Integral transformation of Heun's equation and some applications

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Abstract. It is known that the Fuchsian differential equation which produces the sixth Painlevé equation corresponds to the Fuchsian differential equation with different parameters via Euler's integral transformation, and Heun's equation also corresponds to Heun's equation with different parameters, again via Euler's integral transformation. In this paper we study the correspondences in detail. After investigating correspondences with respect to monodromy, it is demonstrated that the existence of polynomial-type solutions corresponds to apparency of a singularity. For the elliptical representation of Heun's equation, correspondence with respect to monodromy implies isospectral symmetry. We apply the symmetry to finite-gap potentials and express the monodromy of Heun's equation with parameters which have not yet been studied.

1. Introduction.

The Gauss hypergeometric differential equation

$$z(1-z)\frac{d^2y}{dz^2} + (\gamma - (\alpha + \beta + 1)z)\frac{dy}{dz} - \alpha\beta y = 0. \tag{1.1}$$

is very famous both in physics and especially so in mathematics; it is a canonical form of the second-order Fuchsian differential equation with three singularities on the Riemann sphere $\mathbb{C} \cup \{\infty\}$. There are several important generalizations of Gauss hypergeometric differential equation. We now treat two examples, Heun's equation and the sixth Painlevé equation.

Heun's differential equation (or Heun's equation) is a canonical form of a second-order Fuchsian equation with four singularities, which is given by

$$\frac{d^2y}{dz^2} + \left(\frac{\epsilon_0}{z} + \frac{\epsilon_1}{z-1} + \frac{\epsilon_t}{z-t}\right)\frac{dy}{dz} + \frac{\alpha\beta z - q}{z(z-1)(z-t)}y = 0, \tag{1.2}$$

with the condition $\epsilon_0 + \epsilon_1 + \epsilon_t = \alpha + \beta + 1$ (see [**12**]). The exponents at $z = 0$ (resp. $z = 1$, $z = t$, $z = \infty$) are 0 and $1 - \epsilon_0$ (resp. 0 and $1 - \epsilon_1$, 0 and $1 - \epsilon_t$, α and β).

The sixth Painlevé system is a system of non-linear ordinary differential equations defined by

$$\frac{d\lambda}{dt} = \frac{\partial H}{\partial \mu}, \qquad \frac{d\mu}{dt} = -\frac{\partial H}{\partial \lambda}, \tag{1.3}$$

with the Hamiltonian

$$H = \frac{1}{t(t-1)} \left\{ \lambda(\lambda-1)(\lambda-t)\mu^2 - \{\theta_0(\lambda-1)(\lambda-t) + \theta_1\lambda(\lambda-t) \right. \tag{1.4}$$
$$\left. + (\theta_t-1)\lambda(\lambda-1)\} \mu + \kappa_1(\kappa_2+1)(\lambda-t) \right\}.$$

By eliminating μ in (1.3), we obtain the sixth Painlevé equation for λ which is a non-linear ordinary differential equation of order two in the independent variable t. It is known that the sixth Painlevé system is related to monodromy preserving deformations of certain Fuchsian differential equations. Let $\lambda \notin \{0,1,t,\infty\}$ and $D_{y_1}(\theta_0,\theta_1,\theta_t,\theta_\infty;\lambda,\mu)$ be the second-order linear differential equation given by

$$\frac{d^2 y_1(z)}{dz^2} + \left(\frac{1-\theta_0}{z} + \frac{1-\theta_1}{z-1} + \frac{1-\theta_t}{z-t} - \frac{1}{z-\lambda} \right) \frac{dy_1(z)}{dz}$$
$$+ \left(\frac{\kappa_1(\kappa_2+1)}{z(z-1)} + \frac{\lambda(\lambda-1)\mu}{z(z-1)(z-\lambda)} - \frac{t(t-1)H}{z(z-1)(z-t)} \right) y_1(z) = 0, \tag{1.5}$$
$$\kappa_1 = (\theta_\infty - \theta_0 - \theta_1 - \theta_t)/2, \quad \kappa_2 = -(\theta_\infty + \theta_0 + \theta_1 + \theta_t)/2,$$

where H is given as in (1.4). Then (1.5) is a Fuchsian differential equation with five singularities $\{0,1,t,\lambda,\infty\}$ on the Riemann sphere. The exponents at $z = p$ ($p \in \{0,1,t\}$) (resp. $z = \lambda$, $z = \infty$) are 0 and θ_p (resp. 0 and 2, κ_1 and $\kappa_2 + 1$), and it follows from the condition given by (1.4) that the singularity $z = \lambda$ is apparent. The sixth Painlevé system (1.3) is derived from a monodromy preserving deformation of (1.5) (for details, see [4]), and the function $y_1(z)$ is obtained from a first order 2×2 Fuchsian differential system with four singularities $\{0,1,t,\infty\}$, denoted by $D_Y(\theta_0,\theta_1,\theta_t,\theta_\infty;\lambda,\mu;k)$ in [22], [23].

Heun's equation and the Fuchsian differential equation $D_{y_1}(\theta_0,\theta_1,\theta_t,\theta_\infty;\lambda,\mu)$ admit integral transformations. We fix a base point o for the integrals in the complex plane $\mathbb{C}$ appropriately. Let $f(z)$ be a holomorphic function locally defined around $z = o$ and $f^\gamma(z)$ be the function analytically continued along a cycle γ whose base point is o. Define

$$\langle \gamma, f \rangle_\kappa = \int_\gamma f(w)(z-w)^\kappa dw. \tag{1.6}$$

This is called Euler's integral transformation (or an Euler transformation). Let p be an singularity of the function $f(w)$ in the Riemann sphere $\mathbb{C} \cup \{\infty\}$, γ_p be a cycle on the Riemann sphere with variable w which starts from $w = o$, goes around $w = p$ in a counter-clockwise direction and ends at $w = o$, and $[\gamma_z, \gamma_p] = \gamma_z \gamma_p \gamma_z^{-1} \gamma_p^{-1}$ be the Pochhammer contour. The following proposition was obtained by Novikov [10], independently by Kazakov and Slavyanov [8], and also derived by considering an explicit form of a middle convolution of a 2×2 Fuchsian differential system [23]:

PROPOSITION 1.1 ([**8**], [**10**], [**23**]).   *If $y_1(z)$ is a solution of $D_{y_1}(\theta_0, \theta_1, \theta_t, \theta_\infty; \lambda, \mu)$, then the function*

$$\tilde{y}(z) = \langle [\gamma_z, \gamma_p], y_1 \rangle_{\kappa_2 - 1} = \int_{[\gamma_z, \gamma_p]} y_1(w)(z - w)^{\kappa_2 - 1} dw, \qquad (1.7)$$

satisfies $D_{y_1}(\tilde{\theta}_0, \tilde{\theta}_1, \tilde{\theta}_t, \tilde{\theta}_\infty; \tilde{\lambda}, \tilde{\mu})$ for $p \in \{0, 1, t, \infty\}$, where

$$\kappa_2 = -(\theta_\infty + \theta_0 + \theta_1 + \theta_t)/2, \quad \tilde{\theta}_p = \kappa_2 + \theta_p \quad (p = 0, 1, t, \infty), \qquad (1.8)$$

$$\tilde{\lambda} = \lambda - \frac{\kappa_2}{\mu - \frac{\theta_0}{\lambda} - \frac{\theta_1}{\lambda - 1} - \frac{\theta_t}{\lambda - t}}, \quad \tilde{\mu} = \frac{\kappa_2 + \theta_0}{\tilde{\lambda}} + \frac{\kappa_2 + \theta_1}{\tilde{\lambda} - 1} + \frac{\kappa_2 + \theta_t}{\tilde{\lambda} - t} + \frac{\kappa_2}{\lambda - \tilde{\lambda}}.$$

Kazakov and Slavyanov established an integral transformation for solutions of Heun's equation in [**7**], and it was also obtained by taking suitable limits in Proposition 1.1, which was discussed in [**23**] by considering the relationship with the space of initial conditions of the sixth Painlevé equation.

PROPOSITION 1.2 ([**7**], [**23**]).   *Let α, β, ϵ_0, ϵ_1, ϵ_t be the parameters in Heun's equation* (1.12). *Set $\eta = \alpha$ or β and*

$$\epsilon'_0 = \epsilon_0 - \eta + 1, \ \epsilon'_1 = \epsilon_1 - \eta + 1, \ \epsilon'_t = \epsilon_t - \eta + 1, \qquad (1.9)$$
$$\{\alpha', \beta'\} = \{2 - \eta, \alpha + \beta - 2\eta + 1\}, \ q' = q + (1 - \eta)(\epsilon_t + \epsilon_1 t + (\epsilon_0 - \eta)(t + 1)).$$

Let $v(w)$ be a solution of

$$\frac{d^2 v}{dw^2} + \left(\frac{\epsilon'_0}{w} + \frac{\epsilon'_1}{w - 1} + \frac{\epsilon'_t}{w - t} \right) \frac{dv}{dw} + \frac{\alpha' \beta' w - q'}{w(w - 1)(w - t)} v = 0. \qquad (1.10)$$

Then the function

$$y(z) = \langle [\gamma_z, \gamma_p], v \rangle_{-\eta} = \int_{[\gamma_z, \gamma_p]} v(w)(z - w)^{-\eta} dw \qquad (1.11)$$

is a solution of

$$\frac{d^2 y}{dz^2} + \left(\frac{\epsilon_0}{z} + \frac{\epsilon_1}{z - 1} + \frac{\epsilon_t}{z - t} \right) \frac{dy}{dz} + \frac{\alpha \beta z - q}{z(z - 1)(z - t)} y = 0, \qquad (1.12)$$

for $p \in \{0, 1, t, \infty\}$.

Note that the inverse correspondence of the parameters is given by setting $\eta = \alpha$ or β and

$$\eta = 2 - \eta', \ \epsilon_0 = \epsilon'_0 - \eta' + 1, \ \epsilon_1 = \epsilon'_1 - \eta' + 1, \ \epsilon_t = \epsilon'_t - \eta' + 1, \qquad (1.13)$$
$$\{\alpha, \beta\} = \{2 - \eta', \alpha' + \beta' - 2\eta' + 1\}, \ q = q' + (1 - \eta')(\epsilon'_t + \epsilon'_1 t + (\epsilon'_0 - \eta')(t + 1)).$$

In this paper, Euler's integral transformations given by (1.7), (1.11) are considered. If we have a solution of the differential equation $D_{y_1}(\theta_0, \theta_1, \theta_t, \theta_\infty; \lambda, \mu)$ (resp. Heun's

differential equation (1.10)), then we may study the solution of $D_{y_1}(\tilde{\theta}_0, \tilde{\theta}_1, \tilde{\theta}_t, \tilde{\theta}_\infty; \tilde{\lambda}, \tilde{\mu})$ (resp. (1.12)) by means of Euler's integral transformations in (1.7) (resp. (1.11)). We apply this strategy for the case where the differential equation $D_{y_1}(\theta_0, \theta_1, \theta_t, \theta_\infty; \lambda, \mu)$ (resp. (1.10)) has a polynomial-type solution. Then it is shown that one of the singularities $\{0, 1, t, \infty\}$ of the differential equation $D_{y_1}(\tilde{\theta}_0, \tilde{\theta}_1, \tilde{\theta}_t, \tilde{\theta}_\infty; \tilde{\lambda}, \tilde{\mu})$ (resp. (1.12)) turns out to be apparent, and the inverse statement also holds (see Theorems 4.3 and 4.2). As a by-product, we have integral representations of solutions of Heun's equation for which one of the singularities $\{0, 1, t, \infty\}$ is apparent (see Theorems 5.2 and 5.4). We also investigate properties of monodromy by means of integral transformations, which are used for the study of solutions.

It is known that Heun's equation has an elliptical representation. Let $\wp(x)$ be the Weierstrass doubly periodic function with periods $(2\omega_1, 2\omega_3)$, $\omega_0(= 0)$, ω_1, $\omega_2(= -\omega_1 - \omega_3)$, ω_3 be the half-periods and $e_i = \wp(\omega_i)$ $(i = 1, 2, 3)$. Heun's equation (1.2) is transformed to

$$\left(-\frac{d^2}{dx^2} + \sum_{i=0}^{3} l_i(l_i + 1)\wp(x + \omega_i) - E \right) f(x) = 0, \qquad (1.14)$$

by setting $z = (\wp(x) - e_1)/(e_2 - e_1)$, $t = (e_3 - e_1)/(e_2 - e_1)$. For details see section 6. Then the integral transformation of (1.11) provides a correspondence of (1.14) with a different parameter described in Proposition 6.2. For the elliptical representation, the invariance of monodromy by the integral transformation with respect to the shift of a period is remarkable. For details see Theorem 6.4. We also obtain correspondences of solutions expressed by quasi-solvability (existence of a polynomial-type solution) and apparency of one of the singularities $\{0, \omega_1, \omega_2, \omega_3\}$. We apply the integral transformation for the case where Heun's equation has the finite-gap property, i.e. the case where $l_0, l_1, l_2, l_3 \in \mathbb{Z}$. For the case $l_0, l_1, l_2, l_3 \in \mathbb{Z}$ we can calculate the monodomy in principle for all E by means of hyperelliptic integrals [18] and by the Hermite–Krichever Ansatz [19]. By applying monodromy invariance, we can calculate the monodromy of Heun's equation for the case $l_0, l_1, l_2, l_3 \in \mathbb{Z} + 1/2$ and $l_0 + l_1 + l_2 + l_3 \in 2\mathbb{Z} + 1$, which have not been studied previously.

This paper is organized as follows: In section 2, we investigate the transformation of the monodromy induced by Euler's integral transformation. In section 3, we obtain some properties of solutions and monodromy of the Fuchsian differential equations $D_{y_1}(\theta_0, \theta_1, \theta_t, \theta_\infty; \lambda, \mu)$, $D_{y_1}(\tilde{\theta}_0, \tilde{\theta}_1, \tilde{\theta}_t, \tilde{\theta}_\infty; \tilde{\lambda}, \tilde{\mu})$ and Heun's equations ((1.10), (1.12)). In section 4, we have correspondences of polynomial-type solutions and solutions such that one of the singularities is apparent. In section 5, we obtain integral representations of solutions of Heun's equation which have apparent singularities by using polynomial-type solutions. In section 6, we translate the results to the elliptical representation of Heun's equation. In section 7, we review results on finite-gap potentials and calculate the monodromy of the elliptical representation of Heun's equation for the case $l_0, l_1, l_2, l_3 \in \mathbb{Z} + 1/2$ and $l_0 + l_1 + l_2 + l_3 \in 2\mathbb{Z} + 1$. In the appendix we provide the technical details.

2. Monodromy and integral transformation.

In this section we investigate the transformation of the monodromy induced by Euler's integral transformation given by (1.6).

We review some facts about cycles in order to discuss the monodromy. Let $a, b \in \mathbb{C} \cup \{\infty\}$ $(a \neq b)$ and p_{ab} be a path linking a and b. We put the base point o of integrals in (1.6) on the left side of the path p_{ab} and the o does not coincide with the singularity of the differential equation or the point z. Let z be a point on p_{ab}. We consider deformations of the cycles γ_a, γ_b and γ_z in the w-plane as the point z turns around the singularity a or b anti-clockwise and the points z, a, b do not cross the cycle. As the point z turns around the singularity $w = a$ anti-clockwise, the cycle γ_a is deformed to $\gamma_a \gamma_z \gamma_a \gamma_z^{-1} \gamma_a^{-1}$, the cycle γ_z is deformed to $\gamma_a \gamma_z \gamma_a^{-1}$ and the cycle γ_b is not deformed (see Figure 1). As the point z turns around the singularity b anti-clockwise, the cycle γ_b is deformed to $\gamma_z \gamma_b \gamma_z^{-1}$, the cycle γ_z is deformed to $\gamma_z \gamma_b \gamma_z \gamma_b^{-1} \gamma_z^{-1}$ and the cycle γ_a is not deformed (see also Figure 1).
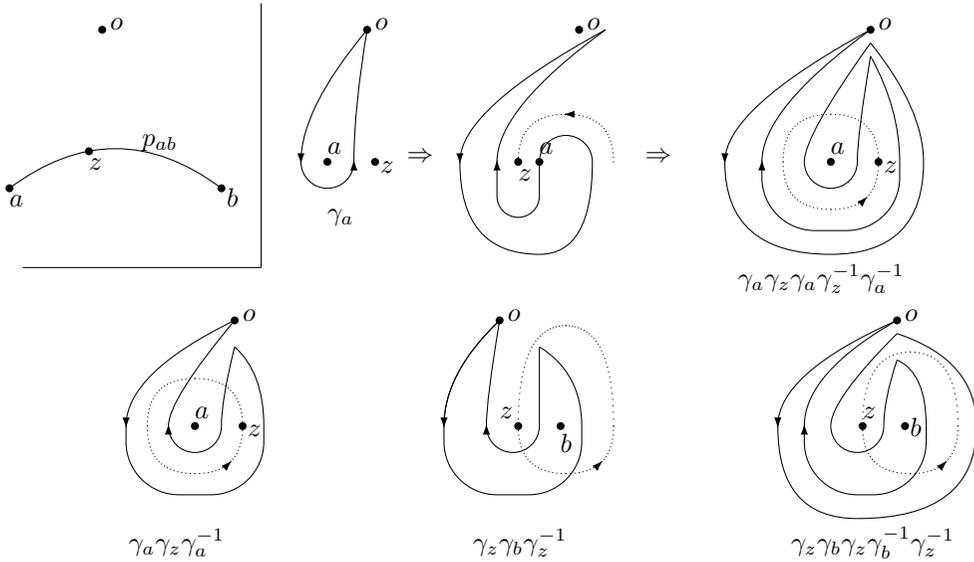

Figure 1. Deformation of the cycles.

The Euler's integral transformation by the Pochhammer contour admits the following expression,

$$\langle [\gamma_z, \gamma_a], f \rangle_\kappa = \int_{\gamma_z \gamma_a \gamma_z^{-1} \gamma_a^{-1}} f(w)(z-w)^\kappa dw = (e^{2\pi\sqrt{-1}\kappa} - 1)\langle \gamma_a, f \rangle_\kappa + \langle \gamma_z, f - f^{\gamma_a} \rangle_\kappa,$$

(2.1)

where we used the relations $0 = \langle \gamma_a, f \rangle_\kappa + \langle \gamma_a^{-1}, f^{\gamma_a} \rangle_\kappa$ and $0 = \langle \gamma_z, f \rangle_\kappa + e^{2\pi\sqrt{-1}\kappa}\langle \gamma_z^{-1}, f \rangle_\kappa$. The formula obtained by replacing a with b also holds true. Then analytic continuation of the transformed functions are calculated as follows;

PROPOSITION 2.1.

$$\langle[\gamma_z,\gamma_b],f\rangle_\kappa^{\gamma_a} = \langle[\gamma_z,\gamma_b],f\rangle_\kappa + \langle[\gamma_z,\gamma_a],f^{\gamma_b}\rangle_\kappa - \langle[\gamma_z,\gamma_a],f\rangle_\kappa, \tag{2.2}$$

$$\langle[\gamma_z,\gamma_a],f\rangle_\kappa^{\gamma_a} = e^{2\pi\sqrt{-1}\kappa}\langle[\gamma_z,\gamma_a],f^{\gamma_a}\rangle_\kappa,$$

$$\langle[\gamma_z,\gamma_a],f\rangle_\kappa^{\gamma_b} = \langle[\gamma_z,\gamma_a],f\rangle_\kappa + e^{2\pi\sqrt{-1}\kappa}\langle[\gamma_z,\gamma_b],f^{\gamma_a}\rangle_\kappa - e^{2\pi\sqrt{-1}\kappa}\langle[\gamma_z,\gamma_b],f\rangle_\kappa,$$

$$\langle[\gamma_z,\gamma_b],f\rangle_\kappa^{\gamma_b} = e^{2\pi\sqrt{-1}\kappa}\langle[\gamma_z,\gamma_b],f^{\gamma_b}\rangle_\kappa.$$

PROOF.   We show the first equality. It follows from (2.1) that

$$\langle[\gamma_z,\gamma_b],f\rangle_\kappa^{\gamma_a} = \langle[\gamma_a\gamma_z\gamma_a^{-1},\gamma_b],f\rangle_\kappa = \langle\gamma_a,f\rangle_\kappa + \langle\gamma_z,f^{\gamma_a}\rangle_\kappa - e^{2\pi\sqrt{-1}\kappa}\langle\gamma_a,f\rangle_\kappa$$
$$+ e^{2\pi\sqrt{-1}\kappa}\langle\gamma_b,f\rangle_\kappa + e^{2\pi\sqrt{-1}\kappa}\langle\gamma_a,f^{\gamma_b}\rangle_\kappa - \langle\gamma_z,f^{\gamma_b\gamma_a}\rangle_\kappa - \langle\gamma_a,f^{\gamma_b}\rangle_\kappa - \langle\gamma_b,f\rangle_\kappa. \tag{2.3}$$

By expanding the right hand side of the first equality, we have the same expression. The second formula follows from

$$\langle[\gamma_z,\gamma_a],f\rangle_\kappa^{\gamma_a} = \langle\gamma_a\gamma_z[\gamma_z,\gamma_a](\gamma_a\gamma_z)^{-1},f\rangle_\kappa = e^{2\pi\sqrt{-1}\kappa}\langle[\gamma_z,\gamma_a],f^{\gamma_a}\rangle_\kappa.$$

The other formulas are shown similarly.   □

From now on, we assume that the function $y(w)$ is a solution of a second-order differential equation, the two points $a,b \in \mathbb{C}$ are regular singularities of the differential equation whose exponents are 0 and θ_a, 0 and θ_b respectively, where the case of Heun's equation and that of the differential equation $D_{y_1}(\theta_0,\theta_1,\theta_t,\theta_\infty;\lambda,\mu)$ are included. We put the base point o of integrals in (1.7) on the left side of the path p_{ab}. Let $y^{(2)}(w)$ be a solution of the second-order differential equation such that the functions $y(w)$, $y^{(2)}(w)$ form a basis of solutions of the differential equation, and we denote the monodromy matrices around the singularity $w = a$ by

$$(y(w)^{\gamma_a}, y^{(2)}(w)^{\gamma_a}) = (y(w), y^{(2)}(w))\begin{pmatrix} a'_{11} & a'_{12} \\ a'_{21} & a'_{22} \end{pmatrix} = (y(w), y^{(2)}(w))M'^{(a)}. \tag{2.4}$$

The eigenvalues of the monodromy matrix $M'^{(a)}$ in (2.4) are 1 and $e^{2\pi\sqrt{-1}\theta_a}$. If one of the exponents around the singularities $w = a$ is zero, then there exists a non-zero solution $f(w)$ that is holomorphic about $w = a$ and it follows from (2.1) that the function $\langle[\gamma_z,\gamma_a],f\rangle_\kappa$ is zero. Let $a'_1 y(w) + a'_2 y^{(2)}(w)$ be a non-zero holomorphic solution of the differential equation about $w = a$. Then we have

$$a'_1\langle[\gamma_z,\gamma_a],y\rangle_\kappa + a'_2\langle[\gamma_z,\gamma_a],y^{(2)}\rangle_\kappa = 0. \tag{2.5}$$

Since $a'_1 y_1(w)^{\gamma_a} + a'_2 y^{(2)}(w)^{\gamma_a} = a'_1 y(w) + a'_2 y^{(2)}(w)$, we have

$$\begin{pmatrix} a'_{11} - 1 & a'_{12} \\ a'_{21} & a'_{22} - 1 \end{pmatrix}\begin{pmatrix} a'_1 \\ a'_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \begin{aligned} (a'_{11}-1)(a'_{22}-1) - a'_{12}a'_{21} = 0, \\ a'_{11} + a'_{22} = \operatorname{tr}M'^{(a)} = 1 + e^{2\pi\sqrt{-1}\theta_a}. \end{aligned} \tag{2.6}$$

We use similar notations for the singularity $w = b$. If $\langle[\gamma_z,\gamma_a],y\rangle_\kappa \neq 0$ and $\langle[\gamma_z,\gamma_b],y\rangle_\kappa \neq 0$, then $y(w)$ is not holomorphic about $w = a,b$ and $a'_2 \neq 0 \neq b'_2$. Combining with (2.2),

we have

$$\langle[\gamma_z,\gamma_a],y\rangle_\kappa^{\gamma_b} = \langle[\gamma_z,\gamma_a],b'_{11}y + b'_{21}y^{(2)}\rangle_\kappa = \left\langle[\gamma_z,\gamma_a],b'_{11}y - \frac{a'_1}{a'_2}b'_{21}y\right\rangle_\kappa, \qquad (2.7)$$

$$\langle[\gamma_z,\gamma_b],y\rangle_\kappa^{\gamma_a} = \langle[\gamma_z,\gamma_b],y\rangle_\kappa + \langle[\gamma_z,\gamma_a],y^{\gamma_b}\rangle_\kappa - \langle[\gamma_z,\gamma_a],y\rangle_\kappa$$

$$= \left(b'_{11} - 1 - \frac{a'_1}{a'_2}b'_{21}\right)\langle[\gamma_z,\gamma_a],y\rangle_\kappa + \langle[\gamma_z,\gamma_b],y\rangle_\kappa,$$

$$\langle[\gamma_z,\gamma_a],y\rangle_\kappa^{\gamma_a} = e^{2\pi\sqrt{-1}\kappa}\langle[\gamma_z,\gamma_a],a'_{11}y + a'_{21}y^{(2)}\rangle_\kappa$$

$$= e^{2\pi\sqrt{-1}\kappa}\left\langle[\gamma_z,\gamma_a],a'_{11}y - \frac{a'_1}{a'_2}a'_{21}y\right\rangle_\kappa$$

$$= e^{2\pi\sqrt{-1}\kappa}(a'_{11} + a'_{22} - 1)\langle[\gamma_z,\gamma_a],y\rangle_\kappa,$$

$$\langle[\gamma_z,\gamma_a],y\rangle_\kappa^{\gamma_b} = \langle[\gamma_z,\gamma_a],y\rangle_\kappa + e^{2\pi\sqrt{-1}\kappa}\left(a'_{11} - 1 - \frac{b'_1}{b'_2}a'_{21}\right)\langle[\gamma_z,\gamma_b],y\rangle_\kappa,$$

$$\langle[\gamma_z,\gamma_b],y\rangle_\kappa^{\gamma_b} = e^{2\pi\sqrt{-1}\kappa}(b'_{11} + b'_{22} - 1)\langle[\gamma_z,\gamma_b],y\rangle_\kappa.$$

For the case that the functions $\langle[\gamma_z,\gamma_a],y\rangle_\kappa$, $\langle[\gamma_z,\gamma_b],y\rangle_\kappa$ are linearly independent, we denote the monodromy matrix on the cycle γ_p with respect to $\langle[\gamma_z,\gamma_a],y\rangle_\kappa$, $\langle[\gamma_z,\gamma_b],y\rangle_\kappa$ by $M_{a,b}^{(p)}$, Then we have

$$M_{a,b}^{(a)} = \begin{pmatrix} e^{2\pi\sqrt{-1}\kappa}(a'_{11} + a'_{22} - 1) & b'_{11} - 1 - \frac{a'_1}{a'_2}b'_{21} \\ 0 & 1 \end{pmatrix}, \qquad (2.8)$$

$$M_{a,b}^{(b)} = \begin{pmatrix} 1 & 0 \\ e^{2\pi\sqrt{-1}\kappa}\left(a'_{11} - 1 - \frac{b'_1}{b'_2}a'_{21}\right) & e^{2\pi\sqrt{-1}\kappa}(b'_{11} + b'_{22} - 1) \end{pmatrix}.$$

Thus $\text{tr}(M_{a,b}^{(a)}) = 1 + e^{2\pi\sqrt{-1}(\kappa+\theta_a)}$, $\text{tr}(M_{a,b}^{(b)}) = 1 + e^{2\pi\sqrt{-1}(\kappa+\theta_b)}$, and we have

$$M_{a,b}^{(a)}M_{a,b}^{(b)} = e^{2\pi\sqrt{-1}\kappa}\begin{pmatrix} \substack{a'_{11}b'_{11}+a'_{12}b'_{21}+a'_{21}b'_{12}+a'_{22}b'_{22} \\ -b'_{11}-b'_{22}+1} & (b'_{11}+b'_{22}-1)(b'_{11}-1-\frac{a'_1}{a'_2}b'_{21}) \\ a'_{11}-1-\frac{b'_1}{b'_2}a'_{21} & b'_{11}+b'_{22}-1 \end{pmatrix},$$

$$(2.9)$$

$$\text{tr}(M_{a,b}^{(a)}M_{a,b}^{(b)}) = e^{2\pi\sqrt{-1}\kappa}\text{tr}(M'^{(a)}M'^{(b)}), \quad \det(M_{a,b}^{(a)}M_{a,b}^{(b)}) = e^{4\pi\sqrt{-1}\kappa}\det(M'^{(a)}M'^{(b)}).$$

Note that values of the trace and determinant are independent of the choice of basis.

We consider the case where $\langle[\gamma_z,\gamma_a],y\rangle_\kappa$, $\langle[\gamma_z,\gamma_b],y\rangle_\kappa$ are linearly dependent. We further assume that the point $w = c$ is also a regular singularity, $\langle[\gamma_z,\gamma_p],y\rangle_\kappa \neq 0$ for $p = a,b,c$ and $\langle[\gamma_z,\gamma_a],y\rangle_\kappa$, $\langle[\gamma_z,\gamma_c],y\rangle_\kappa$ are linearly independent. Then $\langle[\gamma_z,\gamma_b],y\rangle_\kappa = d\langle[\gamma_z,\gamma_a],y\rangle_\kappa$ for some $d \neq 0$. It follows from (2.7) that $\langle[\gamma_z,\gamma_a],y\rangle_\kappa^{\gamma_a} = a_{11}\langle[\gamma_z,\gamma_a],y\rangle_\kappa = (1 + a_{12}/d)\langle[\gamma_z,\gamma_a],y\rangle_\kappa$ and $\langle[\gamma_z,\gamma_a],y\rangle_\kappa^{\gamma_b} = b_{22}\langle[\gamma_z,\gamma_a],y\rangle_\kappa = (1 + db_{21})\langle[\gamma_z,\gamma_a],y\rangle_\kappa$, where $a_{11} = e^{2\pi\sqrt{-1}\kappa}(a'_{11} + a'_{22} - 1)$, $a_{12} = b'_{11} - 1 - b'_{21}a'_1/a'_2$, $b_{21} = e^{2\pi\sqrt{-1}\kappa}(a'_{11} - 1 - a'_{21}b'_1/b'_2)$, $b_{22} = e^{2\pi\sqrt{-1}\kappa}(b'_{11}+b'_{22}-1)$. Hence we have $a_{11} = 1 + a_{12}/d$, $b_{22} = 1 + db_{21}$ and $a_{12}b_{21} = (a_{11}-1)(b_{22}-1)$. By applying (2.6), (2.8), the relation $a_{12}b_{21} = (a_{11}-1)(b_{22}-1)$ can be written as

$$e^{4\pi\sqrt{-1}\kappa}\det(M'^{(a)}M'^{(b)}) - e^{2\pi\sqrt{-1}\kappa}\mathrm{tr}(M'^{(a)}M'^{(b)}) + 1 = 0. \tag{2.10}$$

Assume that the points a, b, c are located anticlockwise with respect to the point o. In a similar way as we obtained (2.8), we have

$$(\langle[\gamma_z,\gamma_b],y\rangle_\kappa^{\gamma_b}, \langle[\gamma_z,\gamma_c],y\rangle_\kappa^{\gamma_b}) = (\langle[\gamma_z,\gamma_b],y\rangle_\kappa, \langle[\gamma_z,\gamma_c],y\rangle_\kappa) \tag{2.11}$$
$$\begin{pmatrix} e^{2\pi\sqrt{-1}\kappa}(b'_{11}+b'_{22}-1) & c'_{11}-1-\frac{b'_1}{b'_2}c'_{21} \\ 0 & 1 \end{pmatrix},$$

$$(\langle[\gamma_z,\gamma_b],y\rangle_\kappa^{\gamma_c}, \langle[\gamma_z,\gamma_c],y\rangle_\kappa^{\gamma_c}) = (\langle[\gamma_z,\gamma_b],y\rangle_\kappa, \langle[\gamma_z,\gamma_c],y\rangle_\kappa)$$
$$\begin{pmatrix} 1 & 0 \\ e^{2\pi\sqrt{-1}\kappa}\left(b'_{11}-1-\frac{c'_1}{c'_2}b'_{21}\right) & e^{2\pi\sqrt{-1}\kappa}(c'_{11}+c'_{22}-1) \end{pmatrix},$$

$$(\langle[\gamma_z,\gamma_c],y\rangle_\kappa^{\gamma_c}, \langle[\gamma_z,\gamma_a],y\rangle_\kappa^{\gamma_c}) = (\langle[\gamma_z,\gamma_c],y\rangle_\kappa, \langle[\gamma_z,\gamma_a],y\rangle_\kappa)$$
$$\begin{pmatrix} e^{2\pi\sqrt{-1}\kappa}(c'_{11}+c'_{22}-1) & a'_{11}-1-\frac{c'_1}{c'_2}a'_{21} \\ 0 & 1 \end{pmatrix},$$

$$(\langle[\gamma_z,\gamma_c],y\rangle_\kappa^{\gamma_a}, \langle[\gamma_z,\gamma_a],y\rangle_\kappa^{\gamma_a}) = (\langle[\gamma_z,\gamma_c],y\rangle_\kappa, \langle[\gamma_z,\gamma_a],y\rangle_\kappa)$$
$$\begin{pmatrix} 1 & 0 \\ e^{2\pi\sqrt{-1}\kappa}\left(c'_{11}-1-\frac{a'_1}{a'_2}c'_{21}\right) & e^{2\pi\sqrt{-1}\kappa}(a'_{11}+a'_{22}-1) \end{pmatrix}.$$

By applying $\langle[\gamma_z,\gamma_b],y\rangle_\kappa = d\langle[\gamma_z,\gamma_a],y\rangle_\kappa$, we obtain

$$(\langle[\gamma_z,\gamma_a],y\rangle_\kappa^{\gamma_a}, \langle[\gamma_z,\gamma_c],y\rangle_\kappa^{\gamma_a}) = (\langle[\gamma_z,\gamma_a],y\rangle_\kappa, \langle[\gamma_z,\gamma_c],y\rangle_\kappa) \tag{2.12}$$
$$\begin{pmatrix} e^{2\pi\sqrt{-1}\kappa}(a'_{11}+a'_{22}-1) & e^{2\pi\sqrt{-1}\kappa}\left(c'_{11}-1-\frac{a'_1}{a'_2}c'_{21}\right) \\ 0 & 1 \end{pmatrix},$$

$$(\langle[\gamma_z,\gamma_a],y\rangle_\kappa^{\gamma_b}, \langle[\gamma_z,\gamma_c],y\rangle_\kappa^{\gamma_b}) = (\langle[\gamma_z,\gamma_a],y\rangle_\kappa, \langle[\gamma_z,\gamma_c],y\rangle_\kappa)$$
$$\begin{pmatrix} e^{2\pi\sqrt{-1}\kappa}(b'_{11}+b'_{22}-1) & (c'_{11}-1-\frac{b'_1}{b'_2}c'_{21})d \\ 0 & 1 \end{pmatrix}.$$

Then $\det(M_{a,c}^{(a)}M_{a,c}^{(b)}) = e^{4\pi\sqrt{-1}\kappa}\det(M'^{(a)}M'^{(b)})$ and $\mathrm{tr}(M_{a,c}^{(a)}M_{a,c}^{(b)}) = e^{4\pi\sqrt{-1}\kappa}\det(M'^{(a)}M'^{(b)}) + 1$. Combining with (2.10), we have $\mathrm{tr}(M_{a,c}^{(a)}M_{a,c}^{(b)}) = e^{2\pi\sqrt{-1}\kappa}\mathrm{tr}(M'^{(a)}M'^{(b)})$, which is also shown for the case where the points a, b, c are located clockwise with respect to the point o.

3. Solutions and monodromy.

In this section, we start with the proposition on the existence of a global simple solution of a second-order Fuchsian differential equation and the reducibility of the monodromy. The monodromy representation of solutions of a Fuchsian differential equation is said to be reducible, iff the monodromy matrices M^γ for fixed basis of solutions have a non-trivial invariant subspace which does not depend on the cycle γ, i.e. there exists a non-trivial subspace of the solutions which is invariant under analytic continuation along any cycle.

PROPOSITION 3.1. *Let $Dy = 0$ $(D = d^2/dw^2 + a_1(w)d/dw + a_2(w))$ be a second-order Fuchsian differential equation with singularities $\{t_1, \ldots, t_n, \infty(= t_{n+1})\}$, and let $\theta_l^{(1)}$ and $\theta_l^{(2)}$ be the exponents at the singularity $w = t_l$ $(l = 1, \ldots, n+1)$.*
(i) If the monodromy representation of solutions of the differential equation $Dy = 0$ is reducible, there exists a non-zero solution $y(w)$ such that $y(w) = h(w) \prod_{l=1}^n (w - t_l)^{\alpha_l}$, $h(w)$ is a polynomial in the variable w and $\prod_{l=1}^n h(t_l) \neq 0$.
(ii) If there exists a non-zero solution $y(w)$ of $Dy = 0$ such that $y(w) = h(w) \prod_{l=1}^n (w - t_l)^{\alpha_l}$, $h(w)$ is a polynomial of degree k and $\prod_{l=1}^n h(t_l) \neq 0$, then $\alpha_l \in \{\theta_l^{(1)}, \theta_l^{(2)}\}$ for each $l = 1, \ldots, n$ and $\theta_{n+1}^{(1)} = -k - \sum_{l=1}^n \alpha_l$ or $\theta_{n+1}^{(2)} = -k - \sum_{l=1}^n \alpha_l$.

PROOF. Assume that the monodromy representation of solutions of the Fuchsian equation $Dy = 0$ is reducible. Let $\tilde{\gamma}_l$ $(l = 1, \ldots, n+1)$ be a cycle on the Riemann sphere which traces a path around the singularity $w = t_l$ anti-clockwise. Since the dimension of the space of solutions of the differential equation $Dy = 0$ is two, it follows from reducibility that there exists a non-zero solution $y(w)$ such that $y^{\tilde{\gamma}_l}(w) = e^{2\pi\sqrt{-1}\tilde{\alpha}_l} y(w)$ for some constants $\tilde{\alpha}_l$ and $l = 1, \ldots, n+1$. The monodromy of the function $y(w) \prod_{l=1}^n (w - t_l)^{-\tilde{\alpha}_l}$ on $\mathbb{C}$ is trivial, because $\tilde{\gamma}_{n+1}$ is written as products of $\tilde{\gamma}_l^{-1}$ $(l = 1, \ldots, n)$. Since $y(w)$ satisfies the Fuchsian equation $Dy = 0$, the function $y(w) \prod_{l=1}^n (w - t_l)^{-\tilde{\alpha}_l}$ does not have any singularities except for $\{t_1 \ldots, t_{n+1}\}$, and the singularity $w = \infty$ is regular at most. Hence the function $y(w) \prod_{l=1}^n (w - t_l)^{-\tilde{\alpha}_l}$ may have poles at $w = t_l$ $(l = 1, \ldots, n)$, it is holomorphic on $\mathbb{C} \setminus \{t_1, \ldots, t_n\}$ and the regular singularity $w = \infty$ is apparent. Therefore we have $y(w) \prod_{l=1}^n (w - t_l)^{-\tilde{\alpha}_l} = p(w) \prod_{l=1}^n (w - t_l)^{-m_l}$ for some integers $m_1, \ldots, m_n$ and a polynomial $p(w)$. Thus $y(w)$ may be written as $y(w) = h(w) \prod_{l=1}^n (w - t_l)^{\alpha_l}$, where $h(w)$ is a polynomial in the variable w and $\prod_{l=1}^n h(t_l) \neq 0$. Therefore we have (i).

If $y(w) = h(w) \prod_{l=1}^n (w - t_l)^{\alpha_l}$ $(h(w)$: a polynomial of degree k, $\prod_{l=1}^n h(t_l) \neq 0)$ satisfies the Fuchsian equation $Dy = 0$, it follows from that the exponents at $w = t_l$ $(l \in \{1, \ldots, n\})$ are $\theta_l^{(1)}, \theta_l^{(2)}$ and $h(p) \neq 0$ that $\alpha_l \in \{\theta_l^{(1)}, \theta_l^{(2)}\}$. Write $h(w) = c_k w^k + c_{k-1} w^{k-1} + \cdots + c_0$ $(c_k \neq 0)$. Then we have the expansion $y(w) = (1/w)^{-k-\sum_{l=1}^n \alpha_l}(c_k + (c_{k-1} - c_k \sum_{l=1}^n t_l \alpha_l)/w + \cdots)$ around $w = \infty$ and the index $-k - \sum_{l=1}^n \alpha_l$ must coincide with one of the exponents at $w = \infty$, which are $\theta_{n+1}^{(1)}$ and $\theta_{n+1}^{(2)}$. Therefore we have $\theta_{n+1}^{(1)} = -k - \sum_{l=1}^n \alpha_l$ or $\theta_{n+1}^{(2)} = -k - \sum_{l=1}^n \alpha_l$. $\qquad\square$

Proposition 3.1 is applicable to the Fuchsian equation $D_{y_1}(\theta_0, \theta_1, \theta_t, \theta_\infty; \lambda, \mu)$ and Heun's equation (1.10) readily.

We introduce a proposition which connects polynomial-type solutions (the ones described in Proposition 3.1) to the ones such that one of the singularities is apparent through the Euler's integral transformation. A regular singularity $z = p$ of a second-order linear differential equation is called apparent, if and only if the monodromy matrix along the singularity $z = p$ is a scalar matrix. Note that it is equivalent that the difference of the exponents is an integer and the logarithmic solution along $z = p$ disappear, which means $A^{\langle p \rangle} = 0$ in (A.2) in the case that one of the exponents is zero.

For a cycle γ, we set

$$\langle \gamma, y \rangle = \begin{cases} \langle \gamma, y \rangle_{\kappa_2 - 1}, & \text{if } y(w) \text{ satisfies } D_{y_1}(\theta_0, \theta_1, \theta_t, \theta_\infty; \lambda, \mu), \\ \langle \gamma, y \rangle_{-\eta}, & \text{if } y(w) \text{ satisfies Heun's equation (1.10)}. \end{cases} \tag{3.1}$$

It follows from Proposition 1.1 (resp. Proposition 1.2) that if $y(w)$ is a solution of $D_{y_1}(\theta_0, \theta_1, \theta_t, \theta_\infty; \lambda, \mu)$ (resp. (1.10)) then $\langle[\gamma_z, \gamma_p], y\rangle$ $(p \in \{0, 1, t, \infty\})$ is a solution of $D_{y_1}(\tilde{\theta}_0, \tilde{\theta}_1, \tilde{\theta}_t, \tilde{\theta}_\infty; \tilde{\lambda}, \tilde{\mu})$ (resp. (1.12)). The local expansion of the function $\langle[\gamma_z, \gamma_p], y\rangle$ $(p = 0, 1, t, \infty)$ about $z = p$ can be calculated using (A.8), (A.13) for the case $\kappa = \kappa_2 - 1$, $\theta_\infty^{(1)} = \kappa_1$, $\theta_\infty^{(2)} = \kappa_2 + 1$ (resp. $\theta_p = 1 - \epsilon'_p$ $(p = 0, 1, t)$, $\kappa = -\eta$, $\theta_\infty^{(1)} = \alpha + \beta - 2\eta + 1$, $\theta_\infty^{(2)} = 2 - \eta$) by using the local expansion of solutions of $D_{y_1}(\theta_0, \theta_1, \theta_t, \theta_\infty; \lambda, \mu)$ (resp. (1.10)) (see (A.1), (A.2), (A.10), (A.11)). They are used to obtain the condition that the function $\langle[\gamma_z, \gamma_p], y\rangle$ $(p \in \{0, 1, t, \infty\})$ is identically zero for all solutions $y(w)$ to $D_{y_1}(\theta_0, \theta_1, \theta_t, \theta_\infty; \lambda, \mu)$ (resp. (1.10)).

PROPOSITION 3.2. (i) *Let* $p \in \{0, 1, t\}$ *and assume* $\kappa_2 \notin \mathbb{Z}$ *(resp.* $\eta \notin \mathbb{Z}$*). The function* $\langle[\gamma_z, \gamma_p], y\rangle$ *is identically zero for all solutions* $y(w)$ *of* $D_{y_1}(\theta_0, \theta_1, \theta_t, \theta_\infty; \lambda, \mu)$ *(resp. (1.10)), if and only if* $\theta_p \in \mathbb{Z}_{\geq 0}$ *(resp.* $\epsilon'_p \in \mathbb{Z}_{\leq 1}$*) and the singularity* $w = p$ *is apparent (i.e.* $A^{\langle p \rangle} = 0$ *in (A.2)), or* $\theta_p + \kappa_2 \in \mathbb{Z}_{\leq -1}$ *(resp.* $\epsilon_p \in \mathbb{Z}_{\geq 2}$*) and the differential equation* $D_{y_1}(\theta_0, \theta_1, \theta_t, \theta_\infty; \lambda, \mu)$ *(resp. (1.10)) has a solution of the form of a product of* $(w - p)^{\theta_p}$ *(resp.* $(w - p)^{1-\epsilon'_p}$*) and a non-zero polynomial of degree no more than* $-\theta_p - \kappa_2 - 1$ *(resp.* $\epsilon_p - 2$*).*
(ii) *Under the assumption* $\kappa_2 \notin \mathbb{Z}$ *(resp.* $\eta \notin \mathbb{Z}$*), the function* $\langle[\gamma_z, \gamma_\infty], y\rangle$ *is identically zero for all solutions* $y(w)$ *of* $D_{y_1}(\theta_0, \theta_1, \theta_t, \theta_\infty; \lambda, \mu)$ *(resp. (1.10)), if and only if* $\theta_\infty \in \mathbb{Z}_{\geq 1}$ *(resp.* $\alpha + \beta - \eta \in \mathbb{Z}_{\geq 1}$*) and the singularity* $w = \infty$ *is apparent, or* $\kappa_1 \in \mathbb{Z}_{\leq 0}$ *(resp.* $\alpha + \beta - 2\eta \in \mathbb{Z}_{\leq -1}$*) and the differential equation* $D_{y_1}(\theta_0, \theta_1, \theta_t, \theta_\infty; \lambda, \mu)$ *(resp. (1.10)) has a non-zero polynomial in the variable* w *of degree* $-\kappa_1$ *(resp.* $2\eta - \alpha - \beta - 1$*).*

Proposition 3.2 will be proved in the appendix.
We investigate a sufficient condition for the functions $\langle[\gamma_z, \gamma_0], y\rangle$, $\langle[\gamma_z, \gamma_1], y\rangle$, $\langle[\gamma_z, \gamma_t], y\rangle$ to span the two-dimensional space of solutions of $D_{y_1}(\tilde{\theta}_0, \tilde{\theta}_1, \tilde{\theta}_t, \tilde{\theta}_\infty; \tilde{\lambda}, \tilde{\mu})$ (resp. (1.12)).

PROPOSITION 3.3. *There exists a solution* $y(w)$ *of* $D_{y_1}(\theta_0, \theta_1, \theta_t, \theta_\infty; \lambda, \mu)$ *(resp. (1.10)) such that* $\langle[\gamma_z, \gamma_0], y\rangle \neq 0$, $\langle[\gamma_z, \gamma_1], y\rangle \neq 0$, $\langle[\gamma_z, \gamma_t], y\rangle \neq 0$ *and the functions* $\langle[\gamma_z, \gamma_0], y\rangle$, $\langle[\gamma_z, \gamma_1], y\rangle$, $\langle[\gamma_z, \gamma_t], y\rangle$ *span the two-dimensional space of solutions of* $D_{y_1}(\tilde{\theta}_0, \tilde{\theta}_1, \tilde{\theta}_t, \tilde{\theta}_\infty; \tilde{\lambda}, \tilde{\mu})$ *(resp. (1.12)), if* $\kappa_2 \notin \mathbb{Z}$ *and* $\theta_p, \tilde{\theta}_p \notin \mathbb{Z}$ *for all* $p \in \{0, 1, t, \infty\}$ *(resp.* $\eta, \epsilon_0, \epsilon_1, \epsilon_t, \alpha - \beta, \epsilon'_0, \epsilon'_1, \epsilon'_t, \alpha' - \beta' \notin \mathbb{Z}$*).*

Although the proposition might be obvious to the experts, we give a proof in the appendix with a more detailed proposition.
By applying the results on the monodromy of integral representations in section 2, we have the following theorem for monodromy matrices:

THEOREM 3.4. *Let* $a, b \in \{0, 1, t\}$ $(a \neq b)$, $M'^{(p)}$ *be a monodromy matrix of a certain basis of solutions of* $D_{y_1}(\theta_0, \theta_1, \theta_t, \theta_\infty; \lambda, \mu)$ *(resp. (1.10)) on the cycle* γ_p $(p \in \{a, b\})$ *and* $M^{(p)}$ *be a monodromy matrix of a certain basis of solutions of* $D_{y_1}(\tilde{\theta}_0, \tilde{\theta}_1, \tilde{\theta}_t, \tilde{\theta}_\infty; \tilde{\lambda}, \tilde{\mu})$ *(resp. (1.12)) on the cycle* γ_p. *Then we have*

$$\mathrm{tr}((M^{(a)} M^{(b)})^n) = e^{2\pi\sqrt{-1}n\kappa_2} \mathrm{tr}((M'^{(a)} M'^{(b)})^n), \qquad (3.2)$$

$$(\text{resp. } \operatorname{tr}((M^{(a)}M^{(b)})^n) = e^{-2\pi\sqrt{-1}n\eta}\operatorname{tr}((M'^{(a)}M'^{(b)})^n)),$$

for $n \in \mathbb{Z}$.

PROOF. Set $\kappa = \kappa_2 - 1$ (resp. $\kappa = -\eta$) for the case of the differential equation $D_{y_1}(\theta_0, \theta_1, \theta_t, \theta_\infty; \lambda, \mu)$ (resp. (1.10)). Firstly we show that $\operatorname{tr}(M^{(a)}M^{(b)}) = e^{2\pi\sqrt{-1}\kappa}\operatorname{tr}(M'^{(a)}M'^{(b)})$ and $\det(M^{(a)}M^{(b)}) = e^{4\pi\sqrt{-1}\kappa}\det(M'^{(a)}M'^{(b)})$ under the assumption of Proposition 3.3. It follows from Proposition 3.3 that there exists a solution $y(w)$ of $D_{y_1}(\theta_0, \theta_1, \theta_t, \theta_\infty; \lambda, \mu)$ (resp. (1.10)) such that $\langle[\gamma_z, \gamma_0], y\rangle \neq 0$, $\langle[\gamma_z, \gamma_1], y\rangle \neq 0$, $\langle[\gamma_z, \gamma_t], y\rangle \neq 0$ and the functions $\langle[\gamma_z, \gamma_0], y\rangle$, $\langle[\gamma_z, \gamma_1], y\rangle$, $\langle[\gamma_z, \gamma_t], y\rangle$ span the two-dimensional space of solutions of $D_{y_1}(\tilde{\theta}_0, \tilde{\theta}_1, \tilde{\theta}_t, \tilde{\theta}_\infty; \tilde{\lambda}, \tilde{\mu})$ (resp. (1.12)). Let c be the element in $\{0, 1, t\}$ which is different from a and b. Then $\langle[\gamma_z, \gamma_a], y\rangle$, $\langle[\gamma_z, \gamma_b], y\rangle$ are linearly independent or $\langle[\gamma_z, \gamma_a], y\rangle$, $\langle[\gamma_z, \gamma_c], y\rangle$ are linearly independent and $\langle[\gamma_z, \gamma_a], y\rangle = d\langle[\gamma_z, \gamma_b], y\rangle$ for some $d(\neq 0)$. Hence it follows from the calculations of monodromy in section 2 that if $\kappa_2 \notin \mathbb{Z}$ and $\theta_p, \tilde{\theta}_p \notin \mathbb{Z}$ for all $p \in \{0, 1, t, \infty\}$ (resp. $\eta, \epsilon_0, \epsilon_1, \epsilon_t, \alpha - \beta, \epsilon'_0, \epsilon'_1, \epsilon'_t, \alpha' - \beta' \notin \mathbb{Z}$) then $\operatorname{tr}(M^{(a)}M^{(b)}) = e^{2\pi\sqrt{-1}\kappa}\operatorname{tr}(M'^{(a)}M'^{(b)})$ and $\det(M^{(a)}M^{(b)}) = e^{4\pi\sqrt{-1}\kappa}\det(M'^{(a)}M'^{(b)})$. It is known the analyticity of the coefficients of the equation leads to the analyticity of the monodromy matrix, when the solutions of the equation which describe the monodromy are analytic with respect to the parameter and form a basis of the space of the equation. Hence we have $\operatorname{tr}(M^{(a)}M^{(b)}) = e^{2\pi\sqrt{-1}\kappa}\operatorname{tr}(M'^{(a)}M'^{(b)})$ and $\det(M^{(a)}M^{(b)}) = e^{4\pi\sqrt{-1}\kappa}\det(M'^{(a)}M'^{(b)})$ for all cases by taking a limit from the case $\kappa_2 \notin \mathbb{Z}$ and $\theta_p, \tilde{\theta}_p \notin \mathbb{Z}$ for all $p \in \{0, 1, t, \infty\}$ (resp. $\eta, \epsilon_0, \epsilon_1, \epsilon_t, \alpha - \beta, \epsilon'_0, \epsilon'_1, \epsilon'_t, \alpha' - \beta' \notin \mathbb{Z}$). Note that the trace and the determinant of a matrix is independent from the choice of the basis. Let l'_1, l'_2 (resp. l_1, l_2) be the solutions of the quadratic equation $x^2 - \operatorname{tr}(M'^{(a)}M'^{(b)})x + \det(M'^{(a)}M'^{(b)}) = 0$ (resp. $x^2 - \operatorname{tr}(M^{(a)}M^{(b)})x + \det(M^{(a)}M^{(b)}) = 0$). Then we have $\{l_1, l_2\} = \{e^{2\pi\sqrt{-1}\kappa}l'_1, e^{2\pi\sqrt{-1}\kappa}l'_2\}$ and $\operatorname{tr}((M^{(a)}M^{(b)})^n) = (l_1)^n + (l_2)^n = e^{2\pi\sqrt{-1}n\kappa}((l'_1)^n + (l'_2)^n) = e^{2\pi\sqrt{-1}n\kappa}\operatorname{tr}((M'^{(a)}M'^{(b)})^n)$ for $n \in \mathbb{Z}$. $\square$

It follows from the relations $M'^{(0)}M'^{(1)}M'^{(t)}M'^{(\infty)} = 1$ and $M^{(0)}M^{(1)}M^{(t)}M^{(\infty)} = 1$ that $\operatorname{tr}((M^{(p)}M^{(\infty)})^n) = e^{-2\pi\sqrt{-1}n\kappa_2}\operatorname{tr}((M'^{(p)}M'^{(\infty)})^n)$ (resp. $\operatorname{tr}((M^{(p)}M^{(\infty)})^n) = e^{2\pi\sqrt{-1}n\eta}\operatorname{tr}((M'^{(p)}M'^{(\infty)})^n))$ for $p \in \{0, 1, t\}$ and $n \in \mathbb{Z}$. It seems that we do not have a simple formula connecting $\operatorname{tr}(M^{(a)}(M^{(b)})^{-1})$ and $\operatorname{tr}(M'^{(a)}(M'^{(b)})^{-1})$ for $a, b \in \{0, 1, t\}$, $a \neq b$. Note that (3.2) can be written as $\operatorname{tr}((M^{(a)}M^{(b)})^n) = \operatorname{tr}((M'^{(a)}M'^{(b)})^n)$ for a 2×2 sl_2-Fuchsian system with four singularities, and it was obtained by Inaba–Iwasaki–Saito [**3**] and Boalch [**1**].

4. Correspondence between polynomial-type solutions and apparency of a singularity.

In this section, we establish correspondences between polynomial-type solutions and apparency of a singularity, which are induced by integral transformations.

Let $y(w)$ be a solution of $D_{y_1}(\theta_0, \theta_1, \theta_t, \theta_\infty; \lambda, \mu)$ (resp. (1.10)), $p \in \{0, 1, t\}$, and consider the local expansion of the solution about $w = p$ as (A.1), (A.2) by setting $\kappa = \kappa_2 - 1$ (resp. $\theta_p = 1 - \epsilon'_p$, $\kappa = -\eta$). It follows from the local expansion of $\langle[\gamma_z, \gamma_p], y\rangle$

about $z = p$ (see (A.8) for the case $\theta_p \in \mathbb{Z}_{\leq -1}, \theta_p + \kappa \notin \mathbb{Z}_{\leq -2}$) that if $\theta_p \in \mathbb{Z}_{\leq -1}$, $\kappa_2 \notin \mathbb{Z}$ (resp. $\epsilon'_p \in \mathbb{Z}_{\geq 2}$, $\eta \notin \mathbb{Z}$) and the singularity $w = p$ of the differential equation $D_{y_1}(\theta_0, \theta_1, \theta_t, \theta_\infty; \lambda, \mu)$ (resp. (1.10)) is apparent, then $A^{\langle p \rangle} = 0$ in (A.2), (A.8), the function $\langle [\gamma_z, \gamma_p], y \rangle$ is non-zero and it is a product of $(z - p)^{\theta_p + \kappa_2}$ (resp. $(z - p)^{2 - \epsilon'_p - \eta}$) and a polynomial of degree no more than $-\theta_p - 1$ (resp. $\epsilon'_p - 2$). Since $\langle [\gamma_z, \gamma_p], y \rangle$ satisfies $D_{y_1}(\tilde{\theta}_0, \tilde{\theta}_1, \tilde{\theta}_t, \tilde{\theta}_\infty; \tilde{\lambda}, \tilde{\mu})$ (resp. (1.12)), we have the following proposition:

PROPOSITION 4.1.   Let $p \in \{0, 1, t\}$. If $\theta_p \in \mathbb{Z}_{\leq -1}$, $\kappa_2 \notin \mathbb{Z}$ (resp. $\epsilon'_p \in \mathbb{Z}_{\geq 2}$, $\eta \notin \mathbb{Z}$) and the singularity $w = p$ of the differential equation $D_{y_1}(\theta_0, \theta_1, \theta_t, \theta_\infty; \lambda, \mu)$ (resp. (1.10)) is apparent, then there exists a non-zero solution of $D_{y_1}(\tilde{\theta}_0, \tilde{\theta}_1, \tilde{\theta}_t, \tilde{\theta}_\infty; \tilde{\lambda}, \tilde{\mu})$ (resp. (1.12)) which can be written as $(z - p)^{\theta_p + \kappa_2} h(z)$ (resp. $(z - p)^{1 - \epsilon_p} h(z)$) where $h(z)$ is a polynomial of degree no more than $-\theta_p - 1$ (resp. $\epsilon'_p - 2$).

The following theorem asserts various correspondences between polynomial-type solutions and apparency of a singularity for Heun's equation.

THEOREM 4.2.   Let a, b, c be elements of $\{0, 1, t\}$ such that $a \neq b \neq c \neq a$ and $\eta, \alpha, \beta, \epsilon_0, \epsilon_1, \epsilon_t, \alpha', \beta', \epsilon'_0, \epsilon'_1, \epsilon'_t$ be the parameters defined in (1.9) or (1.13).
(i) If $\epsilon'_a \in \mathbb{Z}_{\geq 2}$, $\eta \notin \mathbb{Z}$ and the singularity $w = a$ of (1.10) is apparent, then there exists a non-zero solution of (1.12) which can be written as $(z - a)^{1 - \epsilon_a} h(z)$ where $h(z)$ is a polynomial of degree no more than $\epsilon'_a - 2$. Moreover if $\alpha', \beta' \notin \mathbb{Z}$, then $\deg_E h(z) = \epsilon'_a - 2$.
(ii) If $\epsilon'_a \in \mathbb{Z}_{\leq 0}$, $\eta \notin \mathbb{Z}$, $\alpha', \beta', \epsilon_b, \epsilon_c \notin \mathbb{Z}$ and the singularity $w = a$ of (1.10) is apparent, then there exists a non-zero solution of (1.12) which can be written as $(z - b)^{1 - \epsilon_b}(z - c)^{1 - \epsilon_c} h(z)$ where $h(z)$ is a polynomial with $\deg h(z) = -\epsilon'_a$.
(iii) If $\epsilon_a \in \mathbb{Z}_{\geq 2}$, $\alpha, \beta \notin \mathbb{Z}$ and there exists a non-zero solution of (1.10) which can be written as a product of $(w - a)^{1 - \epsilon'_a}$ and a polynomial, then the singularity $z = a$ of (1.12) is apparent.
(iv) If $\epsilon_a \in \mathbb{Z}_{\leq 0}$, $\alpha, \beta, \epsilon'_b, \epsilon'_c \notin \mathbb{Z}$ and there exists a non-zero solution of (1.10) which can be written as $(w - b)^{1 - \epsilon'_b}(w - c)^{1 - \epsilon'_c} h(w)$ where $h(w)$ is a polynomial, then the singularity $z = a$ of (1.12) is apparent.
(v) If $\alpha + \beta - \eta \in \mathbb{Z}_{\leq 0}$, $\eta \notin \mathbb{Z}$ and the singularity $w = \infty$ of (1.10) is apparent, then there exists a non-zero solution of (1.12) which can be written as a polynomial of degree $\eta - \alpha - \beta$.
(vi) If $\alpha + \beta - \eta \in \mathbb{Z}_{\geq 2}$, $\eta, \epsilon_0, \epsilon_1, \epsilon_t \notin \mathbb{Z}$ and the singularity $w = \infty$ of (1.10) is apparent, then there exists a non-zero solution of (1.12) which can be written as $z^{1 - \epsilon_0}(z - 1)^{1 - \epsilon_1}(z - t)^{1 - \epsilon_t} h(z)$ where $h(z)$ is a polynomial of degree $\alpha + \beta - \eta - 2$.
(vii) If $\alpha + \beta - 2\eta \in \mathbb{Z}_{\leq -1}$, $\eta, \epsilon'_0 \notin \mathbb{Z}$ and there exists a non-zero solution of (1.10) written as a polynomial in w, then the singularity $z = \infty$ of (1.12) is apparent.
(viii) If $\alpha + \beta - 2\eta \in \mathbb{Z}_{\geq 1}$, $\eta, \epsilon'_0, \epsilon'_1, \epsilon'_t \notin \mathbb{Z}$ and there exists a non-zero solution of (1.10) which can be written as $w^{1 - \epsilon'_0}(w - 1)^{1 - \epsilon'_1}(w - t)^{1 - \epsilon'_t} h(w)$ where $h(w)$ is a polynomial, then the singularity $z = \infty$ of (1.12) is apparent.

We will prove Theorem 4.2 in the appendix with a more detailed proposition.
Correspondences between polynomial-type solutions and apparency of a singularity for the differential equations $D_{y_1}(\theta_0, \theta_1, \theta_t, \theta_\infty; \lambda, \mu)$ and $D_{y_1}(\tilde{\theta}_0, \tilde{\theta}_1, \tilde{\theta}_t, \tilde{\theta}_\infty; \tilde{\lambda}, \tilde{\mu})$ can also

be described as follows:

THEOREM 4.3. *Set* $\kappa_1 = (\theta_\infty - \theta_0 - \theta_1 - \theta_t)/2$, $\kappa_2 = -(\theta_\infty + \theta_0 + \theta_1 + \theta_t)/2$, $\tilde{\theta}_p = \kappa_2 + \theta_p$ $(p = 0, 1, t, \infty)$ *and*

$$\tilde{\lambda} = \lambda - \frac{\kappa_2}{\mu - (\theta_0/\lambda) - \theta_1/(\lambda-1) - \theta_t/(\lambda-t)}, \quad \tilde{\mu} = \frac{\kappa_2 + \theta_0}{\tilde{\lambda}} + \frac{\kappa_2 + \theta_1}{\tilde{\lambda}-1} + \frac{\kappa_2 + \theta_t}{\tilde{\lambda}-t} + \frac{\kappa_2}{\lambda - \tilde{\lambda}}.$$
(4.1)

Let a, b, c *be elements of* $\{0, 1, t\}$ *such that* $a \neq b \neq c \neq a$. *Assume that* $\lambda, \tilde{\lambda} \notin \{0, 1, t, \infty\}$.
(i) *If* $\theta_a \in \mathbb{Z}_{\leq -1}$, $\kappa_2 \notin \mathbb{Z}$ *and the singularity* $w = a$ *of the differential equation* $D_{y_1}(\theta_0, \theta_1, \theta_t, \theta_\infty; \lambda, \mu)$ *in the variable* w *is apparent, then there exists a non-zero solution of the differential equation* $D_{y_1}(\tilde{\theta}_0, \tilde{\theta}_1, \tilde{\theta}_t, \tilde{\theta}_\infty; \tilde{\lambda}, \tilde{\mu})$ *in the variable* z *which can be written as* $(z - a)^{\tilde{\theta}_a} h(z)$ *where* $h(z)$ *is a polynomial of degree no more than* $-\theta_a - 1$. *Moreover if* $\kappa_1 \notin \mathbb{Z}$, *then* $\deg_E h(z) = -\theta_a - 1$.
(ii) *If* $\theta_a \in \mathbb{Z}_{\geq 0}$, $\kappa_1, \kappa_2, \tilde{\theta}_b, \tilde{\theta}_c \notin \mathbb{Z}$ *and the singularity* $w = a$ *of the differential equation* $D_{y_1}(\theta_0, \theta_1, \theta_t, \theta_\infty; \lambda, \mu)$ *is apparent, then there exists a non-zero solution of the differential equation* $D_{y_1}(\tilde{\theta}_0, \tilde{\theta}_1, \tilde{\theta}_t, \tilde{\theta}_\infty; \tilde{\lambda}, \tilde{\mu})$ *which can be written as* $(z - b)^{\tilde{\theta}_b}(z - c)^{\tilde{\theta}_c} h(z)$ *where* $h(z)$ *is a polynomial with* $\deg h(z) = \theta_a$.
(iii) *If* $\tilde{\theta}_a \in \mathbb{Z}_{\leq 0}$, $\kappa_2, \theta_\infty \notin \mathbb{Z}$ *and there exists a non-zero solution of* $D_{y_1}(\theta_0, \theta_1, \theta_t, \theta_\infty; \lambda, \mu)$ *which can be written as a product of* $(w - a)^{\theta_a}$ *and a polynomial, then the singularity* $z = a$ *of* $D_{y_1}(\tilde{\theta}_0, \tilde{\theta}_1, \tilde{\theta}_t, \tilde{\theta}_\infty; \tilde{\lambda}, \tilde{\mu})$ *is apparent.*
(iv) *If* $\tilde{\theta}_a \in \mathbb{Z}_{\geq 1}$, $\kappa_2, \theta_\infty, \theta_b, \theta_c \notin \mathbb{Z}$ *and there exists a non-zero solution of* $D_{y_1}(\theta_0, \theta_1, \theta_t, \theta_\infty; \lambda, \mu)$ *which can be written as* $(w - b)^{\theta_b}(w - c)^{\theta_c} h(w)$ *where* $h(w)$ *is a polynomial, then the singularity* $z = a$ *of* $D_{y_1}(\tilde{\theta}_0, \tilde{\theta}_1, \tilde{\theta}_t, \tilde{\theta}_\infty; \tilde{\lambda}, \tilde{\mu})$ *is apparent.*
(v) *If* $\theta_\infty \in \mathbb{Z}_{\leq 0}$, $\kappa_2 \notin \mathbb{Z}$ *and the singularity* $w = \infty$ *of* $D_{y_1}(\theta_0, \theta_1, \theta_t, \theta_\infty; \lambda, \mu)$ *is apparent, then there exists a non-zero solution of* $D_{y_1}(\tilde{\theta}_0, \tilde{\theta}_1, \tilde{\theta}_t, \tilde{\theta}_\infty; \tilde{\lambda}, \tilde{\mu})$ *which can be written as a polynomial of degree* $-\theta_\infty$.
(vi) *If* $\theta_\infty \in \mathbb{Z}_{\geq 1}$, $\kappa_2, \tilde{\theta}_0, \tilde{\theta}_1, \tilde{\theta}_t \notin \mathbb{Z}$ *and the singularity* $w = \infty$ *of* $D_{y_1}(\theta_0, \theta_1, \theta_t, \theta_\infty; \lambda, \mu)$ *is apparent, then there exists a non-zero solution of* $D_{y_1}(\tilde{\theta}_0, \tilde{\theta}_1, \tilde{\theta}_t, \tilde{\theta}_\infty; \tilde{\lambda}, \tilde{\mu})$ *which can be written as* $z^{\tilde{\theta}_0}(z - 1)^{\tilde{\theta}_1}(z - t)^{\tilde{\theta}_t} h(z)$ *where* $h(z)$ *is a polynomial of degree* $\theta_\infty - 1$.
(vii) *If* $\kappa_1 \in \mathbb{Z}_{\leq 0}$, $\kappa_2, \theta_0 \notin \mathbb{Z}$ *and there exists a non-zero solution of* $D_{y_1}(\theta_0, \theta_1, \theta_t, \theta_\infty; \lambda, \mu)$ *written as a polynomial in* w, *then the singularity* $z = \infty$ *of* $D_{y_1}(\tilde{\theta}_0, \tilde{\theta}_1, \tilde{\theta}_t, \tilde{\theta}_\infty; \tilde{\lambda}, \tilde{\mu})$ *is apparent.*
(viii) *If* $\kappa_1 \in \mathbb{Z}_{\geq 1}$, $\kappa_2, \theta_0, \theta_1, \theta_t \notin \mathbb{Z}$ *and there exists a non-zero solution of* $D_{y_1}(\theta_0, \theta_1, \theta_t, \theta_\infty; \lambda, \mu)$ *which can be written as* $w^{\theta_0}(w - 1)^{\theta_1}(w - t)^{\theta_t} h(w)$ *where* $h(w)$ *is a polynomial, then the singularity* $z = \infty$ *of* $D_{y_1}(\tilde{\theta}_0, \tilde{\theta}_1, \tilde{\theta}_t, \tilde{\theta}_\infty; \tilde{\lambda}, \tilde{\mu})$ *is apparent.*

5. Quasi-solvability and apparency of a singularity for Heun's equation.

We recall the quasi-solvability of Heun's equation.

PROPOSITION 5.1 ([**12**], [**17**] etc.). *Let* $\nu_j \in \{0, 1 - \epsilon'_j\}$ *for* $j = 0, 1, t$, $\eta' \in \{\alpha', \beta'\}$ *and assume that* $-(\eta' + \nu_0 + \nu_1 + \nu_t) \in \mathbb{Z}_{\geq 0}$. *Set* $n = -(\eta' + \nu_0 + \nu_1 + \nu_t)$. *Then there exists a polynomial* $P(q')$ *of degree* $n + 1$ *in the variable* q' *such that if* q' *satisfies* $P(q') = 0$ *then there exists a solution of* (1.10) *written as* $w^{\nu_0}(w - 1)^{\nu_1}(w - t)^{\nu_t} p(w)$, *where* $p(w)$

is a polynomial of degree no more than n in the variable w.

EXAMPLE 1. We investigate polynomial-type solutions of Heun's equation for the case $\epsilon'_0 - \beta' + 2 = 0$. Set $\nu_0 = 0$, $\nu_1 = 1 - \epsilon'_1$, $\nu_t = 1 - \epsilon'_t$ and $\eta' = \alpha'$ in Proposition 5.1. Then $n = -(\alpha' + 2 - \epsilon'_1 - \epsilon'_t) = -(\epsilon'_0 - \beta' + 1) = 1$. We look for a solution of (1.10) of the form $(w-1)^{1-\epsilon'_1}(w-t)^{1-\epsilon'_t}(c+w)$. By substituting it into (1.10), we have

$$c(q' - \beta'\epsilon'_1 t + \beta' t + 2\epsilon'_1 t - 2t - 2 + \beta' - \beta'\epsilon'_t + 2\epsilon'_t) + 2t - \beta' t = 0, \qquad (5.1)$$
$$c(\beta' - \epsilon'_t - \epsilon'_1 + 2) + q' + \epsilon'_1 t + 2\beta' t - 2t - \beta'\epsilon'_1 t + 2\beta' - 2 - \beta'\epsilon'_t + \epsilon'_t = 0.$$

Hence

$$c = \frac{q' - (\beta' - 1)((\epsilon'_1 - 2)t + \epsilon'_t - 2)}{\epsilon'_t + \epsilon'_1 - \beta' - 2}, \qquad (5.2)$$

$$(q')^2 + ((-2\beta'\epsilon'_1 + 3\beta' + 3\epsilon'_1 - 4)t + (-2\beta'\epsilon'_t + 3\beta' + 3\epsilon'_t - 4))q'$$
$$+ (\beta' - 2)[(\epsilon'_1 - 1)(\epsilon'_1 - 2)(\beta' - 1)t^2 t$$
$$+ \{(\beta' - 1)(2\epsilon'_t\epsilon'_1 - 3\epsilon'_1 - 3\epsilon'_t + 5) - \epsilon'_1 - \epsilon'_t + 3\} + (\epsilon'_t - 1)(\epsilon'_t - 2)(\beta' - 1)] = 0. \qquad (5.3)$$

Therefore, if q' satisfies the quadratic equation in (5.3), then the function $(w-1)^{1-\epsilon'_1}(w-t)^{1-\epsilon'_t}(c+w)$ satisfies (1.10) where c is chosen as (5.2).

We are going to obtain explicit expressions for solutions of Heun's equation which have an apparent singularity by using solutions which are expressed by quasi-solvability.

THEOREM 5.2. *Let a, b, c be elements of $\{0, 1, t\}$ such that $a \neq b \neq c \neq a$ and $\eta, \alpha, \beta, \epsilon_0, \epsilon_1, \epsilon_t, \alpha', \beta', \epsilon'_0, \epsilon'_1, \epsilon'_t$ be the parameters defined in (1.9) or (1.13).*
(i) *If $\epsilon_a \in \mathbb{Z}_{\leq 0}$, $\alpha, \beta, \epsilon'_b, \epsilon'_c \notin \mathbb{Z}$ and the singularity $z = a$ of (1.12) is apparent, then there exists a non-zero solution of (1.10) which can be written as $(w-b)^{1-\epsilon'_b}(w-c)^{1-\epsilon'_c}h(w)$ where $h(w)$ is a polynomial of degree $-\epsilon_a$ and the functions*

$$\int_{[\gamma_z, \gamma_p]} (w-b)^{1-\epsilon'_b}(w-c)^{1-\epsilon'_c}h(w)(z-w)^{-\eta}dw, \quad (p = b, c), \qquad (5.4)$$

are non-zero solutions of (1.12).
(ii) *If $\alpha + \beta - 2\eta \in \mathbb{Z}_{\geq 1}$, $\eta, \epsilon'_0, \epsilon'_1, \epsilon'_t \notin \mathbb{Z}$ and the singularity $z = \infty$ of (1.12) is apparent, then there exists a non-zero solution of (1.10) which can be written as $w^{1-\epsilon'_0}(w-1)^{1-\epsilon'_1}(w-t)^{1-\epsilon'_t}h(w)$ where $h(w)$ is a polynomial of degree $\alpha + \beta - 2\eta - 1$ and the functions*

$$\int_{[\gamma_z, \gamma_p]} w^{1-\epsilon'_0}(w-1)^{1-\epsilon'_1}(w-t)^{1-\epsilon'_t}h(w)(z-w)^{-\eta}dw, \quad (p = 0, 1, t), \qquad (5.5)$$

are non-zero solutions of (1.12).

PROOF. By Theorem 4.2 (ii) (resp. Theorem 4.2 (vi)) and the duality of the parameters $(\alpha, \beta, \epsilon_0, \epsilon_1, \epsilon_t, \eta)$ and $(\alpha', \beta', \epsilon'_0, \epsilon'_1, \epsilon'_t, \eta')$ in (1.9) and (1.13), we obtain the exis-

tence of a non-zero solution of (1.10) which can be written as $(w-b)^{1-\epsilon'_b}(w-c)^{1-\epsilon'_c}h(w)$ (resp. $w^{1-\epsilon'_0}(w-1)^{1-\epsilon'_1}(w-t)^{1-\epsilon'_t}h(w)$) where $h(w)$ is a polynomial of degree $-\epsilon_a$ (resp. $\alpha+\beta-2\eta-1$). It follows from Proposition 1.2 that (5.4) (resp. (5.5)) is a solution of (1.12). We show that (5.4) for $p=b$ is non-zero. We expand $(w-c)^{1-\epsilon'_c}h(w)$ about $w=b$ as $\sum_{j=0}^{\infty}\tilde{c}_j(w-b)^j$. Then there are infinitely many terms such that $\tilde{c}_j\neq 0$, because $\epsilon'_c\notin\mathbb{Z}$. (5.4) for $p=b$ is written as (A.8) for the case $\theta_b=1-\epsilon'_b\notin\mathbb{Z}_{\leq 0}$, $\theta_b+\kappa+1=2-\epsilon'_b-\eta=1-\epsilon_b\in\mathbb{Z}_{\leq -1}$, and it is not identically zero. It is also shown that (5.4) for $p=c$ and (5.5) for $p=0,1,t$ are not identically zero. $\square$

EXAMPLE 2. We investigate solutions of (1.12) for the case $\epsilon_0=-1$ and the singularity $z=0$ of (1.12) is apparent. The condition that the singularity $z=0$ is apparent is described as an algebraic equation of q by following the method in the appendix, and it is written as

$$q^2+(\epsilon_1 t+\epsilon_t-t-1)q+\alpha\beta t=0, \tag{5.6}$$

which is equivalent to (5.3) by applying (1.9) for $\eta=\beta$. If (5.6) is satified, then the function $(w-1)^{1-\epsilon'_1}(w-t)^{1-\epsilon'_t}(w+q/\alpha)$ satisfies (1.10), which follows from Example 1. By applying the integral transformation, the functions

$$\int_{[\gamma_z,\gamma_p]}(w-1)^{1-\epsilon'_1}(w-t)^{1-\epsilon'_t}\left(w+\frac{q}{\alpha}\right)(z-w)^{-\beta}dw, \tag{5.7}$$

are solutions of (1.12), if q satisfies (5.6).

If $\epsilon_a\in\mathbb{Z}_{\geq 2}$, $\alpha,\beta\notin\mathbb{Z}$ and the singularity $z=a$ of (1.12) is apparent, then there exists a non-zero solution of (1.10) which can be written as $(w-a)^{1-\epsilon'_a}h(w)$ where $h(w)$ is a polynomial of degree ϵ_a-2, and the functions

$$\int_{[\gamma_z,\gamma_p]}(w-a)^{1-\epsilon'_a}h(w)(z-w)^{-\eta}dw, \quad (p=0,1,t,\infty), \tag{5.8}$$

are solutions of (1.12). But it is shown that (5.8) is identically zero for $p=0,1,t,\infty$. (For the case $p=a$, it follows from (A.8) on the case $\theta_p\notin\mathbb{Z}$, $\theta_p+\kappa\in\mathbb{Z}_{\leq -2}$. For the case $p=b,c$, it follows from holomorphy of $(w-a)^{1-\epsilon'_a}h(w)$ about $p=b,c$. For the case $p=\infty$, it follows from $\gamma_0\gamma_1\gamma_t\gamma_\infty=1$.) We have a similar situation for the case $\alpha+\beta-2\eta\in\mathbb{Z}_{\leq -1}$, $\eta,\epsilon'_0\notin\mathbb{Z}$. To obtain non-vanishing expressions of integrals, we apply the following proposition.

PROPOSITION 5.3. *Let $\eta,\epsilon_0,\epsilon_1,\epsilon_t,\epsilon'_0,\epsilon'_1,\epsilon'_t$ be the parameters defined in (1.9) or (1.13) and $v(w)$ be a solution of (1.10). Then the function*

$$y(z)=z^{1-\epsilon_0}(z-1)^{1-\epsilon_1}(z-t)^{1-\epsilon_t}\int_{[\gamma_z,\gamma_p]}w^{\epsilon'_0-1}(w-1)^{\epsilon'_1-1}(w-t)^{\epsilon'_t-1}v(w)(z-w)^{\eta-2}dw \tag{5.9}$$

is a solution of (1.12) for $p\in\{0,1,t,\infty\}$.

PROOF.   Let $v(w)$ be a solution of (1.10). Then the function $\tilde{v}(w) = w^{\epsilon'_0 - 1}(w - 1)^{\epsilon'_1 - 1}(w - t)^{\epsilon'_t - 1}v(w)$ is a solution of

$$\frac{d^2\tilde{v}}{dw^2} + \left(\frac{2 - \epsilon'_0}{w} + \frac{2 - \epsilon'_1}{w - 1} + \frac{2 - \epsilon'_t}{w - t}\right)\frac{d\tilde{v}}{dw} + \frac{(2 - \alpha')(2 - \beta')w - \tilde{q}'}{w(w-1)(w-t)}\tilde{v} = 0, \qquad (5.10)$$
$$\tilde{q}' = q' - (\epsilon'_0 + \epsilon'_t - 2) - (\epsilon'_0 + \epsilon'_1 - 2)t.$$

It follows from Proposition 1.2 that the function $\tilde{y}(z) = \int_{[\gamma_z, \gamma_p]} \tilde{v}(w)(z - w)^{-(2-\eta)}dw$ $(p = 0, 1, t, \infty)$ is a solution of

$$\frac{d^2\tilde{y}}{dz^2} + \left(\frac{2 - \epsilon_0}{z} + \frac{2 - \epsilon_1}{z - 1} + \frac{2 - \epsilon_t}{z - t}\right)\frac{d\tilde{y}}{dz} + \frac{(2 - \alpha)(2 - \beta)z - \tilde{q}}{z(z-1)(z-t)}\tilde{y} = 0, \qquad (5.11)$$
$$\tilde{q} = \tilde{q}' + (1 - \eta)\left\{2 - \epsilon'_t + (2 - \epsilon'_1)t + (2 - \epsilon'_0 - \eta)(t + 1)\right\}.$$

By setting $y(z) = z^{1-\epsilon_0}(z - 1)^{1-\epsilon_1}(z - t)^{1-\epsilon_t}\tilde{y}(z)$, it follows that $y(z)$ is a solution of (1.12).   □

THEOREM 5.4.   Let a, b, c be elements of $\{0, 1, t\}$ such that $a \neq b \neq c \neq a$ and $\eta, \alpha, \beta, \epsilon_0, \epsilon_1, \epsilon_t, \epsilon'_0, \epsilon'_1, \epsilon'_t$ be the parameters defined in (1.9) or (1.13).
(i) If $\epsilon_a \in \mathbb{Z}_{\geq 2}$, $\alpha, \beta, \epsilon'_b, \epsilon'_c \notin \mathbb{Z}$ and the singularity $z = a$ of (1.12) is apparent, then there exists a non-zero solution of (1.10) which can be written as $(w - a)^{1-\epsilon'_a}h(w)$ where $h(w)$ is a polynomial of degree $\epsilon_a - 2$, and the functions

$$z^{1-\epsilon_0}(z - 1)^{1-\epsilon_1}(z - t)^{1-\epsilon_t}\int_{[\gamma_z, \gamma_p]}(w - b)^{\epsilon'_b - 1}(w - c)^{\epsilon'_c - 1}h(w)(z - w)^{\eta - 2}dw, \quad (p = b, c),$$
$$(5.12)$$

are non-zero solutions of (1.12).
(ii) If $\alpha + \beta - 2\eta \in \mathbb{Z}_{\leq -1}$, $\eta, \epsilon'_0, \epsilon'_1, \epsilon'_t \notin \mathbb{Z}$ and the singularity $z = \infty$ of (1.12) is apparent, then there exists a non-zero solution of (1.10) which can be written as $h(w)$ where $h(w)$ is a polynomial of degree $2\eta - \alpha - \beta - 1$ and the functions

$$z^{1-\epsilon_0}(z - 1)^{1-\epsilon_1}(z - t)^{1-\epsilon_t}\int_{[\gamma_z, \gamma_p]}w^{\epsilon'_0 - 1}(w - 1)^{\epsilon'_1 - 1}(w - t)^{\epsilon'_t - 1}h(w)(z - w)^{\eta - 2}dw,$$
$$(5.13)$$

$(p = 0, 1, t)$ are non-zero solutions of (1.12).

PROOF.   By Theorem 4.2 (i) (resp. Theorem 4.2 (v)) and the duality of the parameters in (1.9) and (1.13), we obtain the existence of a non-zero solution of (1.10) which can be written as $(w - a)^{1-\epsilon'_a}h(w)$ (resp. $h(w)$) where $h(w)$ is a polynomial of degree $\epsilon_a - 2$ (resp. $2\eta - \alpha - \beta - 1$). It follows from Proposition 5.3 that (5.12) (resp. (5.13)) is a solution of (1.12). It can be shown by a similar argument to that in the proof of Theorem 5.2 that (5.12) for $p = b, c$ and (5.13) for $p = 0, 1, t$ are not identically zero.   □

6.   Elliptical representation of Heun's equation.

Heun's differential equation has an elliptical representation as we mentioned in the introduction. In this section, we rewrite several results on the integral transformation of Heun's equation to the elliptical representation form.

We review the elliptical representation of Heun's differential equation. Set

$$H^{(l_0', l_1', l_2', l_3')} = -\frac{d^2}{dx^2} + \sum_{i=0}^{3} l_i'(l_i' + 1)\wp(x + \omega_i). \tag{6.1}$$

Let α_i' be a number such that $\alpha_i' = -l_i'$ or $\alpha_i' = l_i' + 1$ for each $i \in \{0, 1, 2, 3\}$. By setting

$$z = \frac{\wp(x) - e_1}{e_2 - e_1}, \quad t = \frac{e_3 - e_1}{e_2 - e_1}, \quad f(x) = vz^{\alpha_1'/2}(z - 1)^{\alpha_2'/2}(z - t)^{\alpha_3'/2}; \tag{6.2}$$

the equation

$$\frac{d^2v}{dz^2} + \left(\frac{\epsilon_0'}{z} + \frac{\epsilon_1'}{z - 1} + \frac{\epsilon_t'}{z - t}\right)\frac{dv}{dz} + \frac{\alpha'\beta'z - q'}{z(z - 1)(z - t)}v = 0 \tag{6.3}$$

is transformed to

$$H^{(l_0', l_1', l_2', l_3')} f(x) = E' f(x), \tag{6.4}$$

where

$$\{\alpha', \beta'\} = \left\{\frac{\alpha_1' + \alpha_2' + \alpha_3' + \alpha_0'}{2}, \frac{\alpha_1' + \alpha_2' + \alpha_3' + 1 - \alpha_0'}{2}\right\}, \tag{6.5}$$

$$\epsilon_0' = \alpha_1' + \frac{1}{2}, \quad \epsilon_1' = \alpha_2' + \frac{1}{2}, \quad \epsilon_t' = \alpha_3' + \frac{1}{2},$$

$$q' = \frac{-E'}{4(e_2 - e_1)} + \frac{(-(\alpha' - \beta')^2 + 2(\epsilon_0')^2 - 4\epsilon_0' + 1)(t + 1)}{12}$$

$$+ \frac{6\epsilon_0'\epsilon_t' + 2(\epsilon_t')^2 - 4\epsilon_t' - (\epsilon_1')^2 + 2\epsilon_1'}{12} + \frac{(6\epsilon_0'\epsilon_1' + 2(\epsilon_1')^2 - 4\epsilon_1' - (\epsilon_t')^2 + 2\epsilon_t')t}{12}.$$

We investigate a correspondence of cycles on the Riemann sphere and the torus. For the transformation $z = (\wp(x) - e_1)/(e_2 - e_1)$, the path from x to $-x$ (resp. $-x + 2\omega_1$, $-x + 2\omega_2$, $-x + 2\omega_3$) which traces a semicircle around ω_0 (resp. ω_1, ω_2, ω_3) corresponds to a cycle which surrounds ∞ (resp. 0, 1, t) on the Riemann sphere $\mathbb{C} \cup \{\infty\}$ whose coordinate is z. Let γ_0, (resp. γ_1, γ_t, γ_∞) be a cycle on the Riemann sphere which surrounds the point $z = 0$ (resp. $z = 1$, $z = t$, $z = \infty$) anticlockwise. We choose the cycles so that $\gamma_0\gamma_1\gamma_t\gamma_\infty \sim$ id. Then the shift of the period $x \to x + 2\omega_1$ corresponds to a cycle which is homotopic to $\gamma_t\gamma_1$, $\gamma_1\gamma_t$, $\gamma_t^{-1}\gamma_1^{-1}$ or $\gamma_1^{-1}\gamma_t^{-1}$ on the punctured Riemann sphere, whose choice is dependent on specifying the point x and the zone where the shift $x \to x + 2\omega_1$ passes (see Figure 2).

It is also shown that the shift of the period $x \to x + 2\omega_3$ corresponds to the cycle which is homotopic to $\gamma_0\gamma_1$, $\gamma_1\gamma_0$, $\gamma_0^{-1}\gamma_1^{-1}$ or $\gamma_1^{-1}\gamma_0^{-1}$ on the punctured Riemann sphere, whose choice is dependent on specifying the point x and the zone where the shift $x \to x + 2\omega_3$
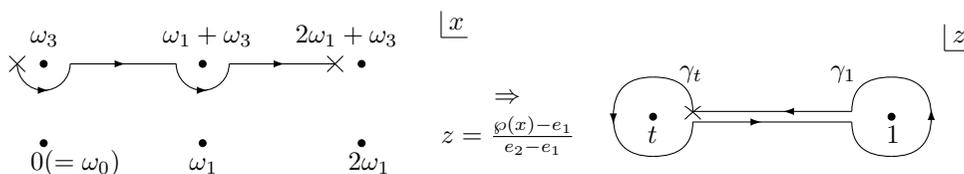
Figure 2.   Correspondence of cycles.

passes.

We rewrite the integral transformation of Heun's equation (i.e. Proposition 1.2) in elliptical representation form, which was announced in [**24**]. It is remarkable that the eigenvalue E is unchanged by the integral transformation.

THEOREM 6.1.   *Let $\sigma(x)$ be the Weierstrass sigma function, $\sigma_i(x)$ $(i = 1, 2, 3)$ be the Weierstrass co-sigma function which has a zero at $x = \omega_i$, and I_i $(i = 0, 1, 2, 3)$ be the cycle on the complex plane with the variable ξ such that points $\xi = x$ and $\xi = -x + 2\omega_i$ are contained and the half-periods $\mathbb{Z}\omega_1 + \mathbb{Z}\omega_3$ are not contained inside the cycle. Let α_i' be a number such that $\alpha_i' = -l_i'$ or $\alpha_i' = l_i' + 1$ for each $i \in \{0, 1, 2, 3\}$. Set $d = -\sum_{i=0}^{3} \alpha_i'/2$ and $\eta = d + 2$. If $\tilde{f}(x)$ satisfies*

$$H^{(l_0', l_1', l_2', l_3')} \tilde{f}(x) = E\tilde{f}(x), \tag{6.6}$$

then the functions

$$f(x) = \sigma(x)^{\alpha_0' + d + 1} \sigma_1(x)^{\alpha_1' + d + 1} \sigma_2(x)^{\alpha_2' + d + 1} \sigma_3(x)^{\alpha_3' + d + 1}$$

$$\cdot \int_{I_i} \tilde{f}(\xi) \sigma(\xi)^{1 - \alpha_0'} \sigma_1(\xi)^{1 - \alpha_1'} \sigma_2(\xi)^{1 - \alpha_2'} \sigma_3(\xi)^{1 - \alpha_3'} (\sigma(x + \xi)\sigma(x - \xi))^{-\eta} d\xi \tag{6.7}$$

$(i \in \{0, 1, 2, 3\})$ satisfy

$$H^{(\alpha_0' + d, \alpha_1' + d, \alpha_2' + d, \alpha_3' + d)} f(x) = Ef(x). \tag{6.8}$$

PROOF.   Let $\tilde{f}(x)$ be a solution of $H^{(l_0', l_1', l_2', l_3')} \tilde{f}(x) = E'\tilde{f}(x)$. By the transformation given by (6.2), the function $\underline{f}(w) = \tilde{f}(\tilde{\wp}^{-1}(w)) w^{-\alpha_1'/2} (w - 1)^{-\alpha_2'/2} (w - t)^{-\alpha_3'/2}$ is a solution of (6.3) where $\tilde{\wp}^{-1}(w)$ is the inverse function of $w = \tilde{\wp}(\xi) = (\wp(\xi) - e_1)/(e_2 - e_1)$, the parameters are given by (6.5) and we choose $\beta' = (\alpha_1' + \alpha_2' + \alpha_3' + \alpha_0')/2 = -d$. Next we apply Proposition 1.2 with the parameter $\eta = 2 - \beta'$. Then the functions $\int_{[\gamma_z, \gamma_p]} \underline{f}(w)(z - w)^{-\eta} dw$ $(p = 0, 1, t, \infty)$ are solutions of (1.12) and we have $\epsilon_0 = \epsilon_0' - \eta' + 1 = \alpha_1' + d + 3/2$, $\epsilon_1 = \alpha_2' + d + 3/2$, $\epsilon_t = \alpha_3' + d + 3/2$ and $\{\alpha, \beta\} = \{2 - \beta', -\alpha' + \beta' + 1\} = \{2 + d, \alpha_0' + 1/2\}$. The value q is expressed in terms of E' and other parameters. We set $\alpha_i = \alpha_i' + d + 1 (\in \{-(\alpha_i' + d), \alpha_i' + d + 1\})$ $(i = 0, 1, 2, 3)$ and transform to the elliptical form by (6.2), (6.5) where the prime $(')$ is omitted. It is shown by a direct calculation that the value E coincides with the original value E', and the functions

$$f(x) = z^{(\alpha_1' + d + 1)/2} (z - 1)^{(\alpha_2' + d + 1)/2} (z - t)^{(\alpha_3' + d + 1)/2}$$

$$\cdot \int_{[\gamma_z,\gamma_p]} \tilde{f}(\tilde{\wp}^{-1}(w))w^{-\alpha_1'/2}(w-1)^{-\alpha_2'/2}(w-t)^{-\alpha_3'/2}(z-w)^{-\eta}dw \qquad (6.9)$$

are solutions of $H^{(\alpha_0'+d,\alpha_1'+d,\alpha_2'+d,\alpha_3'+d)}f(x) = E'f(x)$ for $p \in \{0,1,t,\infty\}$, where $z = (\wp(x)-e_1)/(e_2-e_1)$. On the transformation $w = (\wp(\xi)-e_1)/(e_2-e_1)$, the cycles $[\gamma_z,\gamma_\infty]$, $[\gamma_z,\gamma_0]$, $[\gamma_z,\gamma_1]$, $[\gamma_z,\gamma_t]$ correspond to the cycles I_0, I_1, I_2, I_3. By changing the variable as $w = (\wp(\xi)-e_1)/(e_2-e_1)$ in (6.9) and applying the relations $\sqrt{\wp(\xi)-e_i} = \sigma_i(\xi)/\sigma(\xi)$ $(i=1,2,3)$, $\wp(x)-\wp(\xi) = -\sigma(x+\xi)\sigma(x-\xi)/(\sigma(x)\sigma(\xi))^2$ and $\wp'(\xi) = -2\sigma_1(\xi)\sigma_2(\xi)\sigma_3(\xi)/\sigma(\xi)^3$, we obtain the proposition. $\qquad\square$

PROPOSITION 6.2. *Set*

$$\alpha_0 \in \{-l_0, l_0+1\}, \; l_0' = \frac{-\alpha_0 - l_1 - l_2 - l_3}{2} - 1, \; l_1' = \frac{\alpha_0 + l_1 - l_2 - l_3}{2} - 1, \qquad (6.10)$$

$$l_2' = \frac{\alpha_0 - l_1 + l_2 - l_3}{2} - 1, \; l_3' = \frac{\alpha_0 - l_1 - l_2 + l_3}{2} - 1, \; \eta = \frac{\alpha_0 - l_1 - l_2 - l_3}{2}.$$

If $\tilde{f}(x)$ satisfies $H^{(l_0', l_1', l_2', l_3')}\tilde{f}(x) = E\tilde{f}(x)$, then the functions

$$f(x) = \sigma(x)^{\alpha_0}\sigma_1(x)^{-l_1}\sigma_2(x)^{-l_2}\sigma_3(x)^{-l_3}$$

$$\cdot \int_{I_i} \tilde{f}(\xi)\sigma(\xi)^{l_0'+1}\sigma_1(\xi)^{l_1'+1}\sigma_2(\xi)^{l_2'+1}\sigma_3(\xi)^{l_3'+1}(\sigma(x+\xi)\sigma(x-\xi))^{-\eta}d\xi \qquad (6.11)$$

$(i \in \{0,1,2,3\})$ *satisfy*

$$H^{(l_0, l_1, l_2, l_3)}f(x) = Ef(x). \qquad (6.12)$$

PROOF. We obtain the proposition by applying Theorem 6.1 for $\alpha_0' = 1 + (\alpha_0 + l_1 + l_2 + l_3)/2$, $\alpha_1' = 1 + (-\alpha_0 - l_1 + l_2 + l_3)/2$, $\alpha_2' = 1 + (-\alpha_0 + l_1 - l_2 + l_3)/2$, $\alpha_3' = 1 + (-\alpha_0 + l_1 + l_2 - l_3)/2$. Note that $H^{(-l_0-1, -l_1-1, -l_2-1, -l_3-1)} = H^{(l_0, l_1, l_2, l_3)}$. $\qquad\square$

We review an aspect of the monodromy of a differential equation with periodic potential. Let $q(x)$ be a periodic function with a period T and $\{f_1(x), f_2(x)\}$ be a basis of solutions of the differential equation

$$\left(-\frac{d^2}{dx^2} + q(x)\right)f(x) = Ef(x). \qquad (6.13)$$

Then $f_1(x+T)$ and $f_2(x+T)$ are also solutions. Let M_T be a monodromy matrix for the shift $x \to x+T$ with respect to the basis $\{f_1(x), f_2(x)\}$, i.e.

$$(f_1(x+T), f_2(x+T)) = (f_1(x), f_2(x))M_T = (f_1(x), f_2(x))\begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}. \qquad (6.14)$$

Since the coefficient of the first-order derivative in (6.13) is zero, the following lemma is satisfied;

LEMMA 6.3. *We have* $\det M_T = 1$.

PROOF. Since $f_i''(x) = (q(x) - E)f_i(x)$ $(i = 1, 2)$, we have $(f_1'(x)f_2(x) - f_1(x)f_2'(x))' = 0$. Hence $f_1'(x)f_2(x) - f_1(x)f_2'(x) = C$ for some constant $C \neq 0$ which follows from the linear independence of $f_1(x), f_2(x)$. We have

$$
\begin{aligned}
C &= f_1'(x + T)f_2(x + T) - f_1(x + T)f_2'(x + T) \\
&= (m_{11}m_{22} - m_{12}m_{21})(f_1'(x)f_2(x) - f_1(x)f_2'(x)) = (m_{11}m_{22} - m_{12}m_{21})C. \quad (6.15)
\end{aligned}
$$

Hence $\det M_T = 1$.   □

Note that this situation is applicable to the elliptical representation of Heun's equation by setting $T = 2\omega_1$ or $2\omega_3$ (or any period of the elliptic function $\wp(x)$). If $\text{tr} M_T > 2$ or $\text{tr} M_T < -2$ (resp. $-2 < \text{tr} M_T < 2$), then there exists a basis of solutions $f_+(x), f_-(x)$ such that $f_\pm(x + T) = e^{\pm\nu}f_\pm(x)$ (resp. $f_\pm(x + T) = e^{\pm\sqrt{-1}\nu}f_\pm(x)$) for some $\nu \in \mathbb{R}$ such that $e^{2\nu} - (\text{tr} M_T)e^\nu + 1 = 0$ (resp. $e^{2\sqrt{-1}\nu} - (\text{tr} M_T)e^{\sqrt{-1}\nu} + 1 = 0$). If $\text{tr} M_T = 2$ (resp. $\text{tr} M_T = -2$), then there exists a non-zero periodic (anti-periodic) solution, i.e. a solution $f(x)$ such that $f(x + T) = f(x)$ (resp. $f(x + T) = -f(x)$). It does not simply follow from $\text{tr} M_T = 2$ (resp. $\text{tr} M_T = -2$) that every solution is periodic (resp. anti-periodic). Whether this is the case is determined by the Jordan normal form of M_T.

THEOREM 6.4. Let $k \in \{1, 3\}$ and $M_{2\omega_k}^{(l_0, l_1, l_2, l_3)}(E)$ be the monodromy matrix by the shift of the period $x \to x + 2\omega_k$ with respect to a certain basis of solutions to $H^{(l_0, l_1, l_2, l_3)}f(x) = Ef(x)$. Let α_i' be a number such that $\alpha_i' = -l_i'$ or $\alpha_i' = l_i' + 1$ for each $i \in \{0, 1, 2, 3\}$ and set $d = -\sum_{i=0}^{3}\alpha_i'/2$. Then

$$
\text{tr} M_{2\omega_k}^{(l_0', l_1', l_2', l_3')}(E) = \text{tr} M_{2\omega_k}^{(\alpha_0' + d, \alpha_1' + d, \alpha_2' + d, \alpha_3' + d)}(E). \quad (6.16)
$$

PROOF. We prove the case $k = 1$ such that the shift of the period $x \to x + 2\omega_1$ corresponds to a cycle which is homotopic to $\gamma_t\gamma_1$. Let $\tilde{f}(x)$ (resp. $f(x)$) be a solution of (6.6) (resp. (6.8)). Then the function $\tilde{f}(\tilde{\wp}^{-1}(w))w^{-\alpha_1'/2}(w - 1)^{-\alpha_2'/2}(w - t)^{-\alpha_3'/2}$ (resp. $f(\tilde{\wp}^{-1}(z))z^{-(\alpha'1+d+1)/2}(z-1)^{-(\alpha'2+d+1)/2}(z-t)^{-(\alpha'3+d+1)/2})$ is a solution of (1.10) (resp. (1.12)). Thus we have $\exp(-2\pi\sqrt{-1}(\alpha_2' + \alpha_3')/2)\text{tr} M_{2\omega_1}^{(l_0', l_1', l_2', l_3')}(E) = \text{tr} M'^{(t)}M'^{(1)}$ and $\exp(-2\pi\sqrt{-1}(\alpha_2' + \alpha_3' + 2d + 2)/2)\text{tr} M_{2\omega_1}^{(\alpha_0' + d, \alpha_1' + d, \alpha_2' + d, \alpha_3' + d)}(E) = \text{tr} M^{(t)}M^{(1)}$. It follows from Theorem 3.4 that $\text{tr}(M^{(t)}M^{(1)}) = \exp(-2\pi\sqrt{-1}\eta)\text{tr}(M'^{(t)}M'^{(1)})$. Combining these relations with the relation $\eta = d + 2$ in the proof of Theorem 6.1, we obtain (6.16). The other cases can be proved similarly.   □

COROLLARY 6.5. Assume that the parameters l_0, l_1, l_2, l_3, l_0', l_1', l_2', l_3' satisfy (6.10). Let $k \in \{1, 3\}$. Then

$$
\text{tr} M_{2\omega_k}^{(l_0', l_1', l_2', l_3')}(E) = \text{tr} M_{2\omega_k}^{(l_0, l_1, l_2, l_3)}(E). \quad (6.17)
$$

COROLLARY 6.6. We keep the notations in Theorem 6.4. Let $k \in \{1, 3\}$. If there exists a non-zero solution $\tilde{f}(x, E)$ of $(H^{(l_0', l_1', l_2', l_3')} - E)\tilde{f}(x, E) = 0$ such that $\tilde{f}(x + 2\omega_k, E) = C_k(E)\tilde{f}(x, E)$, then there exists a non-zero solution $f(x, E)$ of $(H^{(\alpha_0' + d, \alpha_1' + d, \alpha_2' + d, \alpha_3' + d)} - E)f(x, E) = 0$ such that $f(x + 2\omega_k, E) = C_k(E)f(x, E)$. In other word, periodicity is preserved by the integral transformation.

PROOF. Let t'_k (resp. t_k) be a solution of the quadratic equation $(t'_k)^2 - \mathrm{tr}M_{2\omega_k}^{(l'_0,l'_1,l'_2,l'_3)}(E)t'_k + 1 = 0$ (resp. $t_k^2 - \mathrm{tr}M_{2\omega_k}^{(\alpha'_0+d,\alpha'_1+d,\alpha'_2+d,\alpha'_3+d)}(E)t_k + 1 = 0$). Since $\det M_{2\omega_k}^{(l'_0,l'_1,l'_2,l'_3)}(E) = \det M_{2\omega_k}^{(\alpha'_0+d,\alpha'_1+d,\alpha'_2+d,\alpha'_3+d)}(E) = 1$, the value t'_k (resp. t_k) is an eigenvalue of the monodromy matrix $M_{2\omega_k}^{(l'_0,l'_1,l'_2,l'_3)}(E)$ (resp. $M_{2\omega_k}^{(\alpha'_0+d,\alpha'_1+d,\alpha'_2+d,\alpha'_3+d)}(E)$) and we have $\{C_k(E),C_k(E)^{-1}\} = \{t'_k,(t'_k)^{-1}\}$. Thus Corollary 6.6 follows from $\mathrm{tr}M_{2\omega_k}^{(l'_0,l'_1,l'_2,l'_3)}(E) = \mathrm{tr}M_{2\omega_k}^{(\alpha'_0+d,\alpha'_1+d,\alpha'_2+d,\alpha'_3+d)}(E)$. $\square$

COROLLARY 6.7. *Assume that the parameters l_0, l_1, l_2, l_3, l'_0, l'_1, l'_2, l'_3 satisfy (6.10). Let $k \in \{1,3\}$. If there exists a non-zero solution $\tilde{f}(x,E)$ of $(H^{(l'_0,l'_1,l'_2,l'_3)} - E)\tilde{f}(x,E) = 0$ such that $\tilde{f}(x+2\omega_k,E) = C_k(E)\tilde{f}(x,E)$, then there exists a non-zero solution $f(x,E)$ of (6.12) such that $f(x+2\omega_k,E) = C_k(E)f(x,E)$.*

If $\omega_1 \in \mathbb{R}_{\neq 0}$ and $\omega_3 \in \sqrt{-1}\mathbb{R}_{\neq 0}$, then the potential $\sum_{i=0}^{3} l_i(l_i+1)\wp(x+\omega_i)$ in (1.14) is real-valued for $x \in \mathbb{R}$. From the viewpoint of quantum mechanics, we are interested in finding square-integrable eigenstates in a suitable Hilbert space for the elliptical representation of Heun's equation, and periodicity with respect to the shift $x \to x + 2\omega_1$ is related to square-integrable eigenstates (see [**17**], [**18**]). Ruijsenaars [**13**] established that the spectrum of (6.6) coincides with that of (6.12) by investigating a certain Hilbert–Schmidt operator. Theorem 6.4 can be regarded as a complex-functional version of Ruijsenaars' result. Khare and Sukhatme [**9**] earlier made a conjecture about correspondences between quasi-solvable solutions of (6.6) and those of (6.12), and Corollary 6.6 gives an approach for a reformulation of their conjecture in terms of monodromy.

For elliptical representations, quasi-solvability is described as follows:

PROPOSITION 6.8 ([**17**, Proposition 5.1]). *Let β'_i be a number such that $\beta'_i = -l'_i$ or $\beta'_i = l'_i + 1$ for each $i \in \{0,1,2,3\}$, and set $\tilde{d} = -\sum_{i=0}^{3}\beta'_i/2$. Suppose that $\tilde{d} \in \mathbb{Z}_{\geq 0}$, and let $V_{\beta'_0,\beta'_1,\beta'_2,\beta'_3}$ be the $\tilde{d}+1$-th dimensional space spanned by*

$$\left\{\widehat{\Phi}(\wp(x))\wp(x)^n\right\}_{n=0,\ldots,\tilde{d}}, \tag{6.18}$$

where $\widehat{\Phi}(z) = (z-e_1)^{\beta'_1/2}(z-e_2)^{\beta'_2/2}(z-e_3)^{\beta'_3/2}$. Then the operator $H^{(l'_0,l'_1,l'_2,l'_3)}$ preserves the space $V_{\beta'_0,\beta'_1,\beta'_2,\beta'_3}$.

To find eigenvalues of the operator $H^{(l'_0,l'_1,l'_2,l'_3)}$ on the space $V_{\beta'_0,\beta'_1,\beta'_2,\beta'_3}$, we obtain an algebraic equation of order $\tilde{d}+1$ in the variable E, which is related to $P(q') = 0$ in Proposition 5.1 for the case $\nu_0 = (\beta'_1 - \alpha'_1)/2$, $\nu_1 = (\beta'_2 - \alpha'_2)/2$, $\nu_t = (\beta'_3 - \alpha'_3)/2$, where $\alpha'_i \in \{-l'_i, l'_i+1\}$ and the transformation between Proposition 5.1 and Proposition 6.8 is determined by (6.2). The eigenvector corresponding to the eigenvalue E can be written as a product of $\widehat{\Phi}(\wp(x))$ and the polynomial in the variable $\wp(x)$ of degree no more than $\tilde{d}$.

Since the functions $\wp(x+2\omega_i)$ ($i = 0,1,2,3$) are even and doubly periodic, the solutions of (6.12) about $x = \omega_i$ ($i = 0,1,2,3$) can be expanded as

$$f(x) = \begin{cases} C^{\langle i \rangle} \displaystyle\sum_{j=0}^{\infty} c_j^{(i)} (x-\omega_i)^{-l_i+2j} + D^{\langle i \rangle} \sum_{j=0}^{\infty} \tilde{c}_j^{(i)} (x-\omega_i)^{l_i+1+2j}, & l_i \notin 1/2 + \mathbb{Z}, \\[2em] C^{\langle i \rangle} \displaystyle\sum_{j=0}^{\infty} c_j^{(i)} (x-\omega_i)^{|l_i+1/2|+1/2+2j} + D^{\langle i \rangle} & l_i \in 1/2 + \mathbb{Z}, \\[1em] \quad \cdot \left(\displaystyle\sum_{j=0}^{\infty} \tilde{c}_j^{(i)} (x-\omega_i)^{-|l_i+1/2|+1/2+2j} + A^{\langle i \rangle} \sum_{j=0}^{\infty} c_j^{(i)} (x-\omega_i)^{|l_i+1/2|+1/2+2j} \right), \end{cases}$$
$$(6.19)$$

where $C^{\langle i \rangle}$ and $D^{\langle i \rangle}$ are constants, $c_0^{(i)} = \tilde{c}_0^{(i)} = 1$, and $c_j^{(i)}$ and $\tilde{c}_j^{(i)}$ $(j = 1, 2, \dots)$ are determined recursively. If $l_i \in 1/2 + \mathbb{Z}$ and $A^{\langle i \rangle} \neq 0$ (resp. $A^{\langle i \rangle} = 0$), then the singularity $x = \omega_i$ is non-apparent (resp. apparent). Note that if $l_i = -1/2$, then the singularity $x = \omega_i$ is always non-apparent, i.e. $A^{\langle i \rangle} \neq 0$. By the transformation given by (6.2), the condition that $l_0 \in 1/2 + \mathbb{Z}$ (resp. $l_1 \in 1/2 + \mathbb{Z}$, $l_2 \in 1/2 + \mathbb{Z}$, $l_3 \in 1/2 + \mathbb{Z}$) and the singularity $x = 0$ (resp. $x = \omega_1$, $x = \omega_2$, $x = \omega_3$) is (non-)apparent is equivalent to that $\alpha - \beta \in \mathbb{Z}$ (resp. $\epsilon_0 \in \mathbb{Z}$, $\epsilon_1 \in \mathbb{Z}$, $\epsilon_t \in \mathbb{Z}$) and the singularity $z = \infty$ (resp. $z = 0$, $z = 1$, $z = t$) is (non-)apparent. The condition that the singularity $x = \omega_i$ $(i \in \{0,1,2,3\})$ is apparent (i.e. $A^{\langle i \rangle} = 0$) for the case $l_i \in -1/2 + \mathbb{Z}_{\neq 0}$ is described as follows: Set $j_0 = -|l_i + 1/2| + 1/2$, $\tilde{c}_0^{(i)} = 1$, $f(x) = \sum_{j=0}^{\infty} \tilde{c}_j^{(i)} (x - \omega_i)^{j_0 + 2j}$. By substituting $f(x)$ into (6.12) and expanding (6.12) as a series in $x - \omega_i$, we obtain an equation for $\tilde{c}_0^{(i)}, \tilde{c}_1^{(i)}, \dots \tilde{c}_j^{(i)}$ on the coefficients of $(x - \omega_i)^{j_0 + 2j - 2}$. We determine $\tilde{c}_{j'}^{(i)}$ $(j' = 1, \dots, |l_i + 1/2| - 1)$ by solving the equation for $\tilde{c}_0^{(i)}, \tilde{c}_1^{(i)}, \dots \tilde{c}_{j'}^{(i)}$ recursively for each j' and we have $\deg_E \tilde{c}_{j'}^{(i)} = j'$. On the coefficient of $(x - \omega_i)^{j_0 + |2l_i + 1| - 2}$, the term concerned with $\tilde{c}_{|l_i + 1/2|}^{(i)}$ disappears and we have an algebraic equation of degree $|l_i + 1/2|$ with respect to the variable E, which we denote by $P^{(i)}(E) = 0$, where $P^{(i)}(E)$ is monic. Then the condition that the singularity $x = \omega_i$ $(i \in \{0,1,2,3\})$ is apparent is equivalent to the eigenvalue E satisfying $P^{(i)}(E) = 0$. The following proposition can be proved by rewriting Theorem 4.2 in its elliptical form.

PROPOSITION 6.9. *Let α_i' be a number such that $\alpha_i' = -l_i'$ or $\alpha_i' = l_i' + 1$ for each $i \in \{0, 1, 2, 3\}$. Set $d = -\sum_{i=0}^{3} \alpha_i'/2$.*
(i) *If $\alpha_0' \in 3/2 + \mathbb{Z}_{\geq 0}$ (resp. $\alpha_1' \in 3/2 + \mathbb{Z}_{\geq 0}$, $\alpha_2' \in 3/2 + \mathbb{Z}_{\geq 0}$, $\alpha_3' \in 3/2 + \mathbb{Z}_{\geq 0}$), $d \notin \mathbb{Z}$ and the singularity $x = 0$ (resp. $x = \omega_1$, $x = \omega_2$, $x = \omega_3$) of (6.6) is apparent, then there exists a non-zero solution of (6.8) which belongs to the space $V_{-\alpha_0'-d, \alpha_1'+d+1, \alpha_2'+d+1, \alpha_3'+d+1}$ (resp. $V_{\alpha_0'+d+1, -\alpha_1'-d, \alpha_2'+d+1, \alpha_3'+d+1}$, $V_{\alpha_0'+d+1, \alpha_1'+d+1, -\alpha_2'-d, \alpha_3'+d+1}$, $V_{\alpha_0'+d+1, \alpha_1'+d+1, \alpha_2'+d+1, -\alpha_3'-d}$).*
(ii) *If $\alpha_0' \in -1/2 + \mathbb{Z}_{\leq 0}$ (resp. $\alpha_1' \in -1/2 + \mathbb{Z}_{\leq 0}$, $\alpha_2' \in -1/2 + \mathbb{Z}_{\leq 0}$, $\alpha_3' \in -1/2 + \mathbb{Z}_{\leq 0}$), $\alpha_1' + d, \alpha_2' + d, \alpha_3' + d \notin 1/2 + \mathbb{Z}$ (resp. $\alpha_0' + d, \alpha_2' + d, \alpha_3' + d \notin 1/2 + \mathbb{Z}$, $\alpha_0' + d, \alpha_1' + d, \alpha_3' + d \notin 1/2 + \mathbb{Z}$, $\alpha_0' + d, \alpha_1' + d, \alpha_2' + d \notin 1/2 + \mathbb{Z}$), $d \notin \mathbb{Z}$ and the singularity $x = 0$ (resp. $x = \omega_1$, $x = \omega_2$, $x = \omega_3$) of (6.6) is apparent, then there exists a non-zero solution of (6.8) which belongs to the space $V_{\alpha_0'+d+1, -\alpha_1'-d, -\alpha_2'-d, -\alpha_3'-d}$ (resp. $V_{-\alpha_0'-d, \alpha_1'+d+1, -\alpha_2'-d, -\alpha_3'-d}$, $V_{-\alpha_0'-d, -\alpha_1'-d, \alpha_2'+d+1, -\alpha_3'-d}$, $V_{-\alpha_0'-d, -\alpha_1'-d, -\alpha_2'-d, \alpha_3'+d+1}$).*
(iii) *If $\alpha_0' + d \in 1/2 + \mathbb{Z}_{\geq 0}$ (resp. $\alpha_1' + d \in 1/2 + \mathbb{Z}_{\geq 0}$, $\alpha_2' + d \in 1/2 + \mathbb{Z}_{\geq 0}$, $\alpha_3' + d \in 1/2 + \mathbb{Z}_{\geq 0}$),*

$l'_1, l'_2, l'_3 \notin 1/2 + \mathbb{Z}$ (*resp.* $l'_0, l'_2, l'_3 \notin 1/2 + \mathbb{Z}$, $l'_0, l'_1, l'_3 \notin 1/2 + \mathbb{Z}$, $l'_0, l'_1, l'_2 \notin 1/2 + \mathbb{Z}$), $d \notin \mathbb{Z}$ *and there exists a non-zero solution of* (6.6) *which belongs to the space* $V_{1-\alpha'_0, \alpha'_1, \alpha'_2, \alpha'_3}$ (*resp.* $V_{\alpha'_0, 1-\alpha'_1, \alpha'_2, \alpha'_3}$, $V_{\alpha'_0, \alpha'_1, 1-\alpha'_2, \alpha'_3}$, $V_{\alpha'_0, \alpha'_1, \alpha'_2, 1-\alpha'_3}$), *then the singularity* $x = 0$ (*resp.* $x = \omega_1$, $x = \omega_2$, $x = \omega_3$) *of* (6.8) *is apparent.*

(iv) *If* $\alpha'_0 + d \in -3/2 + \mathbb{Z}_{\leq 0}$ (*resp.* $\alpha'_1 + d \in -3/2 + \mathbb{Z}_{\leq 0}$, $\alpha'_2 + d \in -3/2 + \mathbb{Z}_{\leq 0}$, $\alpha'_3 + d \in -3/2 + \mathbb{Z}_{\leq 0}$), $l'_1, l'_2, l'_3 \notin 1/2 + \mathbb{Z}$ (*resp.* $l'_0, l'_2, l'_3 \notin 1/2 + \mathbb{Z}$, $l'_0, l'_1, l'_3 \notin 1/2 + \mathbb{Z}$, $l'_0, l'_1, l'_2 \notin 1/2 + \mathbb{Z}$), $d \notin \mathbb{Z}$ *and there exists a non-zero solution of* (6.6) *which belongs to the space* $V_{\alpha'_0, 1-\alpha'_1, 1-\alpha'_2, 1-\alpha'_3}$ (*resp.* $V_{1-\alpha'_0, \alpha'_1, 1-\alpha'_2, 1-\alpha'_3}$, $V_{1-\alpha'_0, 1-\alpha'_1, \alpha'_2, 1-\alpha'_3}$, $V_{1-\alpha'_0, 1-\alpha'_1, 1-\alpha'_2, \alpha'_3}$), *then the singularity* $x = 0$ (*resp.* $x = \omega_1$, $x = \omega_2$, $x = \omega_3$) *of* (6.8) *is apparent.*

With respect to the elliptical representation of Heun's equation, Theorems 5.2 and 5.4 can be rewritten as follows:

PROPOSITION 6.10. *Let* $\alpha_0 \in \{-l_0, l_0 + 1\}$ *and* $l'_0, l'_1, l'_2, l'_3, \eta$ *be the parameters defined in* (6.10).

(i) *If* $-\alpha_0 \in 1/2 + \mathbb{Z}_{\geq 0}$ (*resp.* $l_1 \in 1/2 + \mathbb{Z}_{\geq 0}$, $l_2 \in 1/2 + \mathbb{Z}_{\geq 0}$, $l_3 \in 1/2 + \mathbb{Z}_{\geq 0}$), $l'_1, l'_2, l'_3 \notin 1/2 + \mathbb{Z}$ (*resp.* $l'_0, l'_2, l'_3 \notin 1/2 + \mathbb{Z}$, $l'_0, l'_1, l'_3 \notin 1/2 + \mathbb{Z}$, $l'_0, l'_1, l'_2 \notin 1/2 + \mathbb{Z}$), $\eta \notin \mathbb{Z}$ *and the singularity* $x = 0$ (*resp.* $x = \omega_1$, $x = \omega_2$, $x = \omega_3$) *of* (6.12) *is apparent, then there exists a non-zero solution* $\tilde{f}(x)$ *of* (6.6) *which belongs to the space* $V_{-l'_0, l'_1 + 1, l'_2 + 1, l'_3 + 1}$ (*resp.* $V_{l'_0 + 1, -l'_1, l'_2 + 1, l'_3 + 1}$, $V_{l'_0 + 1, l'_1 + 1, -l'_2, l'_3 + 1}$, $V_{l'_0 + 1, l'_1 + 1, l'_2 + 1, -l'_3}$) *and the functions*

$$f(x) = \sigma(x)^{\alpha_0} \sigma_1(x)^{-l_1} \sigma_2(x)^{-l_2} \sigma_3(x)^{-l_3}$$
$$\cdot \int_{I_i} \tilde{f}(y) \sigma(y)^{l'_0 + 1} \sigma_1(y)^{l'_1 + 1} \sigma_2(y)^{l'_2 + 1} \sigma_3(y)^{l'_3 + 1} (\sigma(x+y) \sigma(x-y))^{-\eta} dy \quad (6.20)$$

for $i = 1, 2, 3$ (*resp.* $i = 2, 3$, $i = 1, 3$, $i = 1, 2$) *are non-zero solutions of* (6.12).

(ii) *If* $-\alpha_0 \in -3/2 + \mathbb{Z}_{\leq 0}$ (*resp.* $l_1 \in -3/2 + \mathbb{Z}_{\leq 0}$, $l_2 \in -3/2 + \mathbb{Z}_{\leq 0}$, $l_3 \in -3/2 + \mathbb{Z}_{\leq 0}$), $l'_1, l'_2, l'_3 \notin 1/2 + \mathbb{Z}$ (*resp.* $l'_0, l'_2, l'_3 \notin 1/2 + \mathbb{Z}$, $l'_0, l'_1, l'_3 \notin 1/2 + \mathbb{Z}$, $l'_0, l'_1, l'_2 \notin 1/2 + \mathbb{Z}$), $\eta \notin \mathbb{Z}$ *and the singularity* $x = 0$ (*resp.* $x = \omega_1$, $x = \omega_2$, $x = \omega_3$) *of* (6.12) *is apparent, then there exists a non-zero solution* $\tilde{f}(x)$ *of* (6.6) *which belongs to the space* $V_{l'_0 + 1, -l'_1, -l'_2, -l'_3}$ (*resp.* $V_{-l'_0, l'_1 + 1, -l'_2, -l'_3}$, $V_{-l'_0, -l'_1, l'_2 + 1, -l'_3}$, $V_{-l'_0, -l'_1, -l'_2, l'_3 + 1}$) *and the functions*

$$f(x) = \sigma(x)^{-\alpha_0 + 1} \sigma_1(x)^{l_1 + 1} \sigma_2(x)^{l_2 + 1} \sigma_3(x)^{l_3 + 1}$$
$$\cdot \int_{I_i} \tilde{f}(y) \sigma(y)^{-l'_0} \sigma_1(y)^{-l'_1} \sigma_2(y)^{-l'_2} \sigma_3(y)^{-l'_3} (\sigma(x+y) \sigma(x-y))^{\eta - 2} dy \quad (6.21)$$

for $i = 1, 2, 3$ (*resp.* $i = 2, 3$, $i = 1, 3$, $i = 1, 2$) *are non-zero solutions of* (6.12).

7. Finite-gap potentials and integral transformations.

We now review the definitions of a finite-gap potential and its properties.

DEFINITION 1. Assume $q(x)$ is real-valued and continuous for $x \in \mathbb{R}$. We set $H = -d^2/dx^2 + q(x)$. Let $\sigma_b(H)$ be the set such that

$$E \in \sigma_b(H) \iff \text{All solutions of } (H - E)f(x) = 0 \text{ are bounded on } x \in \mathbb{R},$$

and $\overline{\sigma_b(H)}$ is the topological closure of $\sigma_b(H)$ in $\mathbb{R}$. If the set $\mathbb{R} \setminus \overline{\sigma_b(H)}$ can be written as

$$\mathbb{R} \setminus \overline{\sigma_b(H)} = (-\infty, E_0) \cup (E_1, E_2) \cup \cdots \cup (E_{2g-1}, E_{2g}), \tag{7.1}$$

with $E_0 < E_1 < \cdots < E_{2g}$ then $q(x)$ is called a finite-gap (g-gap) potential.

If $q(x)$ is real-valued and continuous for $x \in \mathbb{R}$ and periodic with period $T(> 0)$, then $|\mathrm{tr} M_T| > 2 \Rightarrow E \notin \sigma_b(H)$ and $|\mathrm{tr} M_T| < 2 \Rightarrow E \in \sigma_b(H)$, where M_T is a monodromy matrix for the shift $x \to x + T$ with eigenvalue E.

DEFINITION 2.   If there exists an odd-order differential operator $A = (d/dx)^{2g+1} + \sum_{j=0}^{2g-1} b_j(x) (d/dx)^{2g-1-j}$ such that $[A, -d^2/dx^2 + q(x)] = 0$, then $q(x)$ is called an algebro-geometric finite-gap potential.

Note that the equation $[A, -d^2/dx^2 + q(x)] = 0$ is equivalent to the function $q(x)$ being a solution of some stationary higher-order KdV equation. It is known that if $q(x)$ is real-holomorphic on $\mathbb{R}$ and $q(x + T) = q(x)$, then $q(x)$ is a finite-gap potential if and only if $q(x)$ is an algebro-geometric finite-gap potential (see [11]).

For the elliptical representation of Heun's equation, the following theorem is known.

THEOREM 7.1 ([25]).   The potential $\sum_{i=0}^{3} l_i'(l_i' + 1)\wp(x + \omega_i)$ is algebro-geometric finite-gap, if and only if $l_i' \in \mathbb{Z}$ for $i = 0, 1, 2, 3$.

The function $\sum_{i=0}^{3} l_i'(l_i' + 1)\wp(x + \omega_i)$ is called the Treibich–Verdier potential. Subsequently several other researchers have produced results on this subject (see [2], [15], [16], [17], [18], [19], [20]). If $l_0' = l_1' = 0$, $\omega_1 \in \mathbb{R}_{\neq 0}$ and $\omega_3 \in \sqrt{-1}\mathbb{R}_{\neq 0}$, then the potential is real-valued and holomorphic on $\mathbb{R}$, and we have the following corollary:

COROLLARY 7.2.   If $\omega_1 \in \mathbb{R}_{\neq 0}$, $\omega_3 \in \sqrt{-1}\mathbb{R}_{\neq 0}$ and $l_2', l_3' \in \mathbb{Z}$, then the potential $l_2'(l_2' + 1)\wp(x + \omega_2) + l_3'(l_3' + 1)\wp(x + \omega_3)$ is a finite-gap potential.

We review a method for calculating the monodromy for the elliptical representation of Heun's equation for the case $l_0', l_1', l_2', l_3' \in \mathbb{Z}$. Note that (6.6) is invariant under the change $l_i' \leftrightarrow -l_i' - 1$ for each $i \in \{0, 1, 2, 3\}$.

Let $h(x)$ be the product of any pair of solutions of the elliptical representation of Heun's equation. Then the function $h(x)$ satisfies the following third-order differential equation:

$$\left(\frac{d^3}{dx^3} - 4 \left(\sum_{i=0}^{3} l_i'(l_i' + 1)\wp(x + \omega_i) - E \right) \frac{d}{dx} - 2 \left(\sum_{i=0}^{3} l_i'(l_i' + 1)\wp'(x + \omega_i) \right) \right) h(x) = 0. \tag{7.2}$$

It is known that if $l_0', l_1', l_2', l_3' \in \mathbb{Z}$ then (7.2) has a non-zero doubly periodic solution for all E.

PROPOSITION 7.3 ([**16**, Proposition 3.5]). *If $l_0', l_1', l_2', l_3' \in \mathbb{Z}$, then (7.2) has a non-zero doubly periodic solution $\Xi(x, E)$, which has the expansion*

$$\Xi(x, E) = c_0(E) + \sum_{i=0}^{3} \sum_{j=0}^{\max(l_i', -l_i'-1)-1} b_j^{(i)}(E)\wp(x + \omega_i)^{\max(l_i', -l_i'-1)-j}, \qquad (7.3)$$

where the coefficients $c_0(E)$ and $b_j^{(i)}(E)$ are polynomials in E, they do not have common divisors and the polynomial $c_0(E)$ is monic. We set $g = \deg_E c_0(E)$. Then the coefficients satisfy $\deg_E b_j^{(i)}(E) < g$ for all i and j.

Set

$$Q(E) = \Xi(x, E)^2 \left(E - \sum_{i=0}^{3} l_i'(l_i' + 1)\wp(x + \omega_i) \right) + \frac{1}{2}\Xi(x, E)\frac{d^2\Xi(x, E)}{dx^2} - \frac{1}{4}\left(\frac{d\Xi(x, E)}{dx} \right)^2. \qquad (7.4)$$

Then $Q(E)$ is independent of x and it is a monic polynomial in E of degree $2g + 1$ (see [**16**]). Solutions of Heun's equations can be written using $\Xi(x, E)$ and $Q(E)$.

PROPOSITION 7.4 ([**16**, Proposition 3.7]). *The functions*

$$\Lambda(x, E) = \sqrt{\Xi(x, E)} \exp \int \frac{\sqrt{-Q(E)}dx}{\Xi(x, E)} \qquad (7.5)$$

and $\Lambda(-x, E)$ are solutions of (6.6).

Write

$$\Xi(x, E) = c(E) + \sum_{i=0}^{3} \sum_{j=0}^{\max(l_i', -l_i'-1)-1} a_j^{(i)}(E)\left(\frac{d}{dx} \right)^{2j} \wp(x + \omega_i), \qquad (7.6)$$

and set

$$a(E) = \sum_{i=0}^{3} a_0^{(i)}(E). \qquad (7.7)$$

Then the monodromy with respect to the shift of a period can be written in terms of a hyperelliptic integral.

PROPOSITION 7.5 ([**18**], [**19**]). *Assume $l_0', l_1', l_2', l_3' \in \mathbb{Z}$.*
(i) If $Q(E_0) = 0$, then there exists $q_k \in \{0, 1\}$ such that $\Lambda(x + 2\omega_k, E_0) = (-1)^{q_k}\Lambda(x, E_0)$ for each $k \in \{1, 3\}$.
(ii) If $Q(E) \neq 0$, then the functions $\Lambda(x, E)$ and $\Lambda(-x, E)$ are linearly independent and we have

$$\Lambda(\pm(x+2\omega_k),E) = (-1)^{q_k}\Lambda(\pm x,E)\exp\left(\mp\int_{E_0}^{E}\frac{\omega_k c(\tilde{E})-\eta_k a(\tilde{E})}{\sqrt{-Q(\tilde{E})}}d\tilde{E}\right). \qquad (7.8)$$

We introduce another expression of monodromy arising from the Hermite–Krichever Ansatz [**19**]. Set

$$\Phi_i(x,\alpha) = \frac{\sigma(x+\omega_i-\alpha)}{\sigma(x+\omega_i)}\exp(\zeta(\alpha)x), \qquad (i=0,1,2,3), \qquad (7.9)$$

where $\sigma(x)$ (resp. $\zeta(x)$) is the Weierstrass sigma (resp. zeta) function.

PROPOSITION 7.6 ([**19**]).   *Assume* $l'_0, l'_1, l'_2, l'_3 \in \mathbb{Z}$.   *There exist polynomials* $P_1(E),\dots,P_6(E)$ *such that, if the eigenvalue* E *satisfies* $P_2(E) \neq 0$, *then the function* $\Lambda(x,E)$ *in* (7.5) *can be written as*

$$\Lambda(x,E) = \exp(\kappa x)\left(\sum_{i=0}^{3}\sum_{j=0}^{|l'_i+1/2|-3/2}\tilde{b}_j^{(i)}\left(\frac{d}{dx}\right)^j\Phi_i(x,\alpha)\right), \qquad (7.10)$$

and the values α *and* κ *can be expressed as*

$$\wp(\alpha) = \frac{P_1(E)}{P_2(E)}, \quad \wp'(\alpha) = \frac{P_3(E)}{P_4(E)}\sqrt{-Q(E)}, \quad \kappa = \frac{P_5(E)}{P_6(E)}\sqrt{-Q(E)}. \qquad (7.11)$$

The periodicity of the function $\Lambda(\pm x,E)$ *in* (7.10) *is described as*

$$\Lambda(\pm(x+2\omega_k),E) = \exp(\pm(2\omega_k(\zeta(\alpha)+2\kappa)-2\eta_k\alpha))\Lambda(\pm x,E), \quad (k=1,3). \qquad (7.12)$$

If $P_2(E) = 0$, *then the function* $\Lambda(x,E)$ *in* (7.5) *can be expressed as a product of an exponential function and a doubly periodic function.*

We review a relationship between the polynomial $Q(E)$ and finite-dimensional invariant subspaces. We define a vector space V by

$$V = \begin{cases} U_{-l'_0,-l'_1,-l'_2,-l'_3} \oplus U_{-l'_0,-l'_1,l'_2+1,l'_3+1} \oplus U_{-l'_0,l'_1+1,-l'_2,l'_3+1} \oplus U_{-l'_0,l'_1+1,l'_2+1,-l'_3} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (l'_0+l'_1+l'_2+l'_3 : \text{ even}); \\ U_{-l'_0,-l'_1,-l'_2,l'_3+1} \oplus U_{-l'_0,-l'_1,l'_2+1,-l'_3} \oplus U_{-l'_0,l'_1+1,-l'_2,-l'_3} \oplus U_{l'_0+1,-l'_1,-l'_2,-l'_3} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (l'_0+l'_1+l'_2+l'_3 : \text{ odd}), \end{cases} \qquad (7.13)$$

where $U_{\alpha_0,\alpha_1,\alpha_2,\alpha_3}$ are defined by

$$U_{\alpha_0,\alpha_1,\alpha_2,\alpha_3} = \begin{cases} V_{\alpha_0,\alpha_1,\alpha_2,\alpha_3}, & \sum_{i=0}^{3}\alpha_i/2 \in \mathbb{Z}_{\leq 0}; \\ V_{1-\alpha_0,1-\alpha_1,1-\alpha_2,1-\alpha_3}, & \sum_{i=0}^{3}\alpha_i/2 \in \mathbb{Z}_{\geq 2}; \\ \{0\}, & \text{otherwise.} \end{cases} \qquad (7.14)$$

Then $H^{(l'_0,l'_1,l'_2,l'_3)}\cdot V \subset V$ and it can be shown that if $l'_0, l'_1, l'_2, l'_3 \in \mathbb{Z}$ then V is the maxi-

mum finite-dimensional H-invariant subspace of the space spanned by the function $f(x)$ such that $f(x+2\omega_k)/f(x) \in \{\pm 1\}$ for $k = 1, 3$. Let $P(E)$ be the monic characteristic polynomial of the operator $H^{(l_0', l_1', l_2', l_3')}$ on the space V, i.e. $P(E) = \det_V(E \cdot 1 - H^{(l_0', l_1', l_2', l_3')})$.

PROPOSITION 7.7 ([**20**]). *We have $P(E) = Q(E)$.*

The curve $\Gamma : \nu^2 = -Q(E)$ is called the spectral curve, which plays an important role in (7.5) and (7.8). It follows from Proposition 7.7 that edges of the hyperelliptic curve Γ are eigenvalues of the operator $H^{(l_0', l_1', l_2', l_3')}$ on the invariant space V. The genus of the curve Γ is g, where g is defined in Proposition 7.3.

Let us consider the case $Q(E) = 0$. Let E_0 be a zero of $Q(E)$. Then we have $P(E_0) = 0$, $\Lambda(x, E_0) = \sqrt{\Xi(x, E_0)} \in V$ and the functions $\Lambda(x, E_0)$ and $\Lambda(-x, E_0)$ are linearly dependent. Another solution of (6.6) can be derived as $\sqrt{\Xi(x, E_0)} \int \frac{dx}{\Xi(x, E_0)} (= \Lambda_2(x, E_0))$. The monodromy with respect to the shift of a period was calculated in [**21**] and it can be written as

$$(\Lambda(x + 2\omega_k, E_0), \Lambda_2(x + 2\omega_k, E_0))$$

$$= (-1)^{q_k}(\Lambda(x, E_0), \Lambda_2(x, E_0)) \begin{pmatrix} 1 & \frac{2\omega_k c(E) - 2\eta_k a(E)}{\frac{d}{dE} Q(E)} \Big|_{E \to E_0} \\ 0 & 1 \end{pmatrix}. \tag{7.15}$$

EXAMPLE 3. The case $l_0' = 2$, $l_1' = l_2' = l_3' = 0$. The doubly periodic function $\Xi(x, E)$ which satisfies (7.2) and the polynomial $Q(E)$ are evaluated as

$$\Xi(x, E) = 9\wp(x)^2 + 3E\wp(x) + E^2 - 9g_2/4, \tag{7.16}$$

$$Q(E) = (E^2 - 3g_2) \prod_{i=1}^{3}(E - 3e_i). \tag{7.17}$$

The function $\Lambda(x, E)$ defined by (7.5) is a solution of (6.6). For the monodromy with respect to the shift $x \to x + 2\omega_k$ $(k = 1, 3)$, we have a formula described by a hyperelliptic integral of genus two;

$$\Lambda(x + 2\omega_k, E) = \Lambda(x, E) \exp\left(-\frac{1}{2} \int_{\sqrt{3g_2}}^{E} \frac{\omega_k(2\tilde{E}^2 - 3g_2) - 6\eta_k\tilde{E}}{\sqrt{-(\tilde{E}^2 - 3g_2) \prod_{i=1}^{3}(\tilde{E} - 3e_i)}} d\tilde{E}\right). \tag{7.18}$$

The function $\Lambda(x, E)$ can be expressed in the form of the Hermite–Krichever Ansatz

$$\Lambda(x, E) = \exp(\kappa x) \left(\tilde{b}_0^{(0)} \Phi_0(x, \alpha) + \tilde{b}_1^{(0)} \left(\frac{d}{dx}\right) \Phi_0(x, \alpha)\right), \tag{7.19}$$

and α, κ satisfy

$$\wp(\alpha) = e_1 - \frac{(E - 3e_1)(E + 6e_1)^2}{9(E^2 - 3g_2)}, \quad \kappa = \frac{2}{3(E^2 - 3g_2)} \sqrt{-Q(E)}. \tag{7.20}$$

Set

$$V = V_{-2,0,0,0} \oplus V_{0,-1,-1,0} \oplus V_{0,-1,0,-1} \oplus V_{0,0,-1,-1}. \tag{7.21}$$

Then $\dim V = 2 + 1 + 1 + 1 = 5$ and $Q(E)$ is the characteristic polynomial of $H^{(2,0,0,0)}$ on the space V. The characteristic polynomial of $H^{(2,0,0,0)}$ on $V_{-2,0,0,0}$ (resp. $V_{0,-1,-1,0}$, $V_{0,-1,0,-1}$, $V_{0,0,-1,-1}$) is $E^2 - 3g_2$ (resp. $E - 3e_3$, $E - 3e_2$, $E - 3e_1$).

By applying integral transformation to the case of a finite-gap potential (i.e. applying Theorem 6.1 for the case $l'_0, l'_1, l'_2, l'_3 \in \mathbb{Z}$ while choosing $\alpha'_0 \in \{-l'_0, l'_0 + 1\}$ to be $\eta \in 1/2 + \mathbb{Z}$), we obtain Heun's equation for the case $l_0, l_1, l_2, l_3 \in \mathbb{Z} + 1/2$ and $l_0 + l_1 + l_2 + l_3 \in 2\mathbb{Z} + 1$. Conversely we can express solutions and monodromy for the case $l_0, l_1, l_2, l_3 \in \mathbb{Z} + 1/2$ and $l_0 + l_1 + l_2 + l_3 \in 2\mathbb{Z} + 1$ by using solutions and monodromy calculated by the finite-gap potential method for the case $l'_0, l'_1, l'_2, l'_3 \in \mathbb{Z}$. The following proposition is obtained by combining Proposition 6.2, Corollary 6.5, Propositions 7.5 and 7.6.

PROPOSITION 7.8.   *Let $\alpha_0 \in \{-l_0, l_0 + 1\}$ and set*

$$\eta = \frac{-\alpha_0 - l_1 - l_2 - l_3 + 1}{2}, \; l'_0 = \frac{-\alpha_0 + l_1 + l_2 + l_3 + 1}{2}, \tag{7.22}$$
$$l'_1 = \frac{-\alpha_0 + l_1 - l_2 - l_3 - 1}{2}, \; l'_2 = \frac{-\alpha_0 - l_1 + l_2 - l_3 - 1}{2}, \; l'_3 = \frac{-\alpha_0 - l_1 - l_2 + l_3 - 1}{2}.$$

If $l_0, l_1, l_2, l_3 \in \mathbb{Z} + 1/2$ and $l_0 + l_1 + l_2 + l_3 \in 2\mathbb{Z} + 1$, then we have $\eta \in \mathbb{Z} + 1/2$ and $l'_0, l'_1, l'_2, l'_3 \in \mathbb{Z}$. Let $M_{2\omega_k}$ $(k = 1, 3)$ be a monodromy matrix of solutions of (1.14) with respect to the shift $x \to x + 2\omega_k$ for the parameters l_0, l_1, l_2, l_3, E. Then we have

$$\mathrm{tr} M_{2\omega_k} = 2(-1)^{q_k} \cos \left(\int_{E_0}^E \frac{\omega_k c(\tilde{E}) - \eta_k a(\tilde{E})}{\sqrt{Q(\tilde{E})}} d\tilde{E} \right), \tag{7.23}$$

where $c(E)$ and $a(E)$ are defined in (7.6), (7.7) and E_0 is a zero of $Q(E)$ for the parameters $l'_0, l'_1, l'_2, l'_3, E$ such that $\Lambda(x + 2\omega_k, E_0) = (-1)^{q_k} \Lambda(x, E_0)$ for each $k \in \{1, 3\}$. We also have

$$\mathrm{tr} M_{2\omega_k} = 2 \cos \left(\sqrt{-1}(2\omega_k(\zeta(\alpha) + \kappa) - 2\eta_k \alpha) \right), \tag{7.24}$$

where α and κ are determined by (7.11) for the parameters $l'_0, l'_1, l'_2, l'_3, E$.

Note that Heun's equation in Proposition 7.8 for the parameter l'_0, l'_1, l'_2, l'_3 for the case $\alpha_0 = -l_0$ is isomonodromic to the one for the parameter l'_0, l'_1, l'_2, l'_3 for the case $\alpha_0 = l_0 + 1$, and they are linked by the generalized Darboux transformation described in [20]. If we replace the definition of the set $\overline{\sigma_b(H)}$ by the following; $E \in \overline{\sigma_b(H)} \Leftrightarrow -2 \leq \mathrm{tr} M_{2\omega_1} \leq 2$, then the set $\mathbb{R} \backslash \overline{\sigma_b(H)}$ for the case $l_0, l_1, l_2, l_3 \in \mathbb{Z} + 1/2$, $l_0 + l_1 + l_2 + l_3 \in 2\mathbb{Z} + 1$ and $\omega_1, \sqrt{-1}\omega_3 \in \mathbb{R}_{\neq 0}$ has finite gaps, which coincides with the one for the case l'_0, l'_1, l'_2, l'_3 in Proposition 7.8. But the potential for the case $l_0, l_1, l_2, l_3 \in \mathbb{Z} + 1/2$ and $l_0 + l_1 + l_2 + l_3 \in 2\mathbb{Z} + 1$ is not an algebro-geometric finite-gap potential.

It follows from Propositions 6.9 and 6.10 that the eigenvalues of the four spaces for l'_0, l'_1, l'_2, l'_3 in (7.13) corresponds to eigenvalues such that one of the singularities

$\{0, \omega_1, \omega_2, \omega_3\}$ is apparent. By combining these remarks with Proposition 7.7, we have the following proposition:

PROPOSITION 7.9. *Let* $\alpha_0 \in \{-l_0, l_0 + 1\}$ *and define the numbers* l'_0, l'_1, l'_2, l'_3 *by* (7.22). *Assume* $l_0, l_1, l_2, l_3 \in \mathbb{Z} + 1/2$, $l_0 + l_1 + l_2 + l_3 \in 2\mathbb{Z} + 1$ *and let* $Q(E)$ *be the polynomial in* (7.4) *for the parameters* $l'_0, l'_1, l'_2, l'_3 (\in \mathbb{Z})$.
(i) *The condition* $Q(E_0) = 0$ *is equivalent to the condition that there exists* $i \in \{0, 1, 2, 3\}$ *such that the singularity* $x = \omega_i$ *is apparent in* (6.12) *for the parameters* l_0, l_1, l_2, l_3.
(ii) *If* $l'_0 + l'_1 + l'_2 + l'_3$ *is even, then the characteristic polynomial of the operator* $H^{(l'_0, l'_1, l'_2, l'_3)}$ *on the space* $U_{-l'_0, -l'_1, -l'_2, -l'_3}$ *(resp.* $U_{-l'_0, -l'_1, l'_2+1, l'_3+1}$, $U_{-l'_0, l'_1+1, -l'_2, l'_3+1}$, $U_{-l'_0, l'_1+1, l'_2+1, -l'_3}$*)* *coincides with the polynomial* $P^{(0)}(E)$ *(resp.* $P^{(1)}(E)$, $P^{(2)}(E)$, $P^{(3)}(E)$*) for the parameters* l_0, l_1, l_2, l_3 *which are defined between* (6.19) *and Proposition* 6.10.
(iii) *If* $l'_0 + l'_1 + l'_2 + l'_3$ *is odd, then the characteristic polynomial of the operator* $H^{(l'_0, l'_1, l'_2, l'_3)}$ *on the space* $U_{-l'_0, -l'_1, -l'_2, l'_3+1}$ *(resp.* $U_{-l'_0, -l'_1, l'_2+1, -l'_3}$, $U_{-l'_0, l'_1+1, -l'_2, -l'_3}$, $U_{l'_0+1, -l'_1, -l'_2, -l'_3}$*)* *coincides with the polynomial* $P^{(3)}(E)$ *(resp.* $P^{(2)}(E)$, $P^{(1)}(E)$, $P^{(0)}(E)$*) for the parameters* l_0, l_1, l_2, l_3.
(iv) *We have* $Q(E) = P^{(0)}(E)P^{(1)}(E)P^{(2)}(E)P^{(3)}(E)$.

It was shown in [**16**] that if $l'_0, l'_1, l'_2, l'_3 \in \mathbb{Z}$, then any two spaces of the four spaces in (7.13) have no eigenvalues in common. Hence we have

PROPOSITION 7.10. *Assume* $l_0, l_1, l_2, l_3 \in \mathbb{Z} + 1/2$ *and* $l_0 + l_1 + l_2 + l_3 \in 2\mathbb{Z} + 1$. *Then any two of the four equations* $P^{(0)}(E) = 0$, $P^{(1)}(E) = 0$, $P^{(2)}(E) = 0$, $P^{(3)}(E) = 0$ *have no common solutions. In other words, if one of the singularities* $\{0, \omega_1, \omega_2, \omega_3\}$ *is apparent, then the other three singularities are non-apparent.*

Under the assumptions and notations in Proposition 7.9, we have $\deg_E Q(E) = \deg_E P^{(0)}(E) + \deg_E P^{(1)}(E) + \deg_E P^{(2)}(E) + \deg_E P^{(3)}(E) = |l_0 + 1/2| + |l_1 + 1/2| + |l_2 + 1/2| + |l_3 + 1/2|$. Hence the genus of the curve $\Gamma: \nu^2 = -Q(E)$ for $l'_0, l'_1, l'_2, l'_3 (\in \mathbb{Z})$ is obtained by applying Proposition 6.9 and by setting $\alpha'_0 = -l'_0$ (resp. $\alpha'_0 = l'_0 + 1$) for the case that $l'_0 + l'_1 + l'_2 + l'_3$ is even (resp. odd).

PROPOSITION 7.11. *Assume* $l'_0, l'_1, l'_2, l'_3 \in \mathbb{Z}$. *Let* g *be the genus of the curve* $\Gamma:$ $\nu^2 = -Q(E)$.
(i) *If* $l'_0 + l'_1 + l'_2 + l'_3$ *is even, then*

$$g = \frac{1}{2}\left(\left|\frac{l'_0 + l'_1 + l'_2 + l'_3}{2}\right| + \left|\frac{l'_0 + l'_1 - l'_2 - l'_3}{2}\right|\right.$$
$$\left. + \left|\frac{l'_0 - l'_1 + l'_2 - l'_3}{2}\right| + \left|\frac{l'_0 - l'_1 - l'_2 + l'_3}{2}\right|\right). \tag{7.25}$$

(ii) *If* $l'_0 + l'_1 + l'_2 + l'_3$ *is odd, then*

$$g = \frac{1}{2}\left(\left|\frac{-l'_0 + l'_1 + l'_2 + l'_3 + 1}{2}\right| + \left|\frac{l'_0 - l'_1 + l'_2 + l'_3 + 1}{2}\right|\right.$$
$$\left. + \left|\frac{l'_0 + l'_1 - l'_2 + l'_3 + 1}{2}\right| + \left|\frac{l'_0 + l'_1 + l'_2 - l'_3 + 1}{2}\right| - 1\right). \tag{7.26}$$

Note that the expression in Proposition 7.11 is different from the one in [**18**, Proposition 3.3].

EXAMPLE 4. For the case $l'_0 = l'_1 = l'_2 = l'_3 = 0$, (6.6) is written as $(d^2/dx^2 + E)f(x) = 0$ and a basis of solutions can be written as $e^{\kappa x}$, $e^{-\kappa x}$ for the case $E \neq 0$ by writing $E = -\kappa^2$. Hence we have $\mathrm{tr} M'_{2\omega_k} = e^{2\kappa\omega_k} + e^{-2\kappa\omega_k}$ $(k = 1, 3)$, where $M'_{2\omega_k}$ is a monodromy matrix of solutions of (6.6) for the case $l'_0 = l'_1 = l'_2 = l'_3 = 0$. There exists a non-zero periodic (resp. anti-periodic) solution with respect to the period $2\omega_1$, if and only if E can be written as $E = \pi^2 n^2/\omega_1^2$ (resp. $E = \pi^2(2n+1)^2/(2\omega_1)^2$) for some $n \in \mathbb{Z}_{\geq 0}$.

We apply an integral transformation of Theorem 6.1 for the case $\alpha'_0 = 1$, $\alpha'_1 = \alpha'_2 = \alpha'_3 = 0$. By replacing the contour integral I_i by twice of the integral from $-x + 2\omega_i$ to x and setting $E = -\kappa^2$, it follows from (6.11) that the function

$$f(x) = \left(\prod_{i=1}^{3} (\wp(x) - e_i) \right)^{1/4} \int_{-x+2\omega_i}^{x} \frac{e^{\tilde{\kappa}\xi}\sigma(x)\sigma(\xi)}{\sqrt{\sigma(x-\xi)\sigma(x+\xi)}} d\xi \qquad (7.27)$$

is a solution of (6.12) for the case $l_0 = 1/2$, $l_1 = l_2 = l_3 = -1/2$ for $i \in \{0, 1, 2, 3\}$, which reproduces the result in [**22**]. By Corollary 6.5 we have $\mathrm{tr} M_{2\omega_k} = e^{2\kappa\omega_k} + e^{-2\kappa\omega_k}$ $(k = 1, 3)$, where $M_{2\omega_k}$ is a monodromy matrix of solutions of (6.6) for the case $l_0 = 1/2$, $l_1 = l_2 = l_3 = -1/2$. It follows from Corollary 6.7 that there exists a non-zero periodic (resp. anti-periodic) solution with respect to the period $2\omega_1$, if and only of E can be written as $E = \pi^2 n^2/\omega_1^2$ (resp. $E = \pi^2(2n+1)^2/(2\omega_1)^2$) for some $n \in \mathbb{Z}_{\geq 0}$. As a sequel, if $\omega_1 \in \mathbb{R}_{>0}$ and $\omega_3 \in \sqrt{-1}\mathbb{R}_{\neq 0}$, then the spectrum of the operator $H^{(1/2,-1/2,-1/2,-1/2)}$ (see (6.1)) with respect to the interval $[0, \omega_1]$ can be expressed as $L^2([0, \omega_1]) = \{\pi^2 n^2/(2\omega_1)^2 \mid n \in \mathbb{Z}_{\geq 0}\}$, which reproduces the result by Ruijsenaars which was presented at the Bonn conference in 2008 (see [**13**]). Note that Heun's equation for the case $l_0 = 1/2$, $l_1 = l_2 = l_3 = -1/2$ was previously studied by Valent [**26**] to understand an eigenvalue problem related to certain birth and death processes.

EXAMPLE 5. We apply Proposition 7.8 to the case $l_0 = 3/2$, $l_1 = l_2 = l_3 = 1/2$. By setting $\alpha_0 = 5/2$, we have $l'_0 = -3$, $l'_1 = l'_2 = l'_3 = 0$ and $\eta = 1/2$ in Proposition 6.2. Let $M_{2\omega_k}$ $(k = 1, 3)$ be a monodromy matrix of solutions of (1.14) with respect to the shift $x \to x + 2\omega_k$ for the case $l_0 = 3/2$, $l_1 = l_2 = l_3 = 1/2$. Since $H^{(-3,0,0,0)} = H^{(2,0,0,0)}$, it follows from (7.18) and (7.23) that

$$\mathrm{tr} M_{2\omega_k} = 2\cos\left(-\frac{1}{2} \int_{\sqrt{3g_2}}^{E} \frac{\omega_k(2\tilde{E}^2 - 3g_2) - 6\eta_k\tilde{E}}{\sqrt{(\tilde{E}^2 - 3g_2)\prod_{i=1}^{3}(\tilde{E} - 3e_i)}} d\tilde{E} \right), \qquad (7.28)$$

and it follows from Corollary 6.7 that for each $k \in \{1, 3\}$ there exists a non-zero solution $f_k(x, E)$ of (6.12) for the case $l_0 = 3/2$, $l_1 = l_2 = l_3 = 1/2$ such that

$$f_k(x + 2\omega_k, E) = f_k(x, E)\exp\left(-\frac{1}{2} \int_{\sqrt{3g_2}}^{E} \frac{\omega_k(2\tilde{E}^2 - 3g_2) - 6\eta_k\tilde{E}}{\sqrt{-(\tilde{E}^2 - 3g_2)\prod_{i=1}^{3}(\tilde{E} - 3e_i)}} d\tilde{E} \right). \qquad (7.29)$$

We also have

$$\mathrm{tr}M_{2\omega_k} = 2\cos\left(\sqrt{-1}(2\omega_k(\zeta(\alpha)+\kappa)-2\eta_k\alpha)\right), \tag{7.30}$$

where α and κ are defined by

$$\wp(\alpha) = e_1 - \frac{(E-3e_1)(E+6e_1)^2}{9(E^2-3g_2)}, \quad \kappa = \frac{2}{3}\sqrt{\frac{-\prod_{i=1}^{3}(\tilde{E}-3e_i)}{E^2-3g_2}}. \tag{7.31}$$

It follows from Proposition 7.9 that the singularity $x=0$ (resp. $x=\omega_1, \omega_2, \omega_3$) for (1.14) on the case $l_0 = 3/2$, $l_1 = l_2 = l_3 = 1/2$ is apparent if and only if $E = \pm\sqrt{3g_2}$ (resp. $E = 3e_1, 3e_2, 3e_3$).

By setting $\alpha_0 = -3/2$, we have $l'_0 = -1$, $l'_1 = l'_2 = l'_3 = -2$, which can be replaced by $l'_0 = 0$, $l'_1 = l'_2 = l'_3 = 1$. The case $(l'_0, l'_1, l'_2, l'_3) = (0,1,1,1)$ is isomonodromic to the case $(l'_0, l'_1, l'_2, l'_3) = (2,0,0,0)$, and the two cases are linked by the generalized Darboux transformation in [**20**].

8. Summary and concluding remarks.

In this paper, we investigated correspondences of special solutions of Heun's differential equation (1.2) and the second-order linear differential equation $D_{y_1}(\theta_0, \theta_1, \theta_t, \theta_\infty; \lambda, \mu)$ (see (1.5)) by Euler's integral transformation. Namely, polynomial-type solutions are connected to the solutions such that one of the regular singularities is apparent. On the monodromy, the trace of a product of the local monodromy matrices is essentially preserved by the Euler's transformation (see Theorem 3.4), and it is written more clearly on the elliptical representation of Heun's equation (see Theorem 6.1). The monodromy of the elliptical representation of Heun's equation (1.14) in the case $l_0, l_1, l_2, l_3 \in \mathbb{Z} + 1/2$ and $l_0 + l_1 + l_2 + l_3 \in 2\mathbb{Z} + 1$ can be calculated by using the results on finite-gap integration.

The results of this paper would also be valid for confluent families of Heun's equation (see [**12**]) and the linear differential equation related with the other Painlevé equations. They should be presented clearly in a near future.

Appendix. Local expansions and the proof of Propositions 3.2, 3.3 and Theorems 4.2, 4.3.

We investigate local expansions of solutions of a second-order linear differential equation about a regular singularity and the image of the expansion mapped by an integral transformation.

We assume that the function $y(w)$ is a solution of a second-order linear differential equation about a regular singularity $w = p(\neq \infty)$, and the exponents of the second-order linear differential equation at $w = p$ are 0 and θ_p.

Then the function $y(w)$ can be expanded as

$$y(w) = C^{\langle p\rangle}f^{\langle p\rangle}(w) + D^{\langle p\rangle}g^{\langle p\rangle}(w), \quad (C^{\langle p\rangle}, D^{\langle p\rangle} \in \mathbb{C}), \tag{A.1}$$

such that

$$f^{\langle p\rangle}(w) = \begin{cases} \displaystyle\sum_{j=0}^{\infty} c_j^{(p)}(w-p)^j, & \theta_p \notin \mathbb{Z}_{\geq 0}, \\ \displaystyle (w-p)^{\theta_p}\sum_{j=0}^{\infty} c_j^{(p)}(w-p)^j, & \theta_p \in \mathbb{Z}_{\geq 0}, \end{cases} \tag{A.2}$$

$$g^{\langle p\rangle}(w) = \begin{cases} \displaystyle (w-p)^{\theta_p}\sum_{j=0}^{\infty} \tilde{c}_j^{(p)}(w-p)^j, & \theta_p \notin \mathbb{Z}, \\ \displaystyle (w-p)^{\theta_p}\left(\sum_{j=0}^{\infty} \tilde{c}_j^{(p)}(w-p)^j\right) + A^{\langle p\rangle}f^{\langle p\rangle}(w)\log(w-p), & \theta_p \in \mathbb{Z}_{\leq -1}, \\ \displaystyle \left(\sum_{j=0}^{\infty} \tilde{c}_j^{(p)}(w-p)^j\right) + A^{\langle p\rangle}f^{\langle p\rangle}(w)\log(w-p), & \theta_p \in \mathbb{Z}_{\geq 0}, \end{cases}$$

where $c_0^{(p)} = \tilde{c}_0^{(p)} = 1$. The function $f^{\langle p\rangle}(w)$ is holomorphic about $w = p$. The function $g^{\langle p\rangle}(w)$ is branching about $w = p$, if $\theta_p \notin \mathbb{Z}_{\neq 0}$ or $A^{\langle p\rangle} \neq 0$. If $\theta_p \in \mathbb{Z}_{\neq 0}$ and $A^{\langle p\rangle} = 0$, the singularity $w = p$ is apparent

We now describe a criterion that the singularity $w = p$ is apparent for the case $\theta_p \in \mathbb{Z}_{\neq 0}$. We denote the differential equation which the function $y(w)$ satisfies by

$$\frac{d^2y}{dw^2} + \left(\sum_{i=0}^{\infty} r_i(w-p)^{i-1}\right)\frac{dy}{dw} + \left(\sum_{i=0}^{\infty} s_i(w-p)^{i-2}\right)y = 0. \tag{A.3}$$

Let $F(\xi) = \xi^2 + (p_0 - 1)\xi + q_0 = \xi(\xi - \theta_p)$ be the characteristic polynomial about $w = p$. If the function $y = w^\rho(\sum_{i=0}^{\infty} c_i z^i)$ $(c_0 = 1)$ satisfies (A.3), then we have $F(\rho) = 0$ and

$$F(\rho + n)c_n + \sum_{i=1}^{n}\{(n-i+\rho)r_i + s_i\}c_{n-i} = 0. \tag{A.4}$$

If $F(\rho + n) \neq 0$ for all $n \in \mathbb{Z}_{\geq 1}$, then the coefficients c_n are determined recursively. In particular, the coefficients c_n are determined recursively for the case $\theta_p \notin \mathbb{Z}$. We consider the case $\theta_p \in \mathbb{Z}_{\geq 1}$. Set $\rho = 0$. The coefficients $c_1, \ldots c_{\theta_p - 1}$ are determined recursively. We substitute $n = \theta_p$ in (A.4). Then

$$\sum_{i=1}^{\theta_p}\{(\theta_p - i)r_i + s_i\}c_{\theta_p - i} = 0, \tag{A.5}$$

and it gives an equivalent condition to that the singularity $w = p$ is apparent (i.e. $A^{\langle p\rangle} = 0$). A condition of apparency of the singularity $w = p$ for the case $\theta_p \in \mathbb{Z}_{\leq -1}$ is given by (A.4) for the case $\rho = \theta_p$ and $n = -\theta_p$.

We investigate the local expansion of the function $\int_{[\gamma_z,\gamma_p]} y(w)(z-w)^\kappa dw$ about $w = p$ for the case $\kappa \notin \mathbb{Z}$. Set

$$d_{\alpha,\beta} = \begin{cases} (e^{2\pi\sqrt{-1}\alpha} - 1)(e^{2\pi\sqrt{-1}\beta} - 1)\dfrac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}, & \alpha \notin \mathbb{Z}, \\[2mm] 2\pi\sqrt{-1}(e^{2\pi\sqrt{-1}\beta} - 1)\dfrac{(-1)^\alpha \Gamma(\beta)}{(-\alpha)!\,\Gamma(\alpha+\beta)}, & \alpha \in \mathbb{Z}_{\leq 0}, \\[2mm] 2\pi\sqrt{-1}(e^{2\pi\sqrt{-1}\beta} - 1)\dfrac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}, & \alpha \in \mathbb{Z}_{\geq 1}. \end{cases} \tag{A.6}$$

If $\beta \notin \mathbb{Z}$, then $d_{\alpha,\beta} \neq 0 \Leftrightarrow \alpha + \beta \notin \mathbb{Z}_{\leq 0}$ and we have

$$\int_{[\gamma_1,\gamma_0]} s^{\alpha-1}(1-s)^{\beta-1}ds = \begin{cases} d_{\alpha,\beta}, & \alpha \notin \mathbb{Z}_{\geq 1}, \\ 0, & \alpha \in \mathbb{Z}_{\geq 1}, \end{cases} \tag{A.7}$$

$$\int_{[\gamma_1,\gamma_0]} s^{n-1}(1-s)^{\beta-1}(\log s)ds = d_{n,\beta},$$

for $n \in \mathbb{Z}_{\geq 1}$. For the function $y(w)$ in (A.1), we have

$$\langle[\gamma_z,\gamma_p],y\rangle = \int_{[\gamma_z,\gamma_p]} y(w)(z-w)^\kappa dw \tag{A.8}$$

$$= \begin{cases} D^{\langle p\rangle}(z-p)^{\theta_p+\kappa+1}\sum_{j=0}^{\infty} \tilde{c}_j^{(p)} d_{j+\theta_p+1,\kappa+1}(z-p)^j, & \begin{array}{l}\theta_p \notin \mathbb{Z},\\ \theta_p + \kappa \notin \mathbb{Z}_{\leq -2},\end{array} \\[3mm] D^{\langle p\rangle}\sum_{j=0}^{\infty} \tilde{c}_{j-\kappa-\theta_p-1}^{(p)} d_{j-\kappa,\kappa+1}(z-p)^j, & \begin{array}{l}\theta_p \notin \mathbb{Z},\\ \theta_p + \kappa \in \mathbb{Z}_{\leq -2},\end{array} \\[3mm] D^{\langle p\rangle}(z-p)^{\theta_p+\kappa+1}\left\{\begin{array}{l}\sum_{j=0}^{-\theta_p-1} \tilde{c}_j^{(p)} d_{j+\theta_p+1,\kappa+1}(z-p)^j \\[2mm] +A^{\langle p\rangle}\sum_{j=-\theta_p}^{\infty} c_{j+\theta_p}^{(p)} d_{j+\theta_p+1,\kappa+1}(z-p)^j\end{array}\right\}, & \begin{array}{l}\theta_p \in \mathbb{Z}_{\leq -1},\\ \theta_p + \kappa \notin \mathbb{Z}_{\leq -2},\end{array} \\[3mm] D^{\langle p\rangle}A^{\langle p\rangle}(z-p)^{\theta_p+\kappa+1}\sum_{j=0}^{\infty} c_j^{(p)} d_{j+\theta_p+1,\kappa+1}(z-p)^j, & \begin{array}{l}\theta_p \in \mathbb{Z}_{\geq 0},\\ \theta_p + \kappa \notin \mathbb{Z}_{\leq -2},\end{array} \end{cases}$$

by applying the transformation $w = p + (z-p)s$. Hence

$$\langle[\gamma_z,\gamma_p],y\rangle^{\gamma_p} = e^{2\pi\sqrt{-1}(\theta_p+\kappa)}\langle[\gamma_z,\gamma_p],y\rangle. \tag{A.9}$$

If $\theta_p + \kappa \in \mathbb{Z}$, then the function $\langle[\gamma_z,\gamma_p],y\rangle$ is holomorphic about $z = p$. Under the assumption $\kappa \notin \mathbb{Z}$, the function $\langle[\gamma_z,\gamma_p],y\rangle$ is identically zero for any function $y(w)$ written as (A.1), if and only if $\theta_p \in \mathbb{Z}_{\geq 0}$ and the singularity $w = p$ is apparent (i.e. $A^{\langle p\rangle} = 0$), or $\theta_p + \kappa \in \mathbb{Z}_{\leq -2}$ and the function $g^{\langle p\rangle}(w)$ in (A.2) is a product of $(w-p)^{\theta_p}$ and a non-zero polynomial of degree no more than $-\theta_p - \kappa - 2$ (i.e. $\tilde{c}_j^{(p)} = 0$ for $j \geq -\theta_p - \kappa - 1$). By putting $\kappa = \kappa_2 - 1$, (resp. $\theta_p = 1 - \epsilon_p'$ ($p = 0,1,t$), $\kappa = -\eta$), we obtain Proposition 3.2 (i). If $\theta_p \in \mathbb{Z}_{\leq -1}$, $\kappa \notin \mathbb{Z}$ and the singularity $w = p$ is apparent, then $A^{\langle p\rangle} = 0$ and the

function $\langle[\gamma_z, \gamma_p], y\rangle$ is a product of $(z-p)^{\theta_p+\kappa+1}$ and a polynomial of degree no more than $-\theta_p - 1$.

Let us consider the local expansion about $w = \infty$. We assume that the function $y(w)$ is a solution of a second-order linear differential equation about a regular singularity $w = \infty$, and that the exponents of the second-order linear differential equation at $w = \infty$ are $\theta_\infty^{(1)}$ and $\theta_\infty^{(2)}$. Then any solution $y(w)$ can be written as

$$y(w) = C^{\langle\infty\rangle} f^{\langle\infty\rangle}(w) + D^{\langle\infty\rangle} g^{\langle\infty\rangle}(w), \quad (C^{\langle\infty\rangle}, D^{\langle\infty\rangle} \in \mathbb{C}), \tag{A.10}$$

such that

$$f^{\langle\infty\rangle}(w) = \begin{cases} \left(\dfrac{1}{w}\right)^{\theta_\infty^{(2)}} \displaystyle\sum_{j=0}^\infty c_j^{(\infty)} \left(\dfrac{1}{w}\right)^j, & \theta_\infty^{(1)} - \theta_\infty^{(2)} \notin \mathbb{Z}_{\geq 0}, \\[3ex] \left(\dfrac{1}{w}\right)^{\theta_\infty^{(1)}} \displaystyle\sum_{j=0}^\infty c_j^{(\infty)} \left(\dfrac{1}{w}\right)^j, & \theta_\infty^{(1)} - \theta_\infty^{(2)} \in \mathbb{Z}_{\geq 0}, \end{cases} \tag{A.11}$$

$$g^{\langle\infty\rangle}(w) = \begin{cases} \left(\dfrac{1}{w}\right)^{\theta_\infty^{(1)}} \displaystyle\sum_{j=0}^\infty \tilde{c}_j^{(\infty)} \left(\dfrac{1}{w}\right)^j, & \theta_\infty^{(1)} - \theta_\infty^{(2)} \notin \mathbb{Z}, \\[3ex] \left(\dfrac{1}{w}\right)^{\theta_\infty^{(1)}} \left(\displaystyle\sum_{j=0}^\infty \tilde{c}_j^{(\infty)} \left(\dfrac{1}{w}\right)^j\right) + A^{\langle p\rangle} f^{\langle p\rangle}(w) \log\left(\dfrac{1}{w}\right), & \theta_\infty^{(1)} - \theta_\infty^{(2)} \in \mathbb{Z}_{\leq -1}, \\[3ex] \left(\dfrac{1}{w}\right)^{\theta_\infty^{(2)}} \left(\displaystyle\sum_{j=0}^\infty \tilde{c}_j^{(\infty)} \left(\dfrac{1}{w}\right)^j\right) + A^{\langle p\rangle} f^{\langle p\rangle}(w) \log\left(\dfrac{1}{w}\right), & \theta_\infty^{(1)} - \theta_\infty^{(2)} \in \mathbb{Z}_{\geq 0}, \end{cases}$$

where $c_0^{(\infty)} = \tilde{c}_0^{(\infty)} = 1$.

We investigate the local expansion of the function $\int_{[\gamma_z, \gamma_\infty]} y(w)(z-w)^{\theta_\infty^{(2)}-2} dw$ about $w = \infty$ for the case $\theta_\infty^{(2)} \notin \mathbb{Z}$. Since

$$\int_{[\gamma_z, \gamma_\infty]} (1/w)^{\theta_\infty^{(2)}-1+\alpha} (z-w)^{\theta_\infty^{(2)}-2} dw = e^{\pi\sqrt{-1}(\theta_\infty^{(2)}-1)} (1/z)^\alpha d_{\alpha, \theta_\infty^{(2)}-1}, \tag{A.12}$$

$$\int_{[\gamma_z, \gamma_\infty]} (1/w)^{\theta_\infty^{(2)}-1+n} (z-w)^{\theta_\infty^{(2)}-2} (\log(1/w)) dw = e^{\pi\sqrt{-1}(\theta_\infty^{(2)}-1)} (1/z)^n d_{n, \theta_\infty^{(2)}-1},$$

for $n \in \mathbb{Z}_{\geq 1}$, we have

$$e^{\pi\sqrt{-1}(1-\theta_\infty^{(2)})} \langle[\gamma_z, \gamma_\infty], y\rangle$$

$$
= \begin{cases}
D^{\langle\infty\rangle} \left(\dfrac{1}{z}\right)^{\theta_\infty^{(1)}-\theta_\infty^{(2)}+1} \displaystyle\sum_{j=0}^{\infty} \tilde{c}_j^{(\infty)} d_{j+\theta_\infty^{(1)}-\theta_\infty^{(2)}+1,\theta_\infty^{(2)}-1}\left(\dfrac{1}{z}\right)^j, & \theta_\infty^{(1)}-\theta_\infty^{(2)} \notin \mathbb{Z}, \\
& \theta_\infty^{(1)} \notin \mathbb{Z}_{\leq 0}, \\[2mm]
D^{\langle\infty\rangle} \left(\dfrac{1}{z}\right)^{-\theta_\infty^{(2)}+2} \displaystyle\sum_{j=0}^{\infty} \tilde{c}_{j-\theta_\infty^{(1)}+1}^{(\infty)} d_{j-\theta_\infty^{(2)}+2,\theta_\infty^{(2)}-1}\left(\dfrac{1}{z}\right)^j, & \theta_\infty^{(1)}-\theta_\infty^{(2)} \notin \mathbb{Z}, \\
& \theta_\infty^{(1)} \in \mathbb{Z}_{\leq 0}, \\[2mm]
D^{\langle\infty\rangle} \left(\dfrac{1}{z}\right)^{\theta_\infty^{(1)}-\theta_\infty^{(2)}+1} \left\{ \begin{array}{l} \displaystyle\sum_{j=0}^{-\theta_\infty^{(1)}+\theta_\infty^{(2)}-1} \tilde{c}_j^{(\infty)} d_{j+\theta_\infty^{(1)}-\theta_\infty^{(2)}+1,\theta_\infty^{(2)}-1}\left(\dfrac{1}{z}\right)^j \\ +A^{\langle\infty\rangle} \displaystyle\sum_{j=-\theta_\infty^{(1)}+\theta_\infty^{(2)}}^{\infty} c_{j+\theta_\infty^{(1)}-\theta_\infty^{(2)}}^{(\infty)} d_{j+\theta_\infty^{(1)}-\theta_\infty^{(2)}+1,\theta_\infty^{(2)}-1}\left(\dfrac{1}{z}\right)^j \end{array} \right\}, \\
& \theta_\infty^{(1)}-\theta_\infty^{(2)} \in \mathbb{Z}_{\leq -1}, \\
& \theta_\infty^{(1)} \notin \mathbb{Z}_{\leq 0}, \\[2mm]
D^{\langle\infty\rangle} A^{\langle\infty\rangle} \left(\dfrac{1}{z}\right)^{\theta_\infty^{(1)}-\theta_\infty^{(2)}+1} \displaystyle\sum_{j=0}^{\infty} c_j^{(\infty)} d_{j+\theta_\infty^{(1)}-\theta_\infty^{(2)}+1,\theta_\infty^{(2)}-1}\left(\dfrac{1}{z}\right)^j, & \theta_\infty^{(1)}-\theta_\infty^{(2)} \in \mathbb{Z}_{\geq 0}, \\
& \theta_\infty^{(1)} \notin \mathbb{Z}_{\leq 0}.
\end{cases}
\tag{A.13}
$$

Hence

$$
\langle[\gamma_z,\gamma_\infty],y\rangle^{\gamma_\infty} = e^{2\pi\sqrt{-1}(\theta_\infty^{(1)}-\theta_\infty^{(2)})}\langle[\gamma_z,\gamma_\infty],y\rangle.
\tag{A.14}
$$

Under the assumption $\theta_\infty^{(2)} \notin \mathbb{Z}$, the function $\langle[\gamma_z,\gamma_\infty],y\rangle$ is identically zero for any function $y(w)$ written as in (A.10), if and only if $\theta_\infty^{(1)} - \theta_\infty^{(2)} \in \mathbb{Z}_{\geq 0}$ and the singularity $w = \infty$ is apparent (i.e. $A^{\langle\infty\rangle} = 0$), or $\theta_\infty^{(1)} \in \mathbb{Z}_{\leq 0}$ and the function $g^{\langle\infty\rangle}(w)$ in (A.11) is a non-zero polynomial in the variable w of degree $-\theta_\infty^{(1)}$ (i.e. $\tilde{c}_j^{(\infty)} = 0$ for $j \geq 1 - \theta_\infty^{(1)}$). By putting $\theta_\infty^{(1)} = \kappa_1$, $\theta_\infty^{(2)} = \kappa_2 + 1$ (resp. $\theta_\infty^{(1)} = \alpha + \beta - 2\eta + 1$, $\theta_\infty^{(2)} = 2 - \eta$), we obtain Proposition 3.2 (ii). If $\theta_\infty^{(1)} - \theta_\infty^{(2)} + 1 \in \mathbb{Z}_{\leq 0}$ and the singularity $w = \infty$ is apparent, then $A^{\langle\infty\rangle} = 0$ and the function $\langle[\gamma_z,\gamma_\infty],y\rangle$ is a polynomial in the variable z of degree $-\theta_\infty^{(1)} + \theta_\infty^{(2)} - 1$.

We investigate a sufficient condition that the functions $\langle[\gamma_z,\gamma_0],y\rangle$, $\langle[\gamma_z,\gamma_1],y\rangle$, $\langle[\gamma_z,\gamma_t],y\rangle$ span the two-dimensional space of solutions of $D_{y_1}(\tilde{\theta}_0,\tilde{\theta}_1,\tilde{\theta}_t,\tilde{\theta}_\infty;\tilde{\lambda},\tilde{\mu})$ (resp. (1.12)) for some solution $y(w)$ of $D_{y_1}(\theta_0,\theta_1,\theta_t,\theta_\infty;\lambda,\mu)$ (resp. (1.10)) for the case $\kappa_2 \notin \mathbb{Z}$ (resp. $\eta \notin \mathbb{Z}$).

PROPOSITION A.1. *Assume that $\kappa_2 \notin \mathbb{Z}$ (resp. $\eta \notin \mathbb{Z}$), there exists a branching solution of $D_{y_1}(\theta_0,\theta_1,\theta_t,\theta_\infty;\lambda,\mu)$ (resp. (1.10)) for each singularity $w = 0,1,t$ (i.e. $\theta_p \notin \mathbb{Z}$ (resp. $\epsilon'_p \notin \mathbb{Z}$) or $A^{\langle p\rangle} \neq 0$ for $p = 0,1,t$), the differential equation $D_{y_1}(\theta_0,\theta_1,\theta_t,\theta_\infty;\lambda,\mu)$ (resp. (1.10)) does not have a solution written as a product of $(w - p)^{\theta_p}$ (resp. $(w - p)^{1-\epsilon'_p}$) and a non-zero polynomial on the case $\kappa_2 + \theta_p \in \mathbb{Z}_{\leq -1}$ (resp. $2-\eta-\epsilon'_p = 1-\epsilon_p \in \mathbb{Z}_{\leq -1}$, $\epsilon_p \in \mathbb{Z}_{\geq 2}$) for each $p \in \{0,1,t\}$, and the differential equation $D_{y_1}(\tilde{\theta}_0,\tilde{\theta}_1,\tilde{\theta}_t,\tilde{\theta}_\infty;\tilde{\lambda},\tilde{\mu})$ (resp. (1.12)) does not have a solution written as a product of $z^{\kappa_2+\theta_0}(z-1)^{\kappa_2+\theta_1}(z-t)^{\kappa_2+\theta_t}$ (resp. $z^{1-\epsilon_0}(z-1)^{1-\epsilon_1}(z-t)^{1-\epsilon_t}$) and a non-zero polynomial. Then there exists a solution $y(w)$ of the differential equation $D_{y_1}(\theta_0,\theta_1,\theta_t,\theta_\infty;\lambda,\mu)$ (resp. (1.10)) such that the functions $\langle[\gamma_z,\gamma_0],y\rangle$, $\langle[\gamma_z,\gamma_1],y\rangle$, $\langle[\gamma_z,\gamma_t],y\rangle$ span the two-*

dimensional space of solutions of the differential equation $D_{y_1}(\tilde{\theta}_0, \tilde{\theta}_1, \tilde{\theta}_t, \tilde{\theta}_\infty; \tilde{\lambda}, \tilde{\mu})$ (resp. (1.12)), $\langle[\gamma_z, \gamma_0], y\rangle \neq 0$, $\langle[\gamma_z, \gamma_1], y\rangle \neq 0$ and $\langle[\gamma_z, \gamma_t], y\rangle \neq 0$.

PROOF.   Set $\kappa = \kappa_2 - 1$ (resp. $\kappa = -\eta$ and $\epsilon'_p = 1 - \theta_p$ $(p = 0, 1, t)$). If $\langle[\gamma_z, \gamma_p], y\rangle = 0$ $(p \in \{0, 1, t\})$ for all solutions of $D_{y_1}(\theta_0, \theta_1, \theta_t, \theta_\infty; \lambda, \mu)$ (resp. (1.10)). Then it follows from Proposition 3.2 (i) that $\theta_p \in \mathbb{Z}$ and the singularity $w = p$ is apparent, or $\theta_p + \kappa + 1 \in \mathbb{Z}_{\leq -1}$ and the differential equation $D_{y_1}(\theta_0, \theta_1, \theta_t, \theta_\infty; \lambda, \mu)$ (resp. (1.10)) has a solution of the form which is a product of $(w - p)^{\theta_p}$ and a non-zero polynomial of degree no more than $-\theta_p - \kappa - 2$. Hence it follows from the assumptions of Proposition A.1 that $\langle[\gamma_z, \gamma_p], y^{(p)}\rangle \neq 0$ for some solution $y^{(p)}(w)$ for each $p \in \{0, 1, t\}$. By setting $y(w) = c_0 y^{(0)}(w) + c_1 y^{(1)}(w) + c_t y^{(t)}(w)$ and choosing constants c_0, c_1, c_t appropriately, we have $\langle[\gamma_z, \gamma_p], y\rangle \neq 0$ for all $p \in \{0, 1, t\}$. Assume that the functions $\langle[\gamma_z, \gamma_0], y\rangle$, $\langle[\gamma_z, \gamma_1], y\rangle$, $\langle[\gamma_z, \gamma_t], y\rangle$ do not span the space of solutions of $D_{y_1}(\tilde{\theta}_0, \tilde{\theta}_1, \tilde{\theta}_t, \tilde{\theta}_\infty; \tilde{\lambda}, \tilde{\mu})$ (resp. (1.12)). Then $\langle[\gamma_z, \gamma_0], y\rangle = d\langle[\gamma_z, \gamma_1], y\rangle = d'\langle[\gamma_z, \gamma_t], y\rangle$ for some $d \neq 0$ and $d' \neq 0$. Since $\langle[\gamma_z, \gamma_0], y\rangle$ satisfies the differential equation $D_{y_1}(\tilde{\theta}_0, \tilde{\theta}_1, \tilde{\theta}_t, \tilde{\theta}_\infty; \tilde{\lambda}, \tilde{\mu})$ (resp. (1.12)), $\langle[\gamma_z, \gamma_0], y\rangle$ is locally holomorphic in $\mathbb{C}\setminus\{0, 1, t\}$, and it follows from (A.8) that the function $z^{-\theta_0 - \kappa - 1}(z - 1)^{-\theta_1 - \kappa - 1}(z - t)^{-\theta_t - \kappa - 1}\langle[\gamma_z, \gamma_0], y\rangle$ is holomorphic in $\mathbb{C}$, and the singularity $z = \infty$ is regular at most and apparent. Hence $z^{-\theta_0 - \kappa - 1}(z - 1)^{-\theta_1 - \kappa - 1}(z - t)^{-\theta_t - \kappa - 1}\langle[\gamma_z, \gamma_0], y\rangle$ is a polynomial, and $\langle[\gamma_z, \gamma_0], y\rangle = z^{\theta_0 + \kappa + 1}(z - 1)^{\theta_1 + \kappa + 1}(z - t)^{\theta_t + \kappa + 1}h(z)$ for some polynomial $h(z)$. But this contradicts the assumptions of the proposition.   □

COROLLARY A.2 (Proposition 3.3). *There exists a solution $y(w)$ of $D_{y_1}(\theta_0, \theta_1, \theta_t, \theta_\infty; \lambda, \mu)$ (resp. (1.10)) such that $\langle[\gamma_z, \gamma_0], y\rangle \neq 0$, $\langle[\gamma_z, \gamma_1], y\rangle \neq 0$, $\langle[\gamma_z, \gamma_t], y\rangle \neq 0$ and the functions $\langle[\gamma_z, \gamma_0], y\rangle$, $\langle[\gamma_z, \gamma_1], y\rangle$, $\langle[\gamma_z, \gamma_t], y\rangle$ span the two-dimensional space of solutions of $D_{y_1}(\tilde{\theta}_0, \tilde{\theta}_1, \tilde{\theta}_t, \tilde{\theta}_\infty; \tilde{\lambda}, \tilde{\mu})$ (resp. (1.12)), if $\kappa_2 \notin \mathbb{Z}$ and $\theta_p, \tilde{\theta}_p \notin \mathbb{Z}$ for all $p \in \{0, 1, t, \infty\}$ (resp. $\eta, \epsilon_0, \epsilon_1, \epsilon_t, \alpha - \beta, \epsilon'_0, \epsilon'_1, \epsilon'_t, \alpha' - \beta' \notin \mathbb{Z}$).*

PROOF.   It follows from the fact that $\theta_0, \theta_1, \theta_t \notin \mathbb{Z}$ (resp. $\epsilon'_0, \epsilon'_1, \epsilon'_t \notin \mathbb{Z}$) that there exists a branching solution of $D_{y_1}(\theta_0, \theta_1, \theta_t, \theta_\infty; \lambda, \mu)$ (resp. (1.10)). If there exists a solution of $D_{y_1}(\tilde{\theta}_0, \tilde{\theta}_1, \tilde{\theta}_t, \tilde{\theta}_\infty; \tilde{\lambda}, \tilde{\mu})$ (resp. (1.12)) that can be written as $z^{\kappa_2 + \theta_0}(z - 1)^{\kappa_2 + \theta_1}(z - t)^{\kappa_2 + \theta_t}h(z)$ (resp. $z^{1 - \epsilon_0}(z - 1)^{1 - \epsilon_1}(z - t)^{1 - \epsilon_t}h(z)$) for some non-zero polynomial $h(z)$, it follows from Proposition 3.1 (ii) that $\theta_\infty + (\kappa_2 + \theta_0 + \kappa_2 + \theta_1 + \kappa_2 + \theta_t) = -\deg h(z) \in \mathbb{Z}_{\leq 0}$ or $-\kappa_2 + 1 + (\kappa_2 + \theta_0 + \kappa_2 + \theta_1 + \kappa_2 + \theta_t) = -\deg h(z) \in \mathbb{Z}_{\leq 0}$ (resp. $\alpha + (3 - \epsilon_0 - \epsilon_1 - \epsilon_t) = -\deg h(z) \in \mathbb{Z}_{\leq 0}$ or $\beta + (3 - \epsilon_0 - \epsilon_1 - \epsilon_t) = -\deg h(z) \in \mathbb{Z}_{\leq 0}$), i.e. $\kappa_2 \in \mathbb{Z}_{\leq 0}$ or $\theta_\infty \in \mathbb{Z}_{\geq 1}$ (resp. $2 - \beta \in \mathbb{Z}_{\leq 0}$ or $2 - \alpha \in \mathbb{Z}_{\leq 0}$), which contradicts the assumption of the corollary. The condition $\tilde{\theta}_p \in \mathbb{Z}_{\leq -1}$ (resp. $1 - \epsilon_p \in \mathbb{Z}_{\leq -1}$) for $p = 0, 1, t$ is covered in the assumption of the corollary. Thus, the assumption of Proposition A.1 follows from the assumption of the corollary, and the corollary is obtained by applying Proposition A.1.   □

We derive the following proposition which is used to prove Theorem 4.2.

PROPOSITION A.3.   *Let a, b, c be elements of $\{0, 1, t\}$ such that $a \neq b \neq c \neq a$ and $\eta, \alpha, \beta, \epsilon_0, \epsilon_1, \epsilon_t, \alpha', \beta', \epsilon'_0, \epsilon'_1, \epsilon'_t$ be the parameters defined in (1.9) or (1.13).*
(i) If $\epsilon'_a \in \mathbb{Z}_{\geq 2}$, $\eta \notin \mathbb{Z}$ and the singularity $w = a$ of (1.10) is apparent, then there exists a non-zero solution of (1.12) which can be written as $(z - a)^{1 - \epsilon_a}h(z)$ where $h(z)$ is a

polynomial of degree no more than $\epsilon'_a - 2$. Moreover if $\alpha', \beta' \notin \mathbb{Z}$, then $\deg_E h(z) = \epsilon'_a - 2$.
(ii) *If $\epsilon'_a \in \mathbb{Z}_{\leq 0}$, $\eta \notin \mathbb{Z}$, the singularity $w = a$ of (1.10) is apparent and there do not exist any non-zero solutions of (1.10) written in the form $(w-b)^{\alpha_b}(w-c)^{\alpha_c}p(w)$ where $p(w)$ is a polynomial and $(\alpha_b, \alpha_c) = (0,0)$, $(1-\epsilon'_b, 0)$ or $(0, 1-\epsilon'_c)$, then there exists a non-zero solution of (1.12) which can be written as $(z-b)^{1-\epsilon_b}(z-c)^{1-\epsilon_c}h(z)$ where $h(z)$ is a polynomial. Moreover if $\alpha', \beta' \notin \mathbb{Z}$, then $\deg h(z) = -\epsilon'_a$.*
(iii) *If $\epsilon_a \in \mathbb{Z}_{\geq 2}$, $\eta \notin \mathbb{Z}$, there exists a non-zero solution of (1.10) which can be written as $(w-a)^{1-\epsilon'_a}h(w)$ where $h(w)$ is a polynomial and there do not exist any non-zero solutions of (1.12) written as polynomials in z, then the singularity $z = a$ of (1.12) is apparent.*
(iv) *If $\epsilon_a \in \mathbb{Z}_{\leq 0}$, $\eta, \epsilon'_b, \epsilon'_c \notin \mathbb{Z}$, there exists a non-zero solution of (1.10) written as a product of $(w-b)^{1-\epsilon'_b}(w-c)^{1-\epsilon'_c}$ and a polynomial, and there do not exist any non-zero solutions of (1.12) written as a product of $z^{1-\epsilon_0}(z-1)^{1-\epsilon_1}(z-t)^{1-\epsilon_t}$ and a polynomial, then the singularity $z = a$ of (1.12) is apparent.*
(v) *If $\alpha + \beta - \eta \in \mathbb{Z}_{\leq 0}$, $\eta \notin \mathbb{Z}$ and the singularity $w = \infty$ of (1.10) is apparent, then there exists a non-zero solution of (1.12) which can be written as a polynomial of degree $\eta - \alpha - \beta$.*
(vi) *If $\alpha + \beta - \eta \in \mathbb{Z}_{\geq 2}$, $\eta \notin \mathbb{Z}$, the singularity $w = \infty$ of (1.10) is apparent and there do not exist any non-zero solution of (1.10) written as $w^{\alpha_0}(w-1)^{\alpha_1}(w-t)^{\alpha_t}p(w)$ such that $p(w)$ is a polynomial and $(\alpha_0, \alpha_1, \alpha_t) = (1-\epsilon'_0, 0, 0)$, $(0, 1-\epsilon'_1, 0)$ or $(0, 0, 1-\epsilon'_t)$, then there exists a non-zero solution of (1.12) which can be written as $z^{1-\epsilon_0}(z-1)^{1-\epsilon_1}(z-t)^{1-\epsilon_t}h(z)$ where $h(z)$ is a polynomial of degree $\alpha + \beta - \eta - 2$.*
(vii) *If $\alpha + \beta - 2\eta \in \mathbb{Z}_{\leq -1}$, $\eta \notin \mathbb{Z}$, there exists a non-zero solution of (1.10) which is written as a polynomial and there do not exist any non-zero solutions of (1.12) which are written as $(1/z)^\eta p(1/z)$ where $p(1/z)$ is a polynomial in $1/z$, then the singularity $z = \infty$ of (1.12) is apparent.*
(viii) *If $\alpha + \beta - 2\eta \in \mathbb{Z}_{\geq 1}$, $\eta, \epsilon'_0, \epsilon'_1, \epsilon'_t \notin \mathbb{Z}$, there exists a non-zero solution of (1.10) which can be written as a product of $w^{1-\epsilon'_0}(w-1)^{1-\epsilon'_1}(w-t)^{1-\epsilon'_t}$ and a polynomial, then the singularity $z = \infty$ of (1.12) is apparent.*

PROOF. Set $\epsilon'_p = 1 - \theta_p$, $\epsilon_p = 1 - \tilde{\theta}_p$ ($p = 0, 1, t$). Then $\tilde{\theta}_p = \theta_p - \eta + 1$. We apply local expansions in this appendix by setting $\kappa = -\eta$.

To prove (i), it follows from Proposition 4.1 that it remains to show that if $\alpha', \beta' \notin \mathbb{Z}$, then $\deg_E h(z) = \epsilon'_a - 2$. If there exists a non-zero solution of (1.12) which is written as $(z-a)^{1-\epsilon_a}h(z)$ where $h(z)$ is a polynomial, then it follows from Proposition 3.1 (ii) that $\deg_E h(z) = -\alpha - 1 + \epsilon_a$ or $-\beta - 1 + \epsilon_a$, i.e. $\deg_E h(z) = -\eta - 1 + \epsilon_a = \epsilon'_a - 2$ or $-(\alpha + \beta - \eta) - 1 + \epsilon_a = 2\eta - \alpha - \beta + \epsilon'_a - 2$. If $\alpha', \beta' \notin \mathbb{Z}$, then we have $2\eta - \alpha - \beta \notin \mathbb{Z}$ and $\deg_E h(z) = \epsilon'_a - 2$. Hence we obtain (i).

Assume that $\theta_a \in \mathbb{Z}_{\geq 1}$, $\eta \notin \mathbb{Z}$ and the singularity $w = a$ of (1.10) is apparent. Then it follows from Proposition 3.2 (i) that $\langle[\gamma_z, \gamma_a], y\rangle = 0$ for all solutions $y(w)$ of (1.10). Combining this result with (2.2) and a similar equality, we have

$$\langle[\gamma_z, \gamma_b], y\rangle^{\gamma_a} = \langle[\gamma_z, \gamma_b], y\rangle, \quad \langle[\gamma_z, \gamma_c], y\rangle^{\gamma_a} = \langle[\gamma_z, \gamma_c], y\rangle. \tag{A.15}$$

If $\langle[\gamma_z, \gamma_b], y\rangle, \langle[\gamma_z, \gamma_c], y\rangle$ are linearly independent for some solution $y(w)$ of (1.10), then it follows from (A.15) that the monodromy matrix about $z = a$ is a unit and the exponents

of (1.12) at $z = a$ are integers. Hence $\tilde{\theta}_a = \theta_a - \eta + 1 \in \mathbb{Z}$, and this contradicts $\eta \notin \mathbb{Z}$. Therefore $\langle [\gamma_z, \gamma_b], y \rangle$, $\langle [\gamma_z, \gamma_c], y \rangle$ are linearly dependent for any solution $y(w)$. If $\langle [\gamma_z, \gamma_b], y \rangle \neq 0$ for some solution $y(w)$ and $\langle [\gamma_z, \gamma_c], y \rangle \neq 0$ for some solution $y(w)$, then there exists a solution $y(w)$ of (1.10) such that $\langle [\gamma_z, \gamma_c], y \rangle = d' \langle [\gamma_z, \gamma_b], y \rangle \neq 0$ for some constant $d' \neq 0$. It follows from local expansions (A.8) for the case $p = b, c$, (A.15) and the condition $\tilde{\theta}_a \notin \mathbb{Z}$ that the function $h(z) = (z - b)^{-\tilde{\theta}_b}(z - c)^{-\tilde{\theta}_c}\langle [\gamma_z, \gamma_b], y \rangle$ is holomorphic in $\mathbb{C}$. Hence $h(z)$ is a non-branching function in $\mathbb{C} \cup \{\infty\}$ which may have a pole at $z = \infty$, and (1.12) has a non-zero solution $(z - b)^{\tilde{\theta}_b}(z - c)^{\tilde{\theta}_c}h(z)$, where $h(z)$ is a polynomial. Let k be the degree of $h(z)$. It follows from Proposition 3.1 (ii) that $-k - \tilde{\theta}_b - \tilde{\theta}_c = \eta$ or $-\eta + \alpha + \beta$, and by applying the relation $\tilde{\theta}_a + \tilde{\theta}_b + \tilde{\theta}_b + \alpha + \beta = 2$ we have $k = \tilde{\theta}_a + \alpha + \beta - \eta - 2 = \theta_a - 2\eta + \alpha + \beta - 1$ or $k = \tilde{\theta}_a + \eta - 2 = \theta_a - 1$. Hence, if $\alpha', \beta' \notin \mathbb{Z}$, then $-2\eta + \alpha + \beta \notin \mathbb{Z}$ and we have $\deg h(z) = \theta_a - 1 = -\epsilon'_a$. If $\langle [\gamma_z, \gamma_b], y \rangle = 0$ for all solutions $y(w)$, then it follows from Proposition 3.2 (i) that $\theta_b \in \mathbb{Z}_{\geq 0}$ and the singularity $w = b$ is apparent or $\epsilon_b \in \mathbb{Z}_{\geq 2}$ and the (1.10) has a solution of the form which is a product of $(w - b)^{\theta_b}$ and a non-zero polynomial of degree no more than $\epsilon_b - 2$. For the case $\theta_b \in \mathbb{Z}_{\geq 0}$ and the singularity $w = b$ is apparent, by taking a solution $y(w)$ of (1.10) which is holomorphic at $w = c$, the function $y(w)$ is holomorphic on the points $w = a, b, c$ and it is a polynomial in w, because the point $w = \infty$ is an apparent singularity. Hence we have a polynomial solution $y(w)$ of (1.10). By combining this result with a similar statement for the case $\langle [\gamma_z, \gamma_c], y \rangle = 0$ for all solutions $y(w)$, it follows that if $\theta_a \in \mathbb{Z}_{\geq 0}$, the singularity $w = a$ is apparent and $\langle [\gamma_z, \gamma_b], y \rangle = 0$ or $\langle [\gamma_z, \gamma_c], y \rangle = 0$ for all solutions $y(w)$, then there exists a non-zero solution of (1.10) which can be written as $(w - b)^{\alpha_b}(w - c)^{\alpha_c}p(w)$ where $p(w)$ is a polynomial and $(\alpha_b, \alpha_c) = (0, 0)$, $(\theta_b, 0)$ or $(0, \theta_c)$. Therefore we obtain (ii).

We show that if $\theta_a (= 1 - \epsilon'_a) \in \mathbb{Z}_{\leq 0}$, $\eta \notin \mathbb{Z}$, there exists a logarithmic solution of (1.10) about $w = a$ and there do not exist any non-zero solutions of (1.10) written as a polynomial, then there do not exist any non-zero solution of (1.12) written as $(z - a)^{1 - \epsilon_a}p(z)$ such that $p(z)$ is a polynomial. We write a logarithmic solution of (1.10) as in (A.1), (A.2). Then $D^{\langle a \rangle} \neq 0$, $A^{\langle a \rangle} \neq 0$ and it follows from the absence of a non-zero polynomial solution of (1.10) that $\forall K \in \mathbb{Z}$, $\exists j \in \mathbb{Z}_{\geq K}$ such that $c_j^{(a)} \neq 0$. A solution $\langle [\gamma_z, \gamma_a], y \rangle$ of (1.12) can be written as (A.8) for the case $\theta_a \in \mathbb{Z}_{\leq 0}$, $\theta_a + \kappa + 1 \notin \mathbb{Z}_{\leq -1}$, and it cannot be written as $(z - a)^{\theta_a - \eta + 1}p(z)$ such that $p(z)$ is a polynomial because $A^{\langle a \rangle} \neq 0$ and $\forall K \in \mathbb{Z}$, $\exists j \in \mathbb{Z}_{\geq K}$ such that $c_j^{(a)} \neq 0$. Since $(1 - \epsilon_a =)\tilde{\theta}_a = \theta_a - \eta + 1 \notin \mathbb{Z}$, the space of solutions of (1.12) that are written as $(z - a)^{\tilde{\theta}_a}h(z)$ such that $h(z)$ is holomorphic about $z = a$ is one-dimensional. Hence there does not exist a non-zero solution of (1.12) written as $(z - a)^{1 - \epsilon_a}p(z)$ such that $p(z)$ is a polynomial. It follows from the duality of the parameters $(\epsilon_0, \epsilon_1, \epsilon_t, \eta)$ and $(\epsilon'_0, \epsilon'_1, \epsilon'_t, \eta')$ in (1.9) and (1.13) that we obtain (iii) for the case $1 - \epsilon_a \in \mathbb{Z}_{\leq 0}$ by contraposition.

We show that if $\eta \notin \mathbb{Z}$, $\tilde{\theta}_a \in \mathbb{Z}_{\geq 1}$, $\theta_b, \theta_c \notin \mathbb{Z}$, there exists a logarithmic solution of (1.12) about $z = a$, there exists a non-zero solution of (1.10) written as $(w - b)^{\theta_b}(w - c)^{\theta_c}h(w)$ such that $h(w)$ is a polynomial, and there do not exist any non-zero solutions of (1.12) written as $(z - a)^{\tilde{\theta}_a}(z - b)^{\tilde{\theta}_b}(z - c)^{\tilde{\theta}_c}\tilde{p}_0(z)$ where $\tilde{p}_0(z)$ is a polynomial, then we have a contradiction. Assume that there exists a non-zero solution of (1.10) written as $y(w) = (w - b)^{\theta_b}(w - c)^{\theta_c}h(w)$ such that $h(w)$ is a polynomial. Then the functions

$\langle[\gamma_z, \gamma_b], y\rangle$ and $\langle[\gamma_z, \gamma_c], y\rangle$ are solutions of (1.12) and they are non-zero, which follows from $\theta_b, \theta_c \notin \mathbb{Z}$ and (A.8). Since it has been shown that $y^{\gamma_b} = e^{2\pi\sqrt{-1}\theta_b}y$, $y^{\gamma_c} = e^{2\pi\sqrt{-1}\theta_c}y$ and $\langle[\gamma_z, \gamma_a], y\rangle = 0$, we have $\langle[\gamma_z, \gamma_b], y\rangle^{\gamma_a} = \langle[\gamma_z, \gamma_b], y\rangle$ and $\langle[\gamma_z, \gamma_c], y\rangle^{\gamma_a} = \langle[\gamma_z, \gamma_c], y\rangle$. If $\langle[\gamma_z, \gamma_b], y\rangle$, $\langle[\gamma_z, \gamma_c], y\rangle$ are linearly independent, the monodromy matrix about $z = a$ is a unit, and this contradicts the existence of a logarithmic solution. Hence $\langle[\gamma_z, \gamma_b], y\rangle$ and $\langle[\gamma_z, \gamma_c], y\rangle$ are linearly dependent. It follows from a similar argument to the proof of (ii) that there exists a non-zero solution $\tilde{y}(z)$ of (1.12) written as $\tilde{y}(z) = (z-b)^{\tilde{\theta}_b}(z-c)^{\tilde{\theta}_c}\tilde{p}(z)$ such that $\tilde{p}(z)$ is a polynomial. Since $\tilde{\theta}_a \in \mathbb{Z}_{\geq 0}$ and there exists a logarithmic solution of (1.12) about $z = a$, it follows from (A.2) that $y(z)$ can be expressed as $y(z) = (z-a)^{\tilde{\theta}_a}(z-b)^{\tilde{\theta}_b}(z-c)^{\tilde{\theta}_c}\tilde{p}_0(z)$ such that $\tilde{p}_0(z)$ is a polynomial, and we have a contradiction. Hence we obtain that if $\eta \notin \mathbb{Z}$, $\tilde{\theta}_a \in \mathbb{Z}_{\geq 0}$, $\theta_b, \theta_c \notin \mathbb{Z}$, there exists a non-zero solution of (1.10) written as $(w-b)^{\theta_b}(w-c)^{\theta_c}h(w)$ such that $h(w)$ is a polynomial, and there do not exist any non-zero solutions of (1.12) written in the form $(z-a)^{\tilde{\theta}_a}(z-b)^{\tilde{\theta}_b}(z-c)^{\tilde{\theta}_c}\tilde{p}_0(z)$, where $\tilde{p}_0(z)$ is a polynomial, then the singularity $z = a$ of (1.12) is apparent. Therefore we have (iv).

We show (v) and (vi). We apply the expansions in (A.11), (A.13) by setting $\theta_\infty^{(1)} = \alpha + \beta - 2\eta + 1$ and $\theta_\infty^{(2)} = 2 - \eta$. If $\alpha + \beta - \eta = \theta_\infty^{(1)} - \theta_\infty^{(2)} + 1 \in \mathbb{Z}_{\leq 0}$, $\eta \notin \mathbb{Z}$ and the singularity $w = \infty$ of (1.10) is apparent, then $\theta_\infty^{(1)} \notin \mathbb{Z}_{\leq 0}$ and the function $\langle[\gamma_z, \gamma_\infty], y\rangle$ in (A.13) is a product of $(1/z)^{\alpha+\beta-\eta}$ and a polynomial in the variable $1/z$ of degree no more than $\eta - \alpha - \beta$, and it satisfies (1.12). Hence there exists a solution of (1.12) which is a polynomial in z of degree no more than $\eta - \alpha - \beta$. If there exists a solution of (1.12) which is a polynomial in z, then the degree of the polynomial is $-\alpha$ or $-\beta$, i.e., $-\eta (\notin \mathbb{Z})$ or $\eta - \alpha - \beta (\in \mathbb{Z})$. Therefore we have (v).

If $-\eta + \alpha + \beta = \theta_\infty^{(1)} - \theta_\infty^{(2)} \in \mathbb{Z}_{\geq 0}$, $\eta \notin \mathbb{Z}$ and the singularity $w = \infty$ of (1.10) is apparent, then it follows from Proposition 3.2 (ii) and a similar argument to obtain (A.15) that $\langle[\gamma_z, \gamma_\infty], y\rangle = 0$ for all solutions $y(w)$ of (1.10) and

$$\langle[\gamma_z, \gamma_p], y\rangle^{\gamma_\infty} = e^{2\pi\sqrt{-1}\eta}\langle[\gamma_z, \gamma_p], y\rangle, \quad p = 0, 1, t. \tag{A.16}$$

If $\langle[\gamma_z, \gamma_a], y\rangle$ and $\langle[\gamma_z, \gamma_b], y\rangle$ ($a, b \in \{0, 1, t\}$, $a \neq b$) are linearly independent for some solution $y(w)$ of (1.10), it follows from (A.16) that the monodromy matrix of (1.12) about $z = \infty$ is scalar, the difference between the exponents of (1.12) at $z = \infty$ (i.e. $\theta_\infty^{(1)} - \theta_\infty^{(2)} + 1$ and $2 - \theta_\infty^{(2)}$) is an integer, and this contradicts $\theta_\infty^{(1)} - \theta_\infty^{(2)} \in \mathbb{Z}$ and $\eta = 2 - \theta_\infty^{(2)} \notin \mathbb{Z}$. Therefore $\langle[\gamma_z, \gamma_a], y\rangle$ and $\langle[\gamma_z, \gamma_b], y\rangle$ are linearly dependent for any solution $y(w)$ of (1.10) and $a, b \in \{0, 1, t\}$ such that $a \neq b$. If there exists a solution $y^{(p)}(w)$ of (1.10) such that $\langle[\gamma_z, \gamma_p], y^{(p)}\rangle \neq 0$ for each $p \in \{0, 1, t\}$, then there exists a solution $y(w)$ of (1.10) such that $\langle[\gamma_z, \gamma_p], y\rangle \neq 0$ for any $p \in \{0, 1, t\}$ by setting $y(w) = c_0 y^{(0)}(w) + c_1 y^{(1)}(w) + c_t y^{(t)}(w)$ and choosing c_0, c_1, c_t appropriately. Then we have $\langle[\gamma_z, \gamma_0], y\rangle = d\langle[\gamma_z, \gamma_1], y\rangle = d'\langle[\gamma_z, \gamma_t], y\rangle \neq 0$ for some constants $d, d' \neq 0$. It is shown that the function $z^{-\tilde{\theta}_0}(z-1)^{-\tilde{\theta}_1}(z-t)^{-\tilde{\theta}_t}\langle[\gamma_z, \gamma_0], y\rangle$ is holomorphic in $\mathbb{C}$, and (1.12) has a non-zero solution $z^{\tilde{\theta}_0}(z-1)^{\tilde{\theta}_1}(z-t)^{\tilde{\theta}_t}h(z)$ where $h(z)$ is a polynomial. Let k be the degree of $h(z)$. It follows from Proposition 3.1 (ii) that $-k + \alpha + \beta - 2 = \eta$ or $\alpha + \beta - \eta$. Since $\eta \notin \mathbb{Z}$, we have $\deg h(z) = \alpha + \beta - \eta - 2$. If $\langle[\gamma_z, \gamma_p], y\rangle = 0$ for all solutions $y(w)$ and some $p \in \{0, 1, t\}$, then there exists a solution of (1.10) which can

be expressed as a product of $(w - p)^{\theta_p}$ and a polynomial, or $\theta_p \in \mathbb{Z}_{\geq 1}$ and there are no logarithmic solutions about $w = p$. Assume that $\theta_p \in \mathbb{Z}_{\geq 1}$ and there are no logarithmic solutions about $w = p$. Let $p' \in \{0, 1, t\}$ such that $p' \neq p$ and $y(w)$ be a solution of (1.10) which is holomorphic at $w = p'$. Then $y^{\gamma_{p'}}(w) = y(w)$, $y^{\gamma_p}(w) = y(w)$. Since the singularity $w = \infty$ is apparent, we have $y^{\gamma_\infty}(w) = e^{2\pi\sqrt{-1}\theta_\infty^{(2)}} y(w) = e^{-2\pi\sqrt{-1}\eta} y(w)$ and it follows that $y^{\gamma_{p''}}(w) = e^{2\pi\sqrt{-1}\eta} y(w)$ ($p'' \in \{0, 1, t\}$, $p \neq p'' \neq p'$). Since $e^{2\pi\sqrt{-1}\eta} \neq 1$, we have $y^{\gamma_{p''}}(w) = e^{2\pi\sqrt{-1}\theta_{p''}} y(w)$. Hence the function $y(w)$ can be expressed as $y(w) = (w - p'')^{\theta_{p''}} h(w)$ such that $h(w)$ is a polynomial, which follows from the monodromy of $y(w)$. Therefore if $\langle[\gamma_z, \gamma_p], y\rangle = 0$ for all solution $y(w)$ and some $p \in \{0, 1, t\}$ then there exists a solution $y(w)$ of (1.10) such that $y(w) = w^{\alpha_0}(w-1)^{\alpha_1}(w-t)^{\alpha_t} h(w)$, $h(w)$ is a polynomial and $(\alpha_0, \alpha_1, \alpha_t) = (\theta_0, 0, 0)$, $(0, \theta_1, 0)$ or $(0, 0, \theta_t)$, and we have (vi).

We show (vii) and (viii). We apply the expansions in (A.11), (A.13) by setting $\theta_\infty^{(1)} = \alpha + \beta - 2\eta + 1$ and $\theta_\infty^{(2)} = 2 - \eta$. We show that if $\eta(= 2 - \eta' = 2 - \theta_\infty^{(2)}) \notin \mathbb{Z}$, $\alpha + \beta - \eta(= \alpha' + \beta' - 2\eta' + 1 = \theta_\infty^{(1)} - \theta_\infty^{(2)} + 1) \in \mathbb{Z}_{\leq 0}$, there exists a logarithmic solution of (1.10) about $w = \infty$ and there do not exist any non-zero solutions of (1.10) written as a product of $(1/w)^{\eta'}(= (1/w)^{\theta_\infty^{(2)}})$ and a polynomial in $1/w$, then there do not exist any non-zero solutions of (1.12) written as a polynomial. We write a solution of (1.10) as in (A.10), (A.11). Then $D^{\langle\infty\rangle} \neq 0$, $A^{\langle\infty\rangle} \neq 0$ and $\forall K \in \mathbb{Z}$, $\exists j \in \mathbb{Z}_{\geq K}$ such that $c_j^{(\infty)} \neq 0$ in (A.11). It can be shown as in the proof of (iii) that the function $\langle[\gamma_z, \gamma_\infty], y\rangle$ can be written as in (A.13) for the case $\theta_\infty^{(1)} - \theta_\infty^{(2)} \in \mathbb{Z}_{\leq -1}$, $\theta_\infty^{(1)} \notin \mathbb{Z}_{\leq 0}$, and it is not written as $(1/z)^{\theta_\infty^{(1)} - \theta_\infty^{(2)} + 1} p(1/z)$ such that $p(z)$ is a polynomial. It follows from $\theta_\infty^{(1)} - \theta_\infty^{(2)} + 1 - (2 - \theta_\infty^{(2)}) \notin \mathbb{Z}$ that there do not exist any non-zero solutions of (1.12) written as $(1/z)^{\theta_\infty^{(1)} - \theta_\infty^{(2)} + 1} p(1/z)$ such that $p(z)$ is a polynomial. If there exists a non-zero solution of (1.12) written as a polynomial $h(z)$, then it follows from Proposition 3.1 (ii) that $\deg h(z) = \alpha + \beta - \eta$ or η. Because $\eta \notin \mathbb{Z}$, we have $\deg h(z) = \alpha + \beta - \eta = \theta_\infty^{(1)} - \theta_\infty^{(2)} + 1$ and $h(z)$ can be written as $(1/z)^{\theta_\infty^{(1)} - \theta_\infty^{(2)} + 1} p(1/z)$ where $p(z)$ is a polynomial of degree no more than $\theta_\infty^{(1)} - \theta_\infty^{(2)} + 1$. Therefore we obtain that there do not exist any non-zero solutions of (1.12) written as a polynomial. It follows from the duality of the parameters $(\epsilon_0, \epsilon_1, \epsilon_t, \eta)$ and $(\epsilon_0', \epsilon_1', \epsilon_t', \eta')$ in (1.9) and (1.13) that we obtain (vii) by contraposition.

We show that if $\eta, \theta_0, \theta_1, \theta_t \notin \mathbb{Z}$, $\alpha + \beta - 2\eta(= \theta_\infty^{(1)} - \theta_\infty^{(2)} + 1 - (2 - \theta_\infty^{(2)})) \in \mathbb{Z}_{\geq 1}$, there exists a logarithmic solution of (1.12) about $z = \infty$, and there do not exist any non-zero solutions of (1.10) written as $w^{\theta_0}(w - 1)^{\theta_1}(w - t)^{\theta_t} p(w)$ such that $p(w)$ is a polynomial, then we have a contradiction. Assume that there exists a non-zero solution $y(z)$ of (1.10) written as $y(w) = w^{\theta_0}(w-1)^{\theta_1}(w-t)^{\theta_t} p(w)$ such that $p(w)$ is a polynomial. Then the exponent of $y(w)$ at $w = \infty$ is $\theta_\infty^{(1)} + \theta_\infty^{(2)} - \deg p(w) - 2$ and the function $y(w)$ can be expressed as $f^{\langle\infty\rangle}(w)$ in (A.10) for the case $\theta_\infty^{(1)} - \theta_\infty^{(2)} \notin \mathbb{Z}$. Hence $\langle[\gamma_z, \gamma_\infty], y\rangle = 0$ and we have $\langle[\gamma_z, \gamma_p], y\rangle^{\gamma_\infty} = e^{2\pi\sqrt{-1}\eta} \langle[\gamma_z, \gamma_p], y\rangle$ ($p = 0, 1, t$). Since there exists a logarithmic solution about $z = \infty$, any two of $\langle[\gamma_z, \gamma_0], y\rangle$, $\langle[\gamma_z, \gamma_1], y\rangle$, $\langle[\gamma_z, \gamma_t], y\rangle$ are linearly dependent (see the proof of (iv)) and it follows from $\theta_p \notin \mathbb{Z}$ ($p = 0, 1, t$) that there exists a solution $y(z)$ of (1.12) written as $z^{\tilde\theta_0}(z - 1)^{\tilde\theta_1}(z - t)^{\tilde\theta_t} p(z)$ such that $p(z)$ is a polynomial. Then we have $\deg p(z) = \eta - 2$ or $\alpha + \beta - \eta - 2$ and this contradicts $\eta \notin \mathbb{Z}$ and $\alpha + \beta - 2\eta \in \mathbb{Z}$. Hence if $\alpha + \beta - 2\eta \in \mathbb{Z}_{\geq 1}$, $\eta, \theta_0, \theta_1, \theta_t \notin \mathbb{Z}$, there exists a non-zero solution of (1.10) which can be written as a product of $w^{\theta_0}(w - 1)^{\theta_1}(w - t)^{\theta_t}$

and a polynomial, then the singularity $z = \infty$ of (1.12) is apparent. Therefore we have (viii). $\qquad\square$

Theorem 4.2 (i), (v), (viii) follows from Proposition A.3 (i), (v), (viii).

We show Theorem 4.2 (ii). Assume that there exists a non-zero solution of (1.10) which is written as $p(w)$ (resp. $(w - p)^{1-\epsilon'_p}p(w)$) where $p(w)$ is a polynomial. It follows from Proposition 3.1 (ii) that $\deg p(w) = -\alpha'$ or $-\beta'$ (resp. $\deg p(w) = \epsilon'_p - \alpha' - 1$ or $\epsilon'_p - \beta' - 1$). Thus $\alpha' \in \mathbb{Z}_{\leq 0}$ or $\beta' \in \mathbb{Z}_{\leq 0}$ (resp. $\epsilon'_p - \alpha' \in \mathbb{Z}_{\geq 1}$ or $\epsilon'_p - \beta' \in \mathbb{Z}_{\geq 1}$). Therefore, if $\alpha', \beta' \notin \mathbb{Z}$ (resp. $\epsilon'_p - \alpha', \epsilon'_p - \beta' \notin \mathbb{Z}$), then there do not exist any non-zero solutions of (1.10) written in the form $p(w)$ (resp. $(w - p)^{1-\epsilon'_p}p(w)$) where $p(w)$ is a polynomial. It follows from $\alpha', \beta' \notin \mathbb{Z}$ that $\eta' \notin \mathbb{Z}$ and $\eta \notin \mathbb{Z}$. If $\epsilon'_a \in \mathbb{Z}$ and $\epsilon_b = \epsilon'_b - \eta' + 1 \notin \mathbb{Z}$ (resp. $\epsilon_c \notin \mathbb{Z}$), then $\epsilon'_c - (\alpha' + \beta' - \eta') = -\epsilon'_a - \epsilon'_b + \eta' + 1 \notin \mathbb{Z}$ (resp. $\epsilon'_b - (\alpha' + \beta' - \eta') \notin \mathbb{Z}$). By combining with Proposition A.3 (ii), we have Theorem 4.2 (ii).

We show Theorem 4.2 (iii) and (iv). It follows from $\alpha, \beta \notin \mathbb{Z}$ that $\eta \notin \mathbb{Z}$. If there exists a non-zero solution of (1.12) which is written as $p(z)$ (resp. $z^{1-\epsilon_0}(z - 1)^{1-\epsilon_1}(z - t)^{1-\epsilon_t}p(z)$) where $p(z)$ is a polynomial, then $\deg p(z) = -\alpha$ or $-\beta$ (resp. $\deg p(z) = \alpha - 2$ or $\beta - 2$). Hence if $\alpha, \beta \notin \mathbb{Z}$, then there do not exist any non-zero solutions of (1.12) written as a polynomial nor as a product of $z^{1-\epsilon_0}(z - 1)^{1-\epsilon_1}(z - t)^{1-\epsilon_t}p(z)$ and a polynomial. By combining with Proposition A.3 (iii), (iv), we have Theorem 4.2 (iii) and (iv).

If $\alpha + \beta - \eta = \alpha' + \beta' - 2\eta' + 1 \in \mathbb{Z}$ and $\epsilon_p \notin \mathbb{Z}$, then $\epsilon'_p - \eta' = \epsilon_p - 1 \notin \mathbb{Z}$ and $\epsilon'_p - (\alpha' + \beta' - \eta') = \epsilon'_p - \eta' + (\alpha' + \beta' - 2\eta') \notin \mathbb{Z}$. Hence $\epsilon'_p - \alpha', \epsilon'_p - \beta' \notin \mathbb{Z}$ and there do not exist any non-zero solutions of (1.10) written in the form $(w - p)^{1-\epsilon'_p}p(w)$, where $p(w)$ is a polynomial. Hence we have Theorem 4.2 (vi) by combining with Proposition A.3 (vi).

If there exists a non-zero solution of (1.12), written as $(1/z)^\eta p(1/z)$ where $p(1/z)$ is a polynomial in $1/z$, then the exponent of the function $(1/z)^\eta p(1/z)$ is $-\eta - \deg_{1/z} p(1/z)$ and $-\eta - \deg_{1/z} p(1/z) = 0$ or ϵ_0. Hence if $\eta, \epsilon'_0 \notin \mathbb{Z}$, then there do not exist any non-zero solutions of (1.12) written as $(1/z)^\eta p(1/z)$ where $p(1/z)$ is a polynomial in $1/z$. By combining with Proposition A.3 (vii), we have Theorem 4.2 (vii). Thus Theorem 4.2 is proved.

The following proposition concerning solutions of $D_{y_1}(\theta_0, \theta_1, \theta_t, \theta_\infty; \lambda, \mu)$ and $D_{y_1}(\tilde{\theta}_0, \tilde{\theta}_1, \tilde{\theta}_t, \tilde{\theta}_\infty; \tilde{\lambda}, \tilde{\mu})$ is proved similarly to Proposition A.3.

PROPOSITION A.4. *Let a, b, c be elements of $\{0, 1, t\}$ such that $a \neq b \neq c \neq a$. Assume that $\lambda, \tilde{\lambda} \notin \{0, 1, t, \infty\}$.*
(i) If $\theta_a \in \mathbb{Z}_{\leq -1}$, $\kappa_2 \notin \mathbb{Z}$ and the singularity $w = a$ of the differential equation $D_{y_1}(\theta_0, \theta_1, \theta_t, \theta_\infty; \lambda, \mu)$ in the variable w is apparent, then there exists a non-zero solution $\tilde{y}(z)$ of the differential equation $D_{y_1}(\tilde{\theta}_0, \tilde{\theta}_1, \tilde{\theta}_t, \tilde{\theta}_\infty; \tilde{\lambda}, \tilde{\mu})$ in the variable z which can be written as $(z - a)^{\tilde{\theta}_a}h(z)$ where $h(z)$ is a polynomial of degree no more than $-\theta_a - 1$. Moreover if $\kappa_1 \notin \mathbb{Z}$, then $\deg_E h(z) = -\theta_a - 1$.
(ii) If $\theta_a \in \mathbb{Z}_{\geq 0}$, $\kappa_2 \notin \mathbb{Z}$, the singularity $w = a$ of the differential equation $D_{y_1}(\theta_0, \theta_1, \theta_t, \theta_\infty; \lambda, \mu)$ is apparent and there do not exist any non-zero solutions of $D_{y_1}(\theta_0, \theta_1, \theta_t, \theta_\infty; \lambda, \mu)$ written in the form $(w - b)^{\alpha_b}(w - c)^{\alpha_c}p(w)$ where $p(w)$ is a polynomial and $(\alpha_b, \alpha_c) = (0, 0)$, $(\theta_b, 0)$ or $(0, \theta_c)$, then there exists a non-zero solution of the

differential equation $D_{y_1}(\tilde{\theta}_0, \tilde{\theta}_1, \tilde{\theta}_t, \tilde{\theta}_\infty; \tilde{\lambda}, \tilde{\mu})$ which can be written as $(z-b)^{\tilde{\theta}_b}(z-c)^{\tilde{\theta}_c}h(z)$ where $h(z)$ is a polynomial. Moreover if $\kappa_1 \notin \mathbb{Z}$, then we have $\deg h(z) = \theta_a$.

(iii) If $\tilde{\theta}_a \in \mathbb{Z}_{\leq 0}$, $\kappa_2 \notin \mathbb{Z}$, there exists a non-zero solution of $D_{y_1}(\theta_0, \theta_1, \theta_t, \theta_\infty; \lambda, \mu)$ which can be written as $(w-a)^{\theta_a}h(w)$ where $h(w)$ is a polynomial and there do not exist any non-zero solutions of $D_{y_1}(\tilde{\theta}_0, \tilde{\theta}_1, \tilde{\theta}_t, \tilde{\theta}_\infty; \tilde{\lambda}, \tilde{\mu})$ written as a polynomial in z, then the singularity $z = a$ of $D_{y_1}(\tilde{\theta}_0, \tilde{\theta}_1, \tilde{\theta}_t, \tilde{\theta}_\infty; \tilde{\lambda}, \tilde{\mu})$ is apparent.

(iv) If $\tilde{\theta}_a \in \mathbb{Z}_{\geq 1}$, $\kappa_2, \theta_b, \theta_c \notin \mathbb{Z}$, there exists a non-zero solution of $D_{y_1}(\theta_0, \theta_1, \theta_t, \theta_\infty; \lambda, \mu)$ which can be written as a product of $(w-b)^{\theta_b}(w-c)^{\theta_c}$ and a polynomial and there do not exist any solutions of $D_{y_1}(\tilde{\theta}_0, \tilde{\theta}_1, \tilde{\theta}_t, \tilde{\theta}_\infty; \tilde{\lambda}, \tilde{\mu})$ written as a product of $z^{\tilde{\theta}_0}(z-1)^{\tilde{\theta}_1}(z-t)^{\tilde{\theta}_t}$ and a polynomial, then the singularity $z = a$ of $D_{y_1}(\tilde{\theta}_0, \tilde{\theta}_1, \tilde{\theta}_t, \tilde{\theta}_\infty; \tilde{\lambda}, \tilde{\mu})$ is apparent.

(v) If $\theta_\infty \in \mathbb{Z}_{\leq 0}$, $\kappa_2 \notin \mathbb{Z}$ and the singularity $w = \infty$ of $D_{y_1}(\theta_0, \theta_1, \theta_t, \theta_\infty; \lambda, \mu)$ is apparent, then there exists a non-zero solution of $D_{y_1}(\tilde{\theta}_0, \tilde{\theta}_1, \tilde{\theta}_t, \tilde{\theta}_\infty; \tilde{\lambda}, \tilde{\mu})$ which can be written as a polynomial of degree $-\theta_\infty$.

(vi) If $\theta_\infty \in \mathbb{Z}_{\geq 1}$, $\kappa_2 \notin \mathbb{Z}$, the singularity $w = \infty$ of $D_{y_1}(\theta_0, \theta_1, \theta_t, \theta_\infty; \lambda, \mu)$ is apparent and there do not exist any non-zero solutions of $D_{y_1}(\theta_0, \theta_1, \theta_t, \theta_\infty; \lambda, \mu)$ written as $w^{\alpha_0}(w-1)^{\alpha_1}(w-t)^{\alpha_t}p(w)$ such that $p(w)$ is a polynomial and $(\alpha_0, \alpha_1, \alpha_t) = (\theta_0, 0, 0)$, $(0, \theta_1, 0)$ or $(0, 0, \theta_t)$, then there exists a non-zero solution of $D_{y_1}(\tilde{\theta}_0, \tilde{\theta}_1, \tilde{\theta}_t, \tilde{\theta}_\infty; \tilde{\lambda}, \tilde{\mu})$ which can be written as $z^{\tilde{\theta}_0}(z-1)^{\tilde{\theta}_1}(z-t)^{\tilde{\theta}_t}h(z)$, where $h(z)$ is a polynomial of degree $\theta_\infty - 1$.

(vii) If $\kappa_1 \in \mathbb{Z}_{\leq 0}$, $\kappa_2 \notin \mathbb{Z}$, there exists a non-zero solution of $D_{y_1}(\theta_0, \theta_1, \theta_t, \theta_\infty; \lambda, \mu)$ written as a polynomial and there do not exist any non-zero solutions of $D_{y_1}(\tilde{\theta}_0, \tilde{\theta}_1, \tilde{\theta}_t, \tilde{\theta}_\infty; \tilde{\lambda}, \tilde{\mu})$ written in the form $(1/z)^{-\kappa_2+1}p(1/z)$, where $p(1/z)$ is a polynomial in $1/z$, then the singularity $z = \infty$ of $D_{y_1}(\tilde{\theta}_0, \tilde{\theta}_1, \tilde{\theta}_t, \tilde{\theta}_\infty; \tilde{\lambda}, \tilde{\mu})$ is apparent.

(viii) If $\kappa_1 \in \mathbb{Z}_{\geq 1}$, $\kappa_2, \theta_0, \theta_1, \theta_t \notin \mathbb{Z}$, there exists a non-zero solution of $D_{y_1}(\theta_0, \theta_1, \theta_t, \theta_\infty; \lambda, \mu)$ written as a product of $w^{\theta_0}(w-1)^{\theta_1}(w-t)^{\theta_t}$ and a polynomial, then the singularity $z = \infty$ of $D_{y_1}(\tilde{\theta}_0, \tilde{\theta}_1, \tilde{\theta}_t, \tilde{\theta}_\infty; \tilde{\lambda}, \tilde{\mu})$ is apparent.

Theorem 4.3 follows from Proposition A.4.

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