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Joint universality for Lerch zeta-functions

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Abstract. For $0 < \alpha, \lambda \le 1$, the Lerch zeta-function is defined by $L(s;\alpha,\lambda) := \sum_{n=0}^{\infty} e^{2\pi i \lambda n}(n+\alpha)^{-s}$, where $\sigma > 1$. In this paper, we prove joint universality for Lerch zeta-functions with distinct $\lambda_1, \dots, \lambda_m$ and transcendental α.

1. Introduction and statement of main result.

For $0 < \alpha, \lambda \le 1$, we define the Lerch zeta-function by

$$L(s;\alpha,\lambda) := \sum_{n=0}^{\infty} \frac{e(\lambda n)}{(n+\alpha)^s}, \qquad \sigma > 1,$$

where $e(t) = \exp(2\pi i t)$. When $\lambda = 1$, the function $L(s;\alpha,\lambda)$ reduces to the Hurwitz zeta-function $\zeta(s,a)$. If $\lambda \neq 1$, the Lerch zeta-function $L(s;\alpha,\lambda)$ is analytically continuable to an entire function. However, the Hurwitz zeta-function $\zeta(s,a)$ is extended to a meromorphic function, which has a simple pole at $s = 1$.

In this paper, we show the following joint universality theorem expected by Mishou [**6**, Conjecture 1]. In order to state it, put $D := \{s \in \mathbb{C} : 1/2 < \operatorname{Re} s < 1\}$ and let $\operatorname{meas}\{A\}$ be the Lebesgue measure on $\mathbb{R}$ of the set A.

Theorem 1. *Suppose that $L(s;\alpha,\lambda_1), \dots, L(s;\alpha,\lambda_m)$ are Lerch zeta-functions with distinct $\lambda_1, \dots, \lambda_m$ and transcendental α. For $1 \le j \le m$, let $K_j \subset D$ be compact sets with connected complements and $f_j(s)$ be continuous function on K_j and analytic in the interior of K_j. Then, for every $\varepsilon > 0$, we have*

$$\liminf_{T\to\infty} \frac{1}{T} \operatorname{meas}\Big\{\tau \in [0,T] : \max_{1\le j\le m} \max_{s\in K_j} \big|L(s+i\tau;\alpha,\lambda_j) - f_j(s)\big| < \varepsilon\Big\} > 0.$$

Roughly speaking, this theorem implies that any analytic functions can be simultaneously and uniformly approximated by Lerch zeta-functions with distinct $\lambda_1, \dots, \lambda_m$. The proof will be written in Sections 2 and 3. We skip the detail of the proofs of results appeared in Section 2 since they do not contain essentially new ideas. In Section 3, we prove the denseness lemma using an orthogonality of Dirichlet coefficients of the zeta-functions. The main idea of our proof was recently observed in [**5**] by the authors. However, in the present paper we adopt this approach to completely different kind of zeta-functions without Euler product. It proves the conjecture on joint universality for

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Lerch zeta-functions put forward by Mishou in [**6**] and shows that this idea can be applicable to many collections of zeta and L-functions, which independence relies on some orthogonality property of their coefficients.

Now we look back in the history of the joint universality for Lerch zeta-functions. Laurinčikas showed Theorem 1 with $m = 1$ in [**2**, Theorem] (see also [**3**, Theorem 6.1.1]). Laurinčikas and Matsumoto proved Theorem 1 with the condition that $\lambda_j = k_j/l_j$ are distinct rational numbers satisfying $(k_j, l_j) = 1$ and $0 < k_j \le l_j$ in [**4**, Theorem 1] (see also [**3**, Theorem 6.3.1] or [**6**, Theorem 2]). In [**7**, Theorem 17], Nakamura obtained the joint universality of the Lerch zeta-functions with $\lambda_j = \lambda + k_j/l_j$, where $0 < \lambda \le 1$ and λ_j are distinct in mod 1. The method in the both papers [**4**, Theorem 1] and [**7**, Theorem 17] are based on the observation that

$$e((\lambda_i - \lambda_j)n) = e\left(\frac{k_i\ell_j - k_j\ell_i}{\ell_i\ell_j} n\right)$$

is a $(\ell_i\ell_j)$-th root of unity for each $i \ne j$ and $n \in \mathbb{Z}$ so that

$$|e(\lambda_i n) - e(\lambda_j n)| = |1 - e((\lambda_i - \lambda_j)n)| \ge |1 - e(1/(\ell_i\ell_j))| > 0$$

or $e(\lambda_i n) = e(\lambda_j n)$. Recently, Mishou proved in [**6**, Theorem 4], the joint universality of the Lerch zeta-functions for almost all real numbers λ_j, $1 \le j \le m$ such that $1, \lambda_1, \ldots, \lambda_m$ are linearly independent over $\mathbb{Q}$. His proof is based on some results of discrepancy estimate from uniform distribution theory (see [**6**, Section 2]). Obviously, Theorem 1 of the present paper is not only an improvement of Mishou's result [**6**, Theorem 4] but also the final answer to [**6**, Conjecture 1].

By using Theorem 1, we get the following corollaries. We omit their proofs since they follow from the standard argument (see for example [**3**, Section 7.2]).

Corollary 2. *Let $\alpha \in (0,1]$ be transcendental and $\lambda_1, \ldots, \lambda_m \in (0,1]$ be distinct real numbers. For $N \in \mathbb{N}$ and $1/2 < \sigma < 1$, define the mapping $h\colon \mathbb{R} \to \mathbb{C}^{mN}$ by the formula*

$$\begin{aligned} h(t) := \bigl(&L(\sigma + i\tau; \alpha, \lambda_1), L'(\sigma + i\tau; \alpha, \lambda_1), \ldots, L^{(N-1)}(\sigma + i\tau; \alpha, \lambda_1), \\ &\ldots, L(\sigma + i\tau; \alpha, \lambda_m), L'(\sigma + i\tau; \alpha, \lambda_m), \ldots, L^{(N-1)}(\sigma + i\tau; \alpha, \lambda_m)\bigr). \end{aligned}$$

Then the image of $\mathbb{R}$ is dense in $\mathbb{C}^{mN}$.

Corollary 3. *Let $\alpha \in (0,1]$ be transcendental and $\lambda_1, \ldots, \lambda_m \in (0,1]$ be distinct real numbers. Suppose $N \in \mathbb{N}$ and F_l, $0 \le l \le k$ are continuous functions on $\mathbb{C}^{mN}$ and satisfy*

$$\begin{aligned} \sum_{l=0}^{k} s^l F_l\bigl(&L(s; \alpha, \lambda_1), L'(s; \alpha, \lambda_1), \ldots, L^{(N-1)}(s; \alpha, \lambda_1), \\ &\ldots, L(s; \alpha, \lambda_m), L'(s; \alpha, \lambda_m), \ldots, L^{(N-1)}(s; \alpha, \lambda_m)\bigr) \equiv 0. \end{aligned}$$

Then we have $F_l \equiv 0$ for $0 \le l \le k$.

2. Proof of Theorem 1.

Recall that $D := \{s \in \mathbb{C} : 1/2 < \operatorname{Re} s < 1\}$ and denote by $H(D)$ the space of analytic function on D equipped with the topology of uniform convergence on compacta. Let $\mathfrak{B}(X)$ stand for the class of Borel sets of the space X. Define γ as the unit circle on $\mathbb{C}$, and let $\Omega := \prod_{n=0}^{\infty} \gamma_n$, where $\gamma_n = \gamma$ for all $n \in \mathbb{N}_0$. Denoting by m_H the probability Haar measure on $(\Omega, \mathfrak{B}(\Omega))$, we obtain a probability space $(\Omega, \mathfrak{B}(\Omega), m_H)$. For $\sigma > 1$, we define

$$L(s;\alpha,\lambda;\omega) := \sum_{n=0}^{\infty} \frac{e(\lambda n)\omega(n)}{(n+\alpha)^s}, \qquad \omega(n) \in \gamma.$$

Note that for almost all $\omega \in \Omega$ the series above converges uniformly on compact subsets of D (see for instance [**3**, Lemma 5.2.1]).

Let $H(D)^m := H(D) \times \cdots \times H(D)$. We define a probability measure P_T on $(H(D)^m, \mathfrak{B}(H(D)^m))$ by

$$P_T(A) := \frac{1}{T} \operatorname{meas}\Big\{\tau \in [0,T] : \big(L(s+i\tau;\alpha,\lambda_1), \dots, L(s+i\tau;\alpha,\lambda_m)\big) \in A\Big\},$$

where $A \in \mathfrak{B}(H(D)^m)$. Next define the $H(D)^m$-valued random element $\underline{L}(s;\omega)$ by

$$\underline{L}(s;\omega) := \big(L(s;\alpha,\lambda_1;\omega), \dots, L(s;\alpha,\lambda_m;\omega)\big).$$

Denote by $P_{\underline{L}}$ the distribution of the random element $\underline{L}(s;\omega)$, namely,

$$P_{\underline{L}}(A) := m_H\big\{\omega \in \Omega : \underline{L}(s;\omega) \in A\big\}, \qquad A \in \mathfrak{B}(H(D)^m).$$

Then we have the following limit theorem proved by Matsumoto and Laurinčikas [**4**] (see also [**3**, Theorem 5.3.1] or [**6**, Section 5]).

Proposition 4 ([**4**, Lemma 1]). *Let $0 < \alpha < 1$ be transcendental. Then the probability measure P_T converges weakly to $P_{\underline{L}}$ as $T \to \infty$.*

The proof of the next lemma shall be written in Section 3 since it contains the most novel part of the present paper.

Lemma 5. *The set $\{\underline{L}(s;\omega) : \omega \in \Omega\}$ is dense in $H(D)^m$.*

Recall that the minimal closed set $S_{\mathbf{P}} \subset X$ such that $\mathbf{P}(S_{\mathbf{P}}) = 1$ is called the support of a probability space $(X, \mathfrak{B}(X), \mathbf{P})$. The set $S_{\mathbf{P}}$ consists of all $x \in S$ such that for every neighborhood V of x the inequality $\mathbf{P}(V) > 0$ is satisfied. From Lemma 5 and [**3**, Lemma 6.1.3] or [**9**, Lemma 12.7], the support of the probability measure $P_{\underline{L}}$ is $H(D)^m$. First assume that $h_1(s), \dots, h_m(s) \in H(D)$ are polynomials. Let K_j be the same as in Theorem 1 and Φ be the set of functions $\underline{\varphi} \in H(D)^m$ which satisfy

$$\max_{1\le j\le m} \max_{s\in K_j} \big|\varphi_j(s) - h_j(s)\big| < \varepsilon.$$

From Proposition 4, the definition of support, Portmanteau theorem (see for instance [**9**, Theorem 3.1]) and the fact that the support of $P_{\underline{L}}$ is $H(D)^m$, we have

$$\liminf_{T\to\infty} P_T(\Phi) \geq P_{\underline{L}}(\Phi) > 0.$$

Therefore, we obtain

$$\liminf_{T\to\infty} \frac{1}{T}\,\mathrm{meas}\Big\{\tau \in [0,T] : \max_{1\leq j\leq m}\max_{s\in K_j}\big|L(s+i\tau;\alpha,\lambda_j) - h_j(s)\big| < \varepsilon\Big\} > 0.$$

Hence it suffices to show that polynomials $h_j(s)$ can be replaced by $f_j(s)$ appeared in Theorem 1. It is possible by Mergelyan's theorem which implies that any function $f(s)$ which is continuous on K and analytic in the interior of K, where K is a compact subset with connected complement, is uniformly approximative on K by polynomials. Hence we omit the details since this is easily done by the well-known method (see for example [**3**, p. 129] or [**6**, p. 1125]).

3. Proof of Lemma 5.

Let U be a simply connected smooth Jordan domain such that $\overline{U} \subset D$. Let $B^2(U)$ be the Bergman space of all holomorphic square integrable complex functions with respect to the Lebesgue measure on U with the inner product

$$\langle f, g\rangle = \iint_U f(s)\overline{g(s)}d\sigma dt, \qquad f, g \in H(U).$$

The properties below are well-known (see for instance [**8**]).

Lemma 6 ([**8**, Proposition 7.2.2 and Theorem 7.2.3]). *We have the following.*
(a) *Convergence in $B^2(U)$ implies local uniform convergence on U.*
(b) *$B^2(U)$ is a Hilbert space.*
(c) *The set of polynomials is dense in $B^2(U)$.*

Now let $\mathbb{B}^m := B^2(U) \times \cdots \times B^2(U)$ is the Hilbert space with the inner product given, for $\underline{f} = (f_1, \ldots, f_m) \in H(U)^m$ and $\underline{g} = (g_1, \ldots, g_m) \in H(U)^m$ by

$$\langle \underline{f}, \underline{g}\rangle = \sum_{j=1}^{m} \iint_U f_j(s)\overline{g_j(s)}d\sigma dt.$$

In order to prove Lemma 5, we use (b) of Lemma 6 and the following result appeared, for example, in [**9**].

Lemma 7 ([**9**, Theorem 6.1.16]). *Let H be a complex Hilbert space. Assume that a sequence $v_n \in H$, $n \in \mathbb{N}$ satisfies*
(i) *the series $\sum_n \|v_n\|^2 < \infty$;*
(ii) *for any element $0 \neq g \in H$ the series $\sum_n |\langle v_n, g\rangle|$ is divergent.*
Then the set of convergent series

$$\left\{ \sum_n a_n v_n \in H : |a_n| = 1 \right\}$$

is dense in H.

Let $\underline{g} = (g_1, \ldots, g_m) \in \mathbb{B}^m$ be a non-zero element and put

$$\underline{v_n}(s) := \big(v_n(s;\alpha,\lambda_1), \ldots, v_n(s;\alpha,\lambda_m)\big), \qquad v_n(s;\alpha,\lambda_j) := \frac{e(\lambda_j n)}{(n+\alpha)^s}.$$

Then for $\Delta_j(w) := \iint_U e^{-sw}\overline{g_j(s)}d\sigma dt$, one has

$$\big\langle \underline{v_n}(s), \underline{g}(s) \big\rangle = \sum_{j=1}^{m} e(\lambda_j n)\Delta_j(\log(n+\alpha)).$$

We can see that the condition (i) of Lemma 7 is true since $\overline{U} \subset D$ and

$$\big\langle \underline{v_n}(s), \underline{v_n}(s) \big\rangle = \sum_{j=1}^{m} \iint_U (n+\alpha)^{-s}\overline{(n+\alpha)^{-s}}d\sigma dt \ll \sup_{s\in U}\Big|(n+\alpha)^{-2s}\Big|.$$

The truth of the condition (ii) in Lemma 7 easily follows from the following crucial lemma.

Lemma 8. *Assume that $\underline{g}(s) = (g_1(s), \ldots, g_m(s)) \in \mathbb{B}^m$ is a non-zero element and for $j = 1, \ldots m$, put $\Delta_j(z) := \iint_U e^{-sz}\overline{g_j(s)}d\sigma dt$. Then the following series*

$$\sum_{n=0}^{\infty}\big|e(\lambda_1 n)\Delta_1(\log(n+\alpha)) + \cdots + e(\lambda_m n)\Delta_m(\log(n+\alpha))\big|$$

is divergent.

In order to prove the lemma above, we quote the following.

Lemma 9 ([**5**, Corollary 2.7]). *Let $\|g_j\| \neq 0$ for $1 \le j \le m$. Then for every $A > 0$ and every $x > 1$, there exist sequences $B_1 > \cdots > B_m > 0$, $x_0^{(0)} = x, x_0^{(1)}, \ldots, x_0^{(m)}$ and intervals $I_j \subset [x, x+1]$ of length $|I_j| \ge B_j x^{-2j}$ such that $x_0^{(j)} \in I_j$, $I_{j+1} \subset I_j$, and for all $t \in I_j$ we have*

$$\begin{aligned}\frac{1}{2}\big|\Delta_j(x_0^{(j-1)})\big| + O\left(e^{-Ax}\right) &\le \frac{1}{2}\big|\Delta_j(x_0^{(j)})\big| + O\left(e^{-Ax}\right) \\ &\le \big|\Delta_j(t)\big| \le \big|\Delta_j(x_0^{(j)})\big| + O\left(e^{-Ax}\right). \qquad (1)\end{aligned}$$

Proof of Lemma 8. Without loss of generality, we can assume that g_1 is a non-zero element since $\|\underline{g}\| \neq 0$ implies that at least one of g_j's is a non-zero element.

We shall check the conditions in [**1**, Lemma 3] for $\Delta_1(z)$. Obviously, $\Delta_1(z) \ll e^{C|z|}$ for some positive constant C depending on U. Let σ_1 and σ_2 be real numbers with

$1/2 < \sigma_1 < \sigma_2 < 1$ such that the vertical strip $\sigma_1 < \operatorname{Re} s < \sigma_2$ contains the simply connected smooth Jordan domain U. Then for sufficiently small $\eta = \eta(U) > 0$ and for all complex z with $|\arg(-z)| \le \eta$, we have $|e^{\sigma_2 z}\Delta_1(z)| \ll 1$. Furthermore, Δ_1 is not identically zero. If it is, we have

$$0 = \Delta_1^{(k)}(0) = \iint_U (-s)^k \overline{g_1(s)} d\sigma dt$$

for any nonnegative intger k, which implies that g_1 is orthogonal to every polynomial in $B^2(U)$. So $g_1 = 0$ by (c) of Lemma 6, but it contradicts to the assumption $\|g_1\| \neq 0$. Hence, by [**1**, Lemma 3] we can find a real sequence x_k tending to infinity such that

$$|\Delta_1(x_k)| \gg e^{-\sigma_2 x_k}.$$

Fix k and put $x = x_k$. Hence, by using Lemma 9, we can see that for every $A > 0$ and $x = x_k$, there exist sequences $B_1 > \cdots > B_m > 0$, $x_0^{(0)} = x, x_0^{(1)}, \dots, x_0^{(m)}$ and intervals $I_j \subset [x, x+1]$ of length $|I_j| \ge B_j x^{-2j}$ such that $x_0^{(j)} \in I_j$, $I_{j+1} \subset I_j$, and for all $t \in I_j$, the inequalities (1) holds. Now let $I_m := [y, y + B_m y^{-2m}] \subset [x, x+1]$. Since $I_m \subset I_j$ for every $j = 1, 2, \dots, m$, the inequalities (1) holds also for all $t \in I_m$. In particular, since $x_0^{(0)} = x$, for $t \in I_m$ one has

$$\big|\Delta_1(t)\big| \gg \big|\Delta_1(x_0^{(0)})\big| \gg e^{-\sigma_2 x}. \tag{2}$$

Moreover, for every $j = 1, 2, \dots, m$ we have

$$\big|\Delta_j(t)\big| \ll e^{-\sigma_1 x}, \qquad t \in [x, x+1]. \tag{3}$$

We denote by $\sum_n{}^*$ the sum over integers $n + \alpha \in [e^y, e^{y + B_m y^{-2m}}]$ in order to obtain $\log(n+\alpha) \in I_m$.

First we consider the following sum

$$S_1(x) := \sum_n{}^* \sum_{j=1}^m \big|\Delta_j(\log(n+\alpha))\big|^2.$$

Obviously, it holds that

$$e^{y + y^{-2m}} - e^y = e^y\big(e^{y^{-2m}} - 1\big) = \frac{e^y}{y^{2m}} \sum_{n=0}^{\infty} y^{-2mn} \gg \frac{e^y}{y^{2m}}.$$

Let $A > 0$ be sufficiently large. Then by using (1), (2), $x \le y \le x+1$ and the formula above, we have

$$\begin{aligned} S_1(x) &\gg \sum_n{}^* \sum_{j=1}^m \Big(\big|\Delta_j(x_0^{(j)})\big|^2 + \big|\Delta_j(x_0^{(j)})\big| O(e^{-Ax}) + O(e^{-2Ax})\Big) \\ &\gg \sum_n{}^* \sum_{j=1}^m \Big(\big|\Delta_j(x_0^{(j)})\big|^2 + O(e^{-Ax})\Big) \gg \sum_n{}^* \Bigg(\sum_{j=1}^m \big|\Delta_j(x_0^{(j)})\big|\Bigg)^2 \end{aligned}$$

$$\gg \sum_n{}^{*} e^{-\sigma_2 x} \sum_{j=1}^{m} \left|\Delta_j(x_0^{(j)})\right| \gg \frac{e^{x(1-\sigma_2)}}{x^{2m}} \sum_{j=1}^{m} \left|\Delta_j(x_0^{(j)})\right|.$$

Since the λ_k's are assumed to be distinct in the interval $(0,1]$, it is easy to see that for any $1 \le k \ne l \le m$

$$\phi_{k,l}(t) := \sum_{n \le t} e((\lambda_k - \lambda_l)n) \ll \frac{1}{|1 - e(\lambda_k - \lambda_\ell)|} \ll 1.$$

Similarly to (3), one can easily get the estimation

$$\frac{d}{du}\Delta_j(\log u) = \frac{1}{u}\Delta_j'(\log u) \ll u^{-1-\sigma_1}.$$

From $\overline{\Delta_j(\log u)} = \overline{\langle u^{-s}, g_j(s)\rangle} = \langle u^{-\bar{s}}, \overline{g_j(s)}\rangle$, we obtain

$$\frac{d}{du}\overline{\Delta_j(\log u)} = \frac{1}{u}\iint_U -\bar{s}u^{-\bar{s}} g_j(s) d\sigma dt = \frac{1}{u}\overline{\Delta_j'(\log u)} \ll u^{-1-\sigma_1}.$$

Hence, using partial summation, we have

$$\begin{aligned}
&\sum_{X_1 \le n \le X_2} \sum_{1 \le k \ne l \le m} e((\lambda_k - \lambda_l)n)\Delta_k(\log(n+\alpha))\overline{\Delta_l(\log(n+\alpha))} \\
&\quad = \sum_{1 \le k \ne l \le m} \int_{X_1}^{X_2} \Delta_k(\log(u+\alpha))\overline{\Delta_l(\log(u+\alpha))} d\phi_{k,l}(u) \\
&\quad \ll X_1^{-2\sigma_1} + \sum_{1 \le k \ne l \le m} \int_{X_1}^{X_2} \left| \left(\Delta_k(\log(u+\alpha))\overline{\Delta_l(\log(u+\alpha))}\right)' \right| du \\
&\quad \ll X_1^{-2\sigma_1} + \int_{X_1}^{X_2} \frac{du}{u^{1+2\sigma_1}} \ll X_1^{-2\sigma_1}
\end{aligned}$$

for sufficiently large $X_2 > X_1 > 0$. Thus we obtain

$$S_2(x) := \sum_{1 \le k \ne l \le m} \sum_n{}^{*} e((\lambda_l - \lambda_k)n)\Delta_k(\log(n+\alpha))\overline{\Delta_l(\log(n+\alpha))} \ll e^{-2\sigma_1 x}.$$

We can easily see that

$$\begin{aligned}
S(x) &:= \sum_n{}^{*} \left| e(\lambda_1 n)\Delta_1(\log(n+\alpha)) + \cdots + e(\lambda_m n)\Delta_m(\log(n+\alpha)) \right|^2 \\
&= S_1(x) + S_2(x) \gg \frac{e^{x(1-\sigma_2)}}{x^{2m}} \sum_{j=1}^{m} \left|\Delta_j(x_0^{(j)})\right| + O\left(e^{-2\sigma_1 x}\right)
\end{aligned}$$

when A is sufficiently large. On the other hand, one has

$$S(x) \ll \sum_n{}^{*} \left| \sum_{j=1}^{m} e(\lambda_j n)\Delta_j(\log(n+\alpha)) \right| \sum_{j=1}^{m} \left|\Delta_j(\log(n+\alpha))\right|$$

$$\ll \sum_n{}^* \left| \sum_{j=1}^m e(\lambda_j n)\Delta_j(\log(n+\alpha)) \right| \sum_{j=1}^m \left|\Delta_j(x_0^{(j)})\right| + O(e^{-(A+\sigma_1-1)x}).$$

Hence, dividing the last inequalities by $\sum_{j=1}^m |\Delta_j(x_0^{(j)})|$, we have

$$\sum_n{}^* \left| \sum_{j=1}^m e(\lambda_j n)\Delta_j(\log(n+\alpha)) \right| \gg \frac{e^{x(1-\sigma_2)}}{x^{2m}},$$

since $2\sigma_1 - \sigma_2 > 0$. Thus, the last inequality implies Lemma 8. □

We now prove Lemma 5. Put

$$v_n(s,\omega(n);\alpha,\lambda_j) := \frac{e(\lambda_j n)\omega(n)}{(n+\alpha)^s}, \qquad \omega(n) \in \gamma,$$
$$\underline{v_n}(s,\omega(n)) := \big(v_n(s,\omega(n);\alpha,\lambda_1),\ldots,v_n(s,\omega(n);\alpha,\lambda_m)\big).$$

Recall U be a simply connected smooth Jordan domain such that $\overline{U} \subset D$. Then the set of convergent series

$$\left\{ \sum_n \underline{v_n}(s,\omega(n)) : \omega \in \Omega \right\}$$

is dense in the space $\mathbb{B}^m$ by Lemmas 7 and 8. Thus, for every compact subsets $\mathcal{K}_1,\ldots,\mathcal{K}_m \subset U$, we can find $b(n) \in \gamma$ and $M \in \mathbb{N}$ satisfying

$$\max_{1\le j\le m} \max_{s\in\mathcal{K}_j} \left| \sum_{n=0}^M v_n(s,b(n);\alpha,\lambda_j) - h_j(s) \right| < \frac{\varepsilon}{2},$$
$$\max_{1\le j\le m} \max_{s\in\mathcal{K}_j} \left| \sum_{n>M} v_n(s,b(n);\alpha,\lambda_j) \right| < \frac{\varepsilon}{2}$$

from (a) of Lemma 6 and Lemma 8. The inequality above and the assumption $\overline{U} \subset D$ implies Lemma 5.

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