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Contact of a regular surface in Euclidean 3-space with cylinders and cubic binary differential equations

By Toshizumi Fukui, Masaru Hasegawa and Kouichi Nakagawa

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Abstract. We investigate the contact types of a regular surface in the Euclidean 3-space $\mathbb{R}^3$ with right circular cylinders. We present the conditions for existence of cylinders with A_1, A_2, A_3, A_4, A_5, D_4, and D_5 contacts with a given surface. We also investigate the kernel field of $A_{\geq 3}$-contact cylinders on the surface. This is defined by a cubic binary differential equation and we classify singularity types of its flow in the generic context.

1. Introduction.

In his celebrated book [**18**, p. 96], R. Thom pointed out the similarities of the geometric features of D_4 singularities and umbilic points of regular surfaces in $\mathbb{R}^3$. Inspired in this observation, Porteous ([**15**], [**16**]) showed that many differential geometric concepts are rediscovered by looking the singularities of distance-squared functions. He introduced a new notion called by ridge which corresponds to A_3-singularities. After Montaldi ([**14**]) defines the notion of contact between two submanifolds and established the relation to $\mathcal{K}$-equivalence, introduced by Mather ([**13**, Section 2]), in singularity theory, this is justified as an investigation of contact of a surface with spheres. The contact with planes is also recognized as fundamental and the second fundamental form plays crucial role. See [**3**], for example, for several relations of singularities of height functions and differential geometric concepts.

A homogeneous surface in the Euclidean 3-space $\mathbb{R}^3$ is an orbit of a certain subgroup of the Euclidean motion group $O(3) \rtimes \mathbb{R}^3$. They are planes, spheres or cylinders ([**17**]).

Investigating contacts of a surface with planes and spheres has produced a rich field connecting differential geometry with singularity theory and this has been done by many authors (see [**8**], [**9**] for typical research articles, [**2**] for a standard textbook, [**6**], [**7**] for surveys, and their references). In this paper, we will investigate the contact of a surface with cylinders, which has not been investigated in detail yet.

Our computation is based on Monge normal form and this is given in Section 2 with definitions of several concepts. In Section 3, we define the notion of contact due to J. Montaldi and quickly recall the conditions that a surface has a sphere or plane with

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degenerate contact in the generic context. Then we state our main theorems in Section 4.1. The proofs of main theorems will be given in Section 4.2.

On a surface in $\mathbb{R}^3$, there are many articles discussing direction fields defined by quadratic differential equations (see e.g. [**4**] and its references), like principal directions, asymptotic directions and characteristic directions. As far as we know, there are no literature concerning a natural direction field on a surface in $\mathbb{R}^3$ defined by a cubic differential equation in a geometric context. The kernel field of A_2-contact with spheres defines principal directions when the point is not umbilic. The kernel field of A_1^--contact with planes defines asymptotic directions in the hyperbolic region. So it is natural to investigate kernel fields of A_3-contact with cylinders. We call them by *cylindrical fields*, and we show that they are defined by a cubic differential equation.

The last section is dedicated to the analysis of the singularities of integral curves of cylindrical fields in the generic context. We show that singularities of flows of cylindrical fields are on the discriminant of Monge cubic or parabolic lines. To do this, we present a formula for Monge normal form at a non-umbilical point in a surface given by Monge normal form in Section 5.1. We discuss several explicit methods of determination of singularity types of the flow.

2. Monge normal form.

Let $\langle\ ,\ \rangle$ denote the Euclidean inner product. For any regular surface S in the Euclidean 3-space $\mathbb{R}^3$ and for any point $\boldsymbol{p}$ on S, we can find an isometric coordinate change $T : (\mathbb{R}^3, \boldsymbol{p}) \to (\mathbb{R}^3, 0)$ so that $T(S)$ is expressed by the image of the map

$$\boldsymbol{x} : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0),\ (u, v) \mapsto (u, v, f(u, v)), \tag{2.1}$$

at least locally, where $f(u,v) = (k_1u^2 + k_2v^2)/2 + \sum_{s=3}^{k} c_s + o(k)$, $c_s = \sum_{i+j=s}(a_{i,j}/i!j!)u^iv^j$. The expression (2.1) is called by the Monge normal form and the coefficients k_1, k_2, $a_{i,j}$ are differential-geometric invariants of S at $\boldsymbol{p}$, at least when one assume that $k_1 < k_2$. Remark that k_1 and k_2 are principal curvatures at $(0,0)$. We set $\boldsymbol{v}_1 = \partial_u$ and $\boldsymbol{v}_2 = \partial_v$. These are principal vectors at $(0,0)$ with respect to k_1 and k_2 respectively.

For a regular surface S in $\mathbb{R}^3$ defined by (2.1), the second fundamental form of this surface at $(0,0)$ is the bilinear form defined by

$$(\boldsymbol{w}, \boldsymbol{w}') \mapsto \mathrm{II}(\boldsymbol{w}, \boldsymbol{w}') = k_1w_1w_1' + k_2w_2w_2'$$

where $\boldsymbol{w} = w_1\partial_u + w_2\partial_v$, $\boldsymbol{w}' = w_1'\partial_u + w_2'\partial_v$. We say $\boldsymbol{w}$ is asymptotic if $\mathrm{II}(\boldsymbol{w}, \boldsymbol{w}) = 0$.

We say $(0,0)$ is *ridge* with respect to $\boldsymbol{v}_1$ (resp. $\boldsymbol{v}_2$), if $a_{30} = 0$ (resp. $a_{03} = 0$). This condition is equivalent to $\boldsymbol{v}_1\kappa_1 = 0$ (resp. $\boldsymbol{v}_2\kappa_2 = 0$) at $(0,0)$ where $\kappa_i = \kappa_i(u,v)$ is the principal curvature with $\kappa_i(0,0) = k_i$. We say the ridge is of the first order if $\boldsymbol{v}_i\kappa_i(0,0) = 0$, $\boldsymbol{v}_i^2\kappa_i(0,0) \neq 0$. More generally, we say that $(0,0)$ is the k-th order ridge with respect to $\boldsymbol{v}_i$ if $\boldsymbol{v}_i^j\kappa_i(0,0) = 0$ if $j \le k$ and $\boldsymbol{v}_i^{k+1}\kappa_i(0,0) \neq 0$.

We say $(0,0)$ is *subparabolic* with respect to $\boldsymbol{v}_j$ if $\boldsymbol{v}_j\kappa_i(0,0) = 0$ ($\{i,j\} = \{1,2\}$). This is equivalent that $a_{ij} = 0$.

When we assume that $k_1k_2 \neq 0$, we can define the conjugate directions. The *conjugate direction* of the direction generated by a vector $\boldsymbol{w}$ is the direction generated by a non-zero vector $\bar{\boldsymbol{w}}$ with $\mathrm{II}(\boldsymbol{w}, \bar{\boldsymbol{w}}) = 0$. So if $\boldsymbol{w} = w_1\partial_u + w_2\partial_v$, then the conjugate direction is generated by $k_2w_2\partial_u - k_1w_1\partial_v$.

We often refer to $6c_3$ as the *Monge cubic* of S at $(0,0)$. We call $\bar{c}_3(\boldsymbol{w}) = 6c_3(\bar{\boldsymbol{w}})$ the *conjugate Monge cubic* at $(0,0)$, and the root(s) of this cubic $\bar{c}_3$ *cylindrical direction*(*s*) (or, *generatrix direction*(*s*)).

For a function $\varphi(u,v)$, we set $\nabla\varphi(u,v) = \varphi_u(u,v)\partial_u + \varphi_v(u,v)\partial_v$.

3. Contact.

3.1. Definition of contact type.

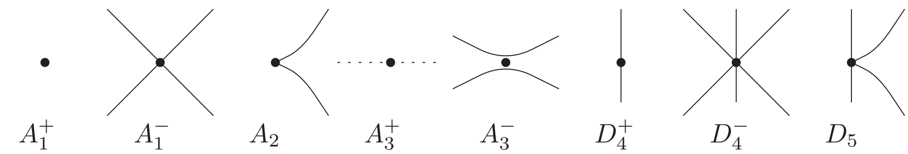
According to [**14**], we define the notion of contact type. Given two pairs of submanifold germs at the origin in $\mathbb{R}^n$, the pairs have the same contact type if there is a diffeomorphism-germ of $(\mathbb{R}^n, 0)$ taking one pair to the other.

THEOREM 3.1 ([**14**]). *For $i = 1, 2$, let $g_i : (X_i, x_i) \to (\mathbb{R}^n, 0)$ be immersion-germs and $f_i : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ be submersion-germs with $Y_i = f_i^{-1}(0)$. Then the pairs (X_1, Y_1) and (X_2, Y_2) have the same contact type if and only if $f_1 \circ g_1$ and $f_2 \circ g_2$ are $\mathcal{K}$-equivalent.*

Here we say that two map-germs $f, g : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ are *$\mathcal{K}$-equivalent* if there are a diffeomorphism $\phi : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ and a smooth map $A : (\mathbb{R}^n, 0) \to GL(\mathbb{R}^n)$ so that $g(\phi(x)) = A(x)f(x)$. In this paper, we consider the following contact types corresponding to $\mathcal{K}$-equivalence classes of $(\mathbb{R}^2, 0) \to (\mathbb{R}, 0)$ represented by

$$A_k \text{ (or } A_k^{\pm}) : x^2 \pm y^{k+1}, \qquad D_k \text{ (or } D_k^{\pm}) : x(y^2 \pm x^{k-2}), (k \geq 4).$$

We show below the zero set of these singularities. Remark that A_k singularities define the kernel field of the Hessian when $k \geq 2$.



3.2. Contact with spheres.

Since spheres in the Euclidean 3-space are determined by their centers and radii the moduli space of spheres is four dimensional. So we expect that there are A_1, A_2, A_3, A_4, D_4-contact spheres for a generic surface.

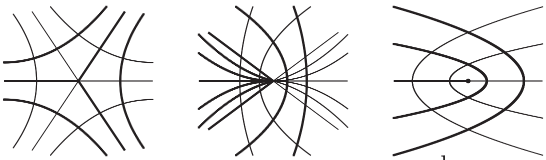
In [**15**], [**16**], Porteous had investigated the singularities of distance squared functions on the surface and these functions measure the contact with sphere. We quickly recall some of his results here.

Let $S_{a,b,c}$ denote the sphere centred at (a,b,c) with radius $\sqrt{a^2+b^2+c^2}$. For a regular surface S in $\mathbb{R}^3$ defined by (2.1), we have the following.

1. The sphere $S_{a,b,c}$ is $A_{\geq 1}$-contact with S at $(0,0,0)$ when (a,b,c) is on the normal line.

2. The sphere $S_{a,b,c}$ is $A_{\geq 2}$-contact with S at $(0,0,0)$ when (a,b,c) is on the focal set of S on the normal line, (i.e., the distance between (a,b,c) and $(0,0,0)$ is $1/k_i$). This sphere is often referred as a curvature sphere with respect to k_i.
3. If $(0,0)$ is ridge with respect to $\boldsymbol{v}_i$ (i.e., $\boldsymbol{v}_i\kappa_i(0,0)=0$), the surface S is $A_{\geq 3}$-contact with the curvature sphere $S_{a,b,c}$ with respect to k_i at $(0,0,0)$.
4. If $(0,0)$ is at least second order ridge with respect to $\boldsymbol{v}_i$ (i.e., $\boldsymbol{v}_i\kappa_i(0,0) = \boldsymbol{v}_i^2\kappa_i(0,0)=0$), the surface S is $A_{\geq 4}$-contact with the curvature sphere $S_{a,b,c}$ with respect to k_i at $(0,0,0)$.
5. If $(0,0)$ is an umbilic, the curvature sphere $S_{a,b,c}$ has $D_{\geq 4}$-contact with S at $(0,0,0)$.

Remark that the kernel field of $A_{\geq 2}$-contact with spheres is defined except at umbilics and it defines principal fields. Classification of singularity types of the curvature curves at generic umbilics are known as Darboux classification ([**5**, Note VII]).



star monstar lemon

3.3. Contact with planes.

Since planes in the Euclidean 3-space are defined by $ax+by+cz=d$, the moduli space of planes is three dimensional. So we expect that there are A_1, A_2, A_3-contact planes for a generic surface.

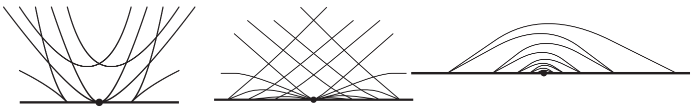
To investigate height functions on the surface, we measure contact with planes. We also recall it quickly. Let $\pi_{a,b,c}$ denote the plane defined by $ax+by+cz=0$ with $a^2+b^2+c^2=1$.

For a regular surface S in $\mathbb{R}^3$ defined by (2.1), we have the following.

1. The plane $\pi_{a,b,c}$ is $A_{\geq 1}$-contact with S at $(0,0,0)$ when (a,b,c) is on the normal line.
2. The plane $\pi_{a,b,c}$ is $A_{\geq 2}$-contact with S at $(0,0,0)$ when $(0,0,0)$ is parabolic.
3. If $(0,0)$ is parabolic (i.e., $k_i=0$) and ridge with respect to $\boldsymbol{v}_i$ (i.e., $\boldsymbol{v}_i\kappa_i(0,0)=0$), the surface S is $A_{\geq 3}$-contact with the plane $\pi_{a,b,c}$ at $(0,0,0)$.

Remark that the kernel fields of A_1^--contact with plane are defined in the hyperbolic region and it defines the asymptotic directions.

Classification of singularity types of asymptotic curves are given by [**1**] as follows.



folded saddle folded node folded focus

4. Contact with cylinders.

Now we consider the contact types of a regular surface with cylinders. Since cylinders in the Euclidean 3-space are determined by their central axes and radii, and central axes are elements of Grassmannian, the moduli space of cylinders is five dimensional. So we expect that there are A_1, A_2, A_3, A_4, A_5, D_4, D_5-contact cylinders for a generic surface.

It is clear that a regular surface S and a cylinder C meet transversely at a point P if they do not have the common normal line at P. So we consider the contact of S and C with common normal line at $P \in S \cap C$.

4.1. Main results.

For a unit vector $\boldsymbol{w} = w_1\partial_u + w_2\partial_v$, we consider the orthogonal projection

$$P_{\boldsymbol{w}} : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0),\ (u, v) \mapsto (w_2u - w_1v, f(u, v)).$$

When $(0, 0)$ is parabolic, the critical locus of $P_{\boldsymbol{w}}$ has a local branch whose tangent is the line generated by $\boldsymbol{w}$. Let $D'_{\boldsymbol{w}}$ denote its image by $P_{\boldsymbol{w}}$.

The main results are stated as follows.

THEOREM 4.1. *Let S be a regular surface defined by* (2.1)*. Let $C_{\boldsymbol{w},\lambda}$ denote the cylinder which contains $(0, 0, 0)$ and whose central axis is parallel to a unit vector $\boldsymbol{w}$, and contains a point $(0, 0, \lambda)$, $\lambda \neq 0$.*

1. *The cylinder $C_{\boldsymbol{w},\lambda}$ has A_1-contact with S at $(0, 0, 0)$ if and only if $k_1k_2\lambda \neq \kappa_n(\boldsymbol{w})$ where $\kappa_n(\boldsymbol{w}) = \mathrm{II}(\boldsymbol{w}, \boldsymbol{w})$.*
2. *The cylinder $C_{\boldsymbol{w},\lambda}$ has A_2-contact with S at $(0, 0, 0)$ if and only if one of the following conditions hold.*

 (A_2a) *$(0, 0)$ is not parabolic (i.e., $k_1k_2 \neq 0$), $\boldsymbol{w}$ is not cylindrical, and $1/\lambda$ is the cylindrical curvature with respect to $\boldsymbol{w}$ (i.e., $\lambda = \kappa_n(\boldsymbol{w})/(k_1k_2)$, see the definition at the beginning of subsection 4.2.1).*

 (A_2b) *$(0, 0)$ is parabolic but not umbilic (i.e., $k_i = 0$, $k_j \neq 0$, $i \neq j$), $w_j = 0$, $\lambda \neq 1/k_j$, and $(0, 0)$ is not ridge with respect to $\boldsymbol{v}_i$.*

 (A_2c) *$(0, 0)$ is a flat umbilic (i.e., $k_1 = k_2 = 0$), and $c_3(\boldsymbol{w}) \neq 0$.*

3. *The cylinder $C_{\boldsymbol{w},\lambda}$ has A_3-contact with S at $(0, 0, 0)$ if and only if one of the following conditions hold.*

 (A_3a) *$(0, 0)$ is not parabolic (i.e., $k_1k_2 \neq 0$), $\boldsymbol{w}$ is cylindrical, $1/\lambda$ is the cylindrical curvature with respect to $\boldsymbol{w}$, and the critical value set of the orthogonal projection with respect to $\boldsymbol{w}$ is nonsingular and is the first order vertex (there is a circle with 4-point contact).*

 (A_3b) *$(0, 0)$ is parabolic but not umbilic (i.e., $k_i = 0$, $k_j \neq 0$, $i \neq j$), $\boldsymbol{w}$ is parallel to the principal vector $\boldsymbol{v}_i$ (i.e., $w_j = 0$), $1/\lambda$ is not a principal curvature (i.e., $1/\lambda \neq k_j$), $(0, 0)$ is ridge with respect to $\boldsymbol{v}_i$, and*

 - *the line generated by $\boldsymbol{w}$ has 4-point contact with S, and $1/\lambda$ is not the limit of curvature $D'_{\boldsymbol{w}}$ at $(0, 0)$, or*
 - *the line generated by $\boldsymbol{w}$ has at least 5-point contact with S, and $(0, 0)$ is not $\boldsymbol{v}_j$-subparabolic.*

 (A_3c) *$(0, 0)$ is a flat umbilic (i.e., $k_1 = k_2 = 0$), $c_3(\boldsymbol{w}) = 0$, and $2c_4(\boldsymbol{w}) + \lambda\langle J\boldsymbol{w}, \nabla c_3(\boldsymbol{w})\rangle^2 \neq 0$ where $J\boldsymbol{w} = w_2\partial_u - w_1\partial_v$.*

Remark 4.2. It is worth to point out that there are no flat umbilics for general surfaces. This means that flat umbilics disappear under small perturbation. In other words, being a flat umbilic is not a generic condition.

Theorem 4.3. *The cylinder* $C_{\boldsymbol{w},\lambda}$ *has* D_4*-contact with* S *at* $(0,0,0)$ *if and only if* $(0,0)$ *is parabolic but not umbilic* (*i.e.*, $k_i = 0$, $k_j \neq 0$, $i \neq j$), $\boldsymbol{w}$ *is parallel to the principal vector* $\boldsymbol{v}_i$ (*i.e.*, $w_j = 0$), $1/\lambda$ *is the other principal curvature* (*i.e.*, $1/\lambda = k_j$), *and* c_3 *does not have multiple roots.*

Theorem 4.4. *The cylinder* $C_{\boldsymbol{w},\lambda}$ *has* D_5*-contact with* S *at* $(0,0,0)$ *if* $(0,0)$ *is parabolic but not umbilic* (*i.e.*, $k_i = 0$, $k_j \neq 0$, $i \neq j$), $\boldsymbol{w}$ *is parallel to the principal vector* $\boldsymbol{v}_i$ (*i.e.*, $w_j = 0$), $1/\lambda$ *is the other principal curvature* (*i.e.*, $1/\lambda = k_j$), c_3 *has a double root* (p_0, q_0), *and* $8c_4(p_0, q_0) \neq k_1^3 p_0^4 + k_2^3 q_0^4$.

Theorem 4.5. *The cylinder* $C_{\boldsymbol{w},\lambda}$ *has* A_4*-contact with* S *at* $(0,0,0)$ *if and only if one of the following conditions hold.*

(A_4a) $(0,0)$ *is not parabolic* (*i.e.*, $k_1 k_2 \neq 0$), $\boldsymbol{w}$ *is cylindrical,* $1/\lambda$ *is the cylindrical curvature with respect to* $\boldsymbol{w}$, *the critical value set of the orthogonal projection with respect to* $\boldsymbol{w}$ *is nonsingular and the second order vertex* (*there is a circle with* 5*-point contact*) *there.*

(A_4b) $(0,0)$ *is parabolic but not umbilic* (*i.e.*, $k_i = 0$, $k_j \neq 0$, $i \neq j$), $\boldsymbol{w}$ *is parallel to the principal vector* $\boldsymbol{v}_i$ (*i.e.*, $w_j = 0$), $1/\lambda$ *is not a principal curvature* (*i.e.*, $\lambda \neq 1/k_j$), $(0,0)$ *is ridge with respect to* $\boldsymbol{v}_i$, *and*

- *the line generated by* $\boldsymbol{w}$ *has* 4*-point contact with* S, *the branch* $D'_{\boldsymbol{w}}$ *has* $(2,5)$ *cusp and* $1/\lambda$ *is the limit of its curvature at* $(0,0)$, *or*
- *the line generated by* $\boldsymbol{w}$ *has* 5*-point contact with* S, *and* $(0,0)$ *is* $\boldsymbol{v}_j$*-subparabolic.*

(A_4c) $(0,0)$ *is a flat umbilic* (*i.e.*, $k_1 = k_2 = 0$), $c_3(\boldsymbol{w}) = 0$,

$$2c_4(\boldsymbol{w}) + \lambda\langle J\boldsymbol{w}, \nabla c_3(\boldsymbol{w})\rangle^2 = 0, \text{ and}$$
$$c_5(\boldsymbol{w}) + \lambda\langle J\boldsymbol{w}, \nabla c_3(\boldsymbol{w})\rangle\langle J\boldsymbol{w}, \nabla c_4(\boldsymbol{w})\rangle + \lambda^2\langle \boldsymbol{w}, \nabla c_3(J\boldsymbol{w})\rangle\langle J\boldsymbol{w}, \nabla c_3(\boldsymbol{w})\rangle^2 \neq 0.$$

Theorem 4.6. *The cylinder* $C_{\boldsymbol{w},\lambda}$ *has* A_5*-contact with* S *at* $(0,0,0)$ *if and only if one of the following conditions hold.*

(A_5a) $(0,0)$ *is not parabolic* (*i.e.*, $k_1 k_2 \neq 0$), $\boldsymbol{w}$ *is cylindrical,* $1/\lambda$ *is the cylindrical curvature with respect to* $\boldsymbol{w}$, *the critical value set of the orthogonal projection with respect to* $\boldsymbol{w}$ *is nonsingular and the third order vertex* (*there is a circle with* 6*-point contact*) *there.*

(A_5b) $k_i = 0$, $k_j \neq 0$, $i \neq j$, $w_j = 0$, $\lambda \neq 1/k_j$, $(0,0)$ *is ridge with respect to* $\boldsymbol{v}_i$.

- *the line generated by* $\boldsymbol{w}$ *has* 4*-point contact with* S, $D'_{\boldsymbol{w}}$ *has* $(2,7)$ *cusp or worse at* $(0,0)$, $1/\lambda$ *is the limit of its curvature at* $(0,0)$ *of* $D'_{\boldsymbol{w}}$, *and the derivative of the curvature of* $D'_{\boldsymbol{w}}$ *by the arc length parameter of* $D'_{\boldsymbol{w}}$ *does not tend to zero at* $(0,0)$, *or*
- *the line generated by* $\boldsymbol{w}$ *has at least* 6*-point contact with* S, *and*

* $a_{60} - 10a_{31}^2(k_2 - 1/\lambda) \neq 0,$ *if* $k_1 = 0,$
* $a_{06} - 10a_{13}^2(k_1 - 1/\lambda) \neq 0,$ *if* $k_2 = 0.$

(A_5c) $(0,0)$ *is a flat umbilic (i.e.,* $k_1 = k_2 = 0$*),* $c_3(\boldsymbol{w}) = 0,$

$$\begin{aligned}
&2c_4(\boldsymbol{w}) + \lambda\langle J\boldsymbol{w}, \nabla c_3(\boldsymbol{w})\rangle^2 = 0,\\
&c_5(\boldsymbol{w}) + \lambda\langle J\boldsymbol{w}, \nabla c_3(\boldsymbol{w})\rangle\langle J\boldsymbol{w}, \nabla c_4(\boldsymbol{w})\rangle + \lambda^2\langle \boldsymbol{w}, \nabla c_3(J\boldsymbol{w})\rangle\langle J\boldsymbol{w}, \nabla c_3(\boldsymbol{w})\rangle^2 = 0, \textit{and}\\
&2c_6(\boldsymbol{w}) + \lambda(\langle J\boldsymbol{w}, \nabla c_4(\boldsymbol{w})\rangle^2/2 + \langle J\boldsymbol{w}, \nabla c_5(\boldsymbol{w})\rangle\langle J\boldsymbol{w}, \nabla c_4(\boldsymbol{w})\rangle)\\
&\quad + \lambda^2\langle J\boldsymbol{w}, \nabla c_3(\boldsymbol{w})\rangle(b_{22}\langle J\boldsymbol{w}, \nabla c_3(\boldsymbol{w})\rangle + \langle J\boldsymbol{w}, \nabla c_4(\boldsymbol{w})\rangle\langle \boldsymbol{w}, \nabla c_3(J\boldsymbol{w})\rangle)\\
&\quad + \lambda^3\langle \boldsymbol{w}, \nabla c_3(J\boldsymbol{w})\rangle^2(2\langle \boldsymbol{w}, \nabla c_3(J\boldsymbol{w})\rangle^2 + c_3(J\boldsymbol{w})\langle \boldsymbol{w}, \nabla c_3(J\boldsymbol{w})\rangle) \neq 0,\\
&\textit{where } b_{22} = (w_2^2/2)(c_4)_{uu}(\boldsymbol{w}) - w_1w_2(c_4)_{uv}(\boldsymbol{w}) + (w_1^2/2)(c_4)_{vv}(\boldsymbol{w}).
\end{aligned}$$

REMARK 4.7. Similar to Remark 4.2, there are no A_4, A_5-contact cylinder at a point in a parabolic line on a generic surface.

4.2. Contact map with cylinders and its singularity type.

First we give the Monge normal form of cylinders $C_{\boldsymbol{w},\lambda}$.

PROPOSITION 4.8. *The Monge normal form of the cylinder which contains* $(0,0,0)$ *and whose central axis contains* $(0,0,\lambda)$ *is given by*

$$(u,v) \mapsto \left(u, v, \sum_{n=1}^{\infty}(-1)^{n-1}\binom{1/2}{n}\frac{(w_1v - w_2u)^{2n}}{\lambda^{2n-1}}\right) \tag{4.1}$$

where $\boldsymbol{w}$ *denotes a unit vector parallel to the central axis of the cylinder.*

PROOF. Set $x_1 = w_1v - w_2u$, $y_1 = w_1u + w_2v$. Then (x_1, y_1, z) form an orthonormal coordinate system, and the cylinder is defined by ${x_1}^2 + (z-\lambda)^2 = \lambda^2$. We thus have

$$z = \lambda\left[1 \pm \left(1 - \left(\frac{x_1}{\lambda}\right)^2\right)\right]^{1/2} = \lambda\left(1 \pm \left(1 + \sum_{n=1}^{\infty}(-1)^n\binom{1/2}{n}\left(\frac{x_1}{\lambda}\right)^{2n}\right)\right),$$

and we obtain the result. □

We investigate contact of a surface given by (2.1) with the cylinder (4.1). To do this, we consider the difference of the third components of (2.1) and (4.1), namely, the contact map defined by

$$F_{\boldsymbol{w},\lambda} = f(u,v) - \sum_{n=1}^{\infty}(-1)^{n-1}\binom{1/2}{n}\frac{(w_1v - w_2u)^{2n}}{\lambda^{2n-1}}.$$

Remark that the Hesse matrix of $F_{\boldsymbol{w},\lambda}$ at $(0,0)$ is

$$\begin{pmatrix} k_1 - \dfrac{{w_2}^2}{\lambda} & \dfrac{w_1w_2}{\lambda} \\ \dfrac{w_1w_2}{\lambda} & k_2 - \dfrac{{w_1}^2}{\lambda} \end{pmatrix}. \tag{4.2}$$

Moreover, the k-jet of $F_{\boldsymbol{w},\lambda}$ at $(0,0)$ is

$$\frac{1}{2}\left(\left(k_1-\frac{w_2^2}{\lambda}\right)u^2+2\frac{w_1w_2}{\lambda}uv+\left(k_2-\frac{w_1^2}{\lambda}\right)v^2\right)$$
$$+\sum_{i+j=3}^{k}\frac{a_{i,j}}{i!j!}u^iv^j-\sum_{n=2}^{2n\le k}(-1)^n\binom{1/2}{n}\frac{(-w_2u+w_1v)^{2n}}{\lambda^{2n-1}}.$$

Proposition 4.9. *The following conditions are equivalent.*

- *The function $F_{\boldsymbol{w},\lambda}$ has a non-degenerate critical point at $(0,0)$.*
- $k_1k_2\lambda\neq\kappa_n(\boldsymbol{w})$.

Proof. They are equivalent that the determinant of (4.2) is not zero. □

Corollary 4.10. *The Hesse matrix* (4.2) *is* 0, *if and only if*

- $k_1=0$, $k_2\neq 0$, $\boldsymbol{w}=\pm\partial_u$, $\lambda=1/k_2$, *or*
- $k_2=0$, $k_1\neq 0$, $\boldsymbol{w}=\pm\partial_v$, $\lambda=1/k_1$.

So the Hesse matrix is of rank 1 *if one of the following conditions hold.*

(a) $k_1k_2\neq 0$, $\lambda=\kappa_n(\boldsymbol{w})/(k_1k_2)$.
(b) $k_i=0$, $k_j\neq 0$, $w_j=0$, $\lambda\neq 1/k_j$ $(i\neq j)$.
(c) $k_1=k_2=0$.

When the Hesse matrix is of rank 1, *its kernel direction is the conjugate direction of* $w_1\partial_u+w_2\partial_v$, *which is generated by* $k_2w_2\partial_u-k_1w_1\partial_v$, *when* $(k_1,k_2)\neq(0,0)$. *When* $k_1=k_2=0$, *the kernel direction is generated by* $w_2\partial_u-w_1\partial_v$.

Proof. Consequence of the expression (4.2). □

To show the results in Section 4.1, it is enough to investigate the conditions that the contact maps $F_{\boldsymbol{w},\lambda}$ have singularities of types A_2, A_3, A_4, A_5, D_4 and D_5 for cases (a), (b) and (c). For convenience to refer, we remark that

$$F_{\boldsymbol{w},\lambda}(u,v)=\begin{cases}\dfrac{(k_1w_1u+k_2w_2v)^2}{2\,\kappa_n(\boldsymbol{w})}+o(2) & \left(k_1k_2\neq 0,\lambda=\dfrac{\kappa_n(\boldsymbol{w})}{k_1k_2}\right),\\[2mm] \dfrac{1}{2}\left(k_2-\dfrac{1}{\lambda}\right)v^2+o(2) & (k_1=0,\boldsymbol{w}=(1,0)),\\[2mm] -\dfrac{1}{2}(w_2u-w_1v)^2+o(2) & (k_1=k_2=0).\end{cases}$$

4.2.1. Case (a): $k_1k_2\neq 0$, $\lambda=\kappa_n(\boldsymbol{w})/(k_1k_2)$.

When $k_1k_2\neq 0$, we call $\lambda=\kappa_n(\boldsymbol{w})/(k_1k_2)$ *cylindrical radius* and $1/\lambda$ *cylindrical curvature* with respect to the direction $\boldsymbol{w}$. The cylindrical radius λ is given by

$$\lambda=\frac{w_2^2}{k_1}+\frac{w_1^2}{k_2}=\frac{1-\cos 2\theta}{2k_1}+\frac{1+\cos 2\theta}{2k_2}=\frac{1/k_1+1/k_2}{2}-\frac{1/k_1-1/k_2}{2}\cos 2\theta$$

where $w_1=\cos\theta$, $w_2=\sin\theta$. We thus have the following properties.

- If $(0,0)$ is not umbilic (i.e., $k_1 \neq k_2$) and is not parabolic point, then the principal directions attain the maximum and the minimum of the cylindrical curvatures.
- If $(0,0)$ is umbilic (i.e., $k_1 = k_2$) and not flat, then the cylindrical curvature does not depend on the tangent vector $\boldsymbol{w}$.

Assume that $k_1k_2 \neq 0$, $\kappa_n(\boldsymbol{w}) \neq 0$ and $\lambda = \kappa_n(\boldsymbol{w})/(k_1k_2)$. Remark that $\kappa_n(\boldsymbol{w}) \neq 0$ since $\lambda \neq 0$. We have

$$F_{\boldsymbol{w},\lambda} = \frac{(k_1w_1u + k_2w_2v)^2}{2\,\kappa_n(\boldsymbol{w})} + \sum_{s=3}^{k} c_s - \sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{\lambda^{2n-1}}(w_1v - w_2u)^{2n} + o(k).$$

PROPOSITION 4.11. *$F_{\boldsymbol{w},\lambda}$ has a singularity of type A_2 at $(0,0)$ if $c_3(\bar{\boldsymbol{w}}) \neq 0$.*

PROOF. Setting $t = k_1w_1u + k_2w_2v$, and $s = w_1v - w_2u$, we have

$$u = \frac{1}{\kappa_n(\boldsymbol{w})}(w_1t - k_2w_2s), \qquad v = \frac{1}{\kappa_n(\boldsymbol{w})}(w_2t + k_1w_1s).$$

Set $\hat{c}_s(t,s) = c_s((1/\kappa_n(\boldsymbol{w}))(w_1t - k_2w_2s), (1/\kappa_n(\boldsymbol{w}))(w_2t + k_1w_1s))$. We obtain that

$$F_{\boldsymbol{w},\lambda} = \frac{t^2}{2\,\kappa_n(\boldsymbol{w})} + \hat{c}_3(t,s) + o(3).$$

This singularity is of type A_2 if $\hat{c}_3(0,1) \neq 0$. This condition is equivalent that $c_3(k_2w_2, -k_1w_1) \neq 0$ and we conclude the proof. □

This shows that the contact map $F_{\boldsymbol{w},\lambda}$ has a singularity of type A_3 or worse when $c_3(\bar{\boldsymbol{w}}) = 0$.

Since $\kappa_n(\boldsymbol{w})$ is the normal curvature $k_1{w_1}^2 + k_2{w_2}^2$ with respect to the direction $\boldsymbol{w} = w_1\partial_u + w_2\partial_v$, the cylindrical curvature times the normal curvature $\kappa_n(\boldsymbol{w})$ is Gauss curvature k_1k_2. This implies the cylindrical curvature is the curvature of the critical value locus of the orthogonal projection to the orthogonal plane to $\boldsymbol{w}$ by Koendrink's theorem ([**11**, Appendix], [**12**, p. 433]).

PROPOSITION 4.12. *Assume that $\kappa_n(\boldsymbol{w}) \neq 0$. The critical value set has a circle with $(k+1)$-point contact (i.e., $(k-2)$-th order vertex) if and only if $F_{\boldsymbol{w},\lambda}$ has A_k singularities at $(0,0)$.*

PROOF. First we remark that the critical locus of the orthogonal projection $(u,v) \mapsto (w_1v - w_2u, f(u,v))$ is zero of

$$-\det(\boldsymbol{x}_u\ \boldsymbol{x}_v\ \boldsymbol{w}) = w_1f_u + w_2f_v = k_1w_1u + k_2w_2v + \sum_{j=3}^{k}\langle \boldsymbol{w}, \nabla c_j\rangle + o(k).$$

Set $\xi = \langle \boldsymbol{w}, \nabla f\rangle$, $\eta = w_1v - w_2u$. Since

$$J = \begin{vmatrix} \dfrac{\partial\xi}{\partial u} & \dfrac{\partial\xi}{\partial v} \\ \dfrac{\partial\eta}{\partial u} & \dfrac{\partial\eta}{\partial v} \end{vmatrix} = \begin{vmatrix} w_1 f_{uu} + w_2 f_{uv} & w_1 f_{uv} + w_2 f_{vv} \\ -w_2 & w_1 \end{vmatrix} = w_1^2 f_{uu} + 2w_1 w_2 f_{uv} + w_2^2 f_{vv}$$

(ξ, η) form a local coordinate at $(0,0)$, whenever $J_0 = \kappa_n(\boldsymbol{w}) \neq 0$. Since

$$\begin{pmatrix} \dfrac{\partial u}{\partial\xi} & \dfrac{\partial u}{\partial\eta} \\ \dfrac{\partial v}{\partial\xi} & \dfrac{\partial v}{\partial\eta} \end{pmatrix} = \begin{pmatrix} \dfrac{\partial\xi}{\partial u} & \dfrac{\partial\xi}{\partial v} \\ \dfrac{\partial\eta}{\partial u} & \dfrac{\partial\eta}{\partial v} \end{pmatrix}^{-1} = \frac{1}{J}\begin{pmatrix} \dfrac{\partial\eta}{\partial v} & -\dfrac{\partial\xi}{\partial v} \\ -\dfrac{\partial\eta}{\partial u} & \dfrac{\partial\xi}{\partial u} \end{pmatrix} = \frac{1}{J}\begin{pmatrix} w_1 & -w_1 f_{uv} - w_2 f_{vv} \\ w_2 & w_1 f_{uu} + w_2 f_{uv} \end{pmatrix},$$

we obtain $\partial_\xi = (\partial u/\partial\xi)\partial_u + (\partial v/\partial\xi)\partial_v = (w_1\partial_u + w_2\partial_v)/J$. Setting $f = (k_1 u^2 + k_2 v^2)/2 + H$, $H_1 = \langle \boldsymbol{w}, \nabla H\rangle$, $H_2 = w_1^2 H_{uu} + 2w_1 w_2 H_{uv} + w_2^2 H_{vv}$, we obtain $\partial_\xi H = H_1/J$, $\partial_\xi H_1 = H_2/J$, and $J = J_0 + H_2$.

Since $\xi = k_1 w_1 u + k_2 w_2 v + H_1$, we have

$$F_{\boldsymbol{w},\lambda} = \frac{(\xi - H_1)^2}{2J_0} + H - \sum_{n=2}^{\infty} (-1)^{n-1} \binom{1/2}{n} \eta^{2n},$$

and we obtain that

$$\begin{aligned} \partial_\xi F_{\boldsymbol{w},\lambda} &= \frac{(\xi - H_1)(1 - \partial_\xi H_1)}{J_0} + \partial_\xi H = \frac{(\xi - H_1)(1 - H_2/J)}{J_0} + \frac{H_1}{J} \\ &= \frac{(\xi - H_1)(J - H_2)}{J_0 J} + \frac{H_1}{J} = \frac{\xi - H_1}{J} + \frac{H_1}{J} = \frac{\xi}{J}. \end{aligned}$$

Thus $\partial_\xi F_{\boldsymbol{w},\lambda}|_{\xi=0}$ is zero. Now consider the Taylor expansion of $F_{\boldsymbol{w},\lambda}$ in the coordinate (ξ, η):

$$F_{\boldsymbol{w},\lambda} = \sum_{i,j} c_{i,j} \xi^i \eta^j.$$

We have $c_{20} = 1/2J_0$ and $c_{1,j} = 0$ for all j. So the singularity type of $F_{\boldsymbol{w},\lambda}$ at $(0,0)$ is A_k if and only if $c_{0,i} = 0$, $i = 1, 2, \ldots, k$, $c_{0,k+1} \neq 0$. This is equivalent that $F_{\boldsymbol{w},\lambda}|_{\xi=0}$ is a unit multiple of η^{k+1}, which means that the critical value set has a circle with $(k+1)$-points contact. □

4.2.2. Case (b): $k_1 = 0$, $k_2 \neq 0$, $w = \partial_u$.

In this case, we have

$$F_{\boldsymbol{w},\lambda} = \frac{1}{2}\Big(k_2 - \frac{1}{\lambda}\Big) v^2 + h(u,v) + \sum_{n=2}^{\infty} (-1)^n \binom{1/2}{n} \frac{v^{2n}}{\lambda^{2n-1}}$$

where $h(u,v) = f(u,v) - k_2 v^2/2$.

Assume that $\lambda \neq 1/k_2$. We see the singularity type of $F_{\boldsymbol{w},\lambda}$ is A_2 if $a_{30} \neq 0$, since the coefficient of u^3 of $F_{\boldsymbol{w},\lambda}$ is $a_{30}/6$. This shows the condition (A_2b) of Theorem 4.1.

Replacing v by $v - a_{21}u^2/2(k_2\lambda - 1)$, the coefficients of $u^2 v$, u^4 become

$$0,\ \frac{a_{40}}{24} - \frac{a_{21}^2}{8(k_2 - 1/\lambda)},$$

respectively. We see the singularity type of $F_{\boldsymbol{w},\lambda}$ is A_3 if $a_{40}(k_2 - 1/\lambda) \neq 3a_{21}^2$. If the line generated by $\boldsymbol{w}$ has at least 5-point contact with S (i.e., $a_{40} = 0$), then this condition becomes $a_{21} \neq 0$, that is, $(0,0)$ is not $\boldsymbol{v}_2$-subparabolic. If the line generated by $\boldsymbol{w}$ has 4-point contact with S (i.e., $a_{40} \neq 0$), then this becomes $1/\lambda \neq k_2 - 3a_{21}^2/a_{40}$ and the right hand side is the limit of the curvature of $D'_{\boldsymbol{w}}$ at $(0,0)$. This shows the condition (A_3b) of Theorem 4.1.

When $a_{40}(k_2 - 1/\lambda) = 3a_{21}^2$ and $a_{21} \neq 0$, the coefficient of u^4 is zero and we see the coefficient of u^5 is

$$\frac{a_{50}}{120} + \frac{a_{40}(a_{40}a_{12} - 2a_{31}a_{21})}{72a_{21}^2}.$$

This shows the condition (A_4b) of Theorem 4.5.

Replacing v by $v - a_{40}(a_{40}a_{12} - a_{31}a_{21})u^3/8a_{21}^3$, the coefficients of u^3v, u^6 become

$$0,\ \frac{a_{60}a_{21}^4 - 45a_{41}a_{40}a_{21}^3 - 5a_{40}^3a_{21}a_{03} - 30a_{40}^3a_{12}^2 - 30a_{40}a_{31}^2a_{21}^2 + 45a_{40}^2a_{22}a_{21}^2 + 60a_{40}^2a_{31}a_{21}a_{12}}{6480a_{21}^4},$$

respectively. So the singularity type of $F_{\boldsymbol{w},\lambda}$ is A_5 if the later number is non zero.

To complete the proof, we describe the branch $D'_{\boldsymbol{w}}$. A parametrization of the branch of the critical set of $P_{\boldsymbol{w}}$ whose tangent is generated by $\boldsymbol{w}$ is given by $t \mapsto (\sqrt{\pm 6a_{21}/a_{40}}t + o(1), \pm t^2)$, and thus $D'_{\boldsymbol{w}}$ is parametrized by

$$t \mapsto \left(\mp t^2,\ \frac{a_{40}k_2 - 3a_{21}^2}{2a_{40}}t^4 + \left[\mp\frac{3a_{21}}{2a_{40}^5}\right]^{1/2}\frac{3a_{21}^2a_{50} + 5a_{12}a_{40}^2 - 10a_{21}a_{31}a_{40}}{5}t^5 + o(5)\right)$$

whenever $a_{40} \neq 0$. This shows the limit of the curvature of $D'_{\boldsymbol{w}}$ is $k_2 - 3a_{21}^2/a_{40}$ at $(0,0)$ and $D'_{\boldsymbol{w}}$ has $(2,5)$ cusp when $3a_{21}^2a_{50} + 5a_{12}a_{40}^2 - 10a_{21}a_{31}a_{40} \neq 0$, which shows Condition (A_4b) in case $a_{21} \neq 0$. Remark that the coefficient of t^6 in the second component is

$$\frac{-1}{240a_{21}a_{40}^4}\begin{bmatrix} 8a_{40}(a_{60}a_{21}^4 - 45a_{41}a_{40}a_{41} - 30a_{40}a_{31}^2a_{21}^2 + 45a_{40}^2a_{22}a_{21}^2 + 60a_{40}^2a_{31}a_{21}a_{12}) \\ -15(3a_{50}a_{21}^2 + 5a_{40}^2a_{12} - 10a_{40}a_{31}a_{21})(3a_{50}a_{21}^2 - 3a_{40}^2a_{12} - 2a_{40}a_{31}a_{21}) \end{bmatrix},$$

which shows Condition (A_5b) in case $a_{21} \neq 0$.

When $a_{40}(k_2 - 1/\lambda) = 3a_{21}^2$ and $a_{21} = 0$, we have $a_{40} = 0$. Thus we have that

$$F_{\boldsymbol{w},\lambda} = \frac{k_2 - 1/\lambda}{2}v^2 + \frac{a_{50}}{120}u^5 + \frac{a_{60} - 10a_{31}^2(k_2 - 1/\lambda)}{720}u^6 + \cdots,$$

and the singularity type of $F_{\boldsymbol{w},\lambda}$ is A_4 (resp. A_5) if $a_{50} \neq 0$ (resp. $a_{50} = 0$, $a_{60} - 10a_{31}^2(k_2 - 1/\lambda) \neq 0$).

Proposition 4.13. *Assume that $\lambda = 1/k_2$.*

- *If c_3 does not have multiple roots, then $F_{\boldsymbol{w},\lambda}$ has a singularity of type D_4 at $(0,0)$.*
- *If c_3 has a double root (p_0, q_0) and a single root (p_1, q_1), then $F_{\boldsymbol{w},\lambda}$ has a singularity of type D_5 at $(0,0)$, whenever $c_4(p_0, q_0) \neq (k_2^3/8)q_0^4$.*

Proof. When $\lambda = 1/k_2$, observe that

$$F_{\boldsymbol{w},\lambda} = \sum_{s=3}^{k} c_s(u,v) - \sum_{n=2}^{\infty}(-1)^n \binom{1/2}{n} {k_2}^{2n-1} v^{2n} + o(k).$$

The first statement is well-known. In the second case we may assume that $c_3 = (q_0 u - p_1 v)^2(q_1 u - p_1 v)$. Setting $u = -p_1 x + p_0 y$, $v = -q_1 x + q_0 y$, we have

$$F_{\boldsymbol{w},\lambda} = \delta^3 x^2 y + \left[c_4(p_0, q_0) - \frac{k_2^3}{8} q_0^4\right] y^4 + \cdots$$

where $\delta = p_0 q_1 - q_0 p_1$. We thus conclude the proof. □

4.2.3. Case (c): $k_1 = k_2 = 0$.

In this case, we have

$$F_{\boldsymbol{w},\lambda} = -\frac{(w_1 v - w_2 u)^2}{2\lambda} + f(u,v) - \sum_{n=2}^{\infty}(-1)^{n-1}\binom{1/2}{n}\frac{(w_1 v - w_2 u)^{2n}}{\lambda^{2n-1}}.$$

Setting $u = w_1 s - w_2 t$, $v = w_2 s + w_1 t$. we have

$$F_{\boldsymbol{w},\lambda} = -\frac{t^2}{2\lambda} + f(w_1 s - w_2 t, w_2 s + w_1 t) - \sum_{n=2}^{\infty}(-1)^{n-1}\binom{1/2}{n}\frac{t^{2n}}{\lambda^{2n-1}}.$$

When we write $f(w_1 s - w_2 t, w_2 s + w_1 t) = \sum_{i,j}^{i+j\le k} b_{i,j} s^i t^j + o(k)$, we conclude that

$$b_{30} = c_3(\boldsymbol{w}),\ b_{21} = \langle J\boldsymbol{w}, \nabla c_3(\boldsymbol{w})\rangle,\ b_{12} = \langle \boldsymbol{w}, \nabla c_3(J\boldsymbol{w})\rangle,\ b_{03} = c_3(J\boldsymbol{w}),\ b_{40} = c_4(\boldsymbol{w}),$$
$$b_{31} = \langle J\boldsymbol{w}, \nabla c_4(\boldsymbol{w})\rangle,\ b_{22} = \frac{w_2^2}{2}(c_4)_{uu}(\boldsymbol{w}) - w_1 w_2 (c_4)_{uv}(\boldsymbol{w}) + \frac{w_1^2}{2}(c_4)_{vv}(\boldsymbol{w}),\ \ldots,$$

$b_{50} = c_5(\boldsymbol{w})$, $b_{41} = \langle J\boldsymbol{w}, \nabla c_5(\boldsymbol{w})\rangle$, $\ldots$, $b_{60} = c_6(\boldsymbol{w})$, $\ldots$, where $J\boldsymbol{w} = w_2\partial_u - w_1\partial_v$. The condition $(A_2\mathrm{c})$ of Theorem 4.1 follows to see the coefficient b_{30}.

Replacing t by $t + \lambda b_{21} s^2$, the coefficient of $s^2 t$ and s^4 become 0, $b_{40} + \lambda b_{21}^2/2$, respectively. The condition $(A_3\mathrm{c})$ follows from Theorem 4.1.

Observe the coefficient of s^5 is $b_{50} + \lambda b_{31} b_{21} + \lambda^2 b_{12} b_{21}^2$, which shows the condition $(A_4\mathrm{c})$ of Theorem 4.5.

Replacing t by $t + \lambda(b_{31} + b_{21}^2\lambda)s^3$, we obtain that the coefficients of $s^3 t$ and s^6 become

$$0,\ b_{60} + \lambda(b_{31}^2/2 + b_{41}b_{21}) + \lambda^2 b_{21}(b_{22}b_{21} + 2b_{31}b_{12}) + \lambda^3 b_{21}^2(2b_{12}^2 + b_{03}b_{21}),$$

respectively. This concludes the condition $(A_5\mathrm{c})$ of Theorem 4.6.

5. Cylindrical directions.

In Theorem 4.1, we observe that the kernel field of $A_{\ge 3}$-contact with cylinders is cylindrical directions whenever $k_1 k_2 \ne 0$. These are also defined at nearby points, and we express it in terms of the Monge normal form. This field is defined by $\bar{C}(du, dv) = 0$ where $\bar{C}(du, dv) = C(\kappa_2 dv, -\kappa_1 du)$, and this can be considered as a counterpart of principal fields. Here $C(p, q)$ denote the Monge cubic defined in Proposition 5.2. This section is dedicated to the analysis of this field.

Let $[p : q]$ denote the homogeneous coordinates of the real projective line P^1. Let M denote the subset defined by $C(p, q) = 0$ in $\mathbb{R}^2 \times P^1$, $\pi : M \to \mathbb{R}^2$ the natural projection, and $\bar{\omega} = p\kappa_1 du + q\kappa_2 dv$.

THEOREM 5.1. *Assume that $(0, 0)$ is not an umbilic of $\boldsymbol{x}$ (i.e., $k_1 \neq k_2$). The flows of $\bar{C}(du, dv) = 0$ is obtained by the projection image of the flows of 1-form $\bar{\omega}|_M$ by the natural projection $M \to \mathbb{R}^2$. The 1-form $\bar{\omega}|_M$ is singular at $(0, 0) \times [p_0, q_0] \in M$ if and only if one of the following conditions holds.*

1. *(p_0, q_0) is a single root of $C(p, q) = 0$ at $(u, v) = (0, 0)$, and*
   (a) *$\kappa_1(0, 0) = 0$, $(0, 0)$ is ridge with respect to $\boldsymbol{v}_1$, and (p_0, q_0) represents the principal direction $\boldsymbol{v}_1$ (i.e., $a_{3,0} = 0$, $q_0 = 0$), or*
   (b) *$\kappa_2(0, 0) = 0$, $(0, 0)$ is ridge with respect to $\boldsymbol{v}_2$, and (p_0, q_0) represents the principal direction $\boldsymbol{v}_2$ (i.e., $a_{0,3} = 0$, $p_0 = 0$).*
2. *(p_0, q_0) is a multiple root of $C(p, q) = 0$ at $(u, v) = (0, 0)$, that is, is in the discriminant set D of $C(p, q) = 0$ (i.e., $C_p(p_0, q_0) = C_q(p_0, q_0) = 0$), and*
   (a) *D is nonsingular at $(0, 0)$ and the tangent direction of D at $(0, 0)$ is conjugate to (p_0, q_0), or*
   (b) *$\alpha = \beta = 0$ (in particular, D is singular at $(0, 0)$) where*

$$\alpha = C_u(p_0, q_0)|_{(u,v)=(0,0)}, \quad \beta = C_v(p_0, q_0)|_{(u,v)=(0,0)}.$$

We conclude that the position of singular points of $\bar{\omega}|_M$ is determined by the 3-jet (resp. 4-jet) of the Monge normal form (2.1) in Case 1 (resp. Case 2). In the generic context, these singularity types are saddle, node or focus, which are determined by the 4-jet (resp. 5-jet) of the Monge normal form (2.1) in Case 1 (resp. Case 2). The phase portraits of singularities of the flows of the equation $\bar{C}(du, dv) = 0$, in the generic context, will be also given. See pictures after Propositions 5.7, 5.11, and Remark 5.9.

5.1. Monge normal form of a surface given by Monge normal form.

We consider the Monge normal form of the surface defined by (2.1) at the point $(u, v, f(u, v))$.

PROPOSITION 5.2. *The Monge normal form of $\boldsymbol{x}$ at $(u, v, f(u, v))$ is given by*

$$z = \kappa_1 \frac{x^2}{2} + \kappa_2 \frac{y^2}{2} + C(x, y) + o(x, y)^3 \tag{5.1}$$

where

$$\kappa_1 = k_1 + c_{3,uu}(u,v) + c_{4,uu}(u,v) - (3k_1^2 u^2 + k_2^2 v^2) + \frac{2c_{3,uv}(u,v)^2}{k_1 - k_2} + o(u,v)^2,$$

$$
\begin{aligned}
\kappa_2 &= k_2 + c_{3,vv}(u,v) + c_{4,vv}(v,v) - (k_1^2u^2 + 3k_2^2v^2) - \frac{2c_{3,uv}(u,v)^2}{k_1-k_2} + o(u,v)^2,\\
C(x,y) &= c_3(x,y) + c_{4,u}(x,y)u + c_{4,v}(x,y)v + \frac{c_{3,uv}(u,v)}{2(k_1-k_2)}C_1\\
&\quad + c_{5,uuu}(u,v)\frac{x^3}{6} + c_{5,uuv}(u,v)\frac{x^2y}{2} + c_{5,uvv}(u,v)\frac{xy^2}{2} + c_{5,vvv}(u,v)\frac{y^3}{6}\\
&\quad + \frac{c_{3,uv}(u,v)}{4(k_1-k_2)^2}(Pu+Qv) - \frac{Y_1}{4} + \frac{X+Y_2}{4(k_1-k_2)} - (k_1^2ux + k_2^2vy)\frac{k_1x^2+k_2y^2}{2}\\
&\quad + o(u,v)^2,\\
C_1 &= a_{21}x^3 + (2a_{12}-a_{30})x^2y + (a_{03}-2a_{21})xy^2 - a_{12}y^3,\\
P &= a_{21}(4a_{12}-3a_{30})x^3 + (4a_{12}^2 + 2a_{03}a_{21} - 7a_{21}^2 - 6a_{12}a_{30} + 2a_{30}^2)x^2y\\
&\quad + (2a_{03}a_{12} - 11a_{12}a_{21} - 2a_{03}a_{30} + 6a_{21}a_{30})xy^2\\
&\quad + (-2a_{12}^2 - a_{03}a_{21} + 2a_{21}^2 + 2a_{12}a_{30})y^3,\\
Q &= (2a_{12}^2 + 2a_{03}a_{21} - 2a_{21}^2 - a_{12}a_{30})x^3 + (6a_{03}a_{12} - 11a_{12}a_{21} - 2a_{03}a_{30} + 2a_{21}a_{30})x^2y\\
&\quad + (2a_{03}^2 - 7a_{12}^2 - 6a_{03}a_{21} + 4a_{21}^2 + 2a_{12}a_{30})xy^2 - a_{12}(3a_{03}-4a_{21})y^3,\\
X &= [3a_{21}a_{31}u^2 + 2(2a_{21}a_{22} + a_{12}a_{31})uv + (a_{13}a_{21} + 2a_{12}a_{22})v^2]x^3\\
&\quad + \begin{bmatrix} (4a_{21}a_{22} + 2a_{12}a_{31} - a_{30}a_{31} - 2a_{21}a_{40})u^2\\ +2(2a_{13}a_{21} + 4a_{12}a_{22} - a_{22}a_{30} - a_{21}a_{31}\\ -a_{12}a_{40})uv + (6a_{12}a_{13} - a_{13}a_{30} - 2a_{12}a_{31})v^2 \end{bmatrix} x^2y\\
&\quad + \begin{bmatrix} (2a_{13}a_{21} + a_{03}a_{31} - 6a_{21}a_{31})u^2 + 2(a_{12}a_{13}\\ +a_{04}a_{21} + a_{03}a_{22} - 4a_{21}a_{22} - 2a_{12}a_{31})uv\\ +(2a_{04}a_{12} + a_{03}a_{13} - 2a_{13}a_{21} - 4a_{12}a_{22})v^2 \end{bmatrix} xy^2\\
&\quad + [(-2a_{21}a_{22} - a_{12}a_{31})u^2 - 2(a_{13}a_{21} + 2a_{12}a_{22})uv - 3a_{12}a_{13}v^2]y^3,\\
Y_1 &= 19a_{30}k_1^2u^2x^3 + 3(3a_{03}k_1k_2 + 4a_{21}k_2^2)v^2x^2y\\
&\quad + 3(4a_{12}k_1^2 + 3a_{30}k_1k_2)u^2xy^2 + 19a_{03}k_2^2v^2y^3,\\
Y_2 &= [-6a_{21}k_1^2(3k_1-k_2)uv + (-3a_{12}k_1^3 - 3a_{12}k_1^2k_2 - a_{30}k_1k_2^2 + a_{30}k_2^3)v^2]x^3\\
&\quad + [-3a_{21}k_1(3k_1^2 - 4k_1k_2 - k_2^2)u^2 + 6k_2(3a_{12}k_1^2 - 2a_{12}k_1k_2 - a_{30}k_2^2)uv]x^2y\\
&\quad + [-6k_1(a_{03}k_1^2 + 2a_{21}k_1k_2 - 3a_{21}k_2^2)uv - 3a_{12}k_2(k_1^2 + 4k_1k_2 - 3k_2^2)v^2]xy^2\\
&\quad + [(-a_{03}k_1^3 + a_{03}k_1^2k_2 + 3a_{21}k_1k_2^2 + 3a_{21}k_2^3)u^2 - 6a_{12}(k_1-3k_2)k_2^2uv]y^3.
\end{aligned}
$$

Remark that

- the 1 (resp. 2)-jet of κ_i is determined by the 3 (resp. 4)-jet of the Monge normal form (2.1), and
- the 1 (resp. 2)-jet of C is determined by the 4 (resp. 5)-jet of the Monge normal form (2.1).

We call the cubic form $C(x,y)$ the *Monge cubic*. To show this proposition, we define an orthonormal frame by

$$\begin{aligned}
\boldsymbol{e}_1(u,v) &= \frac{\boldsymbol{x}_u(u,v)}{\|\boldsymbol{x}_u(u,v)\|},\\
\boldsymbol{e}_2(u,v) &= \frac{[\boldsymbol{x}_u(u,v)\times\boldsymbol{x}_u(u,v)]\times\boldsymbol{x}_u(u,v)}{\|[\boldsymbol{x}_u(u,v)\times\boldsymbol{x}_v(u,v)]\times\boldsymbol{x}_u(u,v)\|},\\
\boldsymbol{e}_3(u,v) &= \frac{\boldsymbol{x}_u(u,v)\times\boldsymbol{x}_v(u,v)}{\|\boldsymbol{x}_u(u,v)\times\boldsymbol{x}_v(u,v)\|}.
\end{aligned}$$

These can be written in the following explicit forms:

$$\begin{aligned}
\boldsymbol{e}_1(u,v) &= \frac{1}{\sqrt{1+f_u(u,v)^2}}(1,0,f_u(u,v)),\\
\boldsymbol{e}_2(u,v) &= \frac{(-f_u(u,v)f_v(u,v),\ 1+f_u(u,v)^2,\ f_v(u,v))}{\sqrt{1+f_u(u,v)^2}\sqrt{1+f_u(u,v)^2+f_v(u,v)^2}},\\
\boldsymbol{e}_3(u,v) &= \frac{(-f_u(u,v),\ -f_v(u,v),\ 1)}{\sqrt{1+f_u(u,v)^2+f_v(u,v)^2}}.
\end{aligned}$$

Setting

$$\begin{aligned}
\bar{x} &= \langle \boldsymbol{x}(u+\bar{u},v+\bar{v})-\boldsymbol{x}(u,v),\ \boldsymbol{e}_1(u,v)\rangle,\\
\bar{y} &= \langle \boldsymbol{x}(u+\bar{u},v+\bar{v})-\boldsymbol{x}(u,v),\ \boldsymbol{e}_2(u,v)\rangle,\\
z &= \langle \boldsymbol{x}(u+\bar{u},v+\bar{v})-\boldsymbol{x}(u,v),\ \boldsymbol{e}_3(u,v)\rangle,
\end{aligned}$$

we obtain

$$\bar{x} = \frac{\bar{u}+[f(u+\bar{u},v+\bar{v})-f(u,v)]f_u(u,v)}{\sqrt{1+f_u(u,v)^2}}, \tag{5.2}$$

$$\bar{y} = \frac{-\bar{u}f_u(u,v)f_v(u,v)+\bar{v}[1+f_u(u,v)^2]+[f(u+\bar{u},v+\bar{v})-f(u,v)]f_v(u,v)}{\sqrt{1+f_u(u,v)^2}\sqrt{1+f_u(u,v)^2+f_v(u,v)^2}}, \tag{5.3}$$

$$z = \frac{-\bar{u}f_u(u,v)-\bar{v}f_v(u,v)+[f(u+\bar{u},v+\bar{v})-f(u,v)]}{\sqrt{1+f_u(u,v)^2+f_v(u,v)^2}}. \tag{5.4}$$

By inverse mapping theorem, $(\bar{u},\bar{v})$ are determined by $(\bar{x},\bar{y})$ when $|(\bar{u},\bar{v})|<\varepsilon\ll 1$.

Lemma 5.3. *Setting $f_{i,j}=f_{u^iv^j}(u,v)$, we have*

$$\bar{x} = \sqrt{1+f_{10}^2}\,\bar{u}+\frac{f_{10}f_{01}}{\sqrt{1+f_{10}^2}}\bar{v}+\frac{f_{10}}{\sqrt{1+f_{10}^2}}\sum_{i+j\ge 2}^{k}\frac{f_{i,j}}{i!j!}\bar{u}^i\bar{v}^j+o(\bar{u},\bar{v})^k,$$

$$\bar{y} = \frac{\sqrt{1+f_{10}^2+f_{01}^2}}{\sqrt{1+f_{10}^2}}\bar{v}+\frac{f_{01}}{\sqrt{1+f_{10}^2}\sqrt{1+f_{10}^2+f_{01}^2}}\sum_{i+j\ge 2}^{k}\frac{f_{i,j}}{i!j!}\bar{u}^i\bar{v}^j+o(\bar{u},\bar{v})^k.$$

Proof. By (5.2) and (5.3), we have

$$\bar{x}_{\bar{u}} = \frac{1+f_u(u+\bar{u},v+\bar{v})f_u(u,v)}{\sqrt{1+f_u(u,v)^2}},\quad \bar{y}_{\bar{u}} = \frac{f_v(u,v)(-f_u(u,v)+f_u(u+\bar{u},v+\bar{v}))}{\sqrt{1+f_u(u,v)^2}\sqrt{1+f_u(u,v)^2+f_v(u,v)^2}},$$

$$\bar{x}_{\bar{v}} = \frac{f_v(u+\bar{u}, v+\bar{v}) f_u(u,v)}{\sqrt{1+f_u(u,v)^2}}, \qquad \bar{y}_{\bar{v}} = \frac{[1+f_u(u,v)^2] + f_v(u+\bar{u}, v+\bar{v}) f_v(u,v)}{\sqrt{1+f_u(u,v)^2}\sqrt{1+f_u(u,v)^2+f_v(u,v)^2}},$$

and, for (i,j) with $i+j \geq 2$,

$$\bar{x}_{\bar{u}^i\bar{v}^j} = \frac{f_{u^iv^j}(u+\bar{u}, v+\bar{v}) f_u(u,v)}{\sqrt{1+f_u(u,v)^2}}, \quad \bar{y}_{\bar{u}^i\bar{v}^j} = \frac{f_{u^iv^j}(u+\bar{u}, v+\bar{v}) f_v(u,v)}{\sqrt{1+f_u(u,v)^2}\sqrt{1+f_u(u,v)^2+f_v(u,v)^2}}.$$

These imply the lemma. □

LEMMA 5.4. *We have*

$$\begin{aligned}
\bar{u} = {} & \frac{1}{\sqrt{1+f_{10}^2}}\bar{x} - \frac{f_{10}f_{01}}{\sqrt{1+f_{10}^2}\sqrt{1+f_{10}^2+f_{01}^2}}\bar{y} \\
& - \frac{f_{10}}{(1+f_{10}^2)(1+f_{10}^2+f_{01}^2)}\biggl(\frac{f_{11}}{2}\bar{x}^2 + \frac{(1+f_{10}^2)f_{11} - f_{10}f_{01}f_{20}}{\sqrt{1+f_{10}^2+f_{01}^2}}\bar{x}\bar{y} \\
& + \frac{f_{10}^2f_{01}^2f_{20} - 2f_{10}f_{01}(1+f_{10}^2)f_{11} + (1+f_{10}^2)f_{02}}{2(1+f_{10}^2+f_{01}^2)}\bar{y}^2\biggr) + o(\bar{x},\bar{y})^2, \\
\bar{v} = {} & \frac{\sqrt{1+f_{10}^2}}{\sqrt{1+f_{10}^2+f_{01}^2}}\bar{y} \\
& + \frac{f_{01}}{(1+f_{10}^2)(1+f_{10}^2+f_{01}^2)}\biggl(\frac{f_{20}}{2}\bar{x}^2 + \frac{-f_{10}f_{01}f_{20} + (1+f_{10}^2)f_{11}}{\sqrt{1+f_{10}^2+f_{01}^2}}\bar{x}\bar{y} \\
& + \frac{f_{10}^2f_{01}^2f_{20} - 2(1+f_{10}^2)f_{11} + (1+f_{10}^2)^2f_{02}}{2(1+f_{10}^2+f_{01}^2)}\bar{y}^2\biggr) + o(\bar{x},\bar{y})^2.
\end{aligned}$$

PROOF. When we write

$$\bar{x} = \sum_{i+j\geq 1}^{k} x_{i,j}\frac{\bar{u}^i\bar{v}^j}{i!j!} + o(\bar{u},\bar{v})^k, \qquad \bar{y} = \sum_{i+j\geq 1}^{k} y_{i,j}\frac{\bar{u}^i\bar{v}^j}{i!j!} + o(\bar{u},\bar{v})^k, \tag{5.5}$$

$$\bar{u} = \sum_{i+j\geq 1}^{k} \bar{u}_{i,j}\frac{\bar{x}^i\bar{y}^j}{i!j!} + o(\bar{u},\bar{v})^k, \qquad \bar{v} = \sum_{i+j\geq 1}^{k} \bar{v}_{i,j}\frac{\bar{x}^i\bar{y}^j}{i!j!} + o(\bar{u},\bar{v})^k, \tag{5.6}$$

we obtain that

$$u_{10} = \frac{y_{01}}{x_{10}y_{01} - x_{01}y_{10}}, \qquad v_{10} = \frac{-y_{10}}{x_{10}y_{01} - x_{01}y_{10}},$$
$$u_{01} = \frac{-x_{01}}{x_{10}y_{01} - x_{01}y_{10}}, \qquad v_{01} = \frac{x_{10}}{x_{10}y_{01} - x_{01}y_{10}}.$$

Moreover, we have

$$\begin{pmatrix} x_{10}I_3 & x_{01}I_3 \\ y_{10}I_3 & y_{01}I_3 \end{pmatrix}\begin{pmatrix} U_2 \\ V_2 \end{pmatrix} + \begin{pmatrix} \Delta_2 & 0 \\ 0 & \Delta_2 \end{pmatrix}\begin{pmatrix} X_2 \\ Y_2 \end{pmatrix} = 0$$

where

$$X_2 = \begin{pmatrix} x_{20} \\ x_{11} \\ x_{02} \end{pmatrix}, \quad Y_2 = \begin{pmatrix} y_{20} \\ y_{11} \\ y_{02} \end{pmatrix}, \quad U_2 = \begin{pmatrix} u_{20} \\ u_{11} \\ u_{02} \end{pmatrix}, \quad V_2 = \begin{pmatrix} v_{20} \\ v_{11} \\ v_{02} \end{pmatrix},$$

$$\Delta_2 = \begin{pmatrix} u_{10}^2 & 2u_{10}v_{10} & v_{10}^2 \\ u_{10}u_{01} & u_{10}v_{01} + u_{01}v_{10} & v_{10}v_{01} \\ u_{01}^2 & 2u_{01}v_{01} & v_{01}^2 \end{pmatrix}.$$

Since $v_{10} = y_{10} = 0$, we obtain that

$$u_{20} = \frac{x_{01}y_{20} - x_{20}y_{01}}{x_{10}^3 y_{01}}, \qquad u_{11} = \frac{-x_{10}x_{11}y_{01} + x_{01}x_{20}y_{01} + x_{01}x_{10}y_{11} - x_{01}^2 y_{20}}{x_{10}^3 y_{01}^2},$$
$$u_{02} = \frac{-x_{02}x_{10}^2 y_{01} + x_{01}(2x_{10}x_{11}y_{01} - x_{01}x_{20}y_{01} + x_{10}^2 y_{02} - 2x_{01}x_{10}y_{11} + x_{01}^2 y_{20})}{x_{10}^3 y_{01}^3},$$
$$v_{20} = \frac{-y_{20}}{x_{10}^2 y_{01}}, \qquad v_{11} = \frac{x_{01}y_{20} - x_{10}y_{11}}{x_{10}^2 y_{01}^2}, \qquad v_{02} = -\frac{x_{10}^2 y_{02} - 2x_{01}x_{10}y_{11} + x_{01}^2 y_{20}}{x_{10}^2 y_{01}^3}.$$

These conclude the proof. □

Proof of Proposition 5.2. By Taylor's theorem, we have

$$f(u+\bar{u}, v+\bar{v}) - f(u,v) = \sum_{i+j\geq 1}^{k} \frac{f_{i,j}}{i!j!} \bar{u}^i \bar{v}^j + o(\bar{u},\bar{v})^k,$$

and thus, by (5.4),

$$z = \frac{1}{\sqrt{1+f_{10}^2+f_{01}^2}} \sum_{i+j\geq 2}^{k} \frac{f_{i,j}}{i!j!} \bar{u}^i \bar{v}^j + o(\bar{u},\bar{v})^k.$$

By (5.6) and $v_{10} = 0$, we have

$$\begin{aligned} z = {} & \frac{1}{\sqrt{1+f_{10}^2+f_{01}^2}} \Big(\tfrac{f_{20}u_{10}^2}{2} \bar{x}^2 + u_{10}(u_{01}f_{20} + v_{01}f_{11}) \bar{x}\bar{y} + \tfrac{u_{01}^2 f_{20} + 2u_{01}v_{01}f_{11} + v_{01}^2 f_{02}}{2} \bar{y}^2 \\ & + \tfrac{u_{10}(f_{3,0}u_{10}^2 + 3f_{20}u_{20} + 3f_{11}v_{20})}{6} \bar{x}^3 \\ & + \tfrac{f_{3,0}u_{01}u_{10}^2 + f_{2,1}u_{10}^2 v_{01} + f_{20}(2u_{10}u_{11} + u_{01}u_{20}) + f_{11}(u_{20}v_{01} + 2u_{10}v_{11} + u_{01}v_{20}) + 2f_{02}v_{01}v_{20}}{2} \bar{x}^2\bar{y} \\ & + \tfrac{f_{3,0}u_{01}^2 u_{10} + 2f_{2,1}u_{01}u_{10}v_{01} + f_{1,2}u_{10}v_{01}^2 + f_{20}(u_{02}u_{10} + 2u_{01}u_{11}) + f_{11}(2u_{11}v_{01} + u_{10}v_{02} + 2u_{01}v_{11}) + 4f_{02}v_{01}v_{11}}{2} \bar{x}\bar{y}^2 \\ & + \tfrac{f_{3,0}u_{01}^3 + 3f_{2,1}u_{01}^2 v_{01} + 3f_{1,2}u_{01}v_{01}^2 + f_{0,3}v_{01}^3 + 3f_{20}u_{01}u_{02} + 3f_{11}(u_{02}v_{01} + u_{01}v_{02}) + 6f_{02}v_{01}v_{02}}{6} \bar{y}^3 \Big) \\ & + o(\bar{x},\bar{y})^3. \end{aligned}$$

We take an orthonormal matrix

$$T = \Big(1 - \frac{c_{3,uv}(u,v)^2}{2(k_1-k_2)^2} + o(u,v)^2\Big) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + c \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

where $c = c_{3,uv}(u,v)/(k_1-k_2) + (a_{2,1}a_{1,2}(u^2-v^2) + (a_{2,1}^2 - a_{1,2}^2 + a_{3,0}a_{1,2} - a_{2,1}a_{0,3})uv) /(k_1-k_2)^2 + (c_{4,uv}(u,v) - k_1^2k_2)/(k_1-k_2) + o(u,v)^2$. Composing the rotation

$$\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = T\begin{pmatrix}x\\ y\end{pmatrix},$$

we obtain the result. □

5.2. Monge cubic.

We assume that $k_1 \neq k_2$ and that the Monge cubic is expressed in a neighbourhood of $(0,0)$ as follows:

$$C(x,y) = A_{3,0}x^3 + 3A_{2,1}x^2y + 3A_{1,2}x\,y^2 + A_{0,3}y^3. \tag{5.7}$$

Remember that D denotes the discriminant set of C, that is, the zero set of

$$\Delta = \begin{vmatrix} A_{3,0} & 2A_{2,1} & A_{1,2} & 0\\ A_{2,1} & 2A_{1,2} & A_{0,3} & 0\\ 0 & A_{3,0} & 2A_{2,1} & A_{1,2}\\ 0 & A_{2,1} & 2A_{1,2} & A_{0,3}\end{vmatrix}. \tag{5.8}$$

Let $\Sigma(M)$ denote the singular set of M.

Lemma 5.5. *The following conditions are equivalent.*

- $(0,0)\times[p_0:q_0]\in\Sigma(M)$.
- $C_u = C_v = C_p = C_q = 0$ *at* $(0,0)\times[p_0:q_0]$, *that is,* $\alpha=\beta=0$,

$$A_{3,0}p_0^2 + 2A_{2,1}p_0q_0 + A_{1,2}q_0^2 = A_{2,1}p_0^2 + 2A_{1,2}p_0q_0 + A_{0,3}q_0^2 = 0 \tag{5.9}$$

at $(0,0)$.

Proof. A consequence of the implicit function theorem. □

The second condition in the lemma holds for some $[p_0:q_0]$ if and only if

$$\operatorname{rank}\begin{pmatrix} \partial_uA_{3,0} & 3\partial_uA_{2,1} & 3\partial_uA_{1,2} & \partial_uA_{0,3} & 0\\ \partial_vA_{3,0} & 3\partial_vA_{2,1} & 3\partial_vA_{1,2} & \partial_vA_{0,3} & 0\\ 0 & \partial_uA_{3,0} & 3\partial_uA_{2,1} & 3\partial_uA_{1,2} & \partial_uA_{0,3}\\ 0 & \partial_vA_{3,0} & 3\partial_vA_{2,1} & 3\partial_vA_{1,2} & \partial_vA_{0,3}\\ A_{3,0} & 2A_{2,1} & A_{1,2} & 0 & 0\\ A_{2,1} & 2A_{1,2} & A_{0,3} & 0 & 0\\ 0 & A_{3,0} & 2A_{2,1} & A_{1,2} & 0\\ 0 & A_{2,1} & 2A_{1,2} & A_{0,3} & 0\\ 0 & 0 & A_{3,0} & 2A_{2,1} & A_{1,2}\\ 0 & 0 & A_{2,1} & 2A_{1,2} & A_{0,3}\end{pmatrix} < 5 \qquad \text{at } (0,0), \tag{5.10}$$

because of the result of [**10**]. Let I denote the ideal generated by maximal minors of

$$\begin{pmatrix} y_0 & y_1 & y_2 & y_3 & 0 \\ z_0 & z_1 & z_2 & z_3 & 0 \\ 0 & y_0 & y_1 & y_2 & y_3 \\ 0 & z_0 & z_1 & z_2 & z_3 \\ x_0 & 2x_1 & x_2 & 0 & 0 \\ x_1 & 2x_2 & x_3 & 0 & 0 \\ 0 & x_0 & 2x_1 & x_2 & 0 \\ 0 & x_1 & 2x_2 & x_3 & 0 \\ 0 & 0 & x_0 & 2x_1 & x_2 \\ 0 & 0 & x_1 & 2x_2 & x_3 \end{pmatrix}$$

in the ring $R = \mathbb{R}[x_i, y_i, z_i; i = 0, 1, 2, 3]$. By computation using Singular, we see the height of I is 3. This means that, for a generic choice of the coefficients in the Monge normal form, we do not have (5.10).

We assume that the rank of the matrix in (5.10) is 5. To investigate the equation $C(du, dv) = 0$, we consider the foliation defined by the restriction of the 1-form $\omega = q\,du - p\,dv$ to M. The singularity of $\omega|_M$ is defined by $pC_u + qC_v = 0$, $C_p = C_q = 0$. Thus the induced flow on M is singular at $(0,0) \times [p_0 : q_0]$ if and only if (5.9) holds, and

$$\alpha p_0 + \beta q_0 = 0 \qquad \text{at } (0,0). \tag{5.11}$$

Remember that D is the discriminant set of the cubic C which is zero of Δ defined by (5.8). We assume $(0,0) \in D$. Then there is a non-zero (p_0, q_0) with (5.9), i.e.,

$$p_0^2(p_0 A_{3,0} + q_0 A_{2,1}) = -p_0 q_0 (p_0 A_{2,1} + q_0 A_{1,2}) = q_0^2 (p_0 A_{1,2} + q_0 A_{0,3}) \tag{5.12}$$

at $(0,0)$.

Let $\Sigma(D)$ denote the singular set of D and let Σ_1 denote the set defined by

$$\operatorname{rank} \begin{pmatrix} A_{3,0} & A_{2,1} & A_{1,2} \\ A_{2,1} & A_{1,2} & A_{0,3} \end{pmatrix} < 2. \tag{5.13}$$

Remark that (5.13) holds if and only if (5.12) is zero for some non-zero (p_0, q_0).

LEMMA 5.6.
- *If $(0,0) \in D \setminus \Sigma(D)$, we have $(\alpha, \beta) \neq (0,0)$, and $\beta\partial_u - \alpha\partial_v$ is tangent to D at $(0,0)$.*
- *$\Sigma(D) = \Sigma_1 \cup \pi(\Sigma(M))$.*
- *Suppose that $(0,0) \in \Sigma(D) \setminus \pi(\Sigma(M))$. Then, the singularity type of $\Sigma(D)$ at $(0,0)$ is cusp if and only if*

$$\begin{vmatrix} \partial_u A_{3,0} & \partial_u A_{2,1} & \partial_u A_{1,2} & \partial_u A_{0,3} \\ \partial_v A_{3,0} & \partial_v A_{2,1} & \partial_v A_{1,2} & \partial_v A_{0,3} \\ q_0^2 & -2p_0q_0 & p_0^2 & 0 \\ 0 & q_0^2 & -2p_0q_0 & p_0^2 \end{vmatrix} + 2p_0^2 q_0^2 A \neq 0 \qquad \text{at } (0,0) \tag{5.14}$$

where $A = -\partial_u A_{3,0}\partial_v A_{0,3} + \partial_u A_{2,1}\partial_v A_{1,2} - \partial_v A_{2,1}\partial_u A_{1,2} + \partial_u A_{0,3}\partial_v A_{3,0}$.

PROOF. Since the linear terms of (5.8) is

$$-\frac{4(p_0q_1-p_1q_0)^3}{27}(\alpha u+\beta v) \tag{5.15}$$

we have the first assertion. $(0,0)\in\Sigma(D)$ if and only if (5.13) or $(\alpha,\beta)=(0,0)$ hold. When $(\alpha,\beta)=(0,0)$ holds, then we have $(0,0)\times[p_0:q_0]\in\Sigma(M)$ by (5.9). This shows the second result. We assume that (5.13) holds. Then the constant term of C is a constant multiple of $(q_0du-p_0dv)^3$. We observe that the quadric part of (5.8) is

$$-(\alpha u+\beta v)^2$$

and the kernel direction for the Hesse matrix is generated by $\beta\partial_u-\alpha\partial_v$. If we evaluate the cubic part of (5.8) over this vector, we obtain a constant multiple of the left hand side of the cube of (5.14), which shows the last result. □

5.3. Conjugate Monge cubic.

Assume that $k_1\neq k_2$. Let $\bar{C}(p,q)$ denote the *conjugate Monge cubic* defined by $\bar{C}(p,q)=C(\kappa_2 q,-\kappa_1 p)$, i.e.,

$$\bar{C}(p,q)=A_{3,0}{\kappa_2}^3q^3-3A_{2,1}\kappa_1{\kappa_2}^2p\,q^2+3A_{1,2}{\kappa_1}^2\kappa_2p^2q-A_{0,2}{\kappa_1}^3p^3.$$

We call $\bar{C}(du,dv)=0$ the *cylindrical cubic differential equation*. Let $\bar{D}$ denote the discriminant of $\bar{C}$, that is, the zero of

$$\bar{\Delta}=\begin{vmatrix}-A_{0,3}{\kappa_1}^3 & 2A_{1,2}{\kappa_1}^2\kappa_2 & -A_{2,1}\kappa_1{\kappa_2}^2 & 0\\ A_{1,2}{\kappa_1}^2\kappa_2 & -2A_{2,1}\kappa_1{\kappa_2}^2 & A_{3,0}{\kappa_2}^3 & 0\\ 0 & -A_{0,3}{\kappa_1}^3 & 2A_{1,2}{\kappa_1}^2\kappa_2 & -A_{2,1}\kappa_1{\kappa_2}^2\\ 0 & A_{1,2}{\kappa_1}^2\kappa_2 & -2A_{2,1}\kappa_1{\kappa_2}^2 & A_{3,0}{\kappa_2}^3\end{vmatrix}=(\kappa_1\kappa_2)^6\Delta.$$

Then $\bar{D}=P_1\cup P_2\cup D$ where P_i denote parabolic lines defined by $\kappa_i=0$. We expect the behavior of the flow could be very degenerate along P_1 and P_2, since the discriminant $\bar{\Delta}$ is divisible by $(\kappa_1\kappa_2)^6$.

We assume that the rank of the matrix in (5.10) is 5. To investigate the equation $\bar{C}(du,dv)=0$, we consider the foliation defined by the restriction of the 1-form $\bar{\omega}=p\kappa_1du+q\kappa_2dv$ to M. The image of a flow of $\bar{\omega}|_M$ by the projection π is a flow of $\bar{C}(du,dv)=0$.

Proof of Theorem 5.1. We show the assertion at a point in $M\cap\{p\neq 0\}$. The case at a point in $M\cap\{q\neq 0\}$ is similar and we omit the details. The equation $C(1,\eta)=0$ is an affine equation for $M\cap\{p\neq 0\}$ where $\eta=q/p$, and this defines

- a function u of (v,η) by the implicit function theorem when $C_u(1,\eta)\neq 0$, and we have

$$u_v=-\frac{C_v(1,\eta)}{C_u(1,\eta)},\qquad u_\eta=-\frac{C_q(1,\eta)}{C_u(1,\eta)}.$$

- a function v of (u,η) by the implicit function theorem when $C_v(1,\eta)\neq 0$, and we have

$$v_u = -\frac{C_u(1,\eta)}{C_v(1,\eta)}, \qquad v_\eta = -\frac{C_q(1,\eta)}{C_v(1,\eta)}.$$

Since $C_u(\xi,1)du + C_v(\xi,1)dv + C_p(\xi,1)d\xi = 0$, we have

$$\begin{aligned}\frac{\bar\omega}{p} &= \kappa_1 du + \eta\kappa_2 dv \\ &= \begin{cases} \dfrac{1}{C_u(1,\eta)}[(C_u(1,\eta)\eta\kappa_2 - C_v(1,\eta)\kappa_1)dv - C_q(1,\eta)\kappa_1 d\eta] & C_u(1,\eta)\neq 0 \\ \dfrac{1}{C_v(1,\eta)}[(C_v(1,\eta)\kappa_1 - C_u(1,\eta)\eta\kappa_2)du - C_q(1,\eta)\eta\kappa_2 d\eta] & C_v(1,\eta)\neq 0. \end{cases}\end{aligned} \tag{5.16}$$

So if (p_0,q_0) is a multiple root of $C(p,q)=0$ at $(u,v)=(0,0)$ (i.e., $C_p(p_0,q_0) = C_q(p_0,q_0) = 0$ at $(u,v)=(0,0)$), $\bar\omega|_M$ is singular at $(0,0)\times(p_0,q_0)$ if and only if $\alpha q_0\kappa_2 - \beta p_0\kappa_1 = 0$. So we obtain the condition 2. If (p_0,q_0) is not a multiple root of $C(p,q)=0$, we have $C_q(p_0,q_0)\neq 0$. Now it is easy to conclude the condition 1. □

Let us consider the case that $(0,0)$ is parabolic (say $k_1=0$). Then the tangent of the flow tend to the direction generated by ∂_u. Remark that ∂_u is not tangent to the parabolic line P_1 whenever $a_{30}\neq 0$. If one writes the flow by $t\mapsto(t, ct^2+\cdots)$, we obtain that

$$a_{30}(k_2^3c^3 - 3k_2^2a_{21}c^2 + 3k_2a_{12}a_{30}c - a_{03}a_{30}^2) = 0.$$

Observe that this has a multiple root when $(0,0)\in D$.

Proposition 5.7. *Consider the case $k_1=0$ and $k_2\neq 0$. Assume that $(0,0)$ is ridge with respect to $\boldsymbol{v}_1$ (i.e., $a_{30}=0$). We assume that $(0,0)$ is not on the discriminant (i.e., $\Delta_0 = -{a_{21}}^2(3{a_{12}}^2 - 4a_{03}a_{21})\neq 0$). For a single root (p_0,q_0) of $c_3(p,q)=0$, we have the following.*

- *If $3{a_{21}}^2 < k_2a_{40}$, then $\bar\omega|_M$ has a saddle at $(0,0)\times[p_0:q_0]$.*
- *If $3{a_{21}}^2 > k_2a_{40}$, and $2(k_2a_{40}/2 - 3a_{21}^2)^2 < (k_2a_{40})^2$, then $\bar\omega|_M$ has a node at $(0,0)\times[p_0:q_0]$.*
- *If $2(k_2a_{40}/2 - 3a_{21}^2)^2 > (k_2a_{40})^2$, then $\bar\omega|_M$ has a focus at $(0,0)\times[p_0:q_0]$.*

Proof. Suppose that $p\neq 0$, $C_u(p_0,q_0)\neq 0$. Then $u_v(0,0) = -\beta/\alpha$, $u_\eta(0,0) = -C_q(1,\eta)/\alpha$. We then observe

$$\begin{aligned} C_u(1,\eta)\eta\kappa_2 - C_v(1,\eta)\kappa_1 &= \frac{a_{21}}{k_2}(3a_{21}a_{12} - k_2a_{31})v + (k_2a_{40} - 3a_{21}^2)\eta + o(1) \\ -C_q(1,\eta)\kappa_1 &= -3a_{21}^2 v + o(1). \end{aligned}$$

The assertions are now concluded using the lemma below, remarking the following:

$$\begin{aligned} \delta &= 3a_{21}^2(3a_{21}^2 - k_2a_{40}), \quad \tau = \frac{k_2a_{40}}{2}, \\ \delta - \tau^2 &= 3a_{21}^2(3a_{21}^2 - k_2a_{40}) - \frac{k_2^2a_{40}^2}{4} = \left(\frac{k_2a_{40}}{2} - 3{a_{21}}^2\right)^2 - \frac{(k_2a_{40})^2}{2}. \end{aligned}$$ □

Lemma 5.8. *Consider 1-form $\omega = a(x,y)dx + b(x,y)dy$ on the xy plane. Assume that $a(0,0) = b(0,0) = 0$. Set $\delta = (a_x b_y - b_x a_y)(0,0)$ and $\tau = ((a_y - b_x)/2)(0,0)$. Then the singularity type of ω at $(0,0)$ is saddle (resp. node, focus) if $\delta < 0$ (resp. $0 < \delta < \tau^2$, $\tau^2 < \delta$).*

Proof. It is enough to consider the singularity type of $-b(x,y)\partial_x + a(x,y)\partial_y$. We just compute the eigenvalue of the matrix

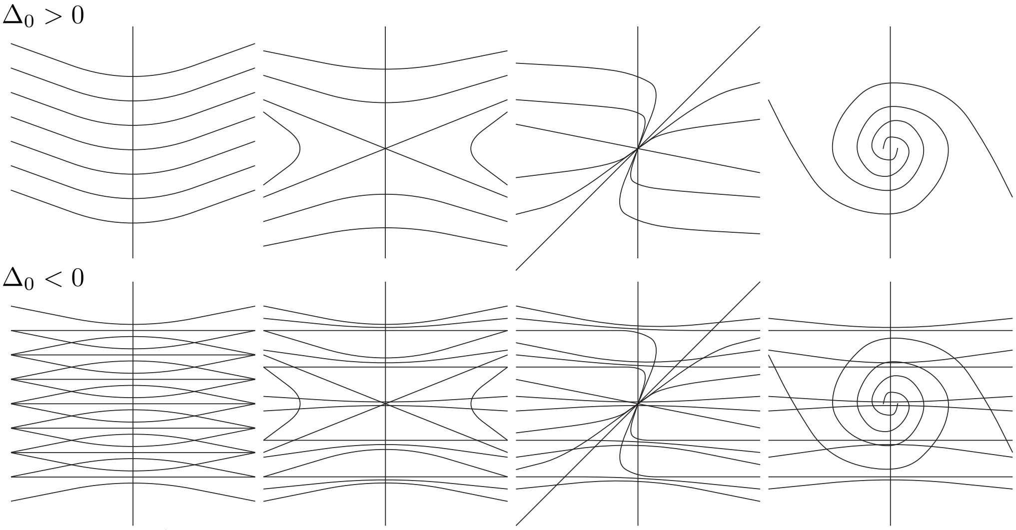
$$\begin{pmatrix} -b_x & -b_y \\ a_x & a_y \end{pmatrix}(0,0)$$

and we have the result. □

We show below phase portraits of flows in the uv plane.

$\Delta_0 > 0$

$\Delta_0 < 0$



$k_1 = 0$, $a_{30} \neq 0$

$k_1 = 0$, $a_{30} = 0$
$0 < 3{a_{21}}^2 < k_2 a_{40}$

$k_1 = 0$, $a_{30} = 0$
$0 < 3{a_{21}}^2 > k_2 a_{40}$
$(k_2 a_{40}/2 - 3a_{21}^2)^2 < (k_2 a_{40})^2/2$

$k_1 = 0$, $a_{30} = 0$
$(k_2 a_{40}/2 - 3a_{21}^2)^2 > (k_2 a_{40})^2/2$

Remark 5.9. We need to look the intersection of the parabolic line with the discriminant, since this can be considered as a degenerate point of cylindrical directions on the surface. We assume that $(0,0) \in P_1 \cap D$, that is, $k_1 = 0$ and

$$a_{30} = q_0^2 q_1, \quad a_{21} = -\frac{1}{3}q_0(p_1 q_0 + 2p_0 q_1), \quad a_{12} = \frac{1}{3}p_0(p_0 q_1 + 2p_1 q_0), \quad a_{03} = -p_0^2 p_1.$$

Whenever $C_u(1,\eta_0) \neq 0$, u is a function of (v,η) by $C(1,\eta) = A_{30} + 3A_{21}\eta + 3A_{12}\eta^2 + A_{03}\eta^3 = 0$. By (5.16), we have

$$\begin{aligned} C_u(1,\eta)\frac{\bar{\omega}}{p} &= [C_u(1,\eta)\eta\kappa_2 - C_v(1,\eta)\kappa_1]dv - C_q(1,\eta)\kappa_1 d\eta \\ &= C_u(1,\eta)\eta\kappa_2 dv + o(1). \end{aligned}$$

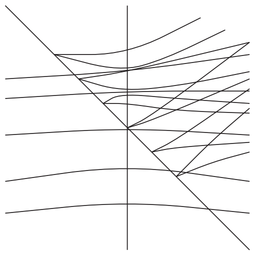
A solution curve tending $(0,0)$ is expressed as

$$t \mapsto \left(t,\ -\frac{p_1 {q_0}^2}{2k_2} t^2 + \cdots\right),$$

or, setting $\epsilon = \text{sign}\ c_{4,u}(p_0, q_0)/(p_0 q_1 - p_1 q_0)$,

$$t \mapsto \left(-\epsilon t^2,\ -\frac{p_0 q_0 q_1}{2k_2} t^4 + \frac{2q_1}{5k_2} \sqrt{\left|\frac{c_{4,u}(p_0, q_0)}{p_0 q_1 - p_1 q_0}\right|}\, t^5 + \cdots\right).$$

A phase portrait is given as follows.



Remark 5.10. Using Lemma 5.8, it should be possible to determine singularity types of $\bar{\omega}|_M$ at points $(0,0) \times [p_0 : q_0]$ where $[p_0 : q_0]$ are multiple roots of $C(p,q) = 0$.

Consider the equation $C_q(1, \eta_0) = 0$, $\eta_0 = q_0/p_0$. We have

$$\begin{aligned}
&\partial_v(C_u(1,\eta)\eta\kappa_2 - C_v(1,\eta)\kappa_1) \\
&= \left(-\frac{C_v}{C_u} C_{uu}(1,\eta) + C_{uv}(1,\eta)\right)\eta\kappa_2 + C_u(1,\eta)\eta\left(-\frac{C_v}{C_u}\kappa_{2,u} + \kappa_{2,v}\right) \\
&\quad - \left(-\frac{C_v}{C_u} C_{uv}(1,\eta) + C_{vv}(1,\eta)\right)\kappa_1 - C_v(1,\eta)\left(-\frac{C_v}{C_u}\kappa_{1,u} + \kappa_{1,v}\right) \\
&= \left(-\frac{\beta}{\alpha} C_{uu} + C_{uv}\right)(1,\eta_0)\eta_0 k_2 - C_u(1,\eta_0)\eta_0\left(-\frac{a_{12}\beta}{\alpha} + a_{03}\right) \\
&\quad - \left(-\frac{C_{uv}\beta}{\alpha} + C_{vv}\right)(1,\eta_0)k_1 - \beta\left(-\frac{\beta}{\alpha}a_{30} + a_{21}\right) + o(0), \\
&\partial_\eta(C_u(1,\eta)\eta\kappa_2 - C_v(1,\eta)\kappa_1) \\
&= C_u(1,\eta)\kappa_2 + \left(-\frac{C_q}{C_u} C_{u,u}(1,\eta) + C_{uq}(1,\eta)\right)\eta\kappa_2 - C_u(1,\eta)\eta\frac{C_q}{C_u}\kappa_{2,u} \\
&\quad - \left(-\frac{C_q}{C_u} C_{uv}(1,\eta) + C_{vq}(1,\eta)\right)\kappa_1 - C_v(1,\eta)\frac{C_q}{C_u}\kappa_{1,u} \\
&= \alpha k_2 + C_{uq}(1,\eta_0)\eta_0 k_2 + \alpha\eta_0 a_{03} - C_{vq}(1,\eta_0)k_1 + o(0),
\end{aligned}$$

$$\begin{aligned}
\partial_v(-C_q(1,\eta)\kappa_1) &= -\left[-\frac{C_v}{C_u} C_{uq}(1,\eta_0) + C_{vq}(1,\eta_0)\right]\kappa_1 + C_q(1,\eta_0)\partial_v\kappa_1 \\
&= -\left(-\frac{\beta}{\alpha} C_{uq} + C_{vq}\right)(1,\eta_0)k_1 + o(0),
\end{aligned}$$

$$\begin{aligned}\partial_\eta(-C_q(1,\eta)\kappa_1) &= -\left(-\frac{C_q}{C_u}C_{uq}+C_{qq}\right)(1,\eta_0)\kappa_1 + C_q(1,\eta_0)\partial_\eta\kappa_1 \\ &= -C_{qq}(1,\eta_0)k_1 + o(0).\end{aligned}$$

So we obtain that

$$\begin{aligned}\delta = &-\left[\left(-\frac{\beta}{\alpha}C_{uu}+C_{uv}\right)(1,\eta_0)\eta_0 k_2 - C_u(1,\eta_0)\eta_0\left(\frac{-a_{12}\beta}{\alpha}+a_{03}\right)\right.\\ &\left.-\left(-\frac{C_{uv}\beta}{\alpha}+C_{vv}\right)(1,\eta_0)k_1 - \beta\left(-\frac{\beta}{\alpha}a_{30}+a_{21}\right)\right] C_{qq}(1,\eta_0)k_1 + C_{vq}(1,\eta_0)k_1 \\ &+\left[\left(-\frac{\beta}{\alpha}C_{uq}+C_{vq}\right)(1,\eta_0)k_1\alpha k_2 + C_{uq}(1,\eta_0)\eta_0 k_2 + \alpha\eta_0 a_{03}\right]\left(-\frac{\beta}{\alpha}\right)C_{uq},\\ \tau = &\ (\alpha + C_{uq}(1,\eta_0))k_2 + \alpha\eta_0 a_{03} - \frac{\beta}{\alpha}C_{uq}(1,\eta_0)k_1.\end{aligned}$$

Now we can apply Lemma 5.8 to have a criterion for singularity types of $\bar{\omega}|_M$. But writing down the explicit conditions would be very complicated unfortunately. It could be better to discuss the normal form of jet and we discuss this method in the next subsection, which works in the generic context of cubic binary differential equations.

5.4. Reduction of jets of cubic differential equations.

Now we discuss the reduction of jets of the cubic binary differential equation $\widehat{C}(du,dv)=0$ where

$$\widehat{C}(p,q) = P(u,v)p^3 + 3Q(u,v)p^2q + 3R(u,v)pq^2 + S(u,v)q^3$$

where $P=\sum_{i,j}(p_{ij}/i!j!)u^iv^j$, $Q=\sum_{i,j}(q_{ij}/i!j!)u^iv^j$, $R=\sum_{i,j}(r_{ij}/i!j!)u^iv^j$, $S=\sum_{i,j}(s_{ij}/i!j!)u^iv^j$. Set

$$\Delta_0 = \Delta(0,0), \quad \Delta = \begin{vmatrix} P & 2Q & R & 0 \\ Q & 2R & S & 0 \\ 0 & P & 2Q & R \\ 0 & Q & 2R & S \end{vmatrix}.$$

Proposition 5.11.

1. *If $\Delta_0 > 0$ (resp. $\Delta_0 < 0$), then the equation $\widehat{C}(du,dv)=0$ reduces to $dx(dx^2+3dy^2)+o(0)=0$ (resp. $dx(dx^2-3dy^2)+o(0)=0$).*
2. *When $\Delta_0 = 0$, then the constant term of $\widehat{C}(du,dv)$ is $(q_0du-p_0dv)^2(q_1du-p_1dv)$. Now we set*

$$\sigma_0 = \alpha p_0 + \beta q_0, \qquad \sigma_1 = \alpha p_1 + \beta q_1, \qquad \sigma_2 = p_0q_1 - p_1q_0,$$

where $\alpha = \widehat{C}_u(p_0,q_0)|_{(u,v)=(0,0)}$ and $\beta = \widehat{C}_v(p_0,q_0)|_{(u,v)=(0,0)}$.

(a) *When $\sigma_0 \neq 0$, $\sigma_1 \neq 0$, and $\sigma_2 \neq 0$, the equation reduces to*

$$[dx^2 + (x+y)dy^2]dy + o(1) = 0. \tag{5.17}$$

(b) *When $\sigma_0 \neq 0$, $\sigma_1 = 0$, and $\sigma_2 \neq 0$, the equation reduces to*

$$[dx^2 + (ax^2 + y)dy^2]dy + o(2) = 0. \tag{5.18}$$

When $a \neq 0$, the discriminant has the first order tangent to the foliation corresponding to the single root of the cylindrical equation. The sign of a coincides with the sign of (5.22) in the proof.

(c) *When $\sigma_0 = 0$, $\sigma_1 \neq 0$, and $\sigma_2 \neq 0$, the equation reduces to*

$$[dx^2 + (x + by^2)dy^2]dy + o(2) = 0. \tag{5.19}$$

The equation (5.19) defines folded saddle (resp. folded node, folded focus) if $b < 0$ (resp. $0 < b < 1/16$, $b > 1/16$). Remark that b is given by (5.23).

3. *If $\Delta_0 = \sigma_2 = 0$, then the equation reduces to*

$$dx^3 + 3(kx + ly)dx\,dy^2 + (ax + by)dy^3 + o(1) = 0 \tag{5.20}$$

and the singularity type of the discriminant is cusp if $al - bk \neq 0$, or equivalently

$$\begin{vmatrix} \widehat{C}_{up}(p_0, q_0) & \widehat{C}_{vp}(p_0, q_0) \\ \widehat{C}_{uq}(p_0, q_0) & \widehat{C}_{vq}(p_0, q_0) \end{vmatrix} \neq 0. \tag{5.21}$$

As we will see in the proof, explicit formulas for a in (5.18) and for b in (5.19) are very complicate when we express them in terms of the coefficients of the Monge normal form (2.1).

Proof. By a suitable linear change of coordinates, the 0-jet of $\widehat{C}$ reduces to $dx(dx^2 + 3dy^2)$ (resp. $dx(dx^2 - 3dy^2)$) when $\Delta_0 > 0$ (resp. $\Delta_0 < 0$). If this determinant is 0, we can assume that the 0-jet of the equation is $(q_0du - p_0dv)^2(q_1du - p_1dv)$ and we reduce the 0-jet to $3{\sigma_2}^3{t_0}^2t_1dx^2dy$ by a coordinates change given by

$$u = t_1p_1x + t_0p_0y + o(1), \; v = t_1q_1x + t_0q_0y + o(1),$$

whenever $\sigma_2 \neq 0$. Here t_0, t_1 are non-zero constants. Multiplying the equation by $1/(3{\sigma_2}^3{t_0}^2t_1)$, we reduce the constant term of the coefficient of dx^2dy is 1. In this case, by a suitable choice of quadratic parts of the coordinate change, we are able to reduce the equation to

$$[dx^2 + (a_1x + a_2y)dy^2]dy + o(1) = 0$$

where $a_1 = {t_0}^2\sigma_1/(3t_1\sigma_2^3)$, $a_2 = {t_0}^3\sigma_0/(3{t_1}^2\sigma_2^3)$. When $\sigma_0 \neq 0$, $\sigma_1 \neq 0$, we reduce this equation to (5.17) setting $(t_0, t_1) = (3\sigma_0\sigma_2^5/\sigma_1^3)(1, \sigma_0\sigma_2^2)$. Remark also that, if $a_1 \neq 0$, then we reduce the coefficient of $x\,dy^3$ to 1 setting $t_1 = {t_0}^2\sigma_1/(3\sigma_2^3)$.

When $\sigma_0 \neq 0$, $\sigma_1 = 0$ (i.e., $a_1 = 0$, $a_2 \neq 0$), we need to look the coefficients of x^2dy^3 which is expressed as follows:

$$
{t_0}^2\left[\frac{3(\widehat{C}_{uu}(p_0,q_0)p_1^2+2\widehat{C}_{uv}(p_0,q_0)p_1q_1+\widehat{C}_{vv}(p_0,q_0)q_1^2)}{4\sigma_2^3}-\frac{\widehat{Q}}{12\sigma_2^6}\right] \tag{5.22}
$$

where $\widehat{Q}=\sum_{i,j}Q_{ij}\,{p_0}^{4-j}{q_0}^j\,{p_1}^{4-i}{q_1}^i$, $Q_{00}=25p_{10}^2$,

$$
\begin{aligned}
&Q_{01}=37p_{01}p_{10}+63p_{10}q_{10},\ Q_{02}=12p_{01}^2+30p_{10}q_{01}+57p_{01}q_{10}+9q_{10}^2+42p_{10}r_{10},\\
&Q_{03}=24p_{01}q_{01}+18q_{01}q_{10}+9p_{10}r_{01}+36p_{01}r_{10}+13p_{10}s_{10},\\
&Q_{04}=9q_{01}^2+3p_{01}r_{01}+2p_{10}s_{01}+11p_{01}s_{10},\ Q_{10}=13p_{01}p_{10}+87p_{10}q_{10},\\
&Q_{11}=13p_{01}^2+114p_{10}q_{01}+60p_{01}q_{10}+171q_{10}^2+42p_{10}r_{10},\\
&Q_{12}=87p_{01}q_{01}+225q_{01}q_{10}+75p_{10}r_{01}+48p_{01}r_{10}+162q_{10}r_{10}+3p_{10}s_{10},\\
&Q_{13}=54q_{01}^2+81p_{01}r_{01}+63q_{10}r_{01}+144q_{01}r_{10}+14p_{10}s_{01}+5p_{01}s_{10}+39q_{10}s_{10},\\
&Q_{14}=45q_{01}r_{01}+16p_{01}s_{01}+6q_{10}s_{01}+33q_{01}s_{10},\\
&Q_{20}=6p_{10}q_{01}+33p_{01}q_{10}+45q_{10}^2+66p_{10}r_{10},\\
&Q_{21}=39p_{01}q_{01}+207q_{01}q_{10}+60p_{10}r_{01}+33p_{01}r_{10}+243q_{10}r_{10}+18p_{10}s_{10},\\
&Q_{22}=162q_{01}^2+27p_{01}r_{01}+324q_{10}r_{01}+162q_{01}r_{10}+162r_{10}^2+18p_{10}s_{01}+18p_{01}s_{10}+27q_{10}s_{10},\\
&Q_{23}=243q_{01}r_{01}+207r_{01}r_{10}+18p_{01}s_{01}+60q_{10}s_{01}+33q_{01}s_{10}+39r_{10}s_{10},\\
&Q_{24}=45r_{01}^2+66q_{01}s_{01}+6r_{10}s_{01}+33r_{01}s_{10},\\
&Q_{30}=6p_{10}r_{01}+33p_{01}r_{10}+45q_{10}r_{10}+16p_{10}s_{10},\\
&Q_{31}=39p_{01}r_{01}+63q_{10}r_{01}+144q_{01}r_{10}+54r_{10}^2+14p_{10}s_{01}+5p_{01}s_{10}+81q_{10}s_{10},\\
&Q_{32}=162q_{01}r_{01}+225r_{01}r_{10}+3p_{01}s_{01}+75q_{10}s_{01}+48q_{01}s_{10}+87r_{10}s_{10},\\
&Q_{33}=171r_{01}^2+42q_{01}s_{01}+114r_{10}s_{01}+60r_{01}s_{10}+13s_{10}^2,\quad Q_{34}=87r_{01}s_{01}+13s_{01}s_{10},\\
&Q_{40}=9r_{10}^2+2p_{10}s_{01}+11p_{01}s_{10}+3q_{10}s_{10},\\
&Q_{41}=18r_{01}r_{10}+13p_{01}s_{01}+9q_{10}s_{01}+36q_{01}s_{10}+24r_{10}s_{10},\ Q_{44}=25s_{01}^2,\\
&Q_{42}=9r_{01}^2+42q_{01}s_{01}+30r_{10}s_{01}+57r_{01}s_{10}+12s_{10}^2,\ Q_{43}=63r_{01}s_{01}+37s_{01}s_{10}.
\end{aligned}
$$

By a suitable choice of cubic parts of the coordinate change we are able to reduce the equation to (5.18). Remark that this coordinate change does not change the coefficient of x^2dy^3. We also remark that the possible singularity of $\omega|_M$ over $(0,0)$ is $[p_0,q_0]=[0:1]$ only if $a_2=0$.

We consider the case $\sigma_0=0$ and $\sigma_1\neq0$ (i.e., $a_1\neq0$, $a_2=0$). We need to look the coefficients of y^2dy^3 which is expressed as follows:

$$
\left(\frac{{t_0}^2}{t_1}\right)^2\left[\frac{\widehat{C}_{uu}(p_0,q_0)p_0^2+2\widehat{C}_{uv}(p_0,q_0)p_0q_0+\widehat{C}_{vv}(p_0,q_0)q_0^2}{6\sigma_2^3}-\frac{\widehat{R}}{4\sigma_2^6}\right]
$$

where $\widehat{R}=\sum_{i,j}R_{ij}\,{p_0}^{6-j}{q_0}^j\,{p_1}^{2-i}{q_1}^i$,

$$
\begin{aligned}
&R_{00}=4p_{10}^2,\ R_{01}=3p_{10}(7p_{01}+q_{10}),\ R_{02}=17p_{01}^2+28p_{10}q_{01}+43p_{01}q_{10}-24q_{10}^2-4p_{10}r_{10},\\
&R_{03}=68p_{01}q_{01}+30q_{01}q_{10}+7p_{10}r_{01}+37p_{01}r_{10}-63q_{10}r_{10}+p_{10}s_{10},\\
&R_{04}=3(18q_{01}^2+16p_{01}r_{01}-9q_{10}r_{01}+6q_{01}r_{10}-12r_{10}^2+5p_{01}s_{10}-4q_{10}s_{10}),\\
&R_{05}=54q_{01}r_{01}-33r_{01}r_{10}+14p_{01}s_{01}-14q_{10}s_{01}+16q_{01}s_{10}-13r_{10}s_{10},\\
&R_{06}=3r_{01}^2+14q_{01}s_{01}-14r_{10}s_{01}+r_{01}s_{10},
\end{aligned}
$$

$R_{10} = -p_{10}(13p_{01} - 21q_{10}), \quad R_{11} = -13p_{01}^2 + 10p_{10}q_{01} -_{20} p_{01}q_{10} + 57q_{10}^2 + 14p_{10}r_{10},$
$R_{12} = -31p_{01}q_{01} + 75q_{01}q_{10} + 31p_{10}r_{01} - 29p_{01}r_{10} + 81q_{10}r_{10} - 7p_{10}s_{10},$
$R_{13} = 2(9q_{01}^2 - 6p_{01}r_{01} + 63q_{10}r_{01} + 18q_{01}r_{10} + 9r_{10}^2 + 4p_{10}s_{01} - 11p_{01}s_{10} - 6q_{10}s_{10}),$
$R_{14} = 81q_{01}r_{01} + 75r_{01}r_{10} - 7p_{01}s_{01} + 31q_{10}s_{01} - 29q_{01}s_{10} - 31r_{10}s_{10},$
$R_{15} = 57r_{01}^2 + 14q_{01}s_{01} + 10r_{10}s_{01} - 20r_{01}s_{10} - 13s_{10}^2, \quad R_{16} = s_{01}(21r_{01} - 13s_{10}),$
$R_{20} = -14p_{10}q_{01} + p_{01}q_{10} + 3q_{10}^2 + 14p_{10}r_{10},$
$R_{21} = -13p_{01}q_{01} - 33q_{01}q_{10} - 14p_{10}r_{01} + 16p_{01}r_{10} + 54q_{10}r_{10} + 14p_{10}s_{10},$
$R_{22} = -3(12q_{01}^2 + 4p_{01}r_{01} + 9q_{10}r_{01} - 6q_{01}r_{10} - 18r_{10}^2 - 5p_{01}s_{10} - 16q_{10}s_{10}),$
$R_{23} = -63q_{01}r_{01} + 30r_{01}r_{10} + p_{01}s_{01} + 7q_{10}s_{01} + 37q_{01}s_{10} + 68r_{10}s_{10},$
$R_{24} = 28r_{10}s_{01} + 43r_{01}s_{10} + 17s_{10}^2 - 24r_{01}^2 - 4q_{01}s_{01}, \ R_{25} = 3s_{01}(r_{01} + 7s_{10}), \ R_{26} = 4s_{01}^2.$

We assume that $a_1 = 1$, since we can choose (t_0, t_1) with ${t_0}^2/t_1 = 3\sigma_2^3/\sigma_1$. Then the coefficient of y^2dy^3 becomes

$$\frac{9}{{\sigma_1}^2}\left[\frac{{\sigma_2}^3(\widehat{C}_{uu}(p_0,q_0)p_0^2 + 2\widehat{C}_{uv}(p_0,q_0)p_0q_0 + \widehat{C}_{vv}(p_0,q_0)q_0^2)}{6} - \frac{\widehat{R}}{4}\right]. \tag{5.23}$$

By a suitable choice of cubic parts of the coordinate change, we are able to reduce the equation to (5.19). Remark that this coordinate change does not change the coefficient of y^2dy^3. Now we can apply Lemma 5.8 to have a criterion for singularity types of the restriction of 1-form $pdu - qdv$ to the subset $\widehat{C}(p,q) = 0$ in $(\mathbb{R}^2, 0) \times P^1$.

Remark that the linear term of Δ is $(4/27){\sigma_2}^3(a_1x + a_2y)$ and $(0,0) \in \Sigma D$ if and only if $\sigma_2 = 0$ or $a_1 = a_2 = 0$.

In the case $\sigma_2 = 0$, the constant part of $\widehat{C}(du, dv)$ is a constant multiple of $(v_0du - u_0dv)^3$ and we are able to reduce the 0-jet of the equation to dx^3. By a suitable choice of quadratic parts of the coordinate change, we are able to reduce the equation to (5.19). We remark that

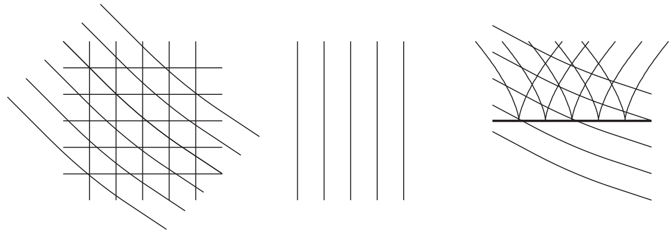
$$\begin{aligned} k &= -\,{p_0}^{-3}t^{-1}\widehat{C}_{vq}(p_0,q_0), & l &= -\,{p_0}^{-3}t^{-2}[\widehat{C}_{uq}(p_0,q_0)p_0 + \widehat{C}_{vq}(p_0,q_0)q_0],\\ a &= -\,{p_0}^{-3}t^{-2}\widehat{C}_{v}(p_0,q_0), & b &= -\,{p_0}^{-3}t^{-3}[\widehat{C}_{u}(p_0,q_0)p_0 + \widehat{C}_{v}(p_0,q_0)q_0]\\ (\text{or } k &= {q_0}^{-3}t^{-1}\widehat{C}_{uq}(p_0,q_0), & l &= {q_0}^{-3}t^{-2}[\widehat{C}_{uq}(p_0,q_0)p_0 + \widehat{C}_{vq}(p_0,q_0)q_0],\\ a &= {q_0}^{-3}t^{-2}\widehat{C}_{u}(p_0,q_0), & b &= {q_0}^{-3}t^{-3}[\widehat{C}_{u}(p_0,q_0)p_0 + \widehat{C}_{v}(p_0,q_0)q_0]), \end{aligned}$$

by a suitable change of coordinate

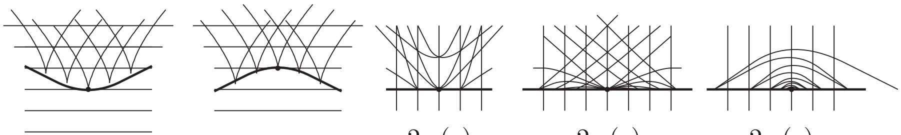
$$u = p_0y + o(1),\ v = tx + q_0y + o(1)\ (\text{or } u = tx + p_0y + o(1),\ v = q_0y + o(1))$$

and the 3-jet of the discriminant is $(ax + by)^2 + (kx + ly)^3$. So the singularity type of the discriminant is cusp if $al - bk \neq 0$, which is equivalent to (5.21). □

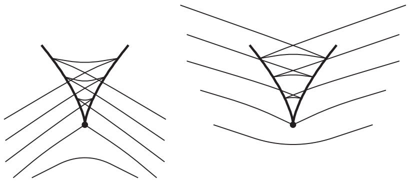
The phase portraits of singularities of the flows are shown by the pictures below.



1. $\Delta_0 < 0$ 1. $\Delta_0 > 0$ 2. (a)



2. (b) $a < 0$ 2. (b) $a > 0$ 2. (c) $b < 0$ 2. (c) $0 < b < 1/16$ 2. (c) $1/16 < b$



3. 3.

We can apply Proposition 5.11 to determine the singularity type of the equation

$$\bar{C}(du, dv) = 0,$$

whenever $k_1 k_2 \neq 0$.

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Toshizumi Fukui
Department of Mathematics
Faculty of Science
Saitama University
Saitama 338-8570, Japan
E-mail: tfukui@rimath.saitama-u.ac.jp

Masaru Hasegawa
Department of Information Science
Center for Liberal Arts and Sciences
Iwate Medical University
2-1-1 Nishitokuda, Yahaba-cho, Shiwa-gun
Iwate 028-3694, Japan
E-mail: mhase@iwate-med.ac.jp

Kouichi Nakagawa
Graduate School of Science and Engineering
Saitama University
Saitama 338-8570, Japan
E-mail: k-nakagawa@h6.dion.ne.jp