J. Math. Soc. Japan
Vol. 69, No. 3 (2017) pp. 1105–1156
doi: 10.2969/jmsj/06931105

Modules over quantized coordinate algebras and PBW-bases

By Toshiyuki TANISAKI

(Received Feb. 13, 2015)
(Revised Sep. 27, 2015)

Abstract. Around 1990 Soibelman constructed certain irreducible modules over the quantized coordinate algebra. A. Kuniba, M. Okado, Y. Yamada [**8**] recently found that the relation among natural bases of Soibelman's irreducible module can be described using the relation among the PBW-type bases of the positive part of the quantized enveloping algebra, and proved this fact using case-by-case analysis in rank two cases. In this paper we will give a realization of Soibelman's module as an induced module, and give a unified proof of the above result of [**8**]. We also verify Conjecture 1 of [**8**] about certain operators on Soibelman's module.

1. Introduction.

1.1. Let G be a connected simply-connected simple algebraic group over the complex number field $\mathbb{C}$ with Lie algebra $\mathfrak{g}$. The coordinate algebra $\mathbb{C}[G]$ of G is a Hopf algebra which is dual to the enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$. So we can naturally define a q-analogue $\mathbb{C}_q[G]$ of $\mathbb{C}[G]$ as the Hopf algebra dual to the quantized enveloping algebra $U_q(\mathfrak{g})$. This paper is concerned with the representation theory of the quantized coordinate algebra $\mathbb{C}_q[G]$.

Since the ordinary coordinate algebra $\mathbb{C}[G]$ is commutative, its irreducible modules are all one-dimensional and are in one-to-one correspondence with the points of G; however, the quantized coordinate algebra $\mathbb{C}_q[G]$ is non-commutative, and its representation theory is much more complicated. In fact, Soibelman [**12**] already pointed out around 1990 that there are not so many one-dimensional $\mathbb{C}_q[G]$-modules and that there really exist infinite dimensional irreducible $\mathbb{C}_q[G]$-modules.

Let us recall Soibelman's result more precisely. He considered the situation where the parameter q is a positive real number with $q \neq 1$. In this case $\mathbb{C}_q[G]$ is endowed with a structure of $*$-algebra, and we have the notion of unitarizable $\mathbb{C}_q[G]$-modules. Soibelman showed that one-dimensional unitarizable $\mathbb{C}_q[G]$-modules are in one-to-one correspondence with the points of the maximal compact subgroup H_{cpt} of the maximal torus H of G. Denote the one-dimensional $\mathbb{C}_q[G]$-module corresponding to $h \in H_{\mathrm{cpt}}$ by $\mathbb{C}_h$. On the other hand infinite-dimensional irreducible unitarizable $\mathbb{C}_q[G]$-modules are constructed as follows. In the case $G = SL_2$ Vaksman and Soibelman [**14**] constructed an irreducible unitarizable $\mathbb{C}_q[SL_2]$-modules $\mathcal{F}$ with basis $\{m_n\}_{n\in\mathbb{Z},n\geq 0}$ using an explicit description of $\mathbb{C}_q[SL_2]$. For general G denote by I the index set of simple roots. For each

2010 *Mathematics Subject Classification.* Primary 20G05, Secondary 17B37.
Key Words and Phrases. quantized coordinate algebra, PBW-basis.
The author was partially supported by Grants-in-Aid for Scientific Research (C) 24540026 from Japan Society for the Promotion of Science.

$i \in I$ we have a natural Hopf algebra homomorphism $\pi_i : \mathbb{C}_q[G] \to \mathbb{C}_{q_i}[SL_2]$, where q_i is some power of q. Via π_i we can regard $\mathcal{F}$ as a $\mathbb{C}_q[G]$-module. Denote this $\mathbb{C}_q[G]$-module by $\mathcal{F}_i$. Let W be the Weyl group of G. For $w \in W$ we denote the length of w by $\ell(w)$. Take $w \in W$ and its reduced expression $w = s_{i_1} \cdots s_{i_{\ell(w)}}$ $(i_r \in I)$ as a product of simple reflections. Soibelman proved that the tensor product $\mathcal{F}_{i_1} \otimes \cdots \otimes \mathcal{F}_{i_{\ell(w)}}$ is a unitarizable irreducible $\mathbb{C}_q[G]$-module. Moreover, he showed that $\mathcal{F}_{i_1} \otimes \cdots \otimes \mathcal{F}_{i_{\ell(w)}}$ depends only on w. So we can denote this $\mathbb{C}_q[G]$-module by $\mathcal{F}_w$. It is also verified in [**12**] that any irreducible unitarizable $\mathbb{C}_q[G]$-module is isomorphic to the tensor product $\mathcal{F}_w \otimes \mathbb{C}_h$ for $w \in W$, $h \in H_{\mathrm{cpt}}$.

As for further development of the theory of $\mathbb{C}_q[G]$-modules we refer to Joseph [**4**], Yakimov [**15**].

Quite recently the above work of Soibelman has been taken up again by Kuniba, Okado, Yamada [**8**]. Let $w_0 \in W$ be the longest element. Note that for each reduced expression $w_0 = s_{i_1} \cdots s_{i_{\ell(w_0)}}$ of w_0 we have a basis

$$\mathcal{B}_{i_1,\dots,i_{\ell(w_0)}} = \{m_{n_1} \otimes \cdots \otimes m_{n_{\ell(w_0)}} \mid n_1, \dots, n_{\ell(w_0)} \geq 0\}$$

of $\mathcal{F}_{w_0} = \mathcal{F}_{i_1} \otimes \cdots \otimes \mathcal{F}_{i_{\ell(w_0)}}$ parametrized by the set of $\ell(w_0)$-tuples $(n_1, \dots, n_{\ell(w_0)})$ of non-negative integers. On the other hand, by Lusztig's result, for each reduced expression $w_0 = s_{i_1} \cdots s_{i_{\ell(w_0)}}$ of w_0 we have a PBW-type basis $\mathcal{B}'_{i_1,\dots,i_{\ell(w_0)}}$ of the positive part $U_q(\mathfrak{n}^+)$ of $U_q(\mathfrak{g})$ parametrized by the set of $\ell(w_0)$-tuples of non-negative integers. Kuniba, Okado, Yamada observed in [**8**] that for two reduced expressions $w_0 = s_{i_1} \cdots s_{i_{\ell(w_0)}} = s_{j_1} \cdots s_{j_{\ell(w_0)}}$ of w_0 the transition matrix between $\mathcal{B}_{i_1,\dots,i_{\ell(w_0)}}$ and $\mathcal{B}_{j_1,\dots,j_{\ell(w_0)}}$ coincides with the transition matrix between $\mathcal{B}'_{i_1,\dots,i_{\ell(w_0)}}$ and $\mathcal{B}'_{j_1,\dots,j_{\ell(w_0)}}$ up to a normalization factor. They proved this fact partly using a case-by-case argument in rank two cases.

In the present paper we give a new approach to the results of Soibelman [**12**] and Kuniba, Okado, Yamada [**8**]. We work over the rational function field $\mathbb{F} = \mathbb{Q}(q)$; however, our arguments also hold in a more general situation (see Section 8 below). Let $\mathfrak{g} = \mathfrak{n}^+ \oplus \mathfrak{h} \oplus \mathfrak{n}^-$ be the triangular decomposition of $\mathfrak{g}$. Let $N^\pm$ and $B^\pm$ be the subgroups of G corresponding to $\mathfrak{n}^\pm$ and $\mathfrak{h} \oplus \mathfrak{n}^\pm$ respectively. For each $w \in W$ we define a $\mathbb{C}_q[G]$-module $\overline{\mathcal{M}}_w$ as the induced module from a one-dimensional representation of a certain subalgebra $\mathbb{C}_q[(N^- \cap wN^+w^{-1})\backslash G]$ of $\mathbb{C}_q[G]$. We will show that $\overline{\mathcal{M}}_w$ is an irreducible $\mathbb{C}_q[G]$-module and that for each reduced expression $w = s_{i_1} \cdots s_{i_{\ell(w)}}$ we have a decomposition $\overline{\mathcal{M}}_w \cong \mathcal{F}_{i_1} \otimes \cdots \otimes \mathcal{F}_{i_{\ell(w)}}$ into tensor product. This gives a new proof of Soibelman's result. We will also show that there exists a natural linear isomorphism

$$\overline{\mathcal{M}}_w \cong U_q(\mathfrak{n}^+ \cap w\mathfrak{n}^-), \tag{1.1}$$

where $U_q(\mathfrak{n}^+ \cap w\mathfrak{n}^-)$ is a certain subalgebra of $U_q(\mathfrak{g})$ defined in terms of Lusztig's braid group action (see De Concini, Kac, Procesi [**2**], Lusztig [**10**]). From this we obtain (in the case $w = w_0$) the result of Kuniba, Okado, Yamada described above. As in [**8**] a certain localization of $\mathbb{C}_q[G]$ plays a crucial role in the proof. More precisely, for each $w \in W$ we consider the localization $\mathbb{C}_q[wN^+B^-]$ of $\mathbb{C}_q[G]$, which is a q-analogue of $\mathbb{C}[wN^+B^-]$. In addition to it, we use the Drinfeld pairing between the positive and negative parts of the quantized enveloping algebra in constructing the isomorphism (1.1). A crucial difference between Soibelman's approach and our approach is that, instead of the decomposition

$$\mathbb{C}_q[G] = \mathbb{C}_q[G/N^+]\mathbb{C}_q[G/N^-]$$

used by Soibelman, we utilize the q-analogue of the decomposition

$$\mathbb{C}[B^- w_0 B^-] \cong \mathbb{C}[B^- w_0 B^-/B^-] \otimes \mathbb{C}[N^- \backslash B^- w_0 B^-]$$

in the case $w = w_0$, and

$$\begin{aligned}\mathbb{C}[wN^+B^-] &\cong \mathbb{C}[(wN^+w^{-1} \cap N^-)] \otimes \mathbb{C}[(wN^+w^{-1} \cap N^+)wB^-] \\ &\cong \mathbb{C}[(wN^+w^{-1} \cap N^-)] \otimes \mathbb{C}[(wN^+w^{-1} \cap N^-)\backslash wN^+B^-],\end{aligned}$$

for general w, which is more natural from geometric point of view. As a consequence of our approach, we can also show easily a conjecture of Kuniba, Okado, Yamada [**8**, Conjecture 1] concerning the action of a certain element of $\mathbb{C}_q[wN^+B^-]$ on $\overline{\mathcal{M}}_w$.

We finally note that our results hold true for any symmetrizable Kac–Moody algebra (see Section 8 below). We hope this fact will be useful in the investigation of 3-dimensional integrable systems, which was the original motivation of [**8**]. After writing up the first draft of this paper Yoshiyuki Kimura pointed out to me that Proposition 2.10 below in the Kac–Moody case is not an obvious fact which is stated as a conjecture in Berenstein and Greenstein [**1**, Conjecture 5.5]. In the present manuscript we have included a proof of Proposition 2.10 which works for the Kac–Moody case. We heard that Kimura also proved it by a different method (see Kimura [**6**]).

After finishing this work we heard that Yoshihisa Saito [**11**] has obtained similar results by a different method.

1.2. We use the following notation for Hopf algebras throughout the paper. For a Hopf algebra H over a field $\mathbb{K}$ we denote its multiplication, comultiplication, counit, antipode by $m_H : H \otimes_{\mathbb{K}} H \to H$, $\Delta_H : H \to H \otimes_{\mathbb{K}} H$, $\varepsilon_H : H \to \mathbb{K}$, $S_H : H \to H$ respectively. The subscript H is often omitted. For left H-modules $V_0, \dots, V_m$ we regard $V_0 \otimes_{\mathbb{K}} \cdots \otimes_{\mathbb{K}} V_m$ as a left H-module via the iterated comultiplication $\Delta_m : H \to H^{\otimes m+1}$. We will occasionally use Sweedler's notation for the comultiplication

$$\Delta(h) = \sum_{(h)} h_{(0)} \otimes h_{(1)} \qquad (h \in H),$$

and the iterated comultiplication

$$\Delta_m(h) = \sum_{(h)_m} h_{(0)} \otimes \cdots \otimes h_{(m)} \qquad (h \in H).$$

1.3. I would like to thank Masato Okado and Yoshiyuki Kimura for some useful discussion.

2. Quantized enveloping algebras.

2.1. Let G be a connected simply-connected simple algebraic group over the complex number field $\mathbb{C}$. We take Borel subgroups B^+ and B^- such that $H = B^+ \cap B^-$ is a maximal torus of G, and set $N^\pm = [B^\pm, B^\pm]$. The Lie algebras of G, $B^\pm$, H, $N^\pm$ are denoted by $\mathfrak{g}$, $\mathfrak{b}^\pm$, $\mathfrak{h}$, $\mathfrak{n}^\pm$ respectively. We denote by P the character group of H. Let Δ^+ and Δ^- be the subsets of P consisting of weights of $\mathfrak{n}^+$ and $\mathfrak{n}^-$ respectively, and set $\Delta = \Delta^+ \cup \Delta^-$. Then Δ is the set of roots of $\mathfrak{g}$ with respect to $\mathfrak{h}$. We denote by $\Pi = \{\alpha_i \mid i \in I\}$ the set of simple roots of Δ such that Δ^+ is the set of positive roots. Let P^+ be the set of dominant weights in P with respect to Π, and set $P^- = -P^+$. We set

$$Q = \sum_{i\in I} \mathbb{Z}\alpha_i, \qquad Q^+ = \sum_{i \in I} \mathbb{Z}_{\geq 0}\alpha_i,$$

where $\mathbb{Z}_{\geq 0}$ denotes the set of non-negative integers. The Weyl group $W = N_G(H)/H$ naturally acts on P and Q. By differentiation we will regard P as a $\mathbb{Z}$-lattice of $\mathfrak{h}^*$ in the following. We denote by

$$(\ ,\ ) : \mathfrak{h}^* \times \mathfrak{h}^* \to \mathbb{C}$$

the W-invariant non-degenerate symmetric bilinear form such that $(\alpha, \alpha) = 2$ for short roots α. For $\alpha \in \Delta$ we set $\alpha^\vee = 2\alpha/(\alpha, \alpha)$. As a subgroup of $GL(\mathfrak{h}^*)$ the Weyl group W is generated by the simple reflections s_i $(i \in I)$ given by $s_i(\lambda) = \lambda - (\lambda, \alpha_i^\vee)\alpha_i$ $(\lambda \in \mathfrak{h}^*)$. We denote by $\ell : W \to \mathbb{Z}_{\geq 0}$ the length function with respect to the generating set $\{s_i \mid i \in I\}$ of W. The longest element of W is denoted by w_0. For $w \in W$ we set

$$\mathcal{I}_w = \{(i_1, \dots, i_{\ell(w)}) \in I^{\ell(w)} \mid w = s_{i_1} \cdots s_{i_{\ell(w)}}\}.$$

2.2. For $n \in \mathbb{Z}$ we set

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}} \in \mathbb{Z}[q, q^{-1}].$$

For $m \in \mathbb{Z}_{\geq 0}$ we set

$$[m]_q! = [m]_q[m-1]_q \cdots [1]_q.$$

For $m, n \in \mathbb{Z}$ with $m \geq 0$ we set

$$\begin{bmatrix} n \\ m \end{bmatrix}_q = \frac{[n]_q[n-1]_q \cdots [n-m+1]_q}{[m]_q[m-1]_q \cdots [1]_q} \in \mathbb{Z}[q, q^{-1}].$$

For $i \in I$ we set $q_i = q^{(\alpha_i, \alpha_i)/2}$, and for $i, j \in I$ we further set $a_{ij} = (\alpha_i^\vee, \alpha_j)$.

We denote by $U = U_q(\mathfrak{g})$ the quantized enveloping algebra of $\mathfrak{g}$. Namely, it is an associative algebra over $\mathbb{F} = \mathbb{Q}(q)$ generated by the elements $k_i^{\pm 1}$, e_i, f_i $(i \in I)$ satisfying the defining relations

$$k_ik_i^{-1}=k_i^{-1}k_i=1 \qquad (i\in I),$$
$$k_ik_j=k_jk_i \qquad (i,j\in I),$$
$$k_ie_jk_i^{-1}=q_i^{a_{ij}}e_j \qquad (i,j\in I),$$
$$k_if_jk_i^{-1}=q_i^{-a_{ij}}e_j \qquad (i,j\in I),$$
$$e_if_j-f_je_i=\delta_{ij}\frac{k_i-k_i^{-1}}{q_i-q_i^{-1}} \qquad (i,j\in I),$$
$$\sum_{m=0}^{1-a_{ij}}(-1)^m e_i^{(1-a_{ij}-m)}e_je_i^{(m)}=0 \qquad (i,j\in I, i\neq j),$$
$$\sum_{m=0}^{1-a_{ij}}(-1)^m f_i^{(1-a_{ij}-m)}f_jf_i^{(m)}=0 \qquad (i,j\in I, i\neq j),$$

where

$$e_i^{(m)}=\frac{1}{[m]_{q_i}!}e_i^m, \qquad f_i^{(m)}=\frac{1}{[m]_{q_i}!}f_i^m \qquad (m\in\mathbb{Z}_{\geq 0}).$$

We endow U with the Hopf algebra structure given by

$$\Delta(k_i^{\pm 1})=k_i^{\pm 1}\otimes k_i^{\pm 1}, \quad \Delta(e_i)=e_i\otimes 1+k_i\otimes e_i, \quad \Delta(f_i)=f_i\otimes k_i^{-1}+1\otimes f_i,$$
$$\varepsilon(k_i^{\pm 1})=1, \quad \varepsilon(e_i)=\varepsilon(f_i)=0,$$
$$S(k_i^{\pm 1})=k_i^{\mp 1}, \quad S(e_i)=-k_i^{-1}e_i, \quad S(f_i)=-f_ik_i.$$

We define subalgebras $U^0=U_q(\mathfrak{h})$, $U^+=U_q(\mathfrak{n}^+)$, $U^-=U_q(\mathfrak{n}^-)$, $U^{\geq 0}=U_q(\mathfrak{b}^+)$, $U^{\leq 0}=U_q(\mathfrak{b}^-)$ by

$$U^0=\langle k_i^{\pm 1}\mid i\in I\rangle, \quad U^+=\langle e_i\mid i\in I\rangle, \qquad U^-=\langle f_i\mid i\in I\rangle,$$
$$U^{\geq 0}=\langle k_i^{\pm 1}, e_i\mid i\in I\rangle, \quad U^{\leq 0}=\langle k_i^{\pm 1}, f_i\mid i\in I\rangle,$$

respectively. Then U^0, $U^{\geq 0}$, $U^{\leq 0}$ are Hopf subalgebras. The multiplication of U induces isomorphisms

$$U\cong U^+\otimes U^0\otimes U^-\cong U^-\otimes U^0\otimes U^+,$$
$$U^{\geq 0}\cong U^0\otimes U^+\cong U^+\otimes U^0, \qquad U^{\leq 0}\cong U^0\otimes U^-\cong U^-\otimes U^0.$$

Remark 2.1. In this paper $\otimes_{\mathbb{F}}$ is often written as $\otimes$.

For $\gamma=\sum_{i\in I}m_i\alpha_i\in Q$ we set

$$k_\gamma=\prod_{i\in I}k_i^{m_i}\in U^0.$$

Then we have $U^0=\bigoplus_{\gamma\in Q}\mathbb{F}k_\gamma$, and hence U^0 is isomorphic to the group algebra of Q. For $\gamma\in Q^+$ we define $U^\pm_{\pm\gamma}$ by

$$U^{\pm}_{\pm\gamma} = \{u \in U^{\pm} \mid k_i u k_i^{-1} = q_i^{\pm(\alpha_i^\vee, \gamma)} u \ (i \in I)\}.$$

Then we have $U^{\pm} = \bigoplus_{\gamma \in Q^+} U^{\pm}_{\pm\gamma}$.

2.3. There exists a unique bilinear map

$$\tau : U^{\geq 0} \times U^{\leq 0} \to \mathbb{F} \tag{2.1}$$

characterized by the properties:

$$(\tau \otimes \tau)(\Delta(x), y_2 \otimes y_1) = \tau(x, y_1 y_2) \quad (x \in U^{\geq 0},\ y_1, y_2 \in U^{\leq 0}), \tag{2.2}$$

$$(\tau \otimes \tau)(x_1 \otimes x_2, \Delta(y)) = \tau(x_1 x_2, y) \quad (x_1, x_2 \in U^{\geq 0},\ y \in U^{\leq 0}), \tag{2.3}$$

$$\tau(e_i, k_\lambda) = \tau(k_\lambda, f_i) = 0 \qquad (i \in I,\ \lambda \in Q), \tag{2.4}$$

$$\tau(k_\lambda, k_\mu) = q^{(\lambda,\mu)} \qquad (\lambda, \mu \in Q), \tag{2.5}$$

$$\tau(e_i, f_j) = \delta_{ij} \frac{1}{q_i - q_i^{-1}} \qquad (i, j \in I). \tag{2.6}$$

We call it the Drinfeld pairing. It also satisfies the following properties:

$$\tau(Sx, Sy) = \tau(x, y) \qquad (x \in U^{\geq 0}, y \in U^{\leq 0}), \tag{2.7}$$

$$\tau(k_\lambda x, k_\mu y) = \tau(x, y) q^{(\lambda,\mu)} \qquad (x \in U^+, y \in U^-), \tag{2.8}$$

$$\gamma, \delta \in Q^+,\ \gamma \neq \delta \implies \tau|_{U^+_\gamma \times U^-_{-\delta}} = 0, \tag{2.9}$$

$$\gamma \in Q^+ \implies \tau|_{U^+_\gamma \times U^-_{-\gamma}} \text{ is non-degenerate.} \tag{2.10}$$

2.4. For a U^0-module M and $\lambda \in P$ we set

$$M_\lambda = \{m \in M \mid k_i m = q_i^{(\lambda, \alpha_i^\vee)} m \ (i \in I)\}.$$

We say that a U^0-module M is a weight module if $M = \bigoplus_{\lambda \in P} M_\lambda$.

For a U-module V we regard $V^* = \mathrm{Hom}_{\mathbb{F}}(V, \mathbb{F})$ as a right U-module by

$$\langle v^* u, v \rangle = \langle v^*, uv \rangle \qquad (v \in V, v^* \in V^*, u \in U).$$

Denote by $\mathrm{Mod}_0(U)$ (resp. $\mathrm{Mod}_0^r(U)$) the category of finite-dimensional left (resp. right) U-modules which is a weight module as a U^0-module. Here, a right U^0-module M is regarded as a left U^0-module by

$$tm := mt \qquad (m \in M, t \in U^0).$$

If $V \in \mathrm{Mod}_0(U)$, then we have $V^* \in \mathrm{Mod}_0^r(U)$. This gives an anti-equivalence $\mathrm{Mod}_0(U) \ni V \mapsto V^\star \in \mathrm{Mod}_0^r(U)$ of categories.

For $\lambda \in P^-$ we denote by $V(\lambda)$ the finite-dimensional irreducible (left) U-module with lowest weight λ. Namely, $V(\lambda)$ is a finite-dimensional U-module generated by a non-zero element $v_\lambda \in V(\lambda)_\lambda$ satisfying $f_i v_\lambda = 0$ $(i \in I)$. Then $\mathrm{Mod}_0(U)$ is a semisimple category with simple objects $V(\lambda)$ for $\lambda \in P^-$ (see Lusztig [**10**]). For $\lambda \in P^-$ we set

$V^*(\lambda)=(V(\lambda))^*$, and define $v_\lambda^*\in V^*(\lambda)_\lambda$ by $\langle v_\lambda^*, v_\lambda\rangle=1$.

The following well known fact will be used occasionally in this paper (see e.g. [**13**, Lemma 2.1]).

PROPOSITION 2.2. *Let $\gamma\in Q^+$.*

(i) *For sufficiently small $\lambda\in P^-$ the linear map $U^+_\gamma\ni x\mapsto xv_\lambda\in V(\lambda)_{\lambda+\gamma}$ is bijective.*

(ii) *For sufficiently small $\lambda\in P^-$ the linear map $U^-_{-\gamma}\ni y\mapsto v^*_\lambda y\in V^*(\lambda)_{\lambda+\gamma}$ is bijective.*

REMARK 2.3. In this paper the expression "for sufficiently small $\lambda\in P^-\dots$" means that "there exists some $\mu\in P^-$ such that for any $\lambda\in\mu+P^-\ \dots$".

2.5. For $i\in I$ and $M\in\mathrm{Mod}_0(U)$ we denote by $\dot T_i,\hat T_i\in GL(M)$ the operators denoted by $T''_{i,1}$ and $T''_{i,-1}$ respectively in [**10**]. We have also algebra automorphisms $\dot T_i$, $\hat T_i$ of U satisfying

$$\dot T_i(um)=\dot T_i(u)\dot T_i(m),\qquad \hat T_i(um)=\hat T_i(u)\hat T_i(m)$$

for $u\in U$, $m\in M\in\mathrm{Mod}_0(U)$. They are given by

$$\dot T_i(e_j)=\begin{cases}-f_ik_i & (j=i)\\ \displaystyle\sum_{r=0}^{-a_{ij}}(-1)^r q_i^{-r} e_i^{(-a_{ij}-r)}e_je_i^{(r)} & (j\neq i),\end{cases}$$

$$\dot T_i(f_j)=\begin{cases}-k_i^{-1}e_i & (j=i)\\ \displaystyle\sum_{r=0}^{-a_{ij}}(-1)^{-a_{ij}-r} q_i^{-a_{ij}-r} f_i^{(-a_{ij}-r)}f_jf_i^{(r)} & (j\neq i),\end{cases}$$

$$\hat T_i(e_j)=\begin{cases}-f_ik_i^{-1} & (j=i)\\ \displaystyle\sum_{r=0}^{-a_{ij}}(-1)^r q_i^{r} e_i^{(-a_{ij}-r)}e_je_i^{(r)} & (j\neq i),\end{cases}$$

$$\hat T_i(f_j)=\begin{cases}-k_ie_i & (j=i)\\ \displaystyle\sum_{r=0}^{-a_{ij}}(-1)^{-a_{ij}-r} q_i^{-(-a_{ij}-r)} f_i^{(-a_{ij}-r)}f_jf_i^{(r)} & (j\neq i),\end{cases}$$

$$\dot T_i(k_\gamma)=\hat T_i(k_\gamma)=k_{s_i\gamma}.$$

By [**10**] both $\{\dot T_i\}_{i\in I}$ and $\{\hat T_i\}_{i\in I}$ satisfy the braid relation, and hence we obtain the operators $\{\dot T_w\}_{w\in W}$, $\{\hat T_w\}_{w\in W}$ given by

$$\dot T_w=\dot T_{i_1}\cdots\dot T_{i_{\ell(w)}},\quad \hat T_w=\hat T_{i_1}\cdots\hat T_{i_{\ell(w)}}\qquad ((i_1,\dots,i_{\ell(w)})\in\mathcal{I}_w).$$

By the description of $\dot T_i$, $\hat T_i$ as automorphisms of U we have

$$\varepsilon(\dot{T}_w(u)) = \varepsilon(\hat{T}_w(u)) = \varepsilon(u) \qquad (w \in W, u \in U). \tag{2.11}$$

For $w \in W$ and $M \in \mathrm{Mod}_0^r(U)$ we define a right action of $\dot{T}_w$ (resp. $\hat{T}_w$) on M by

$$\langle m\dot{T}_w, m^* \rangle = \langle m, \dot{T}_w m^* \rangle \qquad (\text{resp. } \langle m\hat{T}_w, m^* \rangle = \langle m, \hat{T}_w m^* \rangle)$$

for $m \in M$, $m^* \in M^*$. We can easily check the following fact.

LEMMA 2.4. *Let $w \in W$. Then as algebra automorphisms of U we have $\hat{T}_w = S^{-1}\dot{T}_w S$.*

Let $w \in W$ and $\boldsymbol{i} = (i_1, \dots, i_m) \in \mathcal{I}_w$. For $r = 1, \dots, m$ set

$$\begin{aligned}
&k_{\boldsymbol{i},r} = k_{s_{i_1}\cdots s_{i_{r-1}}\alpha_{i_r}},\\
&\dot{e}_{\boldsymbol{i},r} = \dot{T}_{i_1}\cdots\dot{T}_{i_{r-1}}(e_{i_r}), \qquad \dot{f}_{\boldsymbol{i},r} = \dot{T}_{i_1}\cdots\dot{T}_{i_{r-1}}(f_{i_r}),\\
&\tilde{e}_{\boldsymbol{i},r} = \dot{T}_{i_m}^{-1}\cdots\dot{T}_{i_{r+1}}^{-1}(e_{i_r}), \qquad \tilde{f}_{\boldsymbol{i},r} = \dot{T}_{i_m}^{-1}\cdots\dot{T}_{i_{r+1}}^{-1}(f_{i_r}),\\
&\hat{e}_{\boldsymbol{i},r} = \hat{T}_{i_1}\cdots\hat{T}_{i_{r-1}}(e_{i_r}), \qquad \hat{f}_{\boldsymbol{i},r} = \hat{T}_{i_1}\cdots\hat{T}_{i_{r-1}}(f_{i_r}).
\end{aligned}$$

By [**10**] we have $\dot{e}_{\boldsymbol{i},r}, \tilde{e}_{\boldsymbol{i},r}, \hat{e}_{\boldsymbol{i},r} \in U^+$, $\dot{f}_{\boldsymbol{i},r}, \tilde{f}_{\boldsymbol{i},r}, \hat{f}_{\boldsymbol{i},r} \in U^-$. For $n \in \mathbb{Z}_{\geq 0}$ set

$$\begin{aligned}
&\dot{e}_{\boldsymbol{i},r}^{(n)} = \dot{T}_{i_1}\cdots\dot{T}_{i_{r-1}}(e_{i_r}^{(n)}), \qquad \tilde{f}_{\boldsymbol{i},r}^{(n)} = \dot{T}_{i_m}^{-1}\cdots\dot{T}_{i_{r+1}}^{-1}(f_{i_r}^{(n)}),\\
&\hat{e}_{\boldsymbol{i},r}^{(n)} = \hat{T}_{i_1}\cdots\hat{T}_{i_{r-1}}(e_{i_r}^{(n)}),
\end{aligned}$$

and for $\boldsymbol{n} = (n_1, \dots, n_m) \in (\mathbb{Z}_{\geq 0})^m$ set

$$\begin{aligned}
&\dot{e}_{\boldsymbol{i}}^{(\boldsymbol{n})} = \dot{e}_{\boldsymbol{i},m}^{(n_m)}\cdots\dot{e}_{\boldsymbol{i},1}^{(n_1)}, \qquad \dot{f}_{\boldsymbol{i}}^{\boldsymbol{n}} = \dot{f}_{\boldsymbol{i},m}^{n_m}\cdots\dot{f}_{\boldsymbol{i},1}^{n_1},\\
&\tilde{e}_{\boldsymbol{i}}^{\boldsymbol{n}} = \tilde{e}_{\boldsymbol{i},1}^{n_1}\cdots\tilde{e}_{\boldsymbol{i},m}^{n_m}, \qquad \tilde{f}_{\boldsymbol{i}}^{(\boldsymbol{n})} = \tilde{f}_{\boldsymbol{i},1}^{(n_1)}\cdots\tilde{f}_{\boldsymbol{i},m}^{(n_m)},\\
&\hat{e}_{\boldsymbol{i}}^{(\boldsymbol{n})} = \hat{e}_{\boldsymbol{i},m}^{(n_m)}\cdots\hat{e}_{\boldsymbol{i},1}^{(n_1)}, \qquad \hat{f}_{\boldsymbol{i}}^{\boldsymbol{n}} = \hat{f}_{\boldsymbol{i},m}^{n_m}\cdots\hat{f}_{\boldsymbol{i},1}^{n_1}.
\end{aligned}$$

PROPOSITION 2.5 ([**9**]). *Let $w \in W$ and $\boldsymbol{i} \in \mathcal{I}_w$. Then we have*

$$\tau(\hat{e}_{\boldsymbol{i}}^{(\boldsymbol{n})}, \hat{f}_{\boldsymbol{i}}^{\boldsymbol{n}'}) = \delta_{\boldsymbol{n},\boldsymbol{n}'}\prod_{r=1}^{\ell(w)} c_{q_{i_r}}(n_r),$$

where

$$c_q(n) = [n]!q^{-n(n-1)/2}(q - q^{-1})^{-n}.$$

The following result will be used frequently in this paper.

PROPOSITION 2.6 ([**7**], [**9**], [**10**]). *We have*

$$\Delta(\dot{T}_i) = (\dot{T}_i \otimes \dot{T}_i)\exp_{q_i}((q_i - q_i^{-1})f_i \otimes e_i) = \exp_{q_i}((q_i - q_i^{-1})k_i^{-1}e_i \otimes f_i k_i)(\dot{T}_i \otimes \dot{T}_i),$$

where

$$\exp_q(x) = \sum_{n=0}^{\infty} \frac{q^{n(n-1)/2}}{[n]!} x^n.$$

Corollary 2.7. *For $w \in W$ and $\boldsymbol{i} = (i_1, \dots, i_m) \in \mathcal{I}_w$ we have*

$$\begin{aligned}\Delta(\dot{T}_w) &= (\dot{T}_w \otimes \dot{T}_w) \exp_{q_{i_1}}(X_1) \cdots \exp_{q_{i_m}}(X_m) \\ &= \exp_{q_{i_1}}(Y_1) \cdots \exp_{q_{i_m}}(Y_m)(\dot{T}_w \otimes \dot{T}_w), \\ \Delta(\dot{T}_w^{-1}) &= \exp_{q_{i_m}^{-1}}(-X_m) \cdots \exp_{q_{i_1}^{-1}}(-X_1)(\dot{T}_w^{-1} \otimes \dot{T}_w^{-1}) \\ &= (\dot{T}_w^{-1} \otimes \dot{T}_w^{-1}) \exp_{q_{i_m}^{-1}}(-Y_m) \cdots \exp_{q_{i_1}^{-1}}(-Y_1),\end{aligned}$$

where

$$X_r = (q_{i_r} - q_{i_r}^{-1}) \tilde{f}_{\boldsymbol{i},r} \otimes \tilde{e}_{\boldsymbol{i},r}, \qquad Y_r = (q_{i_r} - q_{i_r}^{-1}) k_{\boldsymbol{i},r}^{-1} \dot{e}_{\boldsymbol{i},r} \otimes \dot{f}_{\boldsymbol{i},r} k_{\boldsymbol{i},r}.$$

Lemma 2.8. *For $w \in W$ we have*

$$\begin{aligned}&\Delta(\dot{T}_w(U^+)) \subset U \otimes (\dot{T}_w(U^+))U^0, \quad \Delta(\dot{T}_w^{-1}(U^+)) \subset (\dot{T}_w^{-1}(U^+))U^0 \otimes U, \\ &\Delta(\dot{T}_w(U^-)) \subset (\dot{T}_w(U^-))U^0 \otimes U, \quad \Delta(\dot{T}_w^{-1}(U^+)) \subset U \otimes (\dot{T}_w^{-1}(U^+))U^0.\end{aligned}$$

Proof. We only show the first formula since the proof of other formulas are similar. To show the first formula we need to show that for $y \in \dot{T}_w U^+$ we have $(\dot{T}_w^{-1} \otimes \dot{T}_w^{-1})(\Delta(y)) \in U \otimes U^{\geq 0}$.

For each $\lambda \in P^-$ take $v_{w_0\lambda} \in V(\lambda)_{w_0\lambda} \setminus \{0\}$. Then for $u \in U$ we have $u \in U^{\geq 0}$ if and only if $u(M \otimes v_{w_0\lambda}) \subset M \otimes v_{w_0\lambda}$ for any $\lambda \in P^-$ and $M \in \mathrm{Mod}_0(U)$ (see the proof of [**3**, Proposition 5.11]). By this fact together with [**3**, Proposition 5.11] it is sufficient to show that for $M_1, M_2 \in \mathrm{Mod}_0(U)$ and $\lambda \in P^-$ the element

$$(\mathrm{id} \otimes \Delta)\{(\dot{T}_w^{-1} \otimes \dot{T}_w^{-1})(\Delta(y))\} \in U \otimes U \otimes U$$

sends $M_1 \otimes M_2 \otimes v_{w_0\lambda}$ to itself. As an operator on the tensor product of two integrable modules we have

$$(\dot{T}_w^{-1} \otimes \dot{T}_w^{-1})(\Delta(y)) = (\dot{T}_w^{-1} \otimes \dot{T}_w^{-1}) \circ (\Delta(y)) \circ (\dot{T}_w \otimes \dot{T}_w).$$

Take $\boldsymbol{i} = (i_1, \dots, i_m) \in \mathcal{I}_w$. By Corollary 2.7 we have

$$\dot{T}_w \otimes \dot{T}_w = \Delta(\dot{T}_w) \circ Z^{-1}, \quad Z = \exp_{q_{i_1}}(X_1) \cdots \exp_{q_{i_m}}(X_m),$$

where $X_1, \dots, X_m$ are as in Corollary 2.7. Hence we have

$$(\dot{T}_w^{-1} \otimes \dot{T}_w^{-1}) \circ (\Delta(y)) \circ (\dot{T}_w \otimes \dot{T}_w) = Z \circ \Delta(\dot{T}_w^{-1}(y)) \circ Z^{-1}.$$

Therefore, our assertion is a consequence of $\dot{T}_w^{-1}(y) \in U^+$, and $X_r \in U^- \otimes U^+$. □

For $w \in W$ set

$$U^-[\dot{T}_w] = U^- \cap \dot{T}_w(U^{\geq 0}), \qquad U^+[\dot{T}_w] = U^+ \cap \dot{T}_w(U^{\leq 0}), \tag{2.12}$$
$$U^-[\dot{T}_w^{-1}] = U^- \cap \dot{T}_w^{-1}(U^{\geq 0}), \qquad U^+[\dot{T}_w^{-1}] = U^+ \cap \dot{T}_w^{-1}(U^{\leq 0}), \tag{2.13}$$
$$U^-[\hat{T}_w] = U^- \cap \hat{T}_w(U^{\geq 0}), \qquad U^+[\hat{T}_w] = U^+ \cap \hat{T}_w(U^{\leq 0}). \tag{2.14}$$

We can easily show the following using Lemma 2.4.

Lemma 2.9. *For $w \in W$, $\boldsymbol{i} = (i_1, \dots, i_m) \in \mathcal{I}_w$ we have*

$$S^{-1}\dot{T}_w(\tilde{e}_{\boldsymbol{i}}^{\boldsymbol{n}}) = \hat{f}_{\boldsymbol{i}}^{\boldsymbol{n}}.$$

Hence we have

$$\dot{T}_w^{-1} S(U^-[\hat{T}_w]) = U^+[\dot{T}_w^{-1}].$$

The following well-known result is an easy consequence of the existence of the PBW-type base of $U^{\pm}$. Here, we give another proof which works for the quantized enveloping algebra of any symmetrizable Kac–Moody Lie algebra.

Proposition 2.10. *The multiplication of U induces the isomorphisms*

$$U^{\pm} \cong U^{\pm}[\dot{T}_w] \otimes (U^{\pm} \cap \dot{T}_w(U^{\pm})) \cong (U^{\pm} \cap \dot{T}_w(U^{\pm})) \otimes U^{\pm}[\dot{T}_w], \tag{2.15}$$
$$U^{\pm} \cong U^{\pm}[\dot{T}_w^{-1}] \otimes (U^{\pm} \cap \dot{T}_w^{-1}(U^{\pm})) \cong (U^{\pm} \cap \dot{T}_w^{-1}(U^{\pm})) \otimes U^{\pm}[\dot{T}_w^{-1}], \tag{2.16}$$
$$U^{\pm} \cong U^{\pm}[\hat{T}_w] \otimes (U^{\pm} \cap \hat{T}_w(U^{\pm})) \cong (U^{\pm} \cap \hat{T}_w(U^{\pm})) \otimes U^{\pm}[\hat{T}_w]. \tag{2.17}$$

Proof. We first note that (2.17) follows easily from (2.15) and Lemma 2.4. Consider the ring involution

$$a : U \to U \quad (q \mapsto q^{-1},\ k_\lambda \mapsto k_\lambda^{-1},\ e_i \mapsto -k_i^{-1}e_i,\ f_i \mapsto -f_i k_i),$$

and the ring anti-involution

$$b : U \to U \quad (q \mapsto q^{-1},\ k_\lambda \mapsto k_\lambda^{-1},\ e_i \mapsto f_i,\ f_i \mapsto e_i).$$

By $b\dot{T}_w = \dot{T}_w b$ and $b(U^+) = U^-$ the statements for U^- are consequences of those for U^+. By $a\dot{T}_w a = \dot{T}_{w^{-1}}^{-1}$ and $a(U_\gamma^+) = k_\gamma^{-1}U_\gamma^+$ ($\gamma \in Q^+$) the statements for $\dot{T}_w^{-1}$ are consequences of those for $\dot{T}_{w^{-1}}$. Hence we have only to show

$$U^+ \cong U^+[\dot{T}_w] \otimes (U^+ \cap \dot{T}_w(U^+)), \tag{2.18}$$
$$U^+ \cong (U^+ \cap \dot{T}_w(U^+)) \otimes U^+[\dot{T}_w]. \tag{2.19}$$

We first show (2.18). Note that the injectivity of

$$U^+[\dot{T}_w] \otimes (U^+ \cap \dot{T}_w(U^+)) \to U^+ \tag{2.20}$$

is clear from

$$U^+[\dot{T}_w] \otimes (U^+ \cap \dot{T}_w(U^+)) \subset T_w(U^{\leq 0}) \otimes T_w(U^+)$$

and $U^{\leq 0}\otimes U^+\cong U$. Hence we have only to show the surjectivity of (2.20).

For $\boldsymbol{i}=(i_1,\dots,i_m)\in\mathcal{I}_w$ denote by $U^+[\dot{T}_w;\boldsymbol{i}]$ the subalgebra of U^+ generated by $\dot{e}_{\boldsymbol{i},1},\dots,\dot{e}_{\boldsymbol{i},m}$. By a standard property of $\dot{T}_i$ we have $U^+[\dot{T}_w;\boldsymbol{i}]\subset U^+[\dot{T}_w]$. Hence we have only to show

$$U^+[\dot{T}_w;\boldsymbol{i}](U^+\cap\dot{T}_w(U^+))=U^+. \tag{2.21}$$

We note that our assertion is already known for $w=s_i$. Namely, we have

$$U^+\cong U^+[\dot{T}_i]\otimes(U^+\cap\dot{T}_i(U^+)),\qquad U^+[\dot{T}_i]=\mathbb{F}[e_i], \tag{2.22}$$

$$U^-\cong U^-[\dot{T}_i]\otimes(U^-\cap\dot{T}_i(U^-)),\qquad U^-[\dot{T}_i]=\mathbb{F}[f_i] \tag{2.23}$$

(see [**10**, Chapter 38]).

Now we are going to show (2.21) by induction on $\ell(w)$. Assume that we have $xs_i>x$ for $x\in W$ and $i\in I$. By the above argument we need to show (2.21) for $w=xs_i$ assuming (2.15), (2.16), (2.21) for $w\in W$ with $\ell(w)\leq\ell(x)$. Take $\boldsymbol{i}=(i_1,\dots,i_m)\in\mathcal{I}_x$, and set $\boldsymbol{i}'=(i_1,\dots,i_m,i)\in\mathcal{I}_{xs_i}$. To show our assertion $U^+[\dot{T}_{xs_i};\boldsymbol{i}'](U^+\cap\dot{T}_{xs_i}(U^+))=U^+$, it is sufficient to show

$$U^+\cap\dot{T}_x^{-1}(U^+)=\mathbb{F}[e_i](U^+\cap\dot{T}_i(U^+)\cap\dot{T}_x^{-1}(U^+)). \tag{2.24}$$

Indeed assuming (2.24) we have

$$\begin{aligned}U^+\cap\dot{T}_x(U^+)&=\dot{T}_x(U^+\cap\dot{T}_x^{-1}(U^+))=\mathbb{F}[\dot{T}_x(e_i)](U^+\cap\dot{T}_x(U^+)\cap\dot{T}_{xs_i}(U^+))\\&\subset\mathbb{F}[\dot{T}_x(e_i)](U^+\cap\dot{T}_{xs_i}(U^+)),\end{aligned}$$

and hence

$$\begin{aligned}U^+&=U^+[\dot{T}_x;\boldsymbol{i}](U^+\cap\dot{T}_x(U^+))\subset U^+[\dot{T}_x;\boldsymbol{i}]\mathbb{F}[\dot{T}_x(e_i)](U^+\cap\dot{T}_{xs_i}(U^+))\\&\subset U^+[\dot{T}_{xs_i};\boldsymbol{i}'](U^+\cap\dot{T}_{xs_i}(U^+)).\end{aligned}$$

To verify (2.24) we first show the following.

$$U^+\cap\dot{T}_x^{-1}(U^+)=\{u\in U^+\mid\tau(u,U^-(U^-[\dot{T}_x^{-1}]\cap\operatorname{Ker}(\varepsilon:U^-\to\mathbb{F})))=\{0\}\}, \tag{2.25}$$

$$U^-\cap\dot{T}_x^{-1}(U^-)=\{u\in U^-\mid\tau(U^+(U^+[\dot{T}_x^{-1}]\cap\operatorname{Ker}(\varepsilon:U^+\to\mathbb{F})),u)=\{0\}\}. \tag{2.26}$$

For simplicity set

$$\begin{aligned}V^+&=\{u\in U^+\mid\tau(u,U^-(U^-[\dot{T}_x^{-1}]\cap\operatorname{Ker}(\varepsilon)))=\{0\}\},\\V^-&=\{u\in U^-\mid\tau(U^+(U^+[\dot{T}_x^{-1}]\cap\operatorname{Ker}(\varepsilon)),u)=\{0\}\}.\end{aligned}$$

By (2.9) we have

$$\tau(U^+\cap\dot{T}_x^{-1}(U^+),U^-[\dot{T}_x^{-1}]\cap\operatorname{Ker}(\varepsilon))=\{0\}.$$

Hence by Lemma 2.8 and (2.2) we have $U^+\cap\dot{T}_x^{-1}(U^+)\subset V^+$. By a similar argument we have also $U^- \cap \dot{T}_x^{-1}(U^-) \subset V^-$. On the other hand by the hypothesis of induction we have $U^\pm \cong U^\pm[\dot{T}_x^{-1}] \otimes (U^\pm \cap \dot{T}_x^{-1}(U^\pm))$, and hence $U^\pm = (U^\pm \cap \dot{T}_x^{-1}(U^\pm)) \oplus U^\pm(U^\pm[\dot{T}_x^{-1}] \cap \mathrm{Ker}(\varepsilon))$. Since τ is non-degenerate, we obtain injective linear maps $V^\pm \to (U^\mp \cap \dot{T}_x^{-1}(U^\mp))^*$. Comparing the dimensions of weight spaces we obtain

$$\dim(U^\pm_{\pm\gamma} \cap \dot{T}_x^{-1}(U^\pm)) \le \dim V^\pm_{\pm\gamma} \le \dim(U^\mp_{\mp\gamma} \cap \dot{T}_x^{-1}(U^\mp))$$

for each $\gamma \in Q^+$. This gives (2.25) and (2.26).

By a similar argument we have also

$$U^+ \cap \dot{T}_x(U^+) = \{u \in U^+ \mid \tau(u, (U^-[\dot{T}_x] \cap \mathrm{Ker}(\varepsilon : U^- \to \mathbb{F}))U^-) = \{0\}\}, \tag{2.27}$$

$$U^- \cap \dot{T}_x(U^-) = \{u \in U^- \mid \tau((U^+[\dot{T}_x] \cap \mathrm{Ker}(\varepsilon : U^+ \to \mathbb{F}))U^+, u) = \{0\}\}. \tag{2.28}$$

Now let us show (2.24). Take $u \in U^+ \cap \dot{T}_x^{-1}(U^+)$, and decompose it as

$$u = \sum_n e_i^n u_n \qquad (u_n \in U^+ \cap \dot{T}_i(U^+))$$

(see (2.22)). Then it is sufficient to show $u_n \in \dot{T}_x^{-1}(U^+)$, which is equivalent to

$$\tau(u_n, vz) = 0 \qquad (v \in U^-,\ z \in U^-[\dot{T}_x^{-1}] \cap \mathrm{Ker}(\varepsilon)) \tag{2.29}$$

by (2.25). Let $v \in U^-$, $z \in U^-[\dot{T}_x^{-1}] \cap \mathrm{Ker}(\varepsilon)$. By our assumption we have $\tau(u, vz) = 0$. On the other hand we have

$$\begin{aligned}\tau(u, vz) &= \sum_n \tau(e_i^n u_n, vz) = \sum_n \sum_{(z),(v)} \tau(e_i^n, v_{(0)} z_{(0)}) \tau(u_n, v_{(1)} z_{(1)}) \\ &= \sum_n \sum_{(v)} \tau(e_i^n, v_{(0)}) \tau(u_n, v_{(1)} z)\end{aligned}$$

by Lemma 2.8 and (2.9). Consider the case

$$v = f_i^{(r)} v' \qquad (v' \in U^- \cap \dot{T}_i(U^-)).$$

Then we have

$$\begin{aligned}\tau(u, vz) &= \sum_n \sum_{s=0}^{r} \sum_{(v')} q_i^{-s(r-s)} \tau(e_i^n, f_i^{(r-s)} v'_{(0)}) \tau(u_n, k_i^{-(r-s)} f_i^{(s)} v'_{(1)} z) \\ &= \sum_n \sum_{s=0}^{r} q_i^{-s(r-s)} \tau(e_i^n, f_i^{(r-s)}) \tau(u_n, f_i^{(s)} v' z)\end{aligned}$$

by Lemma 2.8, (2.8), (2.9). By $u_n \in U^+ \cap \dot{T}_i(U^+)$ and (2.27) we have

$$\tau(u_n, f_i U^-) = \{0\}. \tag{2.30}$$

Hence

$$\tau(u, vz) = \sum_n \tau(e_i^n, f_i^{(r)})\tau(u_n, v'z) = \tau(e_i^r, f_i^{(r)})\tau(u_r, v'z).$$

Hence by $\tau(e_i^r, f_i^{(r)}) \neq 0$ we obtain

$$\tau(u_n, v'z) = 0 \qquad (z \in U^-[\dot{T}_x^{-1}] \cap \mathrm{Ker}(\varepsilon), v' \in U^- \cap \dot{T}_i(U^-)) \tag{2.31}$$

for any n. By (2.30), (2.31)

$$\tau(u_n, f_i^r v'z) = 0 \qquad (z \in U^-[\dot{T}_x^{-1}] \cap \mathrm{Ker}(\varepsilon), v' \in U^- \cap \dot{T}_i(U^-), r \geq 0).$$

The proof of (2.18) is complete.

It remains to show (2.19). Similarly to the above proof of (2.18) it is sufficient to show

$$U^+ \cap \dot{T}_x^{-1}(U^+) = (U^+ \cap \dot{T}_i(U^+) \cap \dot{T}_x^{-1}(U^+))\mathbb{F}[e_i]. \tag{2.32}$$

This follows from (2.24) as follows. We can easily show

$$\gamma \in Q^+,\ u \in U^+ \cap \dot{T}_i(U_\gamma^+) \implies ue_i - q_i^{(\gamma, \alpha_i^\vee)} e_i u \in U^+ \cap \dot{T}_i(U^+) \tag{2.33}$$

using [**10**, Proposition 38.1.6]. Hence in view of (2.24) it is sufficient to show that for $u \in U^+ \cap \dot{T}_i(U_\gamma^+) \cap \dot{T}_x^{-1}(U^+)$ we have $ue_i - q_i^{(\gamma, \alpha_i^\vee)} e_i u \in \dot{T}_x^{-1}(U^+)$. This is obvious since $\dot{T}_x(e_i) \in U^+$. □

Remark 2.11. Proposition 2.10 holds true for various $\mathbb{Z}[q, q^{-1}]$-forms of $U^\pm$. To show this it is sufficient to verify (2.22) over $\mathbb{Z}[q, q^{-1}]$. In the case of the De Concini–Kac form $U^{DK,\pm}_{\mathbb{Z}[q,q^{-1}]}$, this follows if we can show that $(U^{DK,+}_{\mathbb{Z}[q,q^{-1}]} \cap U^+[\dot{T}_i])(U^{DK,+}_{\mathbb{Z}[q,q^{-1}]} \cap \dot{T}_i(U^+))$ is stable under the right multiplication of e_j for any $j \in I$. If $j \neq i$, this is obvious. If $j = i$, this follows from (2.33). The argument for the case of the Lusztig form $U^{L,\pm}_{\mathbb{Z}[q,q^{-1}]}$ is similar. Finally, (2.22) over $\mathbb{Z}[q, q^{-1}]$ for the De Concini–Procesi form defined by

$$U^{DP,+}_{\mathbb{Z}[q,q^{-1}]} = \{u \in U^+ \mid \tau(u, U^{L,-}_{\mathbb{Z}[q,q^{-1}]}) \in \mathbb{Z}[q, q^{-1}]\},$$
$$U^{DP,-}_{\mathbb{Z}[q,q^{-1}]} = \{u \in U^- \mid \tau(U^{L,+}_{\mathbb{Z}[q,q^{-1}]}, u) \in \mathbb{Z}[q, q^{-1}]\},$$

is a consequence of that for the Lusztig form by duality.

By our proof of Proposition 2.10 we also obtain the following.

Proposition 2.12. *Let $w \in W$ and $\boldsymbol{i} \in \mathcal{I}_w$.*

(i) *The set $\{\dot{f}_{\boldsymbol{i}}^{\boldsymbol{n}} \mid \boldsymbol{n} \in (\mathbb{Z}_{\geq 0})^m\}$ (resp. $\{\dot{e}_{\boldsymbol{i}}^{(\boldsymbol{n})} \mid \boldsymbol{n} \in (\mathbb{Z}_{\geq 0})^m\}$) forms an $\mathbb{F}$-basis of $U^-[\dot{T}_w]$ (resp. $U^+[\dot{T}_w]$).*

(ii) *The set $\{\tilde{f}_{\boldsymbol{i}}^{(\boldsymbol{n})} \mid \boldsymbol{n} \in (\mathbb{Z}_{\geq 0})^m\}$ (resp. $\{\tilde{e}_{\boldsymbol{i}}^{\boldsymbol{n}} \mid \boldsymbol{n} \in (\mathbb{Z}_{\geq 0})^m\}$) forms an $\mathbb{F}$-basis of $U^-[\dot{T}_w^{-1}]$ (resp. $U^+[\dot{T}_w^{-1}]$).*

(iii) *The set* $\{\hat{f}_{\boldsymbol{i}}^{\boldsymbol{n}} \mid \boldsymbol{n} \in (\mathbb{Z}_{\geq 0})^m\}$ *(resp.* $\{\hat{e}_{\boldsymbol{i}}^{(\boldsymbol{n})} \mid \boldsymbol{n} \in (\mathbb{Z}_{\geq 0})^m\}$*) forms an* $\mathbb{F}$*-basis of* $U^-[\hat{T}_w]$ *(resp.* $U^+[\hat{T}_w]$*).*

For $w \in W$ and $\gamma \in Q^+$ we have

$$U^{\pm}[\dot{T}_w] \cap U^{\pm}_{\pm\gamma} = U^{\pm} \cap \dot{T}_w(k_{\pm w^{-1}\gamma} U^{\mp}_{\pm w^{-1}\gamma}), \tag{2.34}$$

$$U^{\pm}[\dot{T}_w^{-1}] \cap U^{\pm}_{\pm\gamma} = U^{\pm} \cap \dot{T}_w^{-1}(k_{\mp w\gamma} U^{\mp}_{\pm w\gamma}) \tag{2.35}$$

by Proposition 2.12 and the explicit description of $\dot{T}_i$.

3. Quantized coordinate algebras.

3.1. We denote by $\mathbb{C}_q[G]$ the quantized coordinate algebra of U (see, for example, [**4**], [**5**], [**13**] for the basic facts concerning $\mathbb{C}_q[G]$). It is the $\mathbb{F}$-subspace of U^* spanned by the matrix coefficients of U-modules belonging to $\mathrm{Mod}_0(U)$. Namely, for $V \in \mathrm{Mod}_0(U)$ define a linear map

$$\Phi : V^* \otimes V \to U^* \quad (v^* \otimes v \mapsto \Phi_{v^* \otimes v}) \tag{3.1}$$

by

$$\langle \Phi_{v^* \otimes v}, u \rangle = \langle v^*, uv \rangle \qquad (u \in U).$$

Then we have

$$\mathbb{C}_q[G] = \sum_{V \in \mathrm{Mod}_0(U)} \mathrm{Im}(V^* \otimes V \to U^*) \subset U^*. \tag{3.2}$$

It is endowed with a Hopf algebra structure by $m_{\mathbb{C}_q[G]} = {}^t\Delta_U$, $\Delta_{\mathbb{C}_q[G]} = {}^t m_U$. Moreover, (3.2) induces

$$\mathbb{C}_q[G] \cong \bigoplus_{\lambda \in P^-} V^*(\lambda) \otimes V(\lambda). \tag{3.3}$$

We set

$$\mathbb{C}_q[B^+] = \mathrm{Im}(\mathbb{C}_q[G] \to (U^{\geq 0})^*), \quad \mathbb{C}_q[B^-] = \mathrm{Im}(\mathbb{C}_q[G] \to (U^{\leq 0})^*),$$
$$\mathbb{C}_q[H] = \mathrm{Im}(\mathbb{C}_q[G] \to (U^0)^*).$$

For $\lambda \in P$ we denote by $\chi_\lambda : U^0 \to \mathbb{F}$ the algebra homomorphism given by $\chi_\lambda(k_\gamma) = q^{(\lambda,\gamma)}$ for $\gamma \in Q$. Then we have

$$\mathbb{C}_q[H] = \bigoplus_{\lambda \in P} \mathbb{F}\chi_\lambda. \tag{3.4}$$

Note that $\mathbb{C}_q[G]$ is a U-bimodule by

$$\langle u_1 \varphi u_2, u \rangle = \langle \varphi, u_2 u u_1 \rangle \qquad (\varphi \in \mathbb{C}_q[G], u_1, u_2, u \in U).$$

For $\varphi, \psi \in \mathbb{C}_q[G]$ and $u \in U$ we have

$$u(\varphi\psi) = \sum_{(u)} (u_{(0)}\varphi)(u_{(1)}\psi), \tag{3.5}$$

$$(\varphi\psi)u = \sum_{(u)} (\varphi u_{(0)})(\psi u_{(1)}). \tag{3.6}$$

EXAMPLE 3.1. Consider the case where $\mathfrak{g} = \mathfrak{sl}_2$ and $G = SL_2$. In this case $U = U_q(\mathfrak{sl}_2)$ is the $\mathbb{F}$-algebra generated by the elements $k^{\pm 1}$, e, f satisfying

$$kk^{-1} = k^{-1}k = 1, \quad kek^{-1} = q^2 e, \quad kfk^{-1} = q^{-2} f, \quad ef - fe = \frac{k - k^{-1}}{q - q^{-1}}.$$

Let $V = \mathbb{F}v_0 \oplus \mathbb{F}v_1$ be the two-dimensional U-module given by

$$kv_0 = qv_0, \quad kv_1 = q^{-1}v_1, \quad ev_0 = 0, \quad ev_1 = v_0, \quad fv_0 = v_1, \quad fv_1 = 0,$$

and define $a, b, c, d \in \mathbb{C}_q[SL_2]$ by

$$uv_0 = \langle a, u \rangle v_0 + \langle c, u \rangle v_1, \qquad uv_1 = \langle b, u \rangle v_0 + \langle d, u \rangle v_1 \qquad (u \in U).$$

Then $\{a, b, c, d\}$ forms a generator system of the $\mathbb{F}$-algebra $\mathbb{C}_q[SL_2]$ satisfying the fundamental relations

$$ab = qba, \quad cd = qdc, \quad ac = qca, \quad bd = qdb, \quad bc = cb,$$
$$ad - da = (q - q^{-1})bc, \qquad ad - qbc = 1.$$

Its Hopf algebra structure is given by

$$\Delta(a) = a \otimes a + b \otimes c, \qquad \Delta(b) = a \otimes b + b \otimes d,$$
$$\Delta(c) = c \otimes a + d \otimes c, \qquad \Delta(d) = c \otimes b + d \otimes d,$$
$$\varepsilon(a) = \varepsilon(d) = 1, \qquad \varepsilon(b) = \varepsilon(c) = 0,$$
$$S(a) = d, \qquad S(b) = -q^{-1}b, \qquad S(c) = -qc, \qquad S(d) = a.$$

3.2. For $w \in W$ and $\lambda \in P^-$ set

$$v^*_{w\lambda} = v^*_\lambda \dot{T}_w^{-1} \in V^*(\lambda)_{w\lambda},$$

and define $\sigma^w_\lambda \in \mathbb{C}_q[G]$ by

$$\langle \sigma^w_\lambda, u \rangle = \langle v^*_{w\lambda}, uv_\lambda \rangle \qquad (u \in U).$$

Then we have

$$\sigma^w_0 = 1, \quad \sigma^w_\lambda \sigma^w_\mu = \sigma^w_\mu \sigma^w_\lambda = \sigma^w_{\lambda+\mu} \qquad (\lambda, \mu \in P^-). \tag{3.7}$$

Set

$$\mathcal{S}_w = \{\sigma_\lambda^w \mid \lambda \in P^-\} \subset \mathbb{C}_q[G].$$

Proposition 3.2 ([**4**]). *The multiplicative subset $\mathcal{S}_w$ of $\mathbb{C}_q[G]$ satisfies the left and right Ore conditions.*

It follows that we have the localization $\mathcal{S}_w^{-1}\mathbb{C}_q[G] = \mathbb{C}_q[G]\mathcal{S}_w^{-1}$. In the rest of this section we investigate the structure of the algebra $\mathcal{S}_w^{-1}\mathbb{C}_q[G]$. In the course of the arguments we give a new proof of Proposition 3.2.

3.3. In this subsection we consider the case $w = 1$.
Set

$$(U^\pm)^\bigstar = \bigoplus_{\gamma \in Q^+} (U^\pm_{\pm\gamma})^* \subset U^*.$$

Lemma 3.3. *We have*

$$\mathbb{C}_q[G] \subset (U^+)^\bigstar \otimes \mathbb{C}_q[H] \otimes (U^-)^\bigstar \subset (U^+)^* \otimes (U^0)^* \otimes (U^-)^* \subset U^*,$$

where the embedding $(U^+)^* \otimes (U^0)^* \otimes (U^-)^* \subset U^*$ is given by

$$\langle \psi \otimes \chi \otimes \varphi, xty \rangle = \langle \psi, x \rangle \langle \chi, t \rangle \langle \varphi, y \rangle$$
$$(\psi \in (U^+)^*,\ \chi \in (U^0)^*,\ \varphi \in (U^-)^*,\ x \in U^+,\ t \in U^0,\ y \in U^-).$$

Proof. It is easily seen that for any $\varphi \in \mathbb{C}_q[G]$ we have

$$\varphi|_{U^0} \in \mathbb{C}_q[H], \qquad \varphi|_{U^\pm} \in (U^\pm)^\bigstar.$$

Hence the assertion is a consequence of

$$\langle \varphi, xty \rangle = \sum_{(\varphi)_2} \langle \varphi_{(0)}, x \rangle \langle \varphi_{(1)}, t \rangle \langle \varphi_{(2)}, y \rangle \qquad (x \in U^+, t \in U^0, y \in U^-)$$

for $\varphi \in \mathbb{C}_q[G]$. □

Note that U^* is an $\mathbb{F}$-algebra whose multiplication is given by the composite of $U^* \otimes U^* \subset (U \otimes U)^* \xrightarrow{{}^t\Delta} U^*$ and that $\mathbb{C}_q[G]$ is a subalgebra of U^*. We will identify $(U^0)^*$, $(U^\pm)^*$ with subspaces of U^* by

$$(U^+)^* \to U^* \qquad (\psi \mapsto [xty \mapsto \langle \psi, x \rangle \varepsilon(t)\varepsilon(y)]),$$
$$(U^0)^* \to U^* \qquad (\chi \mapsto [xty \mapsto \varepsilon(x) \langle \chi, t \rangle \varepsilon(y)]),$$
$$(U^-)^* \to U^* \qquad (\varphi \mapsto [xty \mapsto \varepsilon(x)\varepsilon(t) \langle \varphi, y \rangle]),$$

where $x \in U^+$, $t \in U^0$, $y \in U^-$. Under this identification we have

$$\chi_\lambda = \sigma_\lambda^1 \qquad (\lambda \in P^-). \tag{3.8}$$

Hence

$$\mathcal{S}_1 = \{\chi_\lambda \mid \lambda \in P^-\} \subset \mathbb{C}_q[H] \subset (U^0)^* \subset U^*. \tag{3.9}$$

LEMMA 3.4. *For $\psi \in (U^+)^*,\ \chi \in (U^0)^*,\ \varphi \in (U^-)^*$ we have*

$$\langle \psi\chi\varphi, xty\rangle = \langle\psi, x\rangle\langle\chi, t\rangle\langle\varphi, y\rangle \qquad (y \in U^-,\ t \in U^0,\ x \in U^+).$$

PROOF. By the definition of the comultiplication of U, for $x \in U^+$, $y \in U^-$ we have

$$\Delta(y) = 1 \otimes y + y', \qquad (\varepsilon \otimes \mathrm{id})(y') = 0,$$
$$\Delta(x) = x \otimes 1 + x', \qquad (\mathrm{id} \otimes \varepsilon)(x') = 0.$$

Hence

$$\begin{aligned}\langle \psi\chi\varphi, xty\rangle &= \sum_{(x)_2,(t)_2,(y)_2} \langle\psi, x_{(0)}t_{(0)}y_{(0)}\rangle\langle\chi, x_{(1)}t_{(1)}y_{(1)}\rangle\langle\varphi, x_{(2)}t_{(2)}y_{(2)}\rangle \\ &= \sum_{(t)_2}\langle\psi, xt_{(0)}\rangle\langle\chi, t_{(1)}\rangle\langle\varphi, t_{(2)}y\rangle = \langle\psi, x\rangle\langle\chi, t\rangle\langle\varphi, y\rangle.\end{aligned}$$

□

LEMMA 3.5. *For $\lambda, \mu \in P$ we have $\chi_\lambda\chi_\mu = \chi_{\lambda+\mu}$.*

PROOF. We have

$$\begin{aligned}\langle \chi_\lambda\chi_\mu, xty\rangle &= \sum_{(x),(t),(y)} \langle\chi_\lambda, x_{(0)}t_{(0)}y_{(0)}\rangle\langle\chi_\mu, x_{(1)}t_{(1)}y_{(1)}\rangle \\ &= \varepsilon(x)\varepsilon(y)\sum_{(t)}\langle\chi_\lambda, t_{(0)}\rangle\langle\chi_\mu, t_{(1)}\rangle \\ &= \varepsilon(x)\varepsilon(y)\langle\chi_{\lambda+\mu}, t\rangle = \langle\chi_{\lambda+\mu}, xty\rangle.\end{aligned}$$

□

LEMMA 3.6. *The subspaces $(U^+)^\bigstar$, $(U^-)^\bigstar$ of U^* are subalgebras of U^*.*

PROOF. For $\varphi, \varphi' \in (U^-)^\bigstar$ we have

$$\begin{aligned}\langle \varphi\varphi', xty\rangle &= \sum_{(x),(t),(y)} \langle\varphi, x_{(0)}t_{(0)}y_{(0)}\rangle\langle\varphi', x_{(1)}t_{(1)}y_{(1)}\rangle \\ &= \varepsilon(x)\sum_{(t),(y)}\langle\varphi, t_{(0)}y_{(0)}\rangle\langle\varphi', t_{(1)}y_{(1)}\rangle \\ &= \varepsilon(x)\varepsilon(t)\sum_{(y)}\langle\varphi, y_{(0)}\rangle\langle\varphi', y_{(1)}\rangle = \varepsilon(x)\varepsilon(t)\langle\varphi\varphi', y\rangle.\end{aligned}$$

The statement for $(U^+)^\bigstar$ is proved similarly. □

LEMMA 3.7. (i) *For $\psi \in (U^+_\gamma)^*$, $\lambda \in P$ we have $\chi_\lambda\psi = q^{(\lambda,\gamma)}\psi\chi_\lambda$.*

(ii) *For $\varphi \in (U^-_{-\gamma})^*$, $\lambda \in P$ we have $\chi_\lambda\varphi = q^{(\lambda,\gamma)}\varphi\chi_\lambda$.*

PROOF. For $x \in U^+_{\gamma'}$, $y \in U^-$, $t \in U^0$ we have

$$\begin{aligned}\langle \chi_\lambda \psi, xty\rangle &= \sum_{(x),(t),(y)} \langle \chi_\lambda, x_{(0)}t_{(0)}y_{(0)}\rangle\langle\psi, x_{(1)}t_{(1)}y_{(1)}\rangle \\ &= \sum_{(t)}\langle\chi_\lambda, k_{\gamma'}t_{(0)}\rangle\langle\psi, xt_{(1)}y\rangle = \varepsilon(t_{(1)})\varepsilon(y)\delta_{\gamma,\gamma'}\sum_{(t)}\langle\chi_\lambda, k_{\gamma'}t_{(0)}\rangle\langle\psi, x\rangle \\ &= \varepsilon(y)\delta_{\gamma,\gamma'}\langle\chi_\lambda, k_\gamma t\rangle\langle\psi, x\rangle = q^{(\lambda,\gamma)}\varepsilon(y)\langle\psi,x\rangle\langle\chi_\lambda, t\rangle.\end{aligned}$$

By a similar calculation we have

$$\langle\psi\chi_\lambda, xty\rangle = \varepsilon(y)\langle\psi,x\rangle\langle\chi_\lambda,t\rangle.$$

The statement (i) is proved. The proof of (ii) is similar. □

Lemma 3.8. (i) *Let $\varphi \in (U^-)^\bigstar$. For sufficiently small $\lambda \in P^-$ we have $\chi_\lambda\varphi, \varphi\chi_\lambda \in \mathbb{C}_q[G]$.*

(ii) *Let $\psi \in (U^+)^\bigstar$. For sufficiently small $\lambda \in P^-$ we have $\chi_\lambda\psi, \psi\chi_\lambda \in \mathbb{C}_q[G]$.*

Proof. (i) We may assume $\varphi \in (U^-_{-\gamma})^*$. By Proposition 2.2 there exists $v \in V(\lambda)_{\lambda+\gamma}$ such that

$$\langle\varphi, y\rangle = \langle v_\lambda^* y, v\rangle \qquad (y \in U^-).$$

Then

$$\langle\Phi_{v_\lambda^*\otimes v}, xty\rangle = \langle v_\lambda^* xty, v\rangle = \varepsilon(x)\langle\chi_\lambda, t\rangle\langle\varphi, y\rangle = \langle\chi_\lambda\varphi, xty\rangle.$$

Hence $\chi_\lambda\varphi = q^{(\lambda,\gamma)}\varphi\chi_\lambda = \Phi_{v_\lambda^*\otimes v} \in \mathbb{C}_q[G]$. The proof of (ii) is similar. □

Corollary 3.9. *Let $f \in (U^+)^\bigstar\mathbb{C}_q[H](U^-)^\bigstar$. Then we have $\chi_\lambda f,\ f\chi_\lambda \in \mathbb{C}_q[G]$ for sufficiently small $\lambda \in P^-$.*

Proof. We may assume $f = \psi\chi_\nu\varphi$ ($\psi \in (U^+_\gamma)^*, \nu \in P, \varphi \in (U^-_{-\delta})^*$). By Lemma 3.8 we have $\chi_{\lambda_1}\psi, \chi_{\lambda_3}\varphi \in \mathbb{C}_q[G]$ when $\lambda_1, \lambda_3 \in P^-$ are sufficiently small. Take $\lambda_2 \in P^-$ such that $\lambda_2 + \nu \in P^-$ and set $\lambda = \lambda_1 + \lambda_2 + \lambda_3$. Then we have

$$\chi_\lambda f = q^{(\lambda_2+\lambda_3,\gamma)}(\chi_{\lambda_1}\psi)\chi_{\lambda_2+\mu}(\chi_{\lambda_3}\varphi) \in \mathbb{C}_q[G].$$

The proof for $f\chi_\lambda$ is similar. □

Proposition 3.10. *Let $f \in \mathbb{C}_q[G]$, $\lambda \in P^-$.*

(i) *If $\sigma_\lambda^1 f = 0$, then $f = 0$.*

(ii) *If $f\sigma_\lambda^1 = 0$, then $f = 0$.*

Proof. In the algebra U^* the element $\sigma_\lambda^1 = \chi_\lambda$ is invertible, and its inverse is given by $\chi_{-\lambda}$. □

We set

$$\mathbb{C}_q[G/N^-] = \{f \in \mathbb{C}_q[G] \mid yf = \varepsilon(y)f \ (y \in U^-)\} = \mathbb{C}_q[G] \cap (U^+)^\bigstar \mathbb{C}_q[H], \tag{3.10}$$

$$\mathbb{C}_q[N^+\backslash G] = \{f \in \mathbb{C}_q[G] \mid fx = \varepsilon(x)f \ (x \in U^+)\} = \mathbb{C}_q[G] \cap \mathbb{C}_q[H](U^-)^\bigstar. \tag{3.11}$$

They are subalgebras of $\mathbb{C}_q[G]$.

Proposition 3.11. *Assume that $\lambda \in P^-$.*

(i) $\forall \psi \in \mathbb{C}_q[G/N^-]\ \exists \mu \in P^-$ *s.t.* $\sigma^1_\mu \psi \in \mathbb{C}_q[G/N^-]\sigma^1_\lambda$, $\psi\sigma^1_\mu \in \sigma^1_\lambda \mathbb{C}_q[G/N^-]$.

(ii) $\forall \varphi \in \mathbb{C}_q[N^+\backslash G]\ \exists \mu \in P^-$ *s.t.* $\sigma^1_\mu \varphi \in \mathbb{C}_q[N^+\backslash G]\sigma^1_\lambda$, $\varphi\sigma^1_\mu \in \sigma^1_\lambda \mathbb{C}_q[G/N^-]$.

(iii) $\forall f \in \mathbb{C}_q[G]\ \exists \mu \in P^-$ *s.t.* $\sigma^1_\mu f \in \mathbb{C}_q[G]\sigma^1_\lambda$, $f\sigma^1_\mu \in \sigma^1_\lambda \mathbb{C}_q[G]$.

Proof. (i) By Lemma 3.7 we have

$$\sigma^1_\lambda \psi \in \sigma^1_\lambda (U^+)^\bigstar \mathbb{C}_q[H] \subset (U^+)^\bigstar \mathbb{C}_q[H]\sigma^1_\lambda.$$

By Corollary 3.9 we have $\sigma^1_{\lambda+\nu}\psi \in \mathbb{C}_q[G/N^-]\sigma^1_\lambda$ for some $\nu \in P^-$. Similarly, we have $\psi\sigma^1_{\lambda+\nu'} \in \sigma^1_\lambda \mathbb{C}_q[G/N^-]$ for some $\nu' \in P^-$.

The statements (ii), (iii) are proved similarly. □

By Proposition 3.10 and Proposition 3.11 we have the following.

Corollary 3.12. *The multiplicative set $\mathcal{S}_1$ satisfies the left and right Ore conditions in all of the three rings $\mathbb{C}_q[G/N^-]$, $\mathbb{C}_q[N^+\backslash G]$, $\mathbb{C}_q[G]$.*

It follows that we have the localizations

$$\mathcal{S}_1^{-1}\mathbb{C}_q[G/N^-] = \mathbb{C}_q[G/N^-]\mathcal{S}_1^{-1}, \tag{3.12}$$

$$\mathcal{S}_1^{-1}\mathbb{C}_q[N^+\backslash G] = \mathbb{C}_q[N^+\backslash G]\mathcal{S}_1^{-1}, \tag{3.13}$$

$$\mathcal{S}_1^{-1}\mathbb{C}_q[G] = \mathbb{C}_q[G]\mathcal{S}_1^{-1}. \tag{3.14}$$

The following result is a special case of [**15**, Theorem 2.6].

Proposition 3.13. (i) *The subset $(U^+)^\bigstar \mathbb{C}_q[H](U^-)^\bigstar$ of U^* is a subalgebra of U^*, which is isomorphic to $\mathcal{S}_1^{-1}\mathbb{C}_q[G]$.*

(ii) *The subset $(U^+)^\bigstar \mathbb{C}_q[H]$ of U^* is a subalgebra of U^*, which is isomorphic to $\mathcal{S}_1^{-1}\mathbb{C}_q[G/N^-]$.*

(iii) *The subset $\mathbb{C}_q[H](U^-)^\bigstar$ of U^* is a subalgebra of U^*, which is isomorphic to $\mathcal{S}_1^{-1}\mathbb{C}_q[N^+\backslash G]$.*

Proof. (i) Since $\mathcal{S}_1$ consists of invertible elements of U^*, we have a canonical homomorphism $\Psi : \mathcal{S}_1^{-1}\mathbb{C}_q[G] \to U^*$ of $\mathbb{F}$-algebras. Since $\mathbb{C}_q[G] \to U^*$ is injective, Ψ is injective by Proposition 3.10. Hence it is sufficient to show that the image of Ψ coincides with $(U^+)^\bigstar \mathbb{C}_q[H](U^-)^\bigstar$. For any $\lambda \in P$ we have

$$\chi_\lambda \mathbb{C}_q[G] \subset \chi_\lambda (U^+)^\bigstar \mathbb{C}_q[H](U^+)^\bigstar = (U^+)^\bigstar \mathbb{C}_q[H](U^-)^\bigstar,$$

and hence $\mathrm{Im}(\Psi) \subset (U^+)^{\bigstar}\mathbb{C}_q[H](U^-)^{\bigstar}$. Another inclusion $\mathrm{Im}(\Psi) \supset (U^+)^{\bigstar}\mathbb{C}_q[H](U^-)^{\bigstar}$ is a consequence of Corollary 3.9.

The proofs of (ii) and (iii) are similar. □

By Proposition 3.13 we obtain the following results.

Proposition 3.14. *The multiplication of $\mathcal{S}_1^{-1}\mathbb{C}_q[G]$ induces the isomorphism*

$$\mathcal{S}_1^{-1}\mathbb{C}_q[G/N^-] \otimes_{\mathbb{C}_q[H]} \mathcal{S}_1^{-1}\mathbb{C}_q[N^+\backslash G] \cong \mathcal{S}_1^{-1}\mathbb{C}_q[G].$$

Proposition 3.15. *For any $f \in \mathbb{C}_q[G]$ there exists some $\lambda \in P^-$ such that*

$$\sigma_\lambda^1 f, f\sigma_\lambda^1 \in \mathbb{C}_q[G/N^-]\mathbb{C}_q[N^+\backslash G].$$

3.4. In this subsection we investigate the localization of $\mathbb{C}_q[G]$ with respect to $\mathcal{S}_w$ for $w \in W$.

As a left (resp. right) U-module, $\mathbb{C}_q[G]$ is a sum of submodules belonging to $\mathrm{Mod}_0(U)$ (resp. $\mathrm{Mod}_0^r(U)$). Hence we have a left (resp. right) action of $\dot{T}_w$ on $\mathbb{C}_q[G]$.

Lemma 3.16. *For $w \in W$ we have*

$$\langle \dot{T}_w\varphi, u\rangle = \langle \varphi\dot{T}_w, \dot{T}_w^{-1}(u)\rangle, \qquad \langle \varphi\dot{T}_w, u\rangle = \langle \dot{T}_w\varphi, \dot{T}_w(u)\rangle$$

for $\varphi \in \mathbb{C}_q[G]$, $u \in U$.

Proof. We may assume that $\varphi = \Phi_{v^*\otimes v}$. Then we have

$$\langle \dot{T}_w\varphi, u\rangle = \langle v^*, u\dot{T}_w v\rangle = \langle v^*\dot{T}_w, (\dot{T}_w^{-1}(u))v\rangle = \langle \varphi\dot{T}_w, \dot{T}_w^{-1}(u)\rangle.$$

The second formula follows from the first. □

Setting $u = 1$ in Lemma 3.16 we obtain the following.

Lemma 3.17. *For $w \in W$ we have*

$$\varepsilon(\varphi\dot{T}_w) = \varepsilon(\dot{T}_w\varphi) \qquad (\varphi \in \mathbb{C}_q[G]).$$

In the rest of this section we fix $w \in W$.

Lemma 3.18. (i) *$\sigma_\lambda^1\dot{T}_w^{-1} = \sigma_\lambda^w$ for $\lambda \in P^-$.*

(ii) *$\mathbb{C}_q[G/N^-]\dot{T}_w^{-1} = \mathbb{C}_q[G/N^-]$.*

(iii) *$\mathbb{C}_q[N^+\backslash G]\dot{T}_w^{-1} = \{\varphi \in \mathbb{C}_q[G] \mid \varphi u = \varepsilon(u)\varphi \ \ (u \in \dot{T}_w(U^+))\}$.*

Proof. The statements (i) and (ii) are obvious. The remaining (iii) is a consequence of

$$(f\dot{T}_w^{-1})(\dot{T}_w(u)) = (fu)\dot{T}_w^{-1} \qquad (f \in \mathbb{C}_q[G], u \in U)$$

and (2.11). □

Lemma 3.19. (i) $(fh)\dot{T}_w^{-1} = (f\dot{T}_w^{-1})(h\dot{T}_w^{-1}) \quad (h \in \mathbb{C}_q[N^+\backslash G],\ f \in \mathbb{C}_q[G])$.

(ii) $(f\sigma_\lambda^1)\dot{T}_w^{-1} = (f\dot{T}_w^{-1})\sigma_\lambda^w \quad (f \in \mathbb{C}_q[G])$.

(iii) $(\sigma_\lambda^1 f)\dot{T}_{w^{-1}} \in \mathbb{F}^\times \sigma_\lambda^w (f\dot{T}_{w^{-1}}) \quad (f \in \mathbb{C}_q[G])$.

Proof. The statement (i) follows from (3.6) and Corollary 2.7. The statement (ii) is a special case of (i). Since $V^*(\lambda)_{w\lambda}$ is one-dimensional, we have $v_{w\lambda}^* \in \mathbb{F}^\times v_\lambda^* \dot{T}_{w^{-1}}$. Hence (iii) also follows from Corollary 2.7. □

Corollary 3.20. *The linear map $\mathbb{C}_q[N^+\backslash G] \ni \varphi \mapsto \varphi\dot{T}_w^{-1} \in \mathbb{C}_q[G]$ is an algebra homomorphism. Hence $\mathbb{C}_q[N^+\backslash G]\dot{T}_w^{-1}$ is a subalgebra of $\mathbb{C}_q[G]$.*

Proposition 3.21. *Let $f \in \mathbb{C}_q[G]$ and $\lambda \in P^-$.*

(i) *If $f\sigma_\lambda^w = 0$, then $f = 0$.*

(ii) *If $\sigma_\lambda^w f = 0$, then $f = 0$.*

Proof. By Lemma 3.19 we have

$$f\sigma_\lambda^w = (f\dot{T}_w\dot{T}_w^{-1})\sigma_\lambda^w = ((f\dot{T}_w)\sigma_\lambda^1)\dot{T}_w^{-1},$$
$$\sigma_\lambda^w f = \sigma_\lambda^w(f\dot{T}_{w^{-1}}^{-1}\dot{T}_{w^{-1}}) \in \mathbb{F}^\times(\sigma_\lambda^1(f\dot{T}_{w^{-1}}^{-1}))\dot{T}_{w^{-1}}.$$

Hence the assertion follows from Proposition 3.11. □

By Proposition 3.11 and Corollary 3.20 we have the following.

Proposition 3.22. *For any $\varphi \in \mathbb{C}_q[N^+\backslash G]\dot{T}_w^{-1}$ and $\lambda \in P^-$ there exists some $\mu \in P^-$ such that $\sigma_\mu^w\varphi \in (\mathbb{C}_q[N^+\backslash G]\dot{T}_w^{-1})\sigma_\lambda^w$ and $\varphi\sigma_\mu^w \in \sigma_\lambda^w(\mathbb{C}_q[N^+\backslash G]\dot{T}_w^{-1})$.*

The following result is proved similarly to [**13**, Proposition 3.4].

Proposition 3.23. *For any $\psi \in \mathbb{C}_q[G/N^-]$ and $\lambda \in P^-$ there exists some $\mu \in P^-$ such that $\sigma_\mu^w\psi \in \mathbb{C}_q[G/N^-]\sigma_\lambda^w$ and $\psi\sigma_\mu^w \in \sigma_\lambda^w\mathbb{C}_q[G/N^-]$.*

Lemma 3.24. *For any $f \in \mathbb{C}_q[G]$ there exists some $\lambda \in P^-$ such that $f\sigma_\lambda^w \in \mathbb{C}_q[G/N^-](\mathbb{C}_q[N^+\backslash G]\dot{T}_w^{-1})$.*

Proof. By Proposition 3.15 there exists some $\lambda \in P^-$ such that $(f\dot{T}_w)\sigma_\lambda^1 \in \mathbb{C}_q[G/N^-]\mathbb{C}_q[N^+\backslash G]$. Hence by Lemma 3.19 we have

$$f\sigma_\lambda^w = ((f\dot{T}_w)\sigma_\lambda^1)\dot{T}_w^{-1} \in (\mathbb{C}_q[G/N^-]\mathbb{C}_q[N^+\backslash G])\dot{T}_w^{-1} = \mathbb{C}_q[G/N^-](\mathbb{C}_q[N^+\backslash G]\dot{T}_w^{-1}). \quad □$$

Proposition 3.25. *For any $f \in \mathbb{C}_q[G]$ and $\lambda \in P^-$ there exists some $\mu \in P^-$ such that $\sigma_\mu^w f \in \mathbb{C}_q[G]\sigma_\lambda^w$ and $f\sigma_\mu^w \in \sigma_\lambda^w\mathbb{C}_q[G]$.*

Proof. We can take $\nu \in P^-$ with $f\sigma_\nu^w \in \mathbb{C}_q[G/N^-](\mathbb{C}_q[N^+\backslash G]\dot{T}_w^{-1})$ by Lemma 3.24 . By Proposition 3.22 and Proposition 3.23 we have $f\sigma_{\nu+\mu'}^w = \sigma_\lambda^w\mathbb{C}_q[G]$ when

$\mu' \in P^-$ is sufficiently small. Similarly we have $\sigma^w_{\mu''} f \sigma^w_\nu = \mathbb{C}_q[G]\sigma^w_{\lambda+\nu}$. when $\mu'' \in P^-$ is sufficiently small. Then we have $\sigma^w_{\mu''} f = \mathbb{C}_q[G]\sigma^w_\lambda$ by Proposition 3.21. □

By Proposition 3.21, Proposition 3.22, Proposition 3.23, and Proposition 3.25 we have the following.

Corollary 3.26. *The multiplicative set $\mathcal{S}_w$ satisfies the left and right Ore conditions in all of the three rings $\mathbb{C}_q[G/N^-]$, $\mathbb{C}_q[N^+\backslash G]\dot{T}_w^{-1}$, $\mathbb{C}_q[G]$.*

It follows that we have the localizations

$$\mathcal{S}_w^{-1}\mathbb{C}_q[G/N^-] = \mathbb{C}_q[G/N^-]\mathcal{S}_w^{-1}, \tag{3.15}$$

$$\mathcal{S}_w^{-1}(\mathbb{C}_q[N^+\backslash G]\dot{T}_w^{-1}) = (\mathbb{C}_q[N^+\backslash G]\dot{T}_w^{-1})\mathcal{S}_w^{-1}, \tag{3.16}$$

$$\mathcal{S}_w^{-1}\mathbb{C}_q[G] = \mathbb{C}_q[G]\mathcal{S}_w^{-1}. \tag{3.17}$$

For $\lambda \in P$ define $\sigma^w_\lambda \in \mathcal{S}_w^{-1}\mathbb{C}_q[G]$ by

$$\sigma^w_\lambda = (\sigma^w_{\lambda_2})^{-1}\sigma^w_{\lambda_1} \qquad (\lambda_1, \lambda_2 \in P^-,\ \lambda = \lambda_1 - \lambda_2), \tag{3.18}$$

and set

$$\tilde{\mathcal{S}}_w = \{\sigma^w_\lambda \mid \lambda \in P\}, \quad \mathbb{F}[\tilde{\mathcal{S}}_w] = \bigoplus_{\lambda \in P} \mathbb{F}\sigma^w_\lambda \subset \mathcal{S}_w^{-1}\mathbb{C}_q[G]. \tag{3.19}$$

Note that $\tilde{\mathcal{S}}_w$ is naturally isomorphic to P as a group.

Proposition 3.27. *We can define a bijective linear map*

$$F_w : \mathcal{S}_1^{-1}\mathbb{C}_q[G] \to \mathcal{S}_w^{-1}\mathbb{C}_q[G] \tag{3.20}$$

by

$$F_w(f(\sigma^1_\lambda)^{-1}) = (f\dot{T}_w^{-1})(\sigma^w_\lambda)^{-1} \qquad (\lambda \in P^-, f \in \mathbb{C}_q[G]).$$

Proof. Assume $f(\sigma^1_\lambda)^{-1} = f'(\sigma^1_\mu)^{-1}$ $(\lambda, \mu \in P^-,\ f, f' \in \mathbb{C}_q[G])$. Then we have $f\sigma^1_\mu = f'\sigma^1_\lambda$, and hence we have $(f\dot{T}_w^{-1})\sigma^w_\mu = (f'\dot{T}_w^{-1})\sigma^w_\lambda$ by Lemma 3.19(ii). It follows that $(f\dot{T}_w^{-1})(\sigma^w_\lambda)^{-1} = (f'\dot{T}_w^{-1})(\sigma^w_\mu)^{-1}$. The bijectivity is obvious. □

Lemma 3.28. (i) *We have*

$$F_w(\mathcal{S}_1^{-1}\mathbb{C}_q[G/N^-]) = \mathcal{S}_w^{-1}\mathbb{C}_q[G/N^-],$$
$$F_w(\mathcal{S}_1^{-1}\mathbb{C}_q[N^+\backslash G]) = \mathcal{S}_w^{-1}(\mathbb{C}_q[N^+\backslash G]\dot{T}_w^{-1}).$$

(ii) *The linear map $\mathcal{S}_1^{-1}\mathbb{C}_q[N^+\backslash G] \ni f \mapsto F_w(f) \in \mathcal{S}_w^{-1}(\mathbb{C}_q[N^+\backslash G]\dot{T}_w^{-1})$ is an algebra isomorphism.*

(iii) *For $\varphi \in \mathcal{S}_1^{-1}\mathbb{C}_q[G/N^-]$ and $\psi \in \mathcal{S}_1^{-1}\mathbb{C}_q[N^+\backslash G]$ we have $F_w(\varphi\psi) = F_w(\varphi)F_w(\psi)$.*

Proof. The statements (i) and (ii) are obvious. The statement (iii) is a consequence of Lemma 3.19. □

By the above arguments we obtain the following results.

Proposition 3.29. *The multiplication induces*

$$\mathcal{S}_w^{-1}\mathbb{C}_q[G/N^-] \otimes_{\mathbb{F}[\tilde{\mathcal{S}}_w]} \mathcal{S}_w^{-1}(\mathbb{C}_q[N^+\backslash G]\dot{T}_w^{-1}) \cong \mathcal{S}_w^{-1}\mathbb{C}_q[G].$$

Proposition 3.30. *For any $f \in \mathbb{C}_q[G]$ there exists some $\lambda \in P^-$ such that*

$$\sigma_\lambda^w f, f\sigma_\lambda^w \in \mathbb{C}_q[G/N^-](\mathbb{C}_q[N^+\backslash G]\dot{T}_w^{-1}).$$

3.5. Set

$$\mathbb{C}_q[N_w^-\backslash G] = \{\varphi \in \mathbb{C}_q[G] \mid \varphi y = \varepsilon(y)\varphi \ \ (y \in U^-[\dot{T}_w])\}. \tag{3.21}$$

Note that

$$\sigma_\lambda^w \in \mathbb{C}_q[G/N^-] \cap \mathbb{C}_q[N_w^-\backslash G] \qquad (\lambda \in P^+).$$

Proposition 3.31. *The subspace $\mathbb{C}_q[N_w^-\backslash G]$ of $\mathbb{C}_q[G]$ is a subalgebra of $\mathbb{C}_q[G]$.*

Proof. Let $\varphi, \psi \in \mathbb{C}_q[N_w^-\backslash G]$. For $y \in U^-[\dot{T}_w] \cap U^-_{-\gamma}$ with $\gamma \in Q^+$ we have

$$(\varphi\psi)y = \sum_{(y)}(\varphi y_{(0)})(\psi y_{(1)}) = (\varphi y)(\psi k_{-\gamma}) = \varepsilon(y)\varphi\psi$$

by Lemma 2.8. Hence $\varphi\psi \in \mathbb{C}_q[N_w^-\backslash G]$. □

Lemma 3.32. *Let $\gamma \in Q^+$ and $y \in U^-[\dot{T}_w] \cap U^-_{-\gamma}$. Then for $\varphi \in \mathbb{C}_q[G]$, $\lambda \in P^-$ we have*

$$(\varphi\sigma_\lambda^w)y = q^{-(w\lambda,\gamma)}(\varphi y)\sigma_\lambda^w.$$

Proof. By Lemma 2.8 we have

$$(\varphi\sigma_\lambda^w)y = (\varphi y)(\sigma_\lambda^w k_{-\gamma}) = q^{-(w\lambda,\gamma)}(\varphi y)\sigma_\lambda^w.$$ □

By Lemma 3.32 we have the following.

Lemma 3.33. *For $\varphi \in \mathbb{C}_q[G]$, $\lambda \in P^-$ we have $\varphi \in \mathbb{C}_q[N_w^-\backslash G]$ if and only if $\varphi\sigma_\lambda^w \in \mathbb{C}_q[N_w^-\backslash G]$.*

Proposition 3.34. *The multiplicative set $\mathcal{S}_w$ satisfies the left Ore condition in $\mathbb{C}_q[N_w^-\backslash G]$.*

Proof. Let $f \in \mathbb{C}_q[N_w^-\backslash G]$, $\lambda \in P^-$. Then we can take $f' \in \mathbb{C}_q[G]$ and $\mu \in P^-$ satisfying $\sigma_\mu^w f = f'\sigma_\lambda^w$. Then by Lemma 3.33 we obtain $f' \in \mathbb{C}_q[N_w^-\backslash G]$. □

We will show later that $\mathcal{S}_w$ also satisfies the right Ore condition in $\mathbb{C}_q[N_w^-\backslash G]$ (see Proposition 3.43 below).

By Proposition 3.34 we have the left localizations

$$\mathcal{S}_w^{-1}\mathbb{C}_q[N_w^-\backslash G], \qquad \mathcal{S}_w^{-1}(\mathbb{C}_q[G/N^-]\cap\mathbb{C}_q[N_w^-\backslash G]).$$

Proposition 3.35. *The multiplication of $\mathcal{S}_w^{-1}\mathbb{C}_q[N_w^-\backslash G]$ induces the isomorphism*

$$\mathcal{S}_w^{-1}\mathbb{C}_q[N_w^-\backslash G] \cong \mathcal{S}_w^{-1}(\mathbb{C}_q[G/N^-]\cap\mathbb{C}_q[N_w^-\backslash G]) \otimes_{\mathbb{F}[\tilde{\mathcal{S}}_w]} \mathcal{S}_w^{-1}(\mathbb{C}_q[N^+\backslash G]\dot{T}_w^{-1}).$$

Proof. It is easily seen that $\mathbb{C}_q[N^+\backslash G]\dot{T}_w^{-1} \subset \mathbb{C}_q[N_w^-\backslash G]$. Let us show that $\mathcal{S}_w^{-1}(\mathbb{C}_q[N^+\backslash G]\dot{T}_w^{-1})$ is a free left $\mathbb{F}[\tilde{\mathcal{S}}_w]$-module. In the case $w=1$ this is a consequence of Proposition 3.13. For general w this follows from the case $w=1$ and Lemma 3.28. Take a basis $\{\psi_j\}_{j\in J}$ of the left free $\mathbb{F}[\tilde{\mathcal{S}}_w]$-module $\mathcal{S}_w^{-1}(\mathbb{C}_q[N^+\backslash G]\dot{T}_w^{-1})$. We may assume that $\psi_j \in \mathbb{C}_q[N^+\backslash G]\dot{T}_w^{-1}$.

Let $f \in \mathcal{S}_w^{-1}\mathbb{C}_q[G]$. By Proposition 3.29 we can uniquely write

$$f = \sum_{j\in J_0} \varphi_j\psi_j \qquad (\varphi_j \in \mathcal{S}_w^{-1}\mathbb{C}_q[G/N^-]),$$

where J_0 is a finite subset of J. Then we need to show

$$f \in \mathcal{S}_w^{-1}\mathbb{C}_q[N_w^-\backslash G] \iff \varphi_j \in \mathcal{S}_w^{-1}(\mathbb{C}_q[G/N^-]\cap\mathbb{C}_q[N_w^-\backslash G]) \quad (\forall j \in J_0).$$

Assume that $\varphi_j \in \mathcal{S}_w^{-1}(\mathbb{C}_q[G/N^-]\cap\mathbb{C}_q[N_w^-\backslash G])$ for any $j \in J_0$. We can take $\lambda \in P^-$ such that $\sigma_\lambda^w\varphi_j \in \mathbb{C}_q[G/N^-]\cap\mathbb{C}_q[N_w^-\backslash G]$ for any $j \in J_0$. Then from

$$\sigma_\lambda^w f = \sum_{j\in J_0}(\sigma_\lambda^w\varphi_j)\psi_j \in \mathbb{C}_q[N_w^-\backslash G]$$

we obtain $f \in \mathcal{S}_w^{-1}\mathbb{C}_q[N_w^-\backslash G]$.

Assume that $f \in \mathcal{S}_w^{-1}\mathbb{C}_q[N_w^-\backslash G]$. Taking $\lambda \in P^-$ such that $\sigma_\lambda^w\varphi_j \in \mathbb{C}_q[G/N^-]$ for any $j \in J_0$, we have

$$\sigma_\lambda^w f = \sum_{j\in J_0}(\sigma_\lambda^w\varphi_j)\psi_j \qquad (\sigma_\lambda^w\varphi_j \in \mathbb{C}_q[G/N^-]).$$

By $f \in \mathcal{S}_w^{-1}\mathbb{C}_q[N_w^-\backslash G]$ we may assume that $\sigma_\lambda^w f \in \mathbb{C}_q[N_w^-\backslash G]$. Then by Lemma 2.8 we have

$$\varepsilon(y)\sigma_\lambda^w f = (\sigma_\lambda^w f)y = \sum_{j\in J_0}((\sigma_\lambda^w\varphi_j)y)\psi_j \qquad (y \in U^-[\dot{T}_w]).$$

By $(\sigma_\lambda^w\varphi_j)y \in \mathbb{C}_q[G/N^-]$ we have $(\sigma_\lambda^w\varphi_j)y = \varepsilon(y)(\sigma_\lambda^w\varphi_j)$ for any $j \in J_0$, and hence $\sigma_\lambda^w\varphi_j \in \mathbb{C}_q[G/N^-]\cap\mathbb{C}_q[N_w^-\backslash G]$. It follows that $\varphi_j \in \mathcal{S}_w^{-1}(\mathbb{C}_q[G/N^-]\cap\mathbb{C}_q[N_w^-\backslash G])$ for any $j \in J_0$. □

3.6. By Proposition 3.13 we have

$$\mathcal{S}_1^{-1}\mathbb{C}_q[G/N^-] \cong (U^+)^\bigstar \otimes \mathbb{C}_q[H].$$

Hence the linear isomorphism $F_w : \mathcal{S}_1^{-1}\mathbb{C}_q[G/N^-] \to \mathcal{S}_w^{-1}\mathbb{C}_q[G/N^-]$ induces an isomorphism

$$\mathcal{S}_w^{-1}\mathbb{C}_q[G/N^-] \cong F_w((U^+)^\bigstar) \otimes_{\mathbb{F}} \mathbb{F}[\tilde{\mathcal{S}}_w] \qquad (f\sigma_\lambda^w \leftrightarrow f \otimes \sigma_\lambda^w) \tag{3.22}$$

of vector spaces.

In this subsection we are going to show the following.

Proposition 3.36. *We have*

$$\mathcal{S}_w^{-1}(\mathbb{C}_q[G/N^-] \cap \mathbb{C}_q[N_w^-\backslash G]) = \left\{F_w((U^+)^\bigstar) \cap \mathcal{S}_w^{-1}\mathbb{C}_q[N_w^-\backslash G]\right\} \otimes_{\mathbb{F}} \mathbb{F}[\tilde{\mathcal{S}}_w].$$

Let $\varphi \in (U^+)^\bigstar$. Then for any sufficiently small $\lambda \in P^-$ there a unique $v^* \in V^*(\lambda)$ such that

$$\langle v^*, xv_\lambda\rangle = \langle \varphi, x\rangle \qquad (x \in U^+)$$

by Proposition 2.2. We denote this v^* by $v^*(\varphi, \lambda)$.

Lemma 3.37. *Let $\varphi \in (U^+)^\bigstar$. Then for sufficiently small $\lambda \in P^-$ we have*

$$F_w(\varphi) = \Phi_{v^*(\varphi,\lambda)\dot{T}_w^{-1}\otimes v_\lambda}(\sigma_\lambda^w)^{-1}.$$

Proof. For $x \in U^+, t \in U^0, y \in U^-$ we have

$$\langle \Phi_{v^*(\varphi,\lambda)\otimes v_\lambda}, xty\rangle = \langle v^*(\varphi,\lambda), xtyv_\lambda\rangle = \langle v^*(\varphi,\lambda), xv_\lambda\rangle\chi_\lambda(t)\varepsilon(y) = \langle\varphi, x\rangle\chi_\lambda(t)\varepsilon(y).$$

Hence we obtain $\Phi_{v^*(\varphi,\lambda)\otimes v_\lambda} = \varphi\chi_\lambda$, or equivalently, $\varphi = \Phi_{v^*(\varphi,\lambda)\otimes v_\lambda}(\chi_\lambda)^{-1}$. It follows that

$$F_w(\varphi) = \{\Phi_{v^*(\varphi,\lambda)\otimes v_\lambda}\dot{T}_w^{-1}\}(\sigma_\lambda^w)^{-1} = \Phi_{v^*(\varphi,\lambda)\dot{T}_w^{-1}\otimes v_\lambda}(\sigma_\lambda^w)^{-1}. \qquad \square$$

Corollary 3.38. $F_w((U^+)^\bigstar) = \bigcup_{\lambda\in P^-}\{\Phi_{v^*\otimes v_\lambda}(\sigma_\lambda^w)^{-1} \mid v^* \in V^*(\lambda)\}$.

Lemma 3.39. *For $\mu \in P$ we have $\sigma_\mu^w F_w((U^+)^\bigstar) = F_w((U^+)^\bigstar)\sigma_\mu^w$.*

Proof. We may assume that $\mu \in P^-$. For $\lambda \in P^-$, $v^* \in V^*(\lambda)$ we have

$$\sigma_\mu^w \Phi_{v^*\otimes v_\lambda} = \Phi_{v^*_{w\mu}\otimes v_\mu}\Phi_{v^*\otimes v_\lambda} = \Phi_{(v^*_{w\mu}\otimes v^*)\otimes(v_\mu\otimes v_\lambda)},$$
$$\Phi_{v^*\otimes v_\lambda}\sigma_\mu^w = \Phi_{v^*\otimes v_\lambda}\Phi_{v^*_{w\mu}\otimes v_\mu} = \Phi_{(v^*\otimes v^*_{w\mu})\otimes(v_\lambda\otimes v_\mu)}.$$

Let

$$p : V^*(\mu) \otimes V^*(\lambda) \to V^*(\lambda+\mu), \quad p' : V^*(\lambda) \otimes V^*(\mu) \to V^*(\lambda+\mu)$$

be the homomorphisms of right U-modules such that $p(v_\lambda^* \otimes v_\mu^*) = v_{\lambda+\mu}^*$, $p'(v_\mu^* \otimes v_\lambda^*) = v_{\lambda+\mu}^*$. Then by [**13**, Lemma 3.5] we have

$$p(v^*_{w\mu}\otimes V^*(\lambda)_{\lambda+\gamma})=V^*(\lambda+\mu)_{w\mu+\lambda+\gamma}=p'(V^*(\lambda)_{\lambda+\gamma}\otimes v^*_{w\mu})$$

for $\gamma\in Q^+$ if $\lambda\in P^-$ is sufficiently small. Hence the assertion follows from Corollary 3.38. □

LEMMA 3.40. *Let $\gamma,\delta\in Q^+$, and let $\varphi\in(U^+_\delta)^*$, $y\in U^-[\dot{T}_w]\cap U^-_{-\gamma}$. Take $z\in U^+_{-w^{-1}\gamma}$ such that $y=\dot{T}_w(k_{-w^{-1}\gamma}z)$ (see (2.34)), and define $\varphi^z\in(U^+)^\bigstar$ by*

$$\langle\varphi^z,x\rangle=\langle\varphi,zx\rangle\qquad(x\in U^+).$$

If $\lambda\in P^-$ is sufficiently small, then we have $F_w(\varphi)\sigma^w_\lambda\in\mathbb{C}_q[G/N^-]$, and

$$(F_w(\varphi)\sigma^w_\lambda)y=q^{-(w^{-1}\gamma,\lambda+\delta)}F_w(\varphi^z)\sigma^w_\lambda.$$

PROOF. If $\lambda\in P^-$ is sufficiently small, then we have $F_w(\varphi)\sigma^w_\lambda=\Phi_{v^*(\varphi,\lambda)\dot{T}_w^{-1}\otimes v_\lambda}\in\mathbb{C}_q[G/N^-]$. For $x\in U^+$ we have

$$\langle v^*(\varphi,\lambda)z,xv_\lambda\rangle=\langle v^*(\varphi,\lambda),zxv_\lambda\rangle=\langle\varphi,zx\rangle=\langle\varphi^z,x\rangle$$

and hence $v^*(\varphi,\lambda)z=v^*(\varphi^z,\lambda)$. It follows that

$$\begin{aligned}(F_w(\varphi)\sigma^w_\lambda)y&=\Phi_{v^*(\varphi,\lambda)\dot{T}_w^{-1}\otimes v_\lambda}y=\Phi_{v^*(\varphi,\lambda)k_{-w^{-1}\gamma}z\dot{T}_w^{-1}\otimes v_\lambda}\\&=q^{-(w^{-1}\gamma,\lambda+\delta)}\Phi_{v^*(\varphi^z,\lambda)\dot{T}_w^{-1}\otimes v_\lambda}=q^{-(w^{-1}\gamma,\lambda+\delta)}F_w(\varphi^z)\sigma^w_\lambda.\end{aligned}$$ □

Let us give a proof of Proposition 3.36. By (3.22) any $f\in\mathcal{S}_w^{-1}\mathbb{C}_q[G/N^-]$ is uniquely written as

$$f=\sum_{\lambda\in P}F_w(\varphi_\lambda)\sigma^w_\lambda\in\mathcal{S}_w^{-1}\mathbb{C}_q[G/N^-]\qquad(\varphi_\lambda\in(U^+)^\bigstar).$$

We need to show that $f\in\mathcal{S}_w^{-1}(\mathbb{C}_q[G/N^-]\cap\mathbb{C}_q[N^-_w\backslash G])$ if and only if $F_w(\varphi_\lambda)\in\mathcal{S}_w^{-1}\mathbb{C}_q[N^-_w\backslash G]$ for any $\lambda\in P$. By Lemma 3.33 we have

$$\begin{aligned}&f\in\mathcal{S}_w^{-1}(\mathbb{C}_q[G/N^-]\cap\mathbb{C}_q[N^-_w\backslash G])\\\Longleftrightarrow\ &\exists\nu\in P^-\text{ s.t. }\sigma^w_\nu f\in\mathbb{C}_q[G/N^-]\cap\mathbb{C}_q[N^-_w\backslash G]\\\Longleftrightarrow\ &\exists\nu\in P^-\text{ s.t. }\sigma^w_\nu f\in\mathbb{C}_q[G/N^-],\ \sigma^w_\nu f\sigma^w_\mu\in\mathbb{C}_q[G/N^-]\cap\mathbb{C}_q[N^-_w\backslash G]\\\Longleftrightarrow\ &f\sigma^w_\mu\in\mathcal{S}_w^{-1}(\mathbb{C}_q[G/N^-]\cap\mathbb{C}_q[N^-_w\backslash G])\end{aligned}$$

for any $\mu\in P^-$. Hence we may assume from the beginning that f is written as

$$f=\sum_{\lambda\in P^-}F_w(\varphi_\lambda)\sigma^w_\lambda\qquad(\varphi_\lambda\in(U^+)^\bigstar).$$

If $F_w(\varphi_\lambda)\in\mathcal{S}_w^{-1}\mathbb{C}_q[N^-_w\backslash G]$ for any $\lambda\in P^-$, there exists some $\mu\in P^-$ such that $\sigma^w_\mu F_w(\varphi_\lambda)\in\mathbb{C}_q[G/N^-]\cap\mathbb{C}_q[N^-_w\backslash G]$ for any $\lambda\in P^-$. It follows that

$$\sigma^w_\mu f = \sum_{\lambda\in P^-} (\sigma^w_\mu F_w(\varphi_\lambda))\sigma^w_\lambda \in \mathbb{C}_q[G/N^-]\cap\mathbb{C}_q[N^-_w\backslash G]$$

by Lemma 3.33, and hence $f \in \mathcal{S}_w^{-1}(\mathbb{C}_q[G/N^-]\cap\mathbb{C}_q[N^-_w\backslash G])$.

It remains to show that if $f \in \mathcal{S}_w^{-1}(\mathbb{C}_q[G/N^-]\cap\mathbb{C}_q[N^-_w\backslash G])$, then $F_w(\varphi_\lambda) \in \mathcal{S}_w^{-1}\mathbb{C}_q[N^-_w\backslash G]$ for any $\lambda \in P^-$. So assume that $f \in \mathcal{S}_w^{-1}(\mathbb{C}_q[G/N^-]\cap\mathbb{C}_q[N^-_w\backslash G])$. Take $\mu\in P^-$ which is sufficiently small. Then we have $\sigma^w_\mu f \in \mathbb{C}_q[G/N^-]\cap\mathbb{C}_q[N^-_w\backslash G]$. By Lemma 3.39 we can write

$$\sigma^w_\mu F_w(\varphi_\lambda) = F_w(\varphi'_\lambda)\sigma^w_\mu \in \mathbb{C}_q[G/N^-] \qquad (\lambda\in P^-, \varphi'_\lambda \in (U^+)^\bigstar),$$

and hence

$$\sigma^w_\mu f = \sum_{\lambda\in P^-} F_w(\varphi'_\lambda)\sigma^w_{\mu+\lambda}, \qquad F_w(\varphi'_\lambda)\sigma^w_{\mu+\lambda} \in \mathbb{C}_q[G/N^-] \quad (\lambda\in P^-).$$

Let $\gamma\in Q^+\setminus\{0\}$ and $y \in U^-[\dot{T}_w]\cap U^-_{-\gamma}$. By $\sigma^w_\mu f \in \mathbb{C}_q[N^-_w\backslash G]$ we have $(\sigma^w_\mu f)y = 0$. On the other hand we have

$$(\sigma^w_\mu f)y = \sum_{\lambda\in P^-} (F_w(\varphi'_\lambda)\sigma^w_{\lambda+\mu})y = \sum_{\lambda\in P^-} q^{-(w\lambda,\gamma)}((F_w(\varphi'_\lambda)\sigma^w_\mu)y)\sigma^w_\lambda.$$

By Lemma 3.40 we have

$$(F_w(\varphi'_\lambda)\sigma^w_\mu)y = F_w(\varphi''_\lambda)\sigma^w_\mu$$

for some $\varphi''_\lambda \in (U^+)^\bigstar$, and hence

$$\sum_{\lambda\in P^-} q^{-(w\lambda,\gamma)} F_w(\varphi''_\lambda)\sigma^w_{\lambda+\mu} = 0.$$

By (3.22) we obtain $F_w(\varphi''_\lambda) = 0$ for any $\lambda\in P^-$. It follows that

$$(\sigma^w_\mu F_w(\varphi_\lambda))y = (F_w(\varphi'_\lambda)\sigma^w_\mu)y = F_w(\varphi''_\lambda)\sigma^w_\mu = 0.$$

We obtain $F_w(\varphi_\lambda) \in \mathcal{S}_w^{-1}\mathbb{C}_q[N^-_w\backslash G]$ for any $\lambda\in P^-$. The proof of Proposition 3.36 is complete.

3.7. Set

$$\begin{aligned}\mathcal{J}_w &= \{\psi \in (U^+)^\bigstar \mid \psi^z = \varepsilon(z)\psi \quad (z\in U^+[\dot{T}_w^{-1}])\}\\ &= \{\psi\in(U^+)^\bigstar \mid \psi|_{\mathrm{Ker}(\varepsilon: U^+[\dot{T}_w^{-1}]\to\mathbb{F})U^+} = 0\}.\end{aligned}$$

In this subsection we are going to show the following.

PROPOSITION 3.41. *$F_w((U^+)^\bigstar)\cap\mathcal{S}_w^{-1}\mathbb{C}_q[N^-_w\backslash G] = F_w(\mathcal{J}_w)$.*

We first show the following result.

Lemma 3.42. *Let $\gamma \in Q^+$, $\psi \in \mathcal{J}_w \cap (U^+_\gamma)^*$ and $\mu \in P$. Then we have $q^{(\mu,\gamma)} F_w(\psi)\sigma^w_\mu = \sigma^w_\mu F_w(\psi)$.*

Proof. We may assume $\mu \in P^-$. When $\lambda \in P^-$ is sufficiently small, we have $F_w(\psi) = \Phi_{v^*(\psi,\lambda)\dot{T}_w^{-1}\otimes v_\lambda}(\sigma^w_\lambda)^{-1}$, and hence it is sufficient to show

$$q^{(\mu,\gamma)}\Phi_{v^*(\psi,\lambda)\dot{T}_w^{-1}\otimes v_\lambda}\sigma^w_\mu = \sigma^w_\mu \Phi_{v^*(\psi,\lambda)\dot{T}_w^{-1}\otimes v_\lambda}.$$

We have

$$\Phi_{v^*(\psi,\lambda)\dot{T}_w^{-1}\otimes v_\lambda}\sigma^w_\mu = \Phi_{v^*(\psi,\lambda)\dot{T}_w^{-1}\otimes v_\lambda}\Phi_{v^*_\mu\dot{T}_w^{-1}\otimes v_\mu} = \Phi_{(v^*(\psi,\lambda)\dot{T}_w^{-1}\otimes v^*_\mu\dot{T}_w^{-1})\otimes(v_\lambda\otimes v_\mu)},$$
$$\sigma^w_\mu\Phi_{v^*(\psi,\lambda)\dot{T}_w^{-1}\otimes v_\lambda} = \Phi_{v^*_\mu\dot{T}_w^{-1}\otimes v_\mu}\Phi_{v^*(\psi,\lambda)\dot{T}_w^{-1}\otimes v_\lambda} = \Phi_{(v^*_\mu\dot{T}_w^{-1}\otimes v^*(\psi,\lambda)\dot{T}_w^{-1})\otimes(v_\mu\otimes v_\lambda)}.$$

Since v^*_μ is the lowest weight vector we have

$$v^*(\psi,\lambda)\dot{T}_w^{-1}\otimes v^*_\mu\dot{T}_w^{-1} = (v^*(\psi,\lambda)\otimes v^*_\mu)\dot{T}_w^{-1}.$$

On the other hand by $\psi \in \mathcal{J}_w$ we have

$$v^*(\psi,\lambda)z = \varepsilon(z)v^*(\psi,\lambda) \qquad (z \in U^+[\dot{T}_w^{-1}]),$$

and hence

$$v^*_\mu\dot{T}_w^{-1}\otimes v^*(\psi,\lambda)\dot{T}_w^{-1} = (v^*_\mu\otimes v^*(\psi,\lambda))\dot{T}_w^{-1}.$$

Therefore, we have only to show

$$q^{(\mu,\gamma)}\Phi_{(v^*(\psi,\lambda)\otimes v^*_\mu)\dot{T}_w^{-1}\otimes(v_\lambda\otimes v_\mu)} = \Phi_{(v^*_\mu\otimes v^*(\psi,\lambda))\dot{T}_w^{-1}\otimes(v_\mu\otimes v_\lambda)}.$$

Let

$$p : V^*(\lambda)\otimes V^*(\mu)\to V^*(\lambda+\mu), \quad p' : V^*(\mu)\otimes V^*(\lambda)\to V^*(\lambda+\mu)$$

be the homomorphisms of U-modules such that $p(v^*_\lambda\otimes v^*_\mu) = v^*_{\lambda+\mu}$ and $p'(v^*_\mu\otimes v^*_\lambda) = v^*_{\lambda+\mu}$. The our assertion is equivalent to

$$q^{(\mu,\gamma)}p(v^*(\psi,\lambda)\otimes v^*_\mu) = q^{(\mu,\gamma)}v^*(\psi,\lambda+\mu) = p'(v^*_\mu\otimes v^*(\psi,\lambda)).$$

This follows from

$$\langle (v^*(\psi,\lambda)\otimes v^*_\mu)x, v_\lambda\otimes v_\mu\rangle = \langle v^*(\psi,\lambda)x\otimes v^*_\mu, v_\lambda\otimes v_\mu\rangle = \langle\psi, x\rangle,$$
$$\langle (v^*_\mu\otimes v^*(\psi,\lambda))x, v_\mu\otimes v_\lambda\rangle = \langle v^*_\mu k_\gamma\otimes v^*(\psi,\lambda)x, v_\mu\otimes v_\lambda\rangle = q^{(\mu,\gamma)}\langle\psi, x\rangle$$

for $x \in U^+$. □

Let us give a proof of Proposition 3.41. Assume that $\varphi \in (U^+)^\bigstar$ satisfies $F_w(\varphi) \in \mathcal{S}_w^{-1}\mathbb{C}_q[N^-_w\backslash G]$. When $\mu \in P^-$ is sufficiently small, we have $\sigma^w_\mu F_w(\varphi) \in \mathbb{C}_q[N^-_w\backslash G]$. By Lemma 3.39 we have

$$\sigma_\mu^w F_w(\varphi) = F_w(\varphi')\sigma_\mu^w. \tag{3.23}$$

By Lemma 3.40 we have

$$(\varphi')^z = \varepsilon(z)\varphi' \qquad (z \in U^+[\dot{T}_w^{-1}]),$$

namely $\varphi' \in \mathcal{J}_w$. Hence (3.23) and Lemma 3.42 implies $\varphi \in \mathcal{J}_w$.

Assume conversely that $\varphi \in \mathcal{J}_w$. Then by Lemma 3.40 and Lemma 3.42 we have $F_w(\varphi) \in \mathcal{S}_w^{-1}\mathbb{C}_q[N_w^-\backslash G]$.

The proof of Proposition 3.41 is complete.

3.8. By Proposition 3.35, Corollary 3.26, Proposition 3.36, Proposition 3.41, and Lemma 3.42 we obtain the following.

PROPOSITION 3.43. *The multiplicative set $\mathcal{S}_w$ satisfies the right Ore condition in $\mathbb{C}_q[N_w^-\backslash G]$.*

Set

$$U^+[\dot{T}_w^{-1}]^\bigstar = \sum_{\gamma \in Q^+} (U^+[\dot{T}_w^{-1}] \cap U_\gamma^+)^* \subset (U^+[\dot{T}_w^{-1}])^*. \tag{3.24}$$

In view of (2.16) we can define an injective linear map

$$i_w^+ : U^+[\dot{T}_w^{-1}]^\bigstar \to (U^+)^\bigstar \tag{3.25}$$

by

$$\langle i_w^+(\varphi), x_1x_2\rangle = \langle \varphi, u_1\rangle\varepsilon(u_2) \quad (x_1 \in U^+[\dot{T}_w^{-1}],\ x_2 \in U^+ \cap \dot{T}_w^{-1}(U^+)).$$

PROPOSITION 3.44. (i) *The multiplication of $(U^+)^\bigstar$ (as a subalgebra of U^*) induces an isomorphism*

$$i_w^+(U^+[\dot{T}_w^{-1}]^\bigstar) \otimes \mathcal{J}_w \cong (U^+)^\bigstar$$

of vector spaces.

(ii) *For $\varphi \in U^+[\dot{T}_w^{-1}]^\bigstar$, $\psi \in \mathcal{J}_w$ we have*

$$F_w(i_w^+(\varphi)\psi) = F_w(i_w^+(\varphi))F_w(\psi) \qquad (\varphi \in U^+[\dot{T}_w^{-1}]^\bigstar,\ \psi \in \mathcal{J}_w).$$

PROOF. (i) For $\varphi \in U^+[\dot{T}_w^{-1}]^\bigstar$, $\psi \in \mathcal{J}_w$, $x \in U^+[\dot{T}_w^{-1}]$, $x' \in U^+ \cap \dot{T}_w^{-1}U^{\geq 0}$ we have

$$\langle i_w^+(\varphi)\psi, xx'\rangle = \sum_{(x),(x')} \langle i_w^+(\varphi), x_{(0)}x'_{(0)}\rangle\langle \psi, x_{(1)}x'_{(1)}\rangle.$$

Hence by Lemma 2.8 we obtain

$$\langle i_w^+(\varphi)\psi, xx'\rangle = \langle i_w^+(\varphi), x\rangle\langle \psi, x'\rangle.$$

(ii) Take $\lambda \in P^-$ such that $i_w^+(\varphi)\chi_\lambda \in \mathbb{C}_q[G/N^-]$. Then we have $\chi_\lambda^{-1}\psi\chi_\lambda = \psi' \in \mathcal{J}_w$. Take $\mu \in P^-$ such that $\psi'\chi_\mu \in \mathbb{C}_q[G/N^-]$. We may assume that $\psi'\chi_\mu = \Phi_{v^*\otimes v_\nu}$ and

$$v^* z = \varepsilon(z) v^* \qquad (z \in U^+[\dot{T}_w^{-1}]).$$

Then we have

$$\begin{aligned} F_w(i_w^+(\varphi)\psi) &= F_w((i_w^+(\varphi)\chi_\lambda)(\psi'\chi_\mu)\chi_{\lambda+\mu}^{-1}) \\ &= \{\{(i_w^+(\varphi)\chi_\lambda)(\psi'\chi_\mu)\}\dot{T}_w^{-1}\}(\sigma_{\lambda+\mu}^w)^{-1} \\ &= \{(i_w^+(\varphi)\chi_\lambda)\dot{T}_w^{-1}\}\{(\psi'\chi_\mu)\dot{T}_w^{-1}\}(\sigma_{\lambda+\mu}^w)^{-1} \\ &= \{F_w(i_w^+(\varphi))\sigma_\lambda^w\}\{F_w(\psi')\sigma_\mu^w\}(\sigma_{\lambda+\mu}^w)^{-1} = F_w(i_w^+(\varphi))F_w(\psi'). \end{aligned}$$

Here, the last equality is a consequence of Lemma 3.42. □

4. Induced modules.

4.1. We fix $w \in W$ in this section.

By Proposition 2.12, Corollary 2.7 and (3.6) we have

$$(\varphi\psi)\dot{T}_w = (\varphi\dot{T}_w)(\psi\dot{T}_w) \qquad (\varphi \in \mathbb{C}_q[G], \psi \in \mathbb{C}_q[N_w^-\backslash G]). \tag{4.1}$$

Define $\eta'_w : \mathbb{C}_q[N_w^-\backslash G] \to \mathbb{C}_q[H]$ by

$$\langle \eta'_w(\varphi), t\rangle = \langle \varphi\dot{T}_w, t\rangle (= \langle \dot{T}_w\varphi, \dot{T}_w(t)\rangle) \qquad (t \in U^0)$$

(see Lemma 3.16).

Lemma 4.1. *The linear map η'_w is an algebra homomorphism. Moreover, for $\lambda \in P^-$ we have $\eta'_w(\sigma_\lambda^w) = \chi_\lambda \in \mathbb{C}_q[H]^\times$.*

Proof. For $\varphi, \psi \in \mathbb{C}_q[N_w^-\backslash G], t \in U^0$ we have

$$\begin{aligned} \langle \eta'_w(\varphi\psi), t\rangle &= \langle (\varphi\psi)\dot{T}_w, t\rangle = \langle (\varphi\dot{T}_w)(\psi\dot{T}_w), t\rangle \\ &= \sum_{(t)} \langle \varphi\dot{T}_w, t_{(0)}\rangle\langle \psi\dot{T}_w, t_{(1)}\rangle = \langle \eta'_w(\varphi)\eta'_w(\psi), t\rangle \end{aligned}$$

by (4.1). For $\lambda \in P^-$ and $t \in U^0$ we have

$$\langle \eta'_w(\sigma_\lambda^w), t\rangle = \langle \sigma_\lambda^w\dot{T}_w, t\rangle = \langle v_\lambda^*, tv_\lambda\rangle = \langle \chi_\lambda, t\rangle.$$ □

Hence we obtain an algebra homomorphism

$$\eta_w : \mathcal{S}_w^{-1}\mathbb{C}_q[N_w^-\backslash G] \to \mathbb{C}_q[H] \tag{4.2}$$

by extending η'_w.

Definition 4.2. Define an $(\mathcal{S}_w^{-1}\mathbb{C}_q[G], \mathbb{C}_q[H])$-bimodule $\mathcal{M}_w$ by

$$\mathcal{M}_w = \mathcal{S}_w^{-1}\mathbb{C}_q[G] \otimes_{\mathcal{S}_w^{-1}\mathbb{C}_q[N_w^-\backslash G]} \mathbb{C}_q[H], \tag{4.3}$$

where $\mathcal{S}_w^{-1}\mathbb{C}_q[N_w^-\backslash G] \to \mathbb{C}_q[H]$ is given by η_w.

By

$$\mathcal{S}_w^{-1}\mathbb{C}_q[G] = \mathbb{C}_q[G] \otimes_{\mathbb{C}_q[N_w^-\backslash G]} \mathcal{S}_w^{-1}\mathbb{C}_q[N_w^-\backslash G]$$

we have

$$\mathcal{M}_w \cong \mathbb{C}_q[G] \otimes_{\mathbb{C}_q[N_w^-\backslash G]} \mathbb{C}_q[H]. \tag{4.4}$$

For $\varphi \in \mathcal{S}_w^{-1}\mathbb{C}_q[G]$ and $\chi \in \mathbb{C}_q[H]$ we write

$$\varphi \star \chi := \varphi \otimes \chi \in \mathcal{M}_w. \tag{4.5}$$

Then we have

$$\varphi\sigma_\lambda^w \star \chi = \varphi \star \chi_\lambda\chi \qquad (\varphi \in \mathcal{S}_w^{-1}\mathbb{C}_q[G], \lambda \in P, \chi \in \mathbb{C}_q[H]). \tag{4.6}$$

By (4.4) $\mathcal{M}_w$ is generated by $\{\varphi \star 1 \mid \varphi \in \mathbb{C}_q[G]\}$ as a $\mathbb{C}_q[H]$-module.

4.2. Set

$$U^{\geq 0}[\dot{T}_w^{-1}] = (U^+[\dot{T}_w^{-1}])U^0 \subset U^{\geq 0}.$$

Define an injective linear map

$$U^+[\dot{T}_w^{-1}]^\bigstar \otimes \mathbb{C}_q[H] \to \mathrm{Hom}_{\mathbb{F}}(U^{\geq 0}[\dot{T}_w^{-1}], \mathbb{F}) \qquad (f \otimes \chi \mapsto c_{f\otimes\chi})$$

by

$$\langle c_{f\otimes\chi}, xt\rangle = \langle f, x\rangle\langle\chi, t\rangle \qquad (x \in U^+[\dot{T}_w^{-1}],\ t \in U^0),$$

and denote its image by $U^{\geq 0}[\dot{T}_w^{-1}]^\bigstar$. Then we have an identification

$$U^{\geq 0}[\dot{T}_w^{-1}]^\bigstar \cong U^+[\dot{T}_w^{-1}]^\bigstar \otimes \mathbb{C}_q[H] \qquad (c_{f\otimes\chi} \leftrightarrow f \otimes \chi) \tag{4.7}$$

of vector spaces. Since $U^+[\dot{T}_w^{-1}]^\bigstar \otimes \mathbb{C}_q[H]$ is naturally a right $\mathbb{C}_q[H]$-module by the multiplication of $\mathbb{C}_q[H]$, $U^{\geq 0}[\dot{T}_w^{-1}]^\bigstar$ is also endowed with a right $\mathbb{C}_q[H]$-module structure via the identification (4.7). Then we have

$$\langle f\chi, xt\rangle = \sum_{(t)}\langle f, xt_{(0)}\rangle\langle\chi, t_{(1)}\rangle$$
$$(f \in U^{\geq 0}[\dot{T}_w^{-1}]^\bigstar,\ \chi \in \mathbb{C}_q[H],\ x \in U^+[\dot{T}_w^{-1}],\ t \in U^0). \tag{4.8}$$

4.3. We construct an isomorphism

$$\Theta_w : \mathcal{M}_w \to U^{\geq 0}[\dot{T}_w^{-1}]^\bigstar$$

of right $\mathbb{C}_q[H]$-modules. We first define $\Theta'_w : \mathbb{C}_q[G] \to U^{\geq 0}[\dot{T}_w^{-1}]^\bigstar$ by

$$\langle \Theta'_w(\varphi), u \rangle = \langle \varphi \dot{T}_w, u \rangle \qquad (\varphi \in \mathbb{C}_q[G],\ u \in U^{\geq 0}[\dot{T}_w^{-1}]).$$

LEMMA 4.3. $\Theta'_w(\varphi\psi) = \Theta'_w(\varphi)\eta'_w(\psi) \qquad (\varphi \in \mathbb{C}_q[G],\ \psi \in \mathbb{C}_q[N_w^- \backslash G])$.

PROOF. Let $\gamma \in Q^+$, $x \in U^+[\dot{T}_w^{-1}] \cap U_\gamma^+$, $t \in U^0$. Then we have

$$\langle \Theta'_w(\varphi), xt \rangle = \langle \varphi \dot{T}_w, xt \rangle = \langle \varphi \dot{T}_w(x) \dot{T}_w, t \rangle.$$

Similarly,

$$\langle \Theta'_w(\varphi\psi), xt \rangle = \langle \{(\varphi\psi)\dot{T}_w(x)\} \dot{T}_w, t \rangle.$$

By (2.35) we can write $\dot{T}_w(x) = yk_{-w\gamma} \quad (y \in U^-[\dot{T}_w] \cap U^-_{w\gamma})$. Thus by Lemma 2.8 and $\psi \in \mathbb{C}_q[N_w^- \backslash G]$ we have

$$(\varphi\psi)(\dot{T}_w(x)) = \left(\sum_{(y)} (\varphi y_{(0)})(\psi y_{(1)}) \right) k_{-w\gamma} = \{(\varphi y)(\psi k_{w\gamma})\}\, k_{-w\gamma} = (\varphi(\dot{T}_w(x)))\psi.$$

Hence by (4.1) we have

$$\begin{aligned} \langle \Theta'_w(\varphi\psi), xt \rangle &= \langle \{(\varphi(\dot{T}_w(x)))\psi\} \dot{T}_w, t \rangle = \langle (\varphi(\dot{T}_w(x))\dot{T}_w)(\psi \dot{T}_w), t \rangle \\ &= \sum_{(t)} \langle \varphi(\dot{T}_w(x))\dot{T}_w, t_{(0)} \rangle \langle \psi \dot{T}_w, t_{(1)} \rangle = \sum_{(t)} \langle \Theta'_w(\varphi), xt_{(0)} \rangle \langle \eta'_w(\psi), t_{(1)} \rangle \\ &= \langle \Theta'_w(\varphi)\eta'_w(\psi), xt \rangle. \end{aligned}$$

□

Hence regarding $U^{\geq 0}[\dot{T}_w^{-1}]^\bigstar$ as a right $\mathbb{C}_q[N_w^- \backslash G]$-module via $\eta'_w : \mathbb{C}_q[N_w^- \backslash G] \to \mathbb{C}_q[H]$, Θ'_w turns out to be a homomorphism of right $\mathbb{C}_q[N_w^- \backslash G]$-modules. Moreover, the right action of the elements of $\mathcal{S}_w (\subset \mathbb{C}_q[N_w^- \backslash G])$ on $U^{\geq 0}[\dot{T}_w^{-1}]^\bigstar$ is invertible. Hence Θ'_w induces

$$\Theta''_w : \mathcal{S}_w^{-1}\mathbb{C}_q[G] = \mathbb{C}_q[G] \otimes_{\mathbb{C}_q[N_w^- \backslash G]} \mathcal{S}_w^{-1}\mathbb{C}_q[N_w^- \backslash G] \to U^{\geq 0}[\dot{T}_w^{-1}]^\bigstar.$$

Then we have

$$\Theta''_w(\varphi\psi) = \Theta''_w(\varphi)\eta_w(\psi) \quad (\varphi \in \mathcal{S}_w^{-1}\mathbb{C}_q[G],\ \psi \in \mathcal{S}_w^{-1}\mathbb{C}_q[N_w^- \backslash G]). \tag{4.9}$$

Therefore, we obtain a homomorphism

$$\Theta_w : \mathcal{M}_w \to U^{\geq 0}[\dot{T}_w^{-1}]^\bigstar \tag{4.10}$$

of right $\mathbb{C}_q[H]$-modules by

$$\Theta_w(\varphi \star \chi) = \Theta''_w(\varphi)\chi \qquad (\varphi \in \mathcal{S}_w^{-1}\mathbb{C}_q[G], \chi \in \mathbb{C}_q[H]).$$

PROPOSITION 4.4. *The linear map*

$$\Upsilon_w : U^+[\dot{T}_w^{-1}]^\bigstar \otimes \mathbb{C}_q[H] \to \mathcal{M}_w \qquad (\varphi \otimes \chi \mapsto F_w(i_w^+(\varphi)) \star \chi)$$

is bijective.

Proof. By Proposition 3.29 and (3.22) we have

$$\mathcal{S}_w^{-1}\mathbb{C}_q[G] \cong F_w((U^+)^\bigstar) \otimes \mathcal{S}_w^{-1}(\mathbb{C}_q[N^+\backslash G]\dot{T}_w^{-1}).$$

On the other hand by Proposition 3.35, Proposition 3.36, Proposition 3.41 we have

$$\mathcal{S}_w^{-1}\mathbb{C}_q[N_w^-\backslash G] \cong F_w(\mathcal{J}_w) \otimes \mathcal{S}_w^{-1}(\mathbb{C}_q[N^+\backslash G]\dot{T}_w^{-1}).$$

Hence by Proposition 3.44 we have

$$\mathcal{S}_w^{-1}\mathbb{C}_q[G] \cong F_w(i_w^+(U^+[\dot{T}_w^{-1}]^\bigstar)) \otimes \mathcal{S}_w^{-1}\mathbb{C}_q[N_w^-\backslash G].$$

It follows that $\mathcal{M}_w \cong U^+[\dot{T}_w^{-1}]^\bigstar \otimes \mathbb{C}_q[H]$. □

Proposition 4.5. *We have* $\Theta_w \circ \Upsilon_w = \mathrm{id}$ *under the identification* (4.7)*. Especially,* Θ_w *is an isomorphism of right* $\mathbb{C}_q[H]$*-modules.*

Proof. Let $\varphi \in U^+[\dot{T}_w^{-1}]^\bigstar$, $\chi \in \mathbb{C}_q[H]$, $x \in U^+[\dot{T}_w^{-1}]$, $t \in U^0$. Then for $\lambda \in P^-$ which is sufficiently small we have

$$\begin{aligned}
\langle (\Theta_w \circ \Upsilon_w)(\varphi \otimes \chi), xt\rangle &= \langle \Theta_w(F_w(i_w^+(\varphi)) \star \chi), xt\rangle \\
&= \langle \Theta_w(\Phi_{v^*(i_w^+(\varphi),\lambda)\dot{T}_w^{-1}\otimes v_\lambda}(\sigma_\lambda^w)^{-1} \star \chi), xt\rangle \\
&= \langle \Theta_w(\Phi_{v^*(i_w^+(\varphi),\lambda)\dot{T}_w^{-1}\otimes v_\lambda} \star \chi_{-\lambda}\chi), xt\rangle \\
&= \langle \Theta'_w(\Phi_{v^*(i_w^+(\varphi),\lambda)\dot{T}_w^{-1}\otimes v_\lambda})(\chi_{-\lambda}\chi), xt\rangle \\
&= \sum_{(t)} \langle \Phi_{v^*(i_w^+(\varphi),\lambda)\otimes v_\lambda}, xt_{(0)}\rangle\langle \chi_{-\lambda}\chi, t_{(1)}\rangle \\
&= \sum_{(t)_2} \langle i_w^+(\varphi), x\rangle\langle \chi_\lambda, t_{(0)}\rangle\langle \chi_{-\lambda}, t_{(1)}\rangle\langle \chi, t_{(2)}\rangle = \langle \varphi, x\rangle\langle \chi, t\rangle \\
&= \langle \varphi \otimes \chi, x \otimes t\rangle.
\end{aligned}$$

4.4. In this subsection we consider the special case where $\mathfrak{g} = \mathfrak{sl}_2$ and $G = SL_2$. We follow the notation of Example 3.1. The Weyl group consists of two elements 1 and s. We give below an explicit description of the $(\mathbb{C}_q[G], \mathbb{C}_q[H])$-bimodule $\mathcal{M}_s$. For $n \in \mathbb{Z}_{\geq 0}$ define $m(n) \in \mathcal{M}_s$ by

$$\langle \Theta(m(n)), e^{n'}k^i\rangle = \delta_{n,n'} \qquad (n' \in \mathbb{Z}_{\geq 0}, i \in \mathbb{Z}).$$

Then we have

$$\mathcal{M}_s = \bigoplus_{n=0}^{\infty} \mathbb{C}_q[H]m(n).$$

LEMMA 4.6. *The action of $\mathbb{C}_q[G]$ on $\mathcal{M}_s$ is given by*

$$am(n) = \chi(q-q^{-1})q^{n-1}m(n-1), \qquad bm(n) = \chi^{-1}q^n m(n),$$
$$cm(n) = -\chi q^{n+1}m(n), \qquad dm(n) = -\chi^{-1}q[n+1]m(n+1).$$

PROOF. By Corollary 2.7 we have

$$\langle(\psi\varphi)\dot{T}_s, e^n k^i\rangle = \sum_{r=0}^{n}\sum_{p=0}^{\infty} q^{-2ri}c(p,n,r)\langle\psi\dot{T}_s f^{(p)}, k^{n-r+i}e^r\rangle\langle\varphi\dot{T}_s, e^{n+p-r}k^i\rangle$$

for $\varphi, \psi \in \mathbb{C}_q[G]$, where

$$c(p,n,r) = q^{p(p-1)/2-r(n-r)}(q-q^{-1})^p \begin{bmatrix} n \\ r \end{bmatrix}.$$

By a direct calculation we have

$$a\dot{T}_s = c, \quad a\dot{T}_s f = a, \quad a\dot{T}_s f^{(2)} = 0.$$

Hence for $\varphi \in \mathbb{C}_q[G]$ we have

$$\begin{aligned}\langle(a\varphi)\dot{T}_s, e^n k^i\rangle &= \sum_{r=0}^{n}\sum_{p=0}^{\infty} q^{-2ri}c(p,n,r)\langle a\dot{T}_s f^{(p)}, k^{n-r+i}e^r\rangle\langle\varphi\dot{T}_s, e^{n+p-r}k^i\rangle \\ &= \sum_{r=0}^{n} q^{-2ri}c(0,n,r)\langle c, k^{n-r+i}e^r\rangle\langle\varphi\dot{T}_s, e^{n-r}k^i\rangle \\ &\qquad + \sum_{r=0}^{n} q^{-2ri}c(1,n,r)\langle a, k^{n-r+i}e^r\rangle\langle\varphi\dot{T}_s, e^{n+1-r}k^i\rangle \\ &= c(1,n,0)q^{n+i}\langle\varphi\dot{T}_s, e^{n+1}k^i\rangle \\ &= q^i(q-q^{-1})q^n\langle\varphi\dot{T}_s, e^{n+1}k^i\rangle.\end{aligned}$$

Taking $\varphi_j \in \mathbb{C}_q[SL_2]$ such that $m(n) = \sum_j \varphi_j \star \chi^j$ we have

$$\begin{aligned}\langle\Theta(am(n)), e^{n'}k^i\rangle &= \sum_j \langle\Theta(a\varphi_j \star \chi_j), e^{n'}k^i\rangle \\ &= \sum_j \langle(a\varphi_j)\dot{T}_s, e^{n'}k^i\rangle q^{ij} = q^i(q-q^{-1})q^{n'}\sum_j\langle\varphi_j\dot{T}_s, e^{n'+1}k^i\rangle q^{ij} \\ &= (q-q^{-1})q^{n'}\sum_j\langle\Theta(\varphi_j \star \chi^{j+1}), e^{n'+1}k^i\rangle \\ &= (q-q^{-1})q^{n'}\langle\Theta(m(n))\chi, e^{n'+1}k^i\rangle = (q-q^{-1})q^{n'}\delta_{n,n'+1}\langle\chi, k^i\rangle \\ &= (q-q^{-1})q^{n-1}\langle\Theta(m(n-1))\chi, e^{n'}k^i\rangle.\end{aligned}$$

Hence $am(n+1) = \chi(q-q^{-1})q^n m(n)$. The proof of other formulas are similar. □

4.5. Let us return to the general situation where $\mathfrak{g}$ is any simple Lie algebra.

Proposition 4.7. *We have*

$$\begin{aligned}\mathcal{M}_w &\cong \mathcal{S}_w^{-1}\mathbb{C}_q[G/N^-] \otimes_{\mathcal{S}_w^{-1}(\mathbb{C}_q[G/N^-]\cap\mathbb{C}_q[N_w^-\backslash G])} \mathbb{C}_q[H] \\ &\cong \mathbb{C}_q[G/N^-] \otimes_{\mathbb{C}_q[G/N^-]\cap\mathbb{C}_q[N_w^-\backslash G]} \mathbb{C}_q[H].\end{aligned}$$

Proof. By (3.22), Proposition 3.36, Proposition 3.41, Proposition 3.44 we have

$$\mathcal{S}_w^{-1}\mathbb{C}_q[G/N^-] \cong F_w(i_w^+(U^+[\dot{T}_w^{-1}]^\bigstar)) \otimes \mathcal{S}_w^{-1}(\mathbb{C}_q[G/N^-]\cap\mathbb{C}_q[N_w^-\backslash G]).$$

Hence we obtain

$$\mathcal{S}_w^{-1}\mathbb{C}_q[G/N^-] \otimes_{\mathcal{S}_w^{-1}(\mathbb{C}_q[G/N^-]\cap\mathbb{C}_q[N_w^-\backslash G])} \mathbb{C}_q[H] \cong U^+[\dot{T}_w^{-1}]^\bigstar \otimes \mathbb{C}_q[H] \cong \mathcal{M}_w$$

by Proposition 4.4. The second isomorphism is a consequence of

$$\mathcal{S}_w^{-1}\mathbb{C}_q[G/N^-] \cong \mathbb{C}_q[G/N^-] \otimes_{\mathbb{C}_q[G/N^-]\cap\mathbb{C}_q[N_w^-\backslash G]} \mathcal{S}_w^{-1}(\mathbb{C}_q[G/N^-]\cap\mathbb{C}_q[N_w^-\backslash G]). \qquad \square$$

We regard $\mathbb{C}_q[H]$ as a subalgebra of $\mathbb{C}_q[B^-]$ via the Hopf algebra homomorphism $U^{\leq 0} \to U^0$ given by $ty \mapsto \varepsilon(y)t$ $(t \in U^0, y \in U^-)$. Define an action of W on $\mathbb{C}_q[H]$ by

$$\langle w\chi, t\rangle = \langle \chi, \dot{T}_w^{-1}(t)\rangle \qquad (w \in W, \chi \in \mathbb{C}_q[H], t \in U^0).$$

For $w \in W$ we define a twisted right $\mathbb{C}_q[H]$-module structure of $\mathbb{C}_q[B^-]$ by

$$\varphi \bullet_w \chi = (Sw\chi)\varphi \qquad (\varphi \in \mathbb{C}_q[B^-],\ \chi \in \mathbb{C}_q[H]). \tag{4.11}$$

We denote by $\mathbb{C}_q[B^-]^{\bullet_w}$ the $\mathbb{F}$-algebra $\mathbb{C}_q[B^-]$ equipped with the twisted right $\mathbb{C}_q[H]$-module structure (4.11).

We are going to construct an embedding

$$\Xi_w : \mathcal{M}_w \hookrightarrow \mathbb{C}_q[B^-]^{\bullet_w}$$

of right $\mathbb{C}_q[H]$-module.

We first define

$$\Xi'_w : \mathbb{C}_q[G/N^-] \to \mathbb{C}_q[B^-]$$

by

$$\langle \Xi'_w(\varphi), u\rangle = \langle \dot{T}_w\varphi, Su\rangle \qquad (\varphi \in \mathbb{C}_q[G/N^-], u \in U^{\leq 0}).$$

Lemma 4.8. *The linear map Ξ'_w is an algebra anti-homomorphism. Moreover, for $\lambda \in P^-$ we have $\Xi'_w(\sigma_\lambda^w) = \chi_{-w\lambda} \in \mathbb{C}_q[B^-]^\times$.*

Proof. For $\varphi, \psi \in \mathbb{C}_q[G/N^-]$, $u \in U^{\leq 0}$ we have

$$\langle \Xi'_w(\varphi\psi), u\rangle = \langle \dot{T}_w(\varphi\psi), Su\rangle = \langle (\dot{T}_w\varphi)(\dot{T}_w\psi), Su\rangle$$

$$=\sum_{(u)}\langle \dot{T}_w\varphi, Su_{(1)}\rangle\langle \dot{T}_w\psi, Su_{(0)}\rangle = \sum_{(u)}\langle \Xi'_w(\varphi), u_{(1)}\rangle\langle \Xi'_w(\psi), u_{(0)}\rangle$$
$$=\langle \Xi'_w(\psi)\Xi'_w(\varphi), u\rangle.$$

Here, the second equality is a consequence of Corollary 2.7. For $\lambda \in P^-$, $t \in U^0$, $y \in U^{\leq 0}$ we have

$$\begin{aligned}\langle \Xi'_w(\sigma^w_\lambda), ty\rangle &= \langle \dot{T}_w\sigma^w_\lambda, (Sy)(St)\rangle = \langle v^*_\lambda \dot{T}_w^{-1}, ((Sy)(St))\dot{T}_w v_\lambda\rangle\\ &= \langle v^*_\lambda, \{\dot{T}_w^{-1}(Sy)\}\{\dot{T}_w^{-1}(St)\}v_\lambda\rangle = \varepsilon(\dot{T}_w^{-1}(Sy))\langle \chi_\lambda, \dot{T}_w^{-1}(St)\rangle\\ &= \varepsilon(y)\langle \chi_{-w\lambda}, t\rangle = \langle \chi_{-w\lambda}, ty\rangle.\end{aligned}$$

□

Hence Ξ'_w induces an algebra anti-homomorphism

$$\Xi''_w : \mathcal{S}_w^{-1}\mathbb{C}_q[G/N^-] \to \mathbb{C}_q[B^-]. \tag{4.12}$$

Lemma 4.9. *For $\varphi \in \mathcal{S}_w^{-1}(\mathbb{C}_q[G/N^-] \cap \mathbb{C}_q[N^-_w\backslash G])$ we have $\Xi''_w(\varphi) = Sw(\eta_w(\varphi))$.*

Proof. We may assume $\varphi \in \mathbb{C}_q[G/N^-]\cap\mathbb{C}_q[N^-_w\backslash G]$. By (2.15) the multiplication of $U^{\leq 0}$ induces an isomorphism

$$U^{\leq 0} \cong S^{-1}(U^- \cap \dot{T}_w(U^-)) \otimes U^0 \otimes S^{-1}(U^-[\dot{T}_w])$$

of vector spaces. Let $y_1 \in U^-[\dot{T}_w]$, $y_2 \in U^- \cap \dot{T}_w(U^-)$, $t \in U^0$. Then we have

$$\begin{aligned}\langle \Xi'_w(\varphi), (S^{-1}y_2)t(S^{-1}y_1)\rangle &= \langle \dot{T}_w\varphi, y_1(St)y_2\rangle = \langle y_2\dot{T}_w\varphi y_1, St\rangle\\ &= \langle \dot{T}_w(\dot{T}_w^{-1}(y_2))\varphi y_1, St\rangle = \varepsilon(y_1)\varepsilon(y_2)\langle \dot{T}_w\varphi, St\rangle\\ &= \varepsilon(y_1)\varepsilon(y_2)\langle \varphi\dot{T}_w, \dot{T}_w^{-1}S(t)\rangle = \langle \eta_w(\varphi), \dot{T}_w^{-1}(St)\rangle\varepsilon(y_1)\varepsilon(y_2)\\ &= \langle Sw(\eta_w(\varphi)), (S^{-1}y_2)t(S^{-1}y_1)\rangle.\end{aligned}$$

□

By Proposition 4.7, Lemma 4.8 and Lemma 4.9 we obtain a homomorphism

$$\Xi_w : \mathcal{M}_w \to \mathbb{C}_q[B^-]^{\bullet_w} \tag{4.13}$$

of right $\mathbb{C}_q[H]$-modules given by

$$\Xi_w(\varphi \star \chi) = \Xi''_w(\varphi) \bullet_w \chi \qquad (\varphi \in \mathcal{S}_w^{-1}\mathbb{C}_q[G/N^-], \chi \in \mathbb{C}_q[H]).$$

Since Ξ''_w is an algebra anti-homomorphism, we have

$$\Xi_w(\varphi\psi \star 1) = \Xi_w(\psi \star 1)\Xi_w(\varphi \star 1) \qquad (\varphi, \psi \in \mathcal{S}_w^{-1}\mathbb{C}_q[G/N^-]). \tag{4.14}$$

4.6. By (2.17) and Lemma 2.9 we can define an injective linear map

$$\Omega_w : U^{\geq 0}[\dot{T}_w^{-1}]^\bigstar \to \mathbb{C}_q[B^-]^{\bullet_w} \tag{4.15}$$

by

$$\langle\Omega_w(f), ty_2y_1\rangle = \varepsilon(y_2)\langle f, \dot{T}_w^{-1}S(ty_1)\rangle$$
$$(f \in U^{\geq 0}[\dot{T}_w^{-1}]^\bigstar,\ y_1 \in U^-[\hat{T}_w],\ y_2 \in U^- \cap \hat{T}_wU^-,\ t \in U^0).$$

LEMMA 4.10. *The linear map Ω_w is a homomorphism of right $\mathbb{C}_q[H]$-modules.*

PROOF. Let $f \in U^{\geq 0}[\dot{T}_w^{-1}]^\bigstar$ and $\chi \in \mathbb{C}_q[H]$. For $y_1 \in U^-[\hat{T}_w],\ y_2 \in U^- \cap \hat{T}_wU^-$, $t \in U^0$ we have

$$\begin{aligned}
\langle\Omega_w(f) \bullet_w \chi, ty_2y_1\rangle &= \langle(Sw\chi)\Omega_w(f), ty_2y_1\rangle \\
&= \sum_{(y_1),(y_2),(t)} \langle Sw\chi, t_{(0)}y_{2(0)}y_{1(0)}\rangle\langle\Omega_w(f), t_{(1)}y_{2(1)}y_{1(1)}\rangle \\
&= \sum_{(t)} \langle Sw\chi, t_{(0)}\rangle\langle\Omega_w(f), t_{(1)}y_2y_1\rangle \\
&= \sum_{(t)} \varepsilon(y_2)\langle Sw\chi, t_{(0)}\rangle\langle f, \dot{T}_w^{-1}S(t_{(1)}y_1)\rangle \\
&= \sum_{(t)} \varepsilon(y_2)\langle \chi, \dot{T}_w^{-1}S(t_{(0)})\rangle\langle f, \{\dot{T}_w^{-1}S(y_1)\}\{\dot{T}_w^{-1}S(t_{(1)})\}\rangle \\
&= \varepsilon(y_2)\langle f\chi, \{\dot{T}_w^{-1}S(y_1)\}\{\dot{T}_w^{-1}S(t)\}\rangle \\
&= \varepsilon(y_2)\langle f\chi, \dot{T}_w^{-1}S(ty_1)\rangle \\
&= \langle\Omega_w(f\chi), ty_2ty_1\rangle.
\end{aligned}$$
□

LEMMA 4.11. $\Omega_w \circ \Theta_w = \Xi_w$.

PROOF. By Proposition 4.7 we have only to show

$$(\Omega_w \circ \Theta_w)(\varphi \star \chi) = \Xi_w(\varphi \star \chi) \qquad (\varphi \in \mathbb{C}_q[G/N^-],\ \chi \in \mathbb{C}_q[H]).$$

By the definitions of Θ_w, Ξ_w and Lemma 4.10 it is sufficient to show

$$(\Omega_w \circ \Theta'_w)(\varphi) = \Xi'_w(\varphi) \qquad (\varphi \in \mathbb{C}_q[G/N^-]).$$

Let $y_1 \in U^-[\hat{T}_w],\ y_2 \in U^-_{-\gamma} \cap \hat{T}_wU^-,\ \delta \in Q$. Then we have

$$\begin{aligned}
\langle(\Omega_w \circ \Theta'_w)(\varphi), k_\delta y_2y_1\rangle &= \varepsilon(y_2)\langle\Theta'_w(\varphi), \dot{T}_w^{-1}S(k_\delta y_1)\rangle \\
&= \varepsilon(y_2)\langle\varphi\dot{T}_w, \dot{T}_w^{-1}S(k_\delta y_1)\rangle = \varepsilon(y_2)\langle\dot{T}_w\varphi, S(k_\delta y_1)\rangle, \\
\langle\Xi'_w(\varphi), k_\delta y_2y_1\rangle &= \langle\dot{T}_w\varphi, (Sy_1)(Sy_2)k_{-\delta}\rangle \\
&= q^{-(\gamma,\delta)}\langle\dot{T}_w\varphi, (Sy_1)k_{-\delta}(Sy_2)\rangle = q^{-(\gamma,\delta)}\langle(Sy_2)\dot{T}_w\varphi, (Sy_1)k_{-\delta}\rangle \\
&= q^{-(\gamma,\delta)}\langle\dot{T}_w(\dot{T}_w^{-1}S(y_2))\varphi, S(k_\delta y_1)\rangle \\
&= q^{-(\gamma,\delta)}\langle\dot{T}_w(S\hat{T}_w^{-1}(y_2))\varphi, S(k_\delta y_1)\rangle = \varepsilon(y_2)\langle\dot{T}_w\varphi, S(k_\delta y_1)\rangle.
\end{aligned}$$
□

We define a $\mathbb{C}_q[H]$-submodule $\mathcal{A}_w$ of $\mathbb{C}_q[B^-]^{\bullet_w}$ by

$$\mathcal{A}_w = \{\varphi \in \mathbb{C}_q[B^-] \mid \varphi y = 0 \quad (y \in U^- \cap \hat{T}_w(U^-))\}.$$

Note that we have an isomorphism

$$(U^-)^\bigstar \otimes \mathbb{C}_q[H] \cong \mathbb{C}_q[B^-]^{\bullet_w} \qquad (f \otimes \chi \leftrightarrow d_{f\otimes\chi}) \tag{4.16}$$

of right $\mathbb{C}_q[H]$-modules given by

$$\langle d_{f\otimes\chi}, ty\rangle = \langle Sw\chi, t\rangle\langle f, y\rangle \qquad (t \in U^0, y \in U^-).$$

Set

$$U^-[\hat{T}_w]^\bigstar = \bigoplus_{\gamma\in Q^+} (U^-[\hat{T}_w] \cap U^-_{-\gamma})^* \subset (U^-[\hat{T}_w])^*. \tag{4.17}$$

By (2.17) we have an injective linear map

$$i_w^- : U^-[\hat{T}_w]^\bigstar \to (U^-)^\bigstar \tag{4.18}$$

given by

$$\langle i_w^-(f), y'y\rangle = \varepsilon(y')\langle f, y\rangle \qquad (y \in U^-[\hat{T}_w], y' \in U^- \cap \hat{T}_w(U^-)).$$

Under the identification (4.16) we have

$$i_w^-(U^-[\hat{T}_w]^\bigstar) \otimes \mathbb{C}_q[H] \cong \mathcal{A}_w.$$

Proposition 4.12. *The linear map Ξ_w is injective and its image coincides with $\mathcal{A}_w$.*

Proof. Note that Θ_w is bijective and Ω_w is injective. Hence by Lemma 4.11 we see that Ξ_w is injective and its image coincides with $\mathrm{Im}(\Omega_w)$. Moreover, by the definition of Ω_w the image of Ω_w coincides with $\mathcal{A}_w$. □

5. The decomposition into tensor product.

5.1. For $i \in I$ define a Hopf subalgebra $U(i)$ of U by

$$U(i) = \langle k_i^{\pm1}, e_i, f_i\rangle \cong \mathbb{F} \otimes_{\mathbb{Q}(q_i)} U_{q_i}(\mathfrak{sl}_2) \subset U.$$

Define subalgebras $U(i)^\flat$ ($\flat = 0, \pm, \geq 0, \leq 0$) by

$$U(i)^0 = \langle k_i^{\pm1}\rangle, \qquad U(i)^+ = \langle e_i\rangle, \qquad U(i)^- = \langle f_i\rangle,$$
$$U(i)^{\geq 0} = \langle k_i^{\pm1}, e_i\rangle, \qquad U(i)^{\leq 0} = \langle k_i^{\pm1}, f_i\rangle.$$

We denote the quantized coordinate algebra of $U(i)$ by $\mathbb{C}_q[G(i)]$ ($\cong \mathbb{F} \otimes_{\mathbb{Q}(q_i)} \mathbb{C}_{q_i}[SL_2]$). As an algebra it is generated by elements a_i, b_i, c_i, d_i satisfying the fundamental relations

$$a_ib_i = q_ib_ia_i, \quad c_id_i = q_id_ic_i, \quad a_ic_i = q_ic_ia_i, \quad b_id_i = q_id_ib_i,$$

$$b_ic_i = c_ib_i, \quad a_id_i - d_ia_i = (q_i - q_i^{-1})b_ic_i, \quad a_id_i - q_ib_ic_i = 1$$

(see Example 3.1).

We have a quotient Hopf algebra $\mathbb{C}_q[H(i)]$ of $\mathbb{C}_q[G(i)]$ corresponding to $U(i)^0$. Then we have

$$\mathbb{C}_q[H(i)] = \mathbb{F}[\chi_i^{\pm 1}], \qquad \chi_i(k_i) = q_i.$$

Denote by

$$r^G_{G(i)} : \mathbb{C}_q[G] \to \mathbb{C}_q[G(i)] \tag{5.1}$$

the Hopf algebra homomorphism corresponding to $U(i) \subset U$.

5.2. Consider the $(\mathbb{C}_q[G(i)], \mathbb{C}_q[H(i)])$-bimodule

$$\mathcal{M}_i = \mathbb{F} \otimes_{\mathbb{Q}(q_i)} \mathcal{M}^{SL_2}_{s_i}, \tag{5.2}$$

where s_i is the generator of the Weyl group of $U(i)$, and $\mathcal{M}^{SL_2}_{s_i}$ is the $\mathbb{C}_q[G]$-module $\mathcal{M}_w$ for $G = SL_2$, $q = q_i$, $w = s_i$. We have an isomorphism

$$\Theta_i : \mathcal{M}_i \to (U(i)^{\geq 0})^\bigstar$$

of right $\mathbb{C}_q[H(i)]$-modules given by

$$\langle \Theta_i(\varphi \star \chi), xt\rangle = \sum_{(t)} \langle \varphi \dot{T}_i, xt_{(0)}\rangle \langle \chi, t_{(1)}\rangle$$
$$(\varphi \in \mathbb{C}_q[G(i)], \chi \in \mathbb{C}_q[H(i)], x \in U(i)^+, t \in U(i)^0).$$

Define $p_i(n) \in \mathcal{M}_i$ by

$$\langle \Theta_i(p_i(n)), e_i^{n'} k_i^j\rangle = \delta_{nn'}(-1)^n q_i^n [n]_{q_i}!.$$

By Lemma 4.6 we obtain the following.

Proposition 5.1. *The set $\{p_i(n) \mid n \in \mathbb{Z}_{\geq 0}\}$ forms a basis of the $\mathbb{C}_q[H(i)]$-module $\mathcal{M}_i$. Moreover, we have*

$$a_ip_i(n) = (1 - q_i^{2n})\chi_i p_i(n-1), \qquad b_ip_i(n) = \chi_i^{-1} q_i^n p_i(n),$$
$$c_ip_i(n) = -\chi_i q_i^{n+1} p_i(n), \qquad d_ip_i(n) = \chi_i^{-1} p_i(n+1).$$

We will regard $\mathcal{M}_i$ as a $(\mathbb{C}_q[G], \mathbb{C}_q[H(i)])$-module via $r^G_{G(i)}$.

5.3. Let $w \in W$ with $\ell(w) = m$. We set

$$z_{\boldsymbol{i},r} = s_{i_{r+1}} s_{i_{r+2}} \cdots s_{i_m} \qquad (r = 0, \dots, m), \tag{5.3}$$

$$\mathbb{C}_q[H(\boldsymbol{i})] = \mathbb{C}_q[H(i_1)] \otimes \cdots \otimes \mathbb{C}_q[H(i_t)]. \tag{5.4}$$

For $\boldsymbol{i} \in \mathcal{I}_w$ consider the $(\mathbb{C}_q[G]^{\otimes t}, \mathbb{C}_q[H(\boldsymbol{i})])$-bimodule $\mathcal{M}_{i_1} \otimes \cdots \otimes \mathcal{M}_{i_m}$. Via the iterated comultiplication $\Delta_{t-1} : \mathbb{C}_q[G] \to \mathbb{C}_q[G]^{\otimes t}$ and the algebra homomorphism

$$\Delta_{\boldsymbol{i}} : \mathbb{C}_q[H] \to \mathbb{C}_q[H(\boldsymbol{i})] \tag{5.5}$$

given by

$$\Delta_{\boldsymbol{i}}(\chi) = \sum_{(\chi)_{m-1}} z_{\boldsymbol{i},1}\chi_{(0)}|_{U(i_1)^0} \otimes \cdots \otimes z_{\boldsymbol{i},m}\chi_{(m-1)}|_{U(i_m)^0},$$

we can regard $\mathcal{M}_{i_1} \otimes \cdots \otimes \mathcal{M}_{i_m}$ as a $(\mathbb{C}_q[G], \mathbb{C}_q[H(\boldsymbol{i})])$-bimodule or a $(\mathbb{C}_q[G], \mathbb{C}_q[H])$-bimodule.

Define a linear map

$$F'_{\boldsymbol{i}} : \mathcal{M}_w \to \mathcal{M}_{i_1} \otimes \cdots \otimes \mathcal{M}_{i_m}$$

by

$$\langle (\Theta_{i_1} \otimes \cdots \otimes \Theta_{i_m})(F'_{\boldsymbol{i}}(m)), u_1 \otimes \cdots \otimes u_m \rangle = \langle \Theta_w(m), (\dot{T}^{-1}_{z_{\boldsymbol{i},1}}(u_1)) \cdots (\dot{T}^{-1}_{z_{\boldsymbol{i},m}}(u_m)) \rangle \quad (m \in \mathcal{M}_w, u_j \in U(i_j)^{\geq 0}).$$

In this subsection we will show the following.

Theorem 5.2. *The linear map $F'_{\boldsymbol{i}}$ is a homomorphism of $(\mathbb{C}_q[G], \mathbb{C}_q[H])$-bimodules, and it induces an isomorphism*

$$F_{\boldsymbol{i}} : \mathcal{M}_w \otimes_{\mathbb{C}_q[H]} \mathbb{C}_q[H(\boldsymbol{i})] \to \mathcal{M}_{i_1} \otimes \cdots \otimes \mathcal{M}_{i_m} \tag{5.6}$$

of $(\mathbb{C}_q[G], \mathbb{C}_q[H(\boldsymbol{i})])$-bimodules, where $\mathbb{C}_q[H] \to \mathbb{C}_q[H(\boldsymbol{i})]$ is given by $\Delta_{\boldsymbol{i}}$.

We first note the following.

Lemma 5.3. *Let $\varphi \in \mathbb{C}_q[G]$, $w, w_1, \dots, w_k \in W$. Then we have*

$$\Delta_k(\varphi \dot{T}_w) = \sum_{(\varphi)_k} (\dot{T}^{-1}_{w_1}\varphi_{(0)}\dot{T}_w) \otimes (\dot{T}^{-1}_{w_2}\varphi_{(1)}\dot{T}_{w_1}) \otimes \cdots \otimes (\dot{T}^{-1}_{w_k}\varphi_{(k-1)}\dot{T}_{w_{k-1}}) \otimes (\varphi_{(k)}\dot{T}_{w_k}).$$

Proof. By induction we may assume that $k = 1$. Set $x = w_1$. We may also assume that $\varphi = \Phi_{v^* \otimes v}$ $(v \in V, v^* \in V^*)$ for some $V \in \mathrm{Mod}_0(U)$. Let $\{v_j\}$ be a basis of V and let $\{v_j^*\}$ be its dual basis. Then we have

$$\Delta(\Phi_{v^* \otimes v}) = \sum_j \Phi_{v^* \otimes v_j} \otimes \Phi_{v_j^* \otimes v}.$$

Since the dual basis of $\{\dot{T}_x^{-1} v_j\}$ is $\{v_j^* \dot{T}_x\}$, we have for $u_0, u_1 \in U$ that

$$\langle \Delta(\Phi_{v^* \otimes v}\dot{T}_w), u_0 \otimes u_1 \rangle = \langle v^* \dot{T}_w, u_0 u_1 v \rangle$$

$$= \sum_j \langle v_j^* \dot{T}_x, u_1 v\rangle \langle v^* \dot{T}_w, u_0 \dot{T}_x^{-1} v_j\rangle = \sum_j \langle \Phi_{v^* \dot{T}_w \otimes \dot{T}_x^{-1} v_j} \otimes \Phi_{v_j^* \dot{T}_x \otimes v}, u_0 \otimes u_1\rangle$$
$$= \sum_j \langle (\dot{T}_x^{-1} \Phi_{v^* \otimes v_j} \dot{T}_w) \otimes (\Phi_{v_j^* \otimes v} \dot{T}_x), u_0 \otimes u_1\rangle. \qquad \square$$

LEMMA 5.4. *For $\varphi \in \mathbb{C}_q[G]$ and $u_r \in U(i_r)^{\geq 0}$ $(r = 1, \ldots, m)$ we have*

$$\langle \Theta_w(\varphi \star 1), (\dot{T}_{z_{\boldsymbol{i},1}}^{-1}(u_1)) \cdots (\dot{T}_{z_{\boldsymbol{i},m}}^{-1}(u_m))\rangle = \sum_{(\varphi)_{m-1}} \prod_{r=1}^m \langle \Theta_{i_r}(r_{G(i_r)}^G(\varphi_{(r-1)}) \star 1), u_r\rangle.$$

PROOF. By Lemma 5.3 we have

$$\Delta_{m-1}(\varphi \dot{T}_w) = \sum_{(\varphi)_m} (\dot{T}_{z_{\boldsymbol{i},1}}^{-1} \varphi_{(0)} \dot{T}_{z_{\boldsymbol{i},0}}) \otimes (\dot{T}_{z_{\boldsymbol{i},2}}^{-1} \varphi_{(1)} \dot{T}_{z_{\boldsymbol{i},1}}) \otimes \cdots \otimes (\dot{T}_{z_{\boldsymbol{i},m}}^{-1} \varphi_{(m-1)} \dot{T}_{z_{\boldsymbol{i},m-1}}).$$

Hence

$$\begin{aligned}
&\langle \Theta_w(\varphi \star 1), (\dot{T}_{z_{\boldsymbol{i},1}}^{-1}(u_1)) \cdots (\dot{T}_{z_{\boldsymbol{i},m}}^{-1}(u_m))\rangle \\
&\quad = \langle \varphi \dot{T}_w, (\dot{T}_{z_{\boldsymbol{i},1}}^{-1}(u_1)) \cdots (\dot{T}_{z_{\boldsymbol{i},m}}^{-1}(u_m))\rangle \\
&\quad = \langle \Delta_{m-1}(\varphi \dot{T}_w), (\dot{T}_{z_{\boldsymbol{i},1}}^{-1}(u_1)) \otimes \cdots \otimes (\dot{T}_{z_{\boldsymbol{i},m}}^{-1}(u_m))\rangle \\
&\quad = \sum_{(\varphi)_{m-1}} \prod_{r=1}^m \langle \dot{T}_{z_{\boldsymbol{i},r}}^{-1} \varphi_{(r-1)} \dot{T}_{z_{\boldsymbol{i},r-1}}, \dot{T}_{z_{\boldsymbol{i},r}}^{-1}(u_r)\rangle \\
&\quad = \sum_{(\varphi)_{m-1}} \left\langle \prod_{r=1}^m \varphi_{(r-1)} \dot{T}_{i_r}, u_r \right\rangle \\
&\quad = \sum_{(\varphi)_{m-1}} \prod_{r=1}^m \langle \Theta_{i_r}(r_{G(i_r)}^G(\varphi_{(r-1)}) \star 1), u_r\rangle
\end{aligned}$$

by Lemma 3.16. $\square$

Now we give a proof of Theorem 5.2.

We first show that $F'_{\boldsymbol{i}}$ is a homomorphism of right $\mathbb{C}_q[H]$-modules. For $m \in \mathcal{M}_w$, $x_j \in U(i_j)^+_{c_j \alpha_{i_j}}$, $\chi \in \mathbb{C}_q[H]$ we have

$$\begin{aligned}
&\langle (\Theta_{i_1} \otimes \cdots \otimes \Theta_{i_m})(F'_{\boldsymbol{i}}(m\chi)), x_1 k_{i_1}^{p_1} \otimes \cdots \otimes x_m k_{i_m}^{p_m}\rangle \\
&\quad = \langle \Theta_w(m\chi), (\dot{T}_{z_{\boldsymbol{i},1}}^{-1}(x_1 k_{i_1}^{p_1})) \cdots (\dot{T}_{z_{\boldsymbol{i},m}}^{-1}(x_m k_{i_m}^{p_m}))\rangle \\
&\quad = q^A \langle \Theta_w(m\chi), (\dot{T}_{z_{\boldsymbol{i},1}}^{-1}(x_1)) \cdots (\dot{T}_{z_{\boldsymbol{i},m}}^{-1}(x_m)) k_{z_{\boldsymbol{i},1}^{-1}\alpha_{i_1}}^{p_1} \cdots k_{z_{\boldsymbol{i},m}^{-1}\alpha_{i_m}}^{p_m}\rangle \\
&\quad = q^A \langle \Theta_w(m), (\dot{T}_{z_{\boldsymbol{i},1}}^{-1}(x_1)) \cdots (\dot{T}_{z_{\boldsymbol{i},m}}^{-1}(x_m)) k_{z_{\boldsymbol{i},1}^{-1}\alpha_{i_1}}^{p_1} \cdots k_{z_{\boldsymbol{i},m}^{-1}\alpha_{i_m}}^{p_m}\rangle \langle \chi, k_{z_{\boldsymbol{i},1}^{-1}\alpha_{i_1}}^{p_1} \cdots k_{z_{\boldsymbol{i},m}^{-1}\alpha_{i_m}}^{p_m}\rangle \\
&\quad = \langle \Theta_w(m), (\dot{T}_{z_{\boldsymbol{i},1}}^{-1}(x_1 k_{i_1}^{p_1})) \cdots (\dot{T}_{z_{\boldsymbol{i},m}}^{-1}(x_m k_{i_m}^{p_m}))\rangle \langle \chi, k_{z_{\boldsymbol{i},1}^{-1}\alpha_{i_1}}^{p_1} \cdots k_{z_{\boldsymbol{i},m}^{-1}\alpha_{i_m}}^{p_m}\rangle \\
&\quad = \langle (\Theta_{i_1} \otimes \cdots \otimes \Theta_{i_m})(F'_{\boldsymbol{i}}(m)), x_1 k_{i_1}^{p_1} \otimes \cdots \otimes x_m k_{i_m}^{p_m}\rangle \langle \chi, k_{z_{\boldsymbol{i},1}^{-1}\alpha_{i_1}}^{p_1} \cdots k_{z_{\boldsymbol{i},m}^{-1}\alpha_{i_m}}^{p_m}\rangle \\
&\quad = \langle \{(\Theta_{i_1} \otimes \cdots \otimes \Theta_{i_m})(F'_{\boldsymbol{i}}(m))\} \Delta_{\boldsymbol{i}}(\chi), x_1 k_{i_1}^{p_1} \otimes \cdots \otimes x_m k_{i_m}^{p_m}\rangle
\end{aligned}$$

$$= \langle (\Theta_{i_1} \otimes \cdots \otimes \Theta_{i_m})(F'_{\boldsymbol{i}}(m)\Delta_{\boldsymbol{i}}(\chi)), x_1 k_{i_1}^{p_1} \otimes \cdots \otimes x_m k_{i_m}^{p_m} \rangle,$$

where

$$A = \sum_{r=1}^{m-1} p_r(z_{\boldsymbol{i},r}^{-1}\alpha_{i_r}, c_{r+1}z_{\boldsymbol{i},r+1}^{-1}\alpha_{i_{r+1}} + \cdots + c_m z_{\boldsymbol{i},m}^{-1}\alpha_{i_m}).$$

Hence $F'_{\boldsymbol{i}}$ is a homomorphism of right $\mathbb{C}_q[H]$-modules.

We next show that $F'_{\boldsymbol{i}}$ is a homomorphism of left $\mathbb{C}_q[G]$-modules. It is sufficient to show $F'_{\boldsymbol{i}}(\varphi m) = \varphi F'_{\boldsymbol{i}}(m)$ for $\varphi \in \mathbb{C}_q[G]$, $m \in \mathcal{M}_w$. Since $F'_{\boldsymbol{i}}$ is a homomorphism of right $\mathbb{C}_q[H]$-module, we may assume that $m = \psi \star 1$ ($\psi \in \mathbb{C}_q[G]$). Then we have

$$\begin{aligned}
&\langle (\Theta_{i_1} \otimes \cdots \otimes \Theta_{i_m})(F'_{\boldsymbol{i}}(\psi \star 1)), u_1 \otimes \cdots \otimes u_m \rangle \\
&\quad = \langle \Theta_w(\psi \star 1), (\dot{T}_{z_{\boldsymbol{i},1}}^{-1}(u_1)) \cdots (\dot{T}_{z_{\boldsymbol{i},m}}^{-1}(u_m)) \rangle \\
&\quad = \sum_{(\psi)_{m-1}} \prod_{r=1}^{m} \langle \Theta_{i_r}(r_{G(i_r)}^G(\psi_{(r-1)}) \star 1), u_r \rangle.
\end{aligned}$$

Hence

$$F'_{\boldsymbol{i}}(\psi \star 1) = \sum_{(\psi)_{m-1}} (r_{G(i_1)}^G(\psi_{(0)}) \star 1) \otimes \cdots \otimes (r_{G(i_m)}^G(\psi_{(m-1)}) \star 1).$$

It follows that

$$\begin{aligned}
F'_{\boldsymbol{i}}(\varphi m) &= F'_{\boldsymbol{i}}(\varphi\psi \star 1) \\
&= \sum_{(\varphi)_{m-1}, (\psi)_{m-1}} (r_{G(i_1)}^G(\varphi_{(0)}\psi_{(0)}) \star 1) \otimes \cdots \otimes (r_{G(i_m)}^G(\varphi_{(m-1)}\psi_{(m-1)}) \star 1) \\
&= \varphi \sum_{(\psi)_{m-1}} (r_{G(i_1)}^G(\psi_{(0)}) \star 1) \otimes \cdots \otimes (r_{G(i_m)}^G(\psi_{(m-1)}) \star 1) = \varphi F'_{\boldsymbol{i}}(m).
\end{aligned}$$

Therefore, $F'_{\boldsymbol{i}}$ is a homomorphism of left $\mathbb{C}_q[G]$-modules.

Since $F'_{\boldsymbol{i}}$ is a homomorphism of $(\mathbb{C}_q[G], \mathbb{C}_q[H])$-bimodules, it induces a homomorphism

$$F_{\boldsymbol{i}} : \mathcal{M}_w \otimes_{\mathbb{C}_q[H]} \mathbb{C}_q[H(\boldsymbol{i})] \to \mathcal{M}_{i_1} \otimes \cdots \otimes \mathcal{M}_{i_m} \qquad (a \otimes \chi \mapsto F'_{\boldsymbol{i}}(a)\chi)$$

of $(\mathbb{C}_q[G], \mathbb{C}_q[H(\boldsymbol{i})])$-bimodules. It remains to show that $F_{\boldsymbol{i}}$ is bijective. Via Θ_w we have

$$\begin{aligned}
\mathcal{M}_w \otimes_{\mathbb{C}_q[H]} \mathbb{C}_q[H(\boldsymbol{i})] &\cong (U^+[\dot{T}_w^{-1}]^\bigstar \otimes \mathbb{C}_q[H]) \otimes_{\mathbb{C}_q[H]} \mathbb{C}_q[H(\boldsymbol{i})] \\
&\cong U^+[\dot{T}_w^{-1}]^\bigstar \otimes \mathbb{C}_q[H(\boldsymbol{i})],
\end{aligned}$$

and via $\Theta_{i_1} \otimes \cdots \otimes \Theta_{i_m}$ we have

$$\begin{aligned}
\mathcal{M}_{i_1} \otimes \cdots \otimes \mathcal{M}_{i_m} &\cong \{(U(i_1)^+)^\bigstar \otimes \mathbb{C}_q[H(i_1)]\} \otimes \cdots \otimes \{(U(i_m)^+)^\bigstar \otimes \mathbb{C}_q[H(i_m)]\} \\
&\cong \{(U(i_1)^+)^\bigstar \otimes \cdots \otimes (U(i_m)^+)^\bigstar\} \otimes \mathbb{C}_q[H(\boldsymbol{i})].
\end{aligned}$$

Hence the assertion follows from

$$(U(i_1)^+)^\bigstar \otimes \cdots \otimes (U(i_m)^+)^\bigstar \cong U^+[\dot{T}_w^{-1}]^\bigstar \qquad (x_1 \otimes \cdots \otimes x_m \leftrightarrow \dot{T}_{z_{\boldsymbol{i},1}}^{-1}(x_1) \otimes \cdots \otimes \dot{T}_{z_{\boldsymbol{i},m}}^{-1}(x_m)).$$

The proof of Theorem 5.2 is complete.

6. Basis elements.

6.1. Let $w \in W$ with $\ell(w) = m$, and fix $\boldsymbol{i} = (i_1, \dots, i_m) \in \mathcal{I}_w$.

For $\boldsymbol{n} = (n_1, \dots, n_m) \in (\mathbb{Z}_{\geq 0})^m$ we denote by $p_{\boldsymbol{i}}(\boldsymbol{n})$ the element of $\mathcal{M}_w \otimes_{\mathbb{C}_q[H]} \mathbb{C}_q[H(\boldsymbol{i})]$ corresponding to

$$p_{i_1}(n_1) \otimes \cdots \otimes p_{i_m}(n_m) \in \mathcal{M}_{i_1} \otimes \cdots \otimes \mathcal{M}_{i_m}$$

under the isomorphism (5.6).

By (4.7) we have

$$U^{\geq 0}[\dot{T}_w^{-1}]^\bigstar \otimes_{\mathbb{C}_q[H]} \mathbb{C}_q[H(\boldsymbol{i})] \cong U^+[\dot{T}_w^{-1}]^\bigstar \otimes \mathbb{C}_q[H(\boldsymbol{i})].$$

Hence Θ_w induces an isomorphism

$$\Theta_{w,\boldsymbol{i}} : \mathcal{M}_w \otimes_{\mathbb{C}_q[H]} \mathbb{C}_q[H(\boldsymbol{i})] \to U^+[\dot{T}_w^{-1}]^\bigstar \otimes \mathbb{C}_q[H(\boldsymbol{i})] \tag{6.1}$$

of right $\mathbb{C}_q[H(\boldsymbol{i})]$-modules. We will regard $U^+[\dot{T}_w^{-1}]^\bigstar \otimes \mathbb{C}_q[H(\boldsymbol{i})]$ as a subset of $\mathrm{Hom}_{\mathbb{F}}(U^+[\dot{T}_w^{-1}], \mathbb{C}_q[H(\boldsymbol{i})])$ in the following.

For $r = 1, \dots, m$ and $\boldsymbol{n} \in (\mathbb{Z}_{\geq 0})^m$ set

$$\beta_{\boldsymbol{i},r} = z_{\boldsymbol{i},r}^{-1}\alpha_{i_r}, \qquad \gamma_{\boldsymbol{i},\boldsymbol{n},r} = n_{r+1}\beta_{\boldsymbol{i},r+1} + \cdots + n_m\beta_{\boldsymbol{i},m}. \tag{6.2}$$

Proposition 6.1. *We have*

$$\langle \Theta_{w,\boldsymbol{i}}(p_{\boldsymbol{i}}(\boldsymbol{n})), \tilde{e}_{\boldsymbol{i}}^{\boldsymbol{n}'} \rangle = \delta_{\boldsymbol{n},\boldsymbol{n}'} \left\{ \prod_{r=1}^{m} (-1)^{n_r} q_{i_r}^{n_r} [n_r]_{q_{i_r}}! \right\} \chi_{i_1}^{\langle \beta_{\boldsymbol{i},1}^\vee, \gamma_{\boldsymbol{i},\boldsymbol{n},1} \rangle} \otimes \cdots \otimes \chi_{i_m}^{\langle \beta_{\boldsymbol{i},m}^\vee, \gamma_{\boldsymbol{i},\boldsymbol{n},m} \rangle}.$$

Proof. Define $a \in \mathcal{M}_w$ by

$$\langle \Theta_w(a), \tilde{e}_{\boldsymbol{i}}^{\boldsymbol{n}'} t \rangle = \delta_{\boldsymbol{n}\boldsymbol{n}'} \varepsilon(t) \qquad (\boldsymbol{n}' \in (\mathbb{Z}_{\geq 0})^m, t \in U^0).$$

Then we have

$$\begin{aligned}
&\langle (\Theta_{i_1} \otimes \cdots \otimes \Theta_{i_m})(F_{\boldsymbol{i}}(a \otimes 1)), e_{i_1}^{n'_1} k_{i_1}^{j_1} \otimes \cdots \otimes e_{i_m}^{n'_m} k_{i_m}^{j_m} \rangle \\
&\quad = \langle (\Theta_{i_1} \otimes \cdots \otimes \Theta_{i_m})(F'_{\boldsymbol{i}}(a)), e_{i_1}^{n'_1} k_{i_1}^{j_1} \otimes \cdots \otimes e_{i_m}^{n'_m} k_{i_m}^{j_m} \rangle \\
&\quad = \langle \Theta_w(a), T_{z_{\boldsymbol{i},1}}^{-1}(e_{i_1}^{n'_1} k_{i_1}^{j_1}) \cdots T_{z_{\boldsymbol{i},m}}^{-1}(e_{i_m}^{n'_m} k_{i_m}^{j_m}) \rangle \\
&\quad = q^{A'} \left\langle \Theta_w(a), \tilde{e}_{\boldsymbol{i}}^{\boldsymbol{n}'} k_{z_{\boldsymbol{i},1}^{-1}\alpha_{i_1}}^{j_1} \cdots k_{z_{\boldsymbol{i},m}^{-1}\alpha_{i_m}}^{j_m} \right\rangle
\end{aligned}$$

$$= \delta_{\boldsymbol{n}\boldsymbol{n}'} q^A$$
$$= \delta_{\boldsymbol{n}\boldsymbol{n}'} \prod_{r=1}^{m} (q_{i_r}^{j_r})^{\langle \beta_{\boldsymbol{i},r}^\vee, \gamma_{\boldsymbol{i},\boldsymbol{n},r}\rangle},$$

where

$$A' = \sum_{r=1}^{m-1} \langle j_r \beta_{\boldsymbol{i},r}, \gamma_{\boldsymbol{i},\boldsymbol{n},r+1}\rangle, \qquad A = \sum_{r=1}^{m-1} \langle j_r \beta_{\boldsymbol{i},r}, \gamma_{\boldsymbol{i},\boldsymbol{n}',r+1}\rangle.$$

On the other hand we have

$$\langle (\Theta_{i_1} \otimes \cdots \otimes \Theta_{i_m})(p_{i_1}(n_1) \otimes \cdots \otimes p_{i_m}(n_m)), e_{i_1}^{n'_1} k_{i_1}^{j_1} \otimes \cdots \otimes e_{i_m}^{n'_m} k_{i_m}^{j_m}\rangle$$
$$= \prod_{r=1}^{m} \delta_{n_r n'_r} (-1)^{n_r} q_{i_r}^{n_r} [n_r]_{q_{i_r}}! = \delta_{\boldsymbol{n}\boldsymbol{n}'} \prod_{r=1}^{m} (-1)^{n_r} q_{i_r}^{n_r} [n_r]_{q_{i_r}}!. \qquad \square$$

6.2. We rewrite Proposition 6.1 using Ξ_w instead of Θ_w. Note that the isomorphism (4.16) induces

$$\mathbb{C}_q[B^-]^{\bullet_w} \otimes_{\mathbb{C}_q[H]} \mathbb{C}_q[H(\boldsymbol{i})] \cong (U^-)^\bigstar \otimes \mathbb{C}_q[H(\boldsymbol{i})] \ \left(\subset \operatorname{Hom}_{\mathbb{F}}(U^-, \mathbb{C}_q[H(\boldsymbol{i})])\right).$$

Hence Ξ_w induces an injection

$$\Xi_{w,\boldsymbol{i}} : \mathcal{M}_w \otimes_{\mathbb{C}_q[H]} \mathbb{C}_q[H(\boldsymbol{i})] \to (U^-)^\bigstar \otimes \mathbb{C}_q[H(\boldsymbol{i})] \ \left(\subset \operatorname{Hom}_{\mathbb{F}}(U^-, \mathbb{C}_q[H(\boldsymbol{i})])\right).$$

Recall that $\{\hat{f}_{\boldsymbol{i}}^{\boldsymbol{n}}\}_{\boldsymbol{n}}$ forms a basis $U^-[\hat{T}_w]$ and the multiplication induces an isomorphism $(U^- \cap \hat{T}_w U^-) \otimes U^-[\hat{T}_w] \cong U^-$ (see Proposition 2.12, (2.17)).

Proposition 6.2. *For $y \in U^- \cap \hat{T}_w U^-$ we have*

$$\langle \Xi_{w,\boldsymbol{i}}(p_{\boldsymbol{i}}(\boldsymbol{n})), y\hat{f}_{\boldsymbol{i}}^{\boldsymbol{n}'}\rangle = \varepsilon(y)\delta_{\boldsymbol{n},\boldsymbol{n}'} \left\{\prod_{r=1}^{m} (-1)^{n_r} q_{i_r}^{n_r} [n_r]_{q_{i_r}}!\right\} \chi_{i_1}^{\langle \beta_{\boldsymbol{i},1}^\vee, \gamma_1\rangle} \otimes \cdots \otimes \chi_{i_m}^{\langle \beta_{\boldsymbol{i},m}^\vee, \gamma_m\rangle}.$$

Proof. Let

$$\Omega_{w,\boldsymbol{i}} : U^+[\dot{T}_w^{-1}]^\bigstar \otimes \mathbb{C}_q[H(\boldsymbol{i})] \to (U^-)^\bigstar \otimes \mathbb{C}_q[H(\boldsymbol{i})]$$

be the homomorphism of right $\mathbb{C}_q[H(\boldsymbol{i})]$-modules induced by Ω_w. For $f \in U^+[\dot{T}_w^{-1}]^\bigstar$ the element of $U^{\geq 0}[\dot{T}_w^{-1}]^\bigstar \otimes_{\mathbb{C}_q[H]} \mathbb{C}_q[H(\boldsymbol{i})]$ corresponding to $f \otimes 1 \in U^+[\dot{T}_w^{-1}]^\bigstar \otimes \mathbb{C}_q[H(\boldsymbol{i})]$ is written as $\tilde{f} \otimes 1$, where $\tilde{f} \in U^{\geq 0}[\dot{T}_w^{-1}]^\bigstar$ is given by

$$\langle \tilde{f}, xt\rangle = \langle f, x\rangle \varepsilon(t) \qquad (x \in U^+[\dot{T}_w^{-1}], t \in U^0).$$

Then for $y_1 \in U^-[\hat{T}_w]$, $y_2 \in U^- \cap \hat{T}_w U^-$, $t \in U^0$ we have

$$\langle \Omega_w(\tilde{f}), ty_2y_1\rangle = \varepsilon(y_2)\langle \tilde{f}, \dot{T}_w^{-1} S(ty_1)\rangle = \varepsilon(y_2 t)\langle f, \dot{T}_w^{-1} S(y_1)\rangle \qquad (\dot{T}_w^{-1} S(y_1) \in U^+[\dot{T}_w^{-1}]).$$

Namely, the element of $(U^-)^\bigstar \otimes \mathbb{C}_q[H(\boldsymbol{i})]$ corresponding to $f \otimes 1$ is written as $\hat{f} \otimes 1$, where $\hat{f} \in (U^-)^\bigstar$ is given by

$$\langle \hat{f}, y_2y_1\rangle = \varepsilon(y_2)\langle f, \dot{T}_w^{-1}S(y_1)\rangle \qquad (y_1 \in U^-[\hat{T}_w], y_2 \in U^- \cap \hat{T}_wU^-).$$

Hence for $y \in U^- \cap \hat{T}_wU^-$ we have

$$\langle \Xi_{w,\boldsymbol{i}}(p_{\boldsymbol{i}}(\boldsymbol{n})), y\hat{f}_{\boldsymbol{i}}^{\boldsymbol{n}'}\rangle = \varepsilon(y)\langle \Theta_{w,\boldsymbol{i}}(p_{\boldsymbol{i}}(\boldsymbol{n})), \tilde{e}_{\boldsymbol{i}}^{\boldsymbol{n}'}\rangle. \qquad \square$$

6.3. Set

$$U^{\geq 0}[\hat{T}_w] = U^+[\hat{T}_w]U^0 \subset U^{\geq 0} \tag{6.3}$$

and define

$$\Psi_w : U^{\geq 0}[\hat{T}_w] \to \mathbb{C}_q[B^-] \tag{6.4}$$

by

$$\langle \Psi_w(x), u\rangle = \tau(x, u) \qquad (x \in U^{\geq 0}[\hat{T}_w], u \in U^{\leq 0}).$$

By Proposition 2.5 Ψ_w is an injective algebra homomorphism and its image is contained in $\mathcal{A}_w$. Hence there exists a unique injective linear map

$$\Gamma_w : U^{\geq 0}[\hat{T}_w] \to \mathcal{M}_w \tag{6.5}$$

such that $\Xi_w \circ \Gamma_w = \Psi_w$.

Theorem 6.3. *We have*

$$p_{\boldsymbol{i}}(\boldsymbol{n}) = d_{\boldsymbol{i}}(\boldsymbol{n})\Gamma_w(\hat{e}_{\boldsymbol{i}}^{(\boldsymbol{n})}) \otimes \left\{\chi_{i_1}^{\langle \beta_{\boldsymbol{i},1}^\vee, \gamma_1\rangle} \otimes \cdots \otimes \chi_{i_m}^{\langle \beta_{\boldsymbol{i},m}^\vee, \gamma_m\rangle}\right\},$$

where

$$d_{\boldsymbol{i}}(\boldsymbol{n}) = \prod_{r=1}^m d_{i_r}(n_r), \qquad d_i(n) = q^{n(n+1)/2}(q^{-1} - q)^n.$$

Proof. For $y \in U^- \cap \hat{T}_wU^-$ we have

$$\begin{aligned}\langle \Xi_w(\Gamma_w(\hat{e}_{\boldsymbol{i}}^{(\boldsymbol{n})})) \otimes 1, y\hat{f}_{\boldsymbol{i}}^{\boldsymbol{n}'}\rangle &= \varepsilon(y)\langle \Psi_w(\hat{e}_{\boldsymbol{i}}^{(\boldsymbol{n})}), \hat{f}_{\boldsymbol{i}}^{\boldsymbol{n}'}\rangle = \varepsilon(y)\tau(\hat{e}_{\boldsymbol{i}}^{(\boldsymbol{n})}, \hat{f}_{\boldsymbol{i}}^{\boldsymbol{n}'}) \\ &= \varepsilon(y)\delta_{\boldsymbol{n}\boldsymbol{n}'}\prod_{t=1}^m c_{q_{i_t}}(n_t),\end{aligned}$$

where

$$c_q(n) = [n]!q^{-n(n-1)/2}(q - q^{-1})^{-n}. \qquad \square$$

6.4. Set $m_0 = \ell(w_0)$. In this subsection we consider the case $w = w_0$.

Lemma 6.4. *Let $i \in I$ and define $i' \in I$ by $w_0\alpha_i = -\alpha_{i'}$. Then we have*

$$\Gamma_{w_0}(e_i)=\frac{1}{1-q_i^2}(\sigma^{w_0}_{-\varpi_{i'}}e_i)(\sigma^{w_0}_{-\varpi_{i'}})^{-1}\star 1.$$

Proof. It is sufficient to show

$$\Psi_{w_0}(e_i)=\frac{1}{1-q_i^2}\Xi_{w_0}((\sigma^{w_0}_{-\varpi_{i'}}e_i)(\sigma^{w_0}_{-\varpi_{i'}})^{-1}\star 1).$$

Set $v^*=v^*_{-\varpi_{i'}}\dot{T}_{w_0}^{-1}$, $v=\dot{T}_{w_0}v_{-\varpi_{i'}}$, so that $v^*\in V^*(-\varpi_{i'})_{\varpi_i}$, $v\in V(-\varpi_{i'})_{\varpi_i}$ with $\langle v^*,v\rangle=1$. For $t\in U^0$, $y\in U^-\cap\hat{T}_{s_i}U^-$, $p\geq 0$ we have

$$\begin{aligned}\langle\Xi'_{w_0}(\sigma^{w_0}_{-\varpi_{i'}}e_i),tyf_i^p\rangle&=\langle\dot{T}_{w_0}(\sigma^{w_0}_{-\varpi_{i'}}e_i),S(tyf_i^p)\rangle\\&=\langle v^*_{-\varpi_{i'}}\dot{T}_{w_0}^{-1}e_i,S(tyf_i^p)\dot{T}_{w_0}v_{-\varpi_{i'}}\rangle=\langle v^*e_i,S(tyf_i^p)v\rangle\\&=\chi_{-\varpi_i}(t)\langle v^*e_i,S(f_i^p)S(y)v\rangle=-\chi_{-\varpi_i}(t)\varepsilon(y)\delta_{p1}\langle v^*e_i,f_ik_iv\rangle\\&=-\chi_{-\varpi_i}(t)\varepsilon(y)\delta_{p1}q_i\langle v^*e_i,f_iv\rangle\\&=-\chi_{-\varpi_i}(t)\varepsilon(y)\delta_{p1}q_i\left\langle v^*,\frac{k_i-k_i^{-1}}{q_i-q_i^{-1}}v\right\rangle=-\chi_{-\varpi_i}(t)\varepsilon(y)\delta_{p1}q_i.\end{aligned}$$

Hence

$$\begin{aligned}&\langle\Xi_{w_0}((\sigma^{w_0}_{-\varpi_{i'}}e_i)(\sigma^{w_0}_{-\varpi_{i'}})^{-1}\star 1),tyf_i^p\rangle=\langle\chi_{\varpi_i}\Xi'_{w_0}(\sigma^{w_0}_{-\varpi_{i'}}e_i),tyf_i^p\rangle\\&\quad=\sum_{(t)}\chi_{\varpi_i}(t_{(0)})(-\chi_{-\varpi_i}(t_{(1)})\varepsilon(y)\delta_{p1}q_i)=-\varepsilon(t)\varepsilon(y)\delta_{p1}q_i.\end{aligned}$$

On the other hand by (2.8) and Proposition 2.5 we have

$$\langle\Psi_{w_0}(e_i),tyf_i^p\rangle=\tau(e_i,tyf_i^p)=\frac{1}{q_i-q_i^{-1}}\varepsilon(t)\varepsilon(y)\delta_{p1}$$

for $t\in U^0$, $y\in U^-\cap\hat{T}_{s_i}U^-$, $p\geq 0$. □

Proposition 6.5. *For $\boldsymbol{i}\in\mathcal{I}_{w_0}$, $i\in I$, $\boldsymbol{n}\in(\mathbb{Z}_{\geq 0})^{m_0}$ write*

$$\hat{e}_{\boldsymbol{i}}^{(\boldsymbol{n})}e_i=\sum_{\boldsymbol{n}'}c_{\boldsymbol{n}\boldsymbol{n}'}\hat{e}_{\boldsymbol{i}}^{(\boldsymbol{n}')}.$$

Then we have

$$\begin{aligned}&\left\{\frac{1}{1-q_i^2}(\sigma^{w_0}_{-\varpi_{i'}}e_i)(\sigma^{w_0}_{-\varpi_{i'}})^{-1}\right\}p_{\boldsymbol{i}}(\boldsymbol{n})\\&\quad=\sum_{\boldsymbol{n}'}c_{\boldsymbol{n}\boldsymbol{n}'}\frac{d_{\boldsymbol{i}}(\boldsymbol{n})}{d_{\boldsymbol{i}}(\boldsymbol{n}')}p_{\boldsymbol{i}}(\boldsymbol{n}')\left(\chi_{i_1}^{\langle\beta^\vee_{\boldsymbol{i},1},\gamma_{\boldsymbol{i},\boldsymbol{n},1}-\gamma_{\boldsymbol{i},\boldsymbol{n}',1}\rangle}\otimes\cdots\otimes\chi_{i_{m_0}}^{\langle\beta^\vee_{\boldsymbol{i},m_0},\gamma_{\boldsymbol{i},\boldsymbol{n},m_0}-\gamma_{\boldsymbol{i},\boldsymbol{n}',m_0}\rangle}\right),\end{aligned}$$

where i' is as in Lemma 6.4.

Proof. Set $\varphi=(1/(1-q_i^2))(\sigma^{w_0}_{-\varpi_{i'}}e_i)(\sigma^{w_0}_{-\varpi_{i'}})^{-1}\in\mathcal{S}_{w_0}^{-1}\mathbb{C}_q[G/N^-]$ so that $\Gamma_{w_0}(e_i)=\varphi\star 1$. For $\boldsymbol{n}'\in(\mathbb{Z}_{\geq 0})^{m_0}$ take $\varphi_{\boldsymbol{n}'}\in\mathcal{S}_{w_0}^{-1}\mathbb{C}_q[G/N^-]$ such that $\Gamma_{w_0}(\hat{e}_{\boldsymbol{i}}^{(\boldsymbol{n}')})=$

$\varphi_{\boldsymbol{n}'} \star 1$. Then we have

$$\begin{aligned}\Xi_{w_0}(\varphi\varphi_{\boldsymbol{n}} \star 1) &= \Xi_{w_0}(\varphi_n \star 1)\Xi_{w_0}(\varphi \star 1) = \Xi_{w_0}(\Gamma_{w_0}(\hat{e}_{\boldsymbol{i}}^{(\boldsymbol{n})}))\Xi_{w_0}(\Gamma_{w_0}(e_i)) \\ &= \Psi_{w_0}(\hat{e}_{\boldsymbol{i}}^{(\boldsymbol{n})})\Psi_{w_0}(e_i) = \Psi_{w_0}(\hat{e}_{\boldsymbol{i}}^{(\boldsymbol{n})}e_i) = \sum_{\boldsymbol{n}'} c_{\boldsymbol{n}\boldsymbol{n}'}\Psi_{w_0}(\hat{e}_{\boldsymbol{i}}^{(\boldsymbol{n}')}) \\ &= \sum_{\boldsymbol{n}'} c_{\boldsymbol{n}\boldsymbol{n}'}\Xi_{w_0}(\Gamma_{w_0}(\hat{e}_{\boldsymbol{i}}^{(\boldsymbol{n}')})) = \sum_{\boldsymbol{n}'} c_{\boldsymbol{n}\boldsymbol{n}'}\Xi_{w_0}(\varphi_{\boldsymbol{n}'} \star 1).\end{aligned}$$

Hence

$$\varphi\varphi_{\boldsymbol{n}} \star 1 = \sum_{\boldsymbol{n}'} c_{\boldsymbol{n}\boldsymbol{n}'}\varphi_{\boldsymbol{n}'} \star 1.$$

It follows that

$$\varphi\Gamma_{w_0}(\hat{e}_{\boldsymbol{i}}^{(\boldsymbol{n})}) = \sum_{\boldsymbol{n}'} c_{\boldsymbol{n}\boldsymbol{n}'}\Gamma_{w_0}(\hat{e}_{\boldsymbol{i}}^{(\boldsymbol{n}')}).$$

Therefore, the assertion is a consequence of Theorem 6.3. □

7. Specialization.

7.1. We denote by $\mathrm{Hom}_{\mathrm{alg}}(\mathbb{C}_q[H], \mathbb{F})$ the set of algebra homomorphisms from $\mathbb{C}_q[H]$ to $\mathbb{F}$. It is endowed with a structure of commutative group via the multiplication

$$(\theta_1\theta_2)(\chi) = \sum_{(\chi)} \theta_1(\chi_{(0)})\theta_2(\chi_{(1)}) \qquad (\theta_1, \theta_2 \in \mathrm{Hom}_{\mathrm{alg}}(\mathbb{C}_q[H], \mathbb{F}), \quad \chi \in \mathbb{C}_q[H]).$$

The identity element is given by ε, and the inverse of θ is given by $\theta \circ S$.

For $\theta \in \mathrm{Hom}_{\mathrm{alg}}(\mathbb{C}_q[H], \mathbb{F})$ we denote by $\mathbb{F}_\theta = \mathbb{F}1_\theta$ the corresponding left $\mathbb{C}_q[H]$-module. For $\theta \in \mathrm{Hom}_{\mathrm{alg}}(\mathbb{C}_q[H], \mathbb{F})$ and $w \in W$ we define an $\mathcal{S}_w^{-1}\mathbb{C}_q[G]$-module $\mathcal{M}_w^\theta$ by

$$\mathcal{M}_w^\theta = \mathcal{M}_w \otimes_{\mathbb{C}_q[H]} \mathbb{F}_\theta.$$

Set $1_w^\theta = (1 \star 1) \otimes 1_\theta \in \mathcal{M}_w^\theta$. We have

$$\mathcal{M}_w^\theta \cong \mathcal{S}_w^{-1}\mathbb{C}_q[G] \otimes_{\mathcal{S}_w^{-1}\mathbb{C}_q[N_w^-\backslash G]} \mathbb{F} \cong \mathbb{C}_q[G] \otimes_{\mathbb{C}_q[N_w^-\backslash G]} \mathbb{F},$$

where $\mathcal{S}_w^{-1}\mathbb{C}_q[N_w^-\backslash G] \to \mathbb{F}$ is given by $\theta \circ \eta_w$.

Note that we have a decomposition

$$\mathbb{C}_q[N_w^-\backslash G] = \bigoplus_{\lambda \in P} \mathbb{C}_q[N_w^-\backslash G]_\lambda$$

with

$$\mathbb{C}_q[N_w^-\backslash G]_\lambda = \{\varphi \in \mathbb{C}_q[N_w^-\backslash G] \mid t\varphi = \chi_\lambda(t)\varphi \quad (t \in U^0)\}.$$

We have

$$(\theta \circ \eta_w)(\varphi) = \varepsilon(\varphi \dot{T}_w)\theta(\chi_\lambda) \qquad (\lambda \in P, \varphi \in \mathbb{C}_q[N_w^- \backslash G]_\lambda). \tag{7.1}$$

Indeed, for $t \in U^0$ we have

$$\langle \eta_w(\varphi), t\rangle = \langle \varphi \dot{T}_w, t\rangle = \langle (t\varphi)\dot{T}_w, 1\rangle = \chi_\lambda(t)\varepsilon(\varphi \dot{T}_w),$$

and hence $\eta_w(\varphi) = \varepsilon(\varphi \dot{T}_w)\chi_\lambda$. Therefore, $(\theta \circ \eta_w)(\varphi) = \varepsilon(\varphi \dot{T}_w)\theta(\chi_\lambda)$.

The $\mathbb{C}_q[H]$-module $\mathbb{F}_\theta$ can also be regarded as a $\mathbb{C}_q[G]$-module via the canonical Hopf algebra homomorphism $r_H^G : \mathbb{C}_q[G] \to \mathbb{C}_q[H]$. We denote this $\mathbb{C}_q[G]$-module by $\mathbb{F}_\theta^G = \mathbb{F}1_\theta^G$.

Proposition 7.1. *Assume that we are given two algebra homomorphisms $\theta_i : \mathbb{C}_q[H] \to \mathbb{F}$ $(i = 1, 2)$. Then as a $\mathbb{C}_q[G]$-module we have*

$$\mathcal{M}_w^{\theta_1} \cong \mathcal{M}_w^{\theta_2} \otimes_{\mathbb{F}} \mathbb{F}^G_{\theta_1(\theta_2 \circ S)}.$$

Here, the right side is regarded as a $\mathbb{C}_q[G]$-module via the comultiplication $\Delta : \mathbb{C}_q[G] \to \mathbb{C}_q[G] \otimes \mathbb{C}_q[G]$.

Proof. Let $\lambda \in P$, $\varphi \in \mathbb{C}_q[N_w^- \backslash G]_\lambda$. For $u \in U$, $t \in U^0$ we have

$$\langle \Delta(\varphi), u \otimes t\rangle = \langle \varphi, ut\rangle = \langle t\varphi, u\rangle = \chi_\lambda(t)\langle \varphi, u\rangle,$$

and hence $(\mathrm{id} \otimes r_H^G)(\varphi) = \varphi \otimes \chi_\lambda$. It follows that

$$\begin{aligned}
\varphi(1_w^{\theta_2} \otimes 1^G_{\theta_1(\theta_2 \circ S)}) &= (\varphi 1_w^{\theta_2}) \otimes (\chi_\lambda 1^G_{\theta_1(\theta_2 \circ S)}) \\
&= \varepsilon(\varphi \dot{T}_w)\theta_2(\chi_\lambda)(\theta_1(\theta_2 \circ S))(\chi_\lambda) 1_w^{\theta_2} \otimes 1^G_{\theta_1(\theta_2 \circ S)} \\
&= \varepsilon(\varphi \dot{T}_w)(\theta_2\theta_1(\theta_2 \circ S))(\chi_\lambda) 1_w^{\theta_2} \otimes 1^G_{\theta_1(\theta_2 \circ S)} \\
&= \varepsilon(\varphi \dot{T}_w)\theta_1(\chi_\lambda) 1_w^{\theta_2} \otimes 1^G_{\theta_1(\theta_2 \circ S)} = (\theta_1 \circ \eta_w)(\varphi) 1_w^{\theta_2} \otimes 1^G_{\theta_1(\theta_2 \circ S)}.
\end{aligned}$$

Hence there exists uniquely a homomorphism $F_{\theta_2}^{\theta_1} : \mathcal{M}_w^{\theta_1} \to \mathcal{M}_w^{\theta_2} \otimes_{\mathbb{F}} \mathbb{F}^G_{\theta_1(\theta_2 \circ S)}$ of $\mathbb{C}_q[G]$-modules sending $1_w^{\theta_1}$ to $1_w^{\theta_2} \otimes 1^G_{\theta_1(\theta_2 \circ S)}$. Similarly, we have a homomorphism $F_{\theta_1}^{\theta_2} : \mathcal{M}_w^{\theta_2} \to \mathcal{M}_w^{\theta_1} \otimes_{\mathbb{F}} \mathbb{F}^G_{\theta_2(\theta_1 \circ S)}$ of $\mathbb{C}_q[G]$-modules sending $1_w^{\theta_2}$ to $1_w^{\theta_1} \otimes 1^G_{\theta_2(\theta_1 \circ S)}$. Applying $(\bullet) \otimes_{\mathbb{F}} \mathbb{F}^G_{\theta_1(\theta_2 \circ S)}$ to $F_{\theta_1}^{\theta_2}$ we obtain a homomorphism

$$\tilde{F}_{\theta_1}^{\theta_2} := F_{\theta_1}^{\theta_2} \otimes_{\mathbb{F}} \mathbb{F}^G_{\theta_1(\theta_2 \circ S)} : \mathcal{M}_w^{\theta_2} \otimes_{\mathbb{F}} \mathbb{F}^G_{\theta_1(\theta_2 \circ S)} \to \mathcal{M}_w^{\theta_1}$$

of $\mathbb{C}_q[G]$-modules sending $1_w^{\theta_2} \otimes 1^G_{\theta_1(\theta_2 \circ S)}$ to $1_w^{\theta_1}$. It remains to show $\tilde{F}_{\theta_1}^{\theta_2} \circ F_{\theta_2}^{\theta_1} = \mathrm{id}$ and $F_{\theta_2}^{\theta_1} \circ \tilde{F}_{\theta_1}^{\theta_2} = \mathrm{id}$. The first identity is a consequence of $(\tilde{F}_{\theta_1}^{\theta_2} \circ F_{\theta_2}^{\theta_1})(1_w^{\theta_1}) = 1_w^{\theta_1}$. The second one follows by applying $(\bullet) \otimes_{\mathbb{F}} \mathbb{F}^G_{\theta_1(\theta_2 \circ S)}$ to $\tilde{F}_{\theta_2}^{\theta_1} \circ F_{\theta_1}^{\theta_2} = \mathrm{id}$. □

7.2. In view of Proposition 7.1 we only consider the $\mathcal{S}_w^{-1}\mathbb{C}_q[G]$-module

$$\overline{\mathcal{M}}_w = \mathcal{M}_w^{\varepsilon} \tag{7.2}$$

in the following. For $\varphi \in \mathcal{S}_w^{-1}\mathbb{C}_q[G]$ we denote by $\overline{\varphi} \in \overline{\mathcal{M}}_w$ the image of $\varphi \star 1$ under $\mathcal{M}_w \to \overline{\mathcal{M}}_w$. Define $\overline{\eta}_w : \mathcal{S}_w^{-1}\mathbb{C}_q[N_w^-\backslash G] \to \mathbb{F}$ as the composite $\varepsilon \circ \eta_w$. Then we have

$$\overline{\mathcal{M}}_w \cong \mathcal{S}_w^{-1}\mathbb{C}_q[G] \otimes_{\mathcal{S}_w^{-1}\mathbb{C}_q[N_w^-\backslash G]} \mathbb{F} \cong \mathbb{C}_q[G] \otimes_{\mathbb{C}_q[N_w^-\backslash G]} \mathbb{F},$$

where $\mathcal{S}_w^{-1}\mathbb{C}_q[N_w^-\backslash G] \to \mathbb{F}$ is given by $\overline{\eta}_w$. Moreover, Θ_w induces a linear isomorphism

$$\overline{\Theta}_w : \overline{\mathcal{M}}_w \to U^+[\dot{T}_w^{-1}]^\bigstar$$

given by

$$\langle \overline{\Theta}_w(\overline{\varphi}), x\rangle = \langle \varphi \dot{T}_w, x\rangle \qquad (\varphi \in \mathbb{C}_q[G], x \in U^+[\dot{T}_w^{-1}]).$$

By the direct sum decomposition

$$U^+[\dot{T}_w^{-1}]^\bigstar = \bigoplus_{\gamma \in Q^+ \cap (-w^{-1}Q^+)} (U^+[\dot{T}_w^{-1}] \cap U_\gamma^+)^*$$

we have a direct sum decomposition

$$\overline{\mathcal{M}}_w = \bigoplus_{\gamma \in Q^+ \cap (-w^{-1}Q^+)} \overline{\mathcal{M}}_{w,\gamma}, \tag{7.3}$$

where

$$\overline{\mathcal{M}}_{w,\gamma} = (\overline{\Theta}_w)^{-1}((U^+[\dot{T}_w^{-1}] \cap U_\gamma^+)^*) \qquad (\gamma \in Q^+ \cap (-w^{-1}Q^+)).$$

Note that

$$\overline{\mathcal{M}}_{w,0} = \mathbb{F}\overline{1}.$$

LEMMA 7.2. *For $m \in \overline{\mathcal{M}}_{w,\gamma}$ and $\lambda \in P$ we have $\sigma_\lambda^w m = q^{-(\lambda,\gamma)} m$.*

PROOF. We may assume $\lambda \in P^-$. Take $\varphi \in \mathbb{C}_q[G]$ such that $\overline{\varphi} = m$. By Corollary 2.7 we have $\dot{T}_w(\sigma_\lambda^w \varphi) = (\dot{T}_w \sigma_\lambda^w)(\dot{T}_w \varphi)$, and hence for $x \in U^+[\dot{T}_w^{-1}]^\bigstar$ we have

$$\langle \overline{\Theta}_w(\sigma_\lambda^w m), x\rangle = \langle \dot{T}_w(\sigma_\lambda^w \varphi), \dot{T}_w(x)\rangle = \langle (\dot{T}_w \sigma_\lambda^w)(\dot{T}_w \varphi), \dot{T}_w(x)\rangle.$$

Assume $x \in U^+[\dot{T}_w^{-1}] \cap U_\delta^+$ with $\delta \in Q^+ \cap (-w^{-1}Q^+)$, and set $y = \dot{T}_w(x)$. Then we have $y \in (U_{w\delta}^-)k_{-w\delta}$ by (2.35). Hence

$$\begin{aligned}\langle \overline{\Theta}_w(\sigma_\lambda^w m), x\rangle &= \sum_{(y)} \langle \dot{T}_w \sigma_\lambda^w, y_{(0)}\rangle \langle \dot{T}_w \varphi, y_{(1)}\rangle = \langle \dot{T}_w \sigma_\lambda^w, k_{-w\delta}\rangle \langle \dot{T}_w \varphi, y\rangle \\ &= q^{-(\lambda,\delta)} \langle \overline{\Theta}_w(m), x\rangle.\end{aligned}$$

□

THEOREM 7.3. *The $\mathbb{C}_q[G]$-module $\overline{\mathcal{M}}_w$ is irreducible.*

PROOF. For any $\mathbb{C}_q[G]$-submodule N of $\overline{\mathcal{M}}_w$ we have

$$N=\bigoplus_{\gamma\in Q^+\cap(-w^{-1}Q^+)}(N\cap\overline{\mathcal{M}}_{w,\gamma})$$

by Lemma 7.2. Since $\overline{\mathcal{M}}_{w,0}$ is one-dimensional and generates the $\mathbb{C}_q[G]$-module $\overline{\mathcal{M}}_w$, it is sufficient to show $N\cap\overline{\mathcal{M}}_{w,0}\neq\{0\}$ for any non-zero $\mathbb{C}_q[G]$-submodule N of $\overline{\mathcal{M}}_w$. By definition the projection $\overline{\mathcal{M}}_w\to\overline{\mathcal{M}}_{w,0}$ with respect to (7.3) is given by $m\mapsto\langle\overline{\Theta}_w(m),1\rangle\overline{1}$, and hence it is sufficient to show that for any $m\in\overline{\mathcal{M}}_w\setminus\{0\}$ there exists some $\psi\in\mathbb{C}_q[G]$ such that $\langle\overline{\Theta}_w(\psi m),1\rangle\neq 0$. Take $\varphi\in\mathbb{C}_q[G]$ such that $\overline{\varphi}=m$. For $z\in U^+$ and $\lambda\in P^-$ we have

$$\langle\overline{\Theta}_w((z\sigma_\lambda^w)m),1\rangle=\langle\{(z\sigma_\lambda^w)\varphi\}\dot{T}_w,1\rangle.$$

Write

$$\Delta\dot{T}_w=(\dot{T}_w\otimes\dot{T}_w)\sum_j u_j^-\otimes u_j^+$$

(see Corollary 2.7). Then we have

$$\begin{aligned}\langle\overline{\Theta}_w((z\sigma_\lambda^w)m),1\rangle&=\sum_j\langle z\sigma_\lambda^w\dot{T}_wu_j^-,1\rangle\langle\varphi\dot{T}_wu_j^+,1\rangle\\&=\sum_j\langle v_\lambda^*u_j^-,zv_\lambda\rangle\langle\overline{\Theta}_w(m),u_j^+\rangle=\langle v_\lambda^*yz,v_\lambda\rangle\end{aligned}$$

with $y=\sum_j\langle\overline{\Theta}_w(m),u_j^+\rangle u_j^-\in U^-\setminus\{0\}$. Hence it is sufficient to show that for any $y\in U^-$ there exists some $\lambda\in P^-$ and $z\in U^+$ such that $\langle v_\lambda^*yz,v_\lambda\rangle\neq 0$. If $\lambda\in P^-$ is sufficiently small, then we have $v_\lambda^*y\neq 0$. Then the assertion is a consequence of the irreducibility of $V^*(\lambda)$ as a right U-module. □

7.3. For $i\in I$ we define a $\mathbb{C}_q[G]$-module $\overline{\mathcal{M}}_i$ by

$$\overline{\mathcal{M}}_i=\mathcal{M}_i\otimes_{\mathbb{C}_q[H(i)]}\mathbb{F}_\varepsilon. \tag{7.4}$$

It is an irreducible $\mathbb{C}_q[G]$-module with basis $\{\overline{p}_i(n)\mid n\in\mathbb{Z}_{\geq 0}\}$ satisfying

$$\begin{aligned}a_i\overline{p}_i(n)&=(1-q_i^{2n})\overline{p}_i(n-1), & b_i\overline{p}_i(n)&=q_i^n\overline{p}_i(n),\\ c_i\overline{p}_i(n)&=-\,q_i^{n+1}\overline{p}_i(n)i, & d_i\overline{p}_i(n)&=\overline{p}_i(n+1).\end{aligned}$$

Fix $w\in W$, and set $\ell(w)=m$. For $\boldsymbol{i}=(i_1,\dots,i_m)\in\mathcal{I}_w$ the isomorphism (5.6) induces an isomorphism

$$\overline{\mathcal{M}}_w\cong\overline{\mathcal{M}}_{i_1}\otimes\cdots\otimes\overline{\mathcal{M}}_{i_m} \tag{7.5}$$

of $\mathbb{C}_q[G]$-modules. For $\boldsymbol{n}=(n_1,\dots,n_m)\in(\mathbb{Z}_{\geq 0})^m$ we denote by $\overline{p}_{\boldsymbol{i}}(\boldsymbol{n})$ the element of $\overline{\mathcal{M}}_w$ corresponding to

$$\overline{p}_{i_1}(n_1)\otimes\cdots\otimes\overline{p}_{i_m}(n_m)\in\overline{\mathcal{M}}_{i_1}\otimes\cdots\otimes\overline{\mathcal{M}}_{i_m}$$

via the isomorphism (7.5).

By Theorem 6.3 we have the following.

THEOREM 7.4. *For $\boldsymbol{i}, \boldsymbol{j} \in \mathcal{I}_w$ we have*

$$\hat{e}_{\boldsymbol{j}}^{(\boldsymbol{n})} = \sum_{\boldsymbol{n}'} a_{\boldsymbol{n}'} \hat{e}_{\boldsymbol{i}}^{(\boldsymbol{n}')} \Longrightarrow \overline{p}_{\boldsymbol{j}}(\boldsymbol{n}) = \sum_{\boldsymbol{n}'} a_{\boldsymbol{n}'} \frac{d_{\boldsymbol{j},\boldsymbol{n}}}{d_{\boldsymbol{i},\boldsymbol{n}'}} \overline{p}_{\boldsymbol{i}}(\boldsymbol{n}'),$$

where $d_{\boldsymbol{i},\boldsymbol{n}}$ is as in Theorem 6.3.

REMARK 7.5. Theorem 7.4 for $w = w_0$ is the main result of Kuniba, Okado, Yamada ([**8**, Theorem 5]).

By Proposition 6.5 we have the following.

PROPOSITION 7.6. *For $\boldsymbol{i} \in \mathcal{I}_{w_0}$, $i \in I$, $\boldsymbol{n} \in (\mathbb{Z}_{\geq 0})^{m_0}$ write*

$$\hat{e}_{\boldsymbol{i}}^{(\boldsymbol{n})} e_i = \sum_{\boldsymbol{n}'} c_{\boldsymbol{n}\boldsymbol{n}'} \hat{e}_{\boldsymbol{i}}^{(\boldsymbol{n}')}.$$

Then we have

$$\left\{ \frac{1}{1-q_i^2} (\sigma_{-\varpi_{i'}}^{w_0}, e_i)(\sigma_{-\varpi_{i'}}^{w_0})^{-1} \right\} \overline{p}_{\boldsymbol{i}}(\boldsymbol{n}) = \sum_{\boldsymbol{n}'} c_{\boldsymbol{n}\boldsymbol{n}'} \frac{d_{\boldsymbol{i}}(\boldsymbol{n})}{d_{\boldsymbol{i}}(\boldsymbol{n}')} \overline{p}_{\boldsymbol{i}}(\boldsymbol{n}'),$$

where i' is as in Lemma 6.4.

REMARK 7.7. Proposition 7.6 is a conjecture of Kuniba, Okado, Yamada ([**8**, Conjecture 1]).

8. Comments.

8.1. In this paper we worked over the base field $\mathbb{F} = \mathbb{Q}(q)$; however, almost all of the arguments work equally well after minor modifications even when $\mathbb{F}$ is an arbitrary field of characteristic zero and $q_i^2 \neq 1$ for any $i \in I$. The only exception is Theorem 7.3, which states that $\overline{\mathcal{M}}_w$ is irreducible. For this result we need to assume that q is not a root of 1.

8.2. Let us consider generalization of our results to the case where $\mathfrak{g}$ is a symmetrizable Kac–Moody Lie algebra. We take $\mathbb{C}_q[G]$ to be the subspace of $U_q(\mathfrak{g})^*$ spanned by the matrix coefficients of integrable lowest weight modules (see [**5**]). Then $\mathbb{C}_q[G]$ is naturally endowed with an algebra structure. A problem is that the comultiplication $\Delta : \mathbb{C}_q[G] \to \mathbb{C}_q[G] \otimes \mathbb{C}_q[G]$ is not defined. Indeed $\Delta(\varphi)$ for $\varphi \in \mathbb{C}_q[G]$ turns out to be an infinite sum which belongs to a completion of $\mathbb{C}_q[G] \otimes \mathbb{C}_q[G]$. However, since we only consider the tensor product modules of type $\mathcal{M}_{i_1} \otimes \cdots \otimes \mathcal{M}_{i_m}$, what we actually need is the homomorphism of the form

$$(r_{G(i_1)}^G \otimes \cdots \otimes r_{G(i_m)}^G) \circ \Delta_{m-1} : \mathbb{C}_q[G] \to \mathbb{C}_q[G(i_1)] \otimes \cdots \otimes \mathbb{C}_q[G(i_m)]. \tag{8.1}$$

We can easily check that (8.1) is well-defined even in the Kac–Moody setting by showing that $(r^G_{G(i_1)} \otimes \cdots \otimes r^G_{G(i_m)}) \circ \Delta_{m-1}$ sends any element of $\mathbb{C}_q[G]$ to a finite sum inside $\mathbb{C}_q[G(i_1)] \otimes \cdots \otimes \mathbb{C}_q[G(i_m)]$. It is easily seen that all of the arguments in this paper also work in the setting where $\mathfrak{g}$ is a symmetrizable Kac–Moody Lie algebra.

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Toshiyuki TANISAKI
Department of Mathematics
Osaka City University
3-3-138, Sugimoto, Sumiyoshi-ku
Osaka 558-8585, Japan
E-mail: tanisaki@sci.osaka-cu.ac.jp