J. Math. Soc. Japan
Vol. 69, No. 1 (2017) pp. 241–262
doi: 10.2969/jmsj/06910241

On the fundamental groups of non-generic $\mathbb{R}$-join-type curves, II

By Christophe Eyral and Mutsuo Oka

(Received Apr. 20, 2015)

Abstract. We study the fundamental groups of (the complements of) plane complex curves defined by equations of the form $f(y) = g(x)$, where f and g are polynomials with real coefficients and real roots (so-called $\mathbb{R}$-join-type curves). For generic (respectively, semi-generic) such polynomials, the groups in question are already considered in [**6**] (respectively, in [**3**]). In the present paper, we compute the fundamental groups of $\mathbb{R}$-join-type curves under a simple arithmetic condition on the multiplicities of the roots of f and g without assuming any (semi-)genericity condition.

1. Introduction.

Let $\nu_1, \ldots, \nu_\ell, \lambda_1, \ldots, \lambda_m$ be positive integers. Denote by ν_0 (respectively, λ_0) the greatest common divisor of $\nu_1, \ldots, \nu_\ell$ (respectively, of $\lambda_1, \ldots, \lambda_m$), and set

$$d := \sum_{j=1}^{\ell} \nu_j \quad \text{and} \quad d' := \sum_{i=1}^{m} \lambda_i.$$

A curve C in $\mathbb{C}^2$ is called a join-type curve with exponents $(\nu_1, \ldots, \nu_\ell; \lambda_1, \ldots, \lambda_m)$ if it is defined by an equation of the form $f(y) = g(x)$, where

$$f(y) := a \cdot \prod_{j=1}^{\ell} (y - \beta_j)^{\nu_j} \quad \text{and} \quad g(x) := b \cdot \prod_{i=1}^{m} (x - \alpha_i)^{\lambda_i}. \tag{1.1}$$

Here, a and b are non-zero complex numbers, and $\beta_1, \ldots, \beta_\ell$ (respectively, $\alpha_1, \ldots, \alpha_m$) are mutually distinct complex numbers. We say that C is an $\mathbb{R}$-join-type curve if the coefficients a, b, α_i and β_j ($1 \leq i \leq m$, $1 \leq j \leq \ell$) are *real* numbers. Hereafter, we shall always assume that C is an $\mathbb{R}$-join-type curve. The singular points of such a curve are the points (x, y) satisfying the equations

$$f(y) = g(x) \quad \text{and} \quad f'(y) = g'(x) = 0.$$

Among these points, those which also satisfy the equations $f(y) = g(x) = 0$—which are nothing but the points (α_i, β_j) with $\lambda_i, \nu_j \geq 2$—will be called *inner* singularities. On the other hand, the singular points for which $f(y) \neq 0$ and $g(x) \neq 0$ will be called *outer* or *exceptional* singularities. Both inner and outer singularities are Brieskorn–Pham singularities $\mathbf{B}_{\lambda,\nu}$ (normal form $y^\nu - x^\lambda$). For instance, the inner singularity at (α_i, β_j)

2010 *Mathematics Subject Classification.* Primary 14H30; Secondary 14H20, 14H45, 14H50.
Key Words and Phrases. plane curves, fundamental group, bifurcation graph, monodromy, Zariski–van Kampen's pencil method.

is of type $\mathbf{B}_{\lambda_i,\nu_j}$. Note that, for an $\mathbb{R}$-join-type curve, outer singularities can be only node singularities (i.e., Brieskorn–Pham singularities of type $\mathbf{B}_{2,2}$). This follows from the discussion in the next paragraph.

Without loss of generality, we can assume that the real numbers α_i $(1 \leq i \leq m)$ and β_j $(1 \leq j \leq \ell)$ are indexed so that $\alpha_1 < \cdots < \alpha_m$ and $\beta_1 < \cdots < \beta_\ell$. Then, by considering the restriction of the function $g(x)$ to the real numbers, we easily see that the equation $g'(x) = 0$ has at least one real root γ_i in the open interval (α_i, α_{i+1}) for each $1 \leq i \leq m-1$. Thus, since the degree of

$$g'(x) \Big/ \prod_{i=1}^{m} (x-\alpha_i)^{\lambda_i - 1}$$

is $m-1$, the roots of $g'(x) = 0$ are exactly $\gamma_1, \ldots, \gamma_{m-1}$ and the roots α_i of $g(x) = 0$ with $\lambda_i \geq 2$. In particular, this implies that $\gamma_1, \ldots, \gamma_{m-1}$ are *simple* roots. Similarly, the equation $f'(y) = 0$ has $\ell - 1$ simple roots $\delta_1, \ldots, \delta_{\ell-1}$ such that $\beta_j < \delta_j < \beta_{j+1}$ for each $1 \leq j \leq \ell - 1$. The other roots of $f'(y) = 0$ are the roots β_j of $f(y) = 0$ with $\nu_j \geq 2$; they are simple for $\nu_j = 2$.

We say that C is *generic* if it has only inner singularities. Thus, C is generic if and only if, for any $1 \leq i \leq m-1$, $g(\gamma_i)$ is a regular value for f, that is $g(\gamma_i) \neq f(\delta_j)$ for any $1 \leq j \leq \ell - 1$. (Of course, this is also equivalent to the condition that, for any $1 \leq j \leq \ell - 1$, $f(\delta_j)$ is a regular value for g.) We say that C is *semi-generic with respect to* g if there exists i_0, $1 \leq i_0 \leq m$, such that $g(\gamma_{i_0-1})$ and $g(\gamma_{i_0})$ are regular values for f. (For $i_0 = 1$ and m, we mean $g(\gamma_1) \notin \mathscr{V}_{\text{crit}}(f)$ and $g(\gamma_{m-1}) \notin \mathscr{V}_{\text{crit}}(f)$ respectively, where $\mathscr{V}_{\text{crit}}(f)$ is the set of critical values of f.) The semi-genericity *with respect to* f is defined similarly by exchanging the roles of f and g. Clearly, any generic curve is semi-generic with respect to both g and f. Of course, the converse is not true. Note that C can be semi-generic with respect to g without being semi-generic with respect to f (for details, see [**3**]).

In [**3**], we proved that if C is semi-generic—with respect to g or with respect to f—then

$$\pi_1(\mathbb{C}^2 \setminus C) \simeq G(\nu_0; \lambda_0),$$

where $G(\nu_0; \lambda_0)$ is the group obtained by taking $p = \nu_0$ and $q = \lambda_0$ in the presentation (2.1) described in Section 2 below. We also showed that, if $\widetilde{C}$ is the projective closure of C, then

$$\pi_1(\mathbb{P}^2 \setminus \widetilde{C}) \simeq \begin{cases} G(\nu_0; \lambda_0; d/\nu_0) & \text{if } d \geq d', \\ G(\lambda_0; \nu_0; d'/\lambda_0) & \text{if } d' \geq d. \end{cases}$$

The groups $G(\nu_0; \lambda_0; d/\nu_0)$ and $G(\lambda_0; \nu_0; d'/\lambda_0)$ are defined in (2.4). In the special case where the curve is generic, the result was first proved in [**6**]. For a survey on this subject, we refer to [**2**].

In the present paper, we compute the fundamental groups of $\mathbb{R}$-join-type curves under a simple arithmetic condition on the exponents ν_j and λ_i of the polynomials f

and g, without assuming any semi-genericity condition (cf. Theorem 1.4). Beforehand, we give an intermediate result (Theorem 1.1 below), half-way between the semi-generic case of [**3**] and Theorem 1.4.

To make our statements simpler, throughout we fix real numbers $\gamma_0 < \alpha_1$ and $\gamma_m > \alpha_m$ (respectively, $\delta_0 < \beta_1$ and $\delta_\ell > \beta_\ell$) so that $g(\gamma_0)$ and $g(\gamma_m)$ are regular values of f (respectively, $f(\delta_0)$ and $f(\delta_\ell)$ are regular values of g).

THEOREM 1.1. *Let C be an $\mathbb{R}$-join-type curve in $\mathbb{C}^2$ defined by the equation $f(y) = g(x)$, where f and g are as in* (1.1), *and let $\hat{\nu}_{j,j+1}$ be the least common multiple of the consecutive exponents ν_j and ν_{j+1}. Suppose that there exists an integer i_0, $1 \leq i_0 \leq m$, such that the following two conditions are satisfied*:

(1) $\lambda_{i_0} > \hat{\nu}_{j,j+1}$ *for any* $1 \leq j \leq \ell - 1$;

(2) *either $g(\gamma_{i_0-1})$ or $g(\gamma_{i_0})$ is a regular value of f.*

Then,

$$\pi_1(\mathbb{C}^2 \setminus C) \simeq G(\nu_0; \lambda_0).$$

Furthermore, if $\widetilde{C}$ is the projective closure of C, then

$$\pi_1(\mathbb{P}^2 \setminus \widetilde{C}) \simeq \begin{cases} G(\nu_0; \lambda_0; d/\nu_0) & \text{if } d \geq d', \\ G(\lambda_0; \nu_0; d'/\lambda_0) & \text{if } d' \geq d. \end{cases}$$

REMARK 1.2. The conclusions of Theorem 1.1 are still valid if, instead of the conditions (1) and (2), we suppose that there exists an integer j_0, $1 \leq j_0 \leq \ell$, such that the following two conditions are satisfied:

(1′) $\nu_{j_0} > \hat{\lambda}_{i,i+1}$ for any $1 \leq i \leq m - 1$;

(2′) either $f(\delta_{j_0-1})$ or $f(\delta_{j_0})$ is a regular value of g.

Here, $\hat{\lambda}_{i,i+1}$ is the least common multiple of λ_i and λ_{i+1}. This remark is an immediate consequence of the theorem itself and Proposition 2.2 below.

Note that if $i_0 = 1$ or m (respectively, if $j_0 = 1$ or ℓ), then the condition (2) (respectively, the condition (2′)) is always satisfied.

EXAMPLE 1.3. Consider the $\mathbb{R}$-join-type curve C defined by the equation $f(y) = g(x)$, where

$$f(y) = \frac{65536}{9765625}\, y^3(y-1)^3 \quad \text{and} \quad g(x) = -\frac{1}{64}\, x^8(x-1)^2.$$

Clearly, the point $(4/5, 1/2)$ is an outer singularity and the curve is not semi-generic. However, as $\lambda_1 = 8 > \hat{\nu}_{1,2} = 3$, Theorem 1.1 applies, and

$$\pi_1(\mathbb{C}^2 \setminus C) \simeq G(3; 2) \simeq B(3),$$

the braid group on three strings, while

$$\pi_1(\mathbb{P}^2 \setminus \widetilde{C}) \simeq G(2;3;5) \simeq \mathbb{Z}_{10}.$$

For the last isomorphism, see Corollary 2.7. Note that $\pi_1(\mathbb{C}^2 \setminus C)$ is not abelian, while $\pi_1(\mathbb{P}^2 \setminus \widetilde{C})$ is. The reason is that the line at infinity is not generic for $\widetilde{C}$.

If we replace the condition (1) in Theorem 1.1 by the condition $[\lambda_{i_0}/2] > \hat{\nu}_{j,j+1}$ for all $1 \leq j \leq \ell - 1$, where $[\lambda_{i_0}/2]$ is the integral part of $\lambda_{i_0}/2$, then the conclusions of the theorem are still valid, even if the condition (2) is not fulfilled. This is stated, more precisely, in the following theorem.

Theorem 1.4. *Again, let C be an $\mathbb{R}$-join-type curve in $\mathbb{C}^2$ defined by the equation $f(y) = g(x)$, where f and g are as in* (1.1), *and let $\hat{\nu}_{j,j+1}$ be the least common multiple of ν_j and ν_{j+1}. Suppose that there exists an integer i_0, $1 \leq i_0 \leq m$, such that*:

$$[\lambda_{i_0}/2] > \hat{\nu}_{j,j+1} \text{ for any } 1 \leq j \leq \ell - 1. \tag{1.2}$$

Then, the conclusions of Theorem 1.1 *still hold true. That is,*

$$\pi_1(\mathbb{C}^2 \setminus C) \simeq G(\nu_0; \lambda_0),$$

and

$$\pi_1(\mathbb{P}^2 \setminus \widetilde{C}) \simeq \begin{cases} G(\nu_0; \lambda_0; d/\nu_0) & \text{if } d \geq d', \\ G(\lambda_0; \nu_0; d'/\lambda_0) & \text{if } d' \geq d. \end{cases}$$

As in Remark 1.2, observe that the conclusions of Theorem 1.4 are still valid if, instead of the condition (1.2), we suppose that there exists an integer j_0, $1 \leq j_0 \leq \ell$, such that:

$$[\nu_{j_0}/2] > \hat{\lambda}_{i,i+1} \text{ for any } 1 \leq i \leq m - 1. \tag{1.3}$$

Example 1.5. Consider polynomials of the form

$$f(y) = a(y - \beta_1)^2(y - \beta_2)(y - \beta_3)^2(y - \beta_4)(y - \beta_5),$$
$$g(x) = b(x - \alpha_1)^2(x - \alpha_2)(x - \alpha_3)^7(x - \alpha_4).$$

Choose the coefficients $a, b, \alpha_1, \ldots, \alpha_4, \beta_1, \ldots, \beta_5$ so that there exist $\gamma_1, \ldots, \gamma_3$ and $\delta_1, \ldots, \delta_4$, with $\alpha_i < \gamma_i < \alpha_{i+1}$ and $\beta_j < \delta_j < \beta_{j+1}$, such that

$$g'(\gamma_i) = f'(\delta_j) = 0 \quad (1 \leq i \leq 3,\ 1 \leq j \leq 4),$$
$$g(\gamma_1) = g(\gamma_3) = f(\delta_1) = f(\delta_4) > 0,$$
$$g(\gamma_2) = f(\delta_2) = f(\delta_3) < 0.$$

The existence of such a polynomial is guaranteed by [**9**]. Then, the $\mathbb{R}$-join-type curve C defined by the equation $f(y) = g(x)$ is not semi-generic and neither Theorem 1.1 nor

Remark 1.2 apply. However, for $i_0 = 3$, we have $\lambda_{i_0} = 7$, and

$$[\lambda_{i_0}/2] = 3 > \hat{\nu}_{j,j+1} \text{ for any } 1 \leq j \leq 4.$$

Thus, Theorem 1.4 applies, and

$$\pi_1(\mathbb{C}^2 \setminus C) \simeq G(\nu_0; \lambda_0) = G(1; 1) \simeq \mathbb{Z},$$

while

$$\pi_1(\mathbb{P}^2 \setminus \widetilde{C}) \simeq G(\lambda_0; \nu_0; \deg(g)/\lambda_0) = G(1; 1; 11) \simeq \mathbb{Z}_{11}.$$

Remark 1.6. The conclusions of Theorem 1.4 are not true without any assumption on the exponents. For example, suppose that $\ell = m \geq 2$, $\nu_j = \lambda_j$, $\beta_j = \alpha_j$ $(1 \leq j \leq \ell)$, $a = b$ and $\nu_0 = 1$. Then, the corresponding join-type curve C is reducible with a line component given by the equation $y = x$, and hence $\pi_1(\mathbb{C}^2 \setminus C) \not\simeq G(1; \lambda_0) \simeq \mathbb{Z}_d$.

Remark 1.7. In [**1**], [**4**], we computed the fundamental groups of all $\mathbb{R}$-join-type curves in the special case where d and d' are both equal to 6 or both equal to 7. Theorem 1.1 and 1.4 above and Theorem 1.1 in [**3**] (the latter concerns the semi-generic case) are complementary to each other. Combined, these three theorems cover many $\mathbb{R}$-join-type curves. However, they do not cover all of them, even in the special cases of sextics and septics treated in [**1**], [**4**].

2. The groups $G(p; q)$ and $G(p; q; r)$.

In this section, we recall the definitions and collect basic properties of the groups $G(p; q)$ and $G(p; q; r)$ introduced in [**6**] and which appear in Theorems 1.1 and 1.4 as the fundamental groups of our curves.

Let p, q, r be positive integers. The group $G(p; q)$ is defined by the presentation

$$\langle\, \omega,\ a_k\ (k \in \mathbb{Z}) \mid \omega = a_{p-1} a_{p-2} \cdots a_0,\ \mathscr{R}_{q,k},\ \mathscr{R}'_{p,k}\ (k \in \mathbb{Z}) \,\rangle, \tag{2.1}$$

where

$$\mathscr{R}_{q,k}\colon a_{k+q} = a_k \text{ (periodicity relation);}$$
$$\mathscr{R}'_{p,k}\colon a_{k+p} = \omega a_k \omega^{-1} \text{ (conjugacy relation).}$$

Remark 2.1 (cf. [**6**]). The group $G(p; q)$ is isomorphic to the fundamental group $\pi_1(\mathbb{C}^2 \setminus C_{p,q})$, where $C_{p,q}$ is the curve given by $y^p - x^q = 0$.

The next proposition has already been used in Remark 1.2. From a purely algebraic point of view, this proposition is not obvious. However, it follows immediately from Remark 2.1 above.

Proposition 2.2. *The groups $G(p; q)$ and $G(q; p)$ are isomorphic.*

The proposition below will be very useful to prove Theorems 1.1 and 1.4.

Proposition 2.3 (cf. [**6**]). *The relations* $\mathscr{R}'_{p,k}$ *(*$k \in \mathbb{Z}$*) and* $\omega = a_{p-1}a_{p-2}\cdots a_0$ *imply the following new relation for any* $k \in \mathbb{Z}$*:*

$$\omega = a_k a_{k-1} \cdots a_{k-p+1}.$$

Now, let $q_1, \ldots, q_n$ be positive integers and $G(p; \{q_1, \ldots, q_n\})$ be the group defined by the presentation

$$\langle\, \omega,\, a_k\ (k \in \mathbb{Z}) \mid \omega = a_{p-1}a_{p-2}\cdots a_0,\ \mathscr{R}_{q_i,k},\ \mathscr{R}'_{p,k}\ (1 \le i \le n,\, k \in \mathbb{Z})\,\rangle,$$

where

$$\mathscr{R}_{q_i,k}\colon a_{k+q_i} = a_k.$$

The following proposition will also be useful in the proofs of Theorems 1.1 and 1.4.

Proposition 2.4 (cf. [**6**]). *The group* $G(p; \{q_1, \ldots, q_n\})$ *is isomorphic to the group* $G(p; q_0)$*, where* $q_0 := \gcd(q_1, \ldots, q_n)$*.*

In the same vein, we also have the next result. Let $p_1, \ldots, p_s$ be positive integers, $p_0 := \gcd(p_1, \ldots, p_s)$ and $G(\{p_1, \ldots, p_s\}; q)$ the group defined by the presentation

$$\langle\, \omega,\, a_k\ (k \in \mathbb{Z}) \mid \omega = a_{p_0-1}a_{p_0-2}\cdots a_0,\ \mathscr{R}_{q,k},\ \mathscr{R}'_{p_i,k}\ (1 \le i \le s,\, k \in \mathbb{Z})\,\rangle, \tag{2.2}$$

where

$$\mathscr{R}'_{p_i,k}\colon a_{k+p_i} = \omega_i a_k \omega_i^{-1},$$

with $\omega_i := a_{p_i-1}a_{p_i-2}\ldots a_0$.

Proposition 2.5. *The group* $G(\{p_1, \ldots, p_s\}; q)$ *is isomorphic to the group* $G(p_0; q)$*.*

Proof. First, we show that the relations of the presentation (2.2) imply the relations of the presentation (2.1) with $p = p_0$. By Proposition 2.3, for each $1 \le i \le s$, the relations $\mathscr{R}'_{p_i,k}$ ($k \in \mathbb{Z}$) and $\omega_i = a_{p_i-1}\cdots a_0$ imply

$$\omega_i = a_{k+p_i-1}\cdots a_k \quad (k \in \mathbb{Z}). \tag{2.3}$$

Write $p_0 = k_1p_1 + \cdots + k_sp_s$, where $k_1, \ldots, k_s \in \mathbb{Z}$. Then, still by $\mathscr{R}'_{p_i,k}$,

$$a_{k+p_0} = (\omega_s^{k_s}\cdots\omega_1^{k_1}) \cdot a_k \cdot (\omega_s^{k_s}\cdots\omega_1^{k_1})^{-1},$$

while (2.3) shows

$$\begin{aligned}\omega &= a_{p_0-1}\cdots a_0\\ &= (a_{k_1p_1+\cdots+k_sp_s-1}\cdots a_{k_1p_1+\cdots+k_{s-1}p_{s-1}})\cdots(a_{k_1p_1-1}\cdots a_0)\\ &= \omega_s^{k_s}\cdots\omega_1^{k_1}.\end{aligned}$$

Conversely, let us show that the relations of the presentation (2.1) with $p = p_0$ imply the relations of the presentation (2.2). By Proposition 2.3, the relations $\mathscr{R}'_{p_0,k}$ $(k \in \mathbb{Z})$ and $\omega = a_{p_0-1} \cdots a_0$ imply

$$\omega_i = \omega^{p_i/p_0}.$$

To conclude, it suffices to observe that the relations $\mathscr{R}'_{p_0,k}$ also imply

$$a_{k+p_i} = \omega^{p_i/p_0} \cdot a_k \cdot \omega^{-p_i/p_0}. \qquad \square$$

The group $G(p;q;r)$ is the quotient of $G(p;q)$ by the normal subgroup generated by ω^r. In other words, $G(p;q;r)$ is given by the presentation

$$\langle\, \omega,\ a_k\ (k \in \mathbb{Z}) \mid \omega = a_{p-1}a_{p-2}\cdots a_0,\ \omega^r = e,\ \mathscr{R}_{q,k},\ \mathscr{R}'_{p,k}\ (k \in \mathbb{Z})\,\rangle, \tag{2.4}$$

where e is the unit element.

Theorem 2.6 (cf. [**6**]). *Let $s := \gcd(p,q)$ and $n := \gcd(q/s, r)$. The center of $G(p;q;r)$ contains the cyclic group $\mathbb{Z}_{r/n}$ generated by ω^n, and*

$$\mathbb{Z}_{r/n} \cap D(G(p;q;r)) = \{e\},$$

where $D(G(p;q;r))$ is the commutator subgroup of $G(p;q;r)$. The latter is equivalent to the injectivity of the composition

$$\mathbb{Z}_{r/n} \hookrightarrow G(p;q;r) \to G(p;q;r)\big/D(G(p;q;r)).$$

Furthermore, the quotient group $G(p;q;r)/\mathbb{Z}_{r/n}$ is isomorphic to the free product

$$\mathbb{Z}_{p/s} * \mathbb{Z}_n * \mathbb{F}(s-1),$$

where $\mathbb{F}(s-1)$ is a free group of rank $s-1$.

This theorem has the following corollary which has already been used in Example 1.3.

Corollary 2.7 (cf. [**6**]). *If p, q, r are mutually coprime, then $G(p;q;r) \simeq \mathbb{Z}_{pr}$.*

Proof. By Theorem 2.6, there is a central extension

$$\{e\} \to \mathbb{Z}_r \to G(p;q;r) \to \mathbb{Z}_p \to \{e\},$$

where $\mathbb{Z}_r$ is generated by ω. Therefore, the relations $\mathscr{R}'_{p,k}$ reduce to $a_{k+p} = a_k$, and the group $G(p;q;r)$ is given by

$$\langle\, \omega,\ a_0 \mid \omega = a_0^p,\ \omega^r = e\,\rangle \simeq \mathbb{Z}_{pr}. \qquad \square$$

Necessary and sufficient conditions for the groups $G(p;q)$ and $G(p;q;r)$ to be abelian are also given in [**6**]. These conditions can be used to test the commutativity of the groups

$\pi_1(\mathbb{C}^2 \setminus C)$ and $\pi_1(\mathbb{P}^2 \setminus \widetilde{C})$ which appear in Theorems 1.1 and 1.4.

3. Bifurcation graphs.

To compute the group $\pi_1(\mathbb{C}^2 \setminus C)$ in Theorems 1.1 and 1.4, we use the Zariski–van Kampen theorem with the pencil $\mathscr{P}$ given by the vertical lines $L(\gamma)\colon x = \gamma$, where $\gamma \in \mathbb{C}$ (cf. [**5**], [**7**], [**10**]).[1] This theorem says that

$$\pi_1(\mathbb{C}^2 \setminus C) \simeq \pi_1(L(\gamma_0) \setminus C)/\mathcal{M},$$

where $L(\gamma_0)$ is a generic line of $\mathscr{P}$ and $\mathcal{M}$ is the normal subgroup of $\pi_1(L(\gamma_0) \setminus C)$ generated by the monodromy relations associated with the 'special' lines of $\mathscr{P}$. Here, a line $L(\gamma)$ of $\mathscr{P}$ is called *special* if it meets the curve C at a point (γ, δ) with intersection multiplicity at least 2. This happens if and only if $f(\delta) = g(\gamma)$ and $f'(\delta) = 0$. As we have seen in Section 1, $f'(\delta) = 0$ if and only if $\delta = \delta_j$ for some j, $1 \leq j \leq \ell - 1$, or $\delta = \beta_j$ for some j, $1 \leq j \leq \ell$, such that $\nu_j \geq 2$. Let $\gamma_{j,1}, \ldots, \gamma_{j,d'}$ be the roots of $g(x) = f(\delta_j)$ for $1 \leq j \leq \ell - 1$. If $g'(\gamma_{j,k}) \neq 0$, then $(\gamma_{j,k}, \delta_j)$ is a smooth point of C. As δ_j is a simple root of $f'(y) = 0$, in a small neighbourhood of this point, C is topologically described by

$$(y - \delta_j)^2 = c(x - \gamma_{j,k}),$$

where $c \neq 0$, and the line $x = \gamma_{j,k}$ is tangent to the curve at $(\gamma_{j,k}, \delta_j)$ with intersection multiplicity 2. In particular, this is the case if $\gamma_{j,k} \in \mathbb{C} \setminus \mathbb{R}$, as $g'(x) = 0$ has only real roots. If $g'(\gamma_{j,k}) = 0$, then $(\gamma_{j,k}, \delta_j)$ is an outer singularity. As $\gamma_{j,k}$ is a simple root too, this singularity is necessarily of type $\boldsymbol{A}_1 = \boldsymbol{B}_{2,2}$. Near this point, the curve is topologically equivalent to

$$(y - \delta_j)^2 = c(x - \gamma_{j,k})^2.$$

For each β_j with $\nu_j \geq 2$, the roots of $g(x) = f(\beta_j)$ are $\alpha_1, \ldots, \alpha_m$. If $\lambda_i = 1$, then (α_i, β_j) is a smooth point of C. In a small neighbourhood of it, C is topologically given by

$$(y - \beta_j)^{\nu_j} = c(x - \alpha_i),$$

and the line $x = \alpha_i$ is tangent to C at (α_i, β_j) with intersection multiplicity ν_j. If $\lambda_i \geq 2$, then the point (α_i, β_j) is an inner singularity of type $\boldsymbol{B}_{\lambda_i,\nu_j}$, and in a small neighbourhood of it, the curve is topologically equivalent to

$$(y - \beta_j)^{\nu_j} = c(x - \alpha_i)^{\lambda_i}.$$

The special lines of the pencil $\mathscr{P}$ correspond to certain vertices of a graph called the 'bifurcation graph'. This graph is defined as follows. Let $\mathscr{V}_{\text{crit}}(f)$ (respectively, $\mathscr{V}_{\text{crit}}(g)$) be the set of critical values of f (respectively, of g), and let

$$\mathscr{V}_{\text{crit}} := \mathscr{V}_{\text{crit}}(f) \cup \mathscr{V}_{\text{crit}}(g).$$

[1] Note that this pencil is 'admissible' in the sense of [**7**].

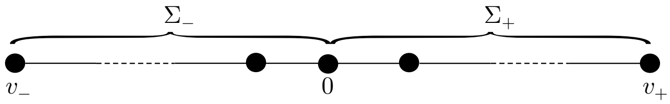

Figure 1. Graph Σ.

If there exists i_0 or j_0 such that $\lambda_{i_0} \geq 2$ or $\nu_{j_0} \geq 2$, then

$$\mathscr{V}_{\text{crit}} = \{0, g(\gamma_1), \dots, g(\gamma_{m-1}), f(\delta_1), \dots, f(\delta_{\ell-1})\}.$$

Otherwise,

$$\mathscr{V}_{\text{crit}} = \{g(\gamma_1), \dots, g(\gamma_{m-1}), f(\delta_1), \dots, f(\delta_{\ell-1})\}.$$

Denote by Σ the bamboo-shaped graph (embedded in the real axis) whose vertices are the points of $\mathscr{V}_{\text{crit}} \cup \{0\}$ (cf. Figure 1). This graph can be decomposed into two connected subgraphs Σ_+ and Σ_-, where Σ_+ (respectively, Σ_-) is the subgraph whose vertices are ≥ 0 (respectively, ≤ 0). Hereafter, we shall denote by $v_+ := \sup\{v \mid v \in \mathscr{V}_{\text{crit}}\}$ and $v_- := \inf\{v \mid v \in \mathscr{V}_{\text{crit}}\}$. The pull-back graph $\Gamma := g^{-1}(\Sigma)$ of Σ by g is called the *bifurcation graph* (*or 'dessin d'enfant'*) *associated with the curve C with respect to g*. Its vertices are the points of the set $g^{-1}(\mathscr{V}_{\text{crit}} \cup \{0\})$.

Observation 3.1. *The special lines $x = \gamma$ of the pencil $\mathscr{P}$ are given by the vertices γ of Γ such that $g(\gamma) \in \mathscr{V}_{\text{crit}}(f)$.*

The bifurcation graph uniquely decomposes as the union of connected subgraphs $\Gamma(\alpha_1), \dots, \Gamma(\alpha_m)$ such that, for $1 \leq i \leq m$, the following properties are satisfied:

(1) $\Gamma(\alpha_i)$ is a star-shaped graph with 'centre' α_i, and with $2\lambda_i$ branches (respectively, λ_i branches) if $v_+ > 0$ and $v_- < 0$ (respectively, if v_+ or v_- is zero);

(2) the restriction of g to $\Gamma(\alpha_i)$ is an λ_i-fold branched covering onto Σ, whose branched locus is $\{0\}$, and $g^{-1}(0) \cap \Gamma(\alpha_i) = \{\alpha_i\}$;

(3) for $i \neq m$, $\Gamma(\alpha_i) \cap \Gamma(\alpha_{i+1}) = \{\gamma_i\}$, and if $g(\gamma_i) \notin \{v_-, v_+\}$, then the branch of $\Gamma(\alpha_i)$ (respectively, $\Gamma(\alpha_{i+1})$) with γ_i as a vertex goes vertically downward (respectively, vertically upward) at γ_i.

The subgraphs $\Gamma(\alpha_i)$ $(1 \leq i \leq m)$ are called the *satellite graphs* of Γ.

Example 3.2. In the special case of Example 1.3 (respectively, Example 1.5), the graphs Σ, Γ and $\Gamma(\alpha_i)$ are as in Figures 2 and 3 (respectively, Figures 4 and 5). In Σ (respectively, in Γ and $\Gamma(\alpha_i)$), the black vertices and the solid lines correspond to the positive branch Σ_+ of Σ (respectively, the part above Σ_+), while the white vertices and the dashed lines correspond to the negative branch Σ_- (respectively, the part above Σ_-). In Γ, the star-style vertices represent the points α_i $(1 \leq i \leq m)$, which are the centres of the satellites. We also use a star-style vertex for $0 = g(\alpha_i) \in \Sigma$. In these two examples, all the vertices γ of the bifurcation graph are such that $g(\gamma) \in \mathscr{V}_{\text{crit}}(f)$, and hence all the vertices correspond to a special line. For more examples, we refer to [**3**].

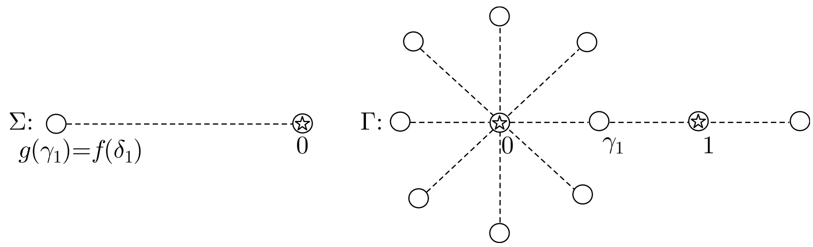

Figure 2. Graphs Σ and Γ of Example 1.3.

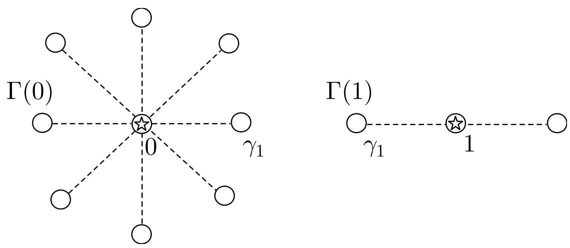

Figure 3. Satellites $\Gamma(0)$ and $\Gamma(1)$ of Example 1.3.

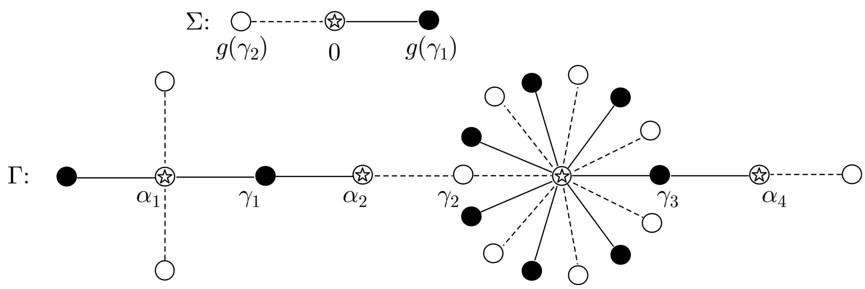

Figure 4. Graphs Σ and Γ of Example 1.5.

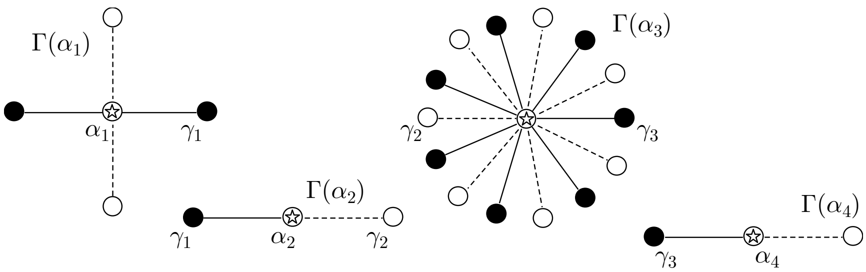

Figure 5. Satellites $\Gamma(\alpha_1), \dots, \Gamma(\alpha_4)$ of Example 1.5.

4. Proof of Theorem 1.1.

It is based on the same pattern as the proof of Theorem 1.3 in [**3**] where the semi-generic case is considered. Hereafter, we shall assume that C is not semi-generic. In this case, the main new difficulty is that all the satellites give rise to at least one outer singularity, so that there is no satellite the branches of which produce only 'tangential'

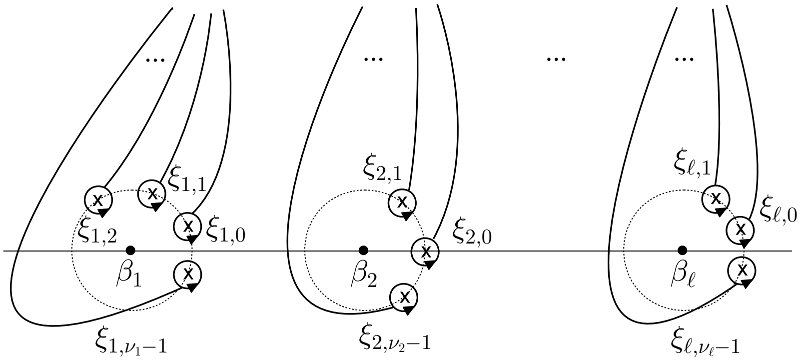

Figure 6. Generators of $\pi_1(L(\alpha_{i_0}^+) \setminus C)$.

monodromy relations—this reduction to tangential relations was a crucial simplification in [**3**]. This 'lack' of tangential relations is compensated by the hypotheses (1) and (2) of the theorem. Here, the key observation is Lemma 4.3.

As mentioned in the previous section, we use the Zariski–van Kampen theorem with the pencil $\mathscr{P}$ given by the vertical lines $L(\gamma)\colon x = \gamma$, where $\gamma \in \mathbb{C}$. We take a sufficiently small positive number ε, and for any real number η, we write $\eta^- := \eta - \varepsilon$ and $\eta^+ := \eta + \varepsilon$. Let i_0 be an integer satisfying the conditions (1) and (2) of the theorem. We consider the generic line $L(\alpha_{i_0}^+)$, and we choose generators

$$\xi_{1,0}, \ldots, \xi_{1,\nu_1-1}, \ldots, \xi_{\ell,0}, \ldots, \xi_{\ell,\nu_\ell-1}$$

of the fundamental group $\pi_1(L(\alpha_{i_0}^+) \setminus C)$ as in Figure 6. (In the figure, we do not respect the numerical scale; we even zoom on the part that collapses to β_j when $\varepsilon \to 0$.) Here, the loops ξ_{j,r_j} $(1 \le j \le \ell,\ 0 \le r_j \le \nu_j - 1)$ are counterclockwise-oriented *lassos* around the intersection points of $L(\alpha_{i_0}^+)$ with C. We shall refer to these generators as the *geometric* generators. For $1 \le j \le \ell$, $0 \le r_j \le \nu_j - 1$ and $n \in \mathbb{Z}$, let

$$\omega_j := \xi_{j,\nu_j-1} \cdots \xi_{j,0} \quad \text{and} \quad \xi_{j,n\nu_j+r_j} := \omega_j^n \cdot \xi_{j,r_j} \cdot \omega_j^{-n}.$$

These relations define elements $\xi_{j,k}$ for any $1 \le j \le \ell$ and any $k \in \mathbb{Z}$. (Indeed, any $k \in \mathbb{Z}$ can be written as $k = n\nu_j + r_j$, with $n \in \mathbb{Z}$ and $0 \le r_j \le \nu_j - 1$.) It is easy to see that

$$\xi_{j,n\nu_j+r} = \omega_j^n \cdot \xi_{j,r} \cdot \omega_j^{-n} \quad \text{for} \quad 1 \le j \le \ell \text{ and } n, r \in \mathbb{Z}. \tag{4.1}$$

As usual, to find the monodromy relations associated with the special lines of the pencil $\mathscr{P}$, we consider a 'standard' system of counterclockwise-oriented generators of the fundamental group $\pi_1(\mathbb{C} \setminus S)$, where S is the set consisting of the vertices α_i $(1 \le i \le m)$ and $\gamma_{j,k}$ $(1 \le j \le \ell - 1,\ 1 \le k \le d')$ in the bifurcation graph Γ. (We recall that the elements $\gamma_{j,k}$ are the roots of the equation $g(x) = f(\delta_j)$, where δ_j is defined as in Section 1.) We choose $\alpha_{i_0}^+$ as base point, and we denote these generators by $\sigma(\alpha_i)$ and $\sigma(\gamma_{j,k})$. Then, $\sigma(\alpha_i)$ (respectively, $\sigma(\gamma_{j,k})$) is a loop in $\mathbb{C} \setminus S$ surrounding the vertex α_i (respectively, $\gamma_{j,k}$). It is based at $\alpha_{i_0}^+$ and it runs along the edges of Γ avoiding the

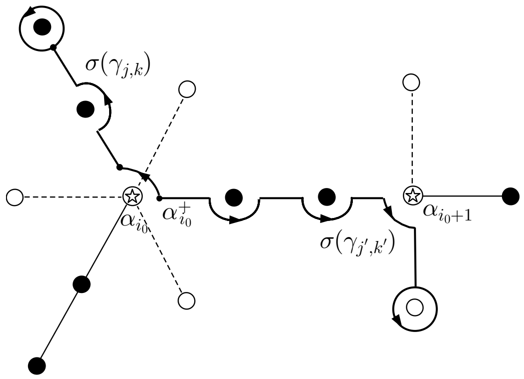

Figure 7. Example of standard generators of $\pi_1(\mathbb{C}\setminus S)$.

vertices corresponding to special lines (cf. Figure 7). The monodromy relations around the special line $L(\alpha_i)$ (respectively, $L(\gamma_{j,k})$) are obtained by dragging the generic fibre $L(\alpha_{i_0}^+)\setminus C$ isotopically along the loop $\sigma(\alpha_i)$ (respectively, $\sigma(\gamma_{j,k})$) and by identifying each generator ξ_{j,r_j} $(1\le j\le \ell,\ 0\le r_j\le \nu_j-1)$ of the group $\pi_1(L(\alpha_{i_0}^+)\setminus C)$ with its image by the terminal homeomorphism of this isotopy. For more details, we refer to [**5**], [**7**], [**10**].

We start with the monodromy relations associated with the special line $L(\alpha_{i_0})$. These relations can be found using the local models

$$y^{\nu_j}=x^{\lambda_{i_0}} \quad \text{for} \quad 1\le j\le \ell.$$

Precisely, if we write $\lambda_{i_0}=n_j\nu_j+r_j$, where $n_j\in\mathbb{Z}$ and $0\le r_j\le \nu_j-1$, they are given by

$$\begin{cases} \xi_{j,0}=\omega_j^{n_j}\cdot \xi_{j,r_j}\cdot \omega_j^{-n_j} \\ \xi_{j,1}=\omega_j^{n_j}\cdot \xi_{j,r_j+1}\cdot \omega_j^{-n_j} \\ \cdots \\ \xi_{j,\nu_j-(r_j+1)}=\omega_j^{n_j}\cdot \xi_{j,\nu_j-1}\cdot \omega_j^{-n_j} \\ \xi_{j,\nu_j-r_j}=\omega_j^{n_j+1}\cdot \xi_{j,0}\cdot \omega_j^{-(n_j+1)} \\ \cdots \\ \xi_{j,\nu_j-1}=\omega_j^{n_j+1}\cdot \xi_{j,r_j-1}\cdot \omega_j^{-(n_j+1)} \end{cases}$$

By (4.1), these relations can be written, more concisely, as

$$\xi_{j,k_j}=\omega_j^{n_j}\cdot \xi_{j,k_j+r_j}\cdot \omega_j^{-n_j}=\xi_{j,k_j+\lambda_{i_0}} \quad \text{for} \quad 1\le j\le \ell \text{ and } 0\le k_j\le \nu_j-1.$$

In fact, (4.1) shows that

$$\xi_{j,k}=\xi_{j,k+\lambda_{i_0}} \quad \text{for} \quad 1\le j\le \ell \text{ and } k\in\mathbb{Z}. \tag{4.2}$$

Remark 4.1. If $\nu_j=1$ for all $1\le j\le \ell$, then $L(\alpha_{i_0})$ is not a special line, and hence, the corresponding monodromy relations are trivial. However, it is clear that the relations (4.2) remain valid, as, in this case, $\xi_{j,k} = \xi_{j,0}$ for all $k \in \mathbb{Z}$.

Next, we look for the monodromy relations along the branches of the satellite $\Gamma(\alpha_{i_0})$. For simplicity, we shall assume $v_- < 0$ and $v_+ > 0$, so that $\Gamma(\alpha_{i_0})$ has $2\lambda_{i_0}$ branches. (The proof can be easily adapted if v_- or v_+ is zero.) For $0 \leq q \leq 2\lambda_{i_0} - 1$, we denote by $B_{i_0,q}$ the q-th branch of $\Gamma(\alpha_{i_0})$. We suppose that the branches $B_{i_0,2q}$ (respectively, $B_{i_0,2q+1}$), $0 \leq q \leq \lambda_{i_0} - 1$, correspond to the positive part Σ_+ (respectively, the negative part Σ_-) of Σ through the correspondence $\Gamma(\alpha_{i_0}) \to \Sigma$ given by the restriction of g. We also suppose that $B_{i_0,0}$ (respectively, $B_{i_0,1}$) contains the line segment $[\alpha_{i_0}, \gamma_{i_0}]$ if $g(\gamma_{i_0}) > 0$ (respectively, if $g(\gamma_{i_0}) < 0$). For simplicity, hereafter, we shall assume $g(\gamma_{i_0}) > 0$. (The argument is similar in the case $g(\gamma_{i_0}) < 0$.) For $0 \leq q \leq \lambda_{i_0} - 1$, let $\alpha_{i_0,2q}$ (respectively, $\alpha_{i_0,2q+1}$) be the unique point of $g^{-1}(g(\alpha_{i_0}^+)) \cap B_{i_0,2q}$ (respectively, of $g^{-1}(g(\alpha_{i_0}^-)) \cap B_{i_0,2q+1}$). See Figure 8. Of course, for $q = 0$, $\alpha_{i_0,0}$ is nothing but the point $\alpha_{i_0}^+$. Finally, recalling the hypothesis (2) of the theorem, let us suppose, for instance, that $g(\gamma_{i_0-1})$ is a regular value of f and $g(\gamma_{i_0})$ is a critical value of f. (The case where $g(\gamma_{i_0})$ is a regular value of f and $g(\gamma_{i_0-1})$ a critical value of f is similar and left to the reader. Note that if both $g(\gamma_{i_0-1})$ and $g(\gamma_{i_0})$ were regular values of f, then the curve would be semi-generic, which is excluded as the result in this case is already proved in [**3**].)

The following observation will be useful.

Observation 4.2. *For any i ($1 \leq i \leq m$), when x moves on the circle $|x - \alpha_i| = \varepsilon$ by the angle $2\pi/\lambda_i$, the centre of each lasso ξ_{j,r_j} ($1 \leq j \leq \ell$, $0 \leq r_j \leq \nu_j - 1$) turns, up to higher order terms, on the circle $|y - \beta_j| = \varepsilon^{\lambda_i/\nu_j}$ by the angle $2\pi/\nu_j$.*

Pick any index j_0, $1 \leq j_0 \leq \ell - 1$. If $f(\delta_{j_0}) > 0$, then, for each $0 \leq q \leq \lambda_{i_0} - 1$, there exists a unique vertex $\gamma_{i_0,j_0,2q} \in B_{i_0,2q}$ such that $g(\gamma_{i_0,j_0,2q}) = f(\delta_{j_0})$. As the only possible critical point for g among the points

$$\gamma_{i_0,j_0,0}, \gamma_{i_0,j_0,2}, \ldots, \gamma_{i_0,j_0,2\lambda_{i_0}-2}$$

is the point $\gamma_{i_0,j_0,0}$, it follows from Observation 4.2 that the monodromy relation associated with the special line $L(\gamma_{i_0,j_0,2q})$, for $0 < q \leq \lambda_{i_0} - 1$, is given by

$$\xi_{j_0,-q} = \xi_{j_0+1,k_{j_0}-q}, \tag{4.3}$$

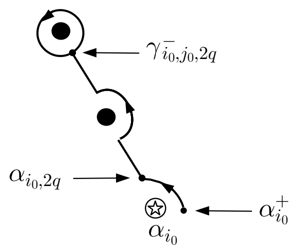

Figure 8. Definitions of $\alpha_{i_0,2q}$ and $\gamma^-_{i_0,j_0,2q}$.

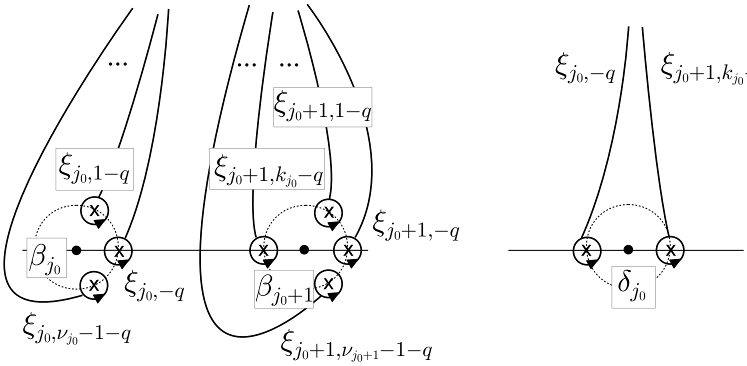

Figure 9. Generators at $x = \alpha_{i_0,2q}$ (left-hand side) and at $x = \gamma^-_{i_0,j_0,2q}$ (right-hand side) when $g(\gamma_{i_0}) > 0$ and $f(\delta_{j_0}) > 0$.

where k_{j_0} is some integer depending only on the first ordering of the elements $\xi_{j_0+1,r_{j_0+1}}$ $(0 \le r_{j_0+1} \le \nu_{j_0+1} - 1)$. See Figure 9. The picture on the left-hand side (respectively, right-hand side) represents the generators in a neighbourhood of β_{j_0} and β_{j_0+1} (respectively, in a neighbourhood of δ_{j_0}) at $x = \alpha_{i_0,2q}$ (respectively, at $x = \gamma^-_{i_0,j_0,2q}$). The complex number $\gamma^-_{i_0,j_0,2q}$ is defined as in Figure 8. For example, if $f(\delta_{j_0}) < g(\gamma_{i_0})$, then $\gamma_{i_0,j_0,0}$ is a real number (so that $\gamma^-_{i_0,j_0,0} := \gamma_{i_0,j_0,0} - \varepsilon$) and $\gamma^-_{i_0,j_0,2q}$ is defined to be the unique point of $g^{-1}(g(\gamma^-_{i_0,j_0,0})) \cap B_{i_0,2q}$. Note that the relation (4.3) is also true for $q = 0$ if $\gamma_{i_0,j_0,0} \neq \gamma_{i_0}$, that is, if $\gamma_{i_0,j_0,0}$ is not a critical point of g.

If $f(\delta_{j_0}) < 0$, then, for each $0 \le q \le \lambda_{i_0} - 1$, there exists a unique vertex $\gamma_{i_0,j_0,2q+1} \in B_{i_0,2q+1}$ such that $g(\gamma_{i_0,j_0,2q+1}) = f(\delta_{j_0})$. As none of the points

$$\gamma_{i_0,j_0,1}, \gamma_{i_0,j_0,3}, \ldots, \gamma_{i_0,j_0,2\lambda_{i_0}-1}$$

is a critical point of g, the monodromy relation associated with the special line $L(\gamma_{i_0,j_0,2q+1})$ is given by

$$\xi_{j_0,h_{j_0}-q} = \xi_{j_0+1,k_{j_0}-q} \tag{4.4}$$

for each $0 \le q \le \lambda_{i_0} - 1$, where h_{j_0} and k_{j_0} are integers depending only on the first ordering of the elements $\xi_{j_0,r_{j_0}}$ $(0 \le r_{j_0} \le \nu_{j_0} - 1)$ and $\xi_{j_0+1,r_{j_0+1}}$ $(0 \le r_{j_0+1} \le \nu_{j_0+1} - 1)$.

Now, by reordering the generators $\xi_{j,k}$ successively for $j = 1, \ldots, \ell$, we can assume that, for any $1 \le j \le \ell - 1$,

$$\xi_{j,k} = \xi_{j+1,k} \quad \text{for} \quad p_j < k \le p_j + \lambda_{i_0} - 1,$$

for some integers $p_1, \ldots, p_{\ell-1}$. These relations, together with (4.2), imply

$$\xi_{j,k} = \xi_{j+1,k} \quad \text{for} \quad k \not\equiv p_j \pmod{\lambda_{i_0}} \tag{4.5}$$

for any $1 \le j \le \ell - 1$. Actually, we are going to prove that the relations (4.5) are true for any $k \in \mathbb{Z}$ (cf. Lemma 4.3). This key observation follows from the relations (4.5) themselves and the hypothesis (1) of the theorem.

LEMMA 4.3. *For any* $1 \leq j \leq \ell - 1$,

$$\xi_{j,k} = \xi_{j+1,k} \quad \text{for} \quad k \in \mathbb{Z}.$$

PROOF. By (4.2) and (4.5), it suffices to show that $\xi_{j,p_j} = \xi_{j+1,p_j}$ for any $1 \leq j \leq \ell - 1$. First of all, observe that (4.1) and Proposition 2.3 imply

$$\omega_j = \xi_{j,k+\nu_j-1} \cdots \xi_{j,k} \tag{4.6}$$

for any $1 \leq j \leq \ell$ and any $k \in \mathbb{Z}$. Now, fix an integer j such that $1 \leq j \leq \ell - 1$. As $\lambda_{i_0} > \hat{\nu}_{j,j+1}$ (hypothesis (1) of the theorem), the relations (4.5) show that

$$\xi_{j,p_j+\hat{\nu}_{j,j+1}} \cdots \xi_{j,p_j+1} = \xi_{j+1,p_j+\hat{\nu}_{j,j+1}} \cdots \xi_{j+1,p_j+1},$$

and hence, by (4.6), we get

$$\omega_j^{\hat{\nu}_{j,j+1}/\nu_j} = \omega_{j+1}^{\hat{\nu}_{j,j+1}/\nu_{j+1}}. \tag{4.7}$$

Besides, by (4.1), we also have

$$\xi_{j,p_j+\hat{\nu}_{j,j+1}} = \omega_j^{\hat{\nu}_{j,j+1}/\nu_j} \cdot \xi_{j,p_j} \cdot \omega_j^{-\hat{\nu}_{j,j+1}/\nu_j} \quad \text{and}$$
$$\xi_{j+1,p_j+\hat{\nu}_{j,j+1}} = \omega_{j+1}^{\hat{\nu}_{j,j+1}/\nu_{j+1}} \cdot \xi_{j+1,p_j} \cdot \omega_{j+1}^{-\hat{\nu}_{j,j+1}/\nu_{j+1}}.$$

Then, as $\xi_{j,p_j+\hat{\nu}_{j,j+1}} = \xi_{j+1,p_j+\hat{\nu}_{j,j+1}}$, the relation (4.7) immediately implies

$$\xi_{j,p_j} = \xi_{j+1,p_j}. \qquad \square$$

Lemma 4.3 tells us that we can take, as generators, the elements

$$\xi_k := \xi_{j_0,k} \quad \text{for} \quad k \in \mathbb{Z}.$$

Then, the relations (4.2) are written as

$$\xi_k = \xi_{k+\lambda_{i_0}} \quad \text{for} \quad k \in \mathbb{Z}, \tag{4.8}$$

and, by (4.1) and Proposition 2.3, we have

$$\xi_{k+\nu_0} = \omega \xi_k \omega^{-1} \quad \text{for} \quad k \in \mathbb{Z}, \tag{4.9}$$

where $\omega := \xi_{\nu_0-1} \cdots \xi_0$.

Now, let us consider the monodromy relations associated with the other satellites. For simplicity, we still assume $g(\gamma_{i_0}) > 0$. We start with the satellite $\Gamma(\alpha_{i_0+1})$ and first look for the relations around the line $L(\alpha_{i_0+1})$. For this purpose, we need to know how the generators are deformed when x moves along the 'modified' line segment $[\alpha_{i_0}^+, \alpha_{i_0+1}^-]$. Here, 'modified' means that x makes a half-turn counterclockwise around each vertex of $\Gamma \cap [\alpha_{i_0}^+, \alpha_{i_0+1}^-]$ corresponding to a special line (cf. Figure 10). Take an element j_0,

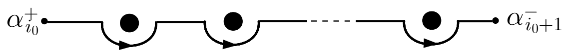

Figure 10. The modified line segment $[\alpha^+_{i_0}, \alpha^-_{i_0+1}]$.

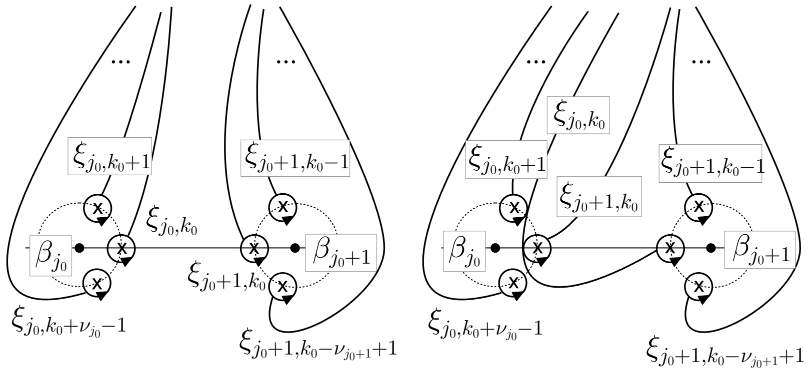

Figure 11. Deformation of the generators when $0 < f(\delta_{j_0}) < g(\gamma_{i_0})$.

$1 \leq j_0 \leq \ell - 1$. If $0 < f(\delta_{j_0}) < g(\gamma_{i_0})$, then there are exactly two vertices $\gamma_{i_0,j_0,0} \neq \gamma_{i_0}$ and $\gamma_{i_0+1,j_0,2q_0} \neq \gamma_{i_0}$ (for some $0 \leq q_0 \leq \lambda_{i_0+1} - 1$) on the line segment $[\alpha^+_{i_0}, \alpha^-_{i_0+1}]$ that correspond to special lines of the pencil associated with the critical value $f(\delta_{j_0})$—that is, $g(\gamma_{i_0,j_0,0}) = f(\delta_{j_0})$ and $g(\gamma_{i_0+1,j_0,2q_0}) = f(\delta_{j_0})$. The first one $\gamma_{i_0,j_0,0}$ is in $\Gamma(\alpha_{i_0})$ and the second one $\gamma_{i_0+1,j_0,2q_0}$ is in $\Gamma(\alpha_{i_0+1})$. Therefore, when x moves along the modified line segment $[\alpha^+_{i_0}, \alpha^-_{i_0+1}]$, the generators are deformed as in Figure 11. The picture on the left-hand side of the figure represents the generators at $x = \alpha^+_{i_0}$ (i.e., before the deformation). The picture on the right-hand side represents the generators at $x = \alpha^-_{i_0+1}$ (i.e., after the deformation). However, by Lemma 4.3, we can suppose that the generators in the fibre $x = \alpha^-_{i_0+1}$ are still the same as in the fibre $x = \alpha^+_{i_0}$. In other words, the picture on the left-hand side of Figure 11 also represents the generators at $x = \alpha^-_{i_0+1}$. Hence, by the same argument as above, the monodromy relations associated with the special line $L(\alpha_{i_0+1})$ give the relations

$$\xi_k = \xi_{k+\lambda_{i_0+1}} \quad \text{for} \quad k \in \mathbb{Z}.$$

We also get the same relations if $f(\delta_{j_0}) = g(\gamma_{i_0})$—that is, if $(\gamma_{i_0}, \delta_{j_0})$ is an outer singularity—or if $g(\gamma_{i_0}) < f(\delta_{j_0})$ or $f(\delta_{j_0}) < 0$. Indeed, in the first case, applying Lemma 4.3 shows that the configuration of the generators is identical on the fibres $x = \alpha^-_{i_0+1}$ and $x = \alpha^+_{i_0}$. It is also identical if $g(\gamma_{i_0}) < f(\delta_{j_0})$ or if $f(\delta_{j_0}) < 0$, as, in these two cases, the set $g^{-1}(f(\delta_{j_0})) \cap [\alpha^+_{i_0}, \alpha^-_{i_0+1}]$ is empty.

The monodromy relations associated with the special lines corresponding to the vertices located on the branches of $\Gamma(\alpha_{i_0+1})$ do not give any new relation. This can be directly shown easily but it is not necessary. In fact, as we shall see below, it suffices to collect the monodromy relations associated with the special lines $L(\alpha_i)$ for all i, $1 \leq i \leq m$. We already know that, for $i = i_0$ and $i_0 + 1$, the monodromy relations

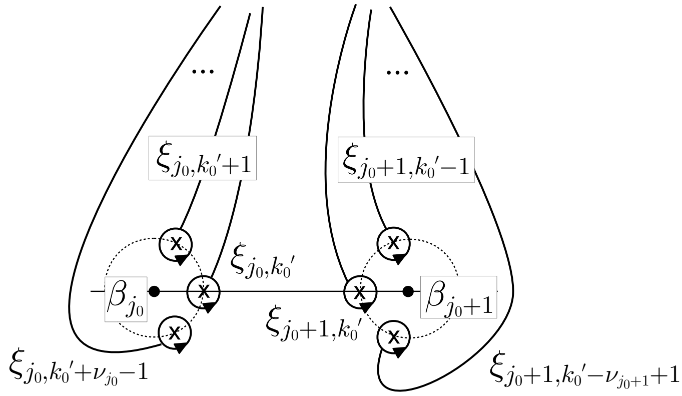

Figure 12. Generators at $x = \alpha^+_{i_0+1}$ and at $x = \alpha^-_{i_0+2}$ when $g(\gamma_{i_0})$, $g(\gamma_{i_0+1})$ and $f(\delta_{j_0})$ are > 0.

around $L(\alpha_i)$ are given by $\xi_k = \xi_{k+\lambda_i}$ for all $k \in \mathbb{Z}$. In fact, this is true for any i. For instance, let us show it for $i = i_0 + 2$. For this purpose, we need to know how the generators are deformed when x makes a half-turn on the circle $|x - \alpha_{i_0+1}| = \varepsilon$ from $\alpha^-_{i_0+1}$ to $\alpha^+_{i_0+1}$, and then moves along the modified line segment $[\alpha^+_{i_0+1}, \alpha^-_{i_0+2}]$. Again, choose an index j_0 with $1 \le j_0 \le \ell - 1$, and, for simplicity, assume that $g(\gamma_{i_0})$, $g(\gamma_{i_0+1})$ and $f(\delta_{j_0})$ are positive. (The other cases are similar.) By Observation 4.2, when x makes a half-turn on the circle $|x-\alpha_{i_0+1}| = \varepsilon$ from $\alpha^-_{i_0+1}$ to $\alpha^+_{i_0+1}$, the generators are deformed as in Figure 12, where $k_0' \in \mathbb{Z}$. That is, the configuration of the generators on the fibre $x = \alpha^+_{i_0+1}$ is just the parallel translation of that on the fibre $x = \alpha^-_{i_0+1}$. Then, as above, by applying Lemma 4.3, when x moves along the modified line segment $[\alpha^+_{i_0+1}, \alpha^-_{i_0+2}]$, we easily see that the generators are still as in Figure 12. It follows that the monodromy relations associated with the special line $L(\alpha_{i_0+2})$ give the relations

$$\xi_k = \xi_{k+\lambda_{i_0+2}} \quad \text{for} \quad k \in \mathbb{Z}.$$

This argument can be repeated for all the other values of i, $1 \le i \le m$, so that the monodromy relations associated with the special line $L(\alpha_i)$ for any i, $1 \le i \le m$, are written as

$$\xi_k = \xi_{k+\lambda_i} \quad \text{for} \quad k \in \mathbb{Z}. \tag{4.10}$$

By Proposition 2.4, the collection of relations (4.10), for $1 \le i \le m$, and the relation (4.9) are equivalent to

$$\begin{cases} \xi_k = \xi_{k+\lambda_0} \\ \xi_{k+\nu_0} = \omega \xi_k \omega^{-1} \end{cases} \quad \text{for} \quad k \in \mathbb{Z}.$$

That is, the fundamental group $\pi_1(\mathbb{C}^2 \setminus C)$ is presented by the generators ξ_k ($k \in \mathbb{Z}$) and ω and by a set of relations that includes the following relations:

$$\omega = \xi_{\nu_0-1} \cdots \xi_0, \tag{4.11}$$
$$\xi_{k+\lambda_0} = \xi_k \quad (k \in \mathbb{Z}), \tag{4.12}$$
$$\xi_{k+\nu_0} = \omega \xi_k \omega^{-1} \quad (k \in \mathbb{Z}). \tag{4.13}$$

In other words, $\pi_1(\mathbb{C}^2 \setminus C)$ is a quotient of the group $G(\nu_0; \lambda_0)$. To show that $\pi_1(\mathbb{C}^2 \setminus C)$ is isomorphic to $G(\nu_0; \lambda_0)$, we consider the family $\{C_t\}_{0 \leq t \ll 1}$ of $\mathbb{R}$-join-type curves, where C_t is defined by the equation

$$f(y) = (1-t)g(x).$$

Clearly, for any $0 < t \ll 1$, the curve C_t has only inner singularities—that is, C_t is generic. Therefore, by the degeneration principle [**8**], [**10**], for any sufficiently small $t > 0$, there is a canonical epimorphism

$$\psi_t \colon \pi_1(\mathbb{C}^2 \setminus C) = \pi_1(\mathbb{C}^2 \setminus C_0) \twoheadrightarrow \pi_1(\mathbb{C}^2 \setminus C_t) \simeq G(\nu_0; \lambda_0).$$

This epimorphism is defined as follows. Take a line L_∞ at infinity, and set $C' := C \cup L_\infty$ and $C'_t := C_t \cup L_\infty$. Pick a sufficiently small regular neighbourhood N of C' in $\mathbb{P}^2$ so that the inclusion

$$\imath \colon \mathbb{P}^2 \setminus N \hookrightarrow \mathbb{P}^2 \setminus C' = \mathbb{C}^2 \setminus C$$

is a homotopy equivalence, and choose a sufficiently small t so that C'_t is contained in N. We may assume that $N \cap L(\alpha^+_{i_0})$ is a copy of d disjoint sufficiently small 2-disks, so that the elements ξ_k $(0 \leq k \leq d-1)$ also give free generators of $\pi_1(L(\alpha^+_{i_0}) \setminus C_t)$ and $\pi_1(L(\alpha^+_{i_0}) \setminus N)$. Then, ψ_t is defined by taking the composition of

$$\imath_\sharp{}^{-1} \colon \pi_1(\mathbb{C}^2 \setminus C) \to \pi_1(\mathbb{P}^2 \setminus N)$$

with the homomorphism induced by the inclusion

$$\mathbb{P}^2 \setminus N \hookrightarrow \mathbb{P}^2 \setminus C'_t = \mathbb{C}^2 \setminus C_t.$$

To distinguish the generators, we write $\xi_k(t)$ $(k \in \mathbb{Z})$ for the generators of $\pi_1(\mathbb{C}^2 \setminus C_t)$, which are represented by the same loops as ξ_k. Note that $\psi_t(\xi_k) = \xi_k(t)$. As C_t is generic, $\pi_1(\mathbb{C}^2 \setminus C_t)$ is presented by the generators $\xi_k(t)$ $(k \in \mathbb{Z})$ and $\omega(t) := \xi_{\nu_0-1}(t) \cdots \xi_0(t)$ and by the relations (4.11)–(4.13), replacing ξ_k by $\xi_k(t)$ and ω by $\omega(t)$. This implies that $\ker \psi_t$ is trivial, and hence

$$\pi_1(\mathbb{C}^2 \setminus C) \simeq G(\nu_0; \lambda_0).$$

(In particular, as announced above, the branches of the satellites $\Gamma(\alpha_i)$, $i \neq i_0$, do not give any new relation.)

As for the fundamental group $\pi_1(\mathbb{P}^2 \setminus \widetilde{C})$, we proceed as follows. If $d \geq d'$, then the base locus of the pencil $X = \gamma Z$ $(\gamma \in \mathbb{C})$ in $\mathbb{P}^2$ does not belong to the curve, and therefore the group $\pi_1(\mathbb{P}^2 \setminus \widetilde{C})$ is obtained from the above presentation of $\pi_1(\mathbb{C}^2 \setminus C)$ by adding the vanishing relation at infinity $\omega_1 \cdots \omega_\ell = e$. By Proposition 2.3, the relations

(4.11) and (4.13) imply $\omega_j = \omega^{\nu_j/\nu_0}$ $(1 \le j \le \ell)$. Therefore, the relation $\omega_1 \cdots \omega_\ell = e$ can also be written as

$$\omega^{d/\nu_0} = e,$$

and hence,

$$\pi_1(\mathbb{P}^2 \setminus \widetilde{C}) \simeq G(\nu_0; \lambda_0; d/\nu_0).$$

If $d' \ge d$, then we consider again the above family $\{C_t\}_{0 \le t \ll 1}$. We use the same regular neighbourhood N and the same isomorphism ψ_t for a sufficiently small $t > 0$. But this time, to compute $\pi_1(\mathbb{C}^2 \setminus C)$, we consider the pencil given by the horizontal lines $y = \delta$, where $\delta \in \mathbb{C}$. We fix a generic line, $y = \delta_0$, and we choose geometric generators ρ_k $(0 \le k \le d'-1)$ as above so that the ρ_k's give generators of the fundamental group of the generic fibre of each complement $\mathbb{P}^2 \setminus N$, $\mathbb{C}^2 \setminus C$ and $\mathbb{C}^2 \setminus C_t$ simultaneously. Then, we define elements τ and ρ_k, for $k \in \mathbb{Z}$, in the same way as we defined the elements ω and ξ_k $(k \in \mathbb{Z})$ above. As ψ_t is an isomorphism and C_t is generic, the generating relations for each group $\pi_1(\mathbb{C}^2 \setminus C_t)$, $\pi_1(\mathbb{C}^2 \setminus C)$ and $\pi_1(\mathbb{P}^2 \setminus N)$ are given by

$$\begin{aligned} &\tau = \rho_{\lambda_0 - 1} \cdots \rho_0, \\ &\rho_{k+\nu_0} = \rho_k \quad (k \in \mathbb{Z}), \\ &\rho_{k+\lambda_0} = \tau \rho_k \tau^{-1} \quad (k \in \mathbb{Z}). \end{aligned}$$

As $d' \ge d$, the base locus of the pencil $Y = \delta Z$ $(\delta \in \mathbb{C})$ in $\mathbb{P}^2$ does not belong to $\widetilde{C}$, and hence the group $\pi_1(\mathbb{P}^2 \setminus \widetilde{C})$ is obtained from the above presentation of $\pi_1(\mathbb{C}^2 \setminus C)$ by adding the vanishing relation at infinity $\tau^{d'/\lambda_0} = e$. Finally, we get

$$\pi_1(\mathbb{P}^2 \setminus \widetilde{C}) \simeq G(\lambda_0; \nu_0; d'/\lambda_0).$$

This completes the proof of Theorem 1.1.

5. Proof of Theorem 1.4.

It is similar to the proof of Theorem 1.1 except that, in the present case, both $g(\gamma_{i_0-1})$ and $g(\gamma_{i_0})$ may be critical values of f. In this case, the satellite $\Gamma(\alpha_{i_0})$ gives rise to two outer singularities instead of one. (Note that a satellite can give at most two such singularities.) The proof therefore requires a special attention when reading the monodromy relations along the branches of $\Gamma(\alpha_{i_0})$. In the present case, the lack of tangential relations is compensated by the hypothesis (1.2) of the theorem. Here, the key observation is Lemma 5.1—counterpart of Lemma 4.3.

We consider the generic line $L(\alpha_{i_0}^+)$, and we choose generators

$$\xi_{1,0}, \ldots, \xi_{1,\nu_1-1}, \ldots, \xi_{\ell,0}, \ldots, \xi_{\ell,\nu_\ell-1}$$

of the fundamental group $\pi_1(L(\alpha_{i_0}^+) \setminus C)$ as in Figure 6. As in the proof of Theorem 1.1, for $1 \le j \le \ell$, $0 \le r_j \le \nu_j - 1$ and $n \in \mathbb{Z}$, we set

$$\omega_j := \xi_{j,\nu_j-1}\cdots\xi_{j,0} \quad \text{and} \quad \xi_{j,n\nu_j+r_j} := \omega_j^n \cdot \xi_{j,r_j} \cdot \omega_j^{-n},$$

and we observe that

$$\xi_{j,n\nu_j+r} = \omega_j^n \cdot \xi_{j,r} \cdot \omega_j^{-n} \quad \text{for} \quad 1 \le j \le \ell \text{ and } n, r \in \mathbb{Z}. \tag{5.1}$$

Then, by the same argument as above, we show that the monodromy relations associated with the special line $L(\alpha_{i_0})$ are given by

$$\xi_{j,k} = \xi_{j,k+\lambda_{i_0}} \quad \text{for} \quad 1 \le j \le \ell \text{ and } k \in \mathbb{Z}. \tag{5.2}$$

Now, we look for the monodromy relations along the branches of $\Gamma(\alpha_{i_0})$. As in the proof of Theorem 1.1, we use the notation $B_{i_0,2q}$ (respectively, $B_{i_0,2q+1}$) for the branches of $\Gamma(\alpha_{i_0})$ corresponding to Σ_+ (respectively, Σ_-), and we suppose that $B_{i_0,0}$ (respectively, $B_{i_0,1}$) contains the line segment $[\alpha_{i_0}, \gamma_{i_0}]$ if $g(\gamma_{i_0}) > 0$ (respectively, if $g(\gamma_{i_0}) < 0$). For simplicity, we also assume $v_- < 0$, $v_+ > 0$ and $g(\gamma_{i_0}) > 0$. Pick an index j_0, $1 \le j_0 \le \ell - 1$. If $f(\delta_{j_0}) > 0$, then, for each $0 \le q \le \lambda_{i_0} - 1$, there exists a unique vertex $\gamma_{i_0,j_0,2q} \in B_{i_0,2q}$ such that $g(\gamma_{i_0,j_0,2q}) = f(\delta_{j_0})$. If λ_{i_0} is odd (respectively, even), then, for each $0 < q \le \lambda_{i_0} - 1$ (respectively, for each $0 < q \le \lambda_{i_0} - 1$ such that $q \ne \lambda_{i_0}/2$), the monodromy relation associated with the special line $L(\gamma_{i_0,j_0,2q})$ is given by

$$\xi_{j_0,-q} = \xi_{j_0+1,k_{j_0}-q},$$

where k_{j_0} is an integer depending only on the first ordering of the elements $\xi_{j_0+1,r_{j_0+1}}$ $(0 \le r_{j_0+1} \le \nu_{j_0+1} - 1)$. If $f(\delta_{j_0}) < 0$, then, for each $0 \le q \le \lambda_{i_0} - 1$, there exists a unique vertex $\gamma_{i_0,j_0,2q+1} \in B_{i_0,2q+1}$ such that $g(\gamma_{i_0,j_0,2q+1}) = f(\delta_{j_0})$. If λ_{i_0} is even (respectively, odd), then, for each $0 \le q \le \lambda_{i_0} - 1$ (respectively, for each $0 \le q \le \lambda_{i_0} - 1$ such that $q \ne (\lambda_{i_0} - 1)/2$), the monodromy relation associated with the special line $L(\gamma_{i_0,j_0,2q+1})$ is given by

$$\xi_{j_0,h_{j_0}-q} = \xi_{j_0+1,k_{j_0}-q},$$

where h_{j_0} and k_{j_0} are integers depending only on the first ordering of the elements $\xi_{j_0,r_{j_0}}$ $(0 \le r_{j_0} \le \nu_{j_0} - 1)$ and $\xi_{j_0+1,r_{j_0+1}}$ $(0 \le r_{j_0+1} \le \nu_{j_0+1} - 1)$. Now, by reordering the generators $\xi_{j,k}$ successively for $j = 1, \ldots, \ell$, we can assume that there are integers $p_1, \ldots, p_{\ell-1}$ such that, for any $1 \le j \le \ell - 1$,

$$\xi_{j,k} = \xi_{j+1,k}$$

for all $p_j \le k \le p_j + \lambda_{i_0} - 1$ such that:

- $k \ne p_j$ if λ_{i_0} is odd and $f(\delta_j) > 0$;
- $k \ne p_j,\ p_j + \lambda_{i_0}/2$ if λ_{i_0} is even and $f(\delta_j) > 0$;
- $k \ne p_j + (\lambda_{i_0} - 1)/2$ if λ_{i_0} is odd and $f(\delta_j) < 0$.

Combined with (5.2), these relations imply

$$\xi_{j,k} = \xi_{j+1,k} \quad \text{for all} \quad k \not\equiv p_j,\, p_j + [\lambda_{i_0}/2] \pmod{\lambda_{i_0}} \tag{5.3}$$

for any $1 \le j \le \ell - 1$. In fact, as in the proof of Theorem 1.1, we are going to prove that the relations (5.3) hold for any $k \in \mathbb{Z}$.

LEMMA 5.1. *For any* $1 \le j \le \ell - 1$,

$$\xi_{j,k} = \xi_{j+1,k} \quad \textit{for} \quad k \in \mathbb{Z}.$$

PROOF. First, note that (5.1) and Proposition 2.3 imply

$$\omega_j = \xi_{j,k+\nu_j-1} \cdots \xi_{j,k} \tag{5.4}$$

for any $1 \le j \le \ell$ and any $k \in \mathbb{Z}$. Now, fix an integer j such that $1 \le j \le \ell - 1$. By (5.2) and (5.3), in order to prove the lemma, it suffices to show:

(i) $\xi_{j,p_j} = \xi_{j+1,p_j}$;

(ii) $\xi_{j,p_j+[\lambda_{i_0}/2]} = \xi_{j+1,p_j+[\lambda_{i_0}/2]}$.

The item (i) is obtained exactly as in the proof of Lemma 4.3, replacing the reference to the hypothesis (1) of Theorem 1.1 by a reference to the hypothesis (1.2) of Theorem 1.4.

As for the item (ii), write $[\lambda_{i_0}/2] = n\hat{\nu}_{j,j+1} + r$ with $n, r \in \mathbb{Z}$ and $0 \le r < \hat{\nu}_{j,j+1} < [\lambda_{i_0}/2]$. Then, by (5.1), we have

$$\begin{aligned} &\xi_{j,p_j+[\lambda_{i_0}/2]} = \omega_j^{n\hat{\nu}_{j,j+1}/\nu_j} \cdot \xi_{j,p_j+r} \cdot \omega_j^{-n\hat{\nu}_{j,j+1}/\nu_j} \quad \text{and} \\ &\xi_{j+1,p_j+[\lambda_{i_0}/2]} = \omega_{j+1}^{n\hat{\nu}_{j,j+1}/\nu_{j+1}} \cdot \xi_{j+1,p_j+r} \cdot \omega_{j+1}^{-n\hat{\nu}_{j,j+1}/\nu_{j+1}}. \end{aligned}$$

As $0 \le r < [\lambda_{i_0}/2]$, the item (i) and the relations (5.3) show that

$$\xi_{j,p_j+r} = \xi_{j+1,p_j+r}.$$

Then, the item (ii) follows immediately from the equality

$$\omega_j^{\hat{\nu}_{j,j+1}/\nu_j} = \omega_{j+1}^{\hat{\nu}_{j,j+1}/\nu_{j+1}}.$$

(This latter equality is obtained as in the proof of Lemma 4.3 by combining the hypothesis (1.2) with the relations (5.3).) □

With Lemma 5.1 in hand—which plays a role analogous to that of Lemma 4.3 in the proof of Theorem 1.1—the remaining of the proof of Theorem 1.4 is identical to that of Theorem 1.1.

ACKNOWLEDGEMENTS. The authors thank the referee for valuable comments and suggestions that led to improvements in the paper.

References

[1] C. Eyral and M. Oka, Fundamental groups of join-type sextics via dessins d'enfants, Proc. Lond. Math. Soc. (3), **107** (2013), 76–120.

[2] C. Eyral and M. Oka, Fundamental groups of join-type curves—achievements and perspectives, Proc. Japan Acad. Ser. A Math. Sci., **90** (2014), 43–47.

[3] C. Eyral and M. Oka, On the fundamental groups of non-generic $\mathbb{R}$-join-type curves, In: Bridging algebra, geometry, and topology, Springer Proc. Math. Stat., **96**, Springer, Cham, 2014, pp. 137–157.

[4] C. Eyral and M. Oka, Classification of the fundamental groups of join-type curves of degree seven, J. Math. Soc. Japan, **67** (2015), 663–698.

[5] E. R. van Kampen, On the fundamental group of an algebraic curve, Amer. J. Math., **55** (1933), 255–260.

[6] M. Oka, On the fundamental group of the complement of certain plane curves, J. Math. Soc. Japan, **30** (1978), 579–597.

[7] M. Oka, Two transforms of plane curves and their fundamental groups, J. Math. Sci. Univ. Tokyo, **3** (1996), 399–443.

[8] M. Oka, A survey on Alexander polynomials of plane curves, In: Singularités Franco–Japonaises, Sémin. Congr., **10**, Soc. Math. France, Paris, 2005, pp. 209–232.

[9] R. Thom, L'équivalence d'une fonction différentiable et d'un polynôme, Topology, **3** (1965), suppl. 2, 297–307.

[10] O. Zariski, On the problem of existence of algebraic functions of two variables possessing a given branch curve, Amer. J. Math., **51** (1929), 305–328.

Christophe Eyral
Institute of Mathematics
Polish Academy of Sciences
ul. Śniadeckich 8
00-656 Warsaw, Poland
E-mail: eyralchr@yahoo.com

Mutsuo Oka
Department of Mathematics
Tokyo University of Science
1-3 Kagurazaka, Shinjuku-ku
Tokyo 162-8601, Japan
E-mail: oka@rs.kagu.tus.ac.jp