quasi-characters $\nu_p^{1/2} \cdot \delta'_p$ and $\nu_p^{-1/2} \cdot \delta'_p$, where ν_p denotes the normalized p-adic valuation of $\mathbb{Q}_p$. Namely, $\pi(f)_p$ is the Steinberg representation of $GL_2(\mathbb{Q}_p)$ twisted by δ'_p.

For a Hecke eigenform $f' \in \mathcal{A}_{\kappa_2}$ let $\pi(f')$ be the irreducible automorphic representation of $H'(\mathbb{A})$ generated by f', and let $\pi(f') = \bigotimes'_{v \leq \infty} \pi(f')_v$ be the decomposition into the restricted tensor product of local representations.

When $p \nmid d_B$, $\pi(f')_p$ is an unramified principal series representation of $B_p^\times \simeq GL_2(\mathbb{Q}_p)$. We let $\chi_{f',p}$ be the unramified character of $\mathbb{Q}_p^\times$ inducing $\pi(f')_p$. As in f, the Hecke eigenvalue $\lambda(f')_p$ of f' for the Hecke operator defined by $GL_2(\mathbb{Z}_p) \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) GL_2(\mathbb{Z}_p)$ can be written similarly as

$$\lambda(f')_p = p^{1/2}(\chi_{f',p}(p) + \chi_{f',p}(p)^{-1}). \tag{4.3}$$

When $p|d_B$, $\pi(f')_p$ is a unramified character of $B_p^\times$ of order at most two. Thus we have

$$\pi(f')_p = \delta_p \circ n$$

with a unramified character δ_p of $\mathbb{Q}_p^\times$ of order at most two, where recall that the notation n stands for the reduced norm of B (cf. Section 2.1). In view of Proposition 3.3 (1),

$$\delta_p(p) = \epsilon'_p = \epsilon_p \tag{4.4}$$

is necessary for $p|D$ in order that $\mathcal{L}(f, f') \not\equiv 0$.

We now consider local components of $\mathrm{JL}(\pi(f'))$. The non-archimedean local component at $p \nmid d_B$ is isomorphic to the unramified principal series representation $\pi(f')_p$ and the local component at $p|d_B$ is the Steinberg representation twisted by δ_p.

4.2.2. Local representations at ∞.

We need to review the archimedean local components of $\pi(f)$ and $\mathrm{JL}(\pi(f'))$ for the proof of Proposition 4.7 (cf. Section 4.4). The archimedean component of $\pi(f)$ (respectively $\mathrm{JL}(\pi(f'))$) is the irreducible admissible representation of $GL_2(\mathbb{R})$ square-integrable modulo center whose restriction to $SL_2(\mathbb{R})$ decomposes into the sum of the discrete series representation of lowest weight $-\kappa_1$ (respectively $-(\kappa_2 + 2)$) and its contragredient.

4.3. The representations generated by $\mathcal{L}(f, f')$.

We now study locally and globally the representation $\pi(f, f')$ of $G(\mathbb{A}) = GSp(1,1)(\mathbb{A})$ or $GSp^*(2)(\mathbb{A})$ generated by $\mathcal{L}(f, f')$. To this end we cite the following proposition (cf. [**36**, Theorem 3.1]).

Proposition 4.3. *Let $\mathcal{G}$ be a reductive algebraic group defined over a number field F. Denote by $\mathbb{A}_\infty$ the ring of archimedean adeles of F.*

Let ϕ be a Hecke eigenform on $\mathcal{G}(\mathbb{A}_F)$ which is square-integrable modulo center and generates an irreducible admissible representation of $\mathcal{G}(\mathbb{A}_\infty)$ at the archimedean component (*for the meaning of "Hecke eigenform" here, see* [**36**, Theorem 3.1 ii)]). *Then the automorphic representation generated by ϕ is irreducible.*

We then have the following:

Proposition 4.4. *Suppose that f and f' are Hecke eigenforms and that $(2\mathbb{Z}_{\geq 0})^2 \ni (\kappa_1, \kappa_2)$ satisfies $1 < \kappa_1 < \kappa_2 + 2$ for $G = G_{\mathrm{nc}}$* (*respectively $1 < \kappa_2 + 2 < \kappa_1$ for $G = G_{\mathrm{c}}$*). *Then the representation $\pi(f, f')$ of $G(\mathbb{A})$ is irreducible.*

Proof. According to Proposition 3.3 (1), the theta lift $\mathcal{L}(f, f')$ is a Hecke eigenform on $G(\mathbb{A})$. The lift $\mathcal{L}(f, f')$ is square-integrable modulo center. In fact, when $G = G_{\mathrm{c}}$ this is obvious and when $G = G_{\mathrm{nc}}$, $\mathcal{L}(f, f')$ is cuspidal as is remarked in the proof of Proposition 3.3 (2). The assertion is thus a consequence of Proposition 3.3 and Proposition 4.3. □

Let (f, f') be Hecke eigenforms and f be primitive. We can therefore decompose $\pi(f, f')$ into the restricted tensor product $\bigotimes'_{v \leq \infty} \pi_v$ and are able to determine each local component π_v, for which we note that the archimedean local component π_∞ has been already determined in Proposition 3.3 (2).

When $p \nmid d_B$ let ϕ, ϕ' and ϕ'' be the Hecke operators for $G(\mathbb{Q}_p)$ defined by the double cosets $K_{1,p}\,\mathrm{diag}(p,p,p,p)K_{1,p}$, $K_{1,p}\,\mathrm{diag}(p,p,1,1)K_{1,p}$ and $K_{1,p}\,\mathrm{diag}(p^2,p,p,1)K_{1,p}$, which form generators of the Hecke algebra with respect to $K_{1,p} \simeq GSp(2)(\mathbb{Z}_p)$. For $p|d_B$ let $\{\varphi_i,\ \varphi'_i\}$ denote the generators of the Hecke algebra with respect to $K_{i,p}$ with $i = 1$ or $i = 2$ (cf. Section A.2). As a consequence of [**30**, Theorem 5.1] and the formulas (4.1)∼(4.4) we have the following:

Lemma 4.5. *Let the notation be as above.*
(1) *Let p do not divide d_B. The Hecke eigenvalues $\pi_p(\phi_0)$ of π_p for Hecke operators $\phi_0 = \phi,\ \phi',\ \phi''$ are given by*

$$\begin{aligned}
\pi_p(\phi) &= 1,\\
\pi_p(\phi') &= p^{3/2}(\chi_{f,p}(p) + \chi_{f,p}(p)^{-1}) + p^{3/2}(\chi_{f',p}(p) + \chi_{f',p}(p)^{-1}),\\
\pi_p(\phi'') &= p^2(\chi_{f,p}(p) + \chi_{f,p}(p)^{-1})(\chi_{f',p}(p) + \chi_{f',p}(p)^{-1}) + p^2 - 1.
\end{aligned}$$

(2) *Let p divide d_B. When $p|D$, the Hecke eigenvalues $\pi_p(\varphi)$ of π_p for $\varphi = \varphi_1,\ \varphi'_1$ are given by*

$$\pi_p(\varphi_1) = \epsilon_p, \quad \pi_p(\varphi_1') = -\epsilon_p(p+1).$$

When $p|d_B/D$, the Hecke eigenvalues $\pi_p(\varphi)$ of π_p for $\varphi = \varphi_2,\ \varphi_2'$ are given by

$$\pi_p(\varphi_2) = \epsilon_p, \quad \pi_p(\varphi_2') = p^{3/2}(\chi_{f,p}(p) + \chi_{f,p}(p)^{-1}) + (p-1)\epsilon_p.$$

Following the notation of the appendix, let ξ be the non-trivial unramified character of $\mathbb{Q}_p^\times$ of order two for $p|d_B$. We further note that, in the appendix, the notation $\chi 1_B \rtimes \sigma$ is used for the induced representation of $GSp(1,1)(\mathbb{Q}_p)$ defined by two quasi-character χ and σ of $\mathbb{Q}_p^\times$ when $p|d_B$. On the other hand, with three unramified quasi-characters $\chi_1,\ \chi_2$ and σ of $\mathbb{Q}_p^\times$, $\chi_1 \times \chi_2 \rtimes \sigma$ denotes a unique irreducible subquotient with a $GSp(2)(\mathbb{Z}_p)$-fixed vector for the unramified principal series representation of $GSp(2)(\mathbb{Q}_p)$ induced from these characters. This representation is referred to as "type I" on the table of Section A.4 of the appendix.

Proposition 4.6. *Let the notation be as above.*
(1) Let $v = p \nmid d_B$. Then π_p is an irreducible admissible representation of type I given by $(\chi_{f',p} \cdot \chi_{f,p}^{-1}) \times (\chi_{f',p}^{-1} \cdot \chi_{f,p}^{-1}) \rtimes \chi_{f,p}$.
(2) Let $v = p|d_B$. When $v = p|d_B/D$, π_p is isomorphic to the irreducible representation of $GSp(1,1)(\mathbb{Q}_p) \simeq GSp^*(2)(\mathbb{Q}_p)$ of type IIa with $\sigma = \chi_{f,p}$ and $\chi = \chi_{f,p}^{-1} \cdot \delta_p$. When $v = p|D$ and δ_p is trivial (respectively non-trivial), π_p is isomorphic to the irreducible representation of $GSp(1,1)(\mathbb{Q}_p) \simeq GSp^*(2)(\mathbb{Q}_p)$ of type Va with $\sigma = \xi$ (respectively $\sigma = 1$). Here, for the representations of $GSp(1,1)(\mathbb{Q}_p) \simeq GSp^*(2)(\mathbb{Q}_p)$ of type IIa and Va, see Section A.4 *of the appendix.*

Proof. For every finite prime p, π_p is a spherical representation of $G_p = GSp(1,1)(\mathbb{Q}_p)$ or $GSp(2)(\mathbb{Q}_p)$ in the sense of Section A.2 of the appendix. As is pointed out there, π_p is uniquely determined by the Hecke eigenvalues. To be precise, up to the conjugation of the Weyl group, such values determine the unramified characters inducing a principal series representation that has π_p as a composition factor.

When a finite prime p does not divide d_B, π_p is an irreducible admissible representation with a $GSp(2)(\mathbb{Z}_p)$-fixed vector and is therefore of type I. In view of the first assertion of Lemma 4.5, we see that π_p is of type I with the explicit characters $(\chi_1, \chi_2, \sigma) = (\chi_{f',p} \cdot \chi_{f,p}^{-1}, \chi_{f',p}^{-1} \cdot \chi_{f,p}^{-1}, \chi_{f,p})$.

Let p divide d_B. Compare the formulas in Lemma 4.5 with the Hecke eigenvalues of the spherical vector for the induced representation $\chi 1_B \rtimes \sigma$ (cf. Section A.2 of the appendix). We can then explicitly determine the representation type of π_p by the table of spherical representations of G_p in Section A.4 of the appendix. □

4.4. Theta lifts to cuspidal representations of G'.

Let $GO(2,2)$ be the reductive $\mathbb{Q}$-algebraic group defined by the orthogonal group of signature $(2,2)$ with similitudes. Let (f, f') be Hecke eigenforms as in Section 4.2. Following Roberts [**38**] and Harris–Kudla [**12**], we introduce the theta lift from the cuspidal representation $\sigma(f, \mathrm{JL}(f'))$ of $GO(2,2)$ associated with $(f, \mathrm{JL}(f'))$ to a cuspidal representation $\pi'(f, \mathrm{JL}(f'))$ of $G'(\mathbb{A})$.

As well as $GO(2,2)$ we consider a subgroup $GSO(2,2)$, the special orthogonal group of signature $(2,2)$ with similitudes. To realize $GSO(2,2)$ let us introduce a quadratic space $(M_2(\mathbb{Q}), 2\det)$. This quadratic space is equivalent to another quadratic space $(\mathbb{Q}^4, Q')$ defined by $Q' = \left(\begin{smallmatrix} 0_2 & w \\ -w & 0_2 \end{smallmatrix}\right)$ with $w = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. Consider the action of $GL_2(\mathbb{Q}) \times GL_2(\mathbb{Q})$ on $M_2(\mathbb{Q})$ defined by

$$h \cdot X = h_1^{-1} X h_2 \quad (X \in M_2(\mathbb{Q}),\ h = (h_1, h_2) \in GL_2(\mathbb{Q}) \times GL_2(\mathbb{Q})).$$

This induces isomorphisms

$$(GL_2(\mathbb{Q}) \times GL_2(\mathbb{Q}))/\{(z,z) \mid z \in \mathbb{Q}^\times\} \simeq GSO(2,2)(\mathbb{Q}).$$

We now note that there is $s \in GO(2,2)(\mathbb{Q}) \setminus GSO(2,2)(\mathbb{Q})$ such that

$$s \cdot (h_1, h_2) = (h_2, h_1)$$

modulo $\{(z,z) \mid z \in \mathbb{Q}^\times\}$.

The outer tensor product $\pi(f) \boxtimes \mathrm{JL}(\pi(f'))$ gives rise to an irreducible cuspidal representation $GSO(2,2)(\mathbb{A})$, where note that the central characters of $\pi(f)$ and $\mathrm{JL}(\pi(f'))$ are trivial. Then, assuming $\kappa_1 \neq \kappa_2 + 2$, there is an irreducible cuspidal representation $\sigma(f, \mathrm{JL}(f'))$ of $GO(2,2)(\mathbb{A})$ such that its restriction to $GSO(2,2)(\mathbb{A})$ decomposes into a direct sum of $\pi(f) \boxtimes \mathrm{JL}(\pi(f'))$ and its conjugation by s, namely $\mathrm{JL}(\pi(f')) \boxtimes \pi(f)$ (cf. [**38**, Theorem 7.1] and [**13**, Section 1]). By $\pi'(f, \mathrm{JL}(f'))$ we denote the theta lift from $\sigma(f, \mathrm{JL}(f'))$ to $G'(\mathbb{A})$ as in [**38**] and [**12**].

Using the notation in Section A.4 of the appendix, we now state the following proposition:

Proposition 4.7. *Let $(f, f') \in S_{\kappa_1}(D) \times \mathcal{A}_{\kappa_2}$ be non-zero Hecke eigenforms and suppose that f is primitive. Assume that $1 < \kappa_1 < \kappa_2 + 2$ or $1 < \kappa_2 + 2 < \kappa_1$ as in Proposition* 3.3 (2). *Then $\pi'(f, \mathrm{JL}(f'))$ is a non-zero irreducible cuspidal representation of $G'(\mathbb{A})$ and thus has the decomposition into the restricted tensor product $\pi'(f, \mathrm{JL}(f')) = \bigotimes'_{v \le \infty} \pi'_v$. Each local component π'_v is determined as follows*:

1. *For $v = p \nmid d_B$, π'_p is isomorphic to π_p, namely the irreducible admissible representation of type I for $G'(\mathbb{Q}_p)$ given by $(\chi_{f',p} \cdot \chi_{f,p}^{-1}) \times (\chi_{f',p}^{-1} \cdot \chi_{f,p}^{-1}) \rtimes \chi_{f,p}$.*

2. *For $v = p | d_B/D$, π'_p is isomorphic to the irreducible representation of $G'(\mathbb{Q}_p)$ of type IIa with $\sigma = \chi_{f,p}$ and $\chi = \chi_{f,p}^{-1} \cdot \delta_p$. When $v = p | D$ and δ_p is trivial (respectively non-trivial), π'_p is isomorphic to the irreducible representation of $G'(\mathbb{Q}_p)$ of type Va with $\sigma = \xi$ (respectively $\sigma = 1$). Here, for the representations of $G'(\mathbb{Q}_p)$ of types IIa and Va, see Section* A.4 *of the appendix.*

3. *For $v = \infty$, π'_∞ is isomorphic to the representation π'_λ as admissible representations of G'_∞ with $\lambda = ((\kappa_1 + \kappa_2)/2, |\kappa_1 - \kappa_2 - 2|/2)$ (for π'_λ see Proposition* 2.1).

Proof. Since $\sigma(f, \mathrm{JL}(f'))$ is irreducible we can decompose it into the product $\bigotimes_{v \le \infty} \sigma(f, \mathrm{JL}(f'))_v$. We now note that each local component of $\pi(f)$ and $\mathrm{JL}(\pi(f'))$ are explained in Section 4.2. In view of Gan–Takeda [**9**, Theorem 8.2] and Przebinda [**37**, Chapter III, Section 3] (see also Harris–Kudla [**12**, Theorem 5.2.1]) we then see that each local component $\sigma(f, \mathrm{JL}(f'))_v$ is involved in a local theta correspondence with $G'(\mathbb{Q}_v)$ for $v \le \infty$. We explain the local theta correspondences in detail soon. We further remark that $\sigma(f, \mathrm{JL}(f'))_v$ is tempered for every $v \le \infty$. By [**38**, Theorem 8.3] we thus see that $\pi'(f, \mathrm{JL}(f'))$ is a non-zero irreducible cuspidal representation of $G'(\mathbb{A})$.

An explicit description of π'_p for $v = p < \infty$ is given in the table 2 (d), (e), (f) of Section 14 or Theorem 8.2 (iv), (v), (vi) of [**9**], for which note that the theta lifts from p-components of $\mathrm{JL}(\pi(f')) \boxtimes \pi(f)$ and $\pi(f) \boxtimes \mathrm{JL}(\pi(f'))$ are isomorphic to each other for every finite prime p as is remarked in [**9**, Theorem 8.2]. The determination of π'_∞ is due to [**37**, Chapter III, Theorem 3.3.1] or [**12**, Theorem 5.2.1]. In fact, by virtue of Section 4.2.2, the archimedean component of $\pi(f) \boxtimes \mathrm{JL}(\pi(f'))$ can be regarded as the irreducible admissible representation of $O(2,2)(\mathbb{R})$ given by the discrete series representation with Harish–Chandra parameter $((\kappa_1+\kappa_2)/2, -|\kappa_1-\kappa_2-2|/2)$. Its image of the theta lift to $Sp(2)(\mathbb{R})$ is the discrete series representation with Harish–Chandra parameter $((\kappa_1+\kappa_2)/2, -|\kappa_1-\kappa_2-2|/2)$. We then know that the archimedean component of $\pi'(f, \mathrm{JL}(f'))$ is the irreducible admissible representation of $G'(\mathbb{R})$ whose restriction to $Sp(2)(\mathbb{R})$ is the sum of the aforementioned discrete series and its contragredient. This admissible representation of $G'(\mathbb{R})$ is nothing but π'_λ with $\lambda = ((\kappa_1+\kappa_2)/2, |\kappa_1-\kappa_2-2|/2)$. If we start from $\mathrm{JL}(\pi(f')) \boxtimes \pi(f)$ instead of $\pi(f) \boxtimes \mathrm{JL}(\pi(f'))$ we come to the same conclusion. We are therefore done. $\square$

Remark 4.8. Up to equivalence, the representations of type Va in Propositions 4.6 and 4.7 do not depend on the signature ϵ_p. For the case of $GSp(2)(\mathbb{Q}_p)$ this is remarked in [**9**, Theorem 8.2 (iv)]. For the case of the inner form the two representations of type Va with the different signatures are related by the intertwining operator $-\mathrm{Id}$ with the identity map Id.

4.5. Spinor L-functions for theta lifts.

4.5.1. Spinor L-functions for G.

Recall that, for $D|d_B$, $K_f(D)$ denotes a maximal open compact subgroup of $G(\mathbb{A}_f)$ (cf. Section 2.2). We first define the spinor L-functions for $K_f(D)$-invariant Hecke eigenforms on $G(\mathbb{Q})\backslash G(\mathbb{A})$ that are square-integrable modulo center and that generate irreducible admissible representations at the archimedean place. Let F be such a Hecke eigenform with the trivial central character and $\pi(F)$ the automorphic representation generated by F. Due to Proposition 4.3, $\pi(F)$ is irreducible and thus has the decomposition into the restricted tensor product $\pi(F) \simeq \bigotimes'_{v \le \infty} \pi_v(F)$ of local representations.

To define the non-archimedean local factors of the spinor L-function of F we introduce several polynomials defined with Hecke eigenvalues of F.

In [**30**, Section 5.1] we introduced three Hecke operators $\mathcal{T}_p^i$ with $0 \le i \le 2$ for $p \nmid d_B$. Let Λ_p^i be the Hecke eigenvalue of $\mathcal{T}_p^i$ for F with $0 \le i \le 2$. For $p \nmid d_B$ we put

$$Q_{F,p}(t) := 1 - p^{-3/2}\Lambda_p^1 t + p^{-2}(\Lambda_p^2 + p^2 + 1)t^2 - p^{-3/2}\Lambda_p^1 t^3 + t^4.$$

For this we note that $Q_{F,p}(p^{-s})^{-1}$ coincides with the local spinor L-function for an (irreducible subquotient of) unramified principal series representation of $GSp(2)(\mathbb{Q}_p)$. In addition, we remark that $\pi_p(F)$ is of type I in the table of Section A.4 (cf. appendix).

On the other hand, in [**30**, Section 5.2], we introduced two Hecke operators $\mathcal{T}_p^i$ with $0 \le i \le 1$ for $p | d_B$. Let Λ'^i_p be the Hecke eigenvalue of $\mathcal{T}_p^i$ for F with $0 \le i \le 1$. According to the table of Section A.4, the representation $\pi_p(F)$ for $p | d_B$ is one of the following types:

IIa, IVc, Va, Vb, Vc, VIc.

We introduce

$$Q_{F,p}(t) := \begin{cases} (1+\Lambda'^0_p p^{-1/2} t)(1-\Lambda'^0_p p^{-1/2} t) & (\pi_p(F) \text{ is of type Va}), \\ (1-p^{-3/2}(\Lambda'^1_p-(p-1)\Lambda'^0_p)t+t^2)(1-\Lambda'^0_p p^{-1/2} t) & (\pi_p(F) \text{ is of the other type}). \end{cases}$$

For this, see Section A.2 of the appendix and note that the triviality of the central character of F implies $(\Lambda'^0_p)^2 = 1$. The second one is due to Sugano [**45**, (3.4)]. We then define the local spinor L-function $L_p(F, \mathrm{spin}, s)$ for $v = p < \infty$ by

$$L_p(F, \mathrm{spin}, s) = Q_{F,p}(p^{-s})^{-1}.$$

We now state the following proposition, which justifies this definition.

PROPOSITION 4.9. *For $v = p < \infty$, $L_p(F, \mathrm{spin}, s)$ coincides with the local spinor L-function of the irreducible admissible representation of $GSp(2)(\mathbb{Q}_p)$ with the same L-parameter as that of $\pi_p(F)$.*

PROOF. This follows from Section A.2 of the appendix and [**39**, Section A.6, Table A.8]. □

REMARK 4.10. This proposition tells us that an irreducible spherical representation with a $K_{2,p}$-fixed vector has a local spinor L-function of degree three while one with a $K_{1,p}$-fixed vector and no $K_{2,p}$-fixed vector has a local spinor L-function of degree two (see also Section A.2 and the table of Section A.4), where note that $K_{1,p}$ and $K_{2,p}$ are denoted by K_1 and K_2 respectively in the appendix.. There is only one spherical representation of the latter type, which is enumerated as Va in the table of Section A.4 of the appendix. This representation may occur as a local factor $\pi_p(F)$ of the automorphic representation $\pi(F)$ only for $p | D$.

Let $\lambda = (\lambda_1, \lambda_2) \in \Xi_I$ (for Ξ_I see Section 2.3). Let now F be a Hecke eigenform in $\mathcal{S}^{\mathrm{nc}}_{\tau_\Lambda}(D)$ (respectively $\mathcal{A}^{\mathrm{c}}_{\tau_\Lambda}(D)$) with $\Lambda = (\lambda_1, \lambda_2 - 1)$ (respectively $\Lambda = (\lambda_1 - 2, \lambda_2 - 1)$). We define the global spinor L-function $L(F, \mathrm{spin}, s)$ of a Hecke eigenform F in $\mathcal{S}^{\mathrm{nc}}_{\tau_\Lambda}(D)$ or $\mathcal{A}^{\mathrm{c}}_{\tau_\Lambda}(D)$ with Λ as above by

$$L(F, \mathrm{spin}, s) := \prod_{v \le \infty} L_v(F, \mathrm{spin}, s),$$

where

$$L_\infty(F, \mathrm{spin}, s) = \Gamma_{\mathbb{C}}\left(s + \frac{\lambda_1 - \lambda_2}{2}\right) \Gamma_{\mathbb{C}}\left(s + \frac{\lambda_1 + \lambda_2}{2}\right).$$

For this we remark that the definition of $L_\infty(F, \mathrm{spin}, s)$ is based on the following fact: π_λ and π'_λ have the same L-parameter (cf. Proposition 2.1 (2)) and the local spinor L-function of π'_λ should be defined as above (cf. [**28**, Section 1.4]).

4.5.2. Spinor *L*-functions for theta lifts.

Let $F = \mathcal{L}(f, f')$ and (f, f') be Hecke eigenforms. In view of Proposition 3.3 (2) we have

$$L_\infty(\mathcal{L}(f, f'), \mathrm{spin}, s) := \Gamma_{\mathbb{C}}\left(s + \frac{\kappa_1 - 1}{2}\right)\Gamma_{\mathbb{C}}\left(s + \frac{\kappa_2 + 1}{2}\right).$$

We have the following, which generalizes [**32**, Proposition 2.9]:

Proposition 4.11. *Let (κ_1, κ_2) be as in Proposition 3.3 (2). Suppose that f is a primitive form. The spinor L-function for $\mathcal{L}(f, f')$ decomposes into*

$$L(\mathcal{L}(f, f'), \mathrm{spin}, s) = L(\pi(f), s)L(\mathrm{JL}(\pi(f')), s),$$

where $L(\pi(f), s)$ (respectively $L(\mathrm{JL}(\pi(f')), s)$) denotes the standard L-function of $\pi(f)$ (respectively $\mathrm{JL}(\pi(f'))$).

The spinor L-function $L(\mathcal{L}(f, f'), \mathrm{spin}, s)$ can be viewed also as the L-function of $\pi(f, f')$. In fact, we have justified the definition of each local factor of $L(\mathcal{L}(f, f'), \mathrm{spin}, s)$ in terms of the representation type of $\pi(f, f')_v$ for $v \le \infty$ (cf. (1)). We can thus denote it also by $L(\pi(f, f'), \mathrm{spin}, s)$. This has the analytic continuation and satisfies the functional equation between s and $1 - s$ since so do $L(\pi(f), s)$ and $L(\mathrm{JL}(\pi(f')), s)$.

On the other hand, we can also consider the spinor L-function $L(\pi'(f, \mathrm{JL}(f')), \mathrm{spin}, s)$ of $\pi'(f, \mathrm{JL}(f'))$. Since the archimedean local component π'_∞ of $\pi'(f, \mathrm{JL}(f'))$ is isomorphic to π'_λ with $\lambda = ((\kappa_1 + \kappa_2)/2, |(\kappa_1 - \kappa_2 - 2)/2|)$ (cf. Proposition 4.7), the local factor at $v = \infty$ coincides with $L_\infty(\mathcal{L}(f, f'), \mathrm{spin}, s)$. In view of Propositions 4.6, 4.7 and 4.9, all the other local factors of $L(\pi'(f, \mathrm{JL}(f')), \mathrm{spin}, s)$ also coincide with those of $L(\mathcal{L}(f, f'), \mathrm{spin}, s)$. We can state the following:

Corollary 4.12. *Under the same assumption in Proposition 4.11 we have*

$$L(\mathcal{L}(f, f'), \mathrm{spin}, s) = L(\pi(f, f'), \mathrm{spin}, s) = L(\pi'(f, \mathrm{JL}(f')), \mathrm{spin}, s).$$

4.6. Main theorem.

As we have seen, $\mathcal{L}(f, f')$ is $K_f(D)$-invariant, namely $\pi(f, f') \in \mathcal{A}_G(K_f(D))$. On the other hand, Proposition 4.7 and [**39**, Section A.8, Table A.13] tell us that $\pi'(f, \mathrm{JL}(f'))$ has a paramodular newform of level $d_B D$, namely, $\pi'(f, \mathrm{JL}(f')) \in \mathcal{A}^{\mathrm{new}}_{G'}(K'_f(D))$.

As a consequence of Propositions 3.3, 4.6, 4.7, 4.9 and Corollary 4.12 we are able to state our main theorem, which provides evidence of Conjecture 4.2.

Theorem 4.13. *Suppose that two even integers (κ_1, κ_2) satisfy $1 < \kappa_1 < \kappa_2 + 2$ when $G = G_{\mathrm{nc}}$ (respectively $1 < \kappa_2 + 2 < \kappa_1$ when $G = G_{\mathrm{c}}$). For any given primitive form $f \in S_{\kappa_1}(D)$ and Hecke eigenform $f' \in \mathcal{A}_{\kappa_2}$, the map*

$$\mathcal{A}_G(K_f(D)) \ni \pi(f, f') \mapsto \pi'(f, \mathrm{JL}(f')) \in \mathcal{A}_{G'}^{\mathrm{new}}(K'_f(D))$$

satisfies the coincidence of the global spinor L-functions (cf. Corollary 4.12) and the compatibility with the local Jacquet–Langlands correspondence for G and $G' = GSp(2)$ at any place $v \leq \infty$ (for the correspondence at $v = p < \infty$, see Section A.4).

Acknowledgements. The author would like to thank Takeo Okazaki for his suggestion that Roberts [**38**] and Gan–Takeda [**9**] are useful to find cuspidal representations of $GSp(2)(\mathbb{A})$ involved in the global Jacquet–Langlands correspondence. Author's thanks is also due to the referee for fruitful comments on this paper.

Appendix: The spherical representations of $GSp(1,1)$. by Ralf Schmidt

A.1. Induced representations for $GSp(1,1)$.

Let F be a non-archimedean local field of characteristic zero with the normalized valuation ν. Let B be the non-split quaternion algebra over F, and let $x \mapsto \bar{x}$ be its standard involution. Let $G = GSp(1,1)$ be the algebraic group with F-points

$$G(F) = \{g \in M(2 \times 2, B) : {}^t\bar{g}Jg = \lambda(g)J,\ \lambda(g) \in F^\times\}, \qquad \text{where } J = \begin{bmatrix} & 1 \\ 1 & \end{bmatrix}.$$

Then G has a unique proper parabolic subgroup $P = MN$, where

$$M = \left\{ \begin{bmatrix} a & \\ & \lambda\bar{a}^{-1} \end{bmatrix} : a \in B^\times,\ \lambda \in F^\times \right\} \cong B^\times \times F^\times,$$

and

$$N = \left\{ \begin{bmatrix} 1 & x \\ & 1 \end{bmatrix} : x \in B,\ x + \bar{x} = 0 \right\}.$$

We consider the normalized parabolic induction from P to G. Let (π, V_π) be a finite-dimensional smooth representation of $B^\times$. Let σ be a character of $F^\times$. Then the standard space of the parabolically induced representation $\pi \rtimes \sigma$ consists of smooth functions $f : G(F) \to V_\pi$ with the transformation property

$$f\left(\begin{bmatrix} a & * \\ & \lambda\bar{a}^{-1} \end{bmatrix} g\right) = \nu\left(\delta_P\left(\begin{bmatrix} a & * \\ & \lambda\bar{a}^{-1} \end{bmatrix}\right)\right)^{1/2} \sigma(\lambda)\pi(a)f(g), \quad a \in B^\times, \lambda \in F^\times, g \in G(F),$$

where

$$\delta_P\left(\begin{bmatrix} a & * \\ & \lambda\bar{a}^{-1} \end{bmatrix}\right) = (\lambda^{-1}n(a))^3$$

with the reduced norm $n(a)$ of $a \in B^\times$. Note that the central character of $\pi \rtimes \sigma$ is $\omega_\pi\sigma^2$, where ω_π is the central character of π. In particular, if χ is a character of $F^\times$, then the central character of $\chi 1_{B^\times} \rtimes \sigma$ is $\chi^2\sigma^2$.

It is easy to see that the trivial representation is contained in $\nu^{-3/2}1_{B^\times} \rtimes \nu^{3/2}$, and hence is a quotient of $\nu^{3/2}1_{B^\times} \rtimes \nu^{-3/2}$. In fact, the reducibility points of the induced representations $\pi \rtimes \sigma$ are known by [**11**, Proposition 5.3]. The result is as follows.

Proposition A.1. *Let π be an irreducible, admissible representation of $B^\times(F)$, and let σ be a character of $F^\times$.*

1. *Assume that* $\dim(\pi) > 1$. *Then $\pi \rtimes \sigma$ is irreducible unless $\pi = \nu^{\pm 1/2}\pi_0$, where π_0 has trivial central character. In this case there is a short exact sequence*

$$0 \longrightarrow \delta(\nu^{1/2}\pi_0, \nu^{-1/2}\sigma) \longrightarrow \nu^{1/2}\pi_0 \rtimes \nu^{-1/2}\sigma \longrightarrow L(\nu^{1/2}\pi_0, \nu^{-1/2}\sigma) \longrightarrow 0,$$

with an irreducible, square-integrable representation $\delta(\nu^{1/2}\pi_0, \nu^{-1/2}\sigma)$, and an irreducible non-tempered representation $L(\nu^{1/2}\pi_0, \nu^{-1/2}\sigma)$.

2. *Assume that* $\dim(\pi) = 1$. *Then $\pi \rtimes \sigma$ is irreducible unless one of the following holds.*

- $\pi = \nu^{\pm 1/2}\xi 1_{B^\times}$, *where $\xi^2 = 1$, $\xi \neq 1$. In this case there is a short exact sequence*

$$0 \longrightarrow \delta(\nu^{1/2}\xi 1_{B^\times}, \nu^{-1/2}\sigma) \longrightarrow \nu^{1/2}\xi 1_{B^\times} \rtimes \nu^{-1/2}\sigma \longrightarrow L(\nu^{1/2}\xi 1_{B^\times}, \nu^{-1/2}\sigma) \longrightarrow 0,$$

with an irreducible, square-integrable representation $\delta(\nu^{1/2}\xi 1_{B^\times}, \nu^{-1/2}\sigma)$, and an irreducible non-tempered representation $L(\nu^{1/2}\xi 1_{B^\times}, \nu^{-1/2}\sigma)$.

- $\pi = \nu^{\pm 3/2}1_{B^\times}$. *In this case there is a short exact sequence*

$$0 \longrightarrow \sigma \mathrm{St}_{GSp(1,1)} \longrightarrow \nu^{3/2}1_{B^\times} \rtimes \nu^{-3/2}\sigma \longrightarrow \sigma 1_{GSp(1,1)} \longrightarrow 0,$$

where $\mathrm{St}_{GSp(1,1)}$ *is the Steinberg representation of $G(F)$.*

A.2. Spherical representations.

Let $\mathfrak{o}_B$ be a maximal order in $B(F)$, and let $\mathfrak{p}_B$ be the unique maximal ideal of $\mathfrak{o}_B$. Let

$$K_1 = \left\{ g \in G(F) \cap \begin{bmatrix} \mathfrak{o}_B & \mathfrak{o}_B \\ \mathfrak{o}_B & \mathfrak{o}_B \end{bmatrix} : \lambda(g) \in \mathfrak{o}^\times \right\},$$

$$K_2 = \left\{ g \in G(F) \cap \begin{bmatrix} \mathfrak{o}_B & \mathfrak{p}_B \\ \mathfrak{p}_B^{-1} & \mathfrak{o}_B \end{bmatrix} : \lambda(g) \in \mathfrak{o}^\times \right\}.$$

Then K_1 and K_2 are maximal compact subgroups of $G(F)$, and every maximal compact subgroup is conjugate to either K_1 or K_2. In fact, the classification of the maximal compact subgroups is reduced to that of maximal lattices in the sense of [**42**, Section 2.3]. We see from [**42**, Propositions 3.7 and 3.9] that, up to $G(F)$-conjugate, there are two maximal lattices ${}^t(\mathfrak{o}_B \oplus \mathfrak{o}_B)$ and ${}^t(\mathfrak{o}_B \oplus \mathfrak{p}_B^{-1})$, whose corresponding maximal compact subgroups are K_1 and K_2 respectively.

It is known that the Hecke algebra $\mathcal{H}(G, K_i)$ consisting of compactly supported left and right K_i-invariant functions on $G(F)$ is commutative; see [**40**]. Therefore the formal reasoning of [**6**] 4.4 applies, and we see:

- If (π, V) is an irreducible representation such that the space V^{K_i} of K_i-invariant vectors is non-zero, then $\dim V^{K_i} = 1$.
- In this case, the action of $\mathcal{H}(G, K_i)$ on a non-zero vector of V^{K_i} defines an algebra homomorphism $\mathcal{H}(G, K_i) \to \mathbb{C}$. This algebra homomorphism determines the equivalence class of π.
- Given an algebra homomorphism $\mathcal{H}(G, K_i) \to \mathbb{C}$, there exists an irreducible, admissible representation (π, V) such that the algebra homomorphism comes from the action of $\mathcal{H}(G, K_i)$ on a non-zero vector of V^{K_i} as above.

Let ϖ_B be a prime element of $\mathfrak{o}_B$. For $i = 1$ or $i = 2$ let

$$\varphi_i = \text{characteristic function of } K_i \begin{bmatrix} \varpi_B & \\ & \varpi_B \end{bmatrix} K_i$$

and

$$\varphi_i' = \text{characteristic function of } K_i \begin{bmatrix} 1 & \\ & \varpi_F \end{bmatrix} K_i.$$

Then

$$\mathcal{H}(G, K_i) = \mathbb{C}[\varphi_i, \varphi_i^{-1}, \varphi_i'].$$

For $z \in B$, let $z^- = (z - \bar{z})/2$, and for a subset A of B, let A^- denote the image of A under the map $z \mapsto z^-$. Then, by [**30**, Section 8],

$$\begin{aligned} &K_1 \begin{bmatrix} 1 & \\ & \varpi_F \end{bmatrix} K_1 \\ &\quad = \bigsqcup_{b \in \mathfrak{o}_B^- / \varpi_F \mathfrak{o}_B^-} \begin{bmatrix} 1 & b \\ & 1 \end{bmatrix} \begin{bmatrix} \varpi_F & \\ & 1 \end{bmatrix} K_1 \sqcup \bigsqcup_{c \in (\mathfrak{p}_B^{-1} - \mathfrak{o}_B)^- / \mathfrak{o}_B^-} \begin{bmatrix} 1 & c \\ & 1 \end{bmatrix} \begin{bmatrix} \varpi_B & \\ & \varpi_B \end{bmatrix} K_1 \sqcup \begin{bmatrix} 1 & \\ & \varpi_F \end{bmatrix} K_1 \end{aligned}$$

and

$$\begin{aligned} &K_2 \begin{bmatrix} 1 & \\ & \varpi_F \end{bmatrix} K_2 \\ &\quad = \bigsqcup_{b \in \mathfrak{p}_B^- / \varpi_F \mathfrak{p}_B^-} \begin{bmatrix} 1 & b \\ & 1 \end{bmatrix} \begin{bmatrix} \varpi_F & \\ & 1 \end{bmatrix} K_2 \sqcup \bigsqcup_{c \in (\mathfrak{o}_B - \mathfrak{p}_B)^- / \mathfrak{p}_B^-} \begin{bmatrix} 1 & c \\ & 1 \end{bmatrix} \begin{bmatrix} \varpi_B & \\ & \varpi_B \end{bmatrix} K_2 \sqcup \begin{bmatrix} 1 & \\ & \varpi_F \end{bmatrix} K_2. \end{aligned}$$

On the other hand, according to [**43**, Theorems 5.9 and 5.13], we have the following lemma.

LEMMA A.2. (1) $\#\mathfrak{o}_B/\mathfrak{p}_B = \#\mathfrak{p}_B^{-1}/\mathfrak{o}_B = q^2$.
(2) $\mathrm{tr}(\mathfrak{o}_B) = \mathfrak{o}_F$ *and* $\mathrm{tr}(\mathfrak{p}_B^{-1}) = \mathfrak{o}_F$ *or equivalently* $\mathrm{tr}(\mathfrak{p}_B) = \mathfrak{p}_F$.

We obviously have

$$\#\mathfrak{o}_B^-/\varpi_F\mathfrak{o}_B^- = \mathfrak{p}_B^-/\varpi_F\mathfrak{p}_B^- = q^3$$

and, in view of the lemma above, we verify

$$\#(\mathfrak{p}_B^{-1} - \mathfrak{o}_B)^-/\mathfrak{o}_B^- = q^2 - 1, \qquad \#(\mathfrak{o}_B - \mathfrak{p}_B)^-/\mathfrak{p}_B^- = q - 1.$$

Let χ and σ be unramified characters of $F^\times$. For $i = 1$ or $i = 2$ let f_i denote the K_i-invariant vector in $\pi = \chi 1_{B^\times} \rtimes \sigma$, normalized such that $f_i(1) = 1$. Using the above double coset decompositions and the cardinalities of the four cosets just mentioned, it is easy to check that

$$\pi(\varphi_i)f_i = \Lambda_i f_i, \qquad \Lambda_i := (\chi\sigma)(\varpi_F),$$

and

$$\pi(\varphi_1')f_1 = \Lambda_1' f_1, \qquad \Lambda_1' = q^{3/2}(\sigma\chi^2)(\varpi_F) + (q^2 - 1)(\sigma\chi)(\varpi_F) + q^{3/2}\sigma(\varpi_F),$$

and

$$\pi(\varphi_2')f_2 = \Lambda_2' f_2, \qquad \Lambda_2' = q^{3/2}(\sigma\chi^2)(\varpi_F) + (q - 1)(\sigma\chi)(\varpi_F) + q^{3/2}\sigma(\varpi_F).$$

As we will see in A.4, $G(F)$-representations of types J=IIa, IVc, Va, Vb, Vc and VIc in the table therein exhaust all spherical representations. We define

$$Q_{\mathrm{J}}(t) = (1 + \Lambda_1 q^{-1/2}t)(1 - \Lambda_1 q^{-1/2}t)$$

for J=Va and

$$Q_{\mathrm{J}}(t) = (1 - q^{-3/2}(\Lambda_2' - (q-1)\Lambda_2)t + \Lambda_2^2 t^2)(1 - \Lambda_2 q^{-1/2}t)$$

for the other J's. Then we verify by a direct calculation that $Q_{\mathrm{J}}(q^{-s})^{-1}$ is the local spinor L-function for a $GSp(4,F)$ representation of type J (see [**39**, Table A.8]). For instance, with ξ being the non-trivial, unramified, quadratic character of $F^\times$,

$$Q_{\mathrm{Va}}(q^{-s})^{-1} = L(s, \chi\sigma\nu^{1/2})L(s, \xi\chi\sigma\nu^{1/2}),$$

which is the local spinor L-function for a $GSp(4,F)$ representation of type Va, and

$$Q_{\mathrm{IIa}}(q^{-s})^{-1} = L(s, \chi^2\sigma)L(s, \sigma)L(s, \chi\sigma\nu^{1/2}),$$

which is the local spinor L-function for a $GSp(4,F)$ representation of type IIa.

A.3. The intertwining operator.

Let χ be the non-trivial, quadratic, unramified character of $F^\times$. The induced representation $\nu^{1/2}\chi 1_{B^\times} \rtimes \sigma\nu^{-1/2}$, has a one-dimensional space of K_1 invariant vectors and also a one-dimensional space of K_2-invariant vectors. We will now use an intertwining operator to determine how these spaces are distributed amongst the two irreducible constituents of $\nu^{1/2}\chi 1_{B^\times} \rtimes \sigma\nu^{-1/2}$. Let B_0 be the space of all trace-zero quaternions. For f in the standard model of $\pi \rtimes \sigma$ as above, let

$$(Mf)(g) = \int\limits_{B_0(F)} f\left(\begin{bmatrix} & 1 \\ -1 & \end{bmatrix}\begin{bmatrix} 1 & x \\ & 1 \end{bmatrix} g\right) dx, \tag{A.1}$$

provided this integral converges. A straightforward calculation shows that Mf is an element of $\tilde{\pi} \rtimes \omega_\pi\sigma$, where ω_π is the central character of π, and $\tilde{\pi}(a) = \pi(\bar{a}^{-1})$. Hence we get an intertwining operator

$$M: \pi \rtimes \sigma \longrightarrow \tilde{\pi} \rtimes \omega_\pi\sigma. \tag{A.2}$$

In particular, for a character χ of $F^\times$, we get an intertwining operator

$$M: \chi 1_{B^\times} \rtimes \sigma \longrightarrow \chi^{-1}1_{B^\times} \rtimes \chi^2\sigma. \tag{A.3}$$

Let s be a complex parameter, and consider $\chi\nu^s \rtimes \sigma\nu^{-s}$. In the following we normalize the Haar measure on $B_0(F)$ such that $B_0(F) \cap \{v_B(x) \geq 0\}$ has volume 1. We will need the following result.

Lemma A.3.

$$\mathrm{vol}\big(B_0(F) \cap \{v_B(x) = -m\}\big) = \begin{cases} q^{3m/2}(1-q^{-1}) & \textit{if } m \textit{ is even,} \\ q^{(3m+1)/2}(1-q^{-2}) & \textit{if } m \textit{ is odd.} \end{cases}$$

Proof. We will prove

$$\mathrm{vol}\big(B_0(F) \cap \{v_B(x) \geq 2t\}\big) = q^{-3t} \tag{A.4}$$

and

$$\mathrm{vol}\big(B_0(F) \cap \{v_B(x) \geq 2t+1\}\big) = q^{-3t-1}, \tag{A.5}$$

from which our assertion follows. Equation (A.4) is true for $t = 0$ by our normalization of Haar measure. For other values of t we obtain (A.4) by multiplication with powers of ϖ_F. By a similar argument, it suffices to prove (A.5) for $t = 0$. By Lemma A.2 (2) we get short exact sequences

$$0 \longrightarrow B_0 \cap \mathfrak{o}_B \longrightarrow \mathfrak{o}_B \longrightarrow \mathfrak{o}_F \longrightarrow 0$$

and

$$0 \longrightarrow B_0 \cap \varpi_B\mathfrak{o}_B \longrightarrow \varpi_B\mathfrak{o}_B \longrightarrow \mathfrak{p}_F \longrightarrow 0,$$

and hence

$$0 \longrightarrow (B_0 \cap \mathfrak{o}_B)/(B_0 \cap \varpi_B\mathfrak{o}_B) \longrightarrow \mathfrak{o}_B/\varpi_B\mathfrak{o}_B \longrightarrow \mathfrak{o}_F/\mathfrak{p}_F \longrightarrow 0.$$

By Lemma A.2 (1), $\#\mathfrak{o}_B/\varpi_B\mathfrak{o}_B = q^2$. It follows that $\#(B_0 \cap \mathfrak{o}_B)/(B_0 \cap \varpi_B\mathfrak{o}_B) = q$, and hence $\mathrm{vol}(B_0 \cap \varpi_B \mathfrak{o}_B) = q^{-1}$. This is the statement (A.5) for $t = 0$. □

Calculation for K_1.

Let f_1 be the K_1-invariant vector in $\nu^s \chi 1_{B^\times} \rtimes \sigma\nu^{-s}$. We calculate, for $\mathrm{Re}(s)$ large enough,

$$\begin{aligned}
(Mf_1)(1) &= \int\limits_{B_0(F)} f_1\left(\begin{bmatrix} & 1 \\ -1 & \end{bmatrix}\begin{bmatrix} 1 & x \\ & 1 \end{bmatrix}\right) dx \\
&= \int\limits_{B_0(F)\cap\{v_B(x)\geq 0\}} f_1\left(\begin{bmatrix} & 1 \\ -1 & \end{bmatrix}\begin{bmatrix} 1 & x \\ & 1 \end{bmatrix}\right) dx \\
&\quad + \int\limits_{B_0(F)\cap\{v_B(x)<0\}} f_1\left(\begin{bmatrix} & 1 \\ -1 & \end{bmatrix}\begin{bmatrix} 1 & x \\ & 1 \end{bmatrix}\right) dx \\
&= \int\limits_{B_0(F)\cap\{v_B(x)\geq 0\}} f_1\left(\begin{bmatrix} & 1 \\ -1 & \end{bmatrix}\right) dx + \int\limits_{B_0(F)\cap\{v_B(x)<0\}} f_1\left(\begin{bmatrix} 1 & \\ x & 1 \end{bmatrix}\begin{bmatrix} & 1 \\ -1 & \end{bmatrix}\right) dx \\
&= \int\limits_{B_0(F)\cap\{v_B(x)\geq 0\}} dx + \int\limits_{B_0(F)\cap\{v_B(x)<0\}} f_1\left(\begin{bmatrix} 1 & \\ x & 1 \end{bmatrix}\right) dx \\
&= 1 + \sum_{m=1}^{\infty} \int\limits_{B_0(F)\cap\{v_B(x)=-m\}} f_1\left(\begin{bmatrix} 1 & x^{-1} \\ & 1 \end{bmatrix}\begin{bmatrix} -x^{-1} & \\ & x \end{bmatrix}\begin{bmatrix} & 1 \\ 1 & \end{bmatrix}\begin{bmatrix} 1 & x^{-1} \\ & 1 \end{bmatrix}\right) dx \\
&= 1 + \sum_{m=1}^{\infty} \int\limits_{B_0(F)\cap\{v_B(x)=-m\}} f_1\left(\begin{bmatrix} -x^{-1} & \\ & x \end{bmatrix}\right) dx \\
&= 1 + \sum_{m=1}^{\infty} \int\limits_{B_0(F)\cap\{v_B(x)=-m\}} q^{-m(s+3/2)}\chi(\varpi_F)^m \, dx.
\end{aligned}$$

Splitting the sum into even and odd m's, and using Lemma A.3, one obtains, after a short calculation,

$$(Mf_1)(1) = \frac{(1+\chi(\varpi_F)q^{1/2-s})(1-\chi(\varpi_F)q^{3/2-s})}{(1+\chi(\varpi_F)q^{-s})(1-\chi(\varpi_F)q^{-s})}.$$

Assume that χ is the non-trivial, quadratic, unramified character of $F^\times$. Then $\chi(\varpi_F) = -1$, and we see that $(Mf_1)(1) = 0$ for $s = 1/2$. This means that $Mf_1 = 0$ for $s = 1/2$. Hence f_1 lies in the unique subrepresentation $\delta(\nu^{1/2}\chi 1_{B^\times}, \nu^{-1/2}\sigma)$ of $\nu^{1/2}\chi 1_{B^\times} \rtimes \sigma\nu^{-1/2}$.

Calculation for K_2.

Let f_2 be the K_2-invariant vector in $\nu^s \chi 1_{B^\times} \rtimes \sigma\nu^{-s}$. We have

$$f_2\left(\begin{bmatrix} & 1 \\ -1 & \end{bmatrix}\right) = f_2\left(\begin{bmatrix} \varpi_B^{-1} & \\ & \bar{\varpi}_B \end{bmatrix}\begin{bmatrix} & \varpi_B \\ -\bar{\varpi}_B^{-1} & \end{bmatrix}\right) = q^{3/2+s}\chi(\varpi_F)^{-1}.$$

Using this, a similar calculation as above shows that

$$(Mf_2)(1) = q^{1/2+s}\chi(\varpi_F)\frac{(1+\chi(\varpi_F)q^{-s-1/2})(1-\chi(\varpi_F)q^{-s-3/2})}{(1+\chi(\varpi_F)q^{-s})(1-\chi(\varpi_F)q^{-s})}.$$

Assume that χ is the non-trivial, quadratic, unramified character of $F^\times$. Then $\chi(\varpi_F) = -1$, and we see that $(Mf_2)(1) = 0$ for $s = -1/2$. This means that $Mf_2 = 0$ for $s = -1/2$. Hence f_2 lies in the unique subrepresentation $L(\nu^{1/2}\chi 1_{B^\times}, \nu^{-1/2}\sigma)$ of $\nu^{-1/2}\chi 1_{B^\times} \rtimes \sigma\nu^{1/2}$.

The result of the calculations in this section is that $\delta(\nu^{1/2}\chi 1_{B^\times}, \nu^{-1/2}\sigma)$ contains a one-dimensional space of K_1-invariant vectors, and $L(\nu^{1/2}\chi 1_{B^\times}, \nu^{-1/2}\sigma)$ contains a one-dimensional space of K_2-invariant vectors. For all other spherical representations of $G(F)$ these dimensions are obvious; we summarize the results in the table below.

A.4. Local Langlands parameters for $GSp(4)$.

To every irreducible, admissible representation of $GSp(4, F)$ there is attached an L-parameter, which is a certain homomorphism from the Weil–Deligne group W_F' to the dual group $GSp(4, \mathbb{C})$. The assignment of parameters to representations was carried out in [**39**, Section 2.4], for the non-supercuspidal representations of $GSp(4, F)$, and in [**10**] in general. The dual group of $G = GSp(1, 1)$ is also $GSp(4, \mathbb{C})$, but the Borel and Siegel parabolic subgroup of $GSp(4, \mathbb{C})$ are *irrelevant.* This means that any parameters whose image lies in (a conjugate of) the Siegel parabolic subgroup should be ignored for the local Langlands correspondence. The complete correspondence was achieved in [**11**]; for the non-supercuspidal representations one can also apply the reasoning of [**39**, Section 2.4].

The following table lists all irreducible, admissible representations of $G(F)$ which are constituents of representations of the form $\chi 1_{B^\times} \rtimes \sigma$, where χ and σ are characters of $F^\times$. The table also lists all the irreducible, admissible representations of $GSp(4, F)$ supported in the Borel subgroup, using the notations and classification scheme of [**39**]. Representations with the same L-parameter $W_F' \to GSp(4, \mathbb{C})$ appear in the same row; this is nothing but the Langlands functorial transfer from $GSp(1, 1)$ to $GSp(4)$ coming from the natural inclusion of dual groups. The actual L-parameters can be found in [**39**, Table A.7].

The columns labeled K_1 and K_2 indicate, in the case when the inducing characters are unramified, the dimension of the space of K_1 resp. K_2 invariant vectors in a representation of $G(F)$.

		$GSp(1,1)$	$GSp(4)$	K_1	K_2
I		—	$\chi_1 \times \chi_2 \rtimes \sigma$ (irreducible)		
II	a	$\chi 1_{B^\times} \rtimes \sigma$	$\chi \mathrm{St}_{GL(2)} \rtimes \sigma$	1	1
	b	—	$\chi 1_{GL(2)} \rtimes \sigma$		
III	a	—	$\chi \rtimes \sigma \mathrm{St}_{GSp(2)}$		
	b	—	$\chi \rtimes \sigma 1_{GSp(2)}$		
IV	a	$\sigma \mathrm{St}_{GSp(1,1)}$	$\sigma \mathrm{St}_{GSp(4)}$	0	0
	b	—	$L(\nu^2, \nu^{-1}\sigma \mathrm{St}_{GSp(2)})$		
	c	$\sigma 1_{GSp(1,1)}$	$L(\nu^{3/2}\mathrm{St}_{GL(2)}, \nu^{-3/2}\sigma)$	1	1
	d	—	$\sigma 1_{GSp(4)}$		
V	a	$\delta(\nu^{1/2}\xi 1_{B^\times}, \nu^{-1/2}\sigma)$	$\delta([\xi, \nu\xi], \nu^{-1/2}\sigma)$	1	0
	b	$L(\nu^{1/2}\xi 1_{B^\times}, \nu^{-1/2}\sigma)$	$L(\nu^{1/2}\xi \mathrm{St}_{GL(2)}, \nu^{-1/2}\sigma)$	0	1
	c	$L(\nu^{1/2}\xi 1_{B^\times}, \xi\nu^{-1/2}\sigma)$	$L(\nu^{1/2}\xi \mathrm{St}_{GL(2)}, \xi\nu^{-1/2}\sigma)$	0	1
	d	—	$L(\nu\xi, \xi \rtimes \nu^{-1/2}\sigma)$		
VI	a	—	$\tau(S, \nu^{-1/2}\sigma)$		
	b	—	$\tau(T, \nu^{-1/2}\sigma)$		
	c	$\nu^{1/2} 1_{B^\times} \rtimes \nu^{-1/2}\sigma$	$L(\nu^{1/2}\mathrm{St}_{GL(2)}, \nu^{-1/2}\sigma)$	1	1
	d	—	$L(\nu, 1_{F^\times} \rtimes \nu^{-1/2}\sigma)$		

For the IIa type representation, χ is such that $\chi^2 \neq \nu^{\pm 1}$ and $\chi \neq \nu^{\pm 3/2}$. For the representations in group V, the character ξ is assumed to be non-trivial and quadratic.

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